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Some Methods for Solving Initial and Boundary Value Problems with Nonlocal Conditions

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By

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بسم الله الرحمن الرحيم

﴿ وَعِندَهُ مَفَاتِحُ الْغَيْبِ لَا يَعْلَمُهَا إِلَّا هُوَ وَيَعْلَمُ مَا فِي
الْبَرِّ وَالْبَحْرِ وَمَا تَسْقُطُ مِن وَرَقَةٍ إِلَّا يَعْلَمُهَا وَلَا حَبَّةٍ
فِي ظُلُمَاتِ الْأَرْضِ وَلَا رَطْبٍ وَلَا يَابِسٍ إِلَّا فِي
كِتَابٍ مُّبِينٍ ﴾

صدق الله العظيم

سورة الأنعام الآية ٥٩

الإهــداء

إلى خاتم الانبياء والمرسلين سيدنا ونبينا

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Contents

Abstract

The nonlocal conditions for the boundary or initial value problems appear when values of the function on the boundary or on the initial are connected to values inside the domain. Such problems are known as nonlocal problem.

The aim of this work is to study some types of nonlocal problems.

This study includes the following aspects:

(1) Discuss the existence and uniqueness of the solution with some nonlocal initial value problems for the non-linear ordinary differential equations via some types of fixed point theorems. Also some numerical methods are used to solve special types of nonlocal initial value problems for the non-linear ordinary differential equations.

(2) Give solutions for some types of the nonlocal initial and boundary value problems for linear eigenvalue problems of the ordinary differential equations.

(3) Use some numerical methods to solve the initial-boundary value problem that consists of the one-dimensional hyperbolic and parabolic equations with two nonlocal non-linear integral boundary conditions. These methods depend on Douglas's equation and Crank-Niklson finite difference scheme, Taylor's expansion and some quadrature rules say Simpson's 1/3 rule.

Introduction

In the last decades, the nonlocal initial-boundary value problems have become a rapidly growing area of research. The study of this type of problems is driven not only by a theoretical interest, but also by the fact that several phenomena in engineering, physics and life sciences can be modeled in this way. The nonlocal initial-boundary value problems formulated for the equations of the mathematical physics where instead of the initial or boundary conditions a certain dependence of the values of the unknown function on the boundary on it's values in internal points of the considered domain is given. The problems with the nonlocal boundary conditions are used for the mathematical modeling, for examples pollution processes in rivers, seas, which are caused by sewage.

The nonlocal boundary conditions simulate decreasing of pollution under influence of natural factors of filtration and settling that cause self purification of the environment. Problems with controls of the thermostat, where a controller at one of its ends adds or removes heat, depending upon the temperature registered in another point can be interpreted with a second-order ordinary differential equation subject to discrete nonlocal boundary condition, [3].

Many researchers studied the nonlocal problems, say Chabrowski in 1984, proved the existence and uniqueness of solutions of the nonlocal problem for the linear parabolic equation with the discrete nonlocal boundary condition, [7]. The existence and uniqueness of solutions of the nonlocal problem for the linear elliptic equation with the discrete nonlocal condition investigated by Chabrowski in 1988, [8]. The existence and uniqueness of solutions of the nonlocal problem for the linear parabolic equation with nonlocal initial condition were studied by Dennis in 1992, [13]. The existence of solutions for the one-dimensional heat equation with the nonlocal initial condition, that investigated by converting to a

Fredholm integral equation studied by Olmstead and Roberts in 1997, [28], Pulkina in 1999, used the Schauder fixed point theorem, to prove the existence of the linear second order hyperbolic equation with the linear integral conditions, [30]. Beilin in 2001, proved the existence and uniqueness of the solution for the one-dimensional wave equation with the nonlocal integral condition, [5], the existence and uniqueness of solution for the second order ordinary differential equations with the nonlocal integral boundary condition was proved by George and Tsamatos in 2002, [16]. The solutions of the eigenvalue problem for the one-dimensional ordinary differential operator with the nonlocal integral boundary condition given by Ciupaila and et al. in 2004, [9]. Yongping in 2005 discussed the existence of the positive solution for the second order with the special case of discrete nonlocal boundary value problem, [37]. Paul and Ahmad in 2005, discussed the existence of the positive solutions of the nonlinear n-th order boundary value problem with the special case of discrete nonlocal conditions, [29]. Mehdi and et al. in 2006, used finite difference method to find the solution of the one-dimensional wave equation with the one nonlocal linear integral condition, [24]. Saadatmandi and Dehghan in 2006, used the shifted Legendre technique for solving the one-dimensional wave equation with the one nonlocal linear integral boundary condition, [33]. Mohammad in 2008, studied the existence and uniqueness of special case of the hyperbolic equation with the two nonlocal integral condition, [25]. Li in 2008, [22], Dehghan and Saadatmandi in 2009, [12], used the homotopy perturbation method and the variational iteration method for approximating solutions of the one-dimensional wave equation with the one nonlocal linear integral condition respectively. Svajunas in 2010, used finite difference methods to find the solution of the two-dimensional heat equation with the nonlocal linear integral condition, [35]. Ashyralyev and Necmettin in 2011, used a finite difference scheme for solving the one-dimensional hyperbolic equation with the one nonlocal linear integral boundary condition, [2]. Borhanifar and et al. in 2011, used finite difference scheme to solve the one-dimensional heat equation with the two nonlocal non-linear integral boundary conditions, [6]. Nemati and Ordokhani in 2012, gave the numerical method which depends on the properties of the Chebyshev polynomials of the second kind for solving the one-dimensional wave equation subject to the one nonlocal linear integral boundary condition, [27]. Marasi and et al. in 2012, used the homotopy analysis method for solving the one-dimensional wave equation with the nonlocal integral condition, [23].

The aim of this work is to study special types of the nonlocal initial-boundary value problems. This study includes the existence and uniqueness of these problems and methods for finding the solutions of them.

This thesis consists of three chapters:

In chapter one, Leary-Schauder fixed point theorem, Schauder fixed point theorem and Banach fixed point theorem used to ensure the existence and uniqueness of the discrete nonlocal initial value problem for special types of the non-linear ordinary differential equations. Also some numerical methods are used to solve special types of the discrete nonlocal initial value problems for the non-linear ordinary differential equations.

In chapter two, we discuss the existences of the solutions for some types of the nonlocal linear eigenvalue problems of the ordinary differential equations.

In chapter three, the solutions of the one-dimensional wave equation with the nonlocal linear integral boundary conditions, the one-dimensional hyperbolic equation and the one-dimensional parabolic equation with the two nonlocal non-linear integral boundary conditions are obtained via the numerical method; this method depends on finite difference scheme, Taylor's expansion and quadrature rule which is Simpson 1/3 rule.

Chapter One
Existence and Uniqueness of the Solution for the Nonlocal Initial Value Problems

Introduction:

The initial value problems involving ordinary differential equations arise in physical sciences and applied mathematics. In some of these problems, subsidiary conditions are imposed locally. In some other cases, nonlocal conditions are imposed. It is sometimes better to impose nonlocal conditions since the measurements needed by a nonlocal condition may be more precise than the measurement given by a local condition, [32]. Abdelkader and Radu in 2003, proved the existence of solutions of the nonlocal initial value problems for the first order non-linear ordinary differential equations via Leray-Schauder fixed point theorem, [1].

In this chapter, we discuss the existence and uniqueness of the solutions of the nonlocal initial value problem that consists of n-th order non-linear ordinary differential equation together with discrete nonlocal initial condition by using some fixed point theorems.

This chapter consists of four sections:

In section one, we use Leray-Schauder fixed point theorem, Schauder fixed point theorem and Banach fixed point theorem to prove the existence and uniqueness of the solution of the nonlocal initial value problem that consists of the first order ordinary differential equation.

In sections two and three, we use Schauder and Banach fixed point theorems to prove the existence and uniqueness of the solution of the nonlocal initial value problem that consists of the second and n-th order non-linear ordinary differential equation.

In section four, we use Euler's method and Trapezium method to solve some nonlocal initial value problems for the non-linear ordinary differential equations.

1.1 Existence and Uniqueness of the Solution for the Nonlocal Initial Value Problems for the First Order Ordinary Differential Equations:

In this section we discuss the existence and uniqueness of the solutions for the nonlocal initial value problem that consists of the first order non-linear ordinary differential equation, [1]:

$$u'(x) = f\big(s, u(s)\big), \qquad x \in [0,1] \tag{1.1}$$

together with the homogenous nonlocal initial discrete condition:

$$u(0) + \sum_{k=1}^{m} a_k u(x_k) = 0 \tag{1.2}$$

where $f: [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ and f(x,.) and u(x) are continuous for all $x \in [0,1]$, x_k are given points with $0 < x_1 \le x_2 \le \cdots \le x_m < 1$ and $a_1, a_2, \ldots, a_m$ are real numbers, such that

$$\left[1 + \sum_{k=1}^{m} a_k\right] \neq 0.$$

We start this section by the following lemma.

Lemma (1.1), [15]:

The vector space C[a, b] of the complex-valued continuous functions defined on a closed interval [a, b] is a Banach space with respect to the following norm:

$$\|u\|_{C[a,b]} = \sup_{x\in [a,b]} |u(x)|, \quad u \in C[a,b].$$

Next, we give the following lemma, which appeared in [1], without proof, here we give its proof.

Lemma (1.2), [1]:

The nonlocal initial value problem given by equations (1.1)-(1.2) is equivalent to the integral equation:

$$u(x) = -C \sum_{k=1}^{m} \left[a_k \int_0^{x_k} f\big(s, u(s)\big)\, ds \right] + \int_0^{x} f\big(s, u(s)\big)\, ds, \quad x \in [0,1] \qquad (1.3)$$

where $C = \left[1 + \sum_{k=1}^{m} a_k \right]^{-1}$.

Proof:

Let u be the solution of the nonlocal initial value problem given by equations (1.1)-(1.2). Then by integrating both sides of equation (1.1) from 0 to x, one can get:

$$u(x) = u(0) + \int_0^{x} f(s, u(s))\, ds, \quad x \in [0,1] \qquad (1.4)$$

and by using the homogenous nonlocal initial condition given by equation (1.2), one can have:

$$u(x) = -\sum_{k=1}^{m} a_k u\,(x_k) + \int_0^{x} f(s, u(s))\, ds \qquad (1.5)$$

Then we substitute $x = x_k$ into equation (1.4), to get:

$$u(x_k) = u(0) + \int_0^{x_k} f\big(s, u(s)\big)\, ds, \quad k = 1,2, \dots, m.$$

By substituting the above equation into equation (1.5), one can have:

$$u(x) = -\sum_{k=1}^{m} a_k \left[u(0) + \int_0^{x_k} f\big(s, u(s)\big)\, ds \right] + \int_0^{x} f\big(s, u(s)\big)\, ds.$$

Hence

$$u(0) = -\sum_{k=1}^{m} a_k \left[u(0) + \int_0^{x_k} f\big(s, u(s)\big)\, ds \right].$$

This implies that

$$u(0) = -\frac{1}{[1+\sum_{k=1}^{m} a_k]} \sum_{k=1}^{m} \left[a_k \int_0^{x_k} f\big(s,u(s)\big)\, ds \right]$$
$$= -C \sum_{k=1}^{m} \left[a_k \int_0^{x_k} f\big(s,u(s)\big)\, ds \right].$$

Thus by substituting the above equation into equation (1.4) one can have:

$$u(x) = u(0) + \int_0^{x} f\big(s,u(s)\big)\, ds, \quad x \in [0,1]$$
$$= -C \sum_{k=1}^{m} \left[a_k \int_0^{x_k} f\big(s,u(s)\big)\, ds \right] + \int_0^{x} f\big(s,u(s)\big)\, ds, \quad x \in [0,1].$$

Therefore u is the solution of the integral equation (1.3).

Conversely, let u be the solution of the integral equation (1.3), then by differentiating equation (1.3), one can get equation (1.1). On the other hand, we substitute $x = 0$ and $x = x_i$, $i = 1,2,\ldots,m$ into equation (1.3), to get:

$$u(0) = -C \sum_{k=1}^{m} \left[a_k \int_0^{x_k} f\big(s,u(s)\big)\, ds \right]$$

and

$$a_i u(x_i) = -C a_i \sum_{k=1}^{m} \left[a_k \int_0^{x_k} f\big(s,u(s)\big)\, ds \right] + a_i \int_0^{x_i} f\big(s,u(s)\big)\, ds, \quad i = 1,2,\ldots,m.$$

Hence

$$\sum_{i=1}^{m} a_i u(x_i) = -C \left[\sum_{i=1}^{m} a_i \right] \sum_{k=1}^{m} \left[a_k \int_0^{x_k} f\big(s,u(s)\big)\, ds \right] + \sum_{i=1}^{m} \left[a_i \int_0^{x_i} f\big(s,u(s)\big)\, ds \right]$$
$$= \left[1 - C \sum_{i=1}^{m} a_i \right] \sum_{k=1}^{m} \left[a_k \int_0^{x_k} f\big(s,u(s)\big)\, ds \right]$$
$$= \left[1 - \frac{1}{(1+\sum_{k=1}^{m} a_k)} \sum_{i=1}^{m} a_i \right] \sum_{k=1}^{m} \left[a_k \int_0^{x_k} f\big(s,u(s)\big)\, ds \right].$$

Therefore

$$\sum_{i=1}^{m} a_i u(x_i) = \left[\frac{1}{1+\sum_{k=1}^{m} a_k}\right] \sum_{k=1}^{m} \left[a_k \int_0^{x_k} f\big(s, u(s)\big)\, ds\right]$$

$$= C \sum_{k=1}^{m} \left[a_k \int_0^{x_k} f\big(s, u(s)\big)\, ds\right] = -u(0).$$

Hence u satisfies the homogenous nonlocal initial condition given by equation (1.2). Thus u is a solution of the nonlocal initial value problem given by equations (1.1)-(1.2). ■

Next, the following theorem which is named as Leray-Schauder fixed point theorem and will be used later to ensure the existence of the solutions for the nonlocal initial value problem given by equations (1.1)-(1.2).

Theorem (1.3), (Leray-Schauder Fixed Point Theorem), [10]:

Let E be a Banach space and let $T: E \longrightarrow E$ be a completely continuous operator. Assume that the set

$$M = \{x \in E \mid x = \mu T x \text{ for some } \mu \in [0,1]\}$$

is bounded. Then T has at least one fixed point.

Now, we are in the position to give the following existence theorem; this theorem is appeared literature. Here we give the details of its proof.

Theorem (1.4), [1]:

The nonlocal initial value problem given by equations (1.1)-(1.2) has a solution if the following conditions are satisfied:

(1) $$|f(x,z)| \le \begin{cases} \omega(x, |z|), & x \in [0, x_m] \\ \alpha(x)\beta(|z|), & x \in [x_m, 1] \end{cases}$$

where $\omega: [0, x_m] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing function in its second argument, $\alpha: [x_m, 1] \longrightarrow \mathbb{R}^+$, $\beta: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing functions,

(2) There exist $R_0 > 0$ if $\rho > R_0$ implies

$$\frac{1}{\rho}\int_0^{x_m} \omega(x,\rho)dx < \frac{1}{D} \tag{1.6}$$

and

$$\int_{x_m}^{1} \alpha(x)dx < \int_{R_1}^{\infty} \frac{1}{\beta(z)}dz \tag{1.7}$$

where $D = \left[1 + |C|\sum_{k=1}^{m}|a_k|\right]$ and $R_1 = D\int_0^{x_m} \omega(x, R_0)dx.$

Proof:

Define the operator $T: C[0,1] \longrightarrow C[0,1]$ by

$$Tu(x) = -C\sum_{k=1}^{m}\left[a_k\int_0^{x_k} f(s,u(s))\,ds\right] + \int_0^{x} f(s,u(s))\,ds, \quad x \in [0,1].$$

Let $M = \{u \in C[0,1] |\ u = \mu Tu \ \text{ for some } \mu \in [0,1]\}$, then we must prove M is a bounded set. To do this, let $u \in M$. If $u = 0$ for all $u \in M$, then M = {0} and this implies that M is a bounded set and hence by using Leray-Schauder fixed point theorem. T has a fixed point. This fixed point is a solution of the integral equation (1.3). By using lemma (1.2) this fixed point is a solution of the nonlocal initial value problem given by equations (1.1)-(1.2). On the other hand, if $u \in M$ such that $u \neq 0$. If $u(x) \neq 0$ for some $x \in [0, x_m]$, then

$$|u(x)| = |\mu Tu(x)| = \mu\left|-C\sum_{k=1}^{m}\left[a_k\int_0^{x_k} f(s,u(s))\,ds\right] + \int_0^{x} f(s,u(s))\,ds\right|$$

$$\leq |C|\sum_{k=1}^{m}|a_k|\int_0^{x_k}|f(s,u(s))|\,ds + \int_0^{x}|f(s,u(s))|\,ds$$

$$\leq |C|\sum_{k=1}^{m}|a_k|\int_0^{x_m}|f(s,u(s))|\,ds + \int_0^{x_m}|f(s,u(s))|\,ds.$$

$$\leq \left[1 + |C|\sum_{k=1}^{m}|a_k|\right]\int_0^{x_m}|f(s,u(s))|\,ds$$

$$\leq D\int_0^{x_m} \omega(s,|u(s)|)\,ds, \quad x \in [0, x_m].$$

But $|u(x)| \leq \sup_{x\in [0,x_m]} |u(x)| = \|u\|_{C[0,x_m]}$ and ω is a nondecreasing function in its second argument, therefore

$$|u(x)| \leq D\int_0^{x_m} \omega\big(s, \|u\|_{C[0,x_m]}\big)\, ds$$

and this implies that:

$$\|u\|_{C[0,x_m]} \leq D\int_0^{x_m} \omega\big(s, \|u\|_{C[0,x_m]}\big)\, ds.$$

By dividing the above inequality by $D\|u\|_{C[0,x_m]}$, one can obtain:

$$\frac{1}{D} \leq \frac{1}{\|u\|_{C[0,x_m]}}\int_0^{x_m} \omega\big(s, \|u\|_{C[0,x_m]}\big)\, ds.$$

This contradicts with inequality (1.6), then

$$\|u\|_{C[0,x_m]} \leq R_0 \tag{1.8}$$

If $u(x) = 0$, for all $x \in [0, x_m]$, then one can get inequality (1.8). Moreover, if $u(x) \neq 0$ for some $x \in [x_m, 1]$, then

$$
\begin{aligned}
|u(x)| &= |\mu T u(x)| \\
&\leq |C| \sum_{k=1}^{m} |a_k| \int_0^{x_m} \big|f\big(s, u(s)\big)\big|\, ds + \int_0^{x_m} \big|f\big(s, u(s)\big)\big|\, ds + \int_{x_m}^{x} \big|f\big(s, u(s)\big)\big|\, ds \\
&\leq \left[1 + |C| \sum_{k=1}^{m} |a_k|\right] \int_0^{x_m} \big|f\big(s, u(s)\big)\big|\, ds + \int_{x_m}^{x} \big|f\big(s, u(s)\big)\big|\, ds. \\
&\leq D\int_0^{x_m} \omega(s, |u(s)|)\, ds + \int_{x_m}^{x} \alpha(s)\beta(|u(s)|)\, ds \\
&\leq D\int_0^{x_m} \omega\big(s, \|u\|_{C[0,x_m]}\big)\, ds + \int_{x_m}^{x} \alpha(s)\beta(|u(s)|)\, ds.
\end{aligned}
$$

By using inequality (1.8) and since ω is a nondecreasing function in its second argument, one can obtain:

$$|u(x)| \leq D\int_0^{x_m} \omega(s, R_0)\, ds + \int_{x_m}^{x} \alpha(s)\beta(|u(s)|)\, ds.$$

Let φ be a function that is defined by

$$\varphi(x) = D\int_0^{x_m} \omega(s, R_0)\, ds + \int_{x_m}^{x} \alpha(s)\beta(|u(s)|)\, ds, \quad x \in [x_m, 1], \text{then}$$

$$\varphi'(x) = \alpha(x)\beta(|u(x)|), \quad x \in [x_m, 1].$$

But $|u(x)| \le \varphi(x)$ and since β is a nondecreasing function then

$$\varphi'(x) = \alpha(x)\beta(|u(x)|) \le \alpha(x)\beta(\varphi(x)), \quad x \in [x_m, 1].$$

This implies that

$$\int_{x_m}^{x} \frac{\varphi'(s)}{\beta(\varphi(s))} ds \le \int_{x_m}^{x} \alpha(s)\, ds, \quad x \in [x_m, 1].$$

Let $\varphi(s) = z$, then $\varphi'(s)ds = dz$ and hence the above inequality becomes:

$$\int_{\varphi(x_m)}^{\varphi(x)} \frac{1}{\beta(z)} dz \le \int_{x_m}^{x} \alpha(s)\, ds, \quad x \in [x_m, 1].$$

Hence

$$\int_{R_1=\varphi(x_m)}^{\varphi(x)} \frac{1}{\beta(z)} dz \le \int_{x_m}^{x} \alpha(s)\, ds \le \int_{x_m}^{1} \alpha(s)\, ds.$$

From inequality (1.7) and from the above inequality, we get $\varphi(x)$ must be finite, that is there is $R_2 > 0$, such that:

$$\varphi(x) \le R_2, \quad x \in [x_m, 1].$$

But $|u(x)| \le \varphi(x)$, therefore $|u(x)| \le R_2, \ x \in [x_m, 1]$.

Thus

$$\|u\|_{C[x_m,1]} = \sup_{x\in [x_m,1]} |u(x)| \le R_2 \tag{1.9}$$

If $u(x) = 0$ for some $x \in [x_m, 1]$, then one can get inequality (1.9).

Let R= max $\{R_0, R_2\}$, then from inequalities (1.8)-(1.9), one can have:

$$|u(x)| \le R, \quad x \in [0,1].$$

Hence

$$\|u\|_{C[0,1]} = \sup_{x\in [0,1]} |u(x)| \leq R, \quad u \in M.$$

This implies that M is a bounded set and since T is completely continuous, [11]. By using Leray-Schauder fixed point theorem, T has a fixed point. This fixed point is a solution of the integral equation (1.3). By using lemma (1.2) this fixed point is a solution of the nonlocal initial value problem given by equations (1.1)-(1.2). ■

Next, we give the following theorem. This theorem is named as Schauder fixed point theorem, which is used later to prove the existence of the solutions for the nonlocal initial value problem given by equations (1.1)-(1.2).

