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Some Approximate Solutions of Fractional Integro-Differential Equations

A Thesis

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Mathematics

By

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بسم الله الرحمن الرحيم

(نرْفَع دَرَجَاتٍ مَن نشاءُ وَفَوْقَ كُلِ ذِي عِلْمٍ عَلِيم)

صدق الله العظيم

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الاهداء

إلى الحبيب الذي أضاء قلوب المؤمنين ببركات أنواره ... رسول الله (ص).

إلى توأم روحي ورفيق دربي ... زوجي.

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Samah Mohammed Ali ✍

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Supervisors Certification

We certify that this thesis was prepared under our supervision at the department of Mathematics, College of Science, Al-Nahrain University, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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We certify that we have read this thesis entitled ***"Some Approximate Solutions of Fractional Integro-Differential Equations"*** and as an examining committee examined the student (***Samah Mohammed Ali***) in its contents and in what it connected with, and that is in our opinion it meets the standards of a thesis for the degree of Master of Science in Mathematics.

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A bstract

The main objective of this thesis is to introduce fractional integro-differential equations using a modified type of operators, which consists of the same order fractional differentiation and fractional integration. Also, the objective of this work is to study and prove the existence theorems of a unique solution of the fractional integro-differential equations, then studying the approximate solutions of such type of the equations using the collocation method, the least square method, the Adomian decomposition method and the modification of Adomian's polynomials method which are presented with the illustrative examples and comparised between the results.

Contents

1

I ntroduction

Fractional calculus is that subject of mathematics which grows out of the traditional definitions of the calculus integral and integral operators in which the same by fractional exponents in an outgrowth of exponents with integral value. Consider the physical meaning of the exponent, according to our primary school teacher, exponents provide a short notation for what is essentially a repeated multiplication of numerical value. This concept in itself is easy to grasp and straight forward. However, the physical definitions can clearly become confused when considering exponents of noninteger value, [Loverro, 2004].

Oldham and Spanier [Oldham, 1974], who wrote in this field or subject had begun their work in 1968 with the realization that the use of half-order derivatives and integrals had to a formulation of certain electro chemical problems which is more economical and useful than the classical approaches. This discovering stimulated our interest, not only in the application of notions of the derivative and integral to an ordinary order, but also in the basic mathematical properties of these fascinating operators, [Ali, 2008].

Fractional calculus are old as classical calculus (refer to [Miller and Ross, 1993] or [Oldham and Spanier, 1974] for historical survey). The use of fractional order differential and integral operators in mathematical models had become increasingly wide spread in recent years (see [Diethelm and Ford, 2002] and

[Mainardi, 1997]). Several forms of fractional integro-differential equations had been proposed in

standard models, and there had been significant interest in developing numerical schemes for their solution (see [Edwards, 2002] and [Podlubny, 1999]). The interesting of fractional calculus have been stimulated by using the subject in many applications like the subject of finding the numerical solution of differential equations and in sciences, such as physics and engineering, etc., [Al-Hussieny, 2006].

However, much of the work published to date has been concerned with linear single term equations and, of these, equations of order less than unity have been most often investigated, [El-Sayed, 2004].

In [Momani, 2000], the Shauder's fixed point theorem had been used to obtain local existence of the solution and Tychonov's fixed point theorem to obtain global existence of the solution. In [Momani, 2001], the successive approximations and Arzela-Ascoli lemma were used to obtain existence and uniqueness of solution of fractional integro-differential equations. In [Momani and Hadid, 2003] some important results were proved concerning the corresponding inequalities of fractional integro-differential equations.

In recent years, there have been interests in the study of fractional integro-diffeorential equation of the type:

$$D_*^q u(t) = f(t, u(t)) + \int_0^t k(t, s, u(s))ds, \quad 0 < q \leq 1$$

With the initial condition:

$u(0) = u_0$,

Where f is a continuous function on (t, u) for $u \in \mathbb{R}$, $a > 0$ and $0 < t < a$, k is a continuous function on (t, s, u) for $u \in \mathbb{R}$ and $0 < t, s < a$, u_0 is a real positive constant and D_*^q denotes the Caputo fractional derivative operator, [Momani , 2007].

This thesis consists of three chapters. In chapter one, we give the definitions related to fractional calculus including historical background and the fundamental concepts. Also, in this chapter, we study some basic properties of fractional integro-differential equations, like linearity, scale change, Leibniz's rule and chain rule.

In chapter two, we give the existence and uniqueness theorem of the solutions for the fractional integro-differential equations using Schauder's fixed point theorem.

In chapter three we present some approximate methods for solving fractional integro-differential equations, since approximate methods may be sometimes considered as the most reliable and applicable method for solving fractional integro-differential equations. This chapter include three methods, namely, the collocation method, the least square method and Adomian decomposition method for solving linear fractional integro-differential equations, in which the collocation method give more accurate results in linear case and the Adomian decomposition method in nonlinear case. Also, for illustration purposes, some examples are given.

It is important to notice that, the computer programs are coded in MATHCAD 14 software and the results are presented in a tabulated form.

Chapter One

Fundamental Concepts of Fractional Calculus

Fractional calculus is an important branch of applied mathematics, which seems first to have many vague notions and poor defined concepts to the readers who are interested in this branch of mathematics. This type of differentiation and integration may be considered as a generalization to the useful definition of differentiation and integration, [Oldham and Spanier, 1974]. This chapter presents some basic concepts and notations, which are necessary for defining and illustrating fractional calculus. Therefore, the chapter consists of three sections. In section one, the historical background of fractional calculus is given. In section two, the fundamental concepts of fractional calculus such as the gamma and beta functions, differentiation and integration of fractional order are given. In section three, some properties of fractional integro-differential equations, like linearity, scale change, etc. are presented and discussed.

1.1 Historical Background

Recently fractional derivatives have been used to model physical processes leading to the formulation of fractional differential equations. The fractional calculus may be considered as an old and yet a new topic. It is an old topic since it is starting in 1695. L'Hopital was the first researcher whose ask in a letter to Leibniz on the possibility to

Performing calculations by means of fractional derivatives of order r = 1/2. Leibniz answered this question looked as a paradox to him (see [Madueno, 2002]). In (1697), Leibniz referring to the infinite product of Walls for $\pi/2$ used the notation $d^{1/2}y$ and summarized that the fractional calculus could be used to get the same results.

The earliest more or less systematic studies seem to had been made in the beginning and middle of the 19^{th} century by Liouville (1832), Riemann (1953), and Holmgren (1864), although Euler (1730), Lagrange (1772), and others made contributions even earlier. It was Liouville (1832) who expanded functions in series of exponentials and defined the q^{th} derivative of such a series by operating term-by-term as though q, where a positive integer. Riemann in (1953), proposed a different definition that involved a definite integral and was applicable to power series with no integer exponents. Also, Grunwald in (1867), disturbed by the restriction of Liouville's approach. Then these theoretical beginnings were a development for the applications of the fractional calculus to various problems. The first of these was discovered by Able in (1823), that the solution of the integral equation for the tautochrone may be accomplished via an integral transform. A powerful stimulus to the use of fractional calculus to solve real life problems was provided by the development of Boole in (1844), of symbolic methods for solving linear differential equations with constant coefficients.

In the twentieth century, some notable contributions had been made to both the theory and application of fractional calculus, Weyl (1917), Hardy (1917), Hardy and Littewood (1932), Kober (1940), and Kuttner (1953), examined some rather special, but natural, properties of integro-differential of functions belonging to Lebesgue and Lipschitz classes, Erdely (1954), and Oster (1970), have given definitions of integro-differential with respect to arbitrary functions, and Post (1930) used difference quotient to define generalized integro-differential for fractional operators, Riesz (1949), has developed a theory of fractional integration for functions of more than one variable, Erdely (1965), has applied the fractional calculus to integral equations and Higgins (1967), has used fractional integral operators to solve differential equations.

However, fractional calculus may be considered as an important topic, as well as, since only from a little more than to the later fifty years, it has been an object of specialized conferences and treatises. For the first monograph the merit is ascribed to Oldham and Spanier (1974), who after a joint collaboration started in 1968, published a book devoted to fractional calculus in 1974. The first texts and proceedings devoted solely or partly to fractional calculus and its applications are, [Poldlubny, 1999].

[Mainardi, 1997], has fractional calculus of some basic problems in continuum and statistical mechanics. Samko, Kilbas and Marichev (1993), has used fractional integrals and derivative of theory and applications, Diethelm and Ford (2002), used analysis of fractional differential equations. In recent years, there has been an interest in the study of fractional integro-differential equations.In (2003) Momani and Hadid proved some important results concerning with the corresponding inequalities of fractional integro-differential equations. Rawashdeh (2005) used the collocation method to approximate the solution of fractional integro-differential equations. [Mittal R. C., 2008], used the

Adomian decomposition method to solve fractional integro-differential equations.

1.2 Fundamental Concepts in Fractional Calculus

It is important to recall that fractional calculus is complicated subject to understand and because of that, we shall present in this section some of the most important notions and definitions that are necessary for understanding this subject.

1.2.1 The Gamma and Beta Functions, [Oldham, 1974]:

The complete gamma function Γ(t) plays an important role in the theory of integro-differential. A comprehensive definition of Γ(t) is that provided by the Euler limit:

$$\Gamma(t) = \lim_{N\to\infty}\left(\frac{N!N^t}{t(t+1)(t+2)\ldots(t+N)}\right),\ t>0 \quad\ldots\ldots\ldots\ldots\ldots (1.1)$$

but the integral transform definition is given by:

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx,\ t>0 \quad\ldots\ldots\ldots\ldots\ldots (1.2)$$

is often more useful, although it is restricted to positive x values. An integration by parts applied to the eq. (1.2) leads to the recurrence relationship:

$$\Gamma(t+1) = t\,\Gamma(t) \quad\ldots\ldots\ldots\ldots\ldots (1.3)$$

which may be rewritten as:

$$\Gamma(t-1) = \frac{\Gamma(t)}{t-1}, \; t \neq 1$$

This is the most important property of the gamma function. The same result is a simple consequence of eq. (1.1), since Γ(1) = 1, this recurrence shows that for a positive integer n:

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$= n! \qquad \text{.........(1.4)}$$

The following are the most important properties of the gamma function:

1. $\Gamma\left(\frac{1}{2}-n\right) = \frac{(-4)^n n!\sqrt{\pi}}{(2n)!}, \; n \in \mathbb{N}.$

2. $\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!\sqrt{\pi}}{4^n n!}, \; n \in \mathbb{N}.$

3. $\Gamma(-t) = \frac{-\pi\csc(\pi t)}{\Gamma(t+1)}, \; n \in \mathbb{N}, t > 0.$

4. $\Gamma(nt) = \sqrt{\frac{2\pi}{n}}\left(\frac{n^t}{\sqrt{2\pi}}\right)^n \prod_{k=0}^{n-1}\Gamma\left(t+\frac{k}{n}\right), \; n \in \mathbb{N}, t > 0.$

The following are some frequently encountered examples of gamma functions for different values of t:

$$\Gamma\left(\frac{-3}{2}\right) = \frac{4}{3}\sqrt{\pi}, \; \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}, \; \Gamma\left(\frac{-1}{2}\right) = -2\sqrt{\pi}, \; \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Which enable us to calculate for any positive real t, the gamma function in terms of the fractional part of t. The gamma function expression:

$$\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} \quad \text{.........(1.5)}$$

where j is a nonnegative integer and q may take any value. The procedure may be generalized to give:

$$\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} = \frac{(j-q-1)(j-q-2)...(-q+1)(-q)}{j!}$$

$$= (-1)^j \frac{q(q-1)(q-2)...(q-j+2)(q-j+1)}{j!}$$

$$= (-1)^j \binom{q}{j} \quad \text{.........(1.6)}$$

where $\binom{q}{j} = \frac{q!}{j!(q-j)!}$.

A function that is closely related to the gamma function is the complete beta function B(p,q).For positive values of the two parameters, p and q; the function is defined by the beta integral:

$$B(p,q) = \int_0^1 y^{p-1}(1-y)^{q-1}dy\,,\ p,\ q > 0 \quad \text{.........(1.7)}$$

which is also known as the Euler's integral of the second kind. If either p or q is non-positive, the integral diverges otherwise $B(p,q)$ is defined by the relationship:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad \text{.........(1.8)}$$

where p and q > 0.

Both the beta and gamma functions have "incomplete" analogs. The incomplete beta function of argument t is defined by the integral:

$$B_t(p,q) = \int_0^t y^{p-1}(1-y)^{q-1}dy \quad \text{...(1.9)}$$

and the incomplete gamma function of argument t is defined by:

$$\gamma^*(c, t) = \frac{c^{-t}}{\Gamma(t)}\int_0^c y^{t-1}e^{-y}dy$$

$$= e^{-t}\sum_{j=0}^{\infty}\frac{t^j}{\Gamma(j+c+1)} \quad \text{...(1.10)}$$

$\gamma^*(c, t)$ is a finite single-valued analytic function of t and c.

1.2.2 The Fractional Derivative, [Oldham, 1974]:

The usual formulation of the fractional derivative, given in standard references such as [Oldham, 1974], [Samko, 1993] is the Riemann-Liouville definition of fractional derivatives, which is:

$$D^q u(t) = \frac{1}{\Gamma(m-q)}\frac{d^m}{dx^m}\int_0^t (t-s)^{m-q-1}u(s)ds \quad \text{.............................(1.11)}$$

where m is the integer defined by $m - 1 < q \le m$.

The Grunwald definition of fractional derivatives is given by:

$$\frac{d^q u(t)}{dt^q} = \lim_{N\to\infty}\left\{\frac{\left(\frac{t}{N}\right)^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}u\left(t - j\left(\frac{t}{N}\right)\right)\right\} \quad \text{..................(1.12)}$$

Where $q < 0$ indicates fractional integration and $q > 0$ indicates fractional differentiation.

The Caputo definition of fractional derivative is given by:

$$D_*^q u(t) = \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} u^{(m)}(s) ds \quad \text{................................(1.13)}$$

For $m - 1 < q \le m$, $m \in \mathbb{N}$, $t > 0$.

