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Chebyshev Series Methods for Solving Some Linear Problems

A Thesis

Submitted to the College of Science\ Al-Nahrain University as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

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بِسْمِ اللهِ الرَّحْمَنِ الرَّحِيْمِ

نْ وَالقَلَمِ وَمَا يَسْطِرُوْنَ

صَدَقَ الله العَظِيْم

سورة القلم(الآية ١)

الإهداء

إلى زهرة كانت بشذاها تعطر حياتي وذبلت أوراقها...

إلى شمعة كانت بنورها تضئ دربي وأنطفئ نورها...

إلى أعز الناس أمي "رحمها الله"...

نور نبيل القيسي

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Noor Al-Qayssi
September 2008

Supervisor Certification

I certify that this thesis was prepared under my supervision at the Al-Nahrain University, College of Science, Department of Mathematics and Computer Applications in partial fulfillment of as of the requirements for the degree of Master of Science in Mathematics.

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Date: / /2008

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Examining Committee's Certification

We certify that we read this thesis entitied "***Chebyshev Series Methods for Solving Some Linear Problems***" and as examining committee examined the student, ***Noor Nabeel Mahmood Al-Qayssi*** in its contents and in what it connected with, and that is in our opinion it meet the standard of thesis for the degree of Master of Science, in Applied Mathematics.

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Abstract

The main purpose of this work may be divided into the following aspects:

1. Study the Chebyshev polynomials of the first and second kinds defined on the intervals [0,1] and [-1,1] and modify some of their properties.
2. Use two methods to solve the linear ordinary differential equations with non-constant coefficients, namely, Chebyshev-matrix method and Chebyshev series method.
3. Devote Chebyshev series method to solve system of linear Fredholm integral equations and integro-differential equations.

Contents

Introduction

Most areas of numerical analysis as well as many other areas of mathematics as a whole make use of the Chebyshev polynomials. In several areas of mathematics polynomial approximation, numerical integration, and pseudospectral methods for ordinary and partial differential equations, the Chebyshev polynomials take a significant role, the following quote has been attributed to a number of distinguished mathematicians. Hence a Chebyshev series can be expected to converge more rapidly than any other polynomial series. A Chebyshev series also generally converges more rapidly than Fourier series particularly for a function which is not truly periodic, [Sarra A., 2005].

Many authors and researchers studied the Chebyshev polynomials such as [Van der pol., 1934] who gave the definition of the inverse Chebyshev series. Also has been decomposed into partial fractions, [Clenshaw C., 1957] gave the numerical solutions of the linear differential equations by using Chebyshev series, [Muranghan F., 1959] used the near-minimax properties of the Chebyshev series, [Veidinger L., 1960] studied the numerical determination of the best approximations in the Chebyshev sense, [Clenshaw C., 1960] proposed almost 40 years ago a quadrature scheme for finding the integral of non-singular function defined on a finite range by expanding the integrand in a series of Chebyshev polynomials and integrating this series term by term, [Elliott D., 1961] solved heat equations by using Chebyshev series for the numerical integration, [Elliott D., 1964] discussed the evaluation and estimation of the coefficients in the Chebyshev series expansion of a Legender function, [Fox L., 1965] studied least squares approximation method using Chebyshev polynomials to solve second order ordinary differential equation and gave some properties of Chebyshev series expansion, [Smith L., 1966] gave an algorithm for finding the inverse polynomial with Gauss-type Chebyshev quadratures, [Mason J., 1967] developed a numerical method for solving heat equations by using Chebyshev series, [Fox L., 1968] studied some important properties of the Chebyshev polynomials, [Mason J., 1969] solved special types of linear partial differential equations via Chebyshev series, [Kin L., 1970] studied high-precision Chebyshev series approximation to the exponential integral, [Knibb D., 1971] solved parabolic equations by using Chebyshev series, [Broucke R., 1973] studied some approximated methods by truncation of the Chebyshev series expansion of an inverse polynomial of degree k in power series, [Boateng G., 1975] solved parabolic partial differential equations by using Chebyshev collection method, [Alwar R., 1976] gave some of the application of Chebyshev polynomials to the nonlinear analysis of circular plates, [Gattieb D., 1977] discussed Chebyshev spectral methods, [Doha E., 1979] discussed some of Chebyshev methods for finding the numerical solution of the third boundary value problem parabolic partial differential equations, [Evans D., 1981] solved biharmonic equation in a rectangular region by using Chebyshev series, [Gemignani L., 1997] gave some algorithms for Chebyshev rational interpolation, [Mihaila B., 1998] compared the solution of linear and non-linear second order ordinary differential equation obtained using the proposed Chebyshev method with numerical solution obtained using the finite-difference method, [Pakhshan M., 1999] solved the linear Fredholm integral equation of the second kind via Chebyshev polynomials, [Nath Y., 2000] solved special types of nonlinear partial differential equations by using a quadratic Chebyshev polynomials extrapolation technique, [John P., 2000] studied Chebyshev and Fourier spectral methods, [Glader C., 2001] discussed method for rational Chebyshev approximation of rational functions on the unit disk and on the unit interval, [El-kady M., 2002] solved nonlinear optimal control problems by using Chebyshev expansion method, [Rababah A., 2003] proved that the shifted Chebyshev polynomials are orthogonal over [0,1], [Bulyshev Y., 2003] gave some new properties of the Chebyshev polynomials used analysis and design of dynamic system, [Mace R., 2005] discussed padé method reconstruct the Chebyshev polynomial approximation as a rational approximation, [Ramos H., 2007] presented a method based on Chebyshev approximation for solving the second order ordinary differential equation.

The aim of this work is to study Chebyshev polynomials of the first and second kinds defined on the intervals [0,1] and [-1,1]. Also some properties of such polynomials are discussed and developed. Moreover, we use the integral properties of Chebyshev polynomials of the first kind defined on [0,1] to produce a method for solving the boundary value problems for linear ordinary differential equations with non-constant coefficients and systems of linear Fredholm integral and integro-differential equations.

Another method, named as Chebyshev-matrix method for solving these problems is utilized. It depends on the product property of the Chebyshev polynomials of the first kind defined on [0,1].

This thesis consist of three chapters:

In chapter one, we give two definitions of the Chebyshev polynomials of the first and second kinds defined on [0,1] and [-1,1] with some of their important properties.

In chapter two, two approximate methods; namely Chebyshev series and Chebyshev-matrix methods are discussed to solve boundary value problems of the linear ordinary differential equations with non-constant coefficients.

In chapter three, we use Chebyshev series method to solve systems of linear Fredholm integral and integro-differential equations. This method depend on some properties of the Chebyshev polynomials of the first kind defined on [0,1].

Chapter One

The Chebyshev Polynomials

Introduction:

The aim of this chapter is to give the definitions of the Chebyshev polynomials of the first and second kinds defined on the intervals [0,1] and [-1,1]. Also some important properties of these polynomials are presented and developed. Most of these properties are necessary for practical applications to be discussed later.

Definition (1.1), [Fox L. and Parker I., 1968]:

The Chebyshev polynomial of the first kind defined on [-1,1], denoted by $T_r(x)$, is defined by:

$$T_r(x) = \cos(r\theta),\ \cos\theta = x,\ -1 \le x \le 1$$

where $r = 0, 1, \ldots$

Remark (1.1):

Since $\cos(r\theta) = \cos(-r\theta)$, $r=0,1,\ldots$, then one can define the Chebyshev polynomials of the first kind $T_{-r}(x)$ defined on [-1,1], by:

$$T_{-r}(x) = T_r(x),\ r = 0, 1, \ldots$$

Definition (1.2), [Fox L. and Parker I., 1968]:

The Chebyshev polynomial of the first kind defined on [0,1], denoted by $T_r^*(x)$, is defined by:

$$T_r^*(x) = T_r(2x - 1),\ 0 \le x \le 1$$

where $r = 0, 1, \ldots$

Definition (1.3), [Fox L. and Parker I., 1968]:

The Chebyshev polynomial of the second kind defined on [-1,1], denoted by $U_r(x)$, is defined by:

$$U_r(x) = \frac{\sin[(r+1)\theta]}{\sin\theta},\ \cos\theta = x\,,\ -1 \le x \le 1$$

where $r = 0,1,....$

Remarks (1.2), [Fox L. and Parker I., 1968]:

It is easy to check that

(i) $\frac{d^2T_r}{d\theta^2} + r^2T_r = 0,\ r = 0,1,....$

(ii) $\frac{d^2U_r}{d\theta^2} + \frac{2\cos\theta}{\sin\theta}\frac{dU_r}{d\theta} + (r^2 + 2r)U_r = 0,\ r = 0,1,....$

Now, the following proposition is appeared in [Fox L. and Parker I., 1968] without proof. Here we give its proofs.

Proposition (1.1):

(i) $T_r\,(1) = 1$ for each $r = 0\,,1\,,....$

(ii) $T_r(-1) = \begin{cases} 1 & r \text{ is an even positive integer} \\ -1 & r \text{ is an odd positive integer} \end{cases}$

Proof:

(i) Assume $x=1$, then $\cos\theta=1$ and hence $\theta=2n\pi,\ n = 0\,,\mp 1\,,\mp 2\,,....$

Therefore $T_r\,(1) = \cos(2nr\pi) = 1$ for $r = 0\,,1\,,....$

(ii) Assume $x=-1$, then $cos(\theta)=-1$ and hence $\theta=(2n+1)\pi$,

$n=0,\mp 1,\mp 2,....$

Therefore $T_r(-1)=cos[r(2n+1)\pi]=\begin{cases} 1 & r \text{ is an even positive integer} \\ -1 & r \text{ is an odd positive integer} \end{cases}$

Next, the following proposition is appeared in [Fox L. and Parker I., 1968] without proof. Here we give its proof.

Proposition (1.2):

(i) $T_{r+1}(x)=2x\,T_r(x)-T_{r-1}(x),\ r=1,2,...$ (1.1)

(ii) $T_s(x)T_r(x)=\frac{1}{2}\{T_{s+r}(x)+T_{s-r}(x)\},\ s\geq r,\ r,s=0,1,...$ (1.2)

(iii) $T_s(T_r(x))=T_r(T_s(x))=T_{rs}(x),\ r,s=0,1,....$

Proof:

(i) From the trigonometric identity

$cos[(r+1)\theta]+cos[(r-1)\theta]=2cos(r\theta)\ cos(\theta),\ r=1,2,...$

one can have

$T_{r+1}(x)+T_{r-1}(x)=2T_r(x)\ T_1(x),\ r=1,2,....$

Therefore

$T_{r+1}(x)=2T_1(x)\ T_r(x)-T_{r-1}(x).$

But $T_1(x)=x$, hence

$T_{r+1}(x)=2xT_r(x)-T_{r-1}(x),\ r=1,2,....$

(ii) From the trigonometric identity

$cos[(r+s)\theta]+cos[(r-s)\theta]=2cos(r\theta)cos(s\theta)$

one can have the desired result.

(iii)

$$T_s\left(T_r(x)\right) = \cos\left[s\cos^{-1}\left(\cos(r\cos^{-1}x)\right)\right]$$
$$= \cos\left(sr\cos^{-1}x\right)$$
$$= T_{sr}(x)$$

Remarks (1.3), [Fox L. and Parker I., 1968]:

(1) From the replacement of x by $(2x-1)$ in equations (1.1) and (1.2) one can have:

$$T^*_{r+1}(x) = 2(2x-1)T^*_r(x) - T^*_{r-1}(x),\ r = 1, 2, .. \qquad (1.3)$$

where

$T_0 = 1$, $T_1 = 2x - 1$

and

$$T^*_s(x)T^*_r(x) = \frac{1}{2}\left[T^*_{s+r}(x) + T^*_{s-r}(x)\right],\ s \geq r,\ r, s = 0, 1, .. \qquad (1.4)$$

respectively.

(2) Since $T_r(T_2(x)) = T_{2r}(x)$, then

$T_r(2x^2-1) = T_{2r}(x)$ and this implies that $T_r(2x-1) = T_{2r}(x^{\frac{1}{2}})$.

But $T_r(2x-1) = T^*_r(x)$, therefore $T^*_r(x) = T_{2r}(x^{\frac{1}{2}})$.

Next, the following proposition give an alternative definition of the Chebyshev polynomial of the first kind defined on [-1,1].

Proposition (1.3), [Fox L. and Parker I., 1968]:

$$T_r(x) = \frac{r}{2}\sum_{k=0}^{r/2} \frac{(-1)^k (r-k-1)!}{k!(r-2k)!}(2x)^{r-2k}, \; r = 1, 2, \ldots \qquad (1.5)$$

Proof:

Consider

$$\sum_{r=0}^{\infty} p^r e^{ir\theta} = \sum_{r=0}^{\infty}\left(pe^{i\theta}\right)^r, \; |p| < 1$$

$$= \left(1 - pe^{i\theta}\right)^{-1} = \{1 - p(\cos\theta + i\sin\theta)\}^{-1}$$

$$= \{1 - p\cos\theta - i\,p\sin\theta\}^{-1} = \left\{1 - px - ip(1-x^2)^{\frac{1}{2}}\right\}^{-1}$$

$$= \frac{1 - px + ip\left(1-x^2\right)^{\frac{1}{2}}}{(1-px)^2 + p^2(1-x^2)}$$

By taking the real part of the above equation one can deduce that

$$\sum_{r=0}^{\infty} p^r \cos(r\theta) = \frac{1-px}{(1-px)^2 + p^2(1-x^2)} = \frac{1-px}{1-2px+p^2}$$

But $T_r(x) = cos(r\theta)$, $r=0,1,\ldots$. Therefore

$$\sum_{r=0}^{\infty} p^r T_r(x) = \frac{1-px}{1-2px+p^2}$$

Hence

$$\sum_{r=0}^{\infty} p^r T_r(x) = (1-px)\left[1-\left(2px-p^2\right)\right]^{-1}$$

$$= (1-px)\left[1+\left(2px-p^2\right)+\left(2px-p^2\right)^2+\cdots\right]$$

$$= \left[1-\frac{1}{2}(2xp)\right]\left[1+p(2x-p)+p^2(2x-p)^2+\cdots\right]$$

and by evaluating the coefficient of p^r on the right-hand side one can get the required result.

Next, the following proposition is appeared in [Fox L. and Parker I., 1968] without proof. Here we give its proof.

Proposition (1.4):

$$T_r^*(x) = \frac{1}{2}\left[2^{2r}x^r - \left\{2\binom{2r-1}{1} - \binom{2r-2}{1}\right\}2^{2r-2}x^{r-1} + \left\{2\binom{2r-2}{2} - \binom{2r-3}{2}\right\}2^{2r-4}x^{r-2} - \cdots\right],\ r = 1,2,.. \qquad (1.6)$$

Proof:

By replacing x by $x^{\frac{1}{2}}$ and r by $2r$ in proposition (1.3) one can get:

$$T_{2r}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}\left[(2x^{\frac{1}{2}})^{2r} - \left\{2\binom{2r-1}{1} - \binom{2r-2}{1}\right\}(2x^{\frac{1}{2}})^{2r-2} + \left\{2\binom{2r-2}{2} - \binom{2r-3}{2}\right\}(2x^{\frac{1}{2}})^{2r-4} - \cdots\right]$$

Thus

$$T_{2r}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}\left[2^{2r}x^{r}-\left\{2\binom{2r-1}{1}-\binom{2r-2}{1}\right\}2^{2r-2}x^{r-1}+\right.$$

$$\left.\left\{2\binom{2r-2}{2}-\binom{2r-3}{2}\right\}2^{2r-4}x^{r-2}-\cdots\right]$$

and from the fact that $T_r^*(x)=T_{2r}(x^{\frac{1}{2}})$ one can get the desired result.

Next, the following proposition gives an alternative definition of the Chebyshev polynomial of the second kind defined on [-1,1].

Proposition (1.5):

$$U_r(x)=\sum_{k=0}^{r/2}\frac{(-1)^k(r-k)!}{k!(r-2k)!}(2x)^{r-2k},\ \ r=0,1,\ldots \qquad (1.7)$$

Proof:

Consider

$$\sum_{r=0}^{\infty}p^r e^{ir\theta}=\sum_{r=0}^{\infty}\left(pe^{i\theta}\right)^r=\left(1-pe^{i\theta}\right)^{-1}=\frac{1-px+ip\left(1-x^2\right)^{\frac{1}{2}}}{1-2px+p^2}$$

By taking the imaginary part of the above equation one can deduce that

$$\sum_{r=0}^{\infty}p^r\sin\left(r\theta\right)=\frac{p(1-x^2)^{\frac{1}{2}}}{1-2px+p^2}$$

But

$$U_r(x)=\frac{\sin\left[(r+1)\theta\right]}{\sin\theta},\ r=0,1,\ldots.$$

Therefore

$$\sum_{r=1}^{\infty} p^r U_{r-1}(x) = p\left[1 - p\left(2x - p\right)\right]^{-1}$$

$$= p + p^2\left(2x - p\right) + p^3\left(2x - p\right)^2 + \cdots$$

and by evaluating the coefficient of p^r on the right-hand side one can get the required result.

Next, the following propositions give the relation between the Chebyshev polynomials of the first and second kinds. They appeared in [Fox L. and Parker I., 1968] without proof. Here we give their proofs.