Theorem (1.5), (Schauder Fixed Point Theorem), [10]:

Let B be a nonempty, convex, closed and bounded set in a Banach space E and let $T: B \longrightarrow B$ be a completely continuous operator. Then T has at least one fixed point in B.

Now, we are in the position that we can give the following existence theorem. This theorem ensures the existence of the solutions for the nonlocal initial value problem given by equations (1.1)-(1.2) under another types of conditions. This theorem is a simple modification of the ideas that are appeared in [1].

Theorem (1.6), [1]:

Consider the nonlocal initial value problem given by equations (1.1)-(1.2). If the following conditions are satisfied:

(1) There exist $\alpha_1, \alpha_2 > 0$, such that $|f(x,z)| \leq \alpha_1 |z| + \alpha_2, \; z \in \mathbb{R}$.

(2) $D\alpha_1 < 1$, where $D = \left[1 + x_m |C| \sum_{k=1}^{m} |a_k|\right]$

then the nonlocal initial value problem given by equations (1.1)-(1.2) has at least one solution.

Proof:

Define the operator T as in theorem (1.4). Let $R = \frac{D\alpha_2}{1-D\alpha_1}$ and assume $B_R = \{u \in C[0,1] \mid \|u\|_{C[0,1]} \leq R\}$. Then it is easy to check that B_R is a nonempty, closed, convex and bounded subset of $C[0,1]$. Let $u \in B_R$, then

$$
\begin{aligned}
|Tu(x)| &= \left| -C \sum_{k=1}^{m} a_k \int_0^{x_k} f(s,u(s))\,ds + \int_0^{x} f(s,u(s))\,ds \right| \\
&\leq |C| \sum_{k=1}^{m} |a_k| \int_0^{x_k} [\alpha_1 |u(s)| + \alpha_2]\,ds + \int_0^{x} [\alpha_1 |u(s)| + \alpha_2]\,ds \\
&\leq |C| \sum_{k=1}^{m} |a_k| \int_0^{x_m} \left[\alpha_1 \|u\|_{C[0,1]} + \alpha_2\right] ds + \int_0^{1} \left[\alpha_1 \|u\|_{C[0,1]} + \alpha_2\right] ds \\
&\leq |C| \sum_{k=1}^{m} |a_k| \left[\alpha_1 \|u\|_{C[0,1]} + \alpha_2\right] x_m + \left[\alpha_1 \|u\|_{C[0,1]} + \alpha_2\right] \\
&\leq \left[1 + x_m |C| \sum_{k=1}^{m} |a_k|\right] \left[\alpha_1 \|u\|_{C[0,1]} + \alpha_2\right] \\
&\leq D(\alpha_1 R + \alpha_2) = D\left[\alpha_1 \frac{D\alpha_2}{1 - D\alpha_1} + \alpha_2\right] = \frac{D\alpha_2}{1 - D\alpha_1} = R, x \in [0,1].
\end{aligned}
$$

Therefore

$$\|Tu\|_{C[0,1]} \leq R.$$

This implies that $TB_R \subseteq B_R$. Moreover T is completely continuous operator on C[0,1], [11]. So by using Schauder fixed point theorem, T has a fixed point. This fixed point is also a solution of the nonlocal initial value problem given by equations (1.1)-(1.2). ■

Next, we give the following theorem. This theorem is named as Banach fixed point theorem, which is used later to ensure the existence of the unique solution for the nonlocal initial value problem given by equations (1.1)-(1.2).

Recall that T is operator from normed space N to itself is said to be a contraction operator in case there exist a constant $0 < L < 1$, such that for all $x, y \in N$

$$\|f(x) - f(y)\| \leq L\|x - y\|.$$

Theorem (1.7), (Banach Fixed Point Theorem), [10]:

If $T: E \longrightarrow E$ is a contraction operator defined on a Banach space E then T has a unique fixed point in E.

Now, we are in the position that we can give the following existence and uniqueness theorem, the proof of this theorem depends on the ideas that appeared in [1].

Theorem (1.8), [1]:

Consider the nonlocal initial value problem given by equations (1.1)-(1.2). If the following conditions are satisfied:

(1) f satisfy Lipschitz condition with respect to the second argument, that is there exists $L > 0$, such that

$$|f(x,y) - f(x,z)| \leq L|y - z|, \quad x \in [0,1] \text{ and } y,z \in \mathbb{R}.$$

(2) $LD < 1$, where $D = \left[1 + x_m|C|\sum_{k=1}^{m}|a_k|\right]$

then the nonlocal initial value problem given by equations (1.1)-(1.2) has a unique solution.

Proof:

From lemma (1.1), C[0,1] is a Banach space with respect to the following norm:

$$\|u\|_{C[0,1]} = \sup_{x \in [0,1]} |u(x)|, \quad u \in C[0,1].$$

Define the operator T as in theorem (1.4) and let $u, v \in C[0,1]$, then

$$|Tu(x) - Tv(x)| = \left| -C \sum_{k=1}^{m} \left[a_k \int_0^{x_k} f(s,u(s))\, ds \right] + \int_0^{x} f(s,u(s))\, ds \right. +$$

$$\left. C \sum_{k=1}^{m} \left[a_k \int_0^{x_k} f(s,v(s))\, ds \right] - \int_0^{x} f(s,v(s))\, ds \right|.$$

$$\leq |C| \sum_{k=1}^{m} |a_k| \int_0^{x_k} |f(s,u(s)) - f(s,v(s))|\, ds +$$

$$\int_0^{x} |f(s,u(s)) - f(s,v(s))|\, ds$$

$$\leq |C| \sum_{k=1}^{m} |a_k| \int_0^{x_m} L|u(s) - v(s)|\, ds + \int_0^{1} L|u(s) - v(s)|\, ds$$

$$\leq |C| \sum_{k=1}^{m} |a_k| \int_0^{x_m} L \sup_{s\in [0,1]} |u(s) - v(s)|\, ds +$$

$$L \int_0^{1} \sup_{s\in [0,1]} |u(s) - v(s)|\, ds$$

$$\leq L|C| \|u - v\|_{C[0,1]} \left[\sum_{k=1}^{m} |a_k|\, x_m \right] + L\|u - v\|_{C[0,1]}$$

$$\leq \left[1 + x_m |C| \sum_{k=1}^{m} |a_k| \right] L\|u - v\|_{C[0,1]}$$

and this implies that

$$\|Tu - Tv\|_{C[0,1]} \leq \left[1 + x_m |C| \sum_{k=1}^{m} |a_k| \right] L\|u - v\|_{C[0,1]}$$

$$\leq DL\|u - v\|_{C[0,1]}.$$

Since $DL \leq 1$, then T is a contraction operator. By using Banach fixed point theorem, T has a unique fixed point. This fixed point is the unique solution of the nonlocal initial value problem given by equations (1.1)-(1.2). ■

Remark (1.9), [1] :

Consider the non-linear first order ordinary differential equation (1.1) together with the nonhomogenous nonlocal initial condition:

$$u(0)+\sum_{k=1}^{m} a_k u\,(x_k)=\alpha \qquad (1.10)$$

Then the existence and uniqueness of the solutions for this nonlocal initial value problem can be also discussed by using theorems (1.4), (1.6) and (1.8), by transforming the nonhomogenous nonlocal initial condition into homogenous ones. To do this, let

$$y(x)=u(x)-C\alpha,\ \ x\in[0,1].$$

Then

$$y(0)=u(0)-C\alpha$$

and

$$y(x_k)=u(x_k)-C\alpha.$$

Therefore

$$y'(x)=u'(x)=f(x,y(x)),\ x\in[0,1].$$

Therefore the nonhomogenous nonlocal initial discrete condition becomes:

$$y(0)+C\alpha+\sum_{k=1}^{m} a_k\,[y(x_k)+C\alpha]=\alpha$$

This implies that

$$y(0)+\sum_{k=1}^{m} a_k\,y(x_k)=\alpha-C\alpha-\sum_{k=1}^{m} a_k\,C\alpha$$

$$=\alpha-C\alpha\left[1+\sum_{k=1}^{m} a_k\right]=0.$$

1.2 Existence and Uniqueness of the Solution for the Nonlocal Initial Value Problems for the Second Order Ordinary Differential Equations:

In this section we discuss the existence and uniqueness of the solution of the nonlocal initial value problem that consists of the second order non-linear ordinary differential equation:

$$u''(x) = f\big(x, u(x), u'(x)\big), \quad x \in [0,1] \tag{1.11}$$

together with the homogenous nonlocal initial discrete condition:

$$u(0) + \sum_{k=1}^{m} a_k u\,(x_k) = 0 \tag{1.12}$$

and with the homogenous local initial condition:

$$u'(0) = 0 \tag{1.13}$$

where $f: [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ and f(x,.,.) is continuous for all $x \in [0,1]$, x_k are given points with $0 < x_1 \leq x_2 \leq \cdots \leq x_m < 1$ and $a_1, a_2, \dots, a_m$ are real numbers such that:

$$\left[1 + \sum_{k=1}^{m} a_k\right] \neq 0.$$

The following lemma appeared in literature, without proof. Here we give its proof.

Lemma (1.10), [15]:

The vector space C^1[a,b] of the one-time complex-valued continuously differentiable functions defined on a closed interval [a, b] is Banach space with respect to the following norm:

$$\|u\|_{C^1[a,b]} = \max\big\{\|u\|_{C[a,b]}, \|u'\|_{C[a,b]}\big\}, \quad u \in C^1[a,b].$$

Proof:

To prove $\left(C^1[a,b], \|.\|_{C^1[a,b]}\right)$ is a normed space, we must prove the following conditions:

(1) $\|u\|_{C^1[a,b]} \geq 0, u \in C^1[a,b]$. To do this, let $u \in C^1[a,b]$, then $u \in C[a,b]$ and $u' \in C[a,b]$. Therefore $\|u\|_{C^1[a,b]} = \max\{\|u\|_{C[a,b]}, \|u'\|_{C[a,b]}\} \geq 0$.

(2) $\|u\|_{C^1[a,b]} = 0$ iff $u = 0$. To do this, let $u = 0$ then $\|u\|_{C[a,b]} = \|u'\|_{C[a,b]}=0$, thus $\|u\|_{C^1[a,b]} = 0$. Conversely, let $\|u\|_{C^1[a,b]} = 0$, then $\|u\|_{C[a,b]} = \|u'\|_{C[a,b]} = 0$. Therefore $u(x) = u'(x) = 0,\ x \in [a,b]$. This implies that $\operatorname{Re} u(x) = \operatorname{Im} u(x) = 0,\ x \in [a,b]$. Hence $u = 0$.

(3) $\|\lambda u\|_{C^1[a,b]} = |\lambda|\|u\|_{C^1[a,b]},\ u \in C^1[a,b]$. To do this, let $u \in C^1[a,b]$ and $\lambda \in \mathbb{C}$, then

$$\|\lambda u\|_{C^1[a,b]} = \max\{\|\lambda u\|_{C[a,b]}, \|\lambda u'\|_{C[a,b]}\}$$

$$= \max\{|\lambda|\|u\|_{C[a,b]}, |\lambda|\|u'\|_{C[a,b]}\}$$

$$= |\lambda|\max\{\|u\|_{C[a,b]}, \|u'\|_{C[a,b]}\}$$

$$= |\lambda|\|u\|_{C^1[a,b]}.$$

(4) $\|u+g\|_{C^1[a,b]} \leq \|u\|_{C^1[a,b]} + \|g\|_{C^1[a,b]},\ u,g \in C^1[a,b]$. To do this, let $u,g \in C^1[a,b]$, then

$$\|u+g\|_{C[a,b]} \leq \|u\|_{C[a,b]} + \|g\|_{C[a,b]}$$

and

$$\|u'+g'\|_{C[a,b]} \leq \|u'\|_{C[a,b]} + \|g'\|_{C[a,b]}.$$

Therefore

$$\begin{aligned}\|u+g\|_{C^1[a,b]} &= \max\{\|u+g\|_{C[a,b]}, \|u'+g'\|_{C[a,b]}\} \\ &\leq \max\{\|u\|_{C[a,b]} + \|g\|_{C[a,b]}, \|u'\|_{C[a,b]} + \|g'\|_{C[a,b]}\} \\ &\leq \max\{\|u\|_{C[a,b]}, \|u'\|_{C[a,b]}\} + \max\{\|g\|_{C[a,b]}, \|g'\|_{C[a,b]}\} \\ &= \|u\|_{C^1[a,b]} + \|g\|_{C^1[a,b]}.\end{aligned}$$

Thus $\left(C^1[a,b], \|.\|_{C^1[a,b]}\right)$ is a normed space.

To prove the completeness of $C^1[a,b]$, let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence of continuous functions in $C^1[a,b]$. In other words, given any $\epsilon_1 > 0$, we choose N_1 large enough such that for $n, m \geq N_1$ we have:

$$\|u_n - u_m\|_{C^1[a,b]} < \epsilon_1.$$

That is

$$\max\left\{\|u_n - u_m\|_{C[a,b]}, \|u_n' - u_m'\|_{C[a,b]}\right\} < \epsilon_1.$$

Thus

$$\|u_n - u_m\|_{C[a,b]} \leq \max\left\{\|u_n - u_m\|_{C[a,b]}, \|u_n' - u_m'\|_{C[a,b]}\right\} < \epsilon_1$$

and

$$\|u_n' - u_m'\|_{C[a,b]} \leq \max\left\{\|u_n - u_m\|_{C[a,b]}, \|u_n' - u_m'\|_{C[a,b]}\right\} < \epsilon_1.$$

In other words, $\{u_n\}_{n=1}^{\infty}$ and $\{u_n'\}_{n=1}^{\infty}$ are Cauchy sequences in C[a, b]. By using lemma (1.1), there exists $u, g \in C[a,b]$, such that

$$u(x) = \lim_{n\to\infty} u_n(x).$$

and

$$g(x) = \lim_{n\to\infty} u_n'(x).$$

But $u_n \in C[a,b]$, then

$$u_n(x) - u_n(a) = \int_a^x u_n'(s)\,ds.$$

Hence

$$\lim_{n\to\infty}\int_a^x u_n'(s)\,ds = \int_a^x \lim_{n\to\infty} u_n'(s)\,ds = \int_a^x g(s)\,ds.$$

But

$$\lim_{n\to\infty}\int_a^x u_n'(s)\,ds = \lim_{n\to\infty}\{u_n(x) - u_n(a)\} = u(x) - u(a).$$

Therefore

$$u(x) - u(a) = \int_a^x g(s)\,ds$$

and this implies that

$$u'(x) = g(x), \quad x \in [a,b].$$

Thus $u \in C^1[a,b]$. So, given any ϵ_1, ϵ_2, we choose N_1 and N_2 large enough such that we have $n \geq N = \max\{N_1, N_2\}$, we have

$$\|u_n - u\|_{C^1[a,b]} = \max\{\|u_n - u\|_{C[a,b]}, \|u_n' - u'\|_{C[a,b]}\}$$

$$< \epsilon, \quad n \geq N$$

where $\epsilon = \max\{\epsilon_1, \epsilon_2\}$. Therefore the Cauchy sequence $\{u_n\}_{n=1}^{\infty}$ is convergent to $u \in C^1[a,b]$ and this completes the proof. ■

Now, we give the following lemma which is a generalization of lemma (1.2) to be valid for the nonlocal initial value problem given by equations (1.11)-(1.13).

Lemma (1.11):

The nonlocal initial value problem given by equations (1.11)-(1.13) is equivalent to the integral equation:

$$u(x) = -C\sum_{k=1}^{m} a_k \int_0^{x_k} (x_k - s) f\big(s, u(s), u'(s)\big)\, ds + \int_0^{x} (x - s) f\big(s, u(s), u'(s)\big)\, ds, \quad x \in [0,1] \qquad (1.14)$$

where $C = \left[1 + \sum_{k=1}^{m} a_k\right]^{-1}$.

Proof:

Let u be the solution of the nonlocal initial value problem given by equations (1.11)-(1.13). Then by integrating twice both sides of equation (1.11) from 0 to x, one can get:

$$u(x) = u(0) + u'(0)x + \int_0^x \int_0^t f\big(s, u(s), u'(s)\big)\, dsdt$$

$$= u(0) + \int_0^x \int_0^t f\big(s, u(s), u'(s)\big)\, dsdt$$

By using Cauchy's formula, then the above equation becomes:

$$u(x) = u(0) + \int_0^x (x - s) f\big(s, u(s), u'(s)\big)\, ds, \quad x \in [0,1] \tag{1.15}$$

and by using the nonlocal initial condition given by equation (1.12), one can obtain:

$$u(x) = -\sum_{k=1}^{m} a_k u\,(x_k) + \int_0^x (x - s) f\big(s, u(s), u'(s)\big)\, ds \tag{1.16}$$

By substituting $x = x_k$ into equation (1.15), one can have:

$$u(x_k) = u(0) + \int_0^{x_k} (x_k - s) f\big(s, u(s), u'(s)\big)\, ds, \quad k = 0,1, \dots, m.$$

Next, we substitute the above equation into equation (1.16), to get:

$$u(x) = -\sum_{k=1}^{m} a_k \left[u(0) + \int_0^{x_k} (x_k - s) f\big(s, u(s), u'(s)\big)\, ds\right] + \int_0^x (x - s) f\big(s, u(s), u'(s)\big)\, ds.$$

Hence

$$u(0) = -\sum_{k=1}^{m} a_k \left[u(0) + \int_0^{x_k} (x_k - s) f\big(s, u(s), u'(s)\big)\, ds\right]$$

and this implies that

$$u(0) = -C \sum_{k=1}^{m} \left[a_k \int_0^{x_k} (x_k - s) f\big(s, u(s), u'(s)\big)\, ds\right].$$

Thus

$$u(x) = u(0) + \int_0^x (x-s) f\big(s, u(s), u'(s)\big)\, ds$$
$$= -C \sum_{k=1}^{m} \left[a_k \int_0^{x_k} (x_k - s) f\big(s, u(s), u'(s)\big)\, ds \right] + \int_0^x (x-s) f\big(s, u(s), u'(s)\big)\, ds, \quad x \in [0,1].$$

Therefore u is a solution of the integral equation (1.14).

Conversely, let u be the solution of the integral equation (1.14), then by differentiating equation (1.14), one can get:

$$u'(x) = \int_0^x \frac{\partial}{\partial x} (x-s) f\big(s, u(s), u'(s)\big)\, ds$$
$$= \int_0^x f\big(s, u(s), u'(s)\big)\, ds$$

and this implies that $u'(0) = 0$ and

$$u''(x) = f\big(x, u(x), u'(x)\big), \quad x \in [0,1].$$

Thus u is a solution of equation (1.11) which satisfies the local initial condition given by equation (1.13). By substituting $x = x_i$ into equation (1.14), one can obtain:

$$\sum_{i=1}^{m} a_i u(x_i) = -C \sum_{i=1}^{m} a_i \sum_{k=1}^{m} \left[a_k \int_0^{x_k} (x_k - s) f\big(s, u(s), u'(s)\big)\, ds \right] + \sum_{i=1}^{m} \left[a_i \int_0^{x_i} (x_i - s) f\big(s, u(s), u'(s)\big)\, ds \right]$$
$$= \left[1 - C \sum_{i=1}^{m} a_i \right] \sum_{k=1}^{m} \left[a_k \int_0^{x_k} (x_k - s) f\big(s, u(s), u'(s)\big)\, ds \right]$$
$$= C \sum_{k=1}^{m} \left[a_k \int_0^{x_k} (x_k - s) f\big(s, u(s), u'(s)\big)\, ds \right] = -u(0).$$

This implies that u satisfy the homogenous nonlocal initial condition given by equation (1.12). Thus u is a solution of the nonlocal initial value problem given by equations (1.11)-(1.13). ■

Next, we give the existences theorem. This theorem is a generalization of theorem (1.6) to be valid for the nonlocal initial value problem given by equations (1.11)-(1.13).

Theorem (1.12):

Consider the nonlocal initial value problem given by equations (1.11)-(1.13). If the following conditions are satisfied:

(1) There exist $\alpha_1, \alpha_2, \alpha_3 > 0$, such that $|f(x, z_1, z_2)| \leq \alpha_1 |z_1| + \alpha_2 |z_2| + \alpha_3$.

(2) $D(\alpha_1 + \alpha_2) < 2$ and $D \geq 2$, where $D = \left[1 + x_m^2 |C| \sum_{k=1}^{m} |a_k|\right]$

then the nonlocal initial value problem given by equations (1.11)-(1.13) has at least one solution.