1.2.3 The Fractional Integral, [Oldham, 1974]:

The common formulation for the fractional integral can be derived directly from a traditional expression of the repeated integration of a function. Several definitions of fractional integration may be given, such as:

The Riemann-Liouville definition of any $q > 0$ for a function $u(t)$ with $t \in \mathbb{R}$ is:

$$J^q u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s) ds, \; q > 0 \quad \text{....................................(1.14)}$$

and $J^0 u(t) = I$ is the Identity operator.

The properties of the operator J^q can be found in [Rawashdeh, 2005] for $q \ge 0$, $\alpha > 0$, we have:

1. $J^q J^\alpha u(t) = J^{q+\alpha} u(t)$.
2. $J^q J^\alpha u(t) = J^\alpha J^q u(t)$.

Some properties of the operators J^q and D^q may be found in [Podulbny, 1999], and we mention the following:

$$J^q t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+q)} t^{\gamma+q}$$

$$, \text{ for } t > 0,\ q \geq 0,\ \gamma > -1 \text{.............................(1.15)}$$

$$D^q t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-q)} t^{\gamma-q}$$

One of the basic properties of the Caputo fractional derivatives, are:

$$J^q D_*^q u(t) = u(t) - \sum_{k=0}^{m-1} u^{(k)}(0^+) \frac{t^k}{k!},\ t > 0 \text{..(1.16)}$$

$$D_*^q J^q u(t) = u(t) \text{...(1.17)}$$

The Weyle definition, where u(t) is a periodic function and its mean value for one period is zero, is given by:

$$J^q u(t) = \frac{1}{\Gamma(q)} \int_{-\infty}^{\infty} (t-s)^{q-1} u(s) ds \text{...(1.18)}$$

But the formula (1.18) is used as the definition of the fractional integral without any conditions at the present time. Other types of fractional integrations are given in [Caputo, 1971].

1.3 Properties of Fractional Integro-Differential Equations, [Oldham, 1974]

Some properties of fractional integro-differential equations, which one might expect to generalize the classical formulas of derivatives and integrals, will be examined and studied next in this section. The properties that will be discussed will provide our primary means of understanding and utilizing the fractional differential equations.

1.3.1 Linearity:

The linearity of the integro-differential operators, means that:

$$\frac{D^q[c_1 f_1 + c_2 f_2]}{Dt^q} = c_1 \frac{D^q f_1}{Dt^q} + c_2 \frac{D^q f_1}{Dt^q} \quad \text{...} (1.19)$$

where f_1 and f_2 are linear continuous functions, c_1, $c_2 \in \mathbb{R}$ and $q \geq 0$ is a fractional number.

1.3.2 Scale Change:

By a scale change of the function f with respect to a lower limit, one mean its replacement with $f(\beta t)$, where β is a constant termed the scaling factor, and hence the fractional derivative of order q with $T = t$ and $Y = \beta y$, is given by:

$$\frac{D^q f(\beta T)}{Dt^q} = \frac{D^q(\beta t)}{Dt^q}$$

$$= \frac{1}{\Gamma(-q)} \int_0^t \frac{f(\beta y)}{(t-y)^{q+1}} dy$$

$$=\frac{1}{\Gamma(-q)}\int_0^{\beta T}\frac{f(Y)[dY/\beta]}{\{(\beta T-Y)/\beta\}^{q+1}}$$

$$=\frac{\beta^q}{\Gamma(-q)}\int_0^{\beta T}\frac{f(Y)}{(\beta T-Y)^{q+1}}dY$$

$$=\beta^q\frac{D^q(\beta T)}{[D(\beta T)]^q} \quad \text{...(1.20)}$$

1.3.3 Leibniz's Rule:

The rule for differentiation of a product of two functions is a familiar result in elementary calculus, which states that:

$$\frac{D^n(fg)}{Dt^n}=\sum_{j=0}^{n}\binom{n}{j}\frac{D^{n-j}f}{Dt^{n-j}}\frac{D^jg}{Dt^j} \quad \text{...(1.21)}$$

and is, of course, restricted to nonnegative integer n. The following product rule is for multiple integrals:

$$\frac{D^{-n}(fg)}{Dt^{-n}}=\sum_{j=0}^{\infty}\binom{-n}{j}\frac{D^{-n-j}f}{Dt^{-n-j}}\frac{D^jg}{Dt^j}$$

when we observe that the finite sum in (1.21) could be equally well extend to infinity, since $\binom{n}{j}=0$ for all j > n, we might expect the product rule to be generalized to an arbitrary order q as:

$$\frac{D^q(fg)}{Dt^q}=\sum_{j=0}^{\infty}\binom{q}{j}\frac{D^{q-j}f}{Dt^{q-j}}\frac{D^jg}{Dt^j} \quad \text{...(1.22)}$$

such that a generalization is indeed valid for all real order.

The argument begins with the consideration of the product tf(t) and q > 0. Making use of the Riemann-Liouville definition, yields:

$$\frac{D^q(f)}{Dt^q} = \frac{1}{\Gamma(-q)}\int_0^t \frac{yf(y)}{(t-y)^{q+1}}dy + \frac{t}{\Gamma(-q)}\int_0^t \frac{f(y)}{(t-y)^{q+1}}dy$$

$$-\frac{t}{\Gamma(-q)}\int_0^t \frac{f(y)}{(t-y)^{q+1}}dy \quad \text{...(1.23)}$$

$$= \frac{t}{\Gamma(-q)}\int_0^t \frac{f(y)}{(t-y)^{q+1}}dy - \frac{1}{\Gamma(-q)}\int_0^t \frac{f(y)}{(t-y)^{q+1}}dy$$

$$= t\frac{D^q f}{Dt^q} + q\frac{D^{q-1}f}{Dt^{q-1}}$$

Extension of this result for q ≥ 0 is now quite easy, if n–1 ≤ q < n, n = 1, 2, …; then:

$$\frac{D^q(tf)}{Dt^q} = \frac{D^n}{Dt^n}\left\{\frac{D^{q-n}(tf)}{Dt^{q-n}}\right\}$$

$$= \frac{D^n}{Dt^n}\left\{t\frac{D^{q-n}f}{Dt^{q-n}} + (q-n)\frac{D^{q-n-1}f}{Dt^{q-n-1}}\right\}$$

$$= t\frac{D^q f}{Dt^q} + n\frac{D^{q-1}f}{Dt^{q-1}} + (q-n)\frac{D^{q-n}f}{Dt^{q-n}}$$

$$= t\frac{D^q f}{Dt^q} + q\frac{D^{q-1}f}{Dt^{q-1}}$$

Leibniz's rule has been thoroughly studied by Osler in (1970), (1971) and (1972). He was led to wonder whether eq. (1.21) is a special case of a still more general result in which the interchange ability of f and g is more apparent. The more general result proved by Osler is:

$$\frac{D^q(fg)}{Dt^q} = \sum_{j=-\infty}^{\infty} \frac{\Gamma(q+1)}{\Gamma(q-\gamma-j+1)\Gamma(\gamma+j+1)} \frac{D^{q-\gamma-j}f}{Dt^{q-\gamma-j}} \frac{D^{\gamma+j}g}{Dt^{\gamma+j}} \text{ ....(1.24)}$$

where γ is an arbitrary constant, which reduces to eq.(1.21), when $\gamma = 0$. Watanabe derived eq. (1.23) in 1931, but this method dose not yield to the precise region of convergence in the complex plan. A further generalization of Leibniz's rule due to Osler in (1972) is the integral form:

$$\frac{D^q(fg)}{Dt^q} = \int_{-\infty}^{\infty} \frac{\Gamma(q+1)}{\Gamma(q-\gamma-\lambda+1)\Gamma(\gamma+\lambda+1)} \frac{D^{q-\gamma-\lambda}f}{Dt^{q-\gamma-\lambda}} \frac{D^{\gamma+\lambda}g}{Dt^{\gamma+\lambda}} d\lambda$$

in which a discrete sum is replaced by an integral.

1.3.4 The Chain Rule:

The chain rule for the first order differentiation is given by:

$$\frac{d}{dt} g(f(t)) = \frac{d}{df(t)} g(f(t)) \frac{d}{dt} f(t)$$

lacks a simple counterpart in the integral calculus. Indeed if there were such a counterpart, the process of integration would pose no greater difficulty than does differentiation. Since any general formula for $\frac{D^q g(f(t))}{Dt^q}$ must encompass integration as a special case of little hope that can be held out for a useful chain rule for arbitrary q. Nevertheless, a formal chain rule in fractional order derivatives may be derived quit simply. Starting from the formula:

$$\frac{D^q\Phi}{Dt^q}=\sum_{j=0}^{\infty}\binom{q}{j}\frac{d^{q-j}(1)}{dt^{q-j}}\frac{d^j\Phi}{dt^j}$$

The following formula given by:

$$\frac{D^q(1)}{Dt^q}=\frac{t^{-q}}{\Gamma(1-q)}$$

permits the evaluation of the effect of the integro-differential operator upon unity, allowing us to write:

$$\frac{D^q\Phi}{Dt^q}=\frac{t^{-q}}{\Gamma(1-q)}\Phi+\sum_{j=1}^{\infty}\binom{q}{j}\frac{t^{j-q}}{\Gamma(j-q+1)}\frac{d^j\Phi}{dt^j}$$

Now, consider $\Phi = \Phi(f(t))$ and evaluate $\frac{D^j\Phi(f(t))}{Dt^j}$, in the second term of the last equation as follows:

$$\frac{D^j}{Dt^j}\Phi(f(t))=j!\sum_{m=1}^{j}\Phi^{(m)}\sum\prod_{k=1}^{j}\frac{1}{p_k!}\left(\frac{f^{(k)}}{k!}\right)^{p_k}$$

where $\sum$ is extended over all combinations of nonnegative integer values of $p_1, p_2, ..., p_j$, such that:

$$\sum_{k=1}^{j}kp_k=j \quad \text{and} \quad \sum_{k=1}^{j}p_k=m$$

Thus:

$$\frac{D^q}{Dt^q}\Phi(f(t))=\frac{t^{-q}}{\Gamma(1-q)}\Phi(f(t))+$$

$$\sum_{j=1}^{\infty}\binom{q}{j}\frac{t^{j-q}}{\Gamma(j-q+1)}j!\sum_{m=1}^{j}\Phi^{(m)}\sum\prod_{k=1}^{j}\frac{1}{p_k!}\left(\frac{f^{(k)}}{k!}\right)^{p_k}$$

The complexity of this result will inhibit its general utility. We see from inserting $q = -1$ that even for the case of a single integration:

$$\int_0^t \Phi(f(t))dy = t\Phi(f(t)) +$$

$$\sum_{j=1}^{\infty}(-1)^j \frac{t^{j+1}}{j+1}\sum_{m=1}^{j}\Phi^{(m)}\sum\prod_{k=1}^{j}\frac{1}{p_k!}\left(\frac{f^{(k)}}{k!}\right)^{p_k}$$

A case in which the generalized chain rule may be of limited utility is provided by $f(t) = e^t$. Then:

$$\frac{D^q}{Dt^q}\Phi(e^t) = \frac{t^{-q}}{\Gamma(1-q)}\Phi(e^t) +$$

$$\sum_{j=1}^{\infty}\binom{q}{j}\frac{t^{j-q}}{\Gamma(j-q+1)}e^{jt}\sum_{m=1}^{j}\delta_j^{(m)}\Phi^{(m)}$$

where $\delta_j^{(m)}$ is a stirling number of the second kind. The chain rule gives an infinite series that offers little hope of begin expressible in closed form, except for trivially simple instances of the functions f and Φ.

Chapter Two

The Existence and Uniqueness Theorem of the Solution of the Fractional Integro-Differential Equations

The existence and uniqueness theorems of solutions in ordinary differential equations with initial and boundary conditions plays an important role in the analytical and numerical solutions, since such type of equations may not be solved if it does not satisfy the conditions of this theorem in which the existence of a unique solution is necessary in this topic, [Ali, 2008].

Generally, this chapter presents some of the most basic concepts in fractional integro-differential equations, secondly the statement and the proof of the existence and uniqueness of theorem for the solution of the fractional integro-differential equations by the means of Schauder fixed point theorem is presented.

This chapter consists of three sections, in section one, which is termed as fractional integro-differential equations, we are introduced the fractional integro-differential equations with the illustration that D_*^q and J^q that appeared in the operator is the Caputo derivative operator of eq. (1.13) and Reimann-Liouville fractional integral operator of eq. (1.14), respectively. In section two, we give the statement and the proof of the existence theorem of the solution of fractional integro-differential equations.

In section three, we give the statement and the proof of the uniqueness theorem of the solution of fractional integro-differential equations.

2.1 Fractional Integro-Differential Equations

Consider the linear fractional integro-differential equation:

$$D_*^q u(t) = f(t) + J^q u(t),\ 0 < q < 1 \quad \text{.......(2.1)}$$

$$u(0) = u_0 \quad \text{.......(2.2)}$$

Where D_*^q refers to the Caputo derivative operator of order $0 < q < 1$, which is defined by eq.(1.13):

$$D_*^q u(t)= \frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1} u^{(m)}(s)ds$$

for $m - 1 < q \leq m$, $m \in \mathbb{N}$, $t \in [0, T]$; and J^q denotes the Riemann-Liouville fractional integral operator of order q, which is defined by eq.(1.14):

$$J^q u(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} u(s)ds$$

Then the operator of the fractional integro-differential equation (2.1), becomes:

$$A\, u(t) = f(t) \quad \text{.......(2.3)}$$

Where:

$$Au(t)= \frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1}u^{(m)}(s)ds-\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}u(s)ds$$

.... (2.4)

2.2 The Existence of the Solution of Fractional Integro-Differential Equations

Before we study the existence and uniqueness theorems of the solution of fractional integro-differential equations, we first introduce some basic concepts related to this study.

