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Study for Some of Stochastic Partial Differential Equations

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بسم الله الرحمن الرحيم

(نَرفَعُ دَرَجٰتٍ مَّن نّشَاءُ وفَوقَ كُلِّ ذِي عِلمٍ عَليمٌ)

صدق الله العظيم

سورة يوسف، جزء من الآية (٧٦)

Dedication

To …

My Parents, Sisters, all Friends and all Persons who Encouraged and Supported Me and to Everyone who Cares to Mathematics.

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Douaa Khudheyer Abid

February, 2011

SUPERVISOR'S CERTIFICATION

We certify that this thesis, entitled "General Study of Stochastic Partial Differential Equations", was prepared under our supervision at the Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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We certify that we have read this thesis entitled "*Study for Some of Stochastic Partial Differential Equations*" and as an examining committee examined the student *(Douaa Khudheyer Abid)* in its contents and in what it connected with, and that, in our opinion, it meets the standards of a thesis for the degree of Master of Science in Mathematics.

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LIST OF SYMBOLS

I_A	The indicator function of the set A.
$\delta(.)$	The Dirac's delta function.
$C^m(G)$	The class of all functions f having continuous derivatives up to an order m in G.
$L^p(R)$	The Functions on $\mathbb{R}^d$ that are locally in L^p with respect to the Lebsegue measure.
$S(R)$	The space of tempered distributions on $\mathbb{R}^d$.
E_μ	The expectation with respect to μ.
$L_2^w[\alpha,\beta]$	The class of all nonanticipative functions f satisfying: $$P\left\{\int_\alpha^\beta \lvert f(t)\rvert^p \, dt < \infty\right\} = 1$$
$S(\mathbb{R}^d)$	The Schwartz space of rapidly decreasing smooth functions (d is called the parameter dimensions).
$(c^N)_c$	The set of all finite sequence in c^N.
∂D	The boundary of the set D in $\mathbb{R}^d$.

ABSTRACT

This thesis has three objectives:

The first objective is to give a study of stochastic calculus, including the fundamental concept of stochastic differential equations. The second objective is to introduce stochastic partial differential equations, as well as, study some theoretical results related to the three types of second order stochastic partial differential equations and give the integral form for each type. The third objective is to study the numerical solution of the three type of stochastic partial differential equations (stochastic Poisson equations, stochastic heat equations and stochastic wave equations) using the finite difference method.

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INTRODUCTION

Stochastic partial differential equations (SPDE's), and perhaps partial differential equations (PDE's), as well as, find their primary motivations in science and engineering .Both ordinary differential equations (ODE's) and PDE's play a fundamental role in describing reality. However, any model of the real world must take into account uncertainty or random fluctuations. It is therefore, surprising that while stochastic ordinary differential equations (SODE's) where studied intensively through the twentieth century, SPDE only received attention much later. Firstly work stemmed from the Zakai equation in filtering theory, [38].

On the theoretical side, there was the work of Pardoux [37] ,and Krylov and Rozovskii [26]. Much of this early work centered on foundational questions, such as setting up the appropriate function spaces for studying solutions or using such analytic tools as the method of monotonicity. Later, Walsh [44] introduced the notion of martingale measures as an alternative framework. The diverse origins of SPDE's have led to a lively interplay of view points. Some people feel that SPDE's should be based on such tools as Sobolev space, as is the case for PDE's. Others, with a background in probability feel that an SPDE's describes a special kind of stochastic process. Applied mathematicians may feel that the study of SPDE's should follow the ideas used in their domain. By a historical accident, particle systems, which may be considered as discrete SPDE's, that studied much earlier than SPDE. Such pioneers as Ted Harris and Frank Spizer laid the groundwork for this theory. Their research was also influenced by results in percolation, by such mathematics as Harrykesten. Particle system has changed its emphasis over the years, and some of this early work is being forgotten. However, we believe that the main methods of particle system will always be relevant to SPDE's. Unfortunately, there was no time to discuss percolation, which we also believe has fundamental importance for SPDE's. Both duality and percolation, as well as, many other ideas, are described in more detail in three classes [28], [29] give detailed technical accounts of the field, and [9] provides a lively intuitive treatment.

Secondly, Watanabe and Dawson found that the scaling limit of critical branching Brownian motions give fundamentally an important model, called the Dawson-Watanabe process or super process. Because, this model involves independently moving particles, there are powerful tools for studying its behaviour, and many of these tools help in the study of SPDE's.

For example, the heat equation can be thought of as the density of a cloud of Brownian particles. Any SPDE, which involves a density of particles, can be studied via the Dawson-Watanabe process. There is a huge literature in this area, but two useful surveys are written by Dawson [7] and Perkins [39].

Thirdly, as one might expect, tools from PDE's are useful for SPDE's. Of course, Sobolev space and Hilbert space play a role, as in the work of DaPrato and Zabczyk [8] and Krulov [19]. But her we wish to concentrate on qualitative tools, such as the maximum principle. In particular, comparison theorems hold for many SPDE's.

Finally, tools from probability theory also find applications in SPDE's. For example, the theory of large deviations of dynamical systems developed by Wentzall and Freidlin [14] also applied to SPDE. If the noise is small, we can estimate the probability that the solutions of the SPDE's and the corresponding PDE's (without noise) differ by more than a given amount. Unfortunately, we had no time to discuss questions of coupling and invariant measures, which play a large role in the study of the stochastic Nevier-Stokes equation.

This thesis consists of three chapters.

In chapter one, we state some of the most general concepts and definitions related to the subject of stochastic calculus, and SDE's which are given for completeness.

In chapter two, the statement and the proof of some theoretical results related to SPDE's are given, as well as, with some formulations, which are needed, namely, stochastic Poisson equation, stochastic heat equation and stochastic wave equation.

In chapter three, numerical method for solving SPDE's have been studied and explained with examples, in which the finite difference method was considered for solving the three types of SPDE's

Last, the numerical results are obtained using computer programs written in MATHCAD 14 computer software and the results are given in a tabulated form.

Chapter One

Stochastic Calculus and Stochastic Differential Equations

CHAPTER ONE

STOCHASTIC CALCULUS AND STOCHASTIC DIFFERENTIAL EQUATIONS

Stochastic calculus is concerned with the study of stochastic processes, which involves randomness or noise. Intuitively, this requires knowledge of the background definitions and concepts that will be required later in this study, where only those definitions which are of direct relevance to this exposition are given.

1.1 BASIC CONCEPTS OF PROBABILITY THEORY

Probability theory is that branch of mathematics which is concerned with random (or chance) phenomena. It has attracted people to study both, because of intrinsic interest and successful applications to many areas within the physical, biological, social sciences, engineering, and in the business world, [22]. Randomness and probability are not easy to define precisely, but we certainly recognize random events when we meet them. For example, randomness is an effect when we flip a coin, a lottery ticket, run horse's race, etc., [25].

The probability theory is an essentially abstract science, in which a probability measure is assigned to every set A, of the space of events called the sample space (denoted by Ω). We introduce the probability of A, denoted by P(A), as a set function of A satisfying the following axioms:

1. $P(A) \geq 0, \ \forall A \subset \Omega$.

2. $P(\Omega) = 1$.

3. If A_i, $(i = 1, 2, \ldots)$ is a countable collection of non-overlapping sets in Ω, then:

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i).$$

4. If A^c is the complement of A, then $P(A^c) = 1 - P(A)$.

5. $P(\emptyset) = 0$, where $\emptyset$ is the empty set.

Now, let F be a nonempty class of subsets drawn from the sample space Ω. We say that the class F is a field or an algebra of sets in Ω if it satisfies the following definition.

Definition (1.1), [25]:

A class of subsets $A_j \subset \Omega$, $\forall$ i = 1, 2, …, n ;denoted by F is a field (or Algebra), when the following conditions are satisfied:

1. If $A_i \in F$, then $A_i^c \in F$, $\forall$ i = 1, 2, …, n.

2. If $\{A_i, \forall i = 1, 2, \ldots, n\} \in F$, then $\bigcup_{i=1}^{n} A_i \in F$.

Remark (1.2), [25]:

In the above definition, if $n \longrightarrow \infty$, then F is said to be σ-Field (or σ-Algebra).

Convergence of a sequence of random variables may defined by using different approaches and have different meanings, as it is defined in the next definitions:

<u>Definition (1.3) (Pointwise Convergence), [25]:</u>

A sequence of random variables $\{X_n\}$ converges to a limit X if and only if for any $\varepsilon > 0$, however small, we can find positive integer n_0, such that:

$$|X_n - X| < \varepsilon, \text{ for every } n > n_0.$$

<u>Remark (1.4), [25]:</u>

If we consider a sequence of random variables $\{X_1, X_2, \ldots, X_n, \ldots\}$ and define a pointwise convergence to another random variable X, as in above definition, then we must have at every point w in Ω the sequence of numbers $X_1(w), X_2(w), \ldots, X_n(w), \ldots$ converging to X(w). This type of convergence is called everywhere convergence.

<u>Definition (1.5) (Almost Sure Convergence), [13]:</u>

A sequence of random variables $\{X_n\}$ converges almost surly (abbreviated by a.s) or almost certainly or strongly to X, if for every point w which is not belong the null event A, then:

$$\lim_{n\to\infty} |X_n(w) - X(w)| = 0$$

This type of convergence is known as convergence with probability 1 and is denoted by:

$$X_n(w) \xrightarrow[n\to\infty]{a.s} X(w) = \lim_{n\to\infty} X_n(w) \quad \text{a.s.}$$

Remark (1.6), [13]:

If the limit X is not known as priori, then we can define a mutual convergence almost surely. The sequence X_n converges mutually almost surely if:

$$\sup_{m\geq n}|X_m - X_n| \xrightarrow[n\to\infty]{a.s} 0$$

In which both definitions are equivalent.

Definition (1.7) (Convergence in Probability), [43]:

A sequence of random variables $\{X_n\}$ converges in probability to X if and only if for every $\varepsilon > 0$, however small,

$$\lim_{n\to\infty} P\left(|X_n - X| \geq \varepsilon\right) = 0$$

or equivalently

$$\lim_{n\to\infty} P\left(|X_n - X| < \varepsilon\right) = 1$$

Remark (1.8), [25]:

1. Mutual convergence may be defined in probability as:

$$\lim_{n\to\infty} \mathrm{Sup}\left(P|X_m - X_n| \geq \varepsilon\right) \longrightarrow 0$$

2. If a sequence of random variables $\{X_n\}$ converge almost surely to X, then it converge in probability to the same limit. The converse is not true, as an example, let $\Omega = [0, 1]$ and let $P([a, b]) = |b - a|$ for any subinterval $[a, b] \subseteq [0, 1]$.

   For $n = 1, 2, \ldots$

Let $X_n = \sqrt{n}\, I_{A_n}$, where $A_n = \left\{ w \in [0,1] : 0 \le w \le \frac{1}{n} \right\}$ .Then:

$$P(X_n \ge \varepsilon) = P(\sqrt{n}\, I_{A_n} \ge \varepsilon)$$

$$= P(I_{A_n} \ge \varepsilon / \sqrt{n})$$

$$= \left| \frac{1}{n} - 0 \right|$$

$$= \frac{1}{n}, \text{ for } 0 < \varepsilon \le \sqrt{n}\,.$$

Thus:

$$\lim_{n\to\infty} P(|X_n - 0| \ge \varepsilon) = \lim_{n\to\infty} P(X_n \ge \varepsilon) = 0$$

for all $\varepsilon > 0$, so X_n converges in probability to $X = 0$. However:

$$\lim_{n\to\infty} \left| X_n - X \right| = \lim_{n\to\infty} X_n$$

$$= \lim_{n\to\infty} \sqrt{n}\, I_{A_n} = \infty$$

So X_n dose not converge to X in the sense of almost surely.

3. If $\{X_n\}$ converge in probability to X, then there exist a subsequence $\{X_{n_k}\}$ of $\{X_n\}$, which converges almost surely to the same limit.

4. $\{X_n\}$ converges in probability if and only if it is converges mutually in probability.

1.2 STOCHASTIC PROCESSES

Differential equations for random functions (stochastic processes, random processes) arise in the investigation of numerous physical and engineering problems, [2]. We have looked at single random variables (X_1, X_2, …, X_n), which we termed as random vectors. However, many practical application of probability theory are concerned with random processes evolving in time, or space, or both, without any limit on the time (or space), [1].