Proof:

Let $T: C^1[0,1] \longrightarrow C^1[0,1]$ be an operator that is defined by

$$Tu(x) = -C \sum_{k=1}^{m} a_k \int_0^{x_k} (x_k - s) f\big(s, u(s), u'(s)\big)\, ds + \int_0^{x} (x - s) f\big(s, u(s), u'(s)\big)\, ds, \quad x \in [0,1] \qquad (1.17)$$

Let $R = \frac{D\alpha_3}{2 - D(\alpha_1 + \alpha_2)}$ and assume that $B_R = \{u \in C^1[0,1] \mid \|u\|_{C^1[0,1]} \leq R\}$, then it is easy to check that B_R is a nonempty, closed, convex and bounded subset of $C^1[0,1]$. Let $u \in B_R$, then

$$|Tu(x)| = \left| -C \sum_{k=1}^{m} a_k \int_0^{x_k} (x_k - s) f\big(s, u(s), u'(s)\big)\, ds + \int_0^{x} (x - s) f\big(s, u(s), u'(s)\big)\, ds \right|.$$

Therefore

$$|Tu(x)| \leq |C| \sum_{k=1}^{m} |a_k| \int_0^{x_k} |(x_k - s)| \left|f\big(s, u(s), u'(s)\big)\right| ds + \int_0^{x} |(x - s)| \left|f\big(s, u(s), u'(s)\big)\right| ds$$

$$\leq |C| \sum_{k=1}^{m} |a_k| \int_0^{x_m} |(x_k - s)| \, [\alpha_1 |u(s)| + \alpha_2 |u'(s)| + \alpha_3] ds + \int_0^{1} |(x - s)| \, [\alpha_1 |u(s)| + \alpha_2 |u'(s)| + \alpha_3] ds$$

$$\leq \left[1 + x_m^2 |C| \sum_{k=1}^{m} |a_k|\right] \frac{\left[\alpha_1 \|u\|_{C[0,1]} + \alpha_2 \|u'\|_{C[0,1]} + \alpha_3\right]}{2}$$

$$\leq \left[1 + x_m^2 |C| \sum_{k=1}^{m} |a_k|\right] \frac{\left[(\alpha_1 + \alpha_2) \|u\|_{C^1[0,1]} + \alpha_3\right]}{2}$$

$$\leq \frac{D}{2} [(\alpha_1 + \alpha_2) R + \alpha_3]$$

$$\leq \frac{D}{2} \left[(\alpha_1 + \alpha_2) \frac{D\alpha_3}{2 - D(\alpha_1 + \alpha_2)} + \alpha_3\right] = R.$$

Hence

$$\|Tu\|_{C[0,1]} \leq R.$$

On the other hand,

$$|(Tu)'(x)| = \left|\int_0^{x} f\big(s, u(s), u'(s)\big) \, ds\right|$$

$$\leq \int_0^{x} \left|f\big(s, u(s), u'(s)\big)\right| ds$$

$$\leq \left[\alpha_1 \|u\|_{C[0,1]} + \alpha_2 \|u'\|_{C[0,1]} + \alpha_3\right]$$

$$\leq \left[(\alpha_1 + \alpha_2) \|u\|_{C^1[0,1]} + \alpha_3\right]$$

$$\leq \left[(\alpha_1 + \alpha_2) \frac{D\alpha_3}{2 - D(\alpha_1 + \alpha_2)} + \alpha_3\right] = \frac{R}{2} < R.$$

This implies that

$$\|(Tu)'\|_{C[0,1]} < R$$

and hence

$$\|Tu\|_{C^1[0,1]} \leq R.$$

Therefore $TB_R \subseteq B_R$ and since T is completely continuous, [11]. So by using Schauder fixed point theorem, T has a fixed point. This fixed point is a solution of the nonlocal initial value problem given by equations (1.11)-(1.13). ■

Next, we give the following theorem, which is a generalization of theorem (1.8) to be valid for the nonlocal initial value problem given by equations (1.11)-(1.13).

Theorem (1.13):

Consider the nonlocal initial value problem given by equations (1.11)-(1.13). If the following conditions are satisfied:

(1) f satisfy a Lipschitz condition with respect to the second and third argument , that is there exists $L_i > 0,\ i = 1,2$, such that

$$|f(x, y_1, y_2) - f(x, z_1, z_2)| \leq L_1|y_1 - z_1| + L_2|y_2 - z_2|.$$

(2) $(L_1 + L_2)D < 1,\ \text{ where } D = \left[1 + x_m^2|C|\sum_{k=1}^{m}|a_k|\right].$

Then the nonlocal initial value problem given by equations (1.11)-(1.13) has a unique solution.

Proof:

From lemma (1.10), $C^1[0,1]$ is Banach space with respect to the following norm:

$$\|u\|_{C^1[0,1]} = \max\{\|u\|_{C[0,1]}, \|u'\|_{C[0,1]}\},\ \ u \in C^1[0,1].$$

Define the operator T as in equation (1.17) and let $u, v \in C^1[0,1]$, then

$$|Tu(x) - Tv(x)|$$
$$\leq |C| \sum_{k=1}^{m} |a_k| \int_0^{x_k} |x_k - s| \left| f\big(s, u(s), u'(s)\big) - f\big(s, v(s), v'(s)\big) \right| ds + \int_0^{x} |x - s| \left| f\big(s, u(s), u'(s)\big) - f\big(s, v(s), v'(s)\big) \right| ds$$
$$\leq \left[|C| \sum_{k=1}^{m} |a_k| \int_0^{x_m} |x_k - s| [L_1|u(s) - v(s)| + L_2|u'(s) - v'(s)|] \, ds \right] + \left[\int_0^{1} |x - s| [L_1|u(s) - v(s)| + L_2|u'(s) - v'(s)|] \, ds \right]$$
$$\leq \frac{1}{2} \left[1 + x_m^2 |C| \sum_{k=1}^{m} |a_k| \right] \left[L_1 \|u - v\|_{C[0,1]} + L_2 \|u' - v'\|_{C[0,1]} \right]$$
$$\leq \frac{D}{2} \left[L_1 \|u - v\|_{C^1[0,1]} + L_2 \|u' - v'\|_{C^1[0,1]} \right].$$
$$\leq \frac{D}{2} (L_1 + L_2) \|u - v\|_{C^1[0,1]}$$
$$< D(L_1 + L_2) \|u - v\|_{C^1[0,1]} \qquad (1.18)$$

this implies that

$$\|Tu - Tv\|_{C[0,1]} < D(L_1 + L_2) \|u - v\|_{C^1[0,1]}.$$

Furthermore

$$|(Tu)'(x) - (Tv)'(x)| = \left| \int_0^{x} f\big(s, u(s), u'(s)\big) \, ds - \int_0^{x} f\big(s, v(s), v'(s)\big) \, ds \right|$$
$$\leq \int_0^{x} \left| f\big(s, u(s), u'(s)\big) - f\big(s, v(s), v'(s)\big) \right| ds$$
$$\leq \int_0^{x} [L_1|u(s) - v(s)| + L_2|u'(s) - v'(s)|] \, ds$$
$$\leq \left[L_1 \|u - v\|_{C[0,1]} + L_2 \|u' - v'\|_{C[0,1]} \right]$$
$$\leq \left[(L_1 + L_2) \|u - v\|_{C^1[0,1]} \right]$$

and this implies that

$$\|(Tu)' - (Tv)'\|_{C[0,1]} \le \left[(L_1 + L_2)\|u - v\|_{C^1[0,1]}\right] \qquad (1.19)$$

From inequalities (1.18)-(1.19), one can get:

$$\|Tu - Tv\|_{C^1[0,1]} = \max\left\{D(L_1 + L_2)\|u - v\|_{C^1[0,1]}, (L_1 + L_2)\|u - v\|_{C^1[0,1]}\right\}$$

$$\le D(L_1 + L_2)\|u - v\|_{C^1[0,1]}.$$

Since $(L_1 + L_2)D < 1$, then T is contraction operator. By using the Banach fixed point theorem, T has a unique fixed point. This fixed point is the unique solution of the nonlocal initial value problem given by equations (1.11)-(1.13). ■

Remark (1.14):

Consider the second order non-linear ordinary differential equation (1.11) together with the nonhomogenous nonlocal initial discrete condition:

$$u(0) + \sum_{k=1}^{m} a_k u(x_k) = \alpha$$

and with the homogenous local initial condition:

$$u'(0) = 0.$$

Then the existence and uniqueness of the solutions for this nonlocal initial value problem can be also discussed by using theorems (1.12)-(1.13), by transforming the nonhomogenous nonlocal initial discrete condition into homogenous ones. To do this, let

$$y(x) = u(x) - C\alpha, \quad x \in [0,1],$$

then

$$y(0) = u(0) - C\alpha.$$

$$y(x_k) = u(x_k) - C\alpha, \quad k = 1,2, \dots, m.$$

$$y'(x) = u'(x), \quad x \in [0,1]$$

and

$$y''(x) = u''(x) = f\big(x, y(x), y'(x)\big), \quad x \in [0,1].$$

Therefore the nonhomogenous nonlocal initial condition becomes:

$$y(0) + C\alpha + \sum_{k=1}^{m} a_k\,[y(x_k) + C\alpha] = \alpha$$

and this implies that

$$y(0) + \sum_{k=1}^{m} a_k\, y(x_k) = \alpha - C\alpha - \sum_{k=1}^{m} a_k\, C\alpha$$

$$= \alpha - C\alpha\left[1 + \sum_{k=1}^{m} a_k\right] = 0.$$

1.3 Existence and Uniqueness of the Solution for the Nonlocal Initial Value Problems for the n-th Order Ordinary Differential Equations:

In this section we discuss the existence and uniqueness of the solution of the nonlocal initial value problems that consists of the n-th order non-linear ordinary differential equation:

$$u^{(n)}(x) = f\left(x, u(x), u'(x), \ldots, u^{(n-1)}(x)\right), \quad x \in [0,1] \tag{1.20}$$

together with the homogenous nonlocal initial discrete condition:

$$u(0) + \sum_{k=1}^{m} a_k u\,(x_k) = 0 \tag{1.21}$$

and with the homogenous local initial conditions:

$$u'(0) = u''(0) = \cdots = u^{(n-1)}(0) = 0 \tag{1.22}$$

where $f: [0,1] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ and f(x,.,...,.) is continuous for all x∈ [0,1], x_k are given points with $0 < x_1 \leq x_2 \leq \cdots \leq x_m < 1$ and $a_1, a_2, \ldots, a_m$ are real numbers such that:

$$\left[1 + \sum_{k=1}^{m} a_k\right] \neq 0.$$

We start this section by the following lemma. This lemma is generalization of lemma (1.10). For the sake of completeness we give the details of its proof.

Lemma (1.15):

The vector space $C^k[a,b]$ of the k-times complex-valued continuously differentiable functions defined on a closed interval [a,b] is Banach space with respect to the following norm

$$\|u\|_{C^k[a,b]} = \max\left\{\|u\|_{C[a,b]}, \|u'\|_{C[a,b]}, \ldots, \left\|u^{(k)}\right\|_{C[a,b]}\right\}, \quad u \in C^k[a,b].$$

Proof:

To prove $\left(C^k[a,b], \|.\|_{C^k[a,b]}\right)$ is a normed space, we must prove the following conditions:

(1) $\|u\|_{C^k[a,b]} \geq 0$, $u \in C^k[a,b]$. To do this, let $u \in C^k[a,b]$, then $u, u', \dots, u^{(k)} \in C[a,b]$. Therefore

$$\|u\|_{C^k[a,b]} = \max\left\{\|u\|_{C[a,b]}, \|u'\|_{C[a,b]}, \dots, \left\|u^{(k)}\right\|_{C[a,b]}\right\} \geq 0.$$

(2) $\|u\|_{C^k[a,b]} = 0$ iff $u = 0$. To do this, let $u = 0$ then $\left\|u^{(i)}\right\|_{C[a,b]} = 0$, $i = 0,1,\dots,k$. Thus $\|u\|_{C^k[a,b]} = 0$.

Conversely, let $\|u\|_{C^k[a,b]} = 0$, then $\left\|u^{(i)}\right\|_{C[a,b]} = 0$, $i = 0,1,\dots,k$. Therefore $\left|u^{(i)}(x)\right| = 0$, for $i = 0,1,\dots,k$, $x \in [a,b]$. This implies that $\operatorname{Re} u(x) = \operatorname{Im} u(x) = 0$, $x \in [a,b]$. Hence $u = 0$.

(3) $\|\lambda u\|_{C^k[a,b]} = |\lambda|\|u\|_{C^k[a,b]}$, $u \in C^k[a,b]$. To do this, let $u \in C^k[a,b]$ and $\lambda \in \mathbb{C}$, then

$$\|\lambda u\|_{C^k[a,b]} = \max\left\{\|\lambda u\|_{C[a,b]}, \|\lambda u'\|_{C[a,b]}, \dots, \left\|\lambda u^{(k)}\right\|_{C[a,b]}\right\}$$

$$= \max\left\{|\lambda|\|u\|_{C[a,b]}, |\lambda|\|u'\|_{C[a,b]}, \dots, |\lambda|\left\|u^{(k)}\right\|_{C[a,b]}\right\}$$

$$= |\lambda|\max\left\{\|u\|_{C[a,b]}, \|u'\|_{C[a,b]}, \dots, \left\|u^{(k)}\right\|_{C[a,b]}\right\}$$

$$= |\lambda|\|u\|_{C^k[a,b]}$$

(4) $\|u + g\|_{C^k[a,b]} \leq \|u\|_{C^k[a,b]} + \|g\|_{C^k[a,b]}$, $u, g \in C^k[a,b]$. To do this, let $u, g \in C^k[a,b]$, then

$$\left\|u^{(i)} + g^{(i)}\right\|_{C[a,b]} \leq \left\|u^{(i)}\right\|_{C[a,b]} + \left\|g^{(i)}\right\|_{C[a,b]}, \quad i = 0,1,\dots,k.$$

$\|u+g\|_{C^k[a,b]}$

$= \max\left\{\|u+g\|_{C[a,b]}, \|u'+g'\|_{C[a,b]}, \dots, \left\|u^{(k)}+g^{(k)}\right\|_{C[a,b]}\right\}$

$\leq \max\left\{\|u\|_{C[a,b]}+\|g\|_{C[a,b]}, \|u'\|_{C[a,b]}+\|g'\|_{C[a,b]}, \dots, \left\|u^{(k)}\right\|_{C[a,b]}+\left\|g^{(k)}\right\|_{C[a,b]}\right\}$

$\leq \max\left\{\|u\|_{C[a,b]}, \|u'\|_{C[a,b]}, \dots, \left\|u^{(k)}\right\|_{C[a,b]}\right\}$

$+\max\left\{\|g\|_{C[a,b]}, \|g'\|_{C[a,b]}, \dots, \left\|g^{(k)}\right\|_{C[a,b]}\right\}$

$\leq \|u\|_{C^k[a,b]}+\|g\|_{C^k[a,b]}, \quad u, g \in C^k[a,b].$

Thus $\left(C^k[a,b], \|.\|_{C^k[a,b]}\right)$ is a normed space.

To prove the completeness of C^k[a,b], let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence of continuous functions in C^k[a,b]. In other words, given any $\epsilon_1 > 0$, we choose N_1 large enough such that for $n, m \geq N_1$, we have

$\|u_n - u_m\|_{C^k[a,b]} < \epsilon_1$. That is $\max\left\{\|u\|_{C[a,b]}, \|u'\|_{C[a,b]}, \dots, \left\|u^{(k)}\right\|_{C[a,b]}\right\} < \epsilon_1$.

Thus

$$\left\|u_n^{(i)} - u_m^{(i)}\right\|_{C[a,b]} \leq \max\left\{\|u_n-u_m\|_{C[a,b]}, \|u_n'-u_m'\|_{C[a,b]}, \dots, \left\|u_n^{(k)}-u_m^{(k)}\right\|_{C[a,b]}\right\}$$

$$< \epsilon_1, \quad i = 0,1,\dots,k.$$

In other words, $\{u_n\}_{n=1}^{\infty}, \{u_n'\}_{n=1}^{\infty}, \dots, \left\{u_n^{(k)}\right\}_{n=1}^{\infty}$ are Cauchy sequences in C[a, b]. By using lemma (1.1), there exists $u,\ g_i \in C[a,b],\ i = 1,2,\dots,k$, such that

$$u(x) = \lim_{n\to\infty} u_n(x), \quad x \in [a,b]$$

and

$$g_i(x) = \lim_{n\to\infty} u_n^{(k)}(x), \quad i = 1,2,\dots,k, \quad x \in [a,b].$$

But $u_n \in C[a,b]$, then

$$u_n(x) - u_n(a) = \int_a^x u_n'(s)\, ds.$$

Hence

$$\lim_{n\to\infty}\int_a^x u_n'(s)\,ds = \int_a^x \lim_{n\to\infty} u_n'(s)\,ds = \int_a^x g_1(s)\,ds.$$

But

$$\lim_{n\to\infty}\int_a^x u_n'(s)\,ds = \lim_{n\to\infty}\{u_n(x)-u_n(a)\} = u(x)-u(a).$$

Therefore

$$u(x)-u(a) = \int_a^x g_1(s)\,ds$$

and this implies that

$$u'(x) = g_1(x), \quad x \in [a,b].$$

Moreover

$$\lim_{n\to\infty}\int_a^x u_n''(s)\,ds = \int_a^x \lim_{n\to\infty} u_n''(s)\,ds = \int_a^x g_2(s)\,ds.$$

But

$$\lim_{n\to\infty}\int_a^x u_n''(s)\,ds = \lim_{n\to\infty}\{u_n'(x)-u_n'(a)\} = g_1(x)-g_1(a).$$

Therefore

$$g_1(x)-g_1(a) = \int_a^x g_2(s)\,ds.$$

This implies that:

$$u''(x) = g_2(x), \quad x \in [a,b].$$

By continuing in this manner, one can get:

$$u^{(i)}(x) = g_i(x), \quad i = 1,2,\dots,k, \quad x \in [a,b].$$

thus $u \in C^K[a,b]$. So, given any $\epsilon_1, \epsilon_2, \dots, \epsilon_{k+1}$, we choose $N_1, N_2, \dots, N_k$ large enough such that for $n \geq N = \max\{N_1, N_2, \dots, N_{k+1}\}$, we have

$$\|u_n - u\|_{C^K[a,b]} = \max\left\{\|u_n - u\|_{C[a,b]}, \|u_n' - u'\|_{C[a,b]}, \dots, \left\|u_n^{(k)} - u^{(k)}\right\|_{C[a,b]}\right\} \leq \epsilon, \quad \text{for } n \geq N$$

where $\epsilon = \max\{\epsilon_1, \epsilon_2, \dots, \epsilon_{k+1}\}$.Therefore $\{u_n\}_{n=1}^{\infty}$ is convergent sequence to $u \in C^k[a,b]$ and this complete the proof. ■

Next, we give the following equivalent lemma. This lemma is a generalization of lemma (1.11).

Lemma (1.16):

The nonlocal initial value problem given by equation (1.20)-(1.22) is equivalent to the integral equation:

$$u(x) = -\frac{C}{(n-1)!}\sum_{k=1}^{m} a_k \int_0^{x_k} (x_k - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds + \frac{1}{(n-1)!}\int_0^{x} (x - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds, \quad x \in [0,1] \quad (1.23)$$

where $C = \left[1 + \sum_{k=1}^{m} a_k\right]^{-1}$.

Proof:

Let u be the solution of the nonlocal initial value problem given by equations (1.20)-(1.22). Then by integrating n-times both sides of equation (1.20) from 0 to x, one can get:

$$u(x) = \sum_{k=0}^{n-1} \frac{x^k}{k!} u^{(k)}(0) + \int_0^{x}\int_0^{t_{n-1}} \dots \int_0^{t_1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds dt_1 \dots dt_{n-1}, \quad x \in [0,1].$$

From the local initial conditions given by equation (1.22) and by using the generalization of Cauchy's formula for an n-fold integrals, [26], then the above equation becomes:

$$u(x) = u(0) + \frac{1}{(n-1)!}\int_0^{x} (x - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds \quad (1.24)$$

and by using the homogenous nonlocal initial condition given by equation (1.21), one can get:

$$u(x) = -\sum_{k=1}^{m} a_k u(x_k) + \frac{1}{(n-1)!}\int_0^{x} (x - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds$$

Then we substitute $x = x_k$ into equation (1.24), to get:

$$u(x_k) = u(0) + \frac{1}{(n-1)!}\int_0^{x_k} (x_k - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds,$$

$$k = 1,2,\dots,m.$$

Hence

$$u(x) = -\sum_{k=1}^{m} a_k \left[u(0) + \frac{1}{(n-1)!}\int_0^{x_k} (x_k - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds\right]$$
$$+ \frac{1}{(n-1)!}\int_0^{x} (x - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds.$$

and

$$u(0) = -\sum_{k=1}^{m} a_k \left[u(0) + \frac{1}{(n-1)!}\int_0^{x_k} (x_k - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds\right].$$

So

$$u(0) = \frac{-1}{(1+\sum_{k=1}^{m} a_k)(n-1)!}\sum_{k=1}^{m} a_k \int_0^{x_k} (x_k - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds$$
$$= \frac{-C}{(n-1)!}\sum_{k=1}^{m} a_k \int_0^{x_k} (x_k - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds.$$

Thus

$$u(x) = \frac{-C}{(n-1)!}\sum_{k=1}^{m} a_k \int_0^{x_k} (x_k - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds +$$
$$\frac{1}{(n-1)!}\int_0^{x} (x - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds.$$

Therefore u is a solution of the integral equation (1.23).

Conversely, let u be a solution of the integral equation (1.23), then by differentiating equation (1.23) (i)-times, one can get:

$$u^{(i)}(x) = \frac{1}{(n-i-1)!}\int_0^{x} \frac{\partial^i}{\partial x^i}(x - s)^{n-i-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds,$$

$$i = 1,2,\dots,n-1.$$

Thus

$$u'(0) = u''(0) = \cdots = u^{(n-1)}(0) = 0$$

and

$$u^{(n-1)}(x) = (n-n)!\frac{1}{(n-n)!}\int_0^x f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds.$$

$$= \int_0^x f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds.$$

Therefore

$$u^{(n)}(x) = f\left(x, u(x), u'(x), \dots, u^{(n-1)}(x)\right), \quad x \in [0,1].$$

This implies that u is a solution of equation (1.20) that satisfies the local initial condition given by equation (1.21). By substituting $x = x_i$ into equation (1.23), one can obtain:

$$\sum_{i=1}^{m} a_i u(x_i)$$

$$= \frac{-C}{(n-1)!}\sum_{i=1}^{m} a_i \sum_{k=1}^{m}\left[a_k \int_0^{x_k} (x_k - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds\right] + \frac{1}{(n-1)!}\sum_{i=1}^{m}\left[a_i \int_0^{x_i} (x_i - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds\right]$$

$$= \frac{1}{(n-1)!}\left[1 - C\sum_{i=1}^{m} a_i\right]\sum_{k=1}^{m}\left[a_k \int_0^{x_k} (x_k - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds\right]$$

$$= \frac{C}{(n-1)!}\sum_{k=1}^{m}\left[a_k \int_0^{x_k} (x_k - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds\right] = -u(0).$$

This implies that u satisfy the homogenous nonlocal initial discrete condition given by equation (1.21). Thus u is a solution of the nonlocal initial value problem given by equations (1.20)-(1.22). ■

Next, we give theorem. This theorem is generalization of theorem (1.6) and (1.12) to be valid for the nonlocal initial value problem given by equations (1.20)-(1.22).