Theorem (2.1) (Schauder Fixed Point Theorem), [Zeidler, 1986]:

Let U be a nonempty, closed, bounded and convex subset of Banach space B and $T : U \longrightarrow U$ is a compact operator. Then T has at least one fixed point in U.

Now, we shall give the definition of an important concept which is called equicontinuous functions.

Definition (2.1), [Marsden, 1995]:

A subset S of C[0, T] is said to be equicontinuous, if for each $\varepsilon > 0$, there is a $\delta > 0$, such that:

$$| t - t_1 | < \delta \quad \text{and} \quad u \in M \quad \text{imply} \qquad \|u (t) - u (t_1) \|_{C[0, T]} < \varepsilon .$$

Theorem (2.2) (Arzela-Ascoli Theorem), [Dieudonne, 1960]:

Suppose F is a Banach space and E is a compact metric space. In order that a subset H of the Banach space $\mathcal{T}_F(E)$ be relatively compact, if and only if H be equicontinuous and that, for each $x \in E$, the set $H(x) = \{f(x): f \in H\}$ be relatively compact in F.

The next theorem will plays an important role in the proof of the uniqueness theorem, which is called Bihari's inequality.

Theorem (2.3) (Bihari's Inequality), [Momani, 2007]:

Let g be a monotone continuous function in an interval I, containing a point u_0, which vanishes nowhere in I. Let u and k be continuous functions in an interval J = [0, T] such that $u(J) \subset I$, and suppose that k is of fixed sign in J. Let $a \in I$, suppose that

$$u(t) \le a + \int_0^t k(s)g(u(s))ds, \qquad t \in J.$$

Then

$$u(t) \le G^{-1}\left[G(a) + \int_0^t k(s)ds\right], \qquad t \in J.$$

where G(a) is a primitive of $\frac{1}{g(x)}$, i.e. $G(u) = \int_{u_0}^{u} \frac{dx}{g(x)}$, $u \in I$.

Theorem (2.4) (The Existence Theorem):

Let u and $u^{(m)}$ be a real nonnegative function in C[0, T], and that $t \in [0, T]$, $0 < q < 1$. Then eqs. (2.1)-(2.2) has a solution u.

Proof:

In order to discuss the conditions for the existence for the solution

of eqs. (2.1)-(2.2), let us define B = C[0, T] to be the Banach space with the supremum norm, let us define the set:

$$U = \{u \in C[0, T] : \|u\| \le c_1, \|u^{(km)}\| \le c_2, c_1, c_2 > 0, k \in \mathbb{N}\}$$

Now, since our proof depends on the Schauder fixed point theorem, then it is sufficient to prove that U is a nonempty, closed, bounded and convex subset of the Banach space B and then the operator $A : U \longrightarrow U$ is compact operator.

It is easy to see that the set U is nonempty since from the properties of the norm we have $0 \in U$ and also bounded and closed (from the definition of U).

To prove U is convex subset of B. Let $u_1, u_2 \in U$, $\| u_1 \| \le c_1$, $\| u_1^{(km)} \| \le c_2$, $\| u_2 \| \le c_1$, $\| u_2^{(km)} \| \le c_2$, such that:

let $u(t) = \lambda u_1 + (1-\lambda)u_2(t)$, $\lambda \in [0,1]$

To prove $u \in U$, $\| u \| \le c_1$, $\| u^{(km)} \| \le c_2$,

$$\|u\| = \| \lambda u_1 + (1-\lambda)u_2 \|$$

$$\le |\lambda| \, \| u_1 \| + |(1-\lambda)| \, \| u_2 \|$$

$\leq \lambda c_1 + (1-\lambda)\, c_1$

$= c_1$

$\| u^{(km)} \| = \| [\lambda u_1 + (1-\lambda) u_2]^{(km)} \|$

$= \| \lambda u_1^{(km)} \| + \|(1-\lambda)\, u_2^{(km)} \|$

$\leq |\lambda|\, \| u_1^{(km)} \| + |(1-\lambda)|\, \| u_2^{(km)} \|$

$\leq \lambda c_2 + (1-\lambda)\, c_2$

$= c_2$

Hence, $u \in U$, U is convex set.

Now, in order to show that eqs. (2.1)-(2.2) has a solution, we have to show that the operator A in eq. (2.4) is completely continuous.

Let v(t) = Au(t), to prove that $v(t) \in U$

$$\|v\| = \| \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} u^{(m)}(s)ds - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s)ds \|$$

$$\leq \frac{\|u^{(m)}\|}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} ds + \frac{\|u\|}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds$$

$$\leq \frac{c_2 T^{m-q}}{\Gamma(m-q+1)} + \frac{c_1 T^q}{\Gamma(q+1)}$$

$$\leq c,$$

That is v(t) is bounded.

$$\|v^{(km)}\| = \| \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} u^{((k+1)m)}(s)ds$$

$$-\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}u^{(km)}(s)ds\,\|$$

$$\leq \frac{\left\|u^{((k+1)m)}\right\|T^{m-q}}{\Gamma(m-q+1)}+\frac{\left\|u^{(km)}\right\|T^{q}}{\Gamma(q+1)}$$

$$\leq \frac{c_2T^{m-q}}{\Gamma(m-q+1)}+\frac{c_2T^{q}}{\Gamma(q+1)}$$

$$\leq c^*.$$

that is $v^{(km)}(t)$ is bounded, $v(t) \in U$. Then the operator A maps U into itself.

Since for all $u \in U$ we have $A(u) \leq c$,then A(U) is bounded operator.

To prove that A is continuous operator. Let $u, v \in U$, then we have:

$$\|Au-Av\|=\|$$

$$\frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1}u^{(m)}(s)ds-\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}u(s)ds$$

$$-\left[\frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1}v^{(m)}(s)ds-\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v(s)ds\right]\|$$

$$=\|\frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1}(u^{(m)}(s)-v^{(m)}(s))ds-$$

$$\frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1}(u(s)-v(s))ds\,\|$$

$$\leq \frac{\left\|u^{(m)} - v^{(m)}\right\|}{\Gamma(m-q+1)} T^{m-q} + \frac{\|u-v\|}{\Gamma(q+1)} T^{q}$$

$$\leq \frac{\left\|(u-v)^{(m)}\right\|}{\Gamma(m-q+1)} T^{m-q} + \frac{\|u-v\|}{\Gamma(q+1)} T^{q}$$

Let $w = u - v$

$$\leq \frac{\left\|w^{(m)}\right\|}{\Gamma(m-q+1)} T^{m-q} + \frac{\|w\|}{\Gamma(q+1)} T^{q}$$

$$\leq c$$

That is Au is bounded operator, Au is continuous operator.

Now, we shall prove that A is equicontinuous operator. Let $u \in U$ and t_1, $t_2 \in [0, T]$, then:

$$\|Au(t_1) - Av(t_2)\| = \left\| \left(\frac{1}{\Gamma(m-q)} \int_0^{t_1} (t_1 - s)^{m-q-1} u^{(m)}(s) ds - \right.\right.$$

$$\left. \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} u(s) ds \right) - \left(\frac{1}{\Gamma(m-q)} \int_0^{t_2} (t_2 - s)^{m-q-1} u^{(m)}(s) ds - \right.$$

$$\left.\left. \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} u(s) ds \right) \right\|$$

$$\leq \frac{\left\|u^{(m)}\right\|}{\Gamma(m-q)} \left| \int_0^{t_1} (t_1 - s)^{m-q-1} ds - \int_0^{t_2} (t_2 - s)^{m-q-1} ds \right| +$$

$$\le \frac{\|u\|}{\Gamma(q)}\left|\int_0^{t_1}(t_1-s)^{q-1}ds - \int_0^{t_2}(t_2-s)^{q-1}ds\right|$$

$$\le \frac{c_2}{\Gamma(m-q+1)}\left|(t_1^{m-q}-t_2^{m-q})\right| + \frac{c_1}{\Gamma(q+1)}\left|(t_1^{q}-t_2^{q})\right|$$

$$\le \frac{2c_2}{\Gamma(m-q+1)}T^{m-q} + \frac{2c_1}{\Gamma(q+1)}T^{q}$$

$$\le c.$$

Au is equicontinuous operator. A is relatively compact. Now from Arzela-Ascoli theorem, A is completely continuous operator,then A is compact. Then Schauder fixed point theorem gives that the operator A has fixed point, which corresponds to the solution of eq. (2.3). ■

2.3 The Uniqueness of the Solution of the Fractional Integro-Differential Equations

Consider the initial value problem, which consists of the fractional integro-differential equation eqs. (2.1)-(2.2) of the type:

$$D_*^q u(t) = f(t) + J^q u(t)$$

with the initial condition:

$u(0)= u_0$

where f is a continuous function on t for $u \in \mathbb{R}$, $t \in [0, T]$, u_0 is a real positive constant and D_*^q denotes the Caputo fractional derivative operator. We shall use Bihari's inequality to obtain the uniqueness theorem to equations given by eqs.(2.1)-(2.2). Eq.(2.1) can be transformed in the next lemma.

Now, some additional properties are given for completeness purposes.

Lemma (2.1):

The solution of the initial value problem given by eqs. (2.1)- (2.2) has the form:

$$u(t) = u_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left\{\frac{1}{\Gamma(q)}\int_0^s (s-\sigma)^{q-1} u(\sigma)d\sigma\right\}ds$$

Proof:

From eqs. (2.1)-(2.2)

$$D_*^q u(t) = f(t) + J^q u(t)$$

With the initial condition:

$$u(0) = u_0$$

Applying the integral:

$$J^q D_*^q u(t) = J^q f(t) + J^q \left[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} u(s)\, ds\right]$$

$$u(t)-u_0=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)\,ds$$

$$+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left\{\frac{1}{\Gamma(q)}\int_0^s (s-\sigma)^{q-1} u(\sigma)d\sigma\right\}ds. \quad \blacksquare$$

Theorem (2.5) (The Uniqueness's Theorem):

The initial value problem given by eqs.(2.1)-(2.2) has a unique solution on the interval [0, T] if u is continuous function in the region:

$D = \{(t, u) \mid 0 < t < T, |u - u_0| \leq b\}$

and satisfy the condition:

$$\int_0^s \left|\frac{1}{\Gamma(q)}(s-\sigma)^{q-1}u(\sigma)-\frac{1}{\Gamma(q)}(s-\sigma)^{q-1}y(\sigma)\right| d\sigma \leq M\phi(|u-y|) \quad \text{...........(2.5)}$$

where M is a positive constant and ϕ is a nondecreasing continuous function and satisfy

$$\frac{1}{\alpha}\phi(x) \leq \phi\left(\frac{x}{\alpha}\right)$$

For $x \geq 0$, $\alpha > 0$ and the following integral

$$\Phi(x) = \int_0^A \frac{dx}{\phi(x)} \quad \text{...(2.6)}$$

Where $\Phi(x)$ is a primitive of the function $\frac{1}{\phi(x)}$, and Φ^{-1} denotes the inverse of Φ.

Proof:

Let that there exists two solutions u and y of eqs. (2.1)-(2.2), then:

$$u(t) = u_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)ds \ +$$

$$\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left\{\frac{1}{\Gamma(q)}\int_0^s (s-\sigma)^{q-1} u(\sigma)\, d\sigma\right\} ds$$

$$y(t)=u_0+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)ds +$$

$$\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left\{\frac{1}{\Gamma(q)}\int_0^s (s-\sigma)^{q-1} y(\sigma)d\sigma\right\} ds$$

this implies to:

$$|u(t)-y(t)| \le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left\{\frac{1}{\Gamma(q)}\int_0^s (s-\sigma)^{q-1} |u(\sigma)-y(\sigma)|\, d\sigma\right\} ds$$

It follows from ineq. (2.5) that:

$$|u(t)-y(t)| \le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} M\phi\left(|u-y|\right) ds$$

thus

$$|u(t)-y(t)| \le \varepsilon + \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1} \phi\left(|u-y|\right) ds$$

for any $\varepsilon > 0$, $0 < t < T$.

By using theorem (2.3), then:

$$|u(t) - y(t)| \leq \Phi^{-1}\left[\Phi(\varepsilon) + \frac{MT^q}{\Gamma(q+1)}\right], \text{for any fixed } t \in [0, T] \quad \ldots (2.7)$$

We shall proof that the right-hand side of ineq. (2.7) tend towards zero as $\varepsilon \to 0$. Since $|u(t) - y(t)|$ is independent of ε, it follows that $u(t) \equiv y(t)$, which we need. Let us remark that condition (2.6) implies that $\Phi(\varepsilon) \to -\infty$ as $\varepsilon \to 0$, no matter how we choose the primitive of $\frac{1}{\phi(x)}$. Thus $\Phi^{-1}(x) \to 0$ as $x \to -\infty$. Consequently, when $\varepsilon \to 0$ in ineq. (2.7), the right-hand side tends towards zero (for all finite t). Therefore, $u(t) = y(t)$, for $t \in [0, T]$. ■

Chapter Three

Numerical and Approximate Solution of Integro-Differential Equations of Fractional Order

In some cases, the analytical solution may be so difficult to be evaluated, therefore numerical and approximate methods may be necessary to be used which cover the problem under consideration. Hence, in this chapter, some approximate methods are considered to solve fractional integro-differential equations. These methods are the collocation method, the least square method and the Adomian decomposition method, which are given in sections one, two and three, respectively with some illustrative examples.