<u>Definition (1.9) (Stochastic Process), [43]:</u>

A stochastic processes is a collection of random variables $\{X(t): t \in T\}$, where t is a parameter that runs over an index set T (T is open, closed, or half closed). In general we call t the time-parameter (or simply the time), and $T \subset \mathbb{R}^+$. Each X(t) takes values in some set $S \subset \mathbb{R}$ called the state space, then X(t) is the state of the process at time t.

<u>Definition (1.10) (Stationary), [1]:</u>

A stochastic processes X(t) is said to be stationary if:

$$P\{X_1(t) \le x_1, X_2(t) \le x_2, \ldots, X_m(t) \le x_m\} = P\{X(t_1 + \theta) \le x_1, X(t_2 + \theta) \le x_2, \ldots, X(t_m + \theta) \le x_m\}$$

for all $t_1, t_2, \ldots, t_m > 0$ and real values $x_1, x_2, \ldots, x_m$. For every natural number m and for all $\theta \in \mathbb{R}^+$.

Remarks (1.11), [43]:

1. If the index set T is a countable set, we call X a discrete time stochastic process, otherwise, we call it a continuous time stochastic process.
2. A continuous time stochastic process $\{X(t): t \in T\}$ is said to have an independent increment if for all $t_0 < t_1 < t_2 < \ldots < t_n$, the random variables $X(t_1) - X(t_0)$, $X(t_2) - X(t_1)$, …, $X(t_n) - X(t_{n-1})$ are independent. The process is stationary if $X(t + s) - X(t)$ has the same distribution X(t) for all t and s > 0. That is, is posses independent increments if the changes in the processes values over non overlapping time intervals are independent of each other, and it process stationary increments if the distribution of the change in the value between any two points depends only on the distance between those points.

1.3 BROWNIAN MOTION AND WHITE NOISE

Brownian motion was introduced by Robert Brown in 1827, when he observed the motion of a pollen grain as it is moved randomly in a glass of water. Because the water molecules collide with pollen grain in a random fashion, the pollen grain moves randomly. The motion of pollen is stochastic, because its position from one point in time to the next position can only be defined in terms of a probability density function, [21]. In 1900 Bachelier used the Brownian motion as a model for studying the movement of stock prices in his mathematical theory of speculation. The mathematical foundation for Brownian motion as stochastic process was introduced by Wiener in 1931, and this process is also called the Wiener process [24], but we shill use separate terminology to distinguish between the mathematical and physical processes.

Definition (1.12) (Brownian Motion), [17]:

A Brownian motion or Wiener process is a stochastic process $W(t), t \geq 0$ satisfying:

1. $W(0) = 0$.
2. For any $0 \leq t_0 < t_1 < \ldots < t_n$, the random variables:

   $$\Delta W_k = W(t_{k+1}) - W(t_k),\ 0 \leq k \leq n-1$$

   are independent.
3. If $0 \leq s < t$, $W(t) - W(s)$ is normally distributed with mean μ_t and variance σ_t^2 then:

   $$E[W(t) - W(s)] = (t-s)\mu_t$$

   $$E[(W(t) - W(s)]^2 = (t-s)\sigma_t^2$$

   where μ_t and σ_t^2 are real constant, $\sigma_t > 0$, and called the drift and variance respectively.

Remarks (1.13), [17], [43]:

1. If $\sigma_t^2 = 1$, the W(t) is said to be the standard Brownian motion. We always make this assumption unless stated otherwise.

2. If $\mu_t = 0$, $\sigma_t^2 = 1$, then we speak about normalized Brownian motion. For any Brownian motion with mean μ_t and σ_t^2, $(W(t) - \mu_t) / \sigma_t$ is a normalized Brownian motion. And if W(t) with drift μ_t , variance σ_t^2, and $0 \leq t_0 < t_1 < \ldots < t_n$, then:

$$\Gamma(t_i, t_k) = E[(W(t_i) - \mu_{t_i})(W(t_k) - \mu_{t_k})]$$

$$= \sigma_t^2 \min\{t_i, t_k\}, \text{ indeed if } t_i > t_k,$$

$$\Gamma(t_i, t_k) \equiv E[(W(t_i) - W(t_k) - \mu(t_i - t_k) + W(t_k) - \mu_{t_k})(W(t_k) - \mu_{t_k})]$$

$$= E(W(t_k) - \mu_{t_k})^2 = t_k \sigma^2.$$

3. In fact, the assumption (2) is not strictly, in that case one can construct (by a limiting procedure) a random process W(t) that obeys (1) and (3) is almost surely continuous of definition (1.12).

<u>Definition (1.14) (n-Dimensional Brownian Motion), [17]:</u>

An n-dimensional process $W(t) = (W_1(t), W_2(t), \ldots, W_n(t))$ is called an n-dimensional Brownian motion, if each process $W_i(t)$ $(i = 1, 2, \ldots, n)$ is a Brownian motion and if the σ-field $\mathcal{F}(W_i(t), t \geq 0)$, $1 \leq i \leq n$, are independent.

Now, we discuss that the notion of white noise as it generally introduced in the science and engineering literature can be thought of as the time derivative of the Wiener process. The non existence of White noise as a stochastic process will never be a problem and we will happily consider noisy observation in their integrated form in order to avoid mathematical unpleasantness. Let us nonetheless to topic of Wiener process we should briefly investigate further the connection between the Wiener process and white noise. In science and engineering white noise is generally defined as follows: it is Gaussian "stochastic process" Y_t with zero mean and covariance $E(Y_sY_t) = \delta(t - s)$, where $\delta(.)$ is Dirac's delta function.

Definition (1.15) (White Noise), [24]:

The white noise process Y(t) is formally defined as the derivative of the Brownian motion:

$$Y(t) = \frac{dW}{dt} = \dot{W}(t)$$

It dose not exist as a function of time t in the usual sense, since a Brownian motion is nowhere differentiable function.

Remark (1.16), [1]:

A special case which is of considerable interest occurs when the process X(t) from which the white noise derives is the Brownian motion. The white noise process then obtained is often referred to as Gaussian white noise.

1.4 STOCHASTIC DIFFERENTIAL EQUATIONS

Stochastic differential equations (SDE's) appears in analysis in various guises. An example from physics will perhaps illuminate the need for this field and give an inkling of its particularities. Consider a physical system whose state at time t is described by vector X_t in $\mathbb{R}^n$, where X_t will refers the stochastic process for the rest of this work. In fact, for concreteness sake imagine that the system is a space probe on the way to the moon. The pertinent quantities are its location and momentum. If x_t its location at time t and p_t its momentum at that instant, that X_t is the t-vector (x_t, p_t) in the phase space $\mathbb{R}^n$. In an ideal world the evolution of the state is governed by the differential equation:

$$\frac{dX_t}{dt} = \begin{pmatrix} dx_t/dt \\ dp_t/dt \end{pmatrix}$$

$$= \begin{pmatrix} p_t/m \\ F(x_t, p_t) \end{pmatrix}$$

Here m is the mass of the probe. The first line merely the definition of p_t: momentum = mass×velocity. The line is Newton’s second law the rate of change of the momentum is the force F. For simplicity of reading we are rewrite this in the form

$$dX_t = a(x_t,, p_t)\, dt \qquad (1.1)$$

which expresses the idea the change of X_t during the time-interval dt is proportional to the time dt elapsed with proportionality constant of coupling coefficient $a(x_t, p_t)$ that depends on the state of the system and is provided by a model for the force acting. We may rewrite eq. (1.1) in the form of an integral equation:

$$X_t = x_0 + \int_0^t a(x_s, p_s)\, ds, \quad X_t(0) = x_0 \qquad (1.2)$$

In the less-than-ideal real world, our system is subject to unknown forces noise. Our recket will travel through gullies in the gravitational field that are due to unknown in homogeneities in the mass distribution of the earth. It will meet guts of winds that cannot be foreseen; it might even run into a gaggle of geese that deflect it. The evolution of the system is better modeled by the differential equation:

$$dX_t = a(x_{t,} p_t)\ dt + dG_t \quad \text{...}(1.3)$$

where G_t is a noise that contributes the differential dG_t to the changes dX_t of X_t during the interval dt. To accommodate idea that the noise comes from without the system one assume that there is back ground noise W_t consisting of gravitational gullies, gusts, and geese in our example-and that its effect on the state during the time-cumulative noise W_t during the time interval dt, with proportionality constant or coupling coefficient b that depends on the state of the system:

$$dG_t = b(x_t, p_t)dW_t$$

For instance, if our probe is at time t half way to the moon, then the effect of the gaggle of geese at that instance should be considered negligible and the effect of the gravitational gullies is small. Equation (1.3) turns into:

$$dX_t = a(x_t, p_t)dt + b(x_t, p_t)dW_t \quad \text{...}(1.4)$$

or as an integral equation of the form:

$$X_t = x_t^0 + \int_0^t a(x_s, p_t)\,ds + \int_0^t b(x_s, p_t)\,dW_s \quad \text{...............................}(1.5)$$

Definition (1.17), [17]:

A stochastic process f(t) defined on $[\alpha, \beta]$ is called a step function if there exists a partition $\alpha = t_0 < t_1 < \ldots < t_r = \beta$ of $[\alpha, \beta]$, such that :

$$f(t) = f(t_i), \text{ if } t_i \le t < t_{i+1}, i = 0, 1, \ldots, r-1$$

Definition (1.18) (Stochastic Integral), [17]:

Let f(t) be a step function in $L^2_W[\alpha, \beta]$, then:

$$f(t) = f(t_i), \text{ if } t_i \le t < t_{i+1}, 0 \le i \le r-1$$

where $\alpha = t_0 < t_1 < t_2 < \ldots < t_r = \beta$. The random variable:

$$\sum_{k=0}^{r-1} f(t_k)\left[W(t_{k+1}) - W(t_k)\right]$$

is denoted by:

$$\int_{\alpha}^{\beta} f(t)\,dW(t)$$

and is called the stochastic integral of f(t) with respect to Brownian motion W, it is also called the Itô integral.

Definition (1.19) (Increasing σ-Field or Filtration σ-Field), [25]:

Let $(\Omega, \mathcal{F})$ be a complete measurable space and let $\{\mathcal{F}_t, t \in \mathbb{R}^+\}$ be a family of sub-σ-fields of $\mathcal{F}$, such that for $s \le t$, $\mathcal{F}_s \subset \mathcal{F}_t$. Then $\{\mathcal{F}_t\}$ is called an increasing family of sub-σ-fields on $(\Omega, \mathcal{F})$ or the filtration σ-field of $(\Omega, \mathcal{F})$. $\mathcal{F}_t$ is called the σ-field of events prior to t. If $\{X_t, t \in$

$\mathbb{R}^+$} is a stochastic process defined on (Ω,F, P), then clearly F_t is given by:

$$F_t = \sigma\{X_s, s \leq t, t \in \mathbb{R}^+\} \text{ is increasing.}$$

Remark (1.20), [25]:

Since the probability space (Ω, F, P) is complete, the σ-field F contains all subsets of Ω having probability measure zero. Here we shall assume that the filtration σ-field $\{F_t, t \in T\}$ also contains all the sets from F having probability measure zero.

Definition (1.21) (Adaptation of {X_t}), [25]:

Let $\{X_t, t \in \mathbb{R}^+\}$ be a stochastic process defined on a probability space (Ω, F, P) and let $\{F_t, t \in \mathbb{R}^+\}$ be filtration σ-field. The stochastic process $\{X_t\}$ is adapted to the family $\{F_t\}$ if X_t is F_t-measurable for every $t \in \mathbb{R}^+$, and:

$$E^{F_t} X_t = X_t, \ t \in \mathbb{R}^+$$

F_t-adapted random processes are also F_t-measurable.