Theorem (1.17):

Consider the nonlocal initial value problem given by equations (1.20)-(1.22). If the following conditions are satisfied:

(1) There exist $\alpha_1, \alpha_2, \dots, \alpha_{n+1} > 0$, such that:

$$|f(x, z_1, z_2, \dots, z_n)| \leq \sum_{i=1}^{n} \alpha_i |z_i| + \alpha_{n+1}.$$

(2) $D\left(\sum_{i=1}^{n} \alpha_i\right) < n!$ and $D \geq n!$, where $D = \left[1 + x_m^n |C| \sum_{k=1}^{m} |a_k|\right]$

then the nonlocal initial value problem given by equations (1.20)-(1.22) has at least one solution.

Proof:

Let $T: C^{n-1}[0,1] \longrightarrow C^{n-1}[0,1]$ be an operator that is defined by

$$Tu(x) = -C \sum_{k=1}^{m} a_k \int_0^{x_k} \frac{(x_k - s)^{n-1}}{(n-1)!} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds$$

$$+ \int_0^{x} \frac{(x - s)^{n-1}}{(n-1)!} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds, \quad x \in [0,1] \quad (1.25)$$

Let $R = \frac{D\alpha_{n+1}}{n! - D(\sum_{i=1}^{n} \alpha_i)}$ and assume that $B_R = \{u \in C^{n-1}[0,1] \mid \|u\|_{C^{n-1}[0,1]} \leq R\}$. Then it is easy to check that B_R is a nonempty, closed, convex and bounded subset of $C^{n-1}[0,1]$. Let $u \in B_R$, then

$$|Tu(x)| = \left| \frac{-C}{(n-1)!} \sum_{k=1}^{m} a_k \int_0^{x_k} (x_k - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds + \right.$$

$$\left. \frac{1}{(n-1)!} \int_0^{x} (x - s)^{n-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds \right|.$$

Hence

$$|Tu(x)| \leq \frac{|C|}{(n-1)!}\sum_{k=1}^{m}|a_k|\int_0^{x_k}|(x_k-s)^{n-1}|\left|f\left(s,u(s),u'(s),\dots,u^{(n-1)}(s)\right)\right|ds$$
$$+\frac{1}{(n-1)!}\int_0^{x}|(x-s)^{n-1}|\left|f\left(s,u(s),u'(s),\dots,u^{(n-1)}(s)\right)\right|ds$$

$$\leq \frac{|C|}{(n-1)!}\sum_{k=1}^{m}|a_k|\int_0^{x_k}|(x_k-s)^{n-1}|\left[\sum_{i=1}^{n}\alpha_i\left|u^{(i-1)}(s)\right|+\alpha_{n+1}\right]ds+$$
$$\frac{1}{(n-1)!}\int_0^{x}|(x-s)^{n-1}|\left[\sum_{i=1}^{n}\alpha_i\left|u^{(i-1)}(s)\right|+\alpha_{n+1}\right]ds$$

$$\leq \frac{|C|}{(n-1)!}\sum_{k=1}^{m}|a_k|\int_0^{x_k}|(x_k-s)^{n-1}|\left[\sum_{i=1}^{n}\alpha_i\left\|u^{(i-1)}\right\|_{C[0,1]}+\alpha_{n+1}\right]ds+$$
$$\frac{1}{(n-1)!}\int_0^{x}|(x-s)^{n-1}|\left[\sum_{i=1}^{n}\alpha_i\left\|u^{(i-1)}\right\|_{C[0,1]}+\alpha_{n+1}\right]ds.$$

$$\leq \frac{|C|}{(n-1)!}\sum_{k=1}^{m}|a_k|\int_0^{x_k}|(x_k-s)^{n-1}|\left[\left(\sum_{i=1}^{n}\alpha_i\right)\|u\|_{C^{n-1}[0,1]}+\alpha_{n+1}\right]ds+$$
$$\frac{1}{(n-1)!}\int_0^{x}|(x-s)^{n-1}|\left[\left(\sum_{i=1}^{n}\alpha_i\right)\|u\|_{C^{n-1}[0,1]}+\alpha_{n+1}\right]ds$$

$$\leq \frac{|C|}{n!}\sum_{k=1}^{m}|a_k|\,x_m^n\left[\left(\sum_{i=1}^{n}\alpha_i\right)\|u\|_{C^{n-1}[0,1]}+\alpha_{n+1}\right]$$
$$+\frac{1}{n!}\left[\left(\sum_{i=1}^{n}\alpha_i\right)\|u\|_{C^{n-1}[0,1]}+\alpha_{n+1}\right]$$

$$\leq \frac{D}{n!}\left[\left(\sum_{i=1}^{n}\alpha_i\right)\frac{D\alpha_{n+1}}{n!-D\left(\sum_{i=1}^{n}\alpha_i\right)}+\alpha_{n+1}\right]=R.$$

Therefore

$$\|Tu\|_{C[0,1]} \leq R \tag{1.26}$$

Furthermore

$$(Tu)^{(i)}(x) = \frac{1}{(n-i-1)!}\int_0^x (x-s)^{n-i-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds$$
$$\leq \frac{1}{(n-i-1)!}\int_0^x \left|(x-s)^{n-i-1}\right| \left[\sum_{j=1}^{n} \alpha_j \left\|u^{(j-1)}\right\|_{C[0,1]} + \alpha_{n+1}\right] ds$$
$$\leq \frac{1}{(n-i-1)!}\left[\left(\sum_{j=1}^{n} \alpha_j\right) R + \alpha_{n+1}\right]$$
$$\leq \frac{1}{(n-i-1)!}\left[\left(\sum_{i=1}^{n} \alpha_i\right)\frac{D\alpha_{n+1}}{n! - D\left(\sum_{i=1}^{n} \alpha_i\right)} + \alpha_{n+1}\right] \leq R.$$

Therefore

$$\left\|(Tu)^{(i)}\right\|_{C[0,1]} \leq R, \quad i = 1,2,\dots,n-1 \tag{1.27}$$

From inequalities (1.26)-(1.27), one can have:

$$\|Tu\|_{C^{n-1}[0,1]} \leq R.$$

Therefore this implies that $TB_R \subseteq B_R$ and since T is completely continuous, [11]. So by using Schauder fixed point theorem, T has a fixed point. This fixed point is a solution of the nonlocal problem given by equations (1.20)-(1.22). ■

Next, we give the generalize theorem of theorem (1.8) and (1.13) to be valid for the nonlocal initial value problem given by equations (1.20)-(1.22).

Theorem (1.18):

Consider the nonlocal initial value problem given by equations (1.20)-(1.22). If the following conditions are satisfied:

(1) f satisfy a Lipschitz condition with respect second and third,…,n-th argument , that is there exists $L_i > 0$, $i = 1,2,\dots,n$, such that that is

$$|f(x, y_1, y_2, \dots, y_n) - f(x, z_1, z_2, \dots, z_n)| \leq \sum_{i=1}^{n} L_i\,|y_i - z_i|$$

(2) $\left[\sum_{i=1}^{n} L_i\right] \max\left\{\frac{D}{n!}, 1\right\} < 1$, where $D = \left[1 + x_m^n |C| \sum_{k=1}^{m} |a_k|\right]$

then the nonlocal initial value problem given by equations (1.20)-(1.22) has a unique solution.

Proof:

From lemma (1.15) $C^{n-1}[0,1]$ is Banach space with respect to the following norm

$$\|u\|_{C^{n-1}[0,1]} = \max\left\{\|u\|_{C[0,1]}, \|u'\|_{C[0,1]}, \dots, \left\|u^{(n-1)}\right\|_{C[0,1]}\right\}, \quad u \in C^{n-1}[0,1].$$

Define the operator T as in equation (1.25) and let $u, v \in C^{n-1}[0,1]$, then

$$|Tu(x) - Tv(x)| \leq \frac{|C|}{(n-1)!} \sum_{k=1}^{m} |a_k| \int_0^{x_k} |(x_k - s)^{n-1}|$$
$$\left|f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) - f\left(s, v(s), v'(s), \dots, v^{(n-1)}(s)\right)\right| ds$$
$$+ \frac{1}{(n-1)!} \sum_{k=1}^{m} \int_0^{x} |(x - s)^{n-1}|$$
$$\left|f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) - f\left(s, v(s), v'(s), \dots, v^{(n-1)}(s)\right)\right| ds.$$
$$\leq \frac{|C|}{(n-1)!} \sum_{k=1}^{m} |a_k| \int_0^{x_m} |(x_k - s)^{n-1}| \sum_{i=1}^{n} L_i \left|u^{(i-1)}(s) - v^{(i-1)}(s)\right| ds$$
$$+ \frac{1}{(n-1)!} \sum_{k=1}^{m} \int_0^{1} |(x - s)^{n-1}| \sum_{i=1}^{n} L_i \left|u^{(i-1)}(s) - v^{(i-1)}(s)\right| ds$$
$$\leq \frac{1}{n!} \left[1 + x_m^n |C| \sum_{k=1}^{m} |a_k|\right] \left[\sum_{i=1}^{n} L_i\right] \left\|u^{(i-1)} - v^{(i-1)}\right\|_{C[0,1]}$$
$$\leq \frac{D}{n!} \left[\sum_{i=1}^{n} L_i\right] \|u - v\|_{C^{n-1}[0,1]}.$$

Therefore

$$\|Tu - Tu\|_{C[0,1]} \leq \frac{D}{n!}\left[\sum_{i=1}^{n} L_i\right]\|u - v\|_{C^{n-1}[0,1]} \tag{1.28}$$

Furthermore

$$(Tu)^{(i)}(x) = \frac{1}{(n-i-1)!}\int_0^x (x-s)^{n-i-1} f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds.$$

This implies that

$$\left|(Tu)^{(i)}(x) - (Tv)^{(i)}(x)\right| \leq \frac{1}{(n-i)!}\left[\sum_{i=1}^{n} L_i\right]\|u - v\|_{C^{n-1}[0,1]}.$$

So

$$\left\|(Tu)^{(i)} - (Tv)^{(i)}\right\|_{C[0,1]} \leq \frac{1}{(n-i)!}\left[\sum_{i=1}^{n} L_i\right]\|u - v\|_{C^{n-1}[0,1]},$$

$$i = 1,2,\dots,n-1 \tag{1.29}$$

From inequalities (1.28)-(1.29), one can have:

$$\|Tu - Tv\|_{C^{n-1}[0,1]} \leq \left[\sum_{i=1}^{n} L_i\right]\max\left\{\frac{D}{n!}, 1\right\}\|u - v\|_{C^{n-1}[0,1]}.$$

Since $[\sum_{i=1}^{n} L_i]\max\left\{\frac{D}{n!}, 1\right\} < 1$, then T is contraction operator. So by using Banach fixed point theorem, T has a fixed point. This fixed point is a unique solution of the nonlocal problem given by equations (1.20)-(1.22). ■

Remark (1.19):

Consider the n-th order non-linear ordinary differential equation (1.20) together with the nonhomogenous nonlocal initial discrete condition:

$$u(0) + \sum_{k=1}^{m} a_k u(x_k) = \alpha$$

and with the homogenous local initial conditions:

$$u^{(i)}(0) = 0, \quad i = 1,2,\dots,n-1.$$

Then the existence of the solutions for this nonlocal problem can be also discussed by using theorems (1.17)-(1.18) by transforming the nonhomogenous nonlocal initial condition into homogenous ones. To do this, let

$$y(x) = u(x) - C\alpha, \quad x \in [0,1],$$

then

$$y(0) = u(0) - C\alpha$$

$$y(x_k) = u(x_k) - C\alpha$$

$$y'(x) = u'(x), \quad x \in [0,1]$$

and

$$y''(x) = u''(x), \quad x \in [0,1].$$

In a similar manner, one can obtain:

$$y^{(n)}(x) = u^{(n)}(x) = f\left(x, y(x), \ldots, y^{(n-1)}(x)\right), \quad x \in [0,1].$$

Therefore the above nonhomogenous nonlocal initial condition becomes:

$$y(0) + C\alpha + \sum_{k=1}^{m} a_k\,[y(x_k) + C\alpha] = \alpha$$

and this implies that:

$$y(0) + \sum_{k=1}^{m} a_k\, y(x_k) = \alpha - C\alpha - \sum_{k=1}^{m} a_k\, C\alpha$$

$$= \alpha - C\alpha\left[1 + \sum_{k=1}^{m} a_k\right] = 0.$$

1.4 Numerical Solutions of Nonlocal Initial Value Problems for the Ordinary Differential Equations:

In this section we use some numerical methods for solving the nonlocal problem given by equations (1.1)-(1.2). To do this, we divide the interval $[0,1]$ into n subintervals with equal step size h, such that $t_i = ih$, i= 0,1, …, n and $x_k = t_{j_k}$, $j_k \in \{0,1,\dots,n\}$, k=1,2, …, m.

1.4.1 Euler's method, [21]:

The simplest example of a one-step method for the numerical solution of the nonlocal initial value problem given by equations (1.1)-(1.2) is Euler's method. A simple derivation of this method proceeds by first integrating the differential equation (1.1) between two consecutive mesh points t_i and t_{i+1} to get:

$$u_{i+1} = u_i + \int_{t_i}^{t_{i+1}} f(s,u(s))\,ds, \quad i = 0,1,\dots,n-1$$

and then applying the rectangle integration rule:

$$\int_{t_i}^{t_{i+1}} f(s,u(s))\,ds \cong hf(t_i,u_i)$$

to get:

$$u_{i+1} = u_i + hf(t_i,u_i), \quad i = 0,1,\dots,n-1 \qquad (1.30)$$

where u_i is the numerical solution of the nonlocal initial value problem given by equations (1.1)-(1.2) at t_i. We evaluate equations (1.30) at each i= 0,1, …, $j_m - 1$, to get the following system of j_m equations:

$$\left.\begin{aligned}
&u_1 - u_0 - hf(t_0, u_0) = 0\\
&u_2 - u_1 - hf(t_1, u_1) = 0\\
&\vdots\\
&u_{j_1} - u_{j_1-1} - hf\left(t_{j_1-1}, u_{j_1-1}\right) = 0\\
&u_{j_1+1} - u_{j_1} - hf\left(t_{j_1}, u_{j_1}\right) = 0\\
&\vdots\\
&u_{j_2-1} - u_{j_2-2} - hf\left(t_{j_2-2}, u_{j_2-2}\right) = 0\\
&u_{j_2} - u_{j_2-1} - hf\left(t_{j_2-1}, u_{j_2-1}\right) = 0\\
&u_{j_2+1} - u_{j_2} - hf\left(t_{j_2}, u_{j_2}\right) = 0\\
&\vdots\\
&u_{j_m} - u_{j_m-1} - hf\left(t_{j_m-1}, u_{j_m-1}\right) = 0
\end{aligned}\right\} \qquad (1.31)$$

with (j_m+1) unknowns $u_0, u_1, \dots, u_{j_1}, u_{j_1+1}, \dots, u_{j_2}, u_{j_2+1}, \dots, u_{j_m}$. By solving the above system of equations (1.31) and equation(1.2), using any suitable method to find $u_0, u_1, \dots, u_{j_1}, u_{j_1+1}, \dots, u_{j_2}, u_{j_2+1}, \dots, u_{j_m}$. Then we evaluate equation (1.30) at each $i = j_m + 1, j_m + 2, \dots, n$, to get the numerical solutions u_i, $i = j_m + 1, j_m + 2, \dots, n$.

To illustrate this method consider the following example.

Example (1.1):

Consider the nonlocal initial value problem that consists of the first order nonlinear ordinary differential equation:

$$u'(x) = u^3(x) + (-25 - 52x - 63x^2 - 44x^3 - 21x^4 - 6x^5 - x^6), x \in [0,1] \qquad (1.32)$$

together with the homogenous nonlocal initial condition:

$$u(0) + u\left(\frac{1}{15}\right) + u\left(\frac{2}{15}\right) + 5u\left(\frac{4}{15}\right) - \frac{247}{34}u\left(\frac{5}{15}\right) = 0. \qquad (1.33)$$

This example is constructed, such that the exact solution is

$$u(x) = x^2 + 2x + 3.$$

In this example let $h = \frac{1}{15}$, then $t_i = \frac{i}{15}$, $i = 0, 1, \dots, 15$. Here $x_1 = t_1$, $x_2 = t_2$, $x_3 = t_4$, $x_4 = t_5$. In this case, the system of equations (1.31) takes the form:

$$u_1 - u_0 - \frac{1}{15}u_0^3 + \frac{25}{15} = 0$$

$$u_2 - u_1 - \frac{1}{15}u_1^3 + \frac{1}{15}\left(25 + 52\left(\frac{1}{15}\right) + 63\left(\frac{1}{15}\right)^2 + 44\left(\frac{1}{15}\right)^3 + 21\left(\frac{1}{15}\right)^4 + 6\left(\frac{1}{15}\right)^5 + \left(\frac{1}{15}\right)^6\right) = 0.$$

$$u_3 - u_2 - \frac{1}{15}u_2^3 + \frac{1}{15}\left(25 + 52\left(\frac{2}{15}\right) + 63\left(\frac{2}{15}\right)^2 + 44\left(\frac{2}{15}\right)^3 + 21\left(\frac{2}{15}\right)^4 + 6\left(\frac{2}{15}\right)^5 + \left(\frac{2}{15}\right)^6\right) = 0.$$

$$u_4 - u_3 - \frac{1}{15}u_3^3 + \frac{1}{15}\left(25 + 52\left(\frac{3}{15}\right) + 63\left(\frac{3}{15}\right)^2 + 44\left(\frac{3}{15}\right)^3 + 21\left(\frac{3}{15}\right)^4 + 6\left(\frac{3}{15}\right)^5 + \left(\frac{3}{15}\right)^6\right) = 0.$$

$$u_5 - u_4 - \frac{1}{15}u_4^3 + \frac{1}{15}\left(25 + 52\left(\frac{4}{15}\right) + 63\left(\frac{4}{15}\right)^2 + 44\left(\frac{4}{15}\right)^3 + 21\left(\frac{4}{15}\right)^4 + 6\left(\frac{4}{15}\right)^5 + \left(\frac{4}{15}\right)^6\right) = 0.$$

and together with the homogenous nonlocal discrete condition (1.33), we get a system with 6 equations and 6 unknowns, To solve the above system, we use MathCAD software package and the results are tabulated in table (1.1).then from equation (1.30), one can get the numerical solutions u_i at x_i, $i = 6,7, \dots, 15$ and the results are tabulated in table (1.2).

Table (1.1) represents the exact and the numerical solutions for h=$\frac{1}{15}$ of example (1.1).

i	x_i	$u(x_i)$	u_i	$\lvert u(x_i) - u_i \rvert$
0	0	3	3.002	2.36×10^{-3}
1	0.067	3.138	3.14	2.17×10^{-3}
2	0.133	3.284	3.286	1.99×10^{-3}
3	0.200	3.440	3.442	1.86×10^{-3}
4	0.267	3.604	3.606	1.84×10^{-3}
5	0.333	3.778	3.78	2.16×10^{-3}

Table (1.2) represents the exact and the numerical solutions for h=$\frac{1}{15}$ of example (1.1).

i	x_i	$u(x_i)$	u_i	$\lvert u(x_i) - u_i \rvert$
6	0.400	3.960	3.964	4.12×10^{-3}
7	0.467	4.151	4.163	0.012
8	0.533	4.351	4.400	0.049
9	0.600	4.560	4.646	0.086
10	0.667	4.778	4.877	0.099
11	0.733	5.004	5.18	0.176
12	0.800	5.240	5.732	0.492
13	0.867	5.484	6.006	0.522
14	0.933	5.738	6.134	0.604
15	1	6	6.627	0.627

1.4.2 The Trapezium Rule, [21]:

The local truncation error can be reduced by using a more accurate quadrature method for finding the integral than Euler's method, for example, the trapezium rule:

$$\int_{t_i}^{t_{i+1}} f(s,u(s))\,ds = \frac{h}{2}[f(t_i,u_i) + f(t_{i+1},u_{i+1})]$$

it cannot be used directly as we do not know u_{i+1}. The solution is to use the forward Euler's method to estimate u_{i+1} as $u_i + hf(t_i,u_i)$ in the trapezium rule. To get:

$$u_{i+1} = u_i + \frac{h}{2}[f(t_i,u_i) + f(t_i + h, u_i + hf(t_i,u_i))], \quad i = 0,1,\ldots,n-1 \qquad (1.34)$$

where u_i is the numerical solution of the nonlocal initial value problem given by equations (1.1)-(1.2) at t_i. We evaluate equations (1.34) at each i= 0,1, …, $j_m - 1$, to get the following system of j_m equations:

$$\left.\begin{array}{l}
u_1 - u_0 - \frac{h}{2}\left[f(t_0, u_0) + f\big(t_0 + h, u_0 + hf(t_0, u_0)\big)\right] = 0 \\
u_2 - u_1 - \frac{h}{2}\left[f(t_1, u_1) + f\big(t_1 + h, u_1 + hf(t_1, u_1)\big)\right] = 0 \\
\vdots \\
u_{j_1} - u_{j_1-1} - \frac{h}{2}\left[f\big(t_{j_1-1}, u_{j_1-1}\big) + f\left(t_{j_1-1} + h, u_{j_1-1} + hf\big(t_{j_1-1}, u_{j_1-1}\big)\right)\right] = 0 \\
u_{j_1+1} - u_{j_1} - \frac{h}{2}\left[f\big(t_{j_1}, u_{j_1}\big) + f\left(t_{j_1} + h, u_{j_1} + hf\big(t_{j_1}, u_{j_1}\big)\right)\right] = 0 \\
\vdots \\
u_{j_2-1} - u_{j_2-2} - \frac{h}{2}\left[f\big(t_{j_2-2}, u_{j_2-2}\big) + f\left(t_{j_2-2} + h, u_{j_2-2} + hf\big(t_{j_2-2}, u_{j_2-2}\big)\right)\right] = 0 \\
u_{j_2} - u_{j_2-1} - \frac{h}{2}\left[f\big(t_{j_2-1}, u_{j_2-1}\big) + f\left(t_{j_2-1} + h, u_{j_2-1} + hf\big(t_{j_2-1}, u_{j_2-1}\big)\right)\right] = 0 \\
u_{j_2+1} - u_{j_2} - \frac{h}{2}\left[f\big(t_{j_2}, u_{j_2}\big) + f\left(t_{j_2} + h, u_{j_2} + hf\big(t_{j_2}, u_{j_2}\big)\right)\right] = 0 \\
\vdots \\
u_{j_m} - u_{j_m-1} - \frac{h}{2}\left[f\big(t_{j_m-1}, u_{j_m-1}\big) + f\left(t_{j_m-1} + h, u_{j_m-1} + hf\big(t_{j_m-1}, u_{j_m-1}\big)\right)\right] = 0
\end{array}\right\} \quad (1.35)$$

with (j_m+1) unknowns $u_0, u_1, \dots, u_{j_1}, u_{j_1+1}, \dots, u_{j_2}, u_{j_2+1}, \dots, u_{j_m}$. By solving this system that consist equations (1.35) and equation(1.2), by using any suitable method to find $u_0, u_1, \dots, u_{j_1}, u_{j_1+1}, \dots, u_{j_2}, u_{j_2+1}, \dots, u_{j_m}$. Then we evaluate equation (1.36) at each $i = j_m + 1,\ j_m + 2, \dots, n,$ to get the numerical solutions $u_i,\ i = j_m + 1,\ j_m + 2, \dots, n.$

To illustrate this method consider the following example.