Proposition (1.6):

(i) $U_{s-1}(x)U_{r-1}(x) = \frac{1}{2(1-x^2)}\left\{T_{s-r}(x) - T_{s+r}(x)\right\}, x \neq \mp 1, s \geq r, r, s = 1, 2, .$

(ii) $T_r(x)U_{s-1}(x) = \frac{1}{2}\left\{U_{s+r-1}(x) + U_{s-r-1}(x)\right\},\ s \geq r,\ r = 0, 1, ..., s = 1, 2, ...$

(iii) $U_{r-1}\left\{T_s(x)\right\}U_{s-1}(x) = U_{s-1}\left\{T_r(x)\right\}U_{r-1}(x) = U_{rs-1}(x),\ r, s = 1, 2, .$

Proof:

(i) From the trigonometric identity

$$\cos\left[\left(s-r\right)\theta\right] - \cos\left[\left(s+r\right)\theta\right] = 2\sin\left(s\theta\right)\ \sin\left(r\theta\right)$$

one can have:

$$T_{s-r}(x) - T_{s+r}(x) = 2sin(s\theta)\ sin(r\theta)$$
$$= \frac{2\sin(s\theta)\sin(r\theta)}{2\sin^2\theta}(2\sin^2\theta)$$
$$= \frac{\sin(s\theta)}{\sin\theta}\frac{\sin(r\theta)}{\sin\theta}\left[2(1-x^2)\right]$$
$$= U_{s-1}(x)U_{r-1}(x)\left[2(1-x^2)\right],\ s \ge r,\ r,s = 1,2,....$$

Hence

$$U_{s-1}(x)U_{r-1}(x) = \frac{1}{2(1-x^2)}\left[T_{s-r}(x) - T_{s+r}(x)\right].\ s \ge r,\ r,s = 1,2,....$$

(ii) From the trigonometric identity

$$sin\left[(s+r)\theta\right] + sin\left[(s-r)\theta\right] = 2sin(s\theta)\ cos(r\theta),\ s \ge r$$

one can have

$$\frac{\sin\left[(s+r)\theta\right]}{\sin\theta} + \frac{\sin\left[(s-r)\theta\right]}{\sin\theta} = 2\frac{\sin(s\theta)}{\sin\theta}\cos(r\theta).$$

Therefore

$$U_{s+r-1}(x) + U_{s-r-1}(x) = 2U_{s-1}(x)T_r(x),\ s \ge r,\ r = 0,1,...,\ s = 1,2,....$$

(iii) Consider

$$U_{r-1}\{T_s(x)\} = \frac{\sin(r\theta)}{\sin\theta},\ \cos\theta = T_s(x)$$

and

$$U_{s-1}(x) = \frac{\sin(s\theta_1)}{\sin\theta_1},\ \cos\theta_1 = x$$

But $T_s(x) = \cos(s\theta_2),\ \cos\theta_2 = x$

Therefore $\theta_1 = \theta_2 + 2n\pi,\ n = 0, \mp 1, \mp 2, ...$ and $\cos\theta = \cos(s\theta_2)$.

Thus $\theta = s\theta_2 + 2n\pi,\ n = 0, \mp 1, \mp 2, ...$

Hence

$$U_{r-1}\{T_s(x)\}U_{s-1}(x)=\left(\frac{\sin\left[r(s\theta_2+2n\pi)\right]}{\sin(s\theta_2+2n\pi)}\right)\left(\frac{\sin\left[s(\theta_2+2n\pi)\right]}{\sin(\theta_2+2n\pi)}\right)$$

$$=\left(\frac{\sin(rs\theta_2)}{\sin(s\theta_2)}\right)\left(\frac{\sin(s\theta_2)}{\sin(\theta_2)}\right)$$

$$=\frac{\sin(rs\theta_2)}{\sin\theta_2}$$

$$=U_{rs-1}(x),\ \cos\theta_2=x,\ r,s=1,2,....$$

In a similar manner one can prove that $U_{s-1}\{T_r(x)\}\,U_{r-1}(x)=U_{rs-1}(x)$.

Proposition (1.7):

(i) $U_{2s-1}(x^{\frac{1}{2}})U_{2r-1}(x^{\frac{1}{2}})=\dfrac{1}{2(1-x)}\left\{T^*_{s-r}(x)-T^*_{s+r}(x)\right\},x\neq 1,s\geq r,r,s=1,2,....$

(ii) $T^*_r(x)U_{s-1}(2x-1)=\dfrac{1}{2}\{U_{s+r-1}(2x-1)+U_{s-r-1}(2x-1)\},s\geq r,r=0,1,...,s=1,2,....$

(iii) $U_{r-1}\left\{T^*_s(x)\right\}U_{s-1}(2x-1)=U_{s-1}\left\{T^*_r(x)\right\}U_{r-1}(2x-1)$

$$=U_{rs-1}(2x-1),\ r,s=1,2,....$$

Proof:

(i) by replacing s with $2s$, r with $2r$ and x with $x^{\frac{1}{2}}$, in proposition (1.6),(i) one can get:

$$U_{2s-1}(x^{\frac{1}{2}})U_{2r-1}(x^{\frac{1}{2}})=\frac{1}{2(1-x)}\left\{T_{2s-2r}(x^{\frac{1}{2}})-T_{2s+2r}(x^{\frac{1}{2}})\right\}.$$

But

$$T_{2r}(x^{\frac{1}{2}})=T^*_r(x),$$

$$T_{2s-2r}(x^{\frac{1}{2}})=T_{2(s-r)}(x^{\frac{1}{2}})=T^*_{s-r}(x)$$

and

$$T_{2s+2r}(x^{\frac{1}{2}}) = T_{2(s+r)}(x^{\frac{1}{2}}) = T^*_{s+r}(x).$$

Therefore

$$U_{2s-1}(x^{\frac{1}{2}})U_{2r-1}(x^{\frac{1}{2}}) = \frac{1}{2(1-x)}\left\{T^*_{s-r}(x) - T^*_{s+r}(x)\right\}$$

(ii) By replacing x by $2x-1$ in proposition (1.6),(ii) we get the result.

(iii) By replacing x by $2x-1$ in proposition (1.16),(iii) we get the result.

Now, the following three propositions describe any positive powers of x as a linear combinations of the Chebyshev polynomials of the first and second kinds respectively. These propositions appeared in [Fox L. and Parker I., 1968] without proofs. Here we give their proofs.

Proposition (1.8):

$$x^s = \frac{1}{2^{s-1}}\left\{T_s(x) + \binom{s}{1}T_{s-2}(x) + \binom{s}{2}T_{s-4}(x) + \cdots\right\},\ s = 1, 2, . \qquad (1.8)$$

with a factor $\frac{1}{2}$ associated with the coefficient of $T_0(x)$ for even s.

Proof:

The proof is followed from the mathematical induction.

For $s=1$, $x^s=x$ and $\frac{1}{2^{s-1}}\{T_s(x)\} = T_1(x) = x$.

Therefore equation (1.8) is true for $s=1$.

For $s=2$, $x^s=x$ and

$$\frac{1}{2^{s-1}}\left\{T_s(x)+\binom{s}{1}T_{s-2}(x)\right\}=\frac{1}{2}\left\{T_2(x)+\frac{1}{2}\binom{2}{1}T_0(x)\right\}$$
$$=\frac{1}{2}\left\{2x^2-1+1\right\}=x^2.$$

Therefore equation (1.8) is true for $s=2$.

Assume equation (1.8) is true for $s=k$. That is

$$x^k=\frac{1}{2^{k-1}}\left\{T_k(x)+\binom{k}{1}T_{k-2}(x)+\binom{k}{2}T_{k-4}(x)+\cdots\right\}$$

Then by using proposition (1.2),(ii) one can have:

$$x^{k+1}=\frac{1}{2^{k-1}}\left\{T_k(x)+\binom{k}{1}T_{k-2}(x)+\binom{k}{2}T_{k-4}(x)+\cdots\right\}T_1(x)$$
$$=\frac{1}{2^k}\left\{T_{k+1}(x)+T_{k-1}(x)+\binom{k}{1}T_{k-1}(x)+\binom{k}{1}T_{k-3}(x)+\binom{k}{2}T_{k-3}(x)+\right.$$
$$\left.\binom{k}{2}T_{k-5}(x)+\cdots\right\}$$
$$=\frac{1}{2^k}\left\{T_{k+1}(x)+\left[1+\binom{k}{1}\right]T_{k-1}(x)+\left[\binom{k}{1}+\binom{k}{2}\right]T_{k-3}(x)+\right.$$
$$\left.\left[\binom{k}{2}+\binom{k}{3}\right]T_{k-5}(x)+\cdots\right\}$$

But

$$\binom{k}{i}+\binom{k}{i+1}=\binom{k+1}{i+1},\ i=0,1,....$$

Therefore

$$x^{k+1}=\frac{1}{2^k}\left\{T_{k+1}(x)+\binom{k+1}{1}T_{k-1}(x)+\binom{k+1}{2}T_{k-3}(x)+\binom{k+1}{3}T_{k-5}(x)+\cdots\right\}$$

and hence equation (1.8) is true for $s=k+1$.

Proposition (1.9):

$$x^s = \frac{1}{2^{2s-1}}\left[T_s^*(x) + \binom{2s}{1}T_{s-1}^*(x) + \binom{2s}{2}T_{s-2}^*(x) + \cdots\right],\ s = 1,2,. \qquad (1.9)$$

with a factor $\frac{1}{2}$ associated with the coefficient of $T_0^*(x)$.

Proof:

From the replacement of x by $x^{\frac{1}{2}}$ and s by $2s$ in equation (1.8) one can have:

$$(x^{\frac{1}{2}})^{2s} = \frac{1}{2^{2s-1}}\left\{T_{2s}(x^{\frac{1}{2}}) + \binom{2s}{1}T_{2s-2}(x^{\frac{1}{2}}) + \binom{2s}{2}T_{2s-4}(x^{\frac{1}{2}}) + \cdots\right\}$$

But

$$T_{2s}(x^{\frac{1}{2}}) = T_s^*(x), T_{2s-2}(x^{\frac{1}{2}}) = T_{2(s-1)}(x^{\frac{1}{2}}) = T_{s-1}^*(x)$$

$T_{2s-4}(x^{\frac{1}{2}}) = T_{2(s-2)}(x^{\frac{1}{2}}) = T_{s-2}^*(x)$ and so on.

Therefore

$$x^s = \frac{1}{2^{2s-1}}\left\{T_s^*(x) + \binom{2s}{1}T_{s-1}^*(x) + \binom{2s}{2}T_{s-2}^*(x) + \cdots\right\},\ s = 1,2,. ..$$

Proposition (1.10):

$$x^s = \frac{1}{2^s}\left[U_s(x) + \left\{\binom{s}{1} - \binom{s}{0}\right\}U_{s-2}(x) + \left\{\binom{s}{2} - \binom{s}{1}\right\}U_{s-4}(x) + \cdots\right],\ s = 0,1,. . . \qquad (1.10)$$

Proof:

The proof is followed by the mathematical induction.

For s=0, x^s=1 and $\frac{1}{2^s}\{U_s(x)\}=U_0(x)=1.$

Therefore equation (1.10) is true for $s=0$.

For s=1, x^s=x and

$$\frac{1}{2^s}\{U_s(x)\}=\frac{1}{2}U_1(x)=x\,.$$

Therefore equation (1.10) is true for $s=1$.

Assume equation (1.10) is true for $s=k$. That is

$$x^k=\frac{1}{2^k}\left[U_k(x)+\left\{\binom{k}{1}-\binom{k}{0}\right\}U_{k-2}(x)+\left\{\binom{k}{2}-\binom{k}{1}\right\}U_{k-4}(x)+\cdots\right]$$

Then by using proposition (1.6),(ii) one can have:

$$x^{k+1}=\frac{1}{2^k}\left[U_k(x)+\left\{\binom{k}{1}-\binom{k}{0}\right\}U_{k-2}(x)+\left\{\binom{k}{2}-\binom{k}{1}\right\}U_{k-4}(x)+\cdots\right]T_1(x)$$

$$=\frac{1}{2^{k+1}}\left[U_{k+1}(x)+U_{k-1}(x)+\left\{\binom{k}{1}-\binom{k}{0}\right\}U_{k-1}(x)+\left\{\binom{k}{1}-\binom{k}{0}\right\}U_{k-3}(x)+\right.$$

$$\left.\left\{\binom{k}{2}-\binom{k}{1}\right\}U_{k-3}(x)+\left\{\binom{k}{2}-\binom{k}{1}\right\}U_{k-5}(x)+\cdots\right]$$

But

$$1+\binom{k}{1}-\binom{k}{0}=k=\binom{k+1}{1}-\binom{k+1}{0}$$

and

$$\binom{k}{i+2}-\binom{k}{i}=\binom{k+1}{i+2}-\binom{k+1}{i+1},\ i=0,1,....$$

Therefore

$$x^{k+1} = \frac{1}{2^{k+1}}\left[U_{k+1}(x) + \left\{\binom{k+1}{1} - \binom{k+1}{0}\right\}U_{k-1}(x) + \left\{\binom{k+1}{2} - \binom{k+1}{1}\right\}U_{k-3}(x) + \cdots\right]$$

and hence equation (1.10) is true for $s=k+1$.

Next, the following three propositions appeared in [Fox L. and Parker I., 1968] without proof. Here we give their proof.

Proposition (1.11):

$$x^s T_r(x) = \frac{1}{2^s}\sum_{i=0}^{s}\binom{s}{i}T_{r-s+2i}(x),\ r,s=0,1,... \qquad (1.11)$$

Proof:

From equation (1.8) and equation (1.2) one can have:

$$x^s T_r(x) = \frac{1}{2^{s-1}}\left\{T_s(x) + \binom{s}{1}T_{s-2}(x) + \binom{s}{2}T_{s-4}(x) + \cdots\right\}T_r(x)$$

$$= \frac{1}{2^{s-1}}\left\{T_s(x)T_r(x) + \binom{s}{1}T_{s-2}(x)T_r(x) + \binom{s}{2}T_{s-4}(x)T_r(x) + \cdots\right\}$$

$$= \frac{1}{2^{s-1}}\left\{\frac{1}{2}\left[T_{r+s}(x) + T_{r-s}(x)\right] + \frac{1}{2}\binom{s}{1}\left[T_{r+s-2}(x) + T_{r-s+2}(x)\right] + \cdots\right\}$$

$$= \frac{1}{2^s}\sum_{i=0}^{s}\binom{s}{i}T_{r-s+2i}(x),\ \ r,s=0,1,....$$

Proposition (1.12):

$$x^s T_r^*(x) = \frac{1}{2^{2s}}\sum_{i=0}^{2s}\binom{2s}{i}T_{r-s+i}^*(x),\ r,s=0,1,...$$

Proof:

Replacing x by $x^{\frac{1}{2}}$, r by 2r and s by 2s in equation (1.11) one can have

$$x^s T_r^*(x) = x^s T_{2r}(x^{\frac{1}{2}}) = \frac{1}{2^{2s}} \sum_{i=0}^{2s} \binom{2s}{i} T_{2r-2s+2i}(x^{\frac{1}{2}})$$

$$= \frac{1}{2^{2s}} \sum_{i=0}^{2s} \binom{2s}{i} T_{2(r-s+i)}(x^{\frac{1}{2}})$$

$$= \frac{1}{2^{2s}} \sum_{i=0}^{2s} \binom{2s}{i} T_{r-s+i}^*(x), \; r,s = 0,1,....$$

Proposition (1.13):

$$x^s U_r(x) = \frac{1}{2^s} \sum_{i=0}^{s} \binom{s}{i} U_{r+s-2i}(x), \; r,s = 0,1,...$$

Proof:

From equation (1.8) and proposition (1.6),(ii) one can get:

$$x^s U_r(x) = \frac{1}{2^{s-1}} \left[T_s(x) + \binom{s}{1} T_{s-2}(x) + \binom{s}{2} T_{s-4}(x) + \cdots \right] U_r(x)$$

$$= \frac{1}{2^{s-1}} \left[T_s(x) U_r(x) + \binom{s}{1} T_{s-2}(x) U_r(x) + \binom{s}{2} T_{s-4}(x) U_r(x) + \cdots \right]$$

$$= \frac{1}{2^s} \Big[U_{r+s}(x) + U_{r-s}(x) + \binom{s}{1} \{ U_{r+s-2}(x) + U_{r-s+2}(x) \} +$$

$$\binom{s}{2} \{ U_{r+s-4}(x) + U_{r-s+4}(x) \} + \cdots \Big]$$

$$= \frac{1}{2^s} \sum_{i=0}^{s} \binom{s}{i} U_{r+s-2i}(x), \; r,s = 0,1,....$$

Now, the following two propositions describe the integral of the Chebyshev polynomials of the first kind defined on [-1,1] and [0,1] in terms f theirselves in case the constant of integration is taken to be zero.

Proposition (1.14), [Fox L. and Parker I., 1968]:

$$\int T_r(x)dx = \begin{cases} \frac{1}{2}\left\{\frac{1}{r+1}T_{r+1}(x) - \frac{1}{r-1}T_{r-1}(x)\right\}, & r \neq 1 \\ \frac{1}{4}\{T_0(x) + T_2(x)\}, & r = 1 \end{cases}$$

Proof:

Consider

$$\int T_r(x)\ dx = -\int cos(r\theta)\ sin\theta\ d\theta$$

$$= -\frac{1}{2}\int\{sin(r+1)\theta - sin(r-1)\theta\}d\theta$$

Thus

$$\int T_r(x)\ dx = \frac{1}{2}\left\{\frac{1}{r+1}\cos\left[(r+1)\theta\right] - \frac{1}{r-1}\cos\left[(r-1)\theta\right]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{r+1}T_{r+1}(x) - \frac{1}{r-1}T_{r-1}(x)\right\},\ r = 2,3,....$$

Proposition (1.15):

$$\int T_r^*(x)\ dx = \begin{cases} \frac{1}{4}\left[\frac{1}{r+1}T_{r+1}^*(x) - \frac{1}{r-1}T_{r-1}^*(x)\right], & r \neq 1 \\ \frac{1}{8}\left[T_2^*(x) - T_0^*(x)\right], & r = 1 \end{cases}$$

Proof:

We replace x by $2x-1$ in proposition (1.14) to get:

$$\int T_r(2x-1)d(2x-1)=\frac{1}{2}\left\{\frac{1}{r+1}T_{r+1}(2x-1)-\frac{1}{r-1}T_{r-1}(2x-1)\right\}$$

Therefore

$$\int T_r(2x-1)dx=\frac{1}{4}\left\{\frac{1}{r+1}T^*_{r+1}(x)-\frac{1}{r-1}T^*_{r-1}(x)\right\}$$

and this implies that

$$\int T^*_r(x)dx=\frac{1}{4}\left\{\frac{1}{r+1}T^*_{r+1}(x)-\frac{1}{r-1}T^*_{r-1}(x)\right\},\ r=2,3,.$$

Next, the following proposition describes the integral of Chebyshev polynomial of the second kind defined on [-1,1] in terms of itself in case the constant of integration is taken to be zero.

Proposition (1.16):

$$\int U_r(x)dx=\frac{T_{r+1}(x)}{r+1},\ r \text{ is an even positive integer}.$$

Proof:

Consider

$$\int U_r(x)\,dx = \int \frac{\sin[(r+1)\theta]}{\sin\theta}dx$$
$$= \int \frac{\sin[(r+1)\theta]}{\sin\theta}\frac{dx}{d\theta}d\theta$$
$$= \int \frac{\sin[(r+1)\theta]}{\sin\theta}(-\sin\theta)\,d\theta$$
$$= -\int \sin[(r+1)\theta]\,d\theta$$
$$= \frac{\cos[(r+1)\theta]}{r+1}$$
$$= \frac{T_{r+1}(x)}{r+1},\ r \text{ is an even positive integer.}$$

Remarks (1.4):

(i) $$\int_{-1}^{1} T_i(x)T_j(x)dx = \begin{cases} \frac{1}{1-(i+j)^2} + \frac{1}{1-(i-j)^2}, & (i+j) \text{ is an even positive integer} \\ 0, & (i+j) \text{ is an odd positive integer} \end{cases}$$

For the proof, see [Fox L. and Parker I., 1968].