3.1 The Collocation Method

The collocation method may be considered as one of the most common used methods in approximating the solution of differential equations and integral equations, [Deleves, 1985]. This method is based on approximating the solution of the problem as a linear combination of certain complete sequence of functions and then solving the linear algebraic system resulting from substituting this approximate solution in the governing equation at a finite set of points from the domain of definition. Here, this method will be used to solve fractional integro-differential equations of the form:

$$D_*^q u(t) = f(t) + J^q u(t),\ u(0) = u_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1)$$

Let u(t) be the approximate solution of eq.(3.1), defined by:

$$u(t) = \psi(t) + \sum_{i=1}^{n} a_i \varphi_i(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.2)$$

Where $\psi(t)$ is a function which satisfies the nonhomogeneous conditions and $\{\varphi_i(t)\}$ is a complete sequence of functions, which satisfies the homogeneous conditions. To find the approximate solution u(t), substitute u(t) in the operator given by eq.(3.1) and hence the problem is reduced to the problem of evaluating the constants a_i's, for all i = 1, 2, …, n; which is as follows:

$$D_*^q \left[\psi(t) + \sum_{i=1}^{n} a_i \varphi_i(t)\right] = f(t) + J^q\left[\psi(t) + \sum_{i=1}^{n} a_i \varphi_i(t)\right] \ldots\ldots \quad (3.3)$$

and therefore, the residue error R(u, t) will be:

$$R(u, t) = D_*^q \left[\psi(t) + \sum_{i=1}^{n} a_i \varphi_i(t)\right] - J^q\left[\psi(t) + \sum_{i=1}^{n} a_i \varphi_i(t)\right] - f(t) \ldots\ldots \quad (3.4)$$

it is clear that R(u, t) becomes a function of the unknowns a_1, a_2, …, a_n and hence R(u, t) may be rewritten as $R(a_1, a_2, \ldots, a_n; t)$ and our purpose is to make:

$$R(a_1, a_2, \ldots, a_n; t) \cong 0,\ \forall\ t \in [0, T] \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.5)$$

To evaluate the coefficients a_i's, i = 1, 2, …, n; evaluate eq. (3.4) at

n-distinct points $t_1, t_2, \ldots, t_n \in [0, T]$, which will produce the following linear system:

$$R(a_1, a_2, \ldots, a_n; t_1) = 0$$

$$R(a_1, a_2, \ldots, a_n; t_2) = 0$$

$$\vdots$$

$$R(a_1, a_2, \ldots, a_n; t_n) = 0$$

which may be written in matrix form, as:

$$Aa = B$$

where:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

where:

$a_{11} = D_*^q \varphi_1(t_1) - J^q \varphi_1(t_1)$, $a_{12} = D_*^q \varphi_2(t_1) - J^q \varphi_2(t_1)$, …,

$a_{1n} = D_*^q \varphi_n(t_1) - J^q \varphi_n(t_1)$, $a_{21} = D_*^q \varphi_1(t_2) - J^q \varphi_1(t_2)$,

$a_{22} = D_*^q \varphi_2(t_2) - J^q \varphi_2(t_2)$, …, $a_{2n} = D_*^q \varphi_n(t_2) - J^q \varphi_n(t_2)$, ...,

$a_{n1} = D_*^q \varphi_1(t_n) - J^q \varphi_1(t_n)$, $a_{n2} = D_*^q \varphi_2(t_n) - J^q \varphi_2(t_n)$, ..., $a_{nn} = D_*^q \varphi_n(t_n) - J^q \varphi_n(t_n)$

and

$$B = \begin{bmatrix} f(t_1) - (D_*^q - J^q)[\psi(t_1)] \\ f(t_2) - (D_*^q - J^q)[\psi(t_2)] \\ \vdots \\ f(t_n) - (D_*^q - J^q)[\psi(t_n)] \end{bmatrix}, \; a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

It is remarkable that the matrix A is nonsingular since φ_i's, for all i = 1, 2, .., n; are selected from a complete sequence of functions, i.e., the φ_i's are linearly independent and the vector B is not identical to the zero vector if either the fractional integro-differential equations (3.1) has non-zero initial condition or if eq.(3.1) is nonhomogeneous or both.

Moreover, the method may be used to the nonlinear fractional integro-differential equations:

$$D_*^q u(t) = f(t) + J^q N(u(t)), \; u(0) = u_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3.6)}$$

where, N(u) is the nonlinear term, let u(t) be the approximate solution of eq.(3.6), defined by eq. (3.2):

$$u(t) = \psi(t) + \sum_{i=1}^{n} a_i \varphi_i(t)$$

where $\psi(t)$ is a function which satisfies the nonhomogeneous conditions and $\{\varphi_i(t)\}$ is a complete sequence of functions, which satisfies the homogeneous conditions. To find the approximate solution u(t), substitute u(t) in the operator given by eq.(3.6) and hence the problem is reduced to the problem of evaluating the constants a_i's, for all i = 1, 2, …, n; which is as follows:

$$D_*^q\ [\psi(t) + \sum_{i=1}^{n} a_i\varphi_i(t)] = f(t) + J^q N([\psi(t) + \sum_{i=1}^{n} a_i\varphi_i(t)]) \quad \ldots\ldots\ldots(3.7)$$

and therefore, the residue error R(u, t) will be:

$$R(u, t)=D_*^q[\psi(t) + \sum_{i=1}^{n} a_i\varphi_i(t)] - J^q N([\psi(t) + \sum_{i=1}^{n} a_i\varphi_i(t)]) - f(t) \quad \ldots.(3.8)$$

it is clear that R(u, t) becomes a function of the unknowns $a_1, a_2, \ldots, a_n$ and hence R(u, t) may be rewritten as $R(a_1, a_2, \ldots, a_n; t)$ and our purpose is to make:

$$R(a_1, a_2, \ldots, a_n; t) \cong 0,\ \forall\ t \in [0, T] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.9)$$

To evaluate the coefficients a_i's, i = 1, 2, …, n;evaluate eq.(3.8) at n-distinct points $t_1, t_2, \ldots, t_n \in [0, T]$, which will produce by the nonlinear system algebraic equations, which solved using Newton-Raphson method.

The following examples for illustrate the above method of solution:

Example (3.1):

Consider the linear fractional integro-differential equation is:

$$D_*^{0.5}u(t) = f(t) + J^{0.5}u(t),\ u(0) = 0,\ t \in [0, 1] \quad \ldots\ldots\ldots\ldots\ldots(3.10)$$

where:

$$f(t) = \frac{6}{\Gamma(3.5)}t^{2.5} - \frac{6}{\Gamma(4.5)}t^{3.5}$$

In order to solve eq.(3.10) according to the colloca on method, we consider the approximate solution u(t) as:

$$u(t) = \psi(t) + \sum_{i=1}^{5} a_i \varphi_i(t)$$

and since u(0) = 0 is the only initial condition, which is homogeneous, then $\psi(t) = 0$. The functions $\varphi_i(t)$, i = 1, 2, …, 5; which satisfy the homogeneous initial condition u(0) = 0, may be chosen as:

$$\varphi_i(t) = t^i,\ i = 1, 2, \ldots, 5$$

and the approximate solution u(t) will take the form:

$$u(t) = a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

Therefore $R(a_1, a_2, \ldots, a_5 ;t) \cong 0$, implies to:

$$D_*^{0.5}(a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5) - J^{0.5}(a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5) = \frac{6}{\Gamma(3.5)} t^{2.5} - \frac{6}{\Gamma(4.5)} t^{3.5}$$

and evaluating $D_*^{0.5}$ and $J^{0.5}$, the following algebraic equation is obtained:

$$a_1(1.12838t^{0.5} - 0.75225t^{1.5}) + a_2(1.50451t^{1.5} - 0.601802t^{2.5}) + a_3(1.80541t^{2.5} - 0.51583t^{3.5}) + a_4(2.06332t^{3.5} - 0.45852t^{4.5}) + a_5(2.29258t^{4.5} - 0.41683t^{5.5}) - (1.80541t^{2.5} - 0.51583t^{3.5}) = 0$$

Hence, at t = 0.1, 0.3, 0.5, 0.7 1, we get the linear system Aa = B, where:

$$A=\begin{bmatrix} 0.33304 & 4.56734\times10^{-2} & 5.54609\times10^{-3} & 6.37979\times10^{-4} & 7.11796\times10^{-5} \\ 0.49443 & 0.21755 & 8.13694\times10^{-2} & 2.84792\times10^{-2} & 9.61637\times10^{-3} \\ 0.53192 & 0.42554 & 0.27356 & 0.16211 & 9.2108\times10^{-2} \\ 0.50351 & 0.63441 & 0.59212 & 0.50001 & 0.40192 \\ 0.37613 & 0.9027 & 1.28958 & 1.60480 & 1.87575 \end{bmatrix}$$

$$a=\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}, B=\begin{bmatrix} 5.54609\times10^{-3} \\ 8.13694\times10^{-2} \\ 0.27356 \\ 0.59212 \\ 1.28958 \end{bmatrix}$$

Solving this system for $a_1, a_2, \ldots, a_5$, yields to:

$$u(t) = 7.91\times10^{-19}t - 1.8\times10^{-18}t^2 + t^3 - 2\times10^{-18}t^4 - 1\times10^{-19}t^5$$

A comparison between the exact solution $u(t) = t^3$ and the approximate solution is given in table (3.1).

Table (3.1)

Exact and approximate results of example (3.1).

T	*Exact solution*	*Approximate solution*	*Absolute error*
0	0	0	0
0.1	1×10^{-3}	1×10^{-3}	6.08×10^{-20}
0.2	8×10^{-3}	8×10^{-3}	8.28×10^{-20}
0.3	2.7×10^{-2}	2.7×10^{-2}	5.9×10^{-20}
0.4	6.4×10^{-2}	6.39999×10^{-2}	2.4×10^{-20}
0.5	0.125	0.12499	1.7×10^{-19}
0.6	0.216	0.21599	4.4×10^{-19}
0.7	0.343	0.34299	8.3×10^{-19}
0.8	0.512	0.51199	1.37×10^{-18}
0.9	0.729	0.72899	2.12×10^{-18}
1	1	0.99999	3.11×10^{-18}

Example (3.2):

Consider the nonlinear fractional integro-differential equation is:

$$D_{*}^{0.75}u(t) = \frac{1}{\Gamma(1.25)}t^{0.25} - \frac{2}{\Gamma(3.75)}t^{2.75} + J^{0.75}u^{2}(t),\ t\in[0, 1] \text{.........}(3.11)$$

$u(0) = 0$

where the exact solution is given by $u(t) = t$.

In order to solve eq.(3.11) according to the collocation method, we consider the approximate solution u(t) as:

$$u(t) = \psi(t) + \sum_{i=1}^{5} a_i \varphi_i(t)$$

and since u(0) = 0 is the only initial condition, which is homogeneous, then $\psi(t) = 0$. The functions $\varphi_i(t)$, i = 1, 2, …, 5; which satisfy the homogeneous initial condition u(0) = 0, may be chosen as:

$$\varphi_i(t) = t^i,\ i = 1, 2, \ldots, 5$$

and the approximate solution u(t) will take the form:

$$u(t) = a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

Therefore $R(a_1, a_2, \ldots, a_5; t) \cong 0$, implies to:

$$D_*^{0.75}(a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5) - J^{0.75}(a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5)^2 - (1.10326 t^{0.25} - 0.45218 t^{2.75}) = 0$$

and evaluating $D_*^{0.75}$ and $J^{0.75}$, the following algebraic equation is obtained:

$$1.10326 a_1 t^{0.25} + 1.76522\, a_2 t^{1.25} + 2.35363\, a_3 t^{2.25} + 2.89677\, a_4 t^{3.25} + 3.40797\, a_5 t^{4.25} - 0.45218 a_1^2 t^{2.75} - 0.30463 a_2^2 t^{4.75} - 0.23546 a_3^2 t^{6.75} - 0.19445 a_4^2 t^{8.75} - 0.16697 a_5^2 t^{10.75} - 0.36175(2a_1a_2)t^{3.75} - 0.30463(2a_1a_3)t^{4.75} - 0.26489(2a_1a_4)t^{5.75} - 0.23546(2a_1a_5)t^{6.75} - 0.26489(2a_2a_3)t^{5.75} - 0.23546(2a_2a_4)t^{6.75} - 0.21268(2a_2a_5)t^{7.75} - 0.21268(2a_3a_4)t^{7.75} - 0.19445(2a_3a_5)t^{8.75} - 0.17949(2a_4a_5)t^{9.75} = 1.10326 t^{0.25} - 0.45218 t^{2.75} \quad \text{.....................(3.12)}$$

Hence, at t = 0.1, 0.3, 0.5, 0.7 1, we get a nonlinear system of algebraic equations, which solved using Newton-Raphson method, which yields to:

$a_1 = 1$, $a_2 = -2.551\times10^{-5}$, $a_3 = 8.819\times10^{-5}$, $a_4 = -1.204\times10^{-4}$,

$a_5 = 5.393\times10^{-5}$

Therefore:

$$u(t) = t - 2.551\times10^{-5}t^2 + 8.819\times10^{-5}t^3 - 1.204\times10^{-4}t^4 + 5.393\times10^{-5}t^5$$

A comparison between the exact and approximate results is given in table (3.2).

Table (3.2)

Exact and approximate results of example (3.2).

t	*Exact solution*	*Approximate solution*	*Absolute error*
0	0	0	0
0.1	0.1	0.099999821	1.78411×10^{-7}
0.2	0.2	0.19999951	4.90262×10^{-7}
0.3	0.3	0.29999924	7.5896×10^{-7}
0.4	0.4	0.39999903	9.67437×10^{-7}
0.5	0.5	0.49999831	1.19344×10^{-6}
0.6	0.6	0.59999846	1.544803×10^{-6}
0.7	0.7	0.69999791	2.09475×10^{-6}
0.8	0.8	0.79999718	2.81718×10^{-6}
0.9	0.9	0.89999648	3.521904×10^{-6}
1	1	0.99999	3.79×10^{-6}

Example (3.3):

Consider the nonlinear fractional integro-differential equation is:

$$D_*^{0.5}u(t) = f(t) + J^{0.5}u(t),\ u(0) = 0,\ t \in [0, 1] \ \ldots\ldots\ldots\ldots\ldots\ldots (3.13)$$

Where:

$$f(t) = 1.12838t^{0.5} + 0.75225t^{1.5} + 0.3009t^{2.5} + 8.59717t^{3.5}$$

and the exact solution is given by $u(t) = te^t$.