Example (1.2):

Consider example (1.1) let $h = \frac{1}{30}$, then $t_i = \frac{i}{30}$, $i = 0, 1, \dots, 30$. Here $x_1 = t_2$, $x_2 = t_4, x_3 = t_8,\ x_4 = t_{10}$. In this case, In this case, the system of equations (1.37) takes the form:

$$u_1 - u_0 - \frac{1}{60}u_1^3 - \frac{1}{60}u_0^3 + 0.863 = 0$$

$$u_2 - u_1 - \frac{1}{60}u_2^3 - \frac{1}{60}u_1^3 + 0.926 = 0.$$

$$u_3 - u_2 - \frac{1}{60}u_3^3 - \frac{1}{60}u_2^3 + 0.994 = 0.$$

$$\vdots$$

$$u_{10} - u_9 - \frac{1}{60}u_{10}^3 - \frac{1}{60}u_9^3 + 1.648 = 0.$$

and together with the homogenous nonlocal condition (1.33), we get a system with 10 equations and 10 unknowns, To solve the above system, we use MathCAD software package and from equation (1.34), one can get the numerical solutions u_i at x_i, $i = 11,12,\dots,30$ and the results are tabulated in table (1.3).

Table (1.3) represents the exact and the numerical solutions for h=$\frac{1}{30}$ of example (1.2).

i	x_i	$u(x_i)$	u_i	$\lvert u(x_i) - u_i \rvert$
0	0	3.000	3.00000000	0
1	0.033	3.068	3.06800000	0
2	0.067	3.138	3.13800000	0
3	0.100	3.210	3.21000000	0
4	0.133	3.284	3.28400000	0
5	0.167	3.361	3.36100000	0
6	0.200	3.440	3.44000000	0
7	0.233	3.521	3.52100000	0
8	0.267	3.604	3.60400000	0
9	0.300	3.690	3.69000000	0
10	0.333	3.778	3.77800000	0
11	0.367	3.868	3.86800012	1.2×10^{-7}
12	0.400	3.960	3.96000027	2.7×10^{-7}
13	0.433	4.054	4.05400998	9.98×10^{-7}
14	0.467	4.151	4.15100140	14×10^{-7}
15	0.500	4.250	4.25000190	1.9×10^{-6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Chapter Two
Solutions of Special Types of the Linear Eigenvalue Problems with the Nonlocal Integral Conditions

Introduction:

The eigenvalue problems with nonlocal conditions are important problems that arise in many real life applications like those with local conditions, [4], [17]. Many researchers studied the eigenvalue problems with nonlocal conditions, say Karakostas and Tsamatos in 2002, proved the existence of the solutions for the nonlinear eigenvalue problems of the second order nonlinear differential equation with the nonlocal integral boundary condition, [19]. The solutions for the linear eigenvalue problems of the second order linear differential equation with nonlocal integral boundary conditions have been studied by Ciupaila and et al. in 2004, [9]. Henderson and Ntouyas in 2007, proved the existence of solutions for the nonlinear eigenvalue problems of the n-th order nonlinear differential equation with the nonlocal discrete boundary conditions, [18]. Fuyi and Jian in 2010, proved the existence of solutions for the nonlinear eigenvalue problems of fourth order nonlinear differential equation with the nonlocal integral boundary conditions, [14].

The purpose of this chapter is to find solutions of some special types of the linear eigenvalue initial and boundary value problems with the nonlocal integral conditions.

This chapter consists of three sections:

In section one; we give the solutions for the linear eigenvalue problem of the first order linear ordinary differential equations together with the nonlocal initial integral condition.

In section two; we give the solutions for the linear eigenvalue problem of the second order linear ordinary differential equations together with the local and the nonlocal initial conditions.

In section three; solutions for the linear eigenvalue problem of the second order linear ordinary differential equations together with the local and the nonlocal boundary conditions are introduced.

2.1 Solutions of Special Types of the Linear Eigenvalue Problems of the First Order Ordinary Differential Equation with the Nonlocal Initial Integral Condition:

In this section we give the solutions for the linear eigenvalue problems of the first order linear ordinary differential equations together with nonlocal initial condition. To do this, consider the linear eigenvalue problem that consists of the first order linear ordinary differential equation:

$$u'(x) + \lambda u(x) = 0, \qquad x \in [0,1] \tag{2.1}$$

together with the nonlocal initial integral condition:

$$u(0) = a\int_{0}^{1} u(s)ds \tag{2.2}$$

where a is any number. It is clear that for any value of λ, $u(x) = 0$ is a solution of the nonlocal initial value problem given by equations (2.1)-(2.2). In this section we find the values of the parameter λ such that the above nonlocal initial value problem has a nonzero solution u. In this case, λ is said to be an eigenvalue and u is the associated eigenfunction u for the nonlocal initial value problem given by equations (2.1)-(2.1). Moreover (λ, u) is said to be an eigenpair for the nonlocal initial value problem given by equations (2.1)-(2.2). To do this we discuss two different cases:

Case 1:

Assume $\lambda = 0$ then by integrating both sides of equation (2.1) from 0 to x, one can get:

$$u(x) = C, \quad x \in [0,1]$$

where C is an arbitrary constant. By substituting the above equation into equation (2.2), one can get:

$$C(1 - a) = 0.$$

Therefore, if $a \neq 1$, then $\lambda = 0$ is not an eigenvalue for the nonlocal initial value problem given by equations (2.1)-(2.2). But, if $a = 1$, then $\lambda = 0$ is an eigenvalue of the nonlocal initial value problem given by equations (2.1)-(2.2), with the corresponding nonzero constant eigenfunction.

Case 2:

Assume $\lambda \neq 0$, then the general solution of the differential equation (2.1), is

$$u(x) = Ce^{-\lambda x},$$

where C is any nonzero number. By substituting the solution into equation (2.2), one can get:

$$ae^{-\lambda} = a - \lambda\,.$$

To solve this equation, one can use any suitable method. To find this root, we use MathCAD software package and the results are tabulated down.

Table (2.1) represents the values of λ for different values of a and the absolute errors.

a	λ	$\lvert ae^{-\lambda} + \lambda - a \rvert$
16	16	1.234×10^{-14}
14	14	3.748×10^{-15}
10	10	5.295×10^{-12}
8	7.997	5.295×10^{-12}
2	1.594	2.191×10^{-10}

2.2 Solutions of Special Types of the Linear Eigenvalue Problems of the Second Order Ordinary Differential Equation with the Nonlocal Initial Integral Condition:

In this section we devote the linear eigenvalue problem that consists of the second order linear ordinary differential equation:

$$u''(x) + \lambda\, u(x) = 0, \quad x \in [0,1] \tag{2.3}$$

together with the nonlocal initial integral condition:

$$u(0) = a \int_0^1 u(s)ds \tag{2.4}$$

and the local initial condition:

$$u'(0) \ = 0 \tag{2.5}$$

where a is any number. It is clear that for any value of λ, $u(x) = 0$ is a solution of the nonlocal initial value problem given by equations (2.3)-(2.5). The linear eigenvalue problem here is to find the parameter λ for which this problem has a nonzero solution u. In this case, λ is said to be an eigenvalue and u is the associated eigenfunction u for the nonlocal initial value problem given by equations (2.3)-(2.5). Moreover (λ, u) is said to be an eigenpair for the nonlocal initial value problem given by equations (2.3)-(2.5). To do this, we use the ideas that appeared in [9]. So one must consider the following cases:

Case 1:

Assume $\lambda = 0$. In this case, the general solution of the differential equations (2.3) that satisfy the local initial condition given by equation (2.5), takes the form:

$$u(x) = C, \quad x \in [0,1]$$

where C is any nonzero number. By substituting the above equation into equation (2.4), one can get:

$$C(1 - a) = 0.$$

Therefore, if $a \neq 1$, then $\lambda = 0$ is not an eigenvalue of the nonlocal initial value problem given by equations (2.3)-(2.5). If $a = 1$, then $\lambda = 0$ is an eigenvalue of the nonlocal initial value problem given by equations (2.3)-(2.5), with the corresponding nonzero constant eigenfunction.

Case 2:

Assume $\lambda < 0$, then the general solution of the linear differential equation (2.3) takes the form:

$$u(x) = C_1 e^{-\alpha x} + C_2 e^{\alpha x}, \quad x \in [0,1],$$

where C_1, C_2 are an arbitrary constants and $\alpha = \sqrt{-\lambda}$. But this solution must satisfy the local initial condition given by equation (2.5), so

$$u(x) = C \cosh(\alpha x), \ \ x \in [0,1],$$

where C any nonzero number. Further this solution must also satisfy the nonlocal initial condition given by equation (2.4) to get:

$$\sinh\alpha = \frac{\alpha}{a}.$$

To solve this equation, one can use any suitable method. To find this root, we use MathCAD software package and the results are tabulated in table (2.2).

Table (2.2) represents the values of α for different values of a and the absolute errors.

a	α	$\left\|\sinh\alpha - \frac{\alpha}{a}\right\|$
1/2	2.177	1.273×10^{-10}
1/3	2.838	3.855×10^{-13}
1/4	3.264	6.359×10^{-13}
1/5	3.578	8.743×10^{-12}
1/9	4.364	7.105×10^{-15}

Case 3:

Assume $\lambda > 0$, then the general solution of the linear differential equation (2.3) takes the form:

$$u(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x), \ \ x \in [0,1],$$

where C_1, C_2 are an arbitrary constants and $\alpha = \sqrt{\lambda}.$ But this solution must satisfy the local initial condition given by equation (2.5), then the above equation becomes:

$$u(x) = C \cos(\alpha x)\,, \ \ x \in [0,1],$$

where C is any nonzero number. Moreover this solution must satisfy the nonlocal initial condition given by equation (2.4) to obtain:

$$\sin\alpha = \frac{\alpha}{a}.$$

To solve this equation, one can use a suitable method. To find this root, we use MathCAD software package and the results are tabulated down.

Table (2.3) represents the values of α for different values of a and the absolute errors.

a	α	$\left\|\sin\alpha - \frac{\alpha}{a}\right\|$
-8	3.61	1.439×10^{-9}
-5	4.906	1.259×10^{-10}
2	1.895	2.642×10^{-11}
4	2.475	4.179×10^{-14}
8	2.786	1.777×10^{-12}

2.3 Solutions of Special Types of the Linear Eigenvalue Nonlocal Boundary Value Problems of the Second Order Ordinary Differential Equation with theNonlocal Initial Integral Condition,[9]:

In this section we give the solutions for the linear eigenvalue problem of the second order linear ordinary differential equation together with the nonlocal initial integral condition and local boundary condition. To do this, consider the linear eigenvalue problem that consists of the second order linear ordinary differential equation:

$$u''(x) + \lambda\, u(x) = 0, \quad x \in [0,1] \qquad (2.6)$$

together with the nonlocal initial integral condition:

$$u(0) = a \int_0^1 u(s)ds \qquad (2.7)$$

and with the local boundary condition:

$$u(1) = 0 \qquad (2.8)$$

where a is any number. It is clear that for any values λ, $u(x) = 0$ is a solution of the nonlocal boundary value problem given by equations (2.6)-(2.8). The linear eigenvalue problem here is to find the parameter λ for which this problem has a nonzero solution u. In this case, λ is said to be an eigenvalue and u is the associated eigenfunction u for the nonlocal boundary value problem given by equations (2.6)-(2.8). Moreover (λ, u) is said to be an eigenpair for the nonlocal boundary value problem given by equations (2.6)-(2.8). To do this, one must consider the following cases:

Case 1:

Assume $\lambda = 0$. In this case, the general solution of the linear differential equations (2.6) that satisfy the local boundary condition given by equation (2.8), takes the form:

$$u(x) = C(1 - x), \quad x \in [0,1],$$

where C is any nonzero number. By substituting the above equation this solution into equation (2.7), one can get:

$$C\left[1-\frac{a}{2}\right]=0\,.$$

Therefore, if $a \neq 2$, then $\lambda = 0$ is not an eigenvalue of the nonlocal boundary value problem given by equations (2.6)-(2.8). If $a = 2$, then $\lambda = 0$ is an eigenvalue of the nonlocal boundary value problem given by equations (2.6)-(2.8), with the corresponding nonzero constant eigenfunction.

Case 2:

Assume $\lambda < 0$, then the general solution of the differential equation (2.6) takes the form:

$$u(x) = C_1 e^{-\alpha x} + C_2 e^{\alpha x}, \quad x \in [0,1],$$

where C_1, C_2 are any arbitrary numbers and $\alpha = \sqrt{-\lambda}$. But this solution must satisfy the local and nonlocal conditions given by equations (2.7)-(2.8), to get the following system:

$$\begin{pmatrix} 1-\frac{a}{\alpha}(e^{\alpha}-1) & 1+\frac{a}{\alpha}(e^{-\alpha}-1) \\ e^{\alpha} & e^{-\alpha} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

this homogeneous system has a nontrivial solution iff homogeneous

$$\begin{vmatrix} 1-\frac{a}{\alpha}(e^{\alpha}-1) & 1+\frac{a}{\alpha}(e^{-\alpha}-1) \\ e^{\alpha} & e^{-\alpha} \end{vmatrix} = 0.$$

So

$$\left[1-\frac{a}{\alpha}(e^{\alpha}-1)\right]e^{-\alpha} - \left[1+\frac{a}{\alpha}(e^{-\alpha}-1)\right]e^{\alpha} = 0$$

and this implies that

$$-\sinh\alpha + \frac{a}{\alpha}\,[\cosh\alpha - 1] = 0.$$

But

$$\sinh\alpha = 2\sinh\left(\frac{\alpha}{2}\right)\cosh\left(\frac{\alpha}{2}\right), \qquad \cosh\alpha - 1 = 2\sinh^2\left(\frac{\alpha}{2}\right).$$

Thus the above equation becomes:

$$\sinh\left(\frac{\alpha}{2}\right)\cosh\left(\frac{\alpha}{2}\right) = \frac{a}{\alpha}\sinh^2\left(\frac{\alpha}{2}\right)$$

and hence

$$\tanh\left(\frac{\alpha}{2}\right) = \frac{\alpha}{a}.$$

To solve this equation, one must consider the following cases:

(i) If $a \leq 2$, then no nonzero root λ of function $\tanh\left(\frac{\alpha}{2}\right) - \frac{\alpha}{a}$ exists.

(ii) If $a > 2$, then there exists two nonzero roots α of the function $\tanh\left(\frac{\alpha}{2}\right) - \frac{\alpha}{a}$, for which $\alpha > 0$. This root can be found by using any suitable method. To find this root, we use MathCAD software package and the results are tabulated down.

Table (1.4) represented the values of α for different values of a and the absolute errors.

a	α	$\left\lvert\tanh\left(\frac{\alpha}{2}\right) - \frac{\alpha}{a}\right\rvert$
5/2	1.776	1.932×10^{-14}
4	3.83	6.493×10^{-12}
8	7.995	1×10^{-16}
10	9.999	3.053×10^{-14}
12	12	1.95×10^{-15}

Case 3:

Assume $\lambda > 0$, then the general solution of the linear differential equation (2.6) takes the form:

$$u(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$$

where C_1, C_2 are any arbitrary numbers and $\alpha = \sqrt{\lambda}$, But this solution must satisfy the local and the nonlocal conditions given by equations (2.7)-(2.8), to get the following system:

$$\begin{pmatrix} 1 - \frac{a}{\alpha}\sin\alpha & \frac{a}{\alpha}[\cos\alpha - 1] \\ \cos\alpha & \sin\alpha \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This homogeneous system has a nontrivial solution iff

$$\begin{vmatrix} 1 - \frac{a}{\alpha}\sin\alpha & \frac{a}{\alpha}[\cos\alpha - 1] \\ \cos\alpha & \sin\alpha \end{vmatrix} = 0.$$

So

$$\left[1 - \frac{a}{\alpha}\sin\alpha\right]\sin\alpha - \frac{a}{\alpha}[\cos\alpha - 1]\cos\alpha = 0$$

and this implies that

$$\sin\alpha - \frac{a}{\alpha}\,[\sin^2\alpha + \cos^2\alpha - \cos\alpha] = 0 \qquad (2.9)$$

But

$$\sin\alpha = 2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right), \qquad 1 - \cos\alpha = 2\sin^2\left(\frac{\alpha}{2}\right).$$

Thus equation (2.9), becomes:

$$\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) = \frac{a}{\alpha}\,\sin^2\left(\frac{\alpha}{2}\right)$$

and hence

$$\tan\left(\frac{\alpha}{2}\right) = \frac{\alpha}{a}.$$

Therefore, for any value of a there exist infinitely roots α of the function $\tan(\alpha) - \frac{\alpha}{a}$. This root can be found by using any suitable method. To find this root, we use MathCAD software package and the results are tabulated down.

Table (2.5) represented the values of α for different values of a and the absolute errors.

a	α	$\left\lvert \tan\alpha - \frac{\alpha}{a} \right\rvert$
-10	9.826	7.618×10^{-11}
-4	4.578	8.438×10^{-15}
-2	10.174	9.025×10^{-11}
2	8.765	7.105×10^{-15}
5	-8.765	8.743×10^{-10}

Chapter Three
Finite Difference Method for Solving Special Types of the Nonlocal Problems for Linear Partial Differential Equations

Introduction:

The nonlocal initial-boundary value problems involving partial differential equations have been a major research area in modern physics, chemistry and engineering when it is impossible to determine the boundary values of the unknown function, [2]. Many researchers studied these types of problems, from them Said and Abdelfatah in 1999, proved the existence and uniqueness of the second order hyperbolic equation with one nonlocal integral boundary condition, [34]. Rehman in 2009, used five point central finite difference scheme to solve the one-dimensional heat equation with two nonlocal linear integral boundary conditions, [31]. Ashyralyev and Necmettin in 2011, used the finite difference scheme for solving the one-dimensional hyperbolic equation with one nonlocal linear integral boundary condition, [2] and Borhanifar and et al. in 2011, used the finite difference scheme to solve the one-dimensional heat equation with two nonlocal non-linear integral boundary conditions, [6].

In this chapter we use the finite difference scheme to solve special types of the nonlocal initial-boundary value problems.

This chapter consists of three sections:

In section one, we use Douglas's equation and Crank-Niklson finite difference scheme for solving the one-dimensional wave equation with two nonlocal linear integral boundary conditions.

In sections two and three, we use Crank-Niklson finite difference scheme for solving the one-dimensional hyperbolic and parabolic equations with nonlocal non-linear integral boundary conditions.

3.1 Solutions of the One-Dimensional Wave Equation with the Nonlocal Linear Integral Boundary Conditions:

Consider the nonlocal initial-boundary value problem that consists of the one-dimensional wave equation:

$$u_{tt} - u_{xx} = f(x,t), \quad x \in [0,1], \quad t \in [0,T] \tag{3.1}$$

together with the local initial conditions:

$$u(x,0) = d(x), \quad x \in [0,1] \tag{3.2}$$

$$u_t(x,0) = r(x), \quad x \in [0,1] \tag{3.3}$$

and the two nonlocal linear integral boundary conditions:

$$u(0,t) = \int_0^1 w_0(x)\, u(x,t)\, dx + g_0(t), \quad t \in [0,T] \tag{3.4}$$

$$u(1,t) = \int_0^1 w_1(x)\, u(x,t)\, dx + g_1(t), \quad t \in [0,T] \tag{3.5}$$

where f is a known continuous function of x and t, d, r, w_0, w_1, g_0 and g_1, are known continuous functions that must satisfy the compatibility conditions:

(1) $d(0) = \int_0^1 w_0(x)\, d(x)dx + g_0(0)$.

(2) $d(1) = \int_0^1 w_1(x)\, d(x)dx + g_1(0)$.

(3) $r(0) = \int_0^1 w_0(x)\, r(x)dx + \left.\frac{dg_0}{dt}\right|_{t=0}$.

(4) $r(1) = \int_0^1 w_1(x)\, r(x)dx + \left.\frac{dg_1}{dt}\right|_{t=0}$.

In this section we use the finite difference scheme for solving this nonlocal initial-boundary value problem given by equations (3.1)-(3.5). To do this, we divide the region $[0,1] \times [0,T]$ into $N \times M$ mesh point with spatial step size $h = 1/N$ in the x-direction and the time step size $k = T/M$ respectively, where M is positive integer and N is an even positive integer.

The mesh points are given by:

$$x_i = ih, \quad i = 0,1,\ldots,N,$$
$$t_j = jk, \quad j = 0,1,\ldots,M.$$

Define the following difference operators, [36]:

$$\delta_t^2 u_{i,j} = u_{i,j-1} - 2u_{i,j} + u_{i,j+1}, \quad i = 0,1,\ldots,N, \ j = 1,2,\ldots,M-1,$$

$$\delta_x^2 u_{i,j} = u_{i-1,j} - 2u_{i,j} + u_{i+1,j}, \quad i = 1,2,\ldots,N-1, \ j = 0,1,\ldots,M,$$

where $u_{i,j}$ is the numerical solution of the nonlocal initial-boundary value problem given by equations (3.1)-(3.5) at the point (x_i, t_j).