(ii) Since $T_r^*(x) = T_r(2x-1),\ r = 0,1,...,$ then

$$\int_0^1 T_i^*(x)T_j^*(x)dx = \frac{1}{2}\int_{-1}^{1} T_i(x)T_j(x)dx$$
$$= \begin{cases} \frac{1}{2}\left[\frac{1}{1-(i+j)^2} + \frac{1}{1-(i-j)^2}\right], & (i+j) \text{ is an even positive integer} \\ 0, & (i+j) \text{ is an odd positive integer} \end{cases}$$

Next, the following proposition gives the derivative of the Chebyshev polynomials of the first and second kinds. It appeared in [Fox L. and Parker I., 1968] without proof. Here we give its proof.

Proposition (1.17):

(i) $T_r'(x) = rU_{r-1}(x),\ r = 1,\ 2, \ldots.$

(ii) $\left(1-x^2\right)U_r'(x) = x\,U_r(x) - (r+1)T_{r+1}(x).$

(iii) $T_r'(x) = (\mp 1)^{r+1} r^2$ at $x = \mp 1$.

(iv) $T_r^{*\prime}(x) = 2rU_{r-1}(x),\ r = 1, 2, \ldots.$

(v) $2\left(x^2 - x\right)U_r'(2x-1) = (r+1)T_{r+1}^*(x) - (2x-1)U_r(2x-1).$

Proof:

(i) Since $T_r(x) = \cos(r\theta)$, then $T_r'(x) = \dfrac{d}{d\theta}[\cos(r\theta)]\dfrac{d\theta}{dx}$. But $x = cos\theta.$

Thus $\dfrac{d\theta}{dx} = \dfrac{-1}{\sin\theta}$. Hence $T_r'(x) = \dfrac{r\sin(r\theta)}{\sin\theta} = rU_{r-1}(x),\ r = 1, 2, \ldots.$

(ii) From $sin\theta U_r(x) = sin[(r+1)\theta]$, one can have:

$$\cos\theta\, U_r(x)\frac{d\theta}{dx} + \sin\theta\, U_r'(x) = (r+1)\cos\left[(r+1)\theta\right]\frac{d\theta}{dx}.$$

But $\dfrac{d\theta}{dx} = \dfrac{-1}{\sin\theta}$, therefore

$$cos\theta\, U_r(x) - sin^2\theta\, U_r'(x) = (r+1)cos[(r+1)\theta]. \qquad (1.12)$$

Since $x = cos\theta$, $sin^2\theta = 1 - cos^2\theta = 1 - x^2$ and $cos[(r+1)\theta] = T_{r+1}(x)$ hence equation (1.12) becomes

$$x\,U_r(x) - \left(1-x^2\right)U_r'(x) = (r+1)T_{r+1}(x).$$

(iii) It is easy to check that

$$T_r'(x) = r\cos\left[(r-1)\theta\right] + r\cos\left[(r-2)\theta\right]\cos\theta + r\cos\left[(r-3)\theta\right]\cos^2\theta + \cdots +$$
$$r\cos(2\theta)\cos^{(r-3)}\theta + 2r\cos^{(r-1)}\theta\ ,\ r = 3,4,\ldots.$$

For $x=1$, $\theta=2n\pi$, $n = 0,\mp1,\mp2,\ldots$. Therefore

$$T_r'(1) = r\cos\left[2(r-1)n\pi\right] + r\cos\left[2(r-2)n\pi\right]\cos(2n\pi) +$$
$$r\cos\left[2(r-3)n\pi\right]\cos^2 2n\pi + \cdots + r\cos(4n\pi)\cos^{(r-3)}(2n\pi) + 2r\cos^{(r-1)}(2n\pi)$$
$$= \underbrace{r + r + r + \cdots + r}_{r-2 times} + 2r = r^2\ ,\ r = 3,4,\ldots$$

On the other hand, since $T_0(x) = 1$, $T_1(x) = x$ and $T_2(x) = 2x^2 - 1$, thus

$T_0'(x) = 0, T_1'(x) = 1$ and $T_2'(x) = 4x$.

Therefore

$$T_r'(1) = r^2,\ r = 0,1,2.$$

On the other hand, for $x=-1$, $\theta=(2n+1)\pi$, $n = 0,\mp1,\mp2,\ldots.$

Therefore

$$T_r'(-1) = r\cos\left[(r-1)(2n+1)\pi\right] + r\cos\left[(r-2)(2n+1)\pi\right]\cos\left[(2n+1)\pi\right] +$$
$$r\cos\left[(r-3)(2n+1)\pi\right]\cos^2\left[(2n+1)\pi\right] + \cdots +$$
$$r\cos\left[2(2n+1)\pi\right]\cos^{(r-3)}\left[(2n+1)\pi\right] + 2r\cos^{(r-1)}\left[(2n+1)\pi\right].$$

Thus

$$T_r'(-1) = \underbrace{r(-1)^{r+1} + r(-1)^{r+1} + \cdots + r(-1)^{r+1}}_{r-2\ times} + 2r(-1)^{r+1}$$
$$= (-1)^{r+1} r^2\ ,\ r = 3,4,\ldots.$$

On the other hand,

$$T_0'(-1) = 0, T_1'(-1) = 1 = (-1)^2.1^2 \text{ and } T_2'(-1) = -4 = (-1)^3.2^2.$$

(iv) Since $T_r^*(x) = T_r(2x-1) = \cos(r\theta)$, and $\cos(r\theta) = 2x-1$, then

$T_r^{*\prime}(x)=\frac{d}{d\theta}\left[\cos(r\theta)\right]\frac{d\theta}{dx}$. But $\frac{d\theta}{dx}=\frac{-2}{\sin\theta}$, hence

$$T_r^{*\prime}(x)=\frac{2r\sin(r\theta)}{\sin\theta}=2rU_{r-1}(x).$$

(v) We replace x by $2x-1$ in $\sin\theta U_r(x)=\sin[(r+1)\theta]$ to get:

$$\sin\theta\,U_r(2x-1)=\sin\left[(r+1)\theta\right].$$

Therefore

$$\cos\theta\,U_r(2x-1)\frac{d\theta}{dx}+\sin\theta\,U_r'(2x-1)=(r+1)\cos\left[(r+1)\theta\right]\frac{d\theta}{dx}$$

But $\frac{d\theta}{dx}=\frac{-2}{\sin\theta}$, thus

$$\cos\theta\,U_r(2x-1)-\frac{1}{2}\sin^2\theta\,U_r'(2x-1)=(r+1)\cos[(r+1)\theta] \tag{1.13}$$

Since $2x-1=\cos\theta$, $\sin^2\theta = 1-(2x-1)^2=-4x^2-4x$

and $T_{r+1}^*(x)=T_{r+1}(2x-1)=\cos[(r+1)\theta]$, hence equation (1.13) becomes

$$(2x-1)\,U_r(2x-1)+2\left(x^2-x\right)U_r'(2x-1)=(r+1)T_{r+1}^*(x).$$

Now, we give some properties of the finite Chebyshev series of degree N that takes the form:

$$y^*(x)=\sum_{r=0}^{N}{}' a_r T_r(x),\ -1\le x\le 1$$

where the prime denotes that the first term is taken with factor $\frac{1}{2}$ and in which $y^*(x)$ is the truncation of the infinite Chebyshev series: $y(x) = {\sum_{r=0}^{\infty}}' a_r T_r(x), \; -1 \le x \le 1$

where $\{a\}_{r=0}^{\infty}$ are the Chebyshev coefficients. We start by the following remark.

Remark (1.5), [Fox L. and Parker I., 1968]:

Let $x \in [-1,1]$ and consider a function y defined on the interval [-1,1]. Then

$$y(x) = y(\cos\theta) = g(\theta), \; 0 \le \theta \le \pi$$

The function g is even and periodic. The cosine Fourier series takes the form

$$g(\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(k\theta) \qquad (1.14)$$

where

$$a_k = \frac{2}{\pi}\int_0^{\pi} g(\theta)\cos(k\theta)d\theta, \; k = 0,1,....$$

Therefore by interpreting equation (1.14) in terms of the original variable x, we produce the infinite Chebyshev series:

$$y(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k T_k(x)$$

$$= {\sum_{k=0}^{\infty}}' a_k T_k(x)$$

where

$$a_k = \frac{2}{\pi}\int_{-1}^{1}\left(1-x^2\right)^{\frac{-1}{2}} y(x)T_k(x)dx\,,\ k = 0,1,....$$

Theorem (1.1), [Fox L. and Parker I., 1968]:

Let

$$y(x) = \sum_{r=0}^{N}{}' a_r T_r(x) \qquad (1.15)$$

Then

$$y'(x) = \sum_{r=0}^{N-1}{}' c_r T_r(x) \qquad (1.16)$$

where

$$c_{N-1} = 2N\, a_N$$
$$c_{N-2} = 2\left(N-1\right)\, a_{N-2}$$
$$c_{r-1} = c_{r+1} + 2r\, a_r\,,\ r = 1,2,\ldots,\ N-2.$$

Proof:

It is easy to check that, from equation (1.15) one can get equation (1.16). Next, by integrating both sides of equation (1.16) and using remarks (1.4) and proposition (1.14) one can get:

$$y(x) = \frac{c_0}{2}T_1(x) + \frac{1}{4}c_1\left[T_0(x)+T_2(x)\right] + \frac{1}{2}\sum_{r=2}^{N-1} c_r\left[\frac{1}{r+1}T_{r+1}(x) - \frac{1}{r-1}T_{r-1}(x)\right] + A \qquad (1.17)$$

where A is the constant of integration.

By equating the coefficients of $T_r\ (x)$ on each side of equation (1.15) and equation (1.17) one can have:

$$\left.\begin{aligned} a_r &= \frac{1}{2r}(c_{r-1} - c_{r+1}),\ r = 1,2,\ldots,N-2 \\ a_{N-1} &= \frac{1}{2(N-1)} c_{N-2} \\ \text{and } a_N &= \frac{1}{2N}(c_{N-1}) \end{aligned}\right\} \qquad (1.18)$$

Remark (1.6), [Fox L. and Parker I., 1968]:

From theorem (1.1) one can calculate c_r in succession, for decreasing r from the general recurrence relation

$$c_{r-1} = c_{r+1} + 2r\, a_r,\ r = 1,2,\ldots,N-2$$

with starting conditions given by the last two equations of equation (1.18) to get:

$$\begin{aligned} c_{N-1} &= 2N\, a_N \\ c_{N-2} &= 2\left(N-1\right) a_{N-1} \\ c_{N-3} &= 2\left(N-2\right) a_{N-2} + 2N\, a_N \\ &\ \vdots \\ c_1 &= 4\, a_2 + 8\, a_4 + 12 a_6 + \cdots \\ c_0 &= 2\, a_1 + 6\, a_3 + 10 a_5 + \cdots \end{aligned} \qquad (1.19)$$

Each series in equation (1.19) being finite, stopping at the term a_N or a_{N-1}.

Next, the following theorem gives the same result as in theorem (1.1) but for the finite Chebyshev series defined on [0,1].

Theorem (1.2), [Fox L. and Parker I., 1968]:

Let

$$y(x)=\sum_{r=0}^{N}{}' a_r T_r^*(x) \tag{1.20}$$

Then

$$y'(x)=\sum_{r=0}^{N-1}{}' c_N T_r^*(x) \tag{1.21}$$

where

$$c_{N-1}=4N\,a_N$$
$$c_{N-2}=4\left(N-1\right)a_{N-2}$$
$$c_{r-1}=c_{r+1}+4r\,a_r,\; r=1,2,\ldots,N-2.$$

Proof:

It is easy to check that, from equation (1.20) one can get equation (1.21). Next, by integrating both sides of equation (1.21) and using remarks (1.4) and proposition (1.15) one can get:

$$\begin{aligned} y(x)=&\frac{c_0}{4}\left[T_0^*(x)+T_1^*(x)\right]+\frac{1}{8}c_1\left[T_2^*(x)-T_0^*(x)\right]+\\ &\frac{1}{4}\sum_{r=2}^{N-1}c_r\left[\frac{1}{r+1}T_{r+1}^*(x)-\frac{1}{r-1}T_{r-1}^*(x)\right]+A \end{aligned} \tag{1.22}$$

where A is the constant of integrations.

By equating the coefficients of $T_r^*(x)$ on each side of equation (1.20) and equation (1.22) one can have:

$$\left.\begin{aligned} a_r&=\frac{1}{4r}(c_{r-1}-c_{r+1}),\; r=1,2,\ldots,N-2\\ a_{N-1}&=\frac{1}{4(N-1)}c_{N-2},\; a_N=\frac{1}{4N}c_{N-1} \end{aligned}\right\} \tag{1.23}$$

Remark (1.7), [Fox L. and Parker I., 1968]:

From theorem (1.2) one can calculate c_r in succession, for decreasing r from the general recurrence relation

$$c_{r-1} = c_{r+1} + 4r\, a_r\,,\; r = 1, 2, ..., N-2$$

with starting conditions given by the last two equations of equation (1.23) to get:

$$\begin{aligned} c_{N-1} &= 4N\, a_N \\ c_{N-2} &= 4\left(N-1\right) a_{N-1} \\ c_{N-3} &= 4\left(N-2\right) a_{N-2} + 4N\, a_N \\ &\vdots \\ c_1 &= 8\, a_2 + 16\, a_4 + 24 a_6 + \cdots \\ c_0 &= 4\, a_1 + 12\, a_3 + 20 a_5 + \cdots \end{aligned} \qquad (1.24)$$

Each series in equation (1.24) being finite, stopping at the term a_N or a_{N-1}.

Next, we generalize the previous theorems for the infinite Chebyshev series. These theorems appeared in [Sezer M. and Kaynak M., 1996] without proofs. Here we give their proof.

Theorem (1.3):

Let

$$y(x) = {\sum_{r=0}^{\infty}}' a_r T_r(x) \qquad (1.25)$$

and

$$y'(x) = {\sum_{r=0}^{\infty}}' c_r T_r(x) \qquad (1.26)$$

Then

$$c_{r-1} = c_{r+1} + 2r\, a_r\,,\ r = 1,2,...$$

Proof:

By integrating both side of equation (1.26) and using remarks (1.4) and proposition (1.14) one can get:

$$y(x) = \frac{c_0}{2}T_1(x) + \frac{1}{4}c_1\left[T_0(x) + T_2(x)\right] +$$
$$\frac{1}{2}\sum_{r=2}^{\infty} c_r\left[\frac{1}{r+1}T_{r+1}(x) - \frac{1}{r-1}T_{r-1}(x)\right] + A$$

where A is the constant of the integration. Therefore

$$y(x) = \frac{1}{4}c_1T_0(x) + \frac{1}{2}\left[c_0 - c_2\right]T_1(x) + \frac{1}{4}\left[c_1 - c_3\right]T_2(x) +$$
$$\frac{1}{6}\left[c_2 - c_4\right]T_3(x) + \cdots + \frac{1}{2r}\left[c_{r-1} - c_{r+1}\right]T_r(x) + \cdots \qquad (1.27)$$

and by equating the coefficients of $T_r\ (x)$ on each side of equation (1.25) and equation (1.27) one can have:

$$a_r = \frac{1}{2r}\left[c_{r-1} - c_{r+1}\right],\ r = 1,2,....$$

Therefore

$$c_{r-1} = c_{r+1} + 2r\, a_r\,,\ \ r = 1,2,....$$

Theorem (1.4):

Let

$$y(x) = {\sum_{r=0}^{\infty}}' a_r T_r^*(x) \qquad (1.28)$$

and

$$y'(x) = \sum_{r=0}^{\infty}{}' c_r T_r^*(x) \tag{1.29}$$

Then

$$c_{r-1} = c_{r+1} + 4r\, a_r ,\ r = 1,2,...$$

Proof:

By integrating both side of equation (1.29) and using remarks (1.4) and proposition (1.15) one can get

$$y(x) = \frac{c_0}{4}\left[T_0^*(x) + T_1^*(x)\right] + \frac{1}{8}c_1\left[T_2^*(x) - T_0^*(x)\right] +$$
$$\frac{1}{4}\sum_{r=2}^{\infty} c_r\left[\frac{1}{r+1}T_{r+1}^*(x) - \frac{1}{r-1}T_{r-1}^*(x)\right] + A$$

where A is the constant of integration. Therefore

$$y(x) = \left[\frac{1}{4}c_0 - \frac{1}{8}c_1\right]T_0^*(x) + \frac{1}{4}\left[c_0 - c_2\right]T_1^*(x) + \frac{1}{8}\left[c_1 - c_3\right]T_2^*(x) +$$
$$\frac{1}{12}\left[c_2 - c_4\right]T_3^*(x) + \cdots + \frac{1}{4r}\left[c_{r-1} - c_{r+1}\right]T_r^*(x) + \cdots \tag{1.30}$$

and by equating the coefficients of $T_r^*(x)$ on each side of equation (1.29) and equation (1.30) one can have:

$$a_r = \frac{1}{4r}\left[c_{r-1} - c_{r+1}\right],\ r = 1,2,....$$

Therefore

$$c_{r-1} = c_{r+1} + 4r\, a_r ,\ r = 1,2,....$$

Remarks (1.8):

From the previous theorems, it is easy to check that:

(1) If $y(x)=\sum_{r=0}^{\infty}{}' a_r T_r(x)$ and $y^{(n)}(x)=\sum_{r=0}^{\infty}{}' a_r^{(n)} T_r(x)$ then

$$a_{r-1}^{(n+1)}=a_{r+1}^{(n+1)}+2ra_r^{(n)},\ r=1,2,....$$

(2) If $y(x)=\sum_{r=0}^{\infty}{}' a_r T_r^*(x)$ and $y^{(n)}(x)=\sum_{r=0}^{\infty}{}' a_r^{(n)} T_r^*(x)$ then

$$a_{r-1}^{(n+1)}=a_{r+1}^{(n+1)}+4ra_r^{(n)},\ r=1,2,...$$

where $a_r^{(n)}$ and a_r are Chebyshev coefficients such that $a_r^{(0)}=a_r,\ r=0,1,....$

Chapter Two

Chebyshev Methods for Solving Linear Ordinary Differential Equations

Introduction

It is known that the ordinary differential equations with non constant coefficients are usually difficult to solve analytically [Rainville E. and Bedient P., 1989]. In many cases, it is required to approximate their solutions. For this propose, the aim of this chapter is to give some methods which are based on Chebyshev polynomials and their properties to solve linear ordinary differential equations with variable coefficients.