In order to solve eq. (3.13) according to the collocation method, we consider the approximate solution u(t) as:

$$u(t) = \psi(t) + \sum_{i=1}^{5} a_i \varphi_i(t)$$

and since u(0) = 0 is the only initial condition, which is homogeneous, then $\psi(t) = 0$. The function $\varphi_i(t)$, i = 1, 2, …, 5; which satisfy the homogeneous initial condition u(0) = 0, may be chosen as:

$$\varphi_i(t) = t^i,\ i = 1, 2, \ldots, 5$$

and the approximate solution u(t) will take the form:

$$u(t) = a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5$$

Therefore $R(a_1, a_2, \ldots, a_5; t) \cong 0$, implies to:

$$D_*^{0.5}(a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5) - J^{0.5}(a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5)$$
$$= 1.12838t^{0.5} + 0.75225t^{1.5} + 0.3009t^{2.5} + 8.59717t^{3.5}$$

and evaluating $D_*^{0.5}$ and $J^{0.5}$, the following algebraic equation is obtained:

$a_1(1.12838t^{0.5} - 0.75225t^{1.5}) + a_2(1.50451t^{1.5} - 0.601802t^{2.5}) + a_3(1.80541t^{2.5} - 0.51583t^{3.5}) + a_4(2.06332t^{3.5} - 0.45852t^{4.5}) + a_5(2.29258t^{4.5} - 0.41683t^{5.5}) - (1.12838t^{0.5} + 0.75225t^{1.5} +0.3009t^{2.5} + 8.59717t^{3.5}) = 0$

Hence, at t = 0.1, 0.3, 0.5, 0.7 1, we get the linear system Aa = B, where:

$$A = \begin{bmatrix} 0.33304 & 4.56734\times10^{-2} & 5.54609\times10^{-3} & 6.37979\times10^{-4} & 7.11796\times10^{-5} \\ 0.49443 & 0.21755 & 8.13694\times10^{-2} & 2.84792\times10^{-2} & 9.61637\times10^{-3} \\ 0.53192 & 0.42554 & 0.27356 & 0.16211 & 9.2108\times10^{-2} \\ 0.50351 & 0.63441 & 0.59212 & 0.50001 & 0.40192 \\ 0.37613 & 0.9027 & 1.28958 & 1.60480 & 1.87575 \end{bmatrix}$$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}, B = \begin{bmatrix} 0.38159 \\ 0.75775 \\ 1.12464 \\ 1.53267 \\ 2.26750 \end{bmatrix}$$

Solving this system for $a_1, a_2, \ldots, a_5$, yields to:

$$u(t)=1.00019t - 0.99737t^2 + 0.51207t^3 - 0.14267t^4 - 5.42085\times10^{-2}\, t^5$$

A comparison between the exact solution $u(t) = te^t$ and the approximate solution is given in table (3.3).

Table (3.3)

Exact and approximate results of example (3.3).

t	*Exact solution*	*Approximate solution*	*Absolute error*
0	0	0	0
0.1	0.11050	0.11052	2.91242×10^{-6}
0.2	0.24400	0.24427	8.31221×10^{-6}
0.3	0.40350	0.40493	8.34559×10^{-5}
0.4	0.59200	0.59664	3.68465×10^{-4}
0.5	0.81250	0.82406	1.14049×10^{-3}
0.6	1.06800	1.09248	2.87964×10^{-3}
0.7	1.36150	1.40785	6.33362×10^{-3}
0.8	1.69600	1.77685	1.25828×10^{-2}
0.9	2.07450	2.20696	2.31054×10^{-2}
1	2.5	2.70651	3.98423×10^{-2}

3.2 The Least Square Method

One of the most widely used methods to approximate the solution of the integro-differential equations, in general, and fractional integro-differential equation, in particular, is that one which has the general idea of minimizing the square of the residue error, which is called the mean square method. To illustrate this method, consider eq.(3.1):

$$D_*^q u(t) = f(t) + J^q u(t),\ u(0) = u_0,\ t \in [0, T]$$

and approximate the solution by:

$$\varphi_n(t) = \sum_{i=1}^{n} a_i t^i \,,\ n \in \mathbb{N} \quad \text{...(3.14)}$$

where a_i's, i = 1, …, n; are constants to be determined. Therefore, substituting eq.(3.14) back into eq.(3.1) and minimizing the square of the residual error defined as:

$$E(a_1, \ldots, a_n) = \int_0^t \{D_*^q \varphi_n(t) - J^q \varphi_n(t) - f(t)\}^2 \, dt$$

$$= \int_0^t \{D_*^q (\sum_{i=1}^{n} a_i t^i) - J^q (\sum_{i=1}^{n} a_i t^i) - f(t)\}^2 \, dt \text{..........(2.15)}$$

Hence, the problem now is reduced to find the coefficients a_i's, i = 1, 2, …, n. A necessary conditions for the coefficients a_i's, which minimizes E, is that:

$$\frac{\partial E}{\partial a_j} = 0, \text{ for each } j = 1, 2, \ldots, n$$

Which will give a linear system of n-equations, or the residue error given by eq.(3.15) may be minimized using the direct minimization techniques. Consider the fractional integro-differential equation (3.15), which will take the form:

$$E(a_1, \ldots, a_n) = \int_0^t \left\{ \sum_{i=1}^{n} \left[D_*^q (a_i t^i) - J^q (a_i t^i) \right] - f(t) \right\}^2 dt \text{........(3.16)}$$

and hence

$$\frac{\partial E}{\partial a_j} = 2\int_0^t \left\{ \sum_{i=1}^{n} \left[D_*^q (a_i t^i) - J^q (a_i t^i) \right] - f(t) \right\} \left\{ D_*^q t^j - J^q t^j \right\} dt$$

which is equivalent to:

$$\int_0^t \left\{\sum_{i=1}^{n}\left[D_*^q(a_i t^i) - J^q(a_i t^i)\right]\right\}\left\{D_*^q t^j - J^q t^j\right\} dt =$$

$$\int_0^t f(t)(D_*^q t^j - J^q t^j)\, dt \quad \text{.......}(3.17)$$

for all j = 1, …, n; which give an algebraic linear system of n-equations into n-unknowns a_1, a_2, …, a_n; which may be solved using any numerical method for solving linear systems. Moreover, the method may be used to solve the nonlinear fractional integro-differential equations (3.6):

$$D_*^q u(t) = f(t) + J^q N(u(t)), \quad u(0) = u_0,\ t \in [0, T]$$

where, N (u) is the nonlinear term, let u(t) be the approximate solution of eq.(3.6), defined by eq. (3.14):

$$\varphi_n(t) = \sum_{i=1}^{n} a_i t^i \quad , \quad n \in \mathbb{N}$$

where a_i's, i = 1, …, n; are constants to be determined. Therefore, substituting eq.(3.14) back into eq.(3.6) and minimizing the square of the residual error defined as:

$$E(a_1, \ldots, a_n) = \int_0^t \{D_*^q \varphi_n(t) - J^q N[\varphi_n(t)] - f(t)\}^2\, dt$$

$$= \int_0^t \{D_*^q (\sum_{i=1}^{n} a_i t^i) - J^q N[(\sum_{i=1}^{n} a_i t^i)] - f(t)\}^2\, dt \quad \text{.......}(2.18)$$

Hence, the problem now is reduced to find the coefficients a_i's, i = 1, 2, …, n. A necessary conditions for the coefficients a_i's, which minimizes E, is that:

$$\frac{\partial E}{\partial a_j} = 0, \text{ for each } j = 1, 2, \ldots, n$$

Which will give a nonlinear system, or the residue error given by eq.(3.18) may be minimized using the direct minimization techniques. Consider the fractional nonlinear integro-differential equation (3.18), which will take the form:

$$E(a_1, \ldots, a_n) = \int_0^t \left\{ \sum_{i=1}^{n} \left[D_*^q (a_i t^i) - J^q N[(a_i t^i)] \right] - f(t) \right\}^2 dt \quad \text{..(3.19)}$$

and hence

$$\frac{\partial E}{\partial a_j} = 2\int_0^t \left\{ \sum_{i=1}^{n} \left[D_*^q (a_i t^i) - J^q N[(a_i t^i)] \right] - f(t) \right\} \left\{ D_*^q t^j - J^q N[a_j t^j] \right\} dt$$

which is equivalent to:

$$\int_0^t \left\{ \sum_{i=1}^{n} \left[D_*^q (a_i t^i) - J^q N[(a_i t^i)] \right] \right\} \left\{ D_*^q t^j - J^q a_j t^j \right\} dt =$$

$$\int_0^t f(t)(D_*^q t^j - J^q a_j t^j)\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3.20)$$

For all j = 1, 2 , …, n; we get a nonlinear system of algebraic equations. To find $a_1, a_2, \ldots, a_n$, either by minimizing E with respect to $a_1, a_2, \ldots, a_n$ or evaluating the nonlinear system obtained from $\frac{\partial E}{\partial a_i} = 0, \forall\, i = 1, 2, \ldots,$ n; which will be solved then by using Newton-Raphson method.

The following examples illustrate the least square method:

Example (3.4):

We again consider the linear fractional integro-differential equation (3.10):

$$D_*^{0.5}u(t) = \frac{6}{\Gamma(3.5)}t^{2.5} - \frac{6}{\Gamma(4.5)}t^{3.5} + J^{0.5}u(t),\ u(0) = 0,\ t \in [0, 1]$$

and according to the least square method, we consider:

$$\varphi_5(t) = \sum_{i=1}^{5} a_i t^i$$

Hence:

$$E(a_1,\ldots,a_5) = \int_0^1 \left\{\sum_{i=1}^{5}\left[D_*^{0.5}(a_i t^i) - J^{0.5}(a_i t^i)\right] - \left(\frac{6}{\Gamma(3.5)}t^{2.5} - \frac{6}{\Gamma(4.5)}t^{3.5}\right)\right\}^2$$

$$= \int_0^1 \left\{ D_*^{0.5}(a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5) - J^{0.5}(a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5) - \left(\frac{6}{\Gamma(3.5)}t^{2.5} - \frac{6}{\Gamma(4.5)}t^{3.5}\right) \right\}^2 dt$$

$$= \int_0^1 \{a_1(1.12838t^{0.5} - 0.75225t^{1.5}) + a_2(1.50451t^{1.5} - 0.601802t^{2.5}) + a_3(1.80541t^{2.5} - 0.51583t^{3.5}) + a_4(2.06332t^{3.5} - 0.45852t^{4.5}) + a_5(2.29258t^{4.5} - 0.41683t^{5.5}) - (1.80541t^{2.5} - 0.51583t^{3.5})\}^2\, dt$$

Therefore, to find $a_1, a_2, \ldots, a_5$, either be minimizing E with respect to $a_1, a_2, \ldots, a_5$ or evaluate the linear system $\frac{\partial E}{\partial a_i} = 0$, i = 1, 2, …, 5. Evaluating the linear system, as:

$$\frac{\partial E}{\partial a_1} = 0 \Rightarrow 0.21221a_1 + 0.20372a_2 + 0.18593a_3 + 0.20695a_4 + 0.15678a_5 = 0.18593$$

$$\frac{\partial E}{\partial a_2} = 0 \Rightarrow 0.20372a_1 + 0.26408a_2 + 0.27717a_3 + 0.27594a_4 + 0.26977a_5 = 0.27717$$

$$\frac{\partial E}{\partial a_3} = 0 \Rightarrow 0.39289a_1 + 0.\ 27717a_2 + 0.31043a_3 + 0.32193a_4 + 0.32387a_5 = 0.31043$$

$$\frac{\partial E}{\partial a_4} = 0 \Rightarrow 0.16999a_1 + 0.27594a_2 + 0.32193a_3 + 0.34295a_4 + 0.35184a_5 = 0.32193$$

$$\frac{\partial E}{\partial a_5} = 0 \Rightarrow 0.15678a_1 + 0.26977a_2 + 0.32387a_3 + 0.35184a_4 + 0.36632a_5 = 0.\ 32387$$

which has the solution:

$$a_1 = 1.756\times10^{-17},\ a_2 = 0,\ a_3 = 1,\ a_4 = -2\times10^{-18},\ a_5 = 0$$

and hence the solution of the integro-differential equation is:

$$u(t) = 1.756\times10^{-17}t + t^3 - 2\times10^{-18}t^4$$

A comparison between the exact and approximate results is given in table (3.4).

Table (3.4)

Exact and approximate results of example (3.4).

T	Exact solution	Approximate solution	Absolute error
0	0	0	0
0.1	1×10^{-3}	1×10^{-3}	1.7558×10^{-18}
0.2	8×10^{-3}	8×10^{-3}	3.5088×10^{-18}
0.3	2.7×10^{-2}	2.7×10^{-2}	5.252×10^{-18}
0.4	6.4×10^{-2}	6.4×10^{-2}	6.973×10^{-18}
0.5	0.125	0.125	8.66×10^{-18}
0.6	0.216	0.216	1.028×10^{-17}
0.7	0.343	0.343	1.181×10^{-17}
0.8	0.512	0.512	1.323×10^{-17}
0.9	0.729	0.729	1.449×10^{-17}
1	1	1	1.56×10^{-17}

Example (3.5):

We again consider the nonlinear equation (3.11), given by:

$$D_*^{0.75}u(t) = \frac{1}{\Gamma(1.25)}t^{0.25} - \frac{2}{\Gamma(3.75)}t^{2.75} + J^{0.75}u^2(t),\ u(0) = 0,\ t \in [0, 1]$$

and according to the least square method, consider:

$$\varphi_5(t) = \sum_{i=1}^{5} a_i t^i$$

Hence:

$$E(a_1, a_2, \ldots, a_5;t) = \int_0^1 \left\{ D_*^{0.75}\left(\sum_{i=1}^{5} a_i t^i\right) - J^{0.75}\left(\sum_{i=1}^{5} a_i t^i\right)^2 - \left(\frac{1}{\Gamma(1.25)} t^{0.25} - \frac{2}{\Gamma(3.75)} t^{2.75}\right)\right\}^2 dt$$

$$= \int_0^1 \left\{ D_*^{0.75}(a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5) - J^{0.75}(a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5)^2 - \left(\frac{1}{\Gamma(1.25)} t^{0.25} - \frac{2}{\Gamma(3.75)} t^{2.75}\right) \right\}^2 dt$$

Then after evaluating the fractional derivatives and fractional integrals under the integral sign. To find $a_1, a_2, \ldots, a_5$, either by minimizing E with respect to $a_1, a_2, \ldots, a_5$ or evaluating the nonlinear system obtained from $\frac{\partial E}{\partial a_i} = 0, \forall\ i = 1, 2, \ldots, 5$; which will be solved then by using Newton-Raphson method, which has the solution:

$$a_1 = 1,\ a_2 = -4.669\times10^{-6},\ a_3 = 1.232\times10^{-5},\ a_4 = -1.332\times10^{-5},$$

$$a_5 = 5.084\times10^{-6}$$

and therefore, the solution of the nonlinear integro-differential equation is given by:

$$u(t) = t - 4.669\times10^{-6}t^2 + 1.232\times10^{-5}t^3 - 1.332\times10^{-5}t^4 + 5.084\times10^{-6}t^5$$

A comparison between the exact solution u(t) = t and the approximate solution are given in table (3.5).