We replaced $\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}$ by Douglas's equation, [36]:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{1}{h^2}\left[{\delta_x}^2 - \frac{1}{12}{\delta_x}^4 + \frac{1}{90}{\delta_x}^6 - \frac{1}{560}{\delta_x}^8 + \cdots\right] u_{i,j},$$
$$i = 1,2,\ldots,N-1, \ j = 0,1,\ldots,M.$$

Then explain equation (3.1) at the point (i,j) by:

$$\frac{1}{k^2}\delta_t^2 u_{i,j} = \frac{1}{2}\left[\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j+1} + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}\right] + f(x_i, t_j),$$
$$i = 1,2,\ldots,N-1, \ j = 1,2,\ldots,M-1.$$

Let $r = \frac{k}{h}$, then the above equation can be rewritten as:

$$\left[1 + \frac{\delta_x^2}{12}\right]\left[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right]$$
$$= \frac{r^2}{2}\left[1 + \frac{\delta_x^2}{12}\right]\left[{\delta_x}^2 - \frac{1}{12}{\delta_x}^4 + \frac{1}{90}{\delta_x}^6 + O({\delta_x}^8)\right]\left[u_{i,j+1} + u_{i,j}\right]$$
$$+k^2\left[1 + \frac{\delta_x^2}{12}\right] f(x_i, t_j).$$
$$= \frac{r^2}{2}\left[{\delta_x}^2 - \frac{1}{12}{\delta_x}^4 + \frac{1}{90}{\delta_x}^6 + \frac{1}{12}{\delta_x}^4 - \frac{1}{144}\delta_x^6 + O({\delta_x}^8)\right]\left[u_{i,j+1} + u_{i,j}\right]$$
$$+k^2\left[1 + \frac{\delta_x^2}{12}\right] f(x_i, t_j).$$

Hence

$$\left[1+\frac{\delta_x^2}{12}\right][u_{i,j+1}-2u_{i,j}+u_{i,j-1}]$$
$$=\frac{r^2}{2}\left[\delta_x{}^2+\left(\frac{1}{240}\right)\delta_x^6+O\left(\delta_x{}^8\right)\right][u_{i,j+1}+u_{i,j}]+k^2\left[1+\frac{\delta_x^2}{12}\right]f(x_i,t_j).$$
$$=\frac{r^2}{2}[\delta_x{}^2+O\left(\delta_x{}^6\right)][u_{i,j+1}+u_{i,j}]+k^2\left[1+\frac{\delta_x^2}{12}\right]f(x_i,t_j).$$

where $O\left(\delta_x{}^6\right)$ denotes terms containing six and higher powers of δ_x. Assuming these terms are negligible, then the above equation becomes,

$$\left[1+\frac{\delta_x^2}{12}\right](u_{i,j+1}-2u_{i,j}+u_{i,j-1})\cong\frac{r^2}{2}\delta_x{}^2(u_{i,j+1}+u_{i,j})+k^2\left[1+\frac{\delta_x^2}{12}\right]f(x_i,t_j).$$

After simple computations, one can have:

$$(1-6r^2)u_{i-1,j+1}+(10+12r^2)u_{i,j+1}+(1-6r^2)u_{i+1,j+1}=C_{i,j} \qquad (3.6)$$

where

$$C_{i,j}=(2+6r^2)u_{i-1,j}+(20-12r^2)u_{i,j}+(2+6r^2)u_{i+1,j}-u_{i-1,j-1}-$$
$$10\,u_{i,j-1}-u_{i+1,j-1}+k^2[f(x_{i-1},t_j)+10f(x_i,t_j)+f(x_{i+1},t_j)],$$

and $i=1,2,\dots,N-1,\ \ j=1,2,\dots,M-1$

By substituting $x=x_i$ in equation (3.2), one can obtain:

$$u_{i,0}=d(x_i), \qquad i=0,1,\dots,N$$

On the other hand, approximate equation (3.3) by using the forward finite difference formula to get:

$$u_{i,1}=kr(x_i)+u_{i,0},\ \ i=0,1,\dots,N \qquad (3.7)$$

The integrals in equations (3.4)-(3.5) can be approximated by using some quadrature rules say Simpson's 1/3 rule, to obtain:

$$\begin{aligned} a_{0,j}u_{0,j+1} + a_{1,j}u_{1,j+1} + \dots + a_{N,j}u_{N,j+1} &= -3\, g_0(t_{j+1}) \\ b_{0,j}u_{0,j+1} + b_{1,j}u_{1,j+1} + \dots + b_{N,j}u_{N,j+1} &= -3\, g_1(t_{j+1}) \end{aligned}, \quad j = 1,2,\dots,M-1 \tag{3.8}$$

where

$a_{0,j} = h\, w_0(x_0) - 3,\qquad b_{0,j} = h\, w_1(x_0),$

$a_{N,j} = h\, w_0(x_N),\qquad b_{N,j} = h\, w_1(x_N) - 3,$

$a_{2i+1,j} = 4h\, w_0(x_{2i+1}),\qquad b_{2i+1,j} = 4h\, w_1(x_{2i+1}),\qquad i = 0,1,\dots,\frac{N}{2}-1,$

$a_{2i,j} = 2h\, w_0(x_{2i})$ and $b_{2i,j} = 2h\, w_1(x_{2i}),\qquad i = 1,2,\dots,\frac{N}{2}-1\,.$

Therefore equations (3.6) and (3.8) can be written in the matrix form:

$$\begin{bmatrix} a_{0,j} & a_{1,j} & a_{2,j} & & a_{N-2,j} & a_{N-1,j} & a_{N,j} \\ \alpha & \beta & \alpha & \cdots & 0 & 0 & 0 \\ 0 & \alpha & \beta & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & \beta & \alpha & 0 \\ 0 & 0 & 0 & \cdots & \alpha & \beta & \alpha \\ b_{0,j} & b_{1,j} & b_{2,j} & & b_{N-2,j} & b_{N-1,j} & b_{N,j} \end{bmatrix} . \begin{bmatrix} u_{0,j+1} \\ u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{N-1,j+1} \\ u_{N,j+1} \end{bmatrix} = \begin{bmatrix} -3\, g_0(t_{j+1}) \\ C_{1,j} \\ C_{2,j} \\ \vdots \\ C_{N-1,j} \\ -3\, g_1(t_{j+1}) \end{bmatrix} \tag{3.9}$$

where $\alpha = 1 - 6r^2$, $\beta = 10 + 12r^2$. This linear system can be solved by using any suitable method to find the numerical solutions $u_{i,j}$, i=0,1,…,N, j=2,3,…,M of the nonlocal problem given by equations (3.1)-(3.5).

To illustrate this method consider the following examples.

Example (3.1):

Consider the nonlocal initial-boundary value problem that consists of the one-dimensional wave equation:

$$u_{tt} - u_{xx} = -2, \quad x \in [0,1], \quad t \in [0,1] \tag{3.10}$$

together with the local initial conditions:

$$u(x,0) = x^2, \qquad x \in [0,1] \tag{3.11}$$

$$u_t(x,0) = 1, \qquad x \in [0,1] \tag{3.12}$$

and the nonlocal linear integral boundary conditions:

$$u(0,t) = \int_0^1 x\,u(x,t)\,dx + \frac{4t - (1+t)^2 + t^2}{4}, \qquad t \in [0,1] \tag{3.13}$$

$$u(1,t) = \int_0^1 \frac{x}{2} u(x,t)\,dx + \frac{8(1+t) - (1+t)^2 + t^2}{8}, \quad t \in [0,1] \tag{3.14}$$

This example is constructed, such that the exact solution is

$$u(x,t) = x^2 + t.$$

Let N= M=4, then we get $h = k = \frac{1}{4}, r = 1,\ x_i = \frac{i}{4},\ t_j = \frac{j}{4},\ i,j = 0,1,2,3,4.$ From equation (3.11), one can have:

$$u_{i,0} = (x_i)^2 = \left(\frac{i}{4}\right)^2 = \frac{i^2}{16}, \qquad i = 0,1,2,3,4.$$

Therefore

$$u_{0,0} = 0,\ u_{1,0} = \frac{1}{16},\ u_{2,0} = \frac{1}{8},\ u_{3,0} = \frac{9}{16} \ \text{and}\ u_{4,0} = 1.$$

By using equation (3.7), equation (3.12), can be approximated as:

$$u_{i,1} = \frac{1}{4} + \frac{i^2}{16}, \qquad i = 0,1,2,3,4.$$

Hence

$$u_{0,1} = \frac{1}{4},\ u_{1,1} = \frac{5}{16},\ u_{2,1} = \frac{1}{2},\ u_{3,1} = \frac{13}{16},\ u_{4,1} = \frac{5}{4}.$$

Now, for j=1, equation (3.9) takes the following form:

$$\begin{bmatrix} -3 & 0.25 & 0.25 & 0.75 & 0.25 \\ -5 & 22 & -5 & 0 & 0 \\ 0 & -5 & 22 & -5 & 0 \\ 0 & 0 & -5 & 22 & -5 \\ 0 & 0.125 & 0.125 & 0.375 & -2.875 \end{bmatrix} \begin{bmatrix} u_{0,2} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{4,2} \end{bmatrix} = \begin{bmatrix} 0.0938 \\ 6.125 \\ 8.3750 \\ 12.125 \\ -3.7500 \end{bmatrix}.$$

This system can be easily solved by using any suitable method to find the numerical solutions $u_{i,2}$, $i = 0,1,2,3,4$ that are tabulated down in table (3.1).
Next, we substitute j=2 in equation (3.9) and solving the resulting linear system of equations to get the numerical solutions $u_{i,3}$, $i = 0,1,2,3,4$. By continuing in this manner one can get the numerical solutions $u_{i,4}$, $i = 0,1,2,3,4$. These numerical solutions are tabulated down in table (3.1).

Table (3.1) represents the exact and the numerical solutions for h=k=$\frac{1}{4}$ of example (3.1).

i	j	x_i	t_j	$u(x_i,t_2)$	$u_{i,2}$
0		0.00		0.5000	0.46781
1		0.25		0.5625	0.5547
2	2	0.50	0.50	0.7500	0.7481
3		0.75		1.0625	1.0260
4		1.00		1.5000	1.4995
0		0.00		0.7500	0.5477
1		0.25		0.8125	0.7399
2	3	0.50	0.75	0.1000	0.9500
3		0.75		1.3125	1.2391
4		1.00		1.7500	1.5395
0		0.00		1.0000	0.7845
1		0.25		1.0625	0.9139
2	4	0.50	1.00	1.2500	1.0791
3		0.75		1.5625	1.3862
4		1.00		2.0000	1.8579

Now if we take N=20 and M=40, then h=$\frac{1}{20}$, $k = \frac{1}{40}$, $r = \frac{1}{2}$. $x_i = \frac{i}{20}$, $t_j = \frac{j}{40}$, $i = 0,1,\dots,20$, $j = 0,1,\dots,40$.. By following the same previous steps one can get some of the numerical solutions that are tabulated in table (3.2). In table (3.3) we take different values for h and k and the values of the absolute errors at some special values.

Table (3.2) represents the exact and the numerical solutions for h=$\frac{1}{20}$, k=$\frac{1}{40}$ of example (3.1).

i	j	x_i	t_j	$u(x_i, t_2)$	$u_{i,2}$
0		0.00		0.0500	0.0494
1		0.05		0. 0525	0. 0525
2	2	0.10	0.05	0. 0600	0. 0600
3		0.15		0.0725	0.0725
4		0.20		0.0900	0.0900
5		0.25		0.1125	0.1125
6		0.30		0.1400	0.1400
7		0.35		0.1725	0.1725
8		0.40		0.2100	0.2100
9		0.45		0.2525	0.2525
10		0.50		0.3000	0.3000

Table (3.3) represent the absolute errors for spatial values for h and k of example (3.1).

h	k	r	x_i	t_2	$\lvert u(x_i, t_2) - u_{i,2} \rvert$
0.1	0.1	1	0.2 0.5 1	0.2	2.56×10^{-6} 6.471×10^{-6} 1.949×10^{-5}
0.005	0.025	5	0.2 0.5 1	0.05	1.753×10^{-15} 4.665×10^{-16} 9.360×10^{-8}

Example (3.2):

Consider the nonlocal initial-boundary value problem that consists of the one-dimensional wave equation*:*

$$u_{tt} - u_{xx} = -x\sin(t), \quad x \in [0,1], \quad t \in [0,1] \qquad (3.15)$$

together with the local initial conditions:

$$u(x,0) = 0, \quad x \in [0,1] \qquad (3.16)$$

$$u_t(x,0) = x, \quad x \in [0,1] \qquad (3.17)$$

and the nonlocal linear integral boundary conditions:

$$u(0,t) = \int_0^1 u(x,t)\ dx - \frac{\sin(t)}{2}, \quad t \in [0,1] \qquad (3.18)$$

$$u(1,t) = \int_0^1 x^2\, u(x,t)\, dx + \frac{3\sin(t)}{4}, \quad t \in [0,1] \qquad (3.19)$$

This example is constructed, such that the exact solution is

$$u(x,t) = x\sin(t).$$

Now if we take N=100 and M=40, then we get h=$\frac{1}{100}$, $k = \frac{1}{40}$, $r = 6.25$, $x_i = \frac{i}{100}$, $t_j = \frac{j}{40}$, $i = 0,1,\dots,100$, $j = 0,1,\dots,40$.

From equation (3.16), one can have:

$$u_{i,0} = 0, \qquad i = 0,1,\dots,100.$$

By using equation (3.7), equation (3.17), can be approximated as:

$$u_{i,1} = \frac{i}{4000}, \qquad i = 0,1,2,\dots,100.$$

Hence

$$u_{0,1} = 0\,,\ u_{1,1} = \frac{1}{4000}\,, \dots ,\ u_{100,1} = \frac{1}{40}\,.$$

By following the same previous steps one can get some of the numerical solutions that are tabulated in table (3.4), in table (3.5) we take different values for h and k and the values of the absolute errors at some special values.

Table (3.4), represents the exact and the numerical solutions for h=$\frac{1}{100}$, k=$\frac{1}{40}$of example (3.2).

i	j	x_i	t_j	$u(x_i,t_2)$	$u_{i,2}$
0		0.00		0.0000	0.0000
1		0.01		0.0005	0. 0005
2	2	0.02	0.05	0.0010	0. 0010
3		0.03		0.0015	0.0015
4		0.04		0.0020	0.0020
5		0.05		0.0025	0.0025
6		0.06		0.0030	0. 0030
7		0.07		0.0040	0.0040
8		0.08		0.0045	0. 0045
9		0.09		0.0050	0.0050
10		0.10		0.0055	0.0055

Table (3.5), represent the absolute errors for spatial values for h and k of example (3.2).

h	k	r	x_i	t_2	$\lvert u(x_i,t_2)-u_{i,2}\rvert$
0.1	0.1	1	0.2 0.5 1	0.2	3.129×10^{-4} 3.276×10^{-4} 1.313×10^{-4}
0.005	0.025	25	0.2 0.5 1	0.05	1.0125×10^{-9} 1.9511×10^{-9} 2.4661×10^{-9}

3.2 Solutions of the One-Dimensional Hyperbolic Equation with the Nonlocal Non-Linear Integral Boundary Conditions:

Consider the nonlocal initial-boundary value problem that consists of the one-dimensional hyperbolic equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left[a(x)\ \frac{\partial u(x,t)}{\partial x}\right] = f(x,t),\ \ x \in [0,1],\ \ t \in [0,T] \qquad (3.20)$$

together with the local initial conditions:

$$u(x,0) = d(x),\ \ x \in [0,1] \qquad (3.21)$$

$$u_t(x,0\ ) = r(x),\ \ x \in [0,1] \qquad (3.22)$$

and the nonlocal non-linear integral boundary conditions:

$$u(0,t) = \int_0^1 w_0\,(x)\ u^p(x,t)\ dx + g_0(t),\ \ t \in [0,T] \qquad (3.23)$$

$$u(1,t) = \int_0^1 w_1\,(x)\ u^q(x,t)\ dx + g_1(t),\ \ t \in [0,T] \qquad (3.24)$$

where $p \geq 1, q \geq 1$ are known constants, f is a known continuous function of x and t , a is a known continuous function of x, d, w_0, w_1, g_0, g_1 and r are known continuous functions that must satisfy the compatibility conditions:

(1) $d(0) = \int_0^1 w_0\,(x)\ [d(x)]^p\ dx + g_0(0).$

(2) $d(1) = \int_0^1 w_1\,(x)\ [d(x)]^q\ dx + g_1(0).$

(3) $r(0) = p\int_0^1 w_0\,(x)[d(x)]^{p-1}r(x)dx + \left.\frac{dg_0}{dt}\right|_{t=0}.$

(4) $r(1) = q\int_0^1 w_1\,(x)[d(x)]^{q-1}r(x)dx + \left.\frac{dg_1}{dt}\right|_{t=0}.$

In this section, we used Crank-Niklson finite difference scheme for finding the solutions of the nonlocal initial-boundary value problem given by equations (3.20)-(3.24). To do this, we divide the region $[0, 1] \times [0, T]$ into $N \times M$ mesh points with spatial step size $h = 1/N$ in the x-direction and the time step size $k = T/M$ respectively, where M is positive integer and N is even positive integers.

The mesh points are given by:

$$x_i = ih, \qquad i = 0,1,\dots,N,$$

$$t_j = jk, \qquad j = 0,1,\dots,M.$$

Define the following difference operators, [36]:

$$\delta_x u(x_i,t_j) = u_{i+1,j} - u_{i,j}, \qquad i = 0,1,\dots,N-1, \ \ j = 0,1,\dots,M,$$

$$\delta_x^2 u(x_i,t_j) = u_{i-1,j} - 2u_{i,j} + u_{i+1,j}, \ \ i = 1,2,\dots,N-1, \ \ j = 0,1,\dots,M,$$

$$\delta_t^2 u(x_i,t_j) = u_{i,j-1} - 2u_{i,j} + u_{i,j+1}, \ \ i = 0,1,\dots,N, \ \ j = 1,2,\dots,M-1,$$

where $u_{i,j}$ is the numerical solution of the nonlocal problem given by equations (3.20)-(3.24) at the point (x_i,t_j). We replaced $\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}$ by the mean of its finite difference representation on the (j+1)-th and j-th time rows:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{1}{2h^2}\left[\delta_x^2 u(x_i,t_j) + \delta_x^2 u(x_i,t_{j+1})\right],$$

$$i = 1,2,\dots,N-1, \ \ j = 1,2,\dots,M-1.$$

Then we approximate equation (3.20) at the point (i,j) by

$$\frac{1}{k^2}\delta_t^2 u_{i,j} = \frac{a'(x_i)}{h}\delta_x u_{i,j} + \frac{a(x_i)}{2h^2}\left[\delta_x^2\left(u_{i,j+1} + u_{i,j}\right)\right] + f(x_i,t_j),$$

$$i = 1,2,\dots,N-1, \ \ j = 1,2,\dots,M-1.$$

Let $r = \frac{k}{h}$, then the above equation becomes:

$$-r^2 a(x_i)u_{i-1,j+1} + 2[1 + a(x_i)r^2]u_{i,j+1} - r^2 a(x_i)u_{i+1,j+1} = C_{i,j},$$

$$i = 1,2,\dots,N-1, \ \ j = 1,2,\dots,M-1 \qquad (3.25)$$

where

$$C_{i,j} = r^2 a(x_i)u_{i-1,j} + 2\left[2 - r^2 a(x_i) + \frac{k^2}{h}a'(x_i)\right]u_{i,j} + \left[r^2 a(x_i) + 2\frac{k^2}{h}a'(x_i)\right]u_{i+1,j} - 2u_{i,j-1} + 2k^2 f(x_i,t_j).$$

By substituting $x=x_i$ in equation (3.21), one can obtain:

$$u_{i,0} = d(x_i), \qquad i = 0,1,\dots,N,$$

then we approximate equation (3.22) by using forward finite difference formula to get:

$$u_{i,1} = kr(x_i) + u_{i,0}\,, \quad i = 0,1,\dots,N \tag{3.26}$$

Now, by using Taylor's expansion for the nonlinear functions $u^p(x_i, t_{j+1})$ and $u^q(x_i, t_{j+1})$ about the point (x_i, t_j), one can have:

$$u^p_{i,j+1} = u^p_{i,j} + kpu^{p-1}_{i,j}\frac{\partial u_{i,j}}{\partial t} + O(k^2)$$

and

$$u^q_{i,j+1} = u^q_{i,j} + kqu^{q-1}_{i,j}\frac{\partial u_{i,j}}{\partial t} + O(k^2)$$

respectively, where $O(k^2)$ denotes terms containing second and higher powers of k. But

$$\frac{\partial u_{i,j}}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k}.$$

Therefore, one can get:

$$\left.\begin{aligned} u^p_{i,j+1} &= (1-p)u^p_{i,j} + pu^{p-1}_{i,j}u_{i,j+1} \\ u^q_{i,j+1} &= (1-q)u^q_{i,j} + qu^{q-1}_{i,j}u_{i,j+1} \end{aligned}\right\}, \quad i = 0,1,\dots,N, \quad j = 0,1,\dots,M-1 \tag{3.27}$$

Moreover the integrals that appeared in equations (3.23)-(3.24) can be approximated by using some quadrature rules say Simpson's 1/3 rule, to obtain:

$$u_{0,j+1} = \tfrac{h}{3}[w_0(x_0)u^p_{0,j+1} + 4\,w_0(x_1)u^p_{1,j+1} + 2w_0(x_2)u^p_{2,j+1} + \cdots + w_0(x_N)u^p_{N,j+1}] + g_0(t_{j+1}), \quad j = 1,2,\dots,M-1 \tag{3.28}$$

and

$$u_{N,j+1} = \tfrac{h}{3}[w_1(x_0)u^q_{0,j+1} + 4\,w_1(x_1)u^q_{1,j+1} + 2w_1(x_2)u^q_{2,j+1} + \cdots + w_1(x_N)u^q_{N,j+1}] + g_1(t_{j+1}), \quad j = 1,2,\dots,M-1 \tag{3.29}$$