This chapter consist of two sections. In section one we give a method to solve linear ordinary differential equations with nonconstant coefficients by transforming them to system of algebraic equations. This method is based on the finite Chebyshev series of the first kind defined on [0,1] with some of their properties and it is a simple modification of the method that appeared in [Fox L. and Parker I., 1968].

In section two, a matrix method which is called Chebyshev-matrix method for finding approximated solutions of linear ordinary differential equations in terms of Chebyshev polynomials is presented. This method is based on taking the truncated Chebyshev series of the first kind defined on [0,1] in the linear ordinary differential equations and then substituting their matrix forms into the given linear ordinary differential equations. Therefore the linear ordinary differential equation reduces to a matrix equation, which corresponds to a system of linear algebraic equations with unknown Chebyshev coefficients.

2.1 Chebyshev Series Method for Solving Linear Ordinary Differential Equations:

In this section, we use the finite Chebyshev series as a method for solving linear ordinary differential equations with non-constant coefficients. To do this, consider first the first-order linear ordinary differential equation with non-constant coefficients:

$$q(x)y'(x)+r(x)y(x)=f(x),\ \ 0\le x\le 1 \tag{2.1.a}$$

together with the boundary condition:

$$\alpha y(0)+\beta y(1)=\gamma \tag{2.1.b}$$

where *q*, *r* and *f* are polynomials of *x*, such that $q(x)\neq 0, \forall 0\le x\le 1$, α, β and γ are known constants and *y* is the unknown function that must be determined.

The Chebyshev series method is begin by integrating both sides of equation (2.1.a) to get:

$$q(x)y(x)+\int[r(x)-q'(x)]y(x)dx=\int f(x)dx+A \tag{2.2}$$

where *A* is the constant of integration.

Assume that the solution of equation (2.1) may be written in the form:

$$y(x)={\sum_{r=0}^{\infty}}' a_r T_r^*(x),\ \ 0\le x\le 1 \tag{2.3}$$

where $\{a\}_{r=0}^{\infty}$ are the Chebyshev coefficients that must be determined, then by substituting equation (2.3) into equation (2.2) one can have:

$$q(x){\sum_{r=0}^{\infty}}' a_r T_r^*(x)+\int[r(x)-q'(x)]{\sum_{r=0}^{\infty}}' a_r T_r^*(x)dx=\int f(x)dx+A \tag{2.4}$$

The known formula for products $x^sT_r^*(x)$ in terms of Chebyshev polynomials (see proposition (1.12)) and the integrals of such quantities (see proposition (1.15)), enable us to express the left-hand side of equation (2.4) as infinite Chebyshev series, in which the coefficient of $T_r^*(x)$ is a finite linear combination of coefficients $\{a\}_{r=0}^{\infty}$. The right- hand side is a finite series and the comparison of corresponding terms produces an infinite set of linear algebraic equations.

Next, consider the finite Chebyshev series:

$$y^*(x) = \sum_{r=0}^{N}{}' a_r T_r^*(x), \; 0 \le x \le 1 \qquad (2.5)$$

Then y^* is the truncation of the infinite Chebyshev series give by equation (2.3).

By substituting equation (2.5) into equation (2.2) one can have:

$$q(x)\sum_{r=0}^{N}{}' a_r T_r^*(x) + \int \left[r(x) - q'(x)\right] \sum_{r=0}^{N}{}' a_r T_r^*(x)\,dx = \int f(x)\,dx + A \qquad (2.6)$$

Also, the known formula for products $x^s T_r^*(x)$ in terms of Chebyshev polynomials and the integrals of such quantities, enable us to express the left-hand side of equation (2.6) as finite Chebyshev series, in which the coefficient of $T_r^*(x)$ is a finite linear combination of coefficients $\{a\}_{r=0}^{N}$. The right-hand side is a finite series and the comparison of corresponding terms produces a finite set of linear algebraic equations.

Moreover, there is an extra equation resulting from the boundary condition given by equation (2.1.b). This equation takes the form:

$$\alpha \sum_{r=0}^{N}{}' a_r T_r^*(0) + \beta \sum_{r=0}^{N}{}' a_r T_r^*(1) = \gamma \qquad (2.7)$$

But

$$T_r^*(0) = T_r(-1) = \begin{cases} 1 & r \text{ is an even positive integer} \\ -1 & r \text{ is an odd positive integer} \end{cases}$$

and

$$T_r^*(1) = T_r(1) = 1, \; r = 0,1,....$$

Hence, if N is an even positive integer then equation (2.7) becomes

$$(\alpha+\beta)\frac{a_0}{2}+(\beta-\alpha)a_1+(\alpha+\beta)a_2+(\beta-\alpha)a_3+\cdots+(\alpha+\beta)a_N=\gamma \qquad (2.8)$$

On the other hand if N is an odd positive integer then equation (2.7) becomes

$$(\alpha+\beta)\frac{a_0}{2}+(\beta-\alpha)a_1+(\alpha+\beta)a_2+(\beta-\alpha)a_3+\cdots+(\beta-\alpha)a_N=\gamma \qquad (2.9)$$

Thus by solving the above system of linear algebraic equations one can get the values of $\{a\}_{r=0}^{N}$ and by substituting these values into equation (2.5) one can get the approximated solution of equation (2.1).

Second, Consider the second-order linear ordinary differential equation with non-constant coefficients

$$p(x)y''(x)+q(x)y'(x)+r(x)y(x)=f(x), \quad 0\le x\le 1 \qquad (2.10.a)$$

together with the boundary conditions:

$$\begin{aligned} y(0)&=\alpha \\ y(1)&=\beta \end{aligned} \qquad (2.10.b)$$

where p, q, r and f are polynomials of x, such that $p(x)\ne 0, \forall 0\le x\le 1$ α and β are known constants and y is the unknown function that must be determined.

The Chebyshev series method is begin by integrating both sides of equation (2.10.a) to get:

$$p(x)y'(x)+\{q(x)-p'(x)\}y(x)+\int\{p''(x)-q'(x)+r(x)\}y(x)d\ x = \int f(x)dx+A$$

and a second integration of both sides of the above equation gives

$$p(x)y(x)+\int\{q(x)-2p'(x)\}y(x)dx+\int\int\{p''(x)-q'(x)+r(x)\}y(x)dx\,dx = \int\int f(x)dx\,dx+Ax+B \qquad (2.11)$$

where A and B are the constants of integrations.

Assume the solution of equation (2.10) can be approximated of a finite Chebyshev series given by equation (2.5) which is the truncation of the infinite Chebyshev series give by equation (2.3).

By substituting equation (2.5) into equation (2.11) one can have

$$p(x)\sum_{r=0}^{N}{}' a_r T_r^*(x) + \int \{q(x) - 2p'(x)\} \sum_{r=0}^{N}{}' a_r T_r^*(x)\, dx +$$
$$\iint \{p''(x) - q'(x) + r(x)\} \sum_{r=0}^{N}{}' a_r T_r^*(x)\, dx\, dx = \iint f(x)\, dx\, dx + Ax + B$$

(2.12)

Also, the known formula for products $x^s T_r^*(x)$ in terms of Chebyshev polynomials and the integrals of such quantities, enable us to express the left-hand side of equation (2.12) as finite Chebyshev series, in which the coefficient of $T_r^*(x)$ is a finite linear combination of coefficients $\{a\}_{r=0}^{N}$. The right-hand side is a finite series and the comparison of corresponding terms produces a finite set of linear algebraic equations.

Moreover, there are two extra equations coming from the boundary conditions given by equation (2.10.b). These equations take the forms:

$$\sum_{r=0}^{N}{}' a_r T_r^*(0) = \alpha \qquad (2.13)$$

and

$$\sum_{r=0}^{N}{}' a_r T_r^*(1) = \beta \qquad (2.14)$$

But

$$T_r^*(0) = T_r(-1) = \begin{cases} 1 & r \text{ is an even positive integer} \\ -1 & r \text{ is an odd positive integer} \end{cases}$$

and

$T_r^*(1) = T_r(1) = 1,\ r = 0,1,\ldots.$

Hence, if N is an even positive integer then equation (2.13) becomes

$$\frac{a_0}{2} - a_1 + a_2 - a_3 + \cdots + a_N = \alpha \qquad (2.15)$$

and if N is an odd positive integer then equation (2.13) becomes

$$\frac{a_0}{2} - a_1 + a_2 - a_3 + \cdots - a_N = \alpha \qquad (2.16)$$

On the other hand if N is an even or odd positive integer then equation (2.14) becomes

$$\frac{a_0}{2} + a_1 + a_2 + a_3 + \cdots + a_N = \beta \qquad (2.17)$$

Thus, by solving of the above system linear algebraic equations one can get the values of $\{a\}_{r=0}^{N}$ and by substituting these values into equation (2.5) one can get the approximated solution of equation (2.10).

The following examples illustrate these methods.

Example (2.1):

Consider the first order linear ordinary differential equation:

$$(1+x)y'(x) + (1+x+x^2)y(x) = x^4 + x^3 + x^2 - 2,\ \ 0 \le x \le 1 \qquad (2.18.a)$$

together with the boundary condition:

$$y(0) - \frac{3}{4}y(1) = \frac{-5}{4} \qquad (2.18.b)$$

By integrating both sides of the above differential equation one can get:

$$\int (1+x)y'(x)dx + \int (1+x+x^2)y(x)dx = \int (x^4 + x^3 + x^2 - 2)\,dx,\ \ 0 \le x \le 1$$

(2.19)

But

$$\int (1+x)y'(x)dx = (1+x)y(x) - \int y(x)dx$$

By substituting the above equation into equation (2.19) yields:

$$y(x) + x\,y(x) + \int x\,y(x)\,dx + \int x^2\,y(x)\,dx = \frac{x^5}{5} + \frac{x^4}{4} + \frac{x^3}{3} - 2x + A \qquad (2.20)$$

Assume that the solution of equation (2.18) can be written as in equation (2.3). Then by substituting equation (2.3) into equation (2.20) one can get:

$$\sum_{r=0}^{\infty}{}' a_r T_r^*(x) + x\sum_{r=0}^{\infty}{}' a_r T_r^*(x) + \int x\sum_{r=0}^{\infty}{}' a_r T_r^*(x)\,dx + \int x^2 \sum_{r=0}^{\infty}{}' a_r T_r^*(x)\,dx$$
$$= \frac{x^5}{5} + \frac{x^4}{4} + \frac{x^3}{3} - 2x + A.$$

therefore

$$\sum_{r=0}^{\infty}{}' a_r T_r^*(x) + \sum_{r=0}^{\infty}{}' a_r [x\,T_r^*(x)] + \sum_{r=0}^{\infty}{}' a_r \int x\,T_r^*(x)\,dx + \sum_{r=0}^{\infty}{}' a_r \int x^2 T_r^*(x)\,dx$$
$$= \frac{x^5}{5} + \frac{x^4}{4} + \frac{x^3}{3} - 2x + A \qquad (2.21)$$

From proposition (1.12) one can have:

$$x\,T_r^*(x) = \frac{1}{2^2}\sum_{i=0}^{2}\binom{2}{i} T_{r-1+i}^*(x)$$
$$= \frac{1}{4}\left\{T_{r-1}^*(x) + 2T_r^*(x) + T_{r+1}^*(x)\right\} \qquad (2.22)$$

and

$$x^2 T_r^*(x) = \frac{1}{2^4}\sum_{i=0}^{4}\binom{4}{i} T_{r-2+i}^*(x)$$
$$= \frac{1}{16}\left\{T_{r-2}^*(x) + 4T_{r-1}^*(x) + 6T_r^*(x) + 4T_{r+1}^*(x) + T_{r+2}^*(x)\right\}$$

and from proposition (1.12) one can get:

$$\int xT_r^*(x)dx = \frac{1}{4}\int\left[T_{r-1}^*(x) + 2T_r^*(x) + T_{r+1}^*(x)\right]dx$$
$$= -\frac{1}{16(r-2)}T_{r-2}^*(x) + \frac{1}{8(r+1)}T_{r+1}^*(x) - \frac{1}{8(r-1)}T_{r-1}^*(x) + \frac{1}{16(r+2)}T_{r+2}^*(x) \qquad (2.23)$$

and

$$\int x^2 T_r^*(x)dx = \frac{1}{16}\int\left[T_{r-2}^*(x) + 4T_{r-1}^*(x) + 6T_r^*(x) + 4T_{r+1}^*(x) + T_{r+2}^*(x)\right]dx$$
$$= \frac{-5}{64(r-1)}T_{r-1}^*(x) - \frac{1}{16(r-2)}T_{r-2}^*(x) - \frac{1}{64(r-3)}T_{r-3}^*(x) +$$
$$\frac{5}{64(r+1)}T_{r+1}^*(x) + \frac{1}{16(r+2)}T_{r+2}^*(x) + \frac{1}{64(r+3)}T_{r+3}^*(x) \qquad (2.24)$$

Moreover, from proposition (1.9) one can get

$$x = \frac{1}{2}\left\{T_1^*(x) + T_0^*(x)\right\} \qquad (2.25)$$

$$\left.\begin{aligned} x^3 &= \frac{1}{32}\left\{10T_0^*(x) + 15T_1^*(x) + 6T_2^*(x) + T_3^*(x)\right\} \\ x^4 &= \frac{1}{128}\left\{35T_0^*(x) + 56T_1^*(x) + 28T_2^*(x) + 8T_3^*(x) + T_4^*(x)\right\} \\ x^5 &= \frac{1}{512}\left\{126T_0^*(x) + 210T_1^*(x) + 120T_2^*(x) + 45T_3^*(x) + 10T_4^*(x) + T_5^*(x)\right\} \end{aligned}\right\} \qquad (2.26)$$

By substituting equations (2.22)-(2.26) into equation (2.21) and after simple computations one can obtain:

$$\left[\frac{1}{64r}a_{r-3} + \frac{1}{8r}a_{r-2} + \left[\frac{1}{4} + \frac{13}{64r}\right]a_{r-1} + \frac{3}{2}a_r + \left[\frac{1}{4} - \frac{13}{64r}\right]a_{r+1} - \frac{1}{8r}a_{r+2} - \frac{1}{64r}a_{r+3}\right]T_r^*(x)$$
$$= \left[\frac{1703}{7680} + A\right]T_0^*(x) - \frac{167}{256}T_1^*(x) + \frac{21}{128}T_2^*(x) + \frac{67}{1536}T_3^*(x) + \frac{3}{512}T_4^*(x) + \frac{1}{2560}T_5^*(x) \qquad (2.27)$$

From the boundary condition given by equation (2.18.b) one can have:

$$\sum_{r=0}^{\infty}{}' a_r T_r^*(0) - \frac{3}{4}\sum_{r=0}^{\infty}{}' a_r T_r^*(1) = \frac{-5}{4} \qquad (2.28)$$

But

$$T_r^*(0) = \begin{cases} 1, & r \text{ is an even positive integer} \\ -1, & r \text{ is an odd positive integer} \end{cases} \qquad (2.29)$$

and

$$T_r^*(1) = 1,\ r = 0,1,.... \qquad (2.30)$$

By substituting equations (2.29)-(2.30) into equation (2.28) and after simple computations one can have:

$$\frac{1}{8}a_0 - \frac{7}{4}a_1 + \frac{1}{4}a_2 - \frac{7}{4}a_3 + \frac{1}{4}a_4 - \frac{7}{4}a_5 + \cdots = \frac{-5}{4} \qquad (2.31)$$

Hence, the following system of linear equations consists of equation (2.31) and the other equations can be obtained by substituting *r=1,2,…* into equation (2.27):

$$\begin{aligned}
&\frac{1}{8}a_0 - \frac{7}{4}a_1 + \frac{1}{4}a_2 - \frac{7}{4}a_3 + \frac{1}{4}a_4 - \frac{7}{4}a_5 + \cdots = \frac{-5}{4} \\
&\frac{29}{64}a_0 + \frac{13}{8}a_1 + \frac{1}{16}a_2 - \frac{1}{8}a_3 - \frac{1}{64}a_4 = \frac{-167}{256} \\
&\frac{1}{16}a_0 + \frac{23}{64}a_1 + \frac{3}{2}a_2 + \frac{19}{128}a_3 - \frac{1}{16}a_4 - \frac{1}{128}a_5 = \frac{21}{128} \\
&\frac{1}{192}a_0 + \frac{1}{24}a_1 + \frac{61}{192}a_2 + \frac{3}{2}a_3 - \frac{35}{192}a_4 - \frac{1}{24}a_5 - \frac{1}{192}a_6 = \frac{67}{1536} \\
&\frac{1}{256}a_1 + \frac{1}{32}a_2 + \frac{77}{256}a_3 + \frac{3}{2}a_4 + \frac{51}{256}a_5 - \frac{1}{32}a_6 - \frac{1}{256}a_7 = \frac{3}{512} \\
&\frac{1}{320}a_2 + \frac{1}{40}a_3 + \frac{93}{320}a_4 + \frac{3}{2}a_5 + \frac{67}{320}a_6 - \frac{1}{40}a_7 - \frac{1}{320}a_8 = \frac{1}{2560} \\
&\frac{1}{384}a_3 + \frac{1}{48}a_4 + \frac{109}{384}a_5 + \frac{3}{2}a_6 + \frac{83}{384}a_7 - \frac{1}{48}a_8 - \frac{1}{384}a_9 = 0 \\
&\vdots
\end{aligned} \qquad (2.32)$$

Now, assume that the solution of equation (2.18) can be approximated as a finite Chebyshev of degree two, then the above system reduces to the following system:

$$\frac{1}{8}a_0 - \frac{7}{4}a_1 + \frac{1}{4}a_2 = \frac{-5}{4}$$
$$\frac{29}{64}a_0 + \frac{13}{8}a_1 + \frac{1}{16}a_2 = \frac{-167}{256}$$
$$\frac{1}{16}a_0 + \frac{23}{64}a_1 + \frac{3}{2}a_2 = \frac{21}{128}$$

which has the solution $a_0 = \frac{-13}{4}$, $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{8}$. Hence

$$y(x)\sum_{r=0}^{2}{}' a_r T_r^*(x) = \frac{a_0}{2}T_0^*(x) + a_1 T_1^*(x) + a_2 T_2^*(x)$$
$$= \frac{-13}{8}T_0^*(x) + \frac{1}{2}T_1^*(x) + \frac{1}{8}T_2^*(x)$$
$$= x^2 - 2$$

is the approximated solution of equation (2.18). By substituting this approximated solution into left-hand side of equation (2.18.a) one can get:

$$(1+x)y^{*\prime}(x) + (1+x+x^2)y^*(x) = 1-x$$

In this case $y^*(x) = x^2 - 2$ is the exact solution of equation (2.18).