Definition (1.22) (The Itô Process), [17]:

A stochastic process X_t, $0 \leq t \leq T$ is called an Itô process with respect to $\{W_t, P, F_t\}$ (where F_t is adapted to W_t) relative to B(t), A(t) if:

$$X_t = X_{t_0} + \int_0^t A(s)\,ds + \int_0^t B(s)\,dW_s, \ 0 \leq t \leq T.$$

<u>Definition (1.23) (Stochastic Differential Equations), [19]:</u>

An n-dimensional Itô process X_t is a process that can be represented as:

$$X_t = X_{t_0} + \int_0^t A(X_s, s)\,ds + \int_0^t B(X_s, s)\,dW_s$$

where W_t is an m-dimensional standard Brownian motion, A and B are n-dimensional and n×m-dimensional F_t-adapted processes, respectively. We often use the notation:

$dX_t = A(X_t, t)\,dt + B(X_t, t)\,dW_t,\ X(t_0) = X_{t_0}$ ....(1.6)

1.5 THE ITÔ'S FORMULA, [18], [43]

Itô formula is the analog of integration by parts in the deterministic calculus. In stochastic calculus this is not possible; the useful range of techniques is practically restricted to those that deal with integral equations. Of these by far the most important is that which is known as Itô's formula, where may be seen as a stochastic chain rule. Let us recall some elementary non-random chain rule; as usual primes may denote differentiation.

1. One-variable chain rule:

If y(t) = f(g(t)), then:

$$y'(t) = \frac{dy}{dt} = f'(g(t))\,g'(t))$$

Assuming that the derivatives f' and g' exists. We may express this in differential notion as:

$$dy = f'(g)\, g'(t)\, dt = f'(g)\, dg.$$

2. Two variables chain rule: If:

$$Y(t) = f(X(t), W(t)).$$

Then:

$$\frac{dy}{dt} = \frac{\partial f}{\partial X}\frac{\partial X}{\partial t} + \frac{\partial f}{\partial W}\frac{\partial W}{\partial t}$$

where differentiation may be denoted by suffices in an obvious way. In particular, if $X = t$, we obtain, for $y = f(t, W(t))$.

$$dy = f_t\, dt + f_w\, dW$$

Itô formula is extremely useful in many topics, particularly in evaluating stochastic integrals.

<u>Theorem (1.24) (Itô Formula), [13]:</u>

Suppose that X_t has a stochastic differential equation:

$$dX_t = A(X_t,t)\, dt + B(X_{t,},t)\, dW_t, \text{ for } A \in L^1(0, T), B \in L^2(0, T)$$

Assume $u : \mathbb{R} \times [0, T] \longrightarrow \mathbb{R}$ is continuous and $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial X_t}, \frac{\partial^2 u}{\partial X_t^2}$ exist and are continuous. Set:

$$Y_t = u(X_t, t)$$

Then Y_t has the stochastic differential:

$$dY_t = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial X_t} dX_t + \frac{1}{2}\frac{\partial^2 u}{\partial X_t^2} B^2\, dt$$

$$= \left[\frac{\partial u}{\partial t} + \frac{\partial u}{\partial X_t} A + \frac{1}{2}\frac{\partial^2 u}{\partial X_t^2} B^2\right] dt + \frac{\partial u}{\partial X_t} B\, dW_t \quad \text{................}(1.7)$$

The eq.(1.7)is called the Itô's formula or Itô chain rule.

Example (1.25), [46]:

Use the Itô formula to solve SDE:

$$\left.\begin{array}{l} dX_t = X_t\, dW_t,\ t \in [0, 1] \\ X_t(0) = 1 \end{array}\right\} \quad \text{..}(1.8)$$

Hence, from the stochastic differential equation, we have:

$$\frac{dX_t}{X_t} = dW_t$$

and therefore:

$$\int_0^t \frac{dX_s}{X_s} = \int_0^t dW_s$$

i.e.,

$$\int_0^t \frac{dX_s}{X_s} = W_t \quad \text{..}(1.9)$$

Using the Itô formula for the function

$g(t, X_t) = \ln X_t$, $X_t > 0$

From eq.(1.7) and obtain that from eq. (1.8) we get:

$$d(\ln X_t) = \frac{1}{X_t} dX_t + \frac{1}{2}\left(-\frac{1}{X_t^2} \right)(dX_t)^2$$

$$= \frac{dX_t}{X_t} + \frac{1}{2}\left(-\frac{1}{X_t^2} \right)(X_t \, dW_t)^2$$

and since $dW_t \approx (dt)^{1/2}$. Hence:

$$d(\ln X_t) = \frac{dX_t}{X_t} + \frac{1}{2}\left(-\frac{1}{X_t^2} \right) X_t^2 \, dt$$

$$= \frac{dX_t}{X_t} - \frac{1}{2} dt$$

or equivalently:

$$\frac{dX_t}{X_t} = d(\ln X_t) + \frac{1}{2} dt$$

So, from eq. (1.9) one can conclude that:

$$\int_0^t \frac{d X_s}{X_s} = \int_o^t d(\ln X_s)\, ds + \int_0^t \frac{1}{2} ds$$

$$W_t = \ln \frac{X_t}{X_t(0)} + \frac{1}{2} t$$

Therefore, the solution is given by

$$X_t = \exp\left(W_t - \frac{1}{2} t \right).$$

Chapter Two

Stochastic Partial Differential Equations

CHAPTER TWO

STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

Stochastic partial differential equations are known to be an effective tool in modeling complex physical and engineering phenomena. Examples including wave propagation [33], diffusion through heterogeneous random media [43], randomly forced Burgers and Navier-Stokes equation (see e.g., [3, 10, 11, 12, 31, 34, 35, 36, 40, 41] and the reference therein). Additional examples can be found in materials science, chemistry, biology, and other areas. In those problems, the large structures and dominate dynamics are governed by deterministic physical laws, which the unresolved small scales, microscopic effects, and other uncertainties can be naturally modeled by stochastic processes. The resulting equations are usually PDE's with either random coefficients, or randomintical conditions, or random forcing. Unlike deterministic PDE's, solutions of SPDE's are random fields. Hence, it is important to able to study their statistical characteristic, e.g., mean, variance and higher order moments, [30].

The goal of this chapter is to set some ideas in SPDE's, which we have found useful. Sometimes, the idea in a simple case will be explained, and leave it to the reader to develop the topic more broadly as a future work.

2.1 FUNDAMENTAL CONCEPTS

In this section, some of the most important definitions and concepts related to the work of this chapter are given.

Definition (2.1), [17]:

The elliptic partial differential equation is defined by Lu = f(x), where f is given function and L is a linear operator defined by:

$$L\cdot = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}\cdot + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}\cdot + c(x)\cdot \quad \text{.....................(2.1)}$$

the real coefficients a_{ij}, b_i and c are defined in an n-dimensional domain D, L is said to be elliptic type (or elliptic) at a point x_0 if the matrix $(a_{ij}(x_0))$ is positive definite, i.e., for any real vector $\zeta \neq 0$, we have:

$$\sum_{i,j=1}^{n} a_{ij}(x_0)\zeta_i\zeta_j > 0$$

Remark (2.2), [16]:

Let D be bounded domain with boundary ∂D in $C^2(D)$, and let L be an elliptic operator given by (2.1) with coefficients defined in D. Given any function f in D and φ on ∂D, the problem is to find a solution u of the boundary value problem:

$$Lu = f(x) \text{ in } D \quad \text{..(2.2)}$$

$$u(x) = \varphi(x) \text{ on } \partial D \quad \text{..(2.3)}$$

this called the Dirichlet problem or first boundary value problem.

Definition (2.3), [17]:

A barrier $W_y(x)$ at the point $y \in \partial D$ is a continuous nonnegative function in $\bar{D}$ that vanishes only at the point y and for which $LW_y(x) \le -1$.

Theorem (2.4), [17]:

If D is a bounded domain with $\partial D \in C^2(D)$, then for any $y \in \partial D$, there exists a closed ball K such that $K \cap D = \varnothing$ and $K \cap \bar{D} = \{y\}$; thus, a barrier exists.

Definition (2.5), [17]:

The parabolic partial differential is defined by $\mu u = f(x, t)$, where M is a linear operator defined by:

$$\mu\cdot = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j}\cdot + \sum_{i=1}^{n} b_i(x,t)\frac{\partial}{\partial x_i}\cdot + c(x,t)\cdot - \frac{\partial}{\partial t}\cdot \quad ......(2.4)$$

where the real coefficients a_{ij}, b_i and c are defined in an (n + 1)-dimensional domain D. If $\sum a_{ij}(x_0, t_0)\,\zeta_i\zeta_j > 0$, for all $\zeta \in$ ¡ ^{n}, $\zeta \neq 0$, then we say that is μ of parabolic type at (x_0, t_0). μ is uniformly parabolic in D if there is a positive constant M, such that:

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\zeta_i\zeta_j \ge M|\zeta|^2, \text{ for all } (x, t) \in D,\ \zeta \in \text{¡}^n \quad ....(2.5)$$

Definition (2.6), [16]:

The function f(t) is said to be uniformly Hölder continuous (exponent α) in $[0, t_0]$ if $\|f(t) - f(\tau)\| \le c|t - \tau|^{\alpha}$, for all t, τ in $[0, t_0]$; where c is a positive constants and $0 \le \alpha \le 1$.

Definition (2.7), [4]:

A function f is uniformly Lipschitz continuous on a domain D if given any c > 0, there exists $\delta > 0$, such that:

$$|f(x_1) - f(x_2)| < c|x_1 - x_2|, \ \forall\ x_1, x_2 \in D$$

with property $|x_1 - x_2| < \delta$.

Definition (2.8) (The Kondrative Spaces of Stochastic Test Function and Stochastic Distributions), [23]:

a) The stochastic test function spaces:

Let N be a natural number. For $0 \le \rho \le 1$, let:

$$(S)_{\rho}^{N} = (S)_{\rho}^{m;N}$$

consist of those functions of form:

$$f = \sum_{\alpha} c_{\alpha} H_{\alpha} \in L^2(\mu_m)$$

such that:

$$\| f \|_{\rho,k}^2 := \sum_{\alpha} c_{\alpha}^2 (\alpha!)^{1+\rho} (2N)^{k\alpha} < \alpha, \text{ for all } k \in \mathbb{N} \quad \text{....................(2.6)}$$

where:

$$c_{\alpha}^2 = |c_{\alpha}|^2 = \sum_{k=1}^{N} \left(c_{\alpha}^{(k)}\right)^2 \quad \text{if} \quad c_{\alpha} = \left(c_{\alpha}^{(1)}, c_{\alpha}^{(2)}, \ldots, c_{\alpha}^{(N)}\right) \in \mathbb{R}^N$$

b) The stochastic distribution spaces

For $0 \le \rho \le 1$, let:

$$(S)^{N}_{-\rho} = (S)^{m;N}_{-\rho}$$

consist of all formula expansions

$$F = \sum_{\alpha} b_{\alpha} H_{\alpha} \text{ , with } b_{\alpha} \in \mathbb{R}^{N}$$

such that

$$\| f \|_{-\rho,-q} = \sum_{\alpha} b_{\alpha}^{2} (\alpha!)^{1-\rho} (2N)^{-q\alpha} < \infty, \text{ for some } q \in \mathbb{N} \text{ ....(2.7)}$$

The family of semi norms $\|f\|_{\rho,k}$; $k \in \mathbb{N}$ give the rise for a topology on $(S)^{N}_{\rho}$, and we can regard $(S)^{N}_{-\rho}$ as the dual by the action:

$$< F,f > = \sum_{\alpha} (b_{\alpha}, c_{\alpha})\ \alpha! \text{ ....(2.8)}$$

If:

$$F = \sum_{\alpha} b_{\alpha} H_{\alpha} \in (S)^{N}_{-\rho};\ f = \sum_{\alpha} c_{\alpha} H_{\alpha} \in (S)^{N}_{\rho}$$

and (b_{α}, c_{α}) is the usual inner product in $\mathbb{R}^{n}$. Note that, the above action is well defined, since:

$$\sum_{\alpha} |(b_{\alpha}, c_{\alpha})|\, \alpha! = \sum_{\alpha} |(b_{\alpha}, c_{\alpha})|\, (\alpha!)^{\left(\frac{1-\rho}{2}\right)} (2N)^{\frac{-q\alpha}{2}} (2N)^{\frac{q\alpha}{2}}$$

$$\le \left(\sum_{\alpha} b_{\alpha}^{2} (\alpha!)^{1-\rho} (2N)^{-q\alpha} \right)^{\frac{1}{2}} \left(\sum_{\alpha} c_{\alpha}^{2} (\alpha!)^{1+\rho} (2N)^{q\alpha} \right)^{\frac{1}{2}}$$

$$< \infty$$

for q large enough. When the value of m is clear, then from the context we simply write $(S)_{\rho}^{N}$, $(S)_{-\rho}^{N}$ instead of $(S)_{\rho}^{m;N}$, $(S)_{-\rho}^{m;N}$, respectively. If N = 1 we write $(S)_{\rho}$, $(S)_{-\rho}$ instead of $(S)_{\rho}^{1}$, $(S)_{-\rho}^{1}$, respectively.