By substituting equations (3.27) in equations (3.28) and (3.29), one can get:

$$a_{0,j}u_{0,j+1} + a_{i,j}\, u_{i,j+1} + \cdots + a_{N,j}\, u_{N,j+1} = L_{N,j}, \quad j = 1,2,\dots,M-1 \quad (3.30)$$

and

$$b_{0,j}u_{0,j+1} + b_{i,j}\, u_{i,j+1} + \cdots + b_{N,j}\, u_{N,j+1} = Q_{N,j}, \quad j = 1,2,\dots,M-1 \quad (3.31)$$

where

$$a_{0,j} = phw_0(x_0)u_{0,j}^{p-1} - 3, \qquad j = 1,2,\dots,M,$$

$$b_{0,j} = qhw_1(x_0)u_{0,j}^{q-1}, \qquad j = 1,2,\dots,M,$$

$$a_{2i+1,j} = 4phw_0(x_i)u_{i,j}^{p-1}, \qquad i = 0,1,\dots,\frac{N}{2}-1, \; j = 1,2,\dots,M,$$

$$b_{2i+1,j} = 4qhw_1(x_i)u_{i,j}^{q-1}, \qquad i = 0,1,\dots,\frac{N}{2}-1, \; j = 1,2,\dots,M,$$

$$a_{2i+1,j} = 2phw_0(x_i)u_{i,j}^{p-1}, \qquad i = 1,2,\dots,\frac{N}{2}-1, \; j = 1,2,\dots,M,$$

$$b_{2i+1,j} = 2qhw_1(x_i)u_{i,j}^{q-1}, \qquad i = 1,2,\dots,\frac{N}{2}-1, \; j = 1,2,\dots,M,$$

$$a_{N,j} = phw_0(x_N)u_{N,j}^{p-1}, \qquad j = 1,2,\dots,M,$$

$$b_{N,j} = qhw_1(x_N)u_{N,j}^{q-1} - 3, \qquad j = 1,2,\dots,M,$$

$$L_{N,j} = (p-1)hw_0(x_0)u_{0,j}^{p} + 4(p-1)hw_0(x_1)u_{1,j}^{p} + 2(p-1)hw_0(x_2)u_{2,j}^{p} + \cdots + (p-1)hw_0(x_N)u_{N,j}^{p} - 3\, g_0(t_{j+1}), \; j = 1,2,\dots,M-1$$

and

$$Q_{N,j} = (q-1)hw_1(x_0)u_{0,j}^{q} + 4(q-1)hw_1(x_1)u_{1,j}^{q} + 2(q-1)hw_1(x_2)u_{2,j}^{q} + \cdots + (q-1)hw_1(x_N)u_{N,j}^{q} - 3\, g_1(t_{j+1}), \; j = 1,2,\dots,M-1.$$

Therefore equations (3.25), (3.30) and (3.31) can be written in the matrix form:

$$\begin{bmatrix} a_{0,j} & a_{1,j} & a_{2,j} & & a_{N-2,j} & a_{N-1,j} & a_{N,j} \\ \alpha_0 & \beta_1 & \alpha_2 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_2 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & \beta_{N-2} & \alpha_{N-1} & 0 \\ 0 & 0 & 0 & \cdots & \alpha_{N-2} & \beta_{N-1} & \alpha_N \\ b_{0,j} & b_{1,j} & b_{2,j} & & b_{N-2,j} & b_{N-1,j} & b_{N,j} \end{bmatrix} . \begin{bmatrix} u_{0,j+1} \\ u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{N-1,j+1} \\ u_{N,j+1} \end{bmatrix} = \begin{bmatrix} L_{N,j} \\ C_{1,j} \\ C_{2,j} \\ \vdots \\ C_{N-1,j} \\ Q_{N,j} \end{bmatrix} \quad (3.32)$$

where $\alpha_i = -r^2 a(x_i)$, $\beta_i = 2[1 + r^2 a(x_i)]$, $i = 0,1,\dots,N$, $j = 1,2,\dots,M-1$.This linear system can be solved by using any suitable method to find the numerical solutions $u_{i,j}$, $i = 0,1,\dots,N$, $j = 2,3,\dots,M$ of the nonlocal problem given by equations (3.20)-(3.24).

To illustrate this method consider the following examples.

Example (3.3):

Consider the nonlocal initial-boundary value problem that consists of the one-dimensional wave equation:

$$u_{tt} - u_{xx} = -2, \quad x \in [0,1], \quad t \in [0,1] \quad (3.33)$$

together with the local initial conditions:

$$u(x,0) = x^2, \quad x \in [0,1] \quad (3.34)$$

$$u_t(x,0) = 1, \quad x \in [0,1] \quad (3.35)$$

and the nonlocal non-linear integral boundary conditions:

$$u(0,t) = \int_0^1 x\, u^2(x,t)\, dx + \frac{6t - (1+t)^3 + t^3}{6}, \quad t \in [0,1] \quad (3.36)$$

$$u(1,t) = \int_0^1 \frac{x}{2} u^3(x,t)\, dx + \frac{16(1+t) - (1+t)^4 + t^4}{16}, \quad t \in [0,1] \quad (3.37)$$

This example is constructed such that the exact solution is

$$u(x,t) = x^2 + t\,.$$

Let N= M=4, then we get $h = k = \frac{1}{4}, r = 1,\ x_i = \frac{i}{4},\ t_j = \frac{j}{4},\ i, j = 0,1,2,3,4$. From equation (3.34), one can get:

$$u_{i,0} = (x_i)^2 = \left(\frac{i}{4}\right)^2 = \frac{i^2}{16}, \qquad i = 0,1,2,3,4.$$

Therefore

$$u_{0,0} = 0,\ u_{1,0} = \frac{1}{16},\ u_{2,0} = \frac{1}{8},\ u_{3,0} = \frac{9}{16} \text{ and } u_{4,0} = 1.$$

By using equation (3.26), equation (3.35), can be approximated as:

$$u_{i,1} = \frac{1}{4} + \frac{i^2}{16}, \qquad i = 0,1,2,3,4.$$

Hence

$$u_{0,1} = \frac{1}{4},\ u_{1,1} = \frac{5}{16},\ u_{2,1} = \frac{1}{2},\ u_{3,1} = \frac{13}{16},\ u_{4,1} = \frac{5}{4}.$$

Now, for j=1, equation (3.32) takes the following form:

$$\begin{bmatrix} -3 & 0.1563 & 0.2500 & 1.2188 & 0.6250 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0.0366 & 0.0938 & 0.7427 & -2.4141 \end{bmatrix} \begin{bmatrix} u_{0,2} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{4,2} \end{bmatrix} = \begin{bmatrix} 1.0977 \\ 1 \\ 1.375 \\ 2 \\ -2.6331 \end{bmatrix}.$$

This system can be easily solved by using any suitable method to find the numerical solutions $u_{i,2}$, $i = 0,1,2,3,4$ that are tabulated down in table (3.6).

Next, we substitute j=2 in equation (3.32) and solving the resulting linear system of equations to get the numerical solutions $u_{i,3}$, $i = 0,1,2,3,4$. By continuing in this manner one can get the numerical solutions $u_{i,4}$, $i = 0,1,2,3,4$. These numerical solutions are tabulated down in table (3.6).

Table (3.6) represents the exact and the numerical solutions for h=k=$\frac{1}{4}$ of example (3.3).

i	j	x_i	t_j	$u(x_i,t_2)$	$u_{i,2}$
0		0.00		0.5000	0.4492
1		0.25		0.5625	0.5478
2	2	0.50	0.5	0.7500	0.7421
3		0.75		1.0625	1.0455
4		1.00		1.5000	1.4398
0		0.00		0.7500	0.6744
1		0.25		0.8125	0.7614
2	3	0.50	0.75	0.1000	0.9592
3		0.75		1.3125	1.2481
4		1.00		1.7500	1.6352
0		0.00		1.0000	0.8001
1		0.25		1.0625	1.1395
2	4	0.50	1	1.2500	1.3169
3		0.75		1.5625	1.8046
4		1.00		2.0000	2.2810

Now if we take N=20 and M=40, then h=$\frac{1}{20}$, $k = \frac{1}{40}$, $r = \frac{1}{2}$. $x_i = \frac{i}{20}$, $t_j = \frac{j}{40}$, $i = 0,1,\dots,20$, $j = 0,1,\dots,40$.. By following the same previous steps one can get some of the numerical solutions that are tabulated in table (3.7).

In table (3.8) we take different values for h and k and the values of the absolute errors at some special values.

Table (3.7) represents the exact and the numerical solutions for $h=\frac{1}{20}$, $k=\frac{1}{40}$ of example (3.3).

i	j	x_i	t_j	$u(x_i, t_2)$	$u_{i,2}$
0		0.00		0.0500	0.0497
1		0.05		0. 0525	0. 0525
2	2	0.10	0.05	0. 0600	0. 0600
3		0.15		0.0725	0.0725
4		0.20		0.0900	0.0900
5		0.25		0.1125	0.1125
6		0.30		0.1400	0.1400
7		0.35		0.1725	0.1725
8		0.40		0.2100	0.2100
9		0.45		0.2525	0.2525
10		0.50		0.3000	0.3000

Table (3.8) represent the absolute errors for spatial values for h and k of example (3.3).

h	k	r	x_i	t_2	$\lvert u(x_i, t_2) - u_{i,2} \rvert$
0.1	0.1	1	0.2	0.2	5.7×10^{-5}
			0.5		1.5×10^{-5}
			1		5.2×10^{-4}
0.005	0.025	5	0.2	0.05	1.516×10^{-14}
			0.5		1.110×10^{-16}
			1		2.565×10^{-7}

Example (3. 4):

Consider the nonlocal initial-boundary value problem that consists of the one-dimensional hyperbolic equation:

$$u_{tt} - (x+1)u_{xx} = -x\,e^{x+t}, \quad x \in [0,1], \quad t \in [0,1] \tag{3.38}$$

together with the local initial conditions:

$$u(x,0) = e^{x}, \quad x \in [0,1] \tag{3.39}$$

$$u_t(x,0) = e^{x}, \quad x \in [0,1] \tag{3.40}$$

and the nonlocal non-linear boundary conditions:

$$u(0,t) = \int_0^1 e^{x}\,u^2(x,t)\,dx + \frac{3e^{t} + e^{2t} - e^{2t+3}}{3}, \quad t \in [0,1] \tag{3.41}$$

$$u(1,t) = \int_0^1 2\,u^4(x,t)\,dx + \frac{2e^{(t+1)} + e^{4t} - e^{(4t+4)}}{2}, \quad t \in [0,1] \tag{3.42}$$

This example is constructed such that the exact solution is

$$u(x,t) = e^{x+t}.$$

Let N=1000 and M=500, then we get h=$\frac{1}{1000}$, $k = \frac{1}{500}$, $r = 2$, $x_i = \frac{i}{1000}$, $t_j = \frac{j}{500}$, $i = 0,1,\dots,1000, j = 0,1,\dots,500$.

From equation (3.39), one can have:

$$u_{i,0} = e^{i/1000}, \qquad i = 0,1,\dots,1000.$$

Therefore

$$u_{0,0} = 1\,,\ u_{0,1} = 1.001, \dots\,,\ u_{100,0} = 2.718.$$

By using equation (3.7), equation (3.40), can be approximated as:

$$u_{i,1} = (1.025)e^{i/1000}, \qquad i = 0,1,2,\dots,1000.$$

Hence

$$u_{0,1} = 1.025\,,\ u_{1,1} = 1.026\,, \dots\,,\ u_{100,1} = 2.786\,.$$

By following the same previous steps one can get some of the numerical solutions that are tabulated in table (3.9), in table (3.10) we take different values for h and k and the values of the absolute errors at some special values.

Table (3.9) represents the exact and the numerical solutions for h=$\frac{1}{1000}$, k=$\frac{1}{500}$ of example (3.4).

i	j	x_i	t_j	$u(x_i,t_2)$	$u_{i,2}$
0		0.000		1.0040	1.0040
1		0.001		1.0050	1.0050
2	2	0.002	0.004	1.0060	1.0060
3		0.003		1.0070	1.0070
4		0.004		1.0080	1.0080
5		0.005		1.0090	1.0090
6		0.006		1.0100	1.0100
7		0.007		1.0110	1.0111
8		0.008		1.0120	1.0121
9		0.009		1.0130	1.0131
10		0.010		1.0141	1.0141

Table (3.10) represent the absolute errors for spatial values for h and k of example (3.4).

h	k	r	x_i	t_2	$\lvert u(x_i,t_2) - u_{i,2} \rvert$
0.1	0.1	1	0.2 0.5 1	0.2	5.35×10^{-4} 1.51×10^{-4} 7.66×10^{-4}
0.001	0.005	5	0.2 0.5 1	0.01	1.145×10^{-8} 6×10^{-8} 7×10^{-8}

3.3 Solutions of the One-Dimensional Parabolic Equation with the Nonlocal Non-Linear Integral Boundary Conditions:

Consider the nonlocal initial-boundary value problem that consists of the one-dimensional parabolic equation:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial}{\partial x}\left[a(x)\,\frac{\partial u(x,t)}{\partial x}\right] = f(x,t), \quad x \in [0,1], \quad t \in [0,T] \tag{3.43}$$

together with the local initial condition:

$$u(x,0) = d(x), \quad x \in [0,1] \tag{3.44}$$

and the nonlocal non-linear integral boundary conditions:

$$u(0,t) = \int_0^1 w_0\,(x)\; u^p(x,t)\,dx + g_0(t), \qquad t \in [0,T] \tag{3.45}$$

$$u(1,t) = \int_0^1 w_1\,(x)\; u^q(x,t)\,dx + g_1(t), \qquad t \in [0,T] \tag{3.46}$$

where $p \geq 1, q \geq 1$ are known constants, f is a known continuous function of x and t , a is a known continuous function of x, d, w_0, w_1, g_0 and g_1 are known continuous functions that must satisfy the compatibility conditions:

$$(1)\; d(0) = \int_0^1 w_0\,(x)\,[d(x)]^p\,dx + g_0(0).$$

$$(2)\; d(1) = \int_0^1 w_1\,(x)\,[d(x)]^q\,dx + g_1(0).$$

In this section, we used Crank-Niklson finite difference scheme for finding the solutions of the nonlocal initial-boundary value problem given by equations (3.43)-(3.46). To do this, we divide the region $[0, 1] \times [0, T]$ into $N \times M$ mesh points with spatial step size $h = 1/N$ in the x-direction and the time step size $k = T/M$ respectively, where M is positive integer and N is even positive integers.

The mesh points are given by:

$$x_i = ih, \qquad i = 0,1,\ldots,N,$$

$$t_j = jk, \qquad j = 0,1,\ldots,M.$$

Define the following difference operators, [36]:

$$\delta_t u(x_i,t_j) = u_{i,j+1} - u_{i,j}, \qquad i = 0,1,\ldots,N,\ \ j = 0,1,\ldots,M-1,$$

$$\delta_x^2 u(x_i,t_j) = u_{i-1,j} - 2u_{i,j} + u_{i+1,j}, \ \ i = 1,2,\ldots,N-1,\ \ j = 0,1,\ldots,M,$$

$$\delta_t^2 u(x_i,t_j) = u_{i,j-1} - 2u_{i,j} + u_{i,j+1}, \ \ i = 0,1,\ldots,N,\ \ j = 1,2,\ldots,M-1,$$

where $u_{i,j}$ is the numerical solution of the nonlocal problem given by equations (3.43)-(3.46) at the point (x_i,t_j). We replaced $\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}$ by the mean of its finite difference representation on the (j+1)-th and j-th time rows:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{1}{2h^2}\left[\delta_x^2 u(x_i,t_j) + \delta_x^2 u(x_i,t_{j+1})\right],$$

$$i = 1,2,\ldots,N-1,\ \ j = 1,2,\ldots,M-1.$$

Then we approximate equation (3.43) at the point (i,j) by:

$$\tfrac{1}{k}\delta_t u_{i,j} = \frac{a'(x_i)}{h}\delta_x u_{i,j} + \frac{a(x_i)}{2h^2}\left[\delta_x^2\left(u_{i,j+1} + u_{i,j}\right)\right] + f(x_i,t_j),$$

$$i = 1,2,\ldots,N-1,\ \ j = 0,1,\ldots,M-1.$$

Let $r = \frac{k}{h}$, then the above equation becomes:

$$-ra(x_i)u_{i-1,j+1} + 2[1 + a(x_i)r]u_{i,j+1} - ra(x_i)u_{i+1,j+1} = C_{i,j},$$

$$i = 1,2,\ldots,N-1,\ \ j = 1,2,\ldots,M-1 \qquad (3.47)$$

where

$$C_{i,j} = ra(x_i)u_{i-1,j} + 2\left[1 - ra(x_i) - \frac{k}{h}a'(x_i)\right]u_{i,j} + \left[ra(x_i) + 2\frac{k}{h}a'(x_i)\right]u_{i+1,j} + 2k\, f(x_i,t_j).$$

By substituting $x=x_i$ in equation (3.44), one can obtain:

$$u_{i,0} = d(x_i), \quad i = 0,1,\dots,N.$$

Now, by using Taylor's expansion for the nonlinear functions $u^p(x_i, t_{j+1})$ and $u^q(x_i, t_{j+1})$ about the point (x_i, t_j), one can have:

$$u^p_{i,j+1} = u^p_{i,j} + kpu^{p-1}_{i,j}\frac{\partial u_{i,j}}{\partial t} + O(k^2)$$

and

$$u^q_{i,j+1} = u^q_{i,j} + kqu^{q-1}_{i,j}\frac{\partial u_{i,j}}{\partial t} + O(k^2)$$

respectively. But

$$\frac{\partial u_{i,j}}{\partial t} = \frac{(u_{i,j+1} - u_{i,j})}{k}$$

Therefore, one can get:

$$\begin{aligned} u^p_{i,j+1} &= (1-p)u^p_{i,j} + pu^{p-1}_{i,j}u_{i,j+1} \\ u^q_{i,j+1} &= (1-q)u^q_{i,j} + qu^{q-1}_{i,j}u_{i,j+1} \end{aligned}, \quad i = 0,1,\dots,N, \quad j = 0,1,\dots,M-1 \qquad (3.48)$$

the integrals in equations (3.45)-(3.46) can be expressed some quadrature rules such as Simpson 1/3 rule, to obtain:

$$u_{0,j+1} = \frac{h}{3}[w_0(x_0)u^p_{0,j+1} + 4\,w_0(x_1)u^p_{1,j+1} + 2w_0(x_2)u^p_{2,j+1} + \cdots +$$

$$w_0(x_N)u^p_{N,j+1}] + g_0(t_{j+1}), \quad j = 1,2,\dots,M-1 \qquad (3.49)$$

and

$$u_{N,j+1} = \frac{h}{3}[w_1(x_0)u^q_{0,j+1} + 4\,w_1(x_1)u^q_{1,j+1} + 2w_1(x_2)u^q_{2,j+1} + \cdots +$$

$$w_1(x_N)u^q_{N,j+1}] + g_1(t_{j+1}), \quad j = 1,2,\dots,M-1 \qquad (3.50)$$

By substituting equations (3.48) in equations (3.49) and (3.50), one can get:

$$a_{0,j}u_{0,j+1} + a_{i,j}\, u_{i,j+1} + \cdots + a_{N,j}\, u_{N,j+1} = L_{N,j},\ \ j = 1,2,\ldots,M-1 \qquad (3.51)$$

and

$$b_{0,j}u_{0,j+1} + b_{i,j}\, u_{i,j+1} + \cdots + b_{N,j}\, u_{N,j+1} = Q_{N,j},\ \ j = 1,2,\ldots,M-1 \qquad (3.52)$$

where

$$a_{0,j} = phw_0(x_0)u_{0,j}^{p-1} - 3\,, \qquad j = 1,2,\ldots,M,$$

$$b_{0,j} = qhw_1(x_0)u_{0,j}^{q-1}, \qquad j = 1,2,\ldots,M,$$

$$a_{2i+1,j} = 4phw_0(x_i)u_{i,j}^{p-1}, \qquad i = 0,1,\ldots,\frac{N}{2}-1,\ \ j = 1,2,\ldots,M,$$

$$b_{2i+1,j} = 4qhw_1(x_i)u_{i,j}^{q-1}, \qquad i = 0,1,\ldots,\frac{N}{2}-1,\ \ j = 1,2,\ldots,M,$$

$$a_{2i+1,j} = 2phw_0(x_i)u_{i,j}^{p-1}, \qquad i = 1,2,\ldots,\frac{N}{2}-1,\ \ j = 1,2,\ldots,M,$$

$$b_{2i+1,j} = 2qhw_1(x_i)u_{i,j}^{q-1}, \qquad i = 1,2,\ldots,\frac{N}{2}-1,\ \ j = 1,2,\ldots,M,$$

$$a_{N,j} = phw_0(x_N)u_{N,j}^{p-1}, \qquad j = 1,2,\ldots,M,$$

$$b_{N,j} = qhw_1(x_N)u_{N,j}^{q-1} - 3, \qquad j = 1,2,\ldots,M,$$

$$L_{N,j} = (p-1)hw_0(x_0)u_{0,j}^{p} + 4(p-1)hw_0(x_1)u_{1,j}^{p} + 2(p-1)hw_0(x_2)u_{2,j}^{p} + \cdots + (p-1)hw_0(x_N)u_{N,j}^{p} - 3\,g_0\big(t_{j+1}\big),\ \ j = 1,2,\ldots,M-1$$

and

$$Q_{N,j} = (q-1)hw_1(x_0)u_{0,j}^{q} + 4(q-1)hw_1(x_1)u_{1,j}^{q} + 2(q-1)hw_1(x_2)u_{2,j}^{q} + \cdots + (q-1)hw_1(x_N)u_{N,j}^{q} - 3\,g_1\big(t_{j+1}\big)\,,\ \ j = 1,2,\ldots,M-1.$$

Therefore equations (3.47), (3.51) and (3.52) can be written in the matrix form:

$$\begin{bmatrix} a_{0,j} & a_{1,j} & a_{2,j} & & a_{N-2,j} & a_{N-1,j} & a_{N,j} \\ \alpha_0 & \beta_1 & \alpha_2 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_2 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & \beta_{N-2} & \alpha_{N-1} & 0 \\ 0 & 0 & 0 & \cdots & \alpha_{N-2} & \beta_{N-1} & \alpha_N \\ b_{0,j} & b_{1,j} & b_{2,j} & & b_{N-2,j} & b_{N-1,j} & b_{N,j} \end{bmatrix} . \begin{bmatrix} u_{0,j+1} \\ u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{N-1,j+1} \\ u_{N,j+1} \end{bmatrix} = \begin{bmatrix} L_{N,j} \\ C_{1,j} \\ C_{2,j} \\ \vdots \\ C_{N-1,j} \\ Q_{N,j} \end{bmatrix} \qquad (3.53)$$

where $\alpha_i = -ra(x_i)$, $\beta_i = 2[1 + ra(x_i)], i = 0, 1, \ldots, N$, $j = 1,2,\ldots, M-1$. This linear system can be solved by using any suitable method to find the numerical solutions $u_{i,j}$, $i = 0, 1, \ldots, N$, $j = 1,2,\ldots, M$, of the nonlocal problem given by equations (3.43)-(3.46).