Example (2.2):

Consider the second order linear ordinary differential equation:

$$y''(x) + x^2 y'(x) + 2x\, y(x) = 4x^3 + 2, \quad 0 \le x \le 1 \qquad (2.33.a)$$

together with the boundary condition:

$$y(0) = 0$$
$$y(1) = 1 \qquad (2.33.b)$$

By integrating both sides of equation (2.33.a) one can get:

$$y'(x)+x^2y(x)=x^4+2x+A$$

and a second integration of both sides of the above equation gives

$$y(x)+\int x^2y(x)dx=\frac{x^5}{5}+x^2+Ax+B \tag{2.34}$$

Assume that the solution of equation (2.33) can be written as in equation (2.3). Then by substituting this solution into equation (2.34) one can have:

$$\sum_{r=0}^{\infty}{}' a_r T_r^*(x)+\sum_{r=0}^{\infty}{}' a_r \int x^2 T_r^*(x)dx=\frac{x^5}{5}+x^2+Ax+B \tag{2.35}$$

From proposition (1.12) one can obtain:

$$x^2T_r^*(x)=\frac{1}{2^4}\sum_{i=0}^{4}\binom{4}{i}T_{r-2+i}^*(x)$$
$$=\frac{1}{16}\left\{T_{r-2}^*(x)+4T_{r-1}^*(x)+6T_r^*(x)+4T_{r+1}^*(x)+T_{r+2}^*(x)\right\}$$

Therefore from proposition (1.15) one can get:

$$\int x^2T_r^*(x)dx=\frac{1}{16}\int\left[T_{r-2}^*(x)+4T_{r-1}^*(x)+6T_r^*(x)+4T_{r+1}^*(x)+T_{r+2}^*(x)\right]dx$$
$$=\frac{-5}{64(r-1)}T_{r-1}^*(x)-\frac{1}{16(r-2)}T_{r-2}^*(x)-\frac{1}{64(r-3)}T_{r-3}^*(x)+$$
$$\frac{5}{64(r+1)}T_{r+1}^*(x)+\frac{1}{16(r+2)}T_{r+2}^*(x)+\frac{1}{64(r+3)}T_{r+3}^*(x) \tag{2.36}$$

Moreover, from proposition (1.9) one can get:

$$x=\frac{1}{2}\left\{T_1^*(x)+T_0^*(x)\right\} \tag{2.37}$$

$$x^2=\frac{1}{8}\left\{T_2^*(x)+4T_1^*(x)+3T_0^*(x)\right\} \tag{2.38}$$

and

$$x^5 = \frac{1}{512}\left[T_5^*(x) + 10T_4^*(x) + 45T_3^*(x) + 120T_2^*(x) + 210T_1^*(x) + 126T_0^*(x)\right] \tag{2.39}$$

By substituting equations (2.36)-(2.39) into equation (2.35) and after simple computations one can obtain:

$$\left[\frac{1}{64r}a_{r-3} + \frac{1}{16r}a_{r-2} + \frac{5}{64r}a_{r-1} + a_r - \frac{5}{64r}a_{r+1} - \frac{1}{16r}a_{r+2} - \frac{1}{64r}a_{r+3}\right]T_r^*(x)$$
$$= \left[\frac{66}{128} + \frac{A}{2} + B\right]T_0^*(x) + \left[\frac{82}{128} + \frac{A}{2}\right]T_1^*(x) + \frac{22}{128}T_2^*(x) + \frac{27}{1536}T_3^*(x) +$$
$$\frac{1}{256}T_4^*(x) + \frac{1}{2560}T_5^*(x) \tag{2.40}$$

From the boundary conditions given by equation (2.33.b) one can have:

$$\sum_{r=0}^{\infty}{}' a_r T_r^*(0) = 0 \tag{2.41}$$

and

$$\sum_{r=0}^{\infty}{}' a_r T_r^*(1) = 1 \tag{2.42}$$

But

$$T_r^*(x) = T_r(2x-1),$$

$$T_r^*(0) = \begin{cases} 1, & r \text{ is an even positive integer} \\ -1, & r \text{ is an odd positive integer} \end{cases} \tag{2.43}$$

and

$$T_r^*(1) = 1, \ r = 0,1,.... \tag{2.44}$$

By substituting equations (2.43)-(2.44) into equations (2.41)-(2.42) one can have:

$$\frac{1}{2}a_0 - a_1 + a_2 - a_3 + \cdots = 0$$

$$\frac{a_0}{2}+a_1+a_2+\cdots=1$$

Hence, the following system of linear equations consists of the above two equations and the other equations can be obtained by substituting *r=2,3,…* into equation (2.40):

$$\frac{1}{2}a_0-a_1+a_2-a_3+\cdots=0$$

$$\frac{a_0}{2}+a_1+a_2+\cdots=1$$

$$\frac{1}{32}a_0+\frac{6}{128}a_1+a_2-\frac{5}{128}a_3-\frac{1}{32}a_4-\frac{1}{128}a_5=\frac{22}{128}$$

$$\frac{1}{192}a_0+\frac{1}{48}a_1+\frac{5}{192}a_2+a_3-\frac{5}{192}a_4-\frac{1}{48}a_5-\frac{1}{192}a_6=\frac{27}{1536}$$

$$\frac{1}{256}a_1+\frac{1}{64}a_2+\frac{5}{128}a_3+a_4-\frac{5}{256}a_5-\frac{1}{64}a_6-\frac{1}{256}a_7=\frac{1}{256}$$

$$\frac{1}{320}a_2+\frac{1}{80}a_3+\frac{5}{320}a_4+a_5-\frac{5}{320}a_6-\frac{1}{80}a_7-\frac{1}{320}a_8=\frac{1}{2560}$$

$$\frac{1}{384}a_3+\frac{1}{96}a_4+\frac{5}{384}a_5+a_6-\frac{5}{384}a_7-\frac{1}{96}a_8+\frac{1}{384}a_9=0$$

$$\vdots$$

Now, assume that the solution of equation (2.33) may be approximated as a finite Chebyshev series of degree three, then the above system reduces to the following system:

$$\frac{1}{2}a_0-a_1+a_2-a_3=0$$

$$\frac{1}{2}a_0+a_1+a_2=1$$

$$\frac{1}{32}a_0+\frac{6}{128}a_1+a_2-\frac{5}{128}a_3=\frac{22}{128}$$

$$\frac{1}{192}a_0+\frac{1}{48}a_1+\frac{5}{192}a_2=\frac{27}{1536}$$

which has the solution

$a_0 = \frac{3}{4},\ a_1 = \frac{1}{2},\ a_2 = \frac{1}{8}$ and $a_3 = 0$

Hence

$$y^*(x) = \frac{3}{8} + \frac{1}{2}(2x - 1) + \frac{1}{8}(8x^2 - 8x + 1)$$
$$= x^2$$

which is the exact solution of equation (2.33).

2.2 Chebyshev-Matrix Method for solving Linear ordinary Differential Equations:

In this section, we use Chebyshev-matrix method to solve linear ordinary differential equations with nonconstant coefficients. This method is a modification of the method that appeared in [Sezer M. and Kaynak M., 1996]. To do this, consider the boundary value problem given by equations (2.1).

This method begin by approximating the solution y as a finite Chebyshev series given by equation (2.5). Also, assume that f can be approximated as a finite Chebyshev series that takes the form:

$$f(x) \approx f^*(x) = \sum_{r=0}^{N}{}' f_r T_r^*(x) \tag{2.45}$$

Therefore

$$y(x) \approx y^*(x) = \begin{bmatrix} T_0^*(x) & T_1^*(x) & . & . & T_N^*(x) \end{bmatrix} \begin{bmatrix} \frac{1}{2}a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}$$

and

$$f^*(x) = \begin{bmatrix} T_0^*(x) & T_1^*(x) & . & . & T_N^*(x) \end{bmatrix} \begin{bmatrix} \frac{1}{2} f_0 \\ f_1 \\ \vdots \\ f_N \end{bmatrix}$$

This implies that

$$y^*(x) = T^* A$$

and

$$f^*(x) = T^* F$$

where

$$A = \begin{bmatrix} \frac{1}{2} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}, \ F = \begin{bmatrix} \frac{1}{2} f_0 \\ f_1 \\ \vdots \\ f_N \end{bmatrix} \text{ and } T = \begin{bmatrix} T_0^*(x) & T_1^*(x) & . & . & T_N^*(x) \end{bmatrix}.$$

Assume that the functions q and r can be expressed in the forms

$$q(x) = \sum_{i=0}^{m} q_i x^i \tag{2.46}$$

$$r(x) = \sum_{i=0}^{m} r_i x^i \tag{2.47}$$

which are Taylor polynomials of degree *m* at *x=0.*

By substituting equations (2.46)-(2.47) into equation (2.1.a) one can have:

$$\sum_{i=0}^{m} \left\{ q_i x^i y^{*\prime}(x) + r_i x^i y^*(x) \right\} = f(x) \tag{2.48}$$

By approximating $y^{*\prime}(x)$ as a finite Chebyshev series one can have:

$$y'(x) \approx y^{*'}(x) = \sum_{r=0}^{N}{}' a_r^{(1)} T_r^*(x), \; 0 \le x \le 1$$

where $\left\{a^{(1)}\right\}_{r=0}^{N}$ are the Chebyshev coefficients defined by remarks (1.9),(2).

Hence

$$4r\, a_r = a_{r-1}^{(1)} - a_{r+1}^{(1)}, \; r = 1,2,...$$

Thus

$$4(r+1)a_{r+1} = a_r^{(1)} - a_{r+2}^{(1)}$$
$$4(r+3)a_{r+3} = a_{r+2}^{(1)} - a_{r+4}^{(1)}$$
$$4(r+5)a_{r+5}^{(1)} = a_{r+4}^{(1)} - a_{r+6}^{(1)}$$
$$\vdots$$

and so on.

Therefore

$$4(r+1)a_{r+1} + 4(r+3)a_{r+3} + 4(r+5)a_{r+5} + \cdots = a_r^{(1)}.$$

and this implies that

$$a_r^{(1)} = 4\sum_{i=0}^{\infty}(r+2i+1)a_{r+2i+1}, \; r = 0,1,... \qquad (2.49)$$

Now, we discuss two cases on N:

***Case 1*:** If N is an even positive integer, then by substituting $r = 0,1,...,N$ into equation (2.49) one can have:

$$a_0^{(1)} = 4\sum_{i=0}^{\infty}(2i+1)a_{2i+1}$$
$$= 4\left[a_1 + 3a_3 + 5a_5 + \cdots + (N-1)a_{N-1}\right],$$
$$a_1^{(1)} = 4\sum_{i=0}^{\infty}(2i+2)a_{2i+2}$$
$$= 4\left[2a_2 + 4a_4 + 6a_6 + \cdots + N\,a_N\right],$$
$$\vdots$$
$$a_{N-1}^{(1)} = 4\sum_{i=0}^{\infty}(2i+N)a_{2i+N}$$
$$= 4N\,a_N$$

and

$$a_N^{(1)} = 4\sum_{i=0}^{\infty}(N+2i+1)\,a_{N+2i+1} = 0.$$

Therefore

$$A^{(1)} = \begin{bmatrix} \frac{1}{2}a_0^{(1)} \\ a_1^{(1)} \\ a_2^{(1)} \\ \vdots \\ a_{N-1}^{(1)} \\ a_N^{(1)} \end{bmatrix} = 4\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \ldots & \frac{N-1}{2} & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & \ldots & 0 & N \\ 0 & 0 & 0 & 3 & 0 & 5 & \ldots & N-1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2}a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-1} \\ a_N \end{bmatrix}.$$

and hence

$$A^{(1)} = 4MA \tag{2.50}$$

where

$$M=\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \dots & \frac{N-1}{2} & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & \dots & 0 & N \\ 0 & 0 & 0 & 3 & 0 & 5 & \dots & N-1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{2.51}$$

Case (2): If N is an odd positive integer, then by substituting $r=0,1,\dots,N$ into equation (2.49) one can have

$$\begin{aligned} a_0^{(1)} &= 4\sum_{i=0}^{\infty}(2i+1)\ a_{2i+1} \\ &= 4[a_1+3a_3+5a_5+\dots+Na_N], \end{aligned}$$

$$\begin{aligned} a_1^{(1)} &= 4\sum_{i=0}^{\infty}(2i+2)\ a_{2i+2} \\ &= 4[2a_2+4a_4+6a_6+\dots+(N-1)a_{N-1}], \\ &\vdots \end{aligned}$$

$$\begin{aligned} a_{N-1}^{(1)} &= 4\sum_{i=0}^{\infty}(2i+N)\ a_{2i+N} \\ &= 4Na_N \end{aligned}$$

and

$$a_N^{(1)} = 4\sum_{i=0}^{\infty}(2i+N+1)\ a_{2i+N+1} = 0.$$

Therefore

$$A^{(1)} = 4\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \dots & 0 & \frac{N}{2} \\ 0 & 0 & 2 & 0 & 4 & 0 & \dots & N-1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 5 & \dots & 0 & N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and hence

$$A^{(1)} = 4MA \tag{2.52}$$

where

$$M = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \dots & 0 & \frac{N}{2} \\ 0 & 0 & 2 & 0 & 4 & 0 & \dots & N-1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 5 & \dots & 0 & N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{2.53}$$

The Chebyshev expansion of terms $x^i y^*(x)$ and $x^i y^{*\prime}(x)$, $i = 0,1,\dots,m$ in equation (2.48) are obtained by means of the formula:

$$\left[x^i y^{(s)}(x)\right] = \sum_{r=0}^{N}{}' \sum_{j=0}^{2i} 2^{-2i} \binom{2i}{j} a^{(s)}_{|r-i+j|} T_r^*(x), \quad s = 0,1 \tag{2.54}$$

The matrix representation of equation (2.54) can be given by

$$\left[x^i y^{(s)}(x)\right] = T^* M_i A^{(s)}, \; s = 0,1$$

where $A^{(0)} = A$.

From equation (2.50) and equation (2.52) the above equation becomes

$$\left[x^i y^{(s)}(x)\right] = 4T^* M_i M^s A, \; i = 0,1,\dots,N, \; s = 0,1 \tag{2.55}$$

where

$$M_0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$M_1 = \frac{1}{2^2}\begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}$$

$$M_2 = \frac{1}{2^4}\begin{bmatrix} 6 & 4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 8 & 7 & 4 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 4 & 6 & 4 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 4 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 6 \end{bmatrix}$$

$$M_3 = \frac{1}{2^6}\begin{bmatrix} 20 & 15 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 30 & 26 & 16 & 6 & 1 & \cdots & 0 & 0 & 0 \\ 12 & 16 & 20 & 15 & 6 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 15 & 20 & 15 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 6 & 15 & 20 \end{bmatrix}$$

and so on.

By substituting equation (2.55) back into equation (2.48) one can have:

$$\sum_{i=0}^{m} \{4q_i \, M_i \, M + r_i \, M_i\} A = F \tag{2.56}$$

which correspond, to a system of *N+1* algebraic equations for the unknown Chebyshev coefficients $\{a\}_{r=0}^{N}$. This equation can be rewritten in the form:

$$W\,A=F \tag{2.57}$$

where

$$w_{nm} = \sum_{i=0}^{m} \{4q_i\, M_i\, M + r_i\, M_i\}, \quad n,m = 0,1,\ldots,N$$

Then the augmented matrix of equation (2.57) is:

$$[W:F] = \left[\begin{array}{cccc|c} w_{0,0} & w_{0,1} & \cdots & w_{0,N} & \frac{1}{2}f_0 \\ w_{1,0} & w_{1,1} & \cdots & w_{1,N} & f_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N-1,0} & w_{N-1,1} & \cdots & w_{N-1,N} & f_{N-1} \\ w_{N,0} & w_{N,1} & \cdots & w_{N,N} & f_N \end{array}\right] \tag{2.58}$$

Now, by substituting equation (2.5) in the boundary condition given by equation (2.1.b) one can have equation (2.8) or equation (2.9).

Thus the boundary condition given by equation (2.1.b) can be written as:

$$AU = \gamma \tag{2.59}$$

where

$$U = [u_0 \quad u_1 \quad \cdots \quad u_{N-1} \quad u_N]$$

If N is an even or odd positive integer then

$$u_i = \begin{cases} \alpha+\beta & i\,\text{is an even positive integer} \\ \beta-\alpha & i\,\text{is an odd positive integer} \end{cases}$$

The augmented matrix given by equation (2.59) is:

$$[u_0 \quad u_1 \quad \cdots \quad u_N \,|\, \gamma] \tag{2.60}$$

Now, by replacing the row matrix given by equation (2.60) by the last row of augmented matrix given by equation (2.58), we have the new augmented matrix:

$$\left[\begin{array}{cccc|c} w_{00} & w_{01} & \cdots & w_{0N} & \frac{1}{2}f_0 \\ w_{10} & w_{11} & \cdots & w_{1N} & f_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N-1,0} & w_{N-1,1} & \cdots & w_{N-1,N} & f_{N-1} \\ u_0 & u_1 & \cdots & u_N & \gamma \end{array}\right]$$

Let

$$W^* = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} \\ w_{10} & w_{11} & \cdots & w_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ w_{N-1,0} & w_{N-1,1} & \cdots & w_{N-1,N} \\ u_0 & u_1 & \cdots & u_N \end{bmatrix}, \quad F^* = \begin{bmatrix} \frac{1}{2}f_0 \\ f_1 \\ \vdots \\ f_{N-1} \\ \gamma \end{bmatrix}$$

If $|W^*| \neq 0$ then

$$A = (W^*)^{-1} F^*$$

and thus the matrix A is a uniquely determined. By substituting the values of $\{a\}_{r=0}^{N}$ into equation (2.5) one can get the approximate solution of equations (2.1).

Second, consider the boundary value problem given by equations (2.10).

This method is begin by approximating the solution as a finite Chebyshev series give by equation (2.5) and f can be approximated as a finite Chebyshev series given by equation (2.45). Also assume that the functions p, q and r can be expressed in the forms:

$$p(x) = \sum_{i=0}^{m} p_i x^i \tag{2.61}$$

$$q(x)=\sum_{i=0}^{m} q_i\, x^i \tag{2.62}$$

$$r(x)=\sum_{i=0}^{m} r_i\, x^i \tag{2.63}$$

which are Taylor polynomials of degree *m* at *x=0*.

By substituting equations (2.61)-(2.63) into equation (2.10.a) one can have:

$$\sum_{i=0}^{m}\left\{p_i\, x^i\, y^{*\prime\prime}(x)+q_i\, x^i\, y^{*\prime}(x)+r_i\, x^i\, y^{*}(x)\right\}=f\,(x) \tag{2.64}$$

By approximating $y^{*\prime\prime}(x)$ as a finite Chebyshev series one can have:

$$y^{*\prime\prime}(x)=\sum_{r=0}^{N}{}' a_r^{(2)}\, T_r^{*}(x),\ \ 0\le x\le 1$$

where $\left\{a^{(2)}\right\}_{r=0}^{N}$ are the Chebyshev coefficients defined by remarks (1.9),(2).