Remark (2.9), (The Hida Test Function Space (S) and the Hida Distribution Space (S)), [23]:

There is an extensive literatures on those spaces (see [20]). According to characterization in [45], we can describe these spaces, generalized to an arbitrary dimension m, as it follows in the next proposition:

Proposition (2.10),:

a) The Hida test function space $(S)^N$:

Consist of those expansions:

$$f = \sum_{\alpha} c_{\alpha} H_{\alpha} \in L^2(\mu_m), \text{ with } c_{\alpha} \in \mathbb{R}^N$$

such that:

$$\sup_{\alpha} \left\{ c_{\alpha}^2 \, \alpha! (2N)^{k\alpha} \right\} < \infty, \text{ for all } k < \infty \quad \text{.................................... (2.9)}$$

b) The Hida distribution space $(S)^{*,N}$:

Consists of all formula expansions:

$$F = \sum_{\alpha} b_{\alpha} H_{\alpha}, \text{ with } b_{\alpha} \in \mathbb{R}^N$$

such that:

$$\sup_{\alpha}\left\{b_\alpha^2\,\alpha!(2N)^{-q\alpha}\right\}<\infty \text{, for some } q<\infty \quad\text{.......................}(2.10)$$

Hence, after comparison with the definition (2.8), one may see that:

$$(S)^N = (S)_0^{m,N} \quad \text{and} \quad (S)^{*,N} = (S)_{-0}^{m,N} \quad\text{.......................}(2.11)$$

If N = 1, we write:

$$(S)^1 = (S) \ \text{ and } \ (S)^{*,1} = (S)^*$$

2.1.1 Singular White Noise, [23]:

One of the many useful properties of $(S)^*$ is that it consists the singular or pointwise white noise.

Definition (2.11):

a) The 1-dimensional (d-parameter) singular white noise process is defined by the formal expansion:

$$W(x) = W(x,w) = \sum_{k=1}^{\infty} \eta_k(x) H_{\varepsilon_k}(w);\ x \in \mathbb{R}^d \quad\text{.......................}(2.12)$$

where $\{\eta\}_{k=1}^{\infty}$ is the basis of $L^2(\mathbb{R}^d)$, defined by:

$$\eta_j := \zeta_{\delta_1(j)} \otimes \ldots \otimes \zeta_{d(j)};\ j = 1, 2, \ldots \quad\text{.......................}(2.13)$$

while $H_\alpha = H_\alpha^{(1)}$ is defined by:

$$H_\alpha(w) = H_\alpha^{(1)}(w) := \prod_{i=1}^{\infty} h_{\alpha i}(<w,\eta_i>),\ w \in S'(\mathbb{R}^d) \text{.............}(2.14)$$

b) The m-dimensional (d-parameter) singular white noise process is defined by:

$$W(x) = W(x, w) = (W_1(x, w), W_2(x, w), \ldots, W_n(x, w))$$

where the i^{th} component $W_i(x)$ of $W(x)$, has the expansion:

$$W_i(x) = \sum_{j=1}^{\infty} \eta_j(x) H_{\varepsilon_{i+(j-1)m}}$$

$$= \eta_1(x) H_{\varepsilon_{(i)}} + \eta_2(x) H_{\varepsilon_{(i+m)}} + \eta_3(x) H_{\varepsilon_{(i+2m)}} + \ldots \quad \ldots\ldots\ldots\ldots(2.15)$$

Definition (2.12):

Let $F = \sum_{\alpha} b_\alpha H_\alpha \in (S)_{-1}^{N}$ with $b_\alpha \in \mathbb{R}^d$ as in definition (2.8). Then the Hermite transform of F, denoted by $H\tilde{F}$ or $\tilde{F}$ for simplicity, is defined by convergent power series:

$$HF = \tilde{F}(z) = \sum_{\alpha} b_\alpha z^\alpha \in \mathbb{C}^{\mathbb{N}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.16)$$

where $z = (z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$ (the set of all sequence of complex numbers).

$$z_\alpha = z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} \ldots z_n^{\alpha_n} \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.17)$$

if $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{J}$, where $z_j^0 = 1$ and $\mathcal{J} = (\mathbb{N}_0^{\mathbb{N}})_c$ the set of all finite sequences in $\mathbb{N}_0^{\mathbb{N}}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition (2.13):

The 1-dimensional (d-parameter) smooth white noise is the map $W : S(\mathbb{R}^d) \times S'(\mathbb{R}^d) \longrightarrow \mathbb{R}$, given by:

$$W(\varphi) = W(\varphi, w) = \langle W, \varphi \rangle \; ; \; w \in S(\mathbb{R}^d), \; \varphi \in S'(\mathbb{R}^d) \quad \ldots\ldots\ldots(2.18)$$

Example (2.14):

The 1-dimensional smooth white noise W(φ) has the form:

$$W(\varphi, w) = <W, \varphi>$$

$$= <W, \sum_{j=1}^{\infty} (\varphi, \eta_j)\eta_j>$$

$$= \sum_{j=1}^{\infty} (\varphi, \eta_j) < W, \eta_j > = \sum_{j=1}^{\infty} (\varphi, \eta_j) H_{\varepsilon_{(j)}}(w) \quad \text{.............(2.19)}$$

where $\varepsilon^{(j)} = (0, 0, \ldots, 1, \ldots)$ with 1 on entry number j and 0 otherwise. The convergence is in $L^2(\mu)$. In other words:

$$W(\varphi, w) = \sum_{\alpha} c_\alpha H_\alpha(w)$$

with

$$c_\alpha = \begin{cases} (\varphi, \eta_i), & \text{if } \alpha = \varepsilon^{(j)} \\ 0, & \text{other wise} \end{cases} \quad \text{..(2.20)}$$

and therefore the Hermite transform $\tilde{W}(\varphi)$ of W(φ) is:

$$\tilde{W}(\varphi) = \sum_{j=1}^{\infty} (\varphi, \eta_j)\, z_j \quad \text{...(2.21)}$$

which is converge for all $z = (z_1, z_2, \ldots) \in (c^N)_c$.

Definition (2.15):

For $0 < R, q < \infty$, define the infinite-dimensional neighborhood $K_q(R)$ of 0 in c^N by:

$$K_q(R) = \left\{(\zeta_1, \zeta_2, \ldots) \in c^N ; \sum_{\alpha \neq 0} |\zeta^\alpha|^2 (2N)^{q\alpha} < R^2 \right\} \ldots\ldots\ldots\ldots (2.21)$$

Note that:

$q \leq Q$, $r \leq R$

which implies to:

$$K_Q(r) \subset K_q(R) \ldots\ldots\ldots\ldots (2.22)$$

for any $q < \infty$, $\delta > 0$ and natural number k, there exists $\varepsilon > 0$, such that:

$z = (z_1, z_2, \ldots, z_k) \in c^k$ and $|z_i| < \varepsilon$; $1 \leq k$

which implies to $z \in K_q(\delta)$.

2.2 ELLIPTIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS, [17]

Let L be an elliptic operator in a bounded domain D which is given by (2.1). Assume that L is uniformly elliptic in D, i.e.,

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \zeta_i \zeta_j \geq M |\zeta|^2, \, x \in \mathbb{R}^n, \, M > 0 \ldots\ldots\ldots\ldots (2.23)$$

Assume also that:

a_{ij}, b_i are uniformly Lipschitz continuous in $\bar{D}$ ....(2.24)

$c \leq 0$, c uniformly Hölder continuous in $\bar{D}$ ....(2.25)

Assume finally that the boundary ∂D of D is in $C^2(D)$, so that the barriers exist at all points of ∂D (see theorem (2.4)). Then as it will be proved later, the Dirchlet problem (2.2), (2.3) has a unique solution u for any given functions f, φ satisfying:

f is uniformly Hölder continues in $\bar{D}$ ....(2.26)

φ is continuous on ∂D ....(2.27)

Consider the system of stochastic differential equations:

$$d\zeta(t) = \sigma(\zeta(t))\, dW(t) + b(\zeta(t))\, dt \quad \text{...(2.28)}$$

Denote by V_ε the closed ε-neighborhood of ∂D and let $D_\varepsilon = D \backslash V_\varepsilon$. Let V be a function in $C^2(\mathbb{R}^n)$ that coincides with the solution u of (2.2), (2.3) in $D_{\varepsilon/2}$, and let τ be Markov time with respect to time-homogeneous Markov process solution of (2.28). By the Itô's formula, then:

$$E_x (V(\zeta(\tau)) \exp\left[\int_0^\tau c(\zeta(s))\, ds\right] - V(x) =$$
$$E_x \int_0^\tau [L\, V(\zeta(t))] \exp\left[\int_0^t c(\zeta(s))\, ds\right] dt \quad \text{.........................(2.29)}$$

take $x \in D_\varepsilon$, and $\tau = \min\{\tau_\varepsilon, T\}$, where τ_ε is the hitting time of V_ε. Then $V\{\zeta(t)\} = u(\zeta(t))$, for all $0 \le t \le \tau$. Hence (2.29) holds for v = u. Taking $\varepsilon \longrightarrow 0$ and using the Lebesgue bounded convergence theorem (see [5]), we get

$$u(x) = E_x\, u(\zeta(\tau^*))\, \exp\left[\int_0^{\tau^*} c(\zeta(s))\, ds\right] -$$

$$E_x \int_0^{\tau^*} f(\zeta(t))\, \exp\left[\int_0^t c(\zeta(s))\, ds\right] dt \quad \text{..................................(2.30)}$$

where τ is the exit time from D, and $\tau^* = \min\{\tau, T\}$.

Theorem (2.16):

Assume that L is uniformly elliptic in D, such that $c(x) \leq 0$, $\forall x \in D$ and that a_{ij}, b_i, c are uniformly Hölder continuous (exponent α) in $\bar{D}$. If every point of ∂D has a barrier and if φ is a continuous function on ∂D, then there exists a unique solution u in $C^2(D) \cap C^0(\bar{D})$ of the Drichlet problem (2.2), (2.3).

For the proof of the existence and the uniqueness the reader is referred to [15], [16], [17].

Theorem (2.17):

Let:

$d\zeta_i(t) = a_i(t)dt + b_i(t)d\zeta$, $i = 1, 2, \ldots, m$

and let $f(x_1, x_2, \ldots, x_m, t)$ be a continuous function in (x, t) where $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$, $t \geq 0$, together with its first t-derivative and second x-derivatives. Then $f(\zeta_1(t), \ldots, \zeta_m(t), t)$ has a stochastic differential, given by:

$$df(X(t), t) = \left[f_t(X(t), t) + \sum_{i=1}^{m} f_{x_i}(X(t), t) a_i(t) + \frac{1}{2} \sum_{i,j=1}^{m} f_{x_i x_j}(X(t), t) b_i(t) b_j(t) \right] dt + \sum_{i=1}^{m} f_{x_i}(X(t), t) b_i(t) dw(t) \qquad (2.31)$$

where $X(t) = (\zeta_1(t), \zeta_2(t), \ldots, \zeta_m(t))$. Formula (2.31) is called the Itô's formula.