Table (3.5)

Exact and approximate results of example (3.5).

t	*Exact solution*	*Approximatesolution*	*Absolute error*
0	0	0	0
0.1	0.1	9.99999×10^{-2}	3.56512×10^{-8}
0.2	0.2	0.199999	1.07885×10^{-7}
0.3	0.3	0.299999	1.83108×10^{-7}
0.4	0.4	0.399999	2.47492×10^{-7}
0.5	0.5	0.499999	3.00875×10^{-7}
0.6	0.6	0.599999	3.506602×10^{-7}
0.7	0.7	0.699999	4.05714×10^{-7}
0.8	0.8	0.799999	4.70267×10^{-7}
0.9	0.9	0.899999	5.37811×10^{-7}
1	1	0.999999	5.85×10^{-7}

Example (3.6):

Consider the nonlinear equation given by eq. (3.13):

$$D_*^{0.5}u(t) = f(t) + J^{0.5}u(t),\ u(0) = 0,\ t \in [0, 1]$$

and according to the least square method, we consider:

$$\varphi_5(t) = \sum_{i=1}^{5} a_i t^i$$

Hence:

$$E(a_1,\ldots,a_5) = \int_0^1 \left\{ \sum_{i=1}^{5} \left[D_*^{0.5}(a_i t^i) - J^{0.5}(a_i t^i) \right] - \right.$$

$$\left(1.12838t^{0.5} + 0.75225t^{1.5} + 0.3009t^{2.5} + 8.59717t^{3.5}\right)\Big\}^2$$

$$= \int_0^1 \Big\{ D_*^{0.5}(a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5) - J^{0.5}(a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5) - (1.12838t^{0.5} + 0.75225t^{1.5} + 0.3009t^{2.5} + 8.59717t^{3.5})\Big\}^2 dt$$

$$= \int_0^1 \{a_1(1.12838t^{0.5} - 0.75225t^{1.5}) + a_2(1.50451t^{1.5} - 0.601802t^{2.5}) + a_3(1.80541t^{2.5} - 0.51583t^{3.5}) + a_4(2.06332t^{3.5} - 0.45852t^{4.5}) + a_5(2.29258t^{4.5} - 0.41683t^{5.5}) - (1.12838t^{0.5} + 0.75225t^{1.5} + 0.3009t^{2.5} + 8.59717t^{3.5})\Big\}^2 dt$$

Therefore, to find $a_1, a_2, \ldots, a_5$, either be minimizing E with respect to $a_1, a_2, \ldots, a_5$ or evaluate the linear system $\frac{\partial E}{\partial a_i} = 0$, $i = 1, 2, \ldots, 5$. Evaluating the linear system, as:

$$\frac{\partial E}{\partial a_1} = 0 \Rightarrow 0.21221a_1 + 0.20372a_2 + 0.18593a_3 + 0.20695a_4 + 0.15678a_5 = 0.54338$$

$$\frac{\partial E}{\partial a_2} = 0 \Rightarrow 0.20372a_1 + 0.26408a_2 + 0.27717a_3 + 0.27594a_4 + 0.26977a_5 = 0.66304$$

$$\frac{\partial E}{\partial a_3} = 0 \Rightarrow 0.39289a_1 + 0.\ 27717a_2 + 0.31043a_3 + 0.32193a_4 + 0.32387a_5 = 0.68484$$

$$\frac{\partial E}{\partial a_4} = 0 \Rightarrow 0.16999a_1 + 0.27594a_2 + 0.32193a_3 + 0.34295a_4 + 0.35184a_5 = 0.67807$$

$$\frac{\partial E}{\partial a_5} = 0 \Rightarrow 0.15678a_1 + 0.26977a_2 + 0.32387a_3 + 0.35184a_4 + 0.36632a_5 = 0.66174$$

and hence the solution of the integro-differential equation is:

$$u(t) = -1.71402\times10^{-3}t + 4.50386t^2 - 3.17152t^3 + 0.22997t^4 - 1.07351t^5$$

A comparison between the exact and approximate results is given in table (3.6).

Table (3.6)

Exact and approximate results of example (3.6).

t	*Exact solution*	*Approximate solution*	*Absolute error*
0	0	0	0
0.1	0.11050	0.04173	6.87873×10^{-2}
0.2	0.24400	0.15515	8.91158×10^{-2}
0.3	0.40350	0.32367	8.11765×10^{-2}
0.4	0.59200	0.53383	6.24319×10^{-2}
0.5	0.81250	0.77659	4.63282×10^{-2}
0.6	1.06800	1.04859	4.10065×10^{-2}
0.7	1.36150	1.35350	4.8015×10^{-2}
0.8	1.69600	1.70325	6.10206×10^{-2}
0.9	2.07450	2.11933	6.45209×10^{-2}
1	2.5	2.63411	3.25559×10^{-2}

3.3 The Adomian Decomposition Method [Ibrahim, 2006]:

Adomian decomposition method (ADM) is one of the new methods that may be used for solving initial value problems in fractional integro-differential equations of various kinds that arising not only in the field of medicine, physical and biological science, but also in the area of engineering. It is important to note that a large amount of researches workers has been devoted to the application of ADM to a wide class of linear and nonlinear integro-differential equations of fractional order. In resent years, the decomposition method has emerged as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, differential, integral, integro-differential, higher order ordinary differential equations, partial differential equations and in this section fractional integro-differential equations. The convergence of this method have investigated by Cherruault and cooperators. In [Cherruault , 1989], Cherruault proposed a new definition of the method and then he insisted that it will become possible to prove the convergence of the decomposition method. In [Cherruault, 1993], Cherruault and Adomian proposed a new convergence proof of Adomian method based on properties of convergence series. In this method, the solution is considered as the sum of an infinite series, rapidly converging to an accurate solution. In [Abbaoui, 2001], Abbaoui et al. proposed a new approach of decomposition which is obtained in a more natural way in the classical representation. Lesnic [Lesnic, 2002] investigated convergence of Adomian's method to periodic temperature fields in heat conductors. The advantage of this method is that it provides a direct scheme for solving the problem without any need for linearization or discritization. Essentially, the method provides a systematic computational procedure for equations of physical significance. Adomian in (1994), (1989); Kaya and El-Sayed in (2003), has used Adomian decomposition method to solve the problems in applied sciences. Decomposition method provides an analytical approximation to linear and nonlinear problems, [Adomian, 1988], [Adomian, 1985].

3.3.1 The Analysis of ADM [Ibrahim, 2007]:

The ADM will be reviewed by following the procedure of Adomain given in 1988 and improved in 1991. However, the same techniques may be applied to other system of equations [Bulut and Evens, 2002]. To introduce this method, first we consider the eq. (3.1) is linear, i.e., the fractional integro-differential equation (3.1) is of the form:

$$D_*^q u(t) = f(t) + J^q u(t),\ u(0) = u0$$

or equivalently:

$$D_*^q u(t) = f(t) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} u(s)\, ds,\ u(0) = u_0 \qquad (3.21)$$

where $0 < q < 1$, $f(t)$ is assumed to be bounded, $\forall\ t \in [0, T]$. Operating with J^q on both sides of eq. (3.21), getting:

$$J^q D_*^q u(t) = J^q f(t) + J^q\left[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} u(s)\, ds\right]$$

Hence:

$$u(t) = u(0) + J^q f(t) + J^q \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s)\, ds \right] \quad \text{.................(3.22)}$$

Adomian's method defines the solution u(t) by the series:

$$u(t) = \sum_{n=0}^{\infty} u_n(t) \quad \text{.........(3.23)}$$

Hence, from (3.22) we obtain that:

$$u_0 = u(0) + J^q f(t) \quad \text{.........(3.24)}$$

$$u_1 = J^q \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u_0(s)\, ds \right) \quad \text{.........(3.25)}$$

$$\vdots$$

$$u_{n+1} = J^q \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u_n(s)\, ds \right) \quad \text{.........(3.26)}$$

where the components $u_n(t)$ will be determined, recursively. Moreover, the method may be used to the nonlinear integro-differential equation of fractional order (3.6):

$$D_*^q u(t) = f(t) + J^q N(u(t)),\ u(0) = 0$$

or equivalently:

$$D_*^q u(t) = f(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} N(u(s))\, ds,\ u(0) = u_0 \quad \text{.............(3.27)}$$

Where $0 < q < 1$, f(t) is assumed to be bounded for all $t \in I = [0, T]$ and

$$\left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds \right| \le M, \text{ for all } 0 \le s \le t \le T$$

where M is finite constant. The nonlinear term N(u) is Lipschitzian with:

$$|N(u) - N(z)| \leq L|u - z|$$

and may be decomposed in the form:

$$N(u(t)) = \sum_{n=0}^{\infty} A_n(t) \quad \text{.......(3.28)}$$

where A_n are the Adomian polynomials, given by:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]\Bigg|_{\lambda=0} \quad \text{.......(3.29)}$$

Then N(u) will be a function of λ, u_0, u_1, … . Now, substituting eq.(3.28) in eq.(3.27), yields to:

$$D_*^q u(t) = f(t) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left(\sum_{n=0}^{\infty} A_n(s)\right) ds \quad \text{.......(3.30)}$$

Operating with J^q on both sides of eq.(3.30), give:

$$u(t) = u(0) + J^q f(t) + J^q\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left(\sum_{n=0}^{\infty} A_n(s)\right) ds\right) \quad \text{.......(3.31)}$$

The components u_0, u_1, … are determined recursively by:

$$u_0 = u(0) + J^q f(t) \quad \text{.......(3.32)}$$

$$u_{k+1} = J^q\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} A_k(s)\, ds\right) \quad \text{.......(3.33)}$$

The Adomian's polynomials can be generated from Taylor expansion of N(u(t)) about the first component u_0, which means that:

$$N(u) = \sum_{n=0}^{\infty} A_n$$

$$= \sum_{n=0}^{\infty} \frac{(u-u_0)^n}{n!} N^{(n)}(u_0)$$

In [Adomian, 1995], Adomian's polynomials are arranged to have the form:

$$\left.\begin{aligned} A_0 &= f(u_0) \\ A_1 &= u_1 f^{(1)}(u_0) \\ A_2 &= u_2 f^{(1)}(u_0) + \frac{u_1^2}{2!} f^{(2)}(u_0) \\ A_3 &= u_3 f^{(1)}(u_0) + u_1 u_2 f^{(2)}(u_0) + \frac{u_1^3}{3!} f^{(3)}(u_0) \\ &\vdots \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.34)$$

3.3.2 Modification of Adomian's Polynomials [Ibrahim, 2007]:

By rearranging the terms in the obtained polynomials (3.34), yields to:

$$\bar{A}_0 = N(u_0)$$

$$\bar{A}_1 = u_1 N^{(1)}(u_0) + \frac{u_1^2}{2!} N^{(2)}(u_0) + \frac{u_1^3}{3!} N^{(3)}(u_0) + \ldots$$

$$\bar{A}_2 = u_2 N^{(1)}(u_0) + \frac{1}{2!}(u_2^2 + 2u_1u_2) N^{(2)}(u_0) + \frac{1}{3!}(3u_1^2 + 3u_1u_2^2 + u_3^2) N^{(3)}(u_0) + \ldots$$

$$\bar{A}_3 = u_3N^{(1)}(u_0) + \frac{1}{2!}(u_3^2 + 2u_1u_3 + 2u_2u_3)N^{(2)}(u_0) + \frac{1}{3!}(u_3^3 + 3u_3^2(u_1+u_2) + 3u_3(u_1+u_2)^2)N^{(3)}(u_0) + \ldots$$

$$\vdots$$

Define the partial sum $S_n = \sum_{i=0}^{n} u_i(t)$, from the rearranged polynomials, then one can write:

$$\bar{A}_0 = N(u_0) = N(s_0)$$

$$\bar{A}_0 + \bar{A}_1 = N(u_0) + u_1N^{(1)}(u_0) + \frac{u_1^2}{2!}N^{(2)}(u_0) + \frac{u_1^3}{3!}N^{(3)}(u_0) + \ldots$$

$$= N(u_0 + u_1)$$

$$= N(s_1)$$

Similarly:

$$\bar{A}_0 + \bar{A}_1 + \bar{A}_2 = N(u_0 + u_1 + u_2)$$

$$= N(s_2)$$

and by induction, the following sum is obtained:

$$\sum_{i=0}^{n} \bar{A}_i(u_0, u_1, \ldots, u_i) = N(s_n)$$

Therefore, in general:

$$\bar{A}_n = N(s_n) - \sum_{i=0}^{n-1} \bar{A}_i \quad \ldots\ldots (3.35)$$

For example, if $N(u) = u^3$, then the first four polynomials using formulae (3.29) and (3.35) are computed to be:

Using formula (3.29):

$$A_0 = u_0^3$$

$$A_1 = 3u_0^2 u_1$$

$$A_2 = 3u_0 u_1^2 + 3u_0^2 u_2$$

$$A_3 = u_1^3 + 6u_0u_1u_2 + 3u_0^2 u_3$$

$$A_4 = 3u_1^2 u_2 + 3u_0 u_2^2 + 6u_0u_1u_3 + 3u_0^2 u_4$$

Using formula (3.35)

$$\bar{A}_0 = u_0^3$$

$$\bar{A}_1 = 3u_0^3 u_1 + 3u_0 u_1^2 + u_1^3$$

$$\bar{A}_2 = 3u_0^2 u_2 + 3u_0 u_2^2 + 3u_1^2 u_2 + 3u_1 u_2^2 + 6u_0u_1u_2 + u_2^3$$

$$\bar{A}_3 = 3u_0^2 u_3 + 3u_0 u_3^2 + 3u_1^2 u_3 + 3u_1 u_3^2 + 3u_2^2 u_3 + 3u_2 u_3^2 + 6u_0u_1u_3 + 6u_0u_2u_3 + 6u_1u_2u_3 + u_3^3$$

$$\bar{A}_4 = 3u_0^2 u_4 + 3u_0 u_4^2 + 3u_1^2 u_4 + 3u_1 u_4^2 + 3u_2^2 u_4 + 3u_2 u_4^2 + 3u_3^2 u_4 + 3u_3 u_4^2 + 6u_0u_1u_4 + 6u_0u_2u_4 + 6u_0u_3u_4 + 6u_1u_2u_4 + 6u_1u_3u_4 + 6u_2u_3u_4 + u_4^3$$

Therefore, the nonlinear part N(u) of eq.(3.27) may be written in the form:

$$D_*^q u(t) = f(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \bar{A}_n(s)\, ds \qquad (3.36)$$

Clearly, the first four polynomials computed using the suggested formula (3.35) include the first four polynomials computed using formula (3.29) in addition to other terms that should appear in A_5, A_6, A_7, … using formula (3.29). Thus, the solution that is obtained using formula (3.35) enforces many terms to the calculation processes earlier, yielding after a faster convergence.