Hence

$$4r\, a_r^{(1)}=a_{r-1}^{(2)}-a_{r+1}^{(2)},\quad r=1,2,\ldots$$

Thus

$$\begin{aligned}
4(r+1)a_{r+1}^{(1)}&=a_r^{(2)}-a_{r+2}^{(2)}\\
4(r+3)a_{r+3}^{(1)}&=a_{r+2}^{(2)}-a_{r+4}^{(2)}\\
4(r+5)a_{r+5}^{(1)}&=a_{r+4}^{(2)}-a_{r+6}^{(2)}\\
&\vdots
\end{aligned}$$

and so on.

Therefore

$$4(r+1)a_{r+1}^{(1)}+4(r+3)a_{r+3}^{(1)}+4(r+5)a_{r+5}^{(1)}+\cdots=a_r^{(2)}.$$

and this implies that

$$a_r^{(2)} = 4\sum_{i=0}^{\infty}(r+2i+1)a_{r+2i+1}^{(1)},\ r=0,1,\ldots \tag{2.65}$$

Now, we shall discuss two cases on N:

Case 1: If N is an even positive integer, then by substituting $r=0,1,\ldots,N$ into equation (2.65) one can have:

$$A^{(2)} = 4M\ A^{(1)}$$

Case 2: If N is an odd positive integer, then by substituting $r=0,1,\ldots,N$ into equation (2.65) one can have

$$A^{(2)} = 4M\ A^{(1)}$$

But $A^{(1)}=4MA$, Therefore

$$A^{(2)}=4^2 M^2 A. \tag{2.66}$$

The Chebyshev expansions of terms $x^i y^*(x), x^i y^{*\prime}(x)$ and $x^i y^{*\prime\prime}(x),\ i=0,1,\ldots,m$ in equation (2.64) are obtained by means of the formula:

$$x^i y^{(s)}(x) = \sum_{r=0}^{N}{}' \sum_{j=0}^{2i} 2^{-2i}\binom{i}{j} a_{|r-i+j|}^{(s)} T_r^*(x),\ s=0,1,2 \tag{2.67}$$

The matrix representation of equation (2.67) can be given by

$$x^i y^{(s)}(x) = T^* M_i A^{(s)},\ s=0,1,2$$

where $A^{(0)}=A$

From equation (2.66) the above equations becomes:

$$x^i y^{(s)}(x) = 4^2 T^* M_i M^{(s)} A,\ i=0,1,\ldots,N,\ s=0,1,2 \tag{2.68}$$

where M_i can be calculated in a similar manner used previously.

By substituting equation (2.68) into equation (2.64) one can have:

$$\sum_{i=0}^{m}\left\{16 p_i M_i M^2 + 4q_i M_i M + r_i M_i\right\} A = F \tag{2.69}$$

which corresponds to a system of *N+1* algebraic equations for the unknown Chebyshev coefficients $\{a\}_{r=0}^{N}$. This equation can be rewritten in the form:

$$WA=F \qquad (2.70)$$

where

$$W_{n,m}=\sum_{i=0}^{m}\left\{16p_iM_iM^2+4q_iM_iM+r_iM_i\right\},\ n,m=0,1,..N$$

Then the augmented matrix of equation (2.70) is:

$$[W_i:F]=\left[\begin{array}{cccc|c} w_{0,0} & w_{0,1} & \cdots & w_{0,N} & \frac{1}{2}f_0 \\ w_{1,0} & w_{1,1} & \cdots & w_{1,N} & f_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N-1,0} & w_{N-1,1} & \cdots & w_{N-1,N} & f_{N-1} \\ w_{N,0} & w_{N,1} & \cdots & w_{N,N} & f_N \end{array}\right] \qquad (2.71)$$

Now, by substituting equation (2.5) in the first and second boundary conditions given by equation (2.10.b) one can have equation (2.15) or equation (2.16) and equation (2.17).

Thus the first boundary condition given by equation (2.10.b) can be written as:

$$AU=\alpha \qquad (2.72)$$

If *N* is an even or odd positive integer then

$$u_i=\begin{cases}1 & i \text{ is an even positive integer}\\ -1 & i \text{ is an odd positive integer}\end{cases}$$

The augmented matrix of equation (2.72) is:

$$\left[u_0 \quad u_1 \quad \cdots \quad u_N \quad |\alpha\right] \qquad (2.73)$$

and the second boundary condition given by equation (2.10.b) can be written as:

$$AV=\beta \qquad (2.74)$$

where

$$\begin{bmatrix} v_0 & v_1 & \cdots & v_{N-1} & v_N \end{bmatrix}$$

If N is an even or odd positive integer

$v_i = 1,\ i = 0,1,\ldots,N$

The augment matrix of equation (2.74) is:

$$\left[\begin{array}{cccc|c} v_0 & v_1 & \cdots & v_N & \beta \end{array}\right] \tag{2.75}$$

Now, by replacing the two rows matrices given by equation (2.73) and equation (2.75) by the last two rows of augmented matrix given by equation (2.71), we have the new augment matrix:

$$\left[\begin{array}{cccc|c} w_{00} & w_{01} & \cdots & w_{0N} & \frac{1}{2}f_0 \\ w_{10} & w_{11} & \cdots & w_{1N} & f_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N-2,0} & w_{N-2,1} & \cdots & w_{N-2,N} & f_{N-2} \\ u_0 & u_1 & \cdots & u_N & \alpha \\ v_0 & v_1 & \cdots & v_N & \beta \end{array}\right]$$

Let

$$W^* = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} \\ w_{10} & w_{11} & \cdots & w_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ w_{N-2,0} & w_{N-2,1} & \cdots & w_{N-2,N} \\ u_0 & u_1 & \cdots & u_N \\ v_0 & v_1 & \cdots & v_N \end{bmatrix}, \quad F^* = \begin{bmatrix} \frac{1}{2}f_0 \\ f_1 \\ \vdots \\ f_{N-2} \\ \alpha \\ \beta \end{bmatrix}$$

If $|W^*| \neq 0$ then

$$A = (W^*)^{-1} F^*$$

and thus the matrix A is a uniquely determined. By substituting the values of $\{a\}_{r=0}^{N}$ into equation (2.5) one can get the approximated solution of equations (2.10).

To illustrate this method consider the following examples:

Example (2.3):

Consider example (2.1). Assume that the solution y and the function f can be approximated by the finite Chebyshev series of degree four.

$$y^*(x) = {\sum_{r=0}^{4}}' a_r T_r^*(x) \tag{2.76}$$

and

$$\begin{aligned} f(x) = x^4 + x^3 + x^2 - 2 &= {\sum_{r=0}^{4}}' f_r T_r^*(x) \\ &= \frac{1}{2} f_0 T_0^*(x) + f_1 T_1^*(x) + f_2 T_2^*(x) + f_3 T_3^*(x) + f_4 T_4^*(x) \\ &= \frac{1}{2} f_0 + f_1(2x-1) + f_2(8x^2 - 8x + 1) + \\ &\quad f_3(32x^3 - 48x^2 + 18x - 1) + \\ &\quad f_4(128x^4 - 256x^3 + 160x^2 - 32x + 1). \end{aligned}$$

This implies that: $f_0 = \frac{-133}{64}, f_1 = \frac{45}{32}, f_2 = \frac{17}{32}, f_3 = \frac{3}{32}$ and $f_4 = \frac{1}{128}$.

Therefore

$$F=\begin{bmatrix}\frac{-133}{128}\\ \frac{45}{32}\\ \frac{17}{32}\\ \frac{3}{32}\\ \frac{1}{128}\end{bmatrix}.$$

Since $q(x)=1+x$ and $r(x)=1+x+x^2$, then from equations (2.46)-(2.47) one can have:

$q_0= q_1=1, q_2=q_3=q_4= 0$

and

$r_0= r_1= r_2=1, r_3=r_4=0$

Then by substituting the above results into equation (2.56) one can get:

$$\{4M_0 M + M_0 + 4M_1 M + M_1 + M_2\}A = F \qquad (2.77)$$

where

$$M=\begin{bmatrix}0 & \frac{1}{2} & 0 & \frac{3}{2} & 0\\ 0 & 0 & 2 & 0 & 4\\ 0 & 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 0 & 4\\ 0 & 0 & 0 & 0 & 0\end{bmatrix},\ M_0=\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix},\ M_1=\frac{1}{4}\begin{bmatrix}2 & 1 & 0 & 0 & 0\\ 2 & 2 & 1 & 0 & 0\\ 0 & 1 & 2 & 1 & 0\\ 0 & 0 & 1 & 2 & 1\\ 0 & 0 & 0 & 1 & 2\end{bmatrix}$$

and $M_2=\frac{1}{16}\begin{bmatrix}6 & 4 & 1 & 0 & 0\\ 8 & 7 & 4 & 1 & 0\\ 2 & 4 & 6 & 4 & 1\\ 0 & 1 & 4 & 6 & 4\\ 0 & 0 & 1 & 4 & 6\end{bmatrix}$

Therefore, equation (2.77) becomes

$$
\left\{\begin{bmatrix} 0 & 2 & 0 & 6 & 0 \\ 0 & 0 & 8 & 0 & 1 \\ 0 & 0 & 0 & 1 & 20 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix} \\ 6 \\ + \\ 6 \\ \\ \end{matrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 6 & 8 \\ 0 & 0 & 2 & 6 & 8 \\ 0 & 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} + \right.
$$

$$
\left.\begin{bmatrix} \frac{3}{8} & \frac{1}{4} & \frac{1}{16} & 0 & 0 \\ \frac{1}{2} & \frac{7}{16} & \frac{1}{4} & \frac{1}{16} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} \\ 0 & \frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} \\ 0 & 0 & \frac{1}{16} & \frac{1}{4} & \frac{3}{8} \end{bmatrix}\right\} \begin{bmatrix} \frac{1}{2}a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \frac{-133}{128} \\ \frac{4}{32} \\ \frac{17}{32} \\ \frac{3}{32} \\ \frac{1}{128} \end{bmatrix} 5
$$

After simple computations, the augmented matrix of equation (2.77) is:

$$
\left[\begin{array}{ccccc|c} \frac{15}{8} & \frac{7}{2} & \frac{33}{16} & 9 & 4 & \frac{-133}{128} \\ 1 & \frac{47}{16} & \frac{25}{2} & \frac{97}{16} & 24 & \frac{45}{32} \\ \frac{1}{8} & \frac{1}{2} & \frac{31}{8} & \frac{37}{2} & \frac{129}{16} & \frac{17}{32} \\ 0 & \frac{1}{16} & \frac{1}{2} & \frac{39}{8} & \frac{49}{2} & \frac{3}{32} \\ 0 & 0 & \frac{1}{16} & \frac{1}{2} & \frac{47}{8} & \frac{1}{128} \end{array}\right] \tag{2.78}
$$

Now, by substituting equation (2.5) in the boundary condition given by equation (2.18.b) one can get the augmented matrix:

$$
\left[\begin{array}{ccccc|c} \frac{1}{4} & \frac{-7}{4} & \frac{1}{4} & \frac{-7}{4} & \frac{1}{4} & \frac{-5}{4} \end{array}\right] \tag{2.79}
$$

By replacing the row matrix given by equation (2.79) by the last row of augmented matrix given by equation (2.78), we have the new augmented matrix:

$$\left[\begin{array}{ccccc|c} \frac{15}{8} & \frac{7}{2} & \frac{33}{16} & 9 & 4 & \frac{-133}{128} \\ 1 & \frac{47}{16} & \frac{25}{2} & \frac{97}{16} & 24 & \frac{45}{32} \\ \frac{1}{8} & \frac{1}{2} & \frac{31}{8} & \frac{37}{2} & \frac{129}{16} & \frac{17}{32} \\ 0 & \frac{1}{16} & \frac{1}{2} & \frac{39}{8} & \frac{49}{2} & \frac{3}{32} \\ \frac{1}{4} & \frac{-7}{4} & \frac{1}{4} & \frac{-7}{4} & \frac{1}{4} & \frac{-5}{4} \end{array}\right] \tag{2.80}$$

Therefore

$$\begin{bmatrix} \frac{1}{2}a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \frac{15}{8} & \frac{7}{2} & \frac{33}{16} & 9 & 4 \\ 1 & \frac{47}{16} & \frac{25}{2} & \frac{97}{16} & 24 \\ \frac{1}{8} & \frac{1}{2} & \frac{31}{8} & \frac{37}{2} & \frac{129}{16} \\ 0 & \frac{1}{16} & \frac{1}{2} & \frac{39}{8} & \frac{49}{2} \\ \frac{1}{4} & \frac{-7}{4} & \frac{1}{4} & \frac{-7}{4} & \frac{1}{4} \end{bmatrix}^{-1} \begin{bmatrix} \frac{-133}{128} \\ \frac{45}{32} \\ \frac{17}{32} \\ \frac{3}{32} \\ \frac{-5}{4} \end{bmatrix} = \begin{bmatrix} \frac{-13}{8} \\ \frac{1}{2} \\ \frac{1}{8} \\ 0 \\ 0 \end{bmatrix}.$$

Hence

$$a_0 = \frac{-13}{8},\ a_1 = \frac{1}{2},\ a_2 = \frac{1}{8},\ a_3 = 0 \text{ and } \ a_4 = 0$$

by substituting these values into equation (2.76), one can obtain:

$$\begin{aligned} y^*(x) &= \frac{-13}{8}T_0^*(x) - \frac{1}{2}T_1^*(x) + \frac{1}{8}T_2^*(x) \\ &= x^2 - 2. \end{aligned}$$

Note that this solution is the exact solution of equation (2.18).

Example (2.4):

Consider the second order linear differential equation:

$$y''(x) - 2x^3 y'(x) + 8x^2 y(x) = 0, \quad 0 \le x \le 1 \qquad (2.81.a)$$

together with the boundary conditions:

$$y(0) = 1, \; y(1) = \frac{1}{3} \qquad (2.81.b)$$

Assume that the solution y can be approximated by the finite Chebyshev series of degree five:

$$y^*(x) = \sum_{r=0}^{5}{}' a_r T_r^*(x) \qquad (2.82)$$

Since $p(x)=1$, $q(x)= -2x^3$ and $r(x)=8x^2$, then from equations (2.61)-(2.63) one can have:

$$p_0 = 1, \; p_1 = p_2 = p_3 = p_4 = p_5 = 0$$
$$q_0 = q_1 = q_2 = q_4 = q_5 = 0, \; q_3 = -2$$

and

$$r_0 = r_1 = r_3 = r_4 = r_5 = 0, \; r_2 = 8$$

Then, by substituting the above expressions into equation (2.69) one can get:

$$\left\{16M_0M^2 + 8M_2 - 8M_3M\right\}A = 0 \qquad (2.83)$$

where

$$M = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} \\ 0 & 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; M_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M_2 = \frac{1}{16}\begin{bmatrix} 6 & 4 & 1 & 0 & 0 & 0 \\ 8 & 7 & 4 & 1 & 0 & 0 \\ 2 & 4 & 6 & 4 & 1 & 0 \\ 0 & 1 & 4 & 6 & 4 & 0 \\ 0 & 0 & 1 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 4 & 6 \end{bmatrix} \text{ and } M_3 = \frac{1}{64}\begin{bmatrix} 20 & 15 & 6 & 1 & 0 & 0 \\ 30 & 26 & 16 & 6 & 1 & 0 \\ 12 & 16 & 20 & 15 & 6 & 0 \\ 2 & 6 & 15 & 20 & 15 & 0 \\ 0 & 1 & 6 & 15 & 20 & 0 \\ 0 & 0 & 1 & 6 & 15 & 20 \end{bmatrix}$$

Therefore

$$\left\{16\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 & 8 & 0 \\ 0 & 0 & 0 & 6 & 0 & 3\;0 \\ 0 & 0 & 0 & 0 & 1 & 2\;0 \\ 0 & 0 & 0 & 0 & 0 & 2\;0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 6 & 4 & 1 & 0 & 0 & 0 \\ 8 & 7 & 4 & 1 & 0 & 0 \\ 2 & 4 & 6 & 4 & 1 & 0 \\ 0 & 1 & 4 & 6 & 4 & 0 \\ 0 & 0 & 1 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 4 & 6 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} 2 & 01 & 5\;6 & 1 & 0 & 0 \\ 3 & 02 & 61 & 6\;6 & 1 & 0 \\ 1 & 21 & 62 & 01 & 5\;6 & 0 \\ 2 & 6 & 15 & 20 & 15 & 0 \\ 0 & 1 & 6 & 15 & 20 & 0 \\ 0 & 0 & 1 & 6 & 15 & 20 \end{bmatrix}\right.$$

$$\left.\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} \\ 0 & 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right\}\begin{bmatrix} \frac{1}{2}a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

after simple computations the augmented matrix of equation (2.83) is:

$$\left[\begin{array}{cccccc|c} 3 & \frac{3}{4} & \frac{51}{4} & -6 & 120 & -10 & 0 \\ 4 & \frac{13}{8} & \frac{-9}{2} & \frac{679}{8} & -16 & 460 & 0 \\ 1 & \frac{5}{4} & -1 & \frac{-13}{4} & 177 & -20 & 0 \\ 0 & \frac{3}{8} & \frac{1}{2} & -3 & -11 & \frac{2405}{8} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{-1}{4} & -5 & \frac{-65}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} & 2 & \frac{19}{8} & 0 \end{array}\right] \qquad (2.84)$$

Now, by substituting equation (2.5) in the boundary conditions given by equation (2.81.b) one can get the augmented matrix:

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{array}\right]$$
$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & \frac{1}{3} \end{array}\right] \qquad (2.85)$$

By replacing the two row matrices of equation (2.85) by the last two row of augmented matrix of equation (2.83), one can get the new augmented matrix:

$$\left[\begin{array}{cccccc|c} 3 & \frac{3}{4} & \frac{51}{4} & -6 & 120 & -10 & 0 \\ 4 & \frac{13}{8} & \frac{-9}{2} & \frac{679}{8} & -16 & 460 & 0 \\ 1 & \frac{5}{4} & -1 & \frac{-13}{4} & 177 & -20 & 0 \\ 0 & \frac{3}{8} & \frac{1}{2} & -3 & -11 & \frac{2405}{8} & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & \frac{1}{3} \end{array}\right] \qquad (2.86)$$

Therefore

$$\begin{bmatrix} \frac{1}{2}a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 3 & \frac{3}{4} & \frac{51}{4} & -6 & 120 & -10 \\ 4 & \frac{13}{8} & \frac{-9}{2} & \frac{679}{8} & -16 & 460 \\ 1 & \frac{5}{4} & -1 & \frac{-13}{4} & 177 & -20 \\ 0 & \frac{3}{8} & \frac{1}{2} & -3 & -11 & \frac{2405}{8} \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{157}{192} \\ \frac{-7}{24} \\ \frac{-7}{48} \\ \frac{-1}{24} \\ \frac{-1}{192} \\ 0 \end{bmatrix}.$$

Hence

$$a_0 = \frac{157}{96},\ a_1 = \frac{-7}{24},\ a_2 = \frac{-7}{48},\ a_3 = \frac{-1}{24},\ a_4 = \frac{-1}{192} \text{ and } a_5 = 0$$

By substituting these values into equation (2.82) one can obtain:

$$y^*(x) = \frac{157}{192}T_0^*(x) - \frac{7}{24}T_1^*(x) - \frac{7}{48}T_2^*(x) - \frac{1}{24}T_3^*(x) - \frac{1}{192}T_4^*(x)$$
$$= 1 - \frac{2}{3}x^4.$$

Note that this solution is the exact solution of equation (2.81).