Remark (2.18):

Itô's formula (2.31) asserts the two processes:

$f(X(t), t) - f(X(0), 0)$

and

$$\int_0^t \left[f_s(X(t),t) + \sum_{i=1}^{m} f_{x_i}(X(t),t)a_i(t) + \frac{1}{2}\sum_{i,j=1}^{m} f_{x_i x_j}(X(t),t)\,b_i(t)b_j(t) \right] ds + \int_0^t \sum_{i=1}^{m} f_{x_i}(X(t),t)\,b_i(s)\,dw(s)$$

are stochastically equivalent. Since they are continuous and their sample paths coincide a.s. Consequently integrating both side of eq.(2.31) from 0 to τ

$$f(X(\tau), \tau) - f(X(0), 0) = \int_0^{\tau} \Big[f_t(X(t),t) + \sum_{i=1}^{m} f_{x_i}(X(t),t)a_i(t) + \frac{1}{2}\sum_{i,j=1}^{m} f_{x_i x_j}(X(t),t)\,b_i(t)b_j(t) \Big] dt + \int_0^{\tau} \sum_{i=1}^{m} f_{x_i}(X(t),t)\,b_i(t)\,dw(t) \quad \text{.......................................}(2.32)$$

for any random variable τ, $0 \le \tau \le T$.

If, in particular, τ is a stopping time, and when, taking the expectation, we find that:

$$E_x f(X(\tau), \tau) - E_x f(X(0), 0) = E_x \int_0^{\tau} L f(X(t),t)dt \quad \text{..............}(2.33)$$

where:

$$Lf = f_t(X(t),t) + \sum_{i=1}^{m} f_{x_i}(X(t),t)a_i(t) + \frac{1}{2}\sum_{i,j=1}^{m} f_{x_i x_j}(X(t),t)\,b_i(t)b_j(t)$$

The next theorem is given in [17] without details of the proof and we give the proof for completeness.

<u>Theorem (2.19):</u>

Let (2.23) – (2.27) hold and let ∂D belong to $C^2(D)$. Then the unique solution u of the Dirichlet problem (2.2), (2.3) is given by:

$$u(x) = E_x\varphi(\zeta(\tau))\ \exp\left[\int_0^{\tau} c(\zeta(s))\,ds\right] -$$

$$E_x\int_0^{\tau} f(\zeta(t))\ \exp\left[\int_0^{t} c(\zeta(s))\,ds\right]dt \quad \text{.................................(2.34)}$$

where τ is the exist time from D.

Proof:

To prove that $E_x\tau < \infty$, for all $x = (x_1, x_2, \ldots, x_n)$ in D.

Consider the function $h(x) = -A e^{\lambda x_1}$

If A, λ are sufficiently large (A depending on λ), then:

$Lh(x) \le -1$ in D

integrate both sides from 0 to $\tau^* = \min\{\tau, T\}$

$$\int_0^{\tau^*} Lh(\zeta(s))\,ds \le -\int_0^{\tau^*} ds$$

and hence taking the expectation

$$E_x\int_0^{\tau^*} Lh(\zeta(s))\,ds \le -E_x\int_0^{\tau^*} ds$$

and using remark (2.18)

$$E_x h(\zeta(\tau^*) - E_x h(\zeta(0)) \leq -E_x(\tau^*)$$

$$E_x h(\zeta(\tau^*) - h(x)) \leq -E_x(\tau^*)$$

Since $h(x) = -A e^{\lambda x_1}$, then:

$$|h(x)| = \left|-Ae^{\lambda x_1}\right|$$

$$\leq \left|-A\right| \left|e^{\lambda x_1}\right|$$

$$\leq AB$$

$$\leq k, \forall \text{ x in D}$$

we have then:

$$E_x h(\zeta(\tau^*)) - h(x) \leq -E_x(\tau^*)$$

$$E_x(\tau^*) \leq h(x) - E_x h(\zeta(\tau^*))$$

$$\left|E_x(\tau^*)\right| \leq \left|h(x) - E_x h(\zeta(\tau^*))\right|$$

$$\leq \left|h(x) + (-E_x h(\zeta(\tau^*)))\right|$$

$$\leq \left|h(x)\right| + \left|-E_x h(\zeta(\tau^*))\right|$$

$$= \left|h(x)\right| + \left|E_x h(\zeta(\tau^*))\right|$$

Since $|h(x)| \leq k$ and from the definition of expectations:

$$\left| E_x\, h(\zeta(\tau^*)) \right| = \left| \int_{-\infty}^{\infty} h(x) f(s)\, ds \right|$$

$$\leq |h(x)| \left| \int_{-\infty}^{\infty} f(s)\, ds \right|$$

$$\leq k \times 1 \leq k$$

This implies:

$$E_x(\tau^*) \leq k + k$$

$$\leq 2k$$

Taking $T \longrightarrow \infty$, we get:

$$E_x\tau \leq 2k$$

Therefore:

$$E_x\tau < \infty$$

and hence the expectation is bounded. Then by taking $T \longrightarrow \infty$ in (2.30) and using Lebesgue bounded convergence theorem [5], we get that assertion (2.34). <

Consider now the initial-boundary value problem (written for t replaced by T – t).

$$\left.\begin{aligned} & Lu + \frac{\partial u}{\partial t} = f(x,t), \quad \text{in } Q = B \times [0,T] \\ & u(x,\tau) = \varphi(x), \qquad \text{on } B \\ & u(x,t) = g(x,t), \qquad \text{on } S \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.35)$$

where B is bounded domain with C^2 boundary ∂B, $S = \partial B \times [0, T]$, and L is defined by:

$$Lu = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u$$

consider also the system of stochastic differential equations:

$$d\zeta(t) = \sigma(\zeta(t), t)dW(t) + b(\zeta(t), t)dt \quad \text{.....................................(2.36)}$$

where $(\sigma(x, t))^2 = a(x, t)$ in $\bar{Q}$. The coefficients σ and b are extensions of $\sigma(x, t)$, $b(x, t)$, originally defined in $\bar{Q}$, such that:

$$|\sigma(x, t) - \sigma(y, s)| \leq c(|x - y| + |t - s|)$$

$$|b(x, t) - b(y, s)| \leq c(|x - y| + |t - s|).$$

we shall assume:

$$\left.\begin{array}{l} \sum a_{ij}\zeta_i\zeta_j \geq \mu|\zeta|^2 \quad \text{if } (x,t)\in Q, \zeta \in \mathbb{R}^n, \mu > 0, \\ a_{ij}, b_i \text{ are uniformly Lipschitz continuous in } (x,t)\in \bar{Q} \\ c \quad \text{is uniformly Hölder continuous in } (x,t)\in \bar{Q} \\ f \quad \text{is uniformly Hölder continuous in } (x,t)\in \bar{Q} \\ g \quad \text{is continuous on } \bar{S}, \varphi \text{ is continuous on } \bar{B} \text{ and} \\ \quad g(x,T) = \varphi(x) \text{ if } x \in \partial B. \end{array}\right\} \text{........(2.37)}$$

By the next theorem (2.20) there exists a unique solution u of (2.35).

Theorem (2.20):

Assume that μ is uniformly parabolic in Q, i.e., if there is a positive constant M such that $\sum a_{ij}\zeta_i\zeta_j \geq M|\zeta|^2$, for all $(x, t) \in Q$, $\zeta \in \mathbb{R}^n$, that a_{ij}, b_i, c, f are uniformly Hölder continuous in $\bar{Q}$ and that g, φ are continuous functions on $\bar{B}, \bar{S}$ respectively and $g = \varphi$ on $\bar{B} \cap \bar{S}$. Assume also that there exists a barrier at every point of S. Then there exists a unique solution u of the initial-boundary value problem (2.35).

Theorem (2.21):

Let D be bounded domain ∂B belong to $C^2(D)$ and let (2.37) hold. Then the unique solution u of the initial-boundary value problem (2.35) is given by:

$$u(x,t) = E_{x,t}\, g(\zeta(\tau)) \exp\left[\int_t^{\tau} c(\zeta(s),s)ds\right]_{\chi_{\tau<T}} + E_{x,t}\varphi(\zeta(T)) \exp\left[\int_t^{\tau} c(\zeta(s),s)ds\right]_{\chi_{\tau=T}} - E_{x,t}\int_t^{\tau} f(\zeta(s),s) \exp\left[\int_t^{s} c(\zeta(\lambda),\lambda)\right] ds \quad \text{......................... (2.38)}$$

where τ is the first time $\lambda \in [t, T]$ that $\zeta(\lambda)$ leaves B if such a time exists and $\tau = T$ otherwise.

Proof:

The proof is similar to the proof of theorem (2.19). <

2.3 THE STOCHASTIC POISSON EQUATION

Let us illustrate the method described in the last section with respect to the following equation, called the stochastic Poisson equation:

$$\left.\begin{aligned} \Delta U(x) &= -W(x); \quad x \in D \\ U(x) &= 0; \quad\quad\quad x \in \partial D \end{aligned}\right\} \quad \text{.. (2.39)}$$

where $\Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$ is the Laplacian operator in $\mathbb{R}^D$, $D \subset \mathbb{R}^d$ is any given bounded domain with regular boundary and where

$W(x) = \sum_{j=1}^{\infty} \eta_j(x) H_{\varepsilon_j}(w)$ is d-parameter white noise. This equation models, for example, the temperature U(x) in D when the boundary temperature is kept equal to zero and there is a white noise heat source in D.

Taking the Hermite transform of (2.39), we get the equation

$$\left.\begin{array}{ll} \Delta u(x,z) = -\tilde{W}(x,z); & x \in D \\ u(x,z) = 0; & x \in \partial D \end{array}\right\} \qquad (2.40)$$

for our candidate u for $\tilde{U}$, where the Hermite transform $\tilde{W}(x,z) = \sum_{j=1}^{\infty} \eta_j(x) z_j$, when $z = (z_1, z_2, \ldots) \in (c^N)_c$ (see Example (2.14).

by considering the real and imaginary parts of the equation separately, we see that the usual solution formula holds:

$$u(x, z) = \int_{\mathbb{R}^d} G(x,y)\,\tilde{W}(y,z)\,dy \qquad (2.41)$$

where G(x, y) is the classical Green function of D (so G = 0 outside D). Suppose that the solution u(x, z) exist for all $x \in (c^N)_c$, $x \in D$, since the integral on the right of (2.41) converges for all such x, z (for this we only need that $G(x, y) \in L^1$ for each x). Moreover, for $z \in (c^N)_c$, we have:

$$|u(x,z)| = \left|\int G(x,y) \sum \eta_j(y) z_j \, dy\right|$$

$$= \left|\sum z_j \int G(x,y) \eta_j(y)\, dy\right|$$

$$\leq \left|\sum z_j\right| \int |G(x,y)|\, |\eta_j(y)|\, dy$$

$$\le \sup_y |\eta_j(y)| \int |G(x,y)| dy \sum |z_j|$$

$$\le c \sum |z_j|$$

$$\le c \sum_j |z^{\varepsilon_j}| \, j!$$

$$\le c \left(\sum_j |z^{\varepsilon_j}|^2 (2N)^{2\varepsilon_j} \right)^{1/2} \left(\sum_j (2N)^{-2\varepsilon_j} \right)^{1/2} < \infty$$

if $z \in k_2(R)$. Since u(x, y) depends analytically on z, it follows from the characterization theorem [11] that there exists $U(x) \in (S)_{-1}$, such that $\tilde{U}(x) = u(x,z)$.

In particular $\partial^2 u / \partial x^2$ is bounded for $(x, z) \in k_2(R)$, since both $\Delta u = -\tilde{W}$ and u are equal. Therefore, from next theorem (2.22) U(x) solves (2.39) we recognize directly from (2.41) that u is the Hermite transform of

$$U(x) = \int_{\mathbb{R}^d} G(x,y) W(y) dy$$

$$= \sum_{j=1}^{\infty} \int_{\mathbb{R}^d} G(x,y) \eta_j(y) dy \, H_{\varepsilon_j}(w),$$

which converges in $(S)^*$ because (see (2.10))

$$\sum_{j=1}^{\infty} \left(\int_{R^d} G(x,y) \eta_j(y) dy \right)^2 (2j)^{-q} \le c^2 \sum_{j=1}^{\infty} (2j)^{-q} < \infty, \ \forall q > 1$$

Theorem (2.22), [23]:

Suppose u(t, x, z) is a solution (in the usual strong, pointwise sense) of the equation

$$\tilde{A}(t, x, \partial t, \nabla x, u, z) = 0 \quad \text{...} (2.42)$$

for (t, x) in some bounded open set $G \subset \mathbb{R} \times \mathbb{R}^d$, and for all $z \in k_q(R)$, for some q, R. Moreover, suppose that, u(t, x, z) and all its partial derivatives, which are involved in (2.42), are (uniformly) bounded for $(t, x, z) \in G \times k_q(R)$, continuous with respect to $(t, x) \in G$, for each $z \in k_q(R)$ and analytic with respect to $z \in k_q(R)$, for all $(t, x) \in G$. Then there exists $U(t, x) \in (S)_{-1}$, such that u(t, x, z) = (H U(t, x)) (z), for all $(t, x, z) \in G \times k_q(R)$ and U(t, x) solves the equation:

$$A(t, x, \partial t, \nabla x, U, W) = 0 \quad \text{in } (S)_{-1} \quad \text{.......................................} (2.43)$$

Theorem (2.23), [23]:

The unique stochastic process $U(x) \in (S)_{-1}$ solving (2.39) is given by

$$U(x) = \int_{\mathbb{R}^d} G(x, y)\, W(y)\, dy$$

$$= \sum_{j=1}^{\infty} \int G(x, y)\, \eta_j(y)\, dy\, H_{\varepsilon_j}(y) \quad \text{.......................................} (2.44)$$

We have $U(x) \in S^*$, $\forall\, x \in \bar{D}$.