3.3.3 Convergence Analysis:

In chapter two, we prove the existence and uniqueness theorem of fractional integro-differential equation (3.1). Now in this section, the convergence of the series solution (3.23) is also proved. Finally, the maximum absolute error of the truncated series (3.23) is estimated.

<u>Theorem (3.1):</u>

The series solution (3.23) of eq.(3.21) using ADM converges whenever where $k = \dfrac{LMT^q}{\Gamma(q+1)}, 0 < k < 1$.

Proof:

Let s_n and s_m be an arbitrary partial sums with $n \geq m$, and to prove that $\{s_n\}$ is a Cauchy sequence in Banach space $B=(C[I],\| \cdot \|)$ of all continuous functions of I.

Therefore:

$$\|s_n - s_m\| = \max_{t \in I} |s_n - s_m|$$

$$= \max_{t\in I}\left|\sum_{i=m+1}^{n} u_i(t)\right|$$

$$= \max_{t\in I}\left|\sum_{i=m+1}^{n} J^q\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\bar{A}_{i-1}(s)\,ds\right)\right|$$

$$= \max_{t\in I}\left|J^q\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\sum_{i=m+1}^{n}\bar{A}_{i-1}(s)\,ds\right)\right|$$

From (3.35), we have:

$$\sum_{i=m}^{n-1}\bar{A}_i = N(s_{n-1}) - N(s_{m-1})$$

$$\|s_n - s_m\| = \max_{t\in I}\left|J^q\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[N(s_{n-1}) - N(s_{m-1})\right]ds\right)\right|$$

$$\leq \max_{t\in I} J^q\left|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds\right|\left|N(s_{n-1}) - N(s_{m-1})\right|$$

$$\leq \frac{1}{\Gamma(q)}\max_{t\in I}\left|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds\right|\left|N(s_{n-1}) - N(s_{m-1})\right|\left|\int_0^t (t-s)^{q-1}ds\right.$$

$$\leq \frac{LMT^q}{\Gamma(q+1)}\max_{t\in I}\left|s_{n-1} - s_{m-1}\right|$$

$$\leq \frac{LMT^q}{\Gamma(q+1)}\|s_{n-1} - s_{m-1}\|$$

Hence:

$$\|s_n - s_m\| \leq k\|s_{n-1} - s_{m-1}\|$$

Let n = m + 1, then:

$$\|s_{m+1} - s_m\| \le k\|s_m - s_{m-1}\|$$
$$\le k^2\|s_{m-1} - s_{m-2}\|$$
$$\vdots$$
$$\le k^m\|s_1 - s_0\|$$

and from the triangle inequality:

$$\|s_n - s_m\| \le \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \ldots + \|s_n - s_{n-1}\|$$
$$\le (k^m + k^{m+1} + \ldots + k^{n-1})\|s_1 - s_0\|$$
$$\le k^m\left(\frac{1-k^{n-m}}{1-k}\right)\|u_1(t)\|$$

Since $0 < k < 1$, so $(1 - k^{n-m}) < 1$, and then:

$$\|s_n - s_m\| \le \frac{k^m}{1-k} \max_{t\in I} |u_1(t)| \qquad (3.37)$$

But $|u_1(t)| < \infty$

Since f(t) is bounded, so as $m \longrightarrow \infty$, then $\|s_n - s_m\| \longrightarrow 0$

So $\{s_n\}$is a Cauchy sequence in B, and therefore the series $\sum_{i=0}^{\infty} u_i(t)$ converges. ■

Theorem (3.2):

The maximum absolute truncation error of the series solution (3.23) to eq. (3.21) is estimated to be:

$$\max_{t\in I}\left|u(t) - \sum_{i=0}^{\infty} u_i(t)\right| \le \frac{k^m}{1-k} \max_{t\in I} |u_1(t)| \qquad (3.38)$$

Proof:

From ineq. (3.37) in theorem (3.1), we have:

$$\|s_n - s_m\| \le \frac{k^m}{1-k} \max_{t\in I} |u_1(t)|$$

and as $n \longrightarrow \infty$, then $s_n \longrightarrow u(t)$, so we have:

$$\|u(t) - s_m\| \le \frac{k^m}{1-k} \max_{t\in I} |u_1(t)|$$

and the maximum absolute truncation error in the interval I is estimated to be:

$$\max_{t\in I} \left| u(t) - \sum_{i=0}^{\infty} u_i(t) \right| \le \frac{k^m}{1-k} \max_{t\in I} |u_1(t)|. \quad \blacksquare$$

<u>Example (3.7):</u>

Consider the fractional integro-differential equation (3.10):

$$D^{0.5}u(t) = f(t) + J^{0.5}u(t), \ u(0) = 0, \ t \in [0, 1]$$

where:

$$f(t) = \frac{6}{\Gamma(3.5)} t^{2.5} - \frac{6}{\Gamma(4.5)} t^{3.5}$$

and the exact solution is given by $u(t) = t^3$.

According to the Adomian decomposition method, the approximate solution:

$$u(t) = u(0) + \frac{6}{\Gamma(3.5)} J^{0.5} t^{2.5} - \frac{6}{\Gamma(4.5)} J^{0.5} t^{3.5} + J^{0.5}\left[\frac{1}{\Gamma(0.5)} \int_a^t (t-s)^{-0.5} u(s)\, ds\right]$$

and therefore:

$$u_0(t) = u(0) + \frac{6}{\Gamma(3.5)} J^{0.5} t^{2.5} - \frac{6}{\Gamma(4.5)} J^{0.5} t^{3.5}$$

$$= 0 + \frac{6}{\Gamma(3.5)} \frac{\Gamma(3.5)}{\Gamma(4)} t^3 - \frac{6}{\Gamma(4.5)} \frac{\Gamma(4.5)}{\Gamma(5)} t^4$$

$$= t^3 - 0.25t^4$$

$$u_1(t) = J^{0.5}\left[\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5} u_0(s)\,ds\right]$$

$$= 0.25t^4 - 0.05t^5$$

$$u_2(t) = J^{0.5}\left[\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5} u_1(s)\,ds\right]$$

$$= 0.05t^5 - 8.33333\times10^{-3}t^6$$

$$\vdots$$

Hence:

$$u(t) = u_0(t) + u_1(t) + \ldots$$

Also, the absolute error is evaluated, and the following notations for the absolute error will be used:

$$E_3 = |u(t) - \varphi_3(t)|,\ E_4 = |u(t) - \varphi_4(t)|$$

where:

$$\varphi_n(t) = \sum_{i=0}^{n-1} u_i(t),\ n \geq 1$$

and the exact solution is $u(t) = \lim_{n\to\infty} \varphi_n(t)$.

A comparison between the approximate and the exact solutions is given in table (3.7).

Table (3.7)

Exact and approximate results of example (3.7).

t	Exact solution	ADM φ_3	E_3	ADM φ_4	E_4
0	0	0	0	0	0
0.1	1×10^{-3}	9.99992×10^{-4}	8.33333×10^{-9}	9.99999×10^{-4}	1.19048×10^{-10}
0.2	8×10^{-3}	7.99947×10^{-3}	5.33333×10^{-7}	7.99998×10^{-3}	1.52381×10^{-8}
0.3	2.7×10^{-2}	2.69939×10^{-2}	6.07500×10^{-6}	2.69997×10^{-2}	2.60357×10^{-7}
0.4	6.4×10^{-2}	6.39659×10^{-2}	3.41333×10^{-5}	6.3998×10^{-2}	1.95048×10^{-6}
0.5	0.125	0.12487	1.30208×10^{-4}	0.12499	9.30010×10^{-6}
0.6	0.216	0.21561	3.88800×10^{-4}	0.21597	3.33257×10^{-5}
0.7	0.343	0.34202	9.80408×10^{-4}	0.34290	9.80408×10^{-5}
0.8	0.512	0.50982	$2.18453\times\times10^{-3}$	0.51175	2.49661×10^{-4}
0.9	0.729	0.72457	4.42868×10^{-3}	0.72843	5.69401×10^{-4}
1	1	0.99167	8.33333×10^{-3}	0.99881	1.19048×10^{-3}

Example (3.8):

Consider the nonlinear fractional integro-differential equation (3.11):

$$D^{0.75}u(t) = \frac{1}{\Gamma(1.25)}t^{0.25} - \frac{2}{\Gamma(3.75)}t^{2.75} + J^{0.75}u^2(t),\ t \in [0, 1]$$

$$u(0) = 0$$

where the exact solution is given by u(t) = t.

Therefore, according to the Adomian decomposition method:

$$u_0(t) = u(0) + J^{0.75} f(t)$$

$$= 0 + J^{0.75}\left[\frac{1}{\Gamma(1.25)} t^{0.25} - \frac{2}{\Gamma(3.75)} t^{2.75}\right]$$

$$= t - \frac{2}{\Gamma(4.5)} t^{3.5}$$

$$u_{n+1}(t) = J^{0.75}\left[\frac{1}{\Gamma(0.75)}\int_0^t (t-s)^{-0.25} A_n(s)\, ds\right], n = 0, 1, \ldots$$

where:

$$A_0 = u_0^2$$

$$A_1 = 2u_0u_1$$

$$A_2 = 2u_0u_2 + u_1^2$$

$$A_3 = 2u_1u_2 + 2u_0u_3$$

$$\vdots$$

are the Adomian polynomials for the nonlinear term $N(u) = u^2$.

$$u_1(t) = J^{0.75}\left[\frac{1}{\Gamma(0.75)}\int_0^t (t-s)^{-0.25}\left(s - \frac{2}{\Gamma(4.5)} s^{3.5}\right) ds\right]$$

$$= -2.5\times10^{-2}t^6 + 0.17194t^{3.5} + 1.24908\times10^{-3}t^{8.5}$$

$$u_2(t) = J^{0.75}\left[\frac{1}{\Gamma(0.75)}\int_0^t (t-s)^{-0.25}\left[2\left(s - \frac{2}{\Gamma(4.5)} s^{3.5}\right)\right.\right.$$

$$(-2.5\times10^{-2}s^{6}+0.17194s^{3.5}+1.24908\times10^{-3}s^{8.5})\Big]\,ds\Big]$$

$$= -1.73157\times10^{-4}t^{11} + 2.5\times10^{-2}t^{6} - 3.85698\times10^{-4}t^{8.5} - 8.90992\times10^{-6}t^{13.5}$$

$$\vdots$$

Hence, the approximate solution using ADM is given by:

$$u(t) = u_0(t) + u_1(t) + \ldots$$

Also, the absolute error is computed and the following notations are used:

$$E_3 = |u(t) - \varphi_3(t)|,\ E_4 = |u(t) - \varphi_4(t)|$$

where:

$$\varphi_n(t) = \sum_{i=0}^{n-1} u_i(t),\ n \geq 1$$

and the exact solution is $u(t) = \lim_{n\to\infty} \varphi_n(t)$.

A comparison between the approximate and the exact solutions is given in table (3.8).