Chapter Three

Chebyshev Series Method for Solving Special Types of Integral Equations and Integro-Differential Equations

Introduction

The aim of this chapter is to use the finite Chebyshev series method of the first kind defined on the interval [0,1] as a method to solve special types of integral equations namely systems of linear Fredholm integral and integro-differential equations.

This chapter consists of three sections.

In section one, we use the Chebyshev series method to solve the homogeneous and nonhomogeneous linear Fredholm integral equations. This method is based on some properties that appeared in chapter one and it is a modification of the method that appeared in [Sezer M. and Dogan S.,1996].

In section two, we use the same method to solve systems of linear Fredholm integral equations of the second kind.

In section three, solutions of the linear Fredholm integro-differential equations via Chebyshev series method are presented.

3.1 Chebyshev Series Method for Solving Linear Fredholm Integral Equations:

In this section we use the finite Chebyshev series as a method to solve the homogeneous and the nonhomogeneous linear Fredholm integral equations. To do this, consider first the nonhomogeneous linear Fredholm integral equation:

$$u(x) = f(x) + \lambda \int_0^1 k(x,t)\, u(t)\, dt, \qquad 0 \le x \le 1 \tag{3.1}$$

where f is a known function of x, named as the driving term, k is a known function of x and t, known as the kernel of the integral equation, λ is a scalar parameter and u is the unknown function that must be determined.

Assume that the solution of the above integral equation may be approximated as a finite Chebyshev series that takes the form:

$$u(x) \approx u^*(x) = \sum_{r=0}^{N}{}^{/} a_r T_r^*(x), \; 0 \le x \le 1 \tag{3.2}$$

where $\{a\}_{r=0}^{N}$ are the Chebyshev coefficients that must be determined. This approximated solution may be written in matrix form as:

$$u^*(x) = T_x^* A \tag{3.3}$$

where $T_x^* = \left[T_0^*(x) \quad T_1^*(x) \quad . \quad . \; T_N^*(x)\right]$ and $A = \left[\frac{1}{2}a_0 \quad a_1 \quad \ldots \quad a_N\right]^T$.

Moreover, assume that the function f may be approximated as a finite Chebyshev series that takes the form:

$$f(x) \approx f^*(x) = \sum_{r=0}^{N}{}^{/} f_r T_r^*(x)$$

where $\{f\}_{r=0}^{N}$ are known Chebyshev coefficients.

Then $f^*(x)$ can be written in the matrix form:

$$f^*(x) = T_x^* F \tag{3.4}$$

where $F = \left[\frac{1}{2} f_0 \quad f_1 \quad \cdots \quad f_N \right]^T$.

Assume that the kernel function k can be approximated by a double finite Chebyshev series of degree N in both x and t that takes the form:

$$k(x,t) \approx k^*(x,t) = \sum_{r=0}^{N}{}' \sum_{s=0}^{N}{}' k_{r,s} T_r^*(x) T_s^*(t)$$

where $\{k_{r,s}\}_{0,0}^{N,N}$ are the known double Chebyshev coefficients.

Then the approximated kernel function k^* can be written in the matrix form:

$$k^*(x,t) = T_x^* K T_t^{*T} \tag{3.5}$$

where

$$T_t^* = \left[T_0^*(t) \quad T_1^*(t) \quad . \; . \; . T_N^*(t) \right], \quad K = \begin{bmatrix} \frac{1}{4}k_{0,0} & \frac{1}{2}k_{0,1} & \cdots & \frac{1}{2}k_{0,N} \\ \frac{1}{2}k_{1,0} & k_{1,1} & \cdots & k_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}k_{N,0} & k_{N,1} & \cdots & k_{N,N} \end{bmatrix}$$

On the other hand, for the unknown function u in the integrand, we write:

$$u(t) \approx u^*(t) = T_t^* A \tag{3.6}$$

By substituting the matrix forms given by equations (3.3)-(3.6) into equation (3.1) and by simplifying the resulting equation one can get:

$$A = F + \lambda K \left\{ \int_0^1 T_t^{*T} T_t^* \, dt \right\} A$$

or

$$\left(I - \lambda KQ\right) A = F \tag{3.7}$$

where

$$Q = \begin{bmatrix} \int_0^1 T_0^*(t) T_0^*(t) dt & \int_0^1 T_0^*(t) T_1^*(t) dt & \cdots & \int_0^1 T_0^*(t) T_N^*(t) dt \\ \int_0^1 T_1^*(t) T_0^*(t) dt & \int_0^1 T_1^*(t) T_1^*(t) dt & \cdots & \int_0^1 T_1^*(t) T_N^*(t) dt \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^1 T_N^*(t) T_0^*(t) dt & \int_0^1 T_N^*(t) T_1^*(t) dt & \cdots & \int_0^1 T_N^*(t) T_N^*(t) dt \end{bmatrix}$$

and I is the $(N+1) \times (N+1)$ identity matrix. The elements of the matrix Q are denoted by q_{ij} and take the form:

Let $q_{ij} = \int_0^1 T_i^*(t)\, T_j^*(t)\, d$

$$q_{ij} = \begin{cases} \dfrac{1}{2}\left[\dfrac{1}{1-(i+j)^2} + \dfrac{1}{1-(i-j)^2}\right] & (i+j) \text{ is an even posetive integer} \\ 0 & (i+j) \text{ is an odd posetive integer} \end{cases}$$

In equation (3.7), if $\left| I - \lambda KQ \right| \neq 0$, then

$$A = \left(I - \lambda KQ\right)^{-1} F \tag{3.8}$$

Thus the unknown coefficients $\{a\}_{r=0}^{N}$ are uniquely determined by equation (3.8) and hence the integral equation (3.1) has a unique approximate solution given by equation (3.2).

Second, consider the homogeneous linear Fredholm integral equation that takes the form:

$$u(x)=\lambda\int_0^1 k(x,t)\,u(t)\,dt, \qquad 0\le x\le 1 \tag{3.9}$$

The problem here is to determine the generalized eigenvalue λ of the pair of operators $\left(I,\int_0^1 k(x,t).dt\right)$ with the corresponding eigenfunction *u*, where *I* is the identity operator.

In this case (λ,*u*) is said to be the generalized eigenpair of the pair of operators $\left(I,\int_0^1 k(x,t).dt\right)$,[Jerri A., 1985].

By substituting the matrix forms given by equations (3.3),(3.5)-(3.6) into equation (3.9) and by simplifying the resulting equation one can get:

$$A=\lambda K\left\{\int_0^1 T_t^{*T}\,T_t^{*}\,dt\right\}A$$

or

$$\left(I-\lambda KQ\right)A=0 \tag{3.10}$$

where *I*, *K* and *Q* are defined previously.

In equation (3.10) if

$$\left| I - \lambda KQ \right| = 0 \tag{3.11}$$

then λ is the algebraic generalized eigenvalue of the pair of matrices (I,KQ). By substituting the values of λ into equation (3.10) and solving the resulting system of equations one can get the corresponding eigenvectors A.

By substituting the values of $\{a_r\}_{r=0}^{N}$ one can get the approximated eigenpairs (λ,u^*) of the pair of operator $\left(I, \int_0^1 k(x,t).dt \right)$.

To illustrate this method consider the following examples:

Example (3.1):

Consider the nonhomogeneous linear Fredholm integral equation:

$$u(x) = \frac{-x}{4} + \int_0^1 (xt + 5x^2t^2)u(t)dt,\ 0 \le x \le 1$$

Assume that the solution u can be approximated by the finite Chebyshev series of degree two. That is:

$$u(x) \approx \sum_{r=0}^{2}{}' a_r T_r^*(x) \tag{3.12}$$

where $\{a_r\}_{r=0}^{2}$ are the unknown Chebyshev coefficients that must be determined

By using proposition (1.9) one can get:

$$f(x)=\frac{-x}{4}$$
$$=\frac{-1}{8}T_0^*(x)-\frac{1}{8}T_1^*(x)$$

and

$$k(x,t)=xt+5x^2t^2=\frac{1}{4}\left[T_0^*(x)+T_1^*(x)\right]\left[T_0^*(t)+T_1^*(t)\right]+$$
$$\frac{5}{64}\left[3T_0^*(x)+4T_1^*(x)+T_2^*(x)\right]\left[3T_0^*(t)+4T_1^*(t)+T_2^*(t)\right]$$
$$=\frac{61}{64}T_0^*(x)T_0^*(t)+\frac{76}{64}T_0^*(x)T_1^*(t)+\frac{76}{64}T_0^*(t)T_1^*(x)+\frac{15}{64}T_0^*(x)T_2^*(t)+$$
$$\frac{15}{64}T_0^*(t)T_2^*(x)+\frac{96}{64}T_1^*(x)T_1^*(t)+\frac{20}{64}T_1^*(x)T_2^*(t)+\frac{20}{64}T_1^*(t)T_2^*(x)+$$
$$\frac{5}{64}T_2^*(x)T_2^*(t)$$

Therefore

$$F=\begin{bmatrix}\frac{-1}{8}\\ \frac{-1}{8}\\ 0\end{bmatrix} \text{ and } K=\begin{bmatrix}\frac{61}{64} & \frac{76}{64} & \frac{15}{64}\\ \frac{76}{64} & \frac{96}{64} & \frac{20}{64}\\ \frac{15}{64} & \frac{20}{64} & \frac{5}{64}\end{bmatrix}.$$

On the other hand

$$Q=\begin{bmatrix}1 & 0 & \frac{-1}{3}\\ 0 & \frac{1}{3} & 0\\ \frac{-1}{3} & 0 & \frac{7}{15}\end{bmatrix}.$$

By substituting these matrices into equation (3.8) one can have:

$$\begin{bmatrix} \frac{1}{2}a_0 \\ a_1 \\ a_2 \end{bmatrix} = \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{61}{64} & \frac{76}{64} & \frac{15}{64} \\ \frac{76}{64} & \frac{96}{64} & \frac{20}{64} \\ \frac{15}{64} & \frac{20}{64} & \frac{5}{64} \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{-1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{-1}{3} & 0 & \frac{7}{15} \end{bmatrix} \right]^{-1} \begin{bmatrix} \frac{-1}{8} \\ \frac{-1}{8} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{8} & \frac{-19}{48} & \frac{5}{24} \\ \frac{-13}{12} & \frac{1}{2} & \frac{1}{4} \\ \frac{-5}{24} & \frac{-5}{48} & \frac{25}{24} \end{bmatrix}^{-1} \begin{bmatrix} \frac{-1}{8} \\ \frac{-1}{8} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ \frac{1}{2} \\ \frac{1}{8} \end{bmatrix}.$$

Therefore

$$\frac{a_0}{2} = \frac{3}{8},\ a_1 = \frac{1}{2} \text{ and } a_2 = \frac{1}{8}.$$

By substituting these values into equation (3.12) one can obtain

$$\begin{aligned} u(x) &\approx \frac{3}{8}T_0^*(x) + \frac{1}{2}T_1^*(x) + \frac{1}{8}T_2^*(x) \\ &= \frac{3}{8} + \frac{1}{2}(2x-1) + \frac{1}{8}(8x^2 - 8x + 1) \\ &= x^2 \end{aligned}$$

is the approximated solution of the above integral equation. In this case, $u(x) \approx x^2$ is the exact solution of this example.

Example (3.2):

Consider the homogeneous linear Fredholm integral equation:

$$u(x)=\lambda\int_0^1 xt\,u(t)dt,\ \ 0\le x\le 1$$

Assume that the approximated solution is given by equation (3.12). By using proposition (1.9) one can have:

$$k(x,t)=xt=\frac{1}{4}\left[T_0^*(x)+T_1^*(x)\right]\left[T_0^*(t)+T_1^*(t)\right]$$
$$=\frac{1}{4}T_0^*(x)T_0^*(t)+\frac{1}{4}T_0^*(x)T_1^*(t)+\frac{1}{4}T_0^*(t)T_1^*(x)+\frac{1}{4}T_1^*(x)T_1^*(t).$$

Hence

$$K=\begin{bmatrix}\frac{1}{4} & \frac{1}{4} & 0\\ \frac{1}{4} & \frac{1}{4} & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Therefore

$$|I-\lambda KQ|=\left|\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\lambda\begin{bmatrix}\frac{1}{4} & \frac{1}{4} & 0\\ \frac{1}{4} & \frac{1}{4} & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & \frac{-1}{3}\\ 0 & \frac{1}{3} & 0\\ \frac{-1}{3} & 0 & \frac{7}{15}\end{bmatrix}\right|$$
$$=\left|\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\lambda\begin{bmatrix}\frac{1}{4} & \frac{1}{12} & \frac{-1}{12}\\ \frac{1}{4} & \frac{1}{12} & \frac{-1}{12}\\ 0 & 0 & 0\end{bmatrix}\right|$$

$$|I-\lambda KQ| = \begin{vmatrix} 1-\frac{\lambda}{4} & \frac{-\lambda}{12} & \frac{\lambda}{12} \\ \frac{-\lambda}{4} & 1-\frac{\lambda}{12} & \frac{\lambda}{12} \\ 0 & 0 & 1 \end{vmatrix} = 0$$

Therefore

$\left(1-\frac{\lambda}{4}\right)\left(1-\frac{\lambda}{12}\right)-\frac{\lambda^2}{48}=0$ and this implies that $\lambda=3$.

By substitution $\lambda=3$ into equation (3.10) one can have:

$$\begin{bmatrix} \frac{1}{4} & \frac{-1}{4} & \frac{1}{4} \\ \frac{-3}{4} & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

By solving the above homogeneous system of linear equations one can have

$a_1 = \frac{1}{2}a_0$ and $a_2 = 0$

By substituting these values into equation (3.12) one can obtain

$$\begin{aligned} u(x) &= \frac{a_0}{2}T_0^*(x) + \frac{1}{2}a_0 T_1^*(x) \\ &= \frac{a_0}{2}[1+2x-1] \\ &= a_0 x,\ a_0 \neq 0. \end{aligned}$$

Hence $(3, a_0 x)$, $a_0 \neq 0$ is the approximated eigenpair of the pair of operators $\left(I, \int_0^1 xt.dt \right)$. In this case $(3, a_0 x)$ is the exact eigenpair of the pair of operations $\left(I, \int_0^1 xt.dt \right)$.

3.2 Chebyshev Series Method for Solving Systems of Linear Fredholm Integral Equations:

In this section we modify the previous method to solve systems of linear Fredholm integral equations that take the form:

$$u_i(x) = f_i(x) + \sum_{j=1}^{n} \lambda_{ij} \int_0^1 k_{ij}(x,t) u_j(t) dt, \quad 0 \leq x \leq 1, \; i = 1, 2, \ldots, n \qquad (3.13)$$

where f_i is a known functions of x, k_{ij} is a known functions of x and t, λ_{ij} is a scalar parameter and $\{u\}_{i=1}^{n}$ are the unknown functions that must be determined.

Assume that the solution $\{u\}_{i=1}^{n}$ of the above system of integral equations can be approximated as a finite Chebyshev series that takes the form:

$$u_i(x) \approx u_i^*(x) = \sum_{r=0}^{N}{}' a_{ir} T_r^*(x), \; i = 1, 2, \ldots, n \qquad (3.14)$$

where $\{a_{ir}\}_{1,0}^{n,N}$ are the Chebyshev coefficients that must be determined. This approximated solution can be written in the matrix form:

$$u_i^*(x) = T_x^* A_i\,,\ i = 1,2,\ldots,n \tag{3.15}$$

where T_x^* is defined previously and $A_i = \begin{bmatrix} \frac{1}{2}a_{i0} & a_{i1} & \ldots & a_{iN} \end{bmatrix}^T$.

Moreover, assume that the function f_i can be approximated as a finite Chebyshev series that takes the form:

$$f_i(x) \approx f_i^*(x) = \sum_{r=0}^{N}{}' f_{i,r} T_r^*(x),\ i = 1,2,\ldots,n$$

where $\{f_{i,r}\}_{1,0}^{n,N}$ are known Chebyshev coefficients. Then $f_i^*(x)$ can be written in the matrix form:

$$f_i^*(x) = T_x^* F_i\,,\ i = 1,2,\ldots,n \tag{3.16}$$

where $F_i = \begin{bmatrix} \frac{1}{2}f_{i,0} & f_{i,1} & \ldots & f_{i,N} \end{bmatrix}^T,\ i = 1,2,\ldots,n.$

Assume that the approximated kernel function k_{ij} can be approximated by a double finite Chebyshev series of degree N in both x and t that takes the form:

$$k_{ij}(x,t) \approx k_{ij}^*(x,t) = \sum_{r=0}^{N}{}' \sum_{s=0}^{N}{}' k_{ij}^{r,s}\, T_r^*(x) T_s^*(t),\ i,j = 1,2,\ldots,n$$

where $\left\{k_{ij}^{r,s}\right\}_{ij=1,r,s=0}^{n,N}$ are the known double Chebyshev coefficients.