2.4 THE STOCHASTIC HEAT EQUATION, [32]

In the present section, we will concern with following one-dimensional heat equation driven by a space-time white noise:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b(u(x,t)) + \sigma \dot{W}(x,t),\ (x,t) \in [0,1]\times[0,T] \ ....(2.45)$$

where T > 0, the initial condition is given by a continuous function u_0: [0, 1] ⟶ ¡ and we consider Dirichlet boundary conditions. That is:

$$\left.\begin{array}{l} u(x,0) = u_0(x)\ ,\ x \in [0,1] \\ u(0,t) = u(1,t) = 0,\ t \in [0,T] \end{array}\right\} \ ....(2.46)$$

The real valued random field solution to eq. (2.45) will be {u(x,t), (x,t)∈ [0, 1]×[0, T]}. The function b : ¡ ⟶ ¡ is of class C having a bounded derivative and σ > 0 is a constant. We assume that {W(x, t), (x,t) ∈ [0, 1]×[0, T]} is Brownian motion on [0, 1]×[0, T], defined in a complete probability space (Ω, *F*, P). The solution to the formal eq. (2.45) is understood in the mild sense: a F_t-adapted stochastic process {u(x, t), (x, t) ∈ [0, 1]×[0, T]} solved (2.45) with initial and boundary conditions (2.46), if for all any (x, t) ∈ [0, 1]×[0, T]

$$u(x,t) = \int_0^1 G_t(x,y)\,u_0(y)dy + \int_0^t\int_0^1 G_{t-s}(x,y)\,b(u(y,s))dyds +$$

$$\sigma\int_0^t\int_0^1 G_{t-s}(x,y)W(ds,dy) \ ....(2.48)$$

where $G_t(x, y)$, $(t, x, y) \in R\times(0, 1)^2$, denotes the Green's function associated to the heat equation on [0, 1] with Dirichlet boundary conditions:

$$G_t(x, y) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-(x-y)^2}{4t}}.$$

2.5 THE STOCHASTIC WAVE EQUATION, [32]

In this section, we study the stochastic wave equation in one and two dimension.

$$\frac{\partial^2 u}{\partial t^2} = \Delta u(x,t) + b(u(x,t) + \sigma \dot{W}(x,t),\ (x,t) \in \mathbb{R}^d \times [0,T] \text{......}(2.49)$$

where $T > 0$, $b : \mathbb{R} \longrightarrow \mathbb{R}$ is a function with bounded derivatives and suppose that we are given the initial conditions of the form:

$$u(x, 0) = u_0(x),\ \frac{\partial u}{\partial t}(x,0) = \nu_0(x),\ x \in \mathbb{R}^d$$

where u_0, ν_0: $\mathbb{R}^d \longrightarrow \mathbb{R}$ are measurable and bounded functions, such that u_0 is of class $C^1(\mathbb{R}^d)$ and has bounded derivatives ∇u_0. The mild solution of (2.49) is given by $\{\mathcal{F}_t\}$-adapted process $\{u(x, t), (x, t) \in \mathbb{R}^d \times [0,T]\}$ such that, for all $(x, t) \in \mathbb{R}^d \times [0,T]$

$$u(x,t) = \int_{\mathbb{R}^d} \nu_0(x-y)\Gamma_t^d dy + \frac{\partial}{\partial t}\left[\int_{\mathbb{R}^d} u_0(x-y)\Gamma_t^d(dy)\right] + \int_0^t \int_{\mathbb{R}^d} b(u(x-y,s))\Gamma_{t-s}^d dy ds + \sigma \int_0^t \int_{\mathbb{R}^d} \Gamma_t^d(x-y)\, W(ds,dy)$$

..............................(2.50)

where Γ_t^d, t > 0, denotes the fundamental solution of the wave equation in one and two dimension

$$\Gamma_t^1 = \frac{1}{2} 1_{\{|x|<t\}}$$

$$\Gamma_t^2 = \frac{1}{2}\left(t^2 - |x|^2\right)_t^{-1/2}$$

Chapter Three

Finite Difference Method for Solving Stochastic Partial Differential Equations

CHAPTER THREE

FINITE DIFFERENCE METHOD FOR SOLVING STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

The gap between the well-developed theory of SPDE’s and its application for solving such type of equations still wide in range. A crucial task in bridging this gap is the development of an efficient numerical methods for solving SPDE's, and in this connection one of such numerical methods is used which is the finite difference method.

In this chapter, the application of the finite difference method for solving the three types of SPDE's was considered and illustrated with examples.

3.1 FINITE DIFFERENCE APPROXIMATION TO DERIVATIVES, [42]

When a function u and its derivatives are single-valued, finite and continuous functions of x, then by Taylor’s Theorem:

$$u(x + h) = u(x) + hu'(x) + \frac{1}{2}h^2u''(x) + \frac{1}{6}h^3u'''(x) + \ldots \quad \ldots\ldots\ldots\ldots(3.1)$$

and

$$u(x - h) = u(x) - hu'(x) + \frac{1}{2}h^2u''(x) - \frac{1}{6}h^3u'''(x) + \ldots \quad \ldots\ldots\ldots\ldots(3.2)$$

Additions of these expansions give:

$$u(x + h) + u(x - h) = 2u(x) + h^2 u''(x) + 0(h^4) + \ldots \quad \text{................(3.3)}$$

where $O(h^4)$ denotes terms containing fourth and higher powers of h. Assuming these are negligible in comparison with power of h, it follows that:

$$u''(x) = \left(\frac{d^2u}{dx^2}\right)_{x=x} \approx \frac{1}{h^2}[u(x+h) - 2u(x) + u(x-h)] \quad \text{.............(3.4)}$$

with a truncation error on the right-hand side of order h^2.

Subtraction of eq. (3.2) from eq. (3.1) and neglect the terms of order h^3, leads to:

$$u'(x) = \left(\frac{du}{dx}\right)_{x=x} \approx \frac{1}{2h}[u(x+h) - u(x-h)] \quad \text{............................(3.5)}$$

with an error of order h^3.

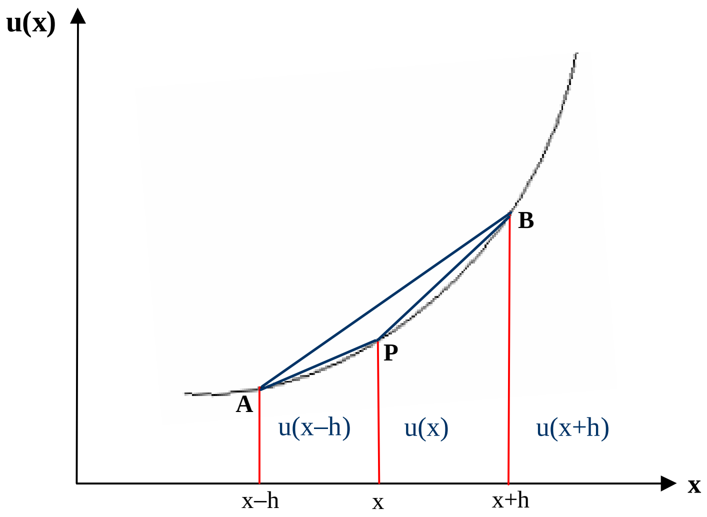

Fig. (3.1).

Equation (3.5) clearly approximates the slope of tangent at P by the slope of the chord AB, and is called a **central-difference approximation**.

We can also approximate the slope of the tangent at P by either the slope of the chord PB, giving the **forward-difference formula**:

$$u'(x) \approx \frac{1}{h}[u(x+h)-u(x)] \qquad (3.6)$$

or the slope of the chord AP giving the **backward-difference formula**:

$$u'(x) \approx \frac{1}{h}[u(x)-u(x-h)] \qquad (3.7)$$

Both eq. (3.6) and eq. (3.7) can be written down immediately from eq. (3.1) and eq. (3.2) respectively, assuming second and higher powers of h are negligible. This shows that the leading errors in these forward and backward-difference formulas are both O(h).

Now, assume u is a function of the independent variables x and t. Subdivide the x-tplane into sets of equal rectangles of sides $\delta x = h$, $\delta t = k$, as shown in Fig.(3.2), and let the coordinates (x, t) of the representative mesh point P be:

$$x = ih, \; t = jk$$

where i and j are integers.

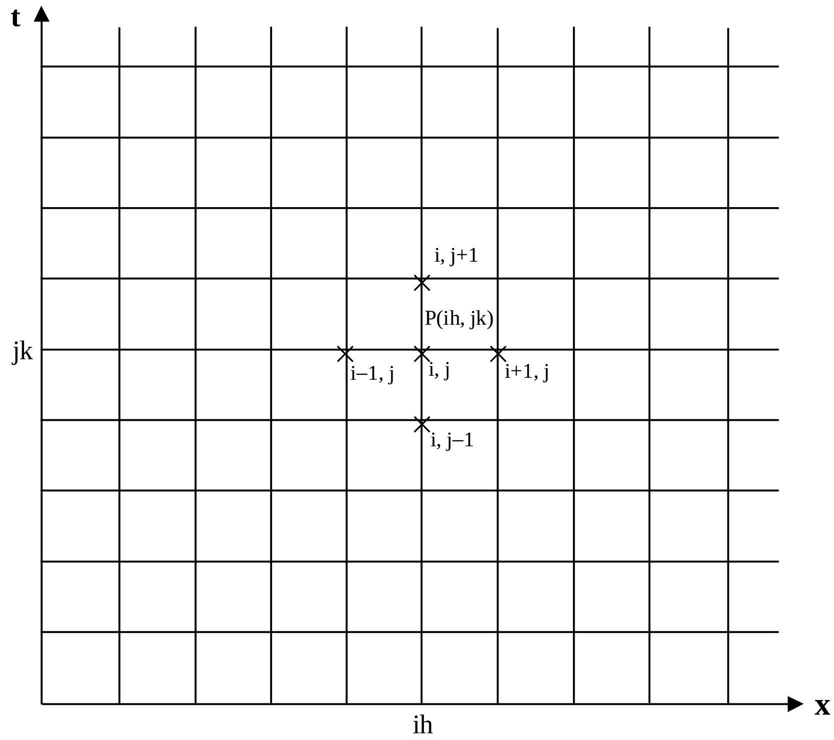

Fig.(3.2) Subdivision of the x-t plane into rectangular mesh points.

Denote the value of u at P by:

$$u_P = u(ih, jk) = u_{i,j}$$

Then by eq. (3.4):

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_P = \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}$$

$$\approx \frac{u[(i+1)h, jk] - 2u[ih, jk] + u[(i-1)h, jk]}{h^2}$$

i.e.

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \quad \text{...(3.8)}$$

with a truncation of order h^2. Similarly:

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \quad \text{.. (3.9)}$$

with a truncation error of order k^2.

With this notation the forward-difference approximation for ∂u/∂t at P is

$$\frac{\partial u}{\partial t} \approx \frac{u_{i,j+1} - u_{i,j}}{k} \quad \text{...(3.10)}$$

with a truncating error or order O(k).