Table (3.8)

Exact and approximate results of example (3.8).

t	Exact solution	ADM φ_3	E_3	ADM φ_4	E_4
0	0	0	0	0	0
0.1	1×10^{-1}	1×10^{-1}	1.06269×10^{-11}	1×10^{-1}	4.34771×10^{-15}
0.2	2×10^{-1}	2×10^{-1}	3.84206×10^{-9}	2×10^{-1}	8.88635×10^{-12}
0.3	3×10^{-1}	3×10^{-1}	1.20243×10^{-7}	3×10^{-1}	7.65381×10^{-10}
0.4	4×10^{-1}	4×10^{-1}	1.38013×10^{-6}	4×10^{-1}	1.79946×10^{-8}
0.5	5×10^{-1}	0.500002	9.13196×10^{-6}	0.500001	2.07341×10^{-7}
0.6	6×10^{-1}	0.600011	4.26005×10^{-5}	0.60005	1.51925×10^{-6}
0.7	7×10^{-1}	0.70004	1.55976×10^{-4}	0.70019	8.13284×10^{-6}
0.8	8×10^{-1}	0.80011	4.77811×10^{-4}	0.80058	3.45493×10^{-5}
0.9	9×10^{-1}	0.9003	1.27607×10^{-3}	0.90151	1.22837×10^{-4}
1	1	1.00068	3.05543×10^{-3}	1.00355	3.78995×10^{-4}

Also, this example may be solved using the modified Adomian's polynomials.

$$u_0(t) = t - \frac{2}{\Gamma(4.5)} t^{3.5}$$

$$u_{n+1}(t) = J^{0.75}\left[\frac{1}{\Gamma(0.75)}\int_0^t (t-s)^{-0.25}\bar{A}_n(s)\,ds\right], n = 0, 1, \ldots$$

where:

$$\bar{A}_0 = u_0^2$$

$$\bar{A}_1 = 2u_0u_1 + u_1^2$$

$$\bar{A}_2 = 2u_0u_2 + 2u_1u_2 + u_2^2$$

$$\bar{A}_3 = 2u_0u_3 + 2u_1u_3 + 2u_2u_3 + u_3^2$$

$$\vdots$$

are the modified Adomian polynomials for the nonlinear term $N(u) = u^2$.

$$u_1(t) = J^{0.75}\left[\frac{1}{\Gamma(0.75)}\int_0^t (t-s)^{-0.25}\bar{A}_0(s)\,ds\right]$$

$$= J^{0.75}\left[\frac{1}{\Gamma(0.75)}\int_0^t (t-s)^{-0.25}\left(s-\frac{2}{\Gamma(4.5)}s^{3.5}\right)^2 ds\right]$$

$$= -2.5\times10^{-2}t^6 + 0.17194t^{3.5} + 1.24908\times10^{-3}t^{8.5}$$

$$u_2(t)=J^{0.75}\left[\frac{1}{\Gamma(0.75)}\int_0^t (t-s)^{-0.25}\left[2\left(s-\frac{2}{\Gamma(4.5)}s^{3.5}\right)(-2.5\times10^{-2}s^6\right.\right.$$

$$\left.\left.+1.24908\times10^{-3}s^{8.5})+(-2.5\times10^{-2}s^6+0.17194s^{3.5}+1.24908\times10^{-3}s^{8.5})^2\right]ds\right]$$

$$= -9.99482\times10^{-17}t^{16} + 2.5\times10^{-2}t^6 + 7.09251\times10^{-5}t^{11} + 2.00164\times10^{-18}t^{18.5} - 3.36153\times10^{-3}t^{8.5} + 1.29643\times10^{-5}t^{13.5}$$

$$\vdots$$

Hence, the approximate solution using ADM is given by:

$u(t) = u_0(t) + u_1(t) + \ldots$

Also, the absolute error is computed and the following notations are used:

$E_3 = |u(t) - \varphi_3(t)|$, $E_4 = |u(t) - \varphi_4(t)|$

where:

$$\varphi_n(t) = \sum_{i=0}^{n-1} u_i(t),\ n \geq 1$$

and the exact solution is $u(t) = t$.

A comparison between the approximate and the exact solutions is given in table (3.9). A comparative study between table (3.8) and table (3.9), shows that the solution using eq.(3.35) converges faster than the solution using eq.(3.29).

Table (3.9)

Exact and approximate results of example (3.8).

t	*Exact solution*	*Modified ADM* φ_3	E_3	*Modified ADM* φ_4	E_4
0	0	0	0	0	0
0.1	1×10^{-1}	0.09999	6.67946×10^{-12}	0.09999	1.199402×10^{-15}
0.2	2×10^{-1}	0.19999	2.41702×10^{-9}	0.19999	2.45548×10^{-12}
0.3	3×10^{-1}	0.29999	7.57866×10^{-8}	0.29999	2.12228×10^{-10}
0.4	4×10^{-1}	0.39999	8.72554×10^{-7}	0.39999	5.01838×10^{-9}
0.5	5×10^{-1}	0.49999	5.79915×10^{-6}	0.49999	5.83087×10^{-8}
0.6	6×10^{-1}	0.59997	2.72134×10^{-5}	0.59999	4.32077×10^{-7}
0.7	7×10^{-1}	0.69989	1.00383×10^{-4}	0.69999	2.34652×10^{-6}
0.8	8×10^{-1}	0.79969	3.10293×10^{-4}	0.79999	1.01467×10^{-5}
0.9	9×10^{-1}	0.89916	8.37477×10^{-4}	0.89996	3.68508×10^{-5}
1	1	0.99797	2.02955×10^{-3}	0.99988	1.16564×10^{-4}

Example (3.9):

Consider the nonlinear integro-differential equation of fractional order given by eq. (3.13):

$$D_*^{0.5}u(t) = f(t) + J^{0.5}u(t),\ u(0) = 0,\ t \in [0, 1]$$

According to the Adomian decomposition method, the approximate solution:

$$u(t) = u(0) + J^{0.5}(1.12838t^{0.5} + 0.75225t^{1.5} + 0.3009t^{2.5} + 8.59717t^{3.5}) +$$

$$J^{0.5}\left[\frac{1}{\Gamma(0.5)}\int_a^t (t-s)^{-0.5}u(s)\,ds\right]$$

and therefore:

$$u_0(t) = u(0) + J^{0.5}(1.12838t^{0.5} + 0.75225t^{1.5} + 0.3009t^{2.5} + 8.59717t^{3.5})$$

$$= t + 0.5\,t^2 + 0.16667\,t^3 + 4.16667\times10^{-2}t^4$$

$$u_1(t) = J^{0.5}\left[\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}u_0(s)\,ds\right]$$

$$= t + 0.5\,t^2 + 0.16667\,t^3 + 4.16667\times10^{-2}t^4 + 8.33333\times10^{-3}t^5$$

$\vdots$

Hence:

$u(t) = u_0(t) + u_1(t) + \ldots$

Also, the absolute error is evaluated, and the following notations for the absolute error will be used:

$E_3 = |u(t) - \varphi_3(t)|,\ E_4 = |u(t) - \varphi_4(t)|$

where:

$$\varphi_n(t) = \sum_{i=0}^{n-1} u_i(t), \; n \geq 1$$

and the exact solution is $u(t) = \lim_{n\to\infty} \varphi_n(t)$.

A comparison between the approximate and the exact solutions is given in table (3.10).

Table (3.10)

Exact and approximate results of example (3.9).

t	*Exact solution*	*ADM φ_3*	*E_3*	*ADM φ_4*	*E_4*
0	0	0	0	0	0
0.1	0.11050	0.11051	399861×10^{-6}	0.11051	2.52798×10^{-7}
0.2	0.24400	0.24420	6.12444×10^{-5}	0.24427	8.18032×10^{-6}
0.3	0.40350	0.40455	2.95988×10^{-4}	0.40491	6.2818×10^{-5}
0.4	0.59200	0.59538	8.90311×10^{-4}	0.59653	2.67703×10^{-4}
0.5	0.81250	0.82086	2.06163×10^{-3}	0.82374	8.26203×10^{-4}
0.6	1.06800	1.08856	4.0392×10^{-3}	1.09168	2.07915×10^{-3}
0.7	1.36150	1.30448	7.0396×10^{-3}	1.40606	4.54489×10^{-3}
0.8	1.69600	1.75302	1.12412×10^{-2}	1.77323	8.96179×10^{-3}
0.9	2.07450	2.161710	1.67579×10^{-2}	2.20018	1.63334×10^{-2}
1	2.50000	2.64306	2.36111×10^{-2}	2.69464	2.79762×10^{-2}

3.3.4 Comparison of the Results:

In this chapter, we have solved the linear fractional integro-differential equation given by eq.(3.10) by applying three approaches; namely, the collocation method, the least square method and the ADM. In table (3.11), the approximate results of the absolute error produced by solving eq.(3.10) using the three approaches are given, and one can see that the collocation give the best results.

Table (3.11)

Comparison of the results of eq.(3.10).

t	The collocation method	The least square method	Adomian decomposition method
0	0	0	0
0.1	6.08×10^{-20}	1.7558×10^{-18}	1.19048×10^{-10}
0.2	8.28×10^{-20}	3.508×10^{-18}	1.52381×10^{-8}
0.3	5.9×10^{-20}	5.252×10^{-18}	2.60357×10^{-7}
0.4	2.4×10^{-20}	6.973×10^{-18}	1.95048×10^{-6}
0.5	1.7×10^{-19}	8.66×10^{-17}	9.30010×10^{-6}
0.6	4.4×10^{-19}	1.028×10^{-17}	3.33257×10^{-5}
0.7	8.3×10^{-19}	1.181×10^{-17}	9.80408×10^{-5}
0.8	1.37×10^{-18}	1.323×10^{-17}	2.49661×10^{-4}
0.9	2.12×10^{-18}	1.449×10^{-17}	5.69401×10^{-4}
1	3.11×10^{-18}	1.56×10^{-17}	1.19048×10^{-3}

Similarly, we have solve the nonlinear fractional integro-differential equa on given by eq.(3.11) by applying the four approaches, which are the collocation method, the least square method, Adomian decomposition method and the modified Adomian decomposition method. In table (3.12), the comparison between the obtained absolute errors are given for each method and one can see that Adomian decomposi on method given by eq.(3.35) and the least square method give the best results.

Table (3.12)

Comparison of the results of eq.(3.11).

t	*The collocation method*	*The least square method*	*Adomian decomposition method*	*Modified Adomian decomposition method*
0	0	0	0	0
0.1	1.78422×10^{-7}	3.56512×10^{-8}	4.34771×10^{-15}	1.199402×10^{-15}
0.2	4.90262×10^{-7}	1.07885×10^{-7}	8.88635×10^{-12}	2.45548×10^{-12}
0.3	7.5896×10^{-7}	1.83108×10^{-7}	7.65381×10^{-10}	2.12228×10^{-10}
0.4	9.67437×10^{-7}	2.47492×10^{-7}	1.79946×10^{-8}	5.01838×10^{-9}
0.5	1.19344×10^{-6}	3.00875×10^{-7}	2.07341×10^{-7}	5.83087×10^{-8}
0.6	1.544803×10^{-6}	3.506602×10^{-7}	1.51925×10^{-6}	4.32077×10^{-7}
0.7	2.09475×10^{-6}	4.05714×10^{-7}	8.13284×10^{-6}	2.34652×10^{-6}
0.8	2.81718×10^{-6}	4.70267×10^{-7}	3.45493×10^{-5}	1.01467×10^{-5}
0.9	3.521904×10^{-6}	5.37811×10^{-7}	1.22837×10^{-4}	3.68508×10^{-5}
1	3.79×10^{-6}	5.85×10^{-7}	3.78995×10^{-4}	1.16564×10^{-4}

Finally, we have solved the nonlinear fractional integro-differential equation given by eq. (3.13) by applying the three approaches, which are the collocation method, the least square method and the Adomian decomposi on method. In table (3.13), the comparison between the obtained absolute errors are given for each method and one can see that the collocation method and the Adomian decomposition method give the best results.

Table (3.13)

Comparison of the results of eq.(3.13).

t	*The collocation method*	*The least square method*	*Adomian decomposition method*
0	0	0	0
0.1	2.91242×10^{-6}	6.87873×10^{-2}	2.52798×10^{-7}
0.2	8.31221×10^{-6}	8.91158×10^{-2}	8.18032×10^{-6}
0.3	8.34559×10^{-5}	8.11765×10^{-2}	6.2818×10^{-5}
0.4	3.68465×10^{-4}	6.24319×10^{-2}	2.67703×10^{-4}
0.5	1.14049×10^{-3}	4.63282×10^{-2}	8.26203×10^{-4}
0.6	2.87964×10^{-3}	4.10065×10^{-2}	2.07915×10^{-3}
0.7	6.33362×10^{-3}	4.8015×10^{-2}	4.54489×10^{-3}
0.8	1.25828×10^{-2}	6.10206×10^{-2}	8.96179×10^{-3}
0.9	2.31054×10^{-2}	6.45209×10^{-2}	1.63334×10^{-2}
1	3.98423×10^{-2}	3.25559×10^{-2}	2.79762×10^{-2}

Conclusions and Recommendations

From the present study, the following conclusions may be drown:

1. In these examples, we solving linear fractional integro-differential equations, the collocation method gives more accurate results than the least square method and Adomian decomposition method.
2. In solving nonlinear fractional integro-differential equations, the Adomian decomposition method gives more accurate results than the collocation method and the least square method.
3. The solved problems in this thesis are all solved for the closed interval [0,1] and with fractional order of integration and differentiation are taken to be equal to q, which may be generalized in a straightforward manner for any interval [a, b] and for arbitrary order of fractional integration an differentiation which may be not equal.

Also, from this study, we can recommend the following open problems for future work:

1. Studying, theoretically and numerically, singular fractional integro-differential equations.
2. Using other numerical and approximate methods for solving fractional integro-differential equations, such as trapezoidal rule, Simpson's rule, Bool's rule, Weddel's rule, Bellman rule, etc.

3. Studying the statement and the proof of the existence and uniqueness theorem of solution of fractional integro-differential equations using other fixed point theorems.
4. Studying fractional integro-differential equations resulting from other types of fractional integro-differential operators, linear or nonlinear.
5. Studying the behaviour and solution of real life problems, which may be modeled as fractional integro-differential equations.

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المستخلص

الهدف الرئيس من هذه الرسالة هودراسة المعادلات التكاملية-التفاضلية الكسرية باستخدام نوع محور من المؤثرات، ***(Fractional Integro-Differential Equations)*** والتي تضمنت نفس الرتب للمشتقات الكسرية والتكاملات الكسرية. كذلك الهدف من هذا العمل هو دراسة و برهنة مبرهنة وجود وحدانية حل المعادلات التكاملية-التفاضلية الكسرية، ثم دراسة الحلول التقريبية لهكذا نوع من المعادلات باستخدام طريقة الحشد ***(The Collocation Method)*** و طريقة المربعات الصغرى ***(The Least Square Method)*** وطريقة ادومين للتحليل ***(Adomian Decomposition Method)*** و طريقة متتاليات ادومين المحورة ***(Modification of Adomian's Polynomials Method)*** قدمناها مع الامثلة التوضيحية وقارنا بين النتائج.