To illustrate this method consider the following examples.

Example (3.5):

Consider the nonlocal initial-boundary value problem that consists of the one-dimensional heat equation:

$$u_t - u_{xx} = 2(t - 3x), \quad x \in [0, 1], \quad t \in [0,1] \qquad (3.54)$$

together with the local initial condition:

$$u(x, 0) = x^3, \quad x \in [0,1] \qquad (3.55)$$

and the nonlocal non-linear integral boundary conditions:

$$u(0, t) = \int_0^1 x^2\, u(x, t)\, dx + \frac{6t^2 - (1 + t^2)^2 + t^4}{6}, \quad t \in [0,1] \qquad (3.56)$$

$$u(1, t) = \int_0^1 3x^2\, u^3(x, t)\, dx + \frac{4(1 + t^2) - (1 + t^2)^4 + t^8}{4}, \quad t \in [0,1] \qquad (3.57)$$

This example is constructed such that the exact solution is

$$u(x, t) = x^3 + t^2.$$

Let N=4 and M=100, then we get $h=\frac{1}{4}$, $k=\frac{1}{100}$, $r=0.16$, $x_i=\frac{i}{4}$, $t_j=\frac{j}{4}$, $i=0,1,2,3,4$, $j=0,1,\ldots,100$. From equation (3.55), we obtain:

$$u_{i,0}=(x_i)^2=\left(\frac{i}{4}\right)^3=\frac{i^3}{64}, \qquad i=0,1,2,3,4.$$

Therefore

$$u_{0,0}=0,\ u_{1,0}=\frac{1}{64},\ u_{2,0}=\frac{1}{8},\ u_{3,0}=\frac{27}{64}\ \text{and}\ u_{4,0}=1.$$

In this case equation (3.53) takes the form:

$$\begin{bmatrix} -3 & 0.0625 & 0.1250 & 0.5625 & 0.2500 \\ -0.16 & 2.32 & -0.16 & 0 & 0 \\ 0 & -0.16 & 2.32 & -0.16 & 0 \\ 0 & 0 & -0.16 & 2.32 & -0.16 \\ 0 & 0.0001 & 0.0176 & 0.9010 & -0.7500 \end{bmatrix}\begin{bmatrix} u_{0,1} \\ u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \end{bmatrix}=\begin{bmatrix} 0.4998 \\ 0.0166 \\ 0.2204 \\ 0.7992 \\ -0.4951 \end{bmatrix}.$$

This system can be easily solved by using any suitable method to find the numerical solutions $u_{i,1}$, $i=0,1,2,3,4$ that are tabulated down in table (3.11).
Next, we substitute j=1 in equation (3.53) and solving the resulting linear system of equations to get the numerical solutions $u_{i,2}$, $i=0,1,2,3,4$. By continuing in this manner one can get the numerical solutions $u_{i,3}$, $i=0,1,2,3,4$. These numerical solutions are tabulated in table (3.11).

In table (3.12) we take different values for h and k and the values of the absolute errors at some special values.

Table (3.11) represents the exact and the numerical solutions for h=$\frac{1}{4}$,k=$\frac{1}{100}$ of example (3.5).

i	j	x_i	t_j	$u(x_i,t_2)$	$u_{i,2}$
0		0.00		0.0001	0.0190
1		0.25		0.0157	0.0172
2	2	0.50	0.01	0. 1251	0.1262
3		0.75		0. 4220	0.4349
4		1.00		1.0001	1.1856
0		0.00		0.0003	0.0195
1		0.25		0.0159	0.0180
2	3	0.50	0.02	0. 1253	0.1264
3		0.75		0. 4222	0.4351
4		1.00		1.0003	1.1858
0		0.00		0.0004	0.0022
1		0.25		0.0160	0.0185
2	4	0.50	0.03	0. 1254	0.1270
3		0.75		0. 4223	0.436
4		1.00		1.0005	1.1862

Table (3.12) represent the absolute errors for spatial values for h and k of example (3.5).

h	k	r	x_i	t_1	$\lvert u(x_i, t_2) - u_{i,2} \rvert$
0.05	0.025	10	0.25 0.75 1	0.025	5.8208×10^{-6} 6.4731×10^{-6} 8.3461×10^{-6}
0.05	0.0025	5	0.25 0.75 1	0.0025	6.2540×10^{-8} 6.4417×10^{-8} 1.4507×10^{-7}

Example (3.6):

Consider the nonlocal initial-boundary value problem that consists of the one-dimensional parabolic equation:

$$u_t - (x+1)u_{xx} = -x\, e^{x+t}, \quad x \in [0,1], \quad t \in [0,1] \qquad (3.58)$$

together with the local initial condition:

$$u(x,0) = e^x, \qquad x \in [0,1] \qquad (3.59)$$

and the nonlocal non-linear boundary conditions:

$$u(0,t) = \int_0^1 e^x\, u^2(x,t)\, dx + \frac{3e^t + e^{2t} - e^{2t+3}}{3}, \quad t \in [0,1] \qquad (3.60)$$

$$u(1,t) = \int_0^1 e^{2x}\, u^3(x,t)\, dx + \frac{5e^{(t+1)} + e^{3t} - e^{(3t+5)}}{2}, \quad t \in [0,1] \qquad (3.61)$$

This example is constructed such that the exact solution is

$$u(x,t) = e^{x+t}.$$

Let N=20 and M=40, then we get $h=\frac{1}{20}, k = \frac{1}{40}$, $r = 10$, $x_i = \frac{i}{20}$, $t_j = \frac{j}{40}$, $i = 0,1,\dots,20$, $j = 0,1,\dots,40$.

From equation (3.59), one can have:

$$u_{i,0} = e^{i/1000}, \qquad i = 0,1,\dots,20.$$

Therefore

$$u_{0,0} = 1\,,\ u_{0,1} = 1.001, \dots,\ u_{20,0} = 2.718.$$

By following the same previous steps one can get some of the numerical solutions that are tabulated in table (3.13), in table (3.14) we take different values for h and k and the values of the absolute errors at some special values.

Table (3.13) represents the exact and the numerical solutions for h=$\frac{1}{20}$, k=$\frac{1}{40}$ of example (3.6).

i	j	x_i	t_j	$u(x_i,t_1)$	$u_{i,1}$
0		0.0000		1.0250	1.0292
1	1	0.0500	0.025	1.0763	1.0808
2		0.1000		1.1302	1.1357
3		0.1500		1.1868	1.1938
4		0.2000		1.2464	1.2549
5		0.2500		1.3090	1.3191
6		0.3000		1.3749	1.3867
7		0.3500		1.4441	1.4577
8		0.4000		1.5168	1.5323
9		0.4500		1.5933	1.6108
10		0.5000		1.6737	1.6933

Table (3.14) represent the absolute errors for spatial values for h and k of example (3.6).

h	k	r	x_i	t_1	$\lvert u(x_i,t_2)-u_{i,2}\rvert$
0.1	0.1	10	0.2 0.5 1	0.1	91×10^{-4} 76×10^{-4} 18×10^{-4}
0.05	0.005	2	0.2 0.5 1	0.005	2.5×10^{-6} 4.0×10^{-6} 1.7×10^{-4}

Remark (3.7):

The finite difference method can be also used to solve the nonlocal initial-boundary value problem that consists of the m-dimensional hyperbolic equation:

$$\frac{\partial^2 u(x_1,x_2,\dots,x_m,t)}{\partial t^2}-\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left[a_i(x_1,x_2,\dots,x_m)\,\frac{\partial u(x_1,x_2,\dots,x_m,t)}{\partial x_i}\right]$$

$$= f(x_1,x_2,\dots,x_m,t),\ \ x_i\in[a_i,b_i], i=1,2,\dots,m,\ \ t\in[0,T]$$

together with the local initial conditions:

$$u(x_1, x_2, \dots, x_m, 0) = d(x_1, x_2, \dots, x_m), \qquad x_i \in [a_i, b_i], \quad i = 1,2,\dots,m$$

$$u_t(x_1, x_2, \dots, x_m, 0) = r(x_1, x_2, \dots, x_m), \qquad x_i \in [a_i, b_i], \quad i = 1,2,\dots,m$$

and the 2m nonlocal non-linear integral boundary conditions:

$$u(x_1, x_2, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_m, t) = \int_{a_i}^{b_i} w_{0,i}(x_1, x_2, \dots, x_m)\, u^{p_i}(x_1, x_2, \dots, x_m, t)\, dx_i + g_{0,i}(t),$$
$$i = 1,2,\dots,m,\ t \in [0,T]$$

$$u(x_1, x_2, \dots, x_{i-1}, b_i, x_{i+1}, \dots, x_m, t) = \int_{a_i}^{b_i} w_{1,i}(x_1, x_2, \dots, x_m)\, u^{q_i}(x_1, x_2, \dots, x_m, t)\, dx_i + g_{1,i}(t),$$
$$i = 1,2,\dots,m,\ t \in [0,T]$$

where $p_i \geq 1, q_i \geq 1$ are known constants, f is a known continuous function of x_i and t , a is a known continuous function of x_i, d, $w_{0,i}$, $w_{1,i}$, $g_{0,i}$, $g_{1,i}$ and r are known continuous functions that must satisfy the compatibility conditions:

(1) $d(x_1, x_2, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_m)$

$$= \int_{a_i}^{b_i} w_{0,i}(x_1, x_2, \dots, x_m)\, [d(x_1, x_2, \dots, x_m)]^{p_i}\, dx_i + g_{0,i}(0)$$

(2) $d(x_1, x_2 \dots, , x_{i-1}, b_i, x_{i+1}, \dots, x_m)$

$$= \int_{a_i}^{b_i} w_{1,i}(x_1, x_2, \dots, x_m)\, [d(x_1, x_2, \dots, x_m)]^{q_i}\, dx_i + g_{1,i}(0)$$

(3) $r(x_1, x_2, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_m) =$

$$p_i \int_{a_i}^{b_i} w_{0,i}(x_1, x_2, \dots, x_m)[d(x_1, x_2, \dots, x_m)]^{p_i - 1} r(x_1, x_2, \dots, x_m) dx_i + \left.\frac{dg_{0,i}}{dt}\right|_{t=0}$$

(4) $r(x_1, x_2, \dots, x_{i-1}, b_i, x_{i+1}, \dots, x_m) =$

$$q_i \int_{a_i}^{b_i} w_{1,i}(x_1, x_2, \dots, x_m)[d(x_1, x_2, \dots, x_m)]^{q_i - 1} r(x_1, x_2, \dots, x_m) dx_i + \left.\frac{dg_{1,i}}{dt}\right|_{t=0}$$

To do this one can see, [20].

Conclusions and Recommendations

From the present study, we can conclude the following:

(1) The nonlocal problems are generalization for the local ones. So the existence and uniqueness theorems given in this work are generalization of the ones that are used for local problems.

(2) It's known that the eigenvalues of the first and second order linear ordinary differential equations together with the homogenous local initial condition does not exist since such problems has only the trivial solution. But with the nonlocal initial condition the eigenvalues exist since such problems it has nontrivial solution.

(3) The finite difference method for solving the initial-boundary value problems that consists of the multi-dimensional hyperbolic and parabolic equations with 2m- nonlocal non-linear integral boundary conditions is a method that based on Crank-Niklson scheme and Taylor’s expansion. It's an effective technique for transforming the nonlinear system of equations to a linear system that can be solved easily by using any suitable method and it gave an acceptable results.

Also, we recommend the following for future work:

(1) Try to use Leray-Schauder fixed point theorem or any other types of fixed point theorems to discuss the existence of the solutions for the nonlocal initial value problem of the n-th order non-linear ordinary differential equations.

(2) Discuss the existence and the uniqueness for the one-dimensional hyperbolic and parabolic differential equations with the nonlocal non-linear integral boundary conditions.

(3) Devote another types for the nonlocal problems say, the nonlocal problems for the delay differential equations and integro-differential equations.

References

[1] Abdelkader B. and Radu P., "On the Nonlocal Initial Value Problem for First Order Differential Equations", Fixed Point Theory, Vol. 4, No. 2, (2003), PP.205-212.

[2] Ashyralyev A. and Necmettin A., "Finite Difference Method for Hyperbolic Equation with the Nonlocal Integral Conditions", Discrete Dynamics in Nature and Society, Vol. 2011, (2011), PP.1-15.

[3] Avalishvili G., Avalishvili M. and Gordeziani D., "On Integral Nonlocal Boundary Value Problems for some Partial Differential Equations", Bull. Georg. Natl. Acad. Sci., Vol. 5, No. 1, (2011), PP.31-37.

[4] Bandyrskii B., Lazurchak I., Makarov V. and Sapagovas M., "Eigenvalue Problem for the Second Order Differential Equation with Nonlocal Conditions", Nonlinear Anal. Modelling and Control, Vol. 11, No. 1, (2006), PP.13-32.

[5] Beilin S. A., "Existence of Solution for One-dimensional Wave Equations with Nonlocal conditions", Electron J. Differential Equations, Vol. 76, (2001), PP.1-8.

[6] Borhanifar A., Pour A. H. and Kabir M. M., "A Numerical Method for Solution of the Heat Equation with Nonlocal Nonlinear Condition", World Appl. Sci. J., Vol. 13, (2011), PP.2405-2409.

[7] Chabrowski J. H., "On Nonlocal Problems for Parabolic Equations", J. Chabrowski Nagoya Math. J., Vol. 93, (1984), PP.109-131.

[8] Chabrowski J. H., "On Solvability of Boundary Value Problem for Elliptic Equations with Bitsadze-samarski Condition", Int. J. Math. and Math. Sci., Vol. 11, No. 1, (1988), PP.101-114.

[9] Ciupaila R., Jeseviciute Z. and Sapagovas M., "On the Eigenvalue Problem for One-Dimensional Differential Operator with Nonlocal Integral Condition", Nonlinear Anal. Modelling and Control, Vol. 9, No. 2, (2004), PP.109–116.

[10] Corduneanu C., "Integral Equations and Applications", Cambridge Univ. press, Cambridge, (1991).

[11] Davis J., "Existence Results for Second Order Three-Point Boundary Value Problems", Differential Equations and Appl., Vol. 4, No. 4, (2012), PP.547-570.

[12] Dehghan M. and Saadatmandi A., "Variational Iteration Method for Solving the Wave Equation Subject to an Integral Conservation Condition", Chaos, Solutions and Fractals, Vol. 41, (2009), PP.1448-1453.

[13] Dennis E. J., "Error Estimates for the Semidiscrete Finite Element Approximation of Linear Nonlocal Parabolic Equations", J. Appl. Math., and Stochastic Anal., Vol. 5, No. 1, (1992), PP.19-28.

[14] Fuyi X. and Jian L., "Symmetric Positive Solutions for Nonlinear Singular Fourth-Order Eigenvalue Problems with Nonlocal Boundary Condition", Hindawi Publishing Corporation Discrete Dynamics in Nature and Society, Vol. 2010, (2010), PP.1-16.

[15] Garrett P., "Banach and Frechet Spaces of Functions", Minnesota Univ., press, (2012).

[16] George L. K. and Tsamatos P. Ch., "Existence of Multiple Positive Solutions for a Nonlocal Boundary Value problem", J. the Juliusz Schauder Center, Vol. 19, (2002), PP.109-121.

[17] Henderson J. and Wang H., "Positive Solutions for Nonlinear Eigenvalue Problems", J. Math. Anal. and Appl., Vol. 208, (1997), PP.252-259.

[18] Henderson J. and Ntouyas S. K., "Positive Solutions for Systems of Nth Order Three-Point Nonlocal Boundary Value Problems", Electronic J. Qualitative Theory of Differential Equations, Vol. 2007, No. 18, (2007), PP.1-12.

[19] Karakostas G. L. and Tsamatos P. C., “Positive Solutions and Nonlinear Eigenvalue Problems for Retarded Second Order Differential Equations”, Electronic J. Differential Equations, Vol. 2002, No. 59, (2002), PP.1-11.

[20] Khaleel A. J. and Kahtan J. A., “A Numerical Method for Solving the Multi-Dimensional Hyperbolic Equations with Nonlocal Non-Linear Conditions”, to be appeared in J. Kerbala Univ..

[21] Lambert D. J., “Numerical Methods in Ordinary Differential Equations”, Wiley Dundee Univ., Scotland, (1991).

[22] Li X., “Numerical Solution of an Initial-Boundary Value Problem with Nonlocal Condition for the Wave Equation”, Math. Sci., Vol. 2, (2008), PP.281-292.

[23] Marasi H. R., Faghiri R. and Hashemi M. S., “A Series Solution of Non-local Hyperbolic Equation”, Communications in Numerical Anal., Vol. 2012, (2012), PP.1-13.

[24] Mehdi R., Mehdi D. and Mohsen R., “Combined Finite Difference and Spectral Methods for the Numerical Solution of Hyperbolic Equation with an Integral Condition”, Wiley Int. Sci., Vol. 10, (2006), PP.1-8.

[25] Mohammad A. A., “A Class of Hyperbolic Equations with Nonlocal Conditions”, Int. J. Math. Anal., Vol. 2, No. 10, (2008), PP.491- 498.

[26] Munkhammar J., “Riemann-Liouville Fractional Derivatives and the Taylor-Riemann Series”, U.U.D.M. Project Report, Vol. 7, (2004).

[27] Nemati S. and Ordokhani Y., “A Second Kind Chebyshev Polynomial Approach for the Wave Equation Subject to an Integral Conservation Condition”, J. Information and Computing Sci., Vol. 7, (2012), PP.163-171.

[28] Olmstead W. E. and Roberts C. A., “The One-Dimensional Heat Equation with a Nonlocal Initial Condition”, Appl. Math. Letters, Vol. 10, No. 3, (1997), PP.89-94.

[29] Paul W. E. and Ahmad B., “Positive Solutions of a Nonlinear nth Order Boundary Value Problem with Nonlocal Conditions”, Appl. Math. Letters Vol. 18, (2005), PP.521-527.

[30] Pulkina L. S., “A Nonlocal Problem with Integral Conditions for Hyperbolic Equations”, Electronic J. Differential Equations, Vol. 1999, No. 45, (1999), PP.1-6.

[31] Rehman M. A., “Fourth-Order Method for Non-Homogeneous Heat Equation with Nonlocal Boundary Conditions”, Appl. Math. Sci., Vol. 3, No. 37, (2009), PP.1811-1821.

[32] Ruyun M., “A Survey On Nonlocal Boundary Value Problems”, Appl. Math., Vol. 7, (2007), PP.257-279.

[33] Saadatmandi A. and Dehghan M., “Numerical Solution of the One-dimensional Wave Equation with Integral Condition”, Wiley Int. Sci., Vol. 23, (2006), PP.282-292.

[34] Said M. and Abdelfatah B., “On a Class of Singular Hyperbolic Equation with a Weighted Integral Condition”, Int. J. Math. and Math. Sci., Vol. 22, No. 3, (1999), PP.511-519.

[35] Svajunas S., “On the Stability of Locally One-Dimensional Method for Two-Dimensional Parabolic Equation with Nonlocal Integral Conditions”, V European Conference on Computational Fluid Dynamics, Lisbon, Portugal, Vol. 14-17, June (2010), PP.1-11.

[36] Smith G. D., “Numerical Solution of Partial Differential Equations”, Brunel Univ. press, Oxford (1978).

[37] Yongping S., “Eigenvalues and Symmetric Positive Solutions for a Three-Point Boundary Value Problem”, Electronic J. Differential Equations, Vol. 2005, No., 127, (2005), PP.1–7.

المستخلص

الشروط اللامحلية للمسائل الحدودية او الابتدائية تظهر عندما تكون قيم الدالة المعطاة في الشروط الحدودية او الابتدائية مرتبطة مع قيم من داخل المجال. وهذه المسائل تعرف بالمسائل اللامحلية.

الهدف من هذا العمل هو دراسة بعض انواع المسائل اللامحلية. هذه الدراسة شملت المحاور التالية :

(١) مناقشة الوجود والوحدانية لحلول بعض المسائل اللامحلية الابتدائية للمعادلات التفاضلية الاعتيادية الخطية و اللاخطية باستخدام بعض نظريات النقطة الصامدة. بالاضافة الى ذلك بعض الطرق العددية استعملت لحل بعض المسائل اللامحلية الابتدائية للمعادلات التفاضلية الاعتيادية الخطية واللاخطية .

(٢) اعطاء الحلول لبعض المسائل اللامحلية الابتدائية والحدودية لمسائل القيم الذاتية للمعادلات التفاضلية الاعتيادية الخطية .

(٣) استعمال بعض الطرق العددية لحل مسائل القيم الحدودية-الابتدائية التي تَشْمل معادلات القطع الزائد والقطع المكافئ ذات البعد الواحد مَع اثنين من الشروط الحدودية التكاملية اللامحلية اللاخطية. هذه الطريقة تعتمد على معادلة دوغلاس ، طريقة كرانك نيكلسن للفروقات المنتهية ، توسيع تايلر و بعض الطرق التربيعية كطريقة سيمبسن $1/3$.

جمهـورية العـراق
وزارة التعليم العالي والبحث العلمي
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كليــة العلوم
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بعض الطرق لحل المسائل الابتدائية والحدودية مع شروط اللامحلية

رسالة
مقدمة إلى كلية العلوم – جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل
جيهان عدنان قحطان

(بكالوريوس رياضيات/ كلية العلوم / جامعة النهرين ٢٠٠٩)

اشراف

أ.م.د. أحلام جميل خليل **أ.م.د. أكرم محمد العبود**

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