3.2 NUMERICAL SOLUTION OF ELLIPTIC STOCHASTIC PARTIAL-DIFFERENTIAL EQUATIONS

The elliptic stochastic partial differential equation that we consider is the Poisson equation:

$$\nabla^2 u(x,y) \equiv \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = -\dot{W}(x,y) \quad \text{....................(3.11)}$$

$u(x, y) = g(x, y)$ for $(x, y) \in \partial D$

$D = \{(x, y) \mid a < x < b, c < y < d\}$,

where $\dot{W}$ is two-dimensional white noise. The first step is to choose integers n and m, and define step sizes h and k by $h = (b - a) / n$, $k = (d - c) / m$, $x_i = a + ih$, $i = 0, 1, \ldots, n$; and $y_i = c + jk$, $j = 0, 1, \ldots, m$.

For each mesh point in the interior (x_i, y_j), $i = 1, 2, \ldots, n - 1$, $j = 1, 2, \ldots, m - 1$, we use eq. (3.8) into eq. (3.11),then obtain

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = -W_{i,j} \quad \text{...............(3.12)}$$

for each i = 1, 2, …, n − 1, j = 1, 2, …, m − 1; and the boundary conditions as:

$u(x_0, y_j) = g(x_0, y_j)$, for each j = 0, 1, …, m

$u(x_n, y_j) = g(x_n, y_j)$, for each j = 0, 1, …, m

$u(x_i, y_0) = g(x_i, y_0)$, for each i = 0, 1, …, n – 1

$u(x_i, y_m) = g(x_i, y_m)$, for each i = 0, 1, …, n – 1

we can write eq. (3.12)

$$2(1 + r^2)u_{i,j} - (u_{i+1,j} + u_{i-1,j}) - r^2(u_{i,j+1} - u_{i,j-1}) = h^2 W_{i,j} \quad \text{...........(3.13)}$$

where $r = \frac{h}{k}$, for each i = 1, …, n – 1 and j = 1, 2, …, m – 1, and:

$u_{0,j} = g(x_0, y_j)$, j = 0, 1, …, m

$u_{n,j} = g(x_n, y_j)$, j = 0, 1, …, m

$u_{i,0} = g(x_i, y_0)$, i = 1, 2, …, n – 1

$u_{i,m} = g(x_i, y_m)$, i = 1, 2, …, n – 1

Algorithm (3.1):

1. Input a, b, c, d, m, n.

2. h = (b – a) / n, k = (d – c) / m, r = h/k.

3. For i = 0 to n; do steps 4 to 7.

4. For j = 0 to m; do steps 5 to 7.

5. $x_i = a + ih, y_j = c + jk$.

6. Generate random vector p(i), which is normally distributed with mean 0 and variance hk.

7. $w^{\langle i \rangle} = p(i)$.

8. For i = 1 to n – 1; for j = 1 to m – 1

$$u_{0,j} = g(x_0, y_j), u_{n,j} = g(x_n, y_j)$$

$$u_{i,0} = g(x_i, y_0), u_{i,m} = g(x_i, y_m)$$

$$u_{i,j} = [(u_{i+1,j} + u_{i-1,j}) + r^2(u_{i,j+1} - u_{i,j-1}) + h^2 w_{i,j}]/2(1 + r^2)$$

$$RE_{i,j} = \{u_{i,j} - [(u_{i+1,j} + u_{i-1,j}) + r^2(u_{i,j+1} - u_{i,j-1}) + h^2 w_{i,j}]/2(1 + r^2)\}^2$$

9. Stop.

Example (3.2):

Consider the stochastic poisson equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\dot{W}, \quad 0 < x < 1, \ 0 < y < 2 \quad \text{.............................(3.14)}$$

With boundary conditions:

$$u(x, 0) = x^2, \quad u(x, 2) = (x - 2)^2, \ 0 \le x \le 1$$

$$u(0, y) = y^2, \quad u(1, y) = (y - 1)^2, \ 0 \le y \le 2$$

with n = m = 10 this implies h = 0.1, k = 0.2 and r = 0.5.

From eq. (3.13) we can write eq. (3.14) as follows:

$$2\left(1+\frac{1}{4}\right)u_{i,j}-(u_{i+1,j}+u_{i-1,j})-\frac{1}{4}(u_{i,j+1}+u_{i,j-1})=\frac{W_{i,j}}{100}$$

$$\frac{5}{2}u_{i,j}-(u_{i+1,j}+u_{i-1,j})-\frac{1}{4}(u_{i,j+1}+u_{i,j-1})=\frac{1}{100}W_{i,j}$$

$$u_{i,j}=\left[u_{i+1,j}+u_{i-1,j}+\frac{1}{4}u_{i,j+1}+\frac{1}{4}u_{i,j-1}+\frac{1}{100}W_{i,j}\right]\frac{2}{5} \quad \text{............}(3.15)$$

for each $i = 1, \ldots, 9,\ j = 1, \ldots, 9$ and

$$\left.\begin{array}{ll} u_{0,j}=y_j^2 & ,\quad j=0,1,\ldots,10 \\ u_{10,j}=(y_{j-1})^2 & ,\quad j=0,1,\ldots,10 \\ u_{i,0}=x_i^2 & ,\quad i=1,\ldots,9 \\ u_{i,10}=(x_{i-1})^2 & ,\quad i=1,\ldots,9 \end{array}\right\} \quad \text{..}(3.16)$$

$$RE_{i,j}=\left[u_{i,j}-\left(u_{i+1,j}+u_{i-1,j}+\frac{1}{4}u_{i,j+1}+\frac{1}{4}u_{i,j-1}+\frac{1}{100}W_{i,j}\right)\frac{2}{5}\right]^2.$$

By equations (3.15) and (3.16) the solution values at the points in example are as shown in table (3.1) and the residue error in table (3.2).

Table (3.1)

Numerical results of example (3.2).

	j =	***0***	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***	***10***
i =	***x*** = \ ***y*** =	***0***	***0.2***	***0.4***	***0.6***	***0.8***	***1***	***1.2***	***1.4***	***1.6***	***1.8***	***2***
0	0	0	0.04	0.16	0.36	0.64	1	1.44	1.96	2.56	3.24	4
1	0.1	0.01	0.017	0.066	0.151	0.271	0.427	0.619	0.846	1.109	1.768	3.61
2	0.2	0.04	0.011	0.027	0.063	0.115	0.182	0.266	0.365	0.48	1.079	3.24
3	0.3	0.09	0.013	0.012	0.026	0.049	0.078	0.114	0.157	0.208	0.741	2.89
4	0.4	0.16	0.021	0.007	0.011	0.021	0.033	0.049	0.068	0.09	0.561	2.56
5	0.5	0.25	0.034	0.006	0.005	0.009	0.014	0.021	0.029	0.039	0.453	2.25
6	0.6	0.36	0.049	0.007	0.003	0.004	0.006	0.009	0.012	0.017	0.379	1.96
7	0.7	0.49	0.069	0.01	0.002	0.002	0.002	0.004	0.005	0.007	0.321	1.69
8	0.8	0.64	0.092	0.013	0.002	0.001	0.001	0.002	0.002	0.003	0.273	1.44
9	0.9	0.81	0.374	0.187	0.083	0.025	0.003	0.017	0.067	0.152	0.501	1.21
10	1	1	0.64	0.36	0.16	0.04	0	0.04	0.16	0.36	0.64	1

Table (3.2)

The residue error of example (3.2).

i* = *1 / ***j* = *1***	***0***	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***
0	0	0	0	0	0	0	0	0	0	0
1	0	0.0001	0.0007	0.0027	0.0078	0.0182	0.0364	0.066	0.1359	0.1863
2	0	0.0001	0.0001	0.0005	0.0014	0.0033	0.0067	0.0123	0.0365	0.0879
3	0	0.0001	0	0.0001	0.0003	0.0006	0.0012	0.0023	0.0121	0.0504
4	0	0.0002	0	0	0	0.0001	0.0002	0.0004	0.0051	0.0329
5	0	0.0004	0	0	0	0	0	0.0001	0.0027	0.023
6	0	0.0008	0	0	0	0	0	0	0.0017	0.0165
7	0	0.0014	0	0	0	0	0	0	0.0011	0.0119
8	0	0.0227	0.0056	0.0011	0.0001	0	0	0.0007	0.0078	0.0402
9	0	0.0003	0.0001	0	0	0	0	0.0002	0.0025	0

3.3 NUMERICAL SOLTUION OF PARABOLIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS, [6]

The parabolic stochastic partial differential equation that we will study is the stochastic heat equation given by:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + \dot{W}(x,t)\ ,\ 0 < x < b, t > 0 \quad \text{....................(3.17)}$$

subject to the initial and boundary conditions:

$u(0, t) = u(b, t) = g(t),\ t > 0$

$u(x, 0) = f(x),\ 0 < x < b$

where $\dot{W}$ is two-dimensional white noise. By eq. (3.10) and (3.8) the finite-difference approximation to eq. (3.17) is:

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + W_{i,j} \quad \text{................................(3.18)}$$

where

$x_i = ih,\ i = 0, 1, \ldots, n$

$t_j = jk,\ j = 0, 1, \ldots, m$

equation (3.18) may be written as:

$$u_{i,j+1} = u_{i,j} + r(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + kW_{i,j} \quad \text{............................(3.19)}$$

$i = 1, 2, \ldots, n - 1$ and $j = 1, 2, \ldots, m - 1$, where $r = k / h^2$ and with error of order h^{-2} and eq. (3.19) is shown to be stable if $r \le \frac{1}{2}$. Since the initial condition $u(x, 0) = f(x)$, for each $i = 0, 1, \ldots, n$, these value can be used in eq. (3.19) to find the value of $u_{i,1}$, for each i = 1, 2, …, n − 1. The boundary conditions $u(0, t_j) = u(b, t_j) = g(t_j)$, for each j = 0, 1, …, m imply that $u_{0,1} = u_{n,1} = g(t_1)$; so all the entries of the form $u_{i,j}$ can be determined, for each i = 1, 2, …, n – 1 and j = 1, 2, …, m – 1.

Algorithm (3.3):

1. Input b, n, m, k.
2. $h = b / n$, $r = k / h^2$.
3. For i = 0 to n; do steps 4 to 7.
4. For j = 0 to m; do steps 5 to 7.
5. $x_i = a + ih$, $t_j = jk$.
6. Generate random vector p(i), which is normally distributed with mean 0 and variance hk.
7. $w^{\langle i \rangle} = p(i)$.
8. For i = 1 to n – 1, for j = 1 to m – 1

   $u_{0,j} = u_{n,j} = g(t_j)$.

   $u_{i,j+1} = u_{i,j} + r(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + kw_{i,j}$

   $RE_{i,j} = \{u_{i,j+1} - [u_{i,j} + r(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + kw_{i,j}]\}^2$.
9. Stop.

Example (3.4):

Consider the stochastic heat equation

$$\dot{u} = \frac{\partial^2 u}{\partial x^2} + \dot{W}, \; 0 < x < 1 \;\text{ and }\; t > 0 \quad \text{.......................................(3.20)}$$

with initial condition u(x, 0) = 1, 0 < x < 1 and u(x, t) = 0 at x = 0 and 1, t ≥ 0., h = 0.1 and k = 0.001, so $r = \frac{1}{10}$

From eq. (3.17) we can write eq. (3.20) as:

$$u_{i,j+1} = u_{i,j} + \frac{1}{10}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] + \frac{1}{1000}W_{i,j}$$

$$u_{i,j+1} = \frac{1}{10}\left[u_{i+1,j} + 8u_{i,j} + u_{i-1,j}\right] + \frac{1}{1000}W_{i,j} \quad \text{.........................(3.21)}$$

with n = m = 10, i = 1, …, 9; j = 0, …, 9; and

$$RE_{i,j} = \left[u_{i,j+1} - \frac{1}{10}\left[u_{i+1,j} + 8u_{i,j} + u_{i-1,j}\right] - \frac{1}{1000}W_{i,j}\right]^2 \quad \text{.........(3.22)}$$

The solution of eqs. (3.21) and (3.22) are shown in Tables (3.3) and (3.4) respectively.

Table (3.3)

Numerical results of example (3.4).

	j =	***0***	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***	***10***
i =	***t*** = / ***r*** =	***0.000***	***0.001***	***0.002***	***0.003***	***0.004***	***0.005***	***0.006***	***0.007***	***0.008***	***0.009***	***1***
0	0.0	0	0	0	0	0	0	0	0	0	0	0
1	0.1	1	0.9	0.72	0.576	0.461	0.369	0.295	0.236	0.189	0.151	0.121
2	0.2	1	1	0.89	0.784	0.685	0.594	0.512	0.439	0.375	0.319	0.27
3	0.3	1	1	0.9	0.809	0.729	0.649	0.579	0.514	0.955	0.402	0.353
4	0.4	1	1	0.9	0.81	0.729	0.656	0.589	0.529	0.475	0.425	0.381
5	0.5	1	1	0.9	0.81	0.729	0.656	0.59	0.531	0.478	0.43	0.386
6	0.6	1	1	0.9	0.81	0.729	0.656	0.59	0.531	0.478	0.43	0.387
7	0.7	1	1	0.9	0.81	0.729	0.656	0.59	0.531	0.478	0.43	0.387
8	0.8	1	1	0.9	0.81	0.729	0.656	0.59	0.531	0.478	0.43	0.387
9	0.9	1	0.9	0.82	0.746	0.678	0.615	0.558	0.505	0.457	0.414	0.374
10	1	0	0	0	0	0	0	0	0	0	0	0

Table (3.4)

The residue error of example (3.4).

i = 1 \ j = 1	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	0.01	0.008	0.006	0.005	0.004	0.003	0.002	0.001	0.001
2	0	0.01	0.008	0.007	0.005	0.004	0.003	0.003	0.002	0.002
3	0	0.01	0.008	0.007	0.005	0.004	0.003	0.003	0.002	0.002
4	0	0.01	0.008	0.007	0.005	0.004	0.003	0.003	0.002	0.002
5	0	0.01	0.008	0.007	0.005	0.004	0.003	0.003	0.002	0.002
6	0	0.01	0.008	0.007	0.005	0.004	0.003	0.003	0.002	0.002
7	0	0.01	0.008	0.007	0.005	0.004	0.003	0.003	0.002	0.002
8	0	0.008	0.007	0.006	0.005	0.004	0.003	0.003	0.002	0.002
9	0	0	0	0	0	0	0	0	0	0

3.4 NUMERICAL SOLUTION OF HYPERBOLIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

In this section, we consider the numerical solution of the stochastic wave equation, which is given by:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \dot{W},\ 0 < x < b,\ t > 0 \quad \text{...(3.23)}$$

subject to the conditions:

$$u(0, t) = u(b, t) = \mathbf{l}(t),\ t > 0$$

$$u(x, 0) = f(x),\ 0 < x < b$$

$$\frac{\partial u}{\partial t}(x,0) = g(x),\ 0 < x < b$$

where $\dot{W}$ is two-dimensional white noise. To set up the finite-difference method, select an integer $n > 0$ and step size $k > 0$ with $h = b/n$, the mesh points (x_i, t_j) are defined by $x_i = ih$, for each $i = 0, 1, \ldots, n$; and $t_j = jk$, for each $j = 0, 1, \ldots$.

The difference method is obtained by using eq. (3.9) and (3.8) we get

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + W_{i,j} \quad \text{...................(3.24)}$$

If r is used to denote k/h we can write eq. (3.24) as:

$$u_{i,\,j+1} - 2u_{i,\,j} + u_{i,\,j-1} = r^2(u_{i+1,\,j} - 2u_{i,\,j} + u_{i-1,\,j}) + k^2 W_{i,\,j}$$

this implies:

$$u_{i,j+1} = 2(1 - r^2)u_{i,j} + r^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1} + k^2W_{i,j} \quad \text{............(3.25)}$$

i = 1, 2, …, n – 1; j = 1, 2, …, n – 1.

The boundary conditions give $u_{0,j}=u_{n,j}=\mathbf{l}(t_j)$ for each j = 1, 2, …; and the initial condition implies that $u_{i,0}=f(x_i)$, for each $i=1,2,\dots,n-1$, a central difference to the initial derivative condition gives that:

$$\frac{1}{2k}(u_{i,1} - u_{i,-1}) = g_{i,0},\ i = 1, 2, \dots, n-1 \quad \text{...............................(3.26)}$$

Putting j = 0 in eq. (3.25) yields

$$u_{i,1} = 2(1-r^2)u_{i,0} + r^2(u_{i+1,0} + u_{i-1,0}) - u_{i,-1} + k^2W_{i,j},\ i = 1, 2,\dots, n-1$$

$$\text{.............................(3.27)}$$

Eliminating $u_{i,-1}$ between these two equations shows that the mesh values t = k can be calculated from the equation

$$u_{i,1} = \frac{1}{2}\left[2(1-r^2)u_{i,0} + r^2(u_{i+1,0} + u_{i-1,0}) + 2k\,g_{i,0} + k^2 w_{i,0}\right] \text{..(3.28)}$$

where i = 1, 2, …, n – 1.

<u>Algorithm (3.5):</u>

1. Input b, n, m, k.
2. h = b / n, r = k / h.
3. For i = 0 to n; do steps 4 to 7.
4. For j = 0 to m; do steps 5 to 7.
5. $x_i = a + ih, y_j = c + jk$.

6. Generate random vector p(i), which is normally distributed with mean 0 and variance hk.

7. $w^{\langle i \rangle} = p(i)$.

8. For i = 1 to n – 1, for j = 1 to m – 1

$$u_{i,0} = f(x_i);\ u_{0,j} = u_{n,j} = \mathbf{l}(t_j)$$

$$u_{i,1} = \frac{1}{2}\left[2(1-r^2)u_{i,0} + r^2(u_{i+1,0} + u_{i-1,j}) + 2kg_{i,0} + k^2 w_{i,0}\right]$$

$$u_{i,j+1} = 2(1-r^2)u_{i,j} + r^2(u_{i+1,j} + u_{i-1,j}) + k^2 w_{i,j}$$

$$RE_{i,j} = \left\{u_{i,j+1} - \left[2(1-r^2)u_{i,j} + r^2(u_{i+1,j} + u_{i-1,j}) + k^2 w_{i,j}\right]\right\}^2$$

9. Stop.

<u>Example (3.6):</u>

Consider the stochastic wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \dot{W},\ 0 < x < 1 \ \text{ and } \ t > 0 \quad \text{.................................(3.29)}$$

the boundary conditions u(0, t) = u(1, t) = 0, t ≥ 0; and the initial conditions u(x, 0) = $\frac{1}{8}\sin(\pi x)$, $\frac{\partial u}{\partial t}(x,0) = 0$, 0 ≤ x ≤ 1; h = k = 0.1, n = m = 10. So r = 1. From eq. (3.25) we can write eq. (3.29) as:

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} + \frac{1}{100}W_{i,j},\ i = 1, 2,\ldots, 9,\ j = 1, 2,\ldots, 9$$

..............................(3.30)

$$\left.\begin{array}{l} \text{with} \quad u_{0,j} = u_{10,j} = 0, \quad j = 0,...,10, \quad \text{and} \\ u_{i,0} = \frac{1}{8}\sin(\pi x_i), \qquad \text{for each } i = 1,2,...,9 \end{array}\right\} \text{......................(3.31)}$$

$$u_{i,1} = \frac{1}{2}\left[u_{i+1,0} + u_{i-1,0} + 2k\,g_{i,0} + k^2 W_{i,0}\right]$$

$$u_{i,1} = \frac{1}{2}\left[u_{i+1,0} + u_{i-1,0} + k^2 W_{i,0}\right],\ i = 1,\ 2,\ \ldots,\ 9 \text{........................(3.32)}$$

$$RE_{i,j} = \left[u_{i,j+1} - \left(u_{i+1,j} + u_{i-1,j} - u_{i,j-1} + \frac{1}{100} W_{i,j}\right)\right]^2 \text{..............(3.33)}$$

The solution is given in Table (3.5) and the residue error in a Table (3.6).

Table (3.5)

Numerical results of example (3.6).

	j =	0	1	2	3	4	5	6	7	8	9	10
i =	t = / x =	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.0	0	0	0	0	0	0	0	0	0	0	0
1	0.1	0.0386	0.0367	0.0313	–0.0368	–0.0313	0.0367	0.0312	–0.0365	–0.0315	0.0366	0.0315
2	0.2	0.0735	0.0699	0.0595	–0.0386	–0.0962	0.0073	0.1328	0.0239	–0.1694	–0.0553	0.2058
3	0.3	0.1011	0.0961	0.0818	–0.0366	–0.1204	–0.0596	0.1281	0.1923	–0.1042	–0.3616	0.0488
4	0.4	0.1189	0.113	0.096	–0.0312	–0.1327	–0.0892	0.0731	0.2172	0.1191	–0.3214	–0.4807
5	0.5	0.125	0.1189	0.101	–0.0228	–0.1323	–0.1098	0.043	0.1827	0.1742	–0.0637	–0.4956
6	0.6	0.1189	0.1131	0.0961	–0.0119	–0.119	–0.1205	0.0093	0.1634	0.1735	0.0107	–0.2373
7	0.7	0.1011	0.0962	0.082	–0.0001	–0.0941	–0.1189	–0.0265	0.1282	0.1899	0.0455	–0.1791
8	0.8	0.0735	0.07	0.0595	0.012	–0.0599	–0.106	–0.0592	0.0795	0.1874	0.1105	–0.142
9	0.9	0.0386	0.0368	0.0313	0.0227	–0.0191	–0.0825	–0.0869	0.0233	0.1663	0.164	–0.0558
10	1	0	0	0	0	0	0	0	0	0	0	0

Table (3.6)

The residue error of example (3.6).

i = \ j =	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0.004	0.001	0.009	0	0.018	0.001	0.029	0.003
2	0	0	0.007	0.001	0.015	0.004	0.016	0.037	0.011	0.131
3	0	0	0.009	0.001	0.018	0.008	0.005	0.047	0.014	0.103
4	0	0	0.01	0.001	0.018	0.012	0.002	0.033	0.03	0.004
5	0	0	0.009	0	0.014	0.015	0	0.027	0.03	0
6	0	0	0.007	0	0.009	0.014	0.001	0.016	0.036	0.002
7	0	0	0.004	0	0.004	0.011	0.004	0.006	0.035	0.012
8	0	0	0.001	0.001	0	0.007	0.008	0.001	0.028	0.027
9	0	0	0	0	0	0	0	0	0	0

CONCLUSIONS AND RECOMMENDATIONS

From the present study, the following conclusions may be drawn:

1. There is no explicitly known analytical method for solving SPDE’s and therefore numerical methods may be considered as the solution for such difficulties.
2. From tables (3.2), (3.4) and (3.6), one can see the accuracy of the obtained results in solving SPDE's.

Also, the following may be considered as recommendations for future work:

1. Use other numerical methods for solving SPDE's, such as the collocation method, the least square method, Adomian decomposition method, differential transform method, etc.
2. Use the implicit finite difference methods for solving SPDE's.
3. Considering the stochastic Taylor series expansion in deriving other models for solving SPDE's.

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المستخلص

لهذه الرسالة ثلاث أهداف رئيسية:

الهدف الأول هو اعطاء دراسة لموضوع التفاضل والتكامل التصادفي، حيث تتضمن الدراسة المفاهيم الأساسية للمعادلات التفاضلية التصادفية.

والهدف الثاني هو لدراسة المعادلات التفاضلية الجزئية التصادفية، بالاضافة الى، دراسة بعض النتائج النظرية للانواع الثلاثة من المعادلات التفاضلية الجزئية التصادفية واعطاء الشكل التكاملي لكل نوع منها.

الهدف الثالث هو دراسة الحلول العددية للانواع الثلاثة للمعادلات التفاضلية الجزئية التصادفية (معادلات باسون التصادفية، معادلات الحرارة التصادفية، ومعادلات الموجة التصادفية) وذلك باستخدام طريقة الفروقات المنتهية.

جمهـورية العـراق
وزارة التعليم العالي والبحث العلمي
جـــامعة النهــــريـن
كليــة العلوم
قسم الريـاضيات وتطبيقات الحاسوب

دراسة بعض المعادلات التفاضلية الجزئية التصادفية

رسالة
مقدمة إلى كلية العلوم - جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل
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