Then the approximated kernel function k_{ij}^* can be written in the matrix form:

$$k_{ij}^*(x,t) = T_x^* K_{ij} T_t^{*T}, \; i,j = 1,2,\dots,n \tag{3.17}$$

where

$$K_{ij} = \begin{bmatrix} \frac{1}{4}k_{ij}^{0,0} & \frac{1}{2}k_{ij}^{0,1} & \cdots & \frac{1}{2}k_{ij}^{0,N} \\ \frac{1}{2}k_{ij}^{1,0} & k_{ij}^{1,1} & \cdots & k_{ij}^{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}k_{ij}^{N,0} & k_{ij}^{N,1} & \cdots & k_{ij}^{N,N} \end{bmatrix}, \; i,j = 1,2,\dots,n$$

On the other hand, for the unknown function u_j in the integrand, we write

$$u_j(t) \approx u_j^*(t) = T_t^{\,*} A_j, \; j = 1,2,\dots,n \tag{3.18}$$

By substituting the matrix forms given by equations (3.15)-(3.18) into equation (3.13) and then by simplifying the resulting equation one can get:

$$A_i = F_i + \sum_{j=1}^{n} \lambda_{ij} K_{ij} Q \, A_j, \; i = 1,2,\dots,n \tag{3.19}$$

Let

$$A = \begin{bmatrix} \frac{1}{2}a_{10} & a_{11} & \cdots & a_{1N} & \frac{1}{2}a_{20} & a_{21} & \cdots & a_{2N} & \cdots & \frac{1}{2}a_{n0} & a_{n1} & \cdots & a_{nN} \end{bmatrix}^T$$

$$F = \begin{bmatrix} \frac{1}{2}f_{1,0} & f_{1,1} & \cdots & f_{1,N} & \frac{1}{2}f_{2,0} & f_{2,1} & \cdots & f_{2,N} & \cdots & \frac{1}{2}f_{n,0} & f_{n,1} & \cdots & f_{n,N} \end{bmatrix}^T$$

and

$$K=\begin{bmatrix}\lambda_{11}k_{11}Q & \lambda_{12}k_{12}Q & \cdots & \lambda_{1n}k_{1n}Q \\ \lambda_{21}k_{21}Q & \lambda_{22}k_{22}Q & \cdots & \lambda_{2n}k_{2n}Q \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1}k_{n1}Q & \lambda_{n2}k_{n2}Q & \cdots & \lambda_{nn}k_{nn}Q\end{bmatrix}$$

then equation (3.19) can be rewritten as:

$$(I-K)A=F$$

where I is the identity matrix.

In the above equation, if

$|I-K|\neq 0$, then

$$A=(I-K)^{-1}F \tag{3.20}$$

Thus the unknown coefficients $\{a_{ir}\}_{1,0}^{n,N}$ are uniquely determined by equation (3.20) and hence the system of integral equations given by equation (3.13) has a unique solution given by equation (3.14).

To illustrate this method consider the following example:

Example (3.3):

Consider the system which consist of two linear Fredholm integral equations:

$$u_1(x) = x^2 - 1 - \frac{6}{5}x + \int_0^1 (3x + 4t)u_1(t)dt + \int_0^1 xt\, u_2(t)dt, \ 0 \le x \le 1$$

$$u_2(x) = x^3 - \frac{1}{3}x^2 - \frac{1}{7}x - \frac{1}{4} + \int_0^1 (x^2 + t)u_1(t)dt + \int_0^1 xt^3 u_2(t)dt, \ 0 \le x \le 1$$

Assume that the solution $\{u\}_{i=1}^{2}$ can be approximated by the finite Chebyshev series of degree three. That is:

$$u_1(x) = \sum_{r=0}^{3}{}' a_{1r} T_r^*(x) \tag{3.21}$$

and

$$u_2(x) = \sum_{r=0}^{3}{}' a_{2r} T_r^*(x) \tag{3.22}$$

By using proposition (1.9) one can have:

$$\begin{aligned} f_1(x) &= x^2 - 1 - \frac{6}{5}x \\ &= \frac{1}{8}\left[3T_0^*(x) + 4T_1^*(x) + T_2^*(x)\right] - T_0^*(x) - \frac{6}{10}\left[T_0^*(x) + T_1^*(x)\right] \\ &= \frac{-49}{40}T_0^*(x) - \frac{1}{10}T_1^*(x) + \frac{1}{8}T_2^*(x) \end{aligned}$$

$$\begin{aligned} f_2(x) &= x^3 - \frac{1}{4} - \frac{1}{3}x^2 - \frac{1}{7}x \\ &= \frac{1}{32}\left[10T_0^*(x) + 15T_1^*(x) + 6T_2^*(x) + T_3^*(x)\right] - \frac{1}{4}T_0^*(x) - \\ &\quad \frac{1}{24}\left[3T_0^*(x) + 4T_1^*(x) + T_2^*(x)\right] - \frac{1}{14}\left[T_0^*(x) + T_1^*(x)\right] \\ &= \frac{-90}{672}T_0^*(x) + \frac{310}{1344}T_1^*(x) + \frac{7}{48}T_2^*(x) + \frac{1}{32}T_3^*(x) \end{aligned}$$

$$
\begin{aligned}
k_{11}(x,t) &= 3x+4t \\
&= \frac{3}{2}\left[T_0^*(x)+T_1^*(x)\right]+2\left[T_0^*(t)+T_1^*(t)\right] \\
&= \frac{7}{2}T_0^*(x)T_0^*(t)+\frac{3}{2}T_0^*(t)T_1^*(x)+2T_1^*(t)T_0^*(x)
\end{aligned}
$$

$$
\begin{aligned}
k_{12}(x,t) &= xt \\
&= \frac{1}{4}\left[T_0^*(x)+T_1^*(x)\right]\left[T_0^*(t)+T_1^*(t)\right] \\
&= \frac{1}{4}T_0^*(x)T_0^*(t)+\frac{1}{4}T_0^*(x)T_1^*(t)+\frac{1}{4}T_1^*(x)T_0^*(t)+\frac{1}{4}T_1^*(x)T_1^*(t),
\end{aligned}
$$

$$
\begin{aligned}
k_{21}(x,t) &= x^2+t \\
&= \frac{3}{8}T_0^*(x)+\frac{1}{2}T_1^*(x)+\frac{1}{8}T_2^*(x)+\frac{1}{2}\left[T_0^*(t)+T_1^*(t)\right] \\
&= \frac{7}{8}T_0^*(x)T_0^*(t)+\frac{1}{2}T_1^*(x)T_0^*(t)+\frac{1}{8}T_2^*(x)T_0^*(t)+\frac{1}{2}T_1^*(t)T_0^*(x),
\end{aligned}
$$

and

$$
\begin{aligned}
k_{22}(x,t) &= xt^3 \\
&= \frac{1}{2}\left[T_0^*(x)+T_1^*(x)\right]\frac{1}{32}\left[10T_0^*(t)+15T_1^*(t)+6T_2^*(t)+T_3^*(t)\right] \\
&= \frac{1}{64}\Big[10T_0^*(x)T_0^*(t)+15T_0^*(x)T_1^*(t)+6T_0^*(x)T_2^*(t)+T_0^*(x)T_3^*(t)+ \\
&\qquad 10T_1^*(x)T_0^*(t)+15T_1^*(x)T_1^*(t)+6T_1^*(x)T_2^*(t)+T_1^*(x)T_3^*(t)\Big]
\end{aligned}
$$

Hence

$$F_1=\begin{bmatrix}\frac{-49}{40}\\ \frac{-1}{10}\\ \frac{1}{8}\\ 0\end{bmatrix},\quad F_2=\begin{bmatrix}\frac{-90}{672}\\ \frac{310}{1344}\\ \frac{7}{48}\\ \frac{1}{32}\end{bmatrix},\quad K_{11}=\begin{bmatrix}\frac{7}{2}&2&0&0\\ \frac{3}{2}&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{bmatrix},\quad K_{12}=\begin{bmatrix}\frac{1}{4}&\frac{1}{4}&0&0\\ \frac{1}{4}&\frac{1}{4}&0&0\\ 0&0&0&0\\ 0&0&0&0\end{bmatrix},$$

$$K_{21}=\begin{bmatrix}\frac{7}{8}&\frac{1}{2}&0&0\\ \frac{1}{2}&\frac{1}{2}&0&0\\ \frac{1}{8}&0&0&0\\ 0&0&0&0\end{bmatrix} \text{ and } K_{22}=\begin{bmatrix}\frac{10}{64}&\frac{15}{64}&\frac{6}{64}&\frac{1}{64}\\ \frac{10}{64}&\frac{15}{64}&\frac{6}{64}&\frac{1}{64}\\ 0&0&0&0\\ 0&0&0&0\end{bmatrix}.$$

On the other hand

$$Q=\begin{bmatrix}1&0&\frac{-1}{3}&0\\ 0&\frac{1}{3}&0&\frac{-1}{5}\\ \frac{-1}{3}&0&\frac{7}{15}&0\\ 0&\frac{-1}{5}&0&\frac{17}{35}\end{bmatrix}.$$

By substituting these matrices into equation (3.19) one can have:

$$\begin{bmatrix} \frac{1}{2}a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} = \begin{bmatrix} \frac{-49}{40} \\ \frac{-1}{10} \\ \frac{1}{8} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{2} & 2 & 0 & 0 \\ \frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{-1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{-1}{5} \\ \frac{-1}{3} & 0 & \frac{7}{15} & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{17}{35} \end{bmatrix} \begin{bmatrix} \frac{1}{2}a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} +$$

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{-1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{-1}{5} \\ \frac{-1}{3} & 0 & \frac{7}{15} & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{17}{35} \end{bmatrix} \begin{bmatrix} \frac{1}{2}a_{20} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{1}{2}a_{20} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = \begin{bmatrix} \frac{-90}{672} \\ \frac{310}{1344} \\ \frac{7}{48} \\ \frac{1}{32} \end{bmatrix} + \begin{bmatrix} \frac{7}{8} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{-1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{-1}{5} \\ \frac{-1}{3} & 0 & \frac{7}{15} & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{17}{35} \end{bmatrix} \begin{bmatrix} \frac{1}{2}a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} +$$

$$\begin{bmatrix} \frac{10}{64} & \frac{15}{64} & \frac{6}{64} & \frac{1}{64} \\ \frac{10}{64} & \frac{15}{64} & \frac{6}{64} & \frac{1}{64} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{-1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{-1}{5} \\ \frac{-1}{3} & 0 & \frac{7}{15} & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{1}{35} \end{bmatrix} \begin{bmatrix} \frac{1}{2}a_{20} \\ a_{21} \\ a_{22} \\ 7a_{23} \end{bmatrix}.$$

The above two systems of equations can be rewritten as:

$$\begin{bmatrix} \frac{1}{2}a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ \frac{1}{2}a_{20} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = \begin{bmatrix} -1.225 \\ -0.1 \\ 0.125 \\ 0 \\ -0.134 \\ 0.231 \\ 0.146 \\ 0.031 \end{bmatrix} + \begin{bmatrix} 3.5 & 0.667 & -1.167 & -0.4 & 0.25 & 0.083 & -0.083 & -0.05 \\ 1.5 & 0 & -0.5 & 0 & 0.25 & 0.083 & -0.083 & -0.05 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.875 & 0.167 & -0.292 & -0.1 & 0.125 & 0.075 & -8.333\times10^{-3} & -0.039 \\ 0.5 & 0 & -0.167 & 0 & 0.125 & 0.075 & -8.333\times10^{-3} & -0.039 \\ 0.125 & 0 & -0.042 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ \frac{1}{2}a_{20} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix}$$

Therefore

$$\begin{bmatrix} \frac{1}{2}a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ \frac{1}{2}a_{20} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = \begin{bmatrix} -0.239 & -0.181 & 0.413 & 0.108 & -0.127 & -0.048 & 0.036 & 0.028 \\ -0.456 & 0.708 & 0.152 & 0.175 & 0.076 & 0.029 & -0.022 & -0.017 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.341 & -0.054 & 0.113 & 0.033 & 1.037 & 0.049 & 0.024 & -0.023 \\ -0.175 & -0.105 & 0.058 & 0.063 & 0.027 & 1.062 & 0.014 & -0.03 \\ -0.03 & -0.023 & 9.654\times10^{-3} & 0.014 & -0.016 & -5.998\times10^{-3} & 1.005 & 3.478\times10^{-3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1.225 \\ -0.1 \\ 0.125 \\ 0 \\ -0.134 \\ 0.231 \\ 0.146 \\ 0.031 \end{bmatrix}$$

and this implies that:

$$\frac{a_{10}}{2} = \frac{3}{8},\ a_{11} = \frac{1}{2},\ a_{12} = \frac{1}{8}, a_{13} = 0, \frac{a_{20}}{2} = \frac{10}{32}, a_{21} = \frac{15}{32}, a_{22} = \frac{6}{32} \text{ and } a_{23} = \frac{1}{32}$$

By substituting these values into equation (3.21) and equation (3.22) one can obtain:

$$
\begin{aligned}
u_1(x) &\approx \frac{3}{8}T_0^*(x) + \frac{1}{2}T_1^*(x) + \frac{1}{8}T_2^*(x) \\
&= \frac{3}{8} + \frac{1}{2}(2x-1) + \frac{1}{8}(8x^2-8x+1) \\
&= x^2
\end{aligned}
$$

and

$$
\begin{aligned}
u_2(x) &\approx \frac{10}{32}T_0^*(x) + \frac{15}{32}T_1^*(x) + \frac{6}{32}T_2^*(x) + \frac{1}{32}T_3^*(x) \\
&= \frac{10}{32} + \frac{15}{32}(2x-1) + \frac{6}{32}(8x^2-8x+1) + \frac{1}{32}(32x^3-48x^2+18x-1) \\
&= x^3.
\end{aligned}
$$

In this case $u_1(x) \approx x^2,\ u_2(x) \approx x^3$ is the exact solution of this example.

3.3 Chebyshev Series Method for Solving Linear Fredholm Integro-Differential Equations:

In this section we use the finite Chebyshev method to solve the first order linear Fredholm integro-differential equation of the second kind:

$$
u'(x) = f(x) + \lambda \int_0^1 k(x,t)\,u(t)\,dt, \qquad 0 \le x \le 1 \tag{3.23}
$$

where f is a known function of x, k is a known function of x and t, known as the kernel of the integro-differential equation, λ is a scalar parameter and u is the unknown function that must be determined.

Assume that the solution u can be approximated as in equation (3.3), f can be approximated as in equation (3.4) and the kernel function k can be approximated as in equation (3.5).

Form equation (2.55) $u'(x)$ can be written in the form:

$$u'(x) \approx T_x^* A^{(1)}$$

$$= 4T^* M A$$

(3.24)

By substituting the matrix forms given by equations (3.4)-(3.6) and equation (3.24) into equation (3.23) and then by simplifying the resulting equation one can get:

$$4MA = F + \lambda KQA$$

or

$$(4M - \lambda KQ)A = F \tag{3.25}$$

where Q is defined previously and M is defined either by equation (2.51) or equation (2.53)

In equation (3.25), if $|4M - \lambda KQ| \neq 0$, then

$$A = (4M - \lambda KQ)^{-1} F \tag{3.26}$$

Thus the unknown coefficients $\{a\}_{r=0}^{N}$ are uniquely determined by equation (3.26) and hence the integro-differential equation (3.23) has a unique approximatesolution given by equation (3.2).

To illustrate this method consider the following example:

Example (3.4):

Consider the nonhomogeneous linear Fredholm integro-differential equation:

$$u'(x) = 2x - \frac{4}{3}x^2 + \frac{13}{12} + \int_0^1 (x^2 + t)u(t)\,dt,\ \ 0 \le x \le 1.$$

Assume that the solution u can be approximated by the finite Chebyshev series of degree two. That is

$$u(x) \approx \sum_{r=0}^{2}{}' a_r T_r^*(x) \tag{3.27}$$

By using proposition (1.9) one can get:

$$\begin{aligned} f(x) &= 2x - \frac{4}{3}x^2 + \frac{13}{12} \\ &= T_0^*(x) + T_1^*(x) - \frac{1}{6}\left[3T_0^*(x) + 4T_1^*(x) + T_2^*(x)\right] + \frac{13}{12} \\ &= \frac{19}{12}T_0^*(x) + \frac{1}{3}T_1^*(x) - \frac{1}{6}T_2^*(x) \end{aligned}$$

and

$$\begin{aligned} k(x,t) &= x^2 + t \\ &= \frac{3}{8}T_0^*(x) + \frac{4}{8}T_1^*(x) + \frac{1}{8}T_2^*(x) + \frac{1}{2}T_0^*(t) + \frac{1}{2}T_1^*(t) \end{aligned}$$

Therefore

$$F = \begin{bmatrix} \frac{19}{12} \\ \frac{1}{3} \\ \frac{-1}{6} \end{bmatrix} \text{ and } K = \begin{bmatrix} \frac{7}{8} & \frac{1}{2} & 0 \\ \frac{4}{8} & 0 & 0 \\ \frac{1}{8} & 0 & 0 \end{bmatrix}.$$

On the other hand

$$Q = \begin{bmatrix} 1 & 0 & \frac{-1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{-1}{3} & 0 & \frac{7}{15} \end{bmatrix} \text{ and } M = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

By substituting these matrices into equation (3.26) one can have:

$$\begin{bmatrix} \frac{1}{2}a_0 \\ a_1 \\ a_2 \end{bmatrix} = \left[\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{7}{8} & \frac{1}{2} & 0 \\ \frac{4}{8} & 0 & 0 \\ \frac{1}{8} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{-1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{-1}{3} & 0 & \frac{7}{15} \end{bmatrix} \right]^{-1} \begin{bmatrix} \frac{19}{12} \\ \frac{1}{3} \\ \frac{-1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-7}{8} & \frac{11}{6} & \frac{7}{24} \\ \frac{-1}{2} & 0 & \frac{49}{6} \\ \frac{-1}{8} & 0 & \frac{1}{24} \end{bmatrix}^{-1} \begin{bmatrix} \frac{19}{12} \\ \frac{1}{3} \\ \frac{-1}{6} \end{bmatrix} = \begin{bmatrix} \frac{11}{8} \\ \frac{3}{2} \\ \frac{1}{8} \end{bmatrix}$$

and this implies that

$$\frac{a_0}{2} = \frac{11}{8},\ a_1 = \frac{3}{2} \text{ and } a_2 = \frac{1}{8}.$$

By substituting these values into equation (3.27) one can obtain:

$$u(x) \approx \frac{11}{8}T_0^*(x) + \frac{3}{2}T_1^*(x) + \frac{1}{8}T_2^*(x)$$
$$= \frac{11}{8} + \frac{3}{2}(2x-1) + \frac{1}{8}(8x^2-8x+1)$$
$$= x^2 + 2x$$

which is the exact solution of this example.

Remark (3.1):

This method can be also extended to solve system of first order linear Fredholm integro-differential equations of the second kind:

$$u_i'(x) = f_i(x) + \sum_{j=1}^{n} \lambda_{ij} \int_0^1 k_{ij}(x,t) u_j(t)\,dt, \quad 0 \le x \le 1, \; i = 1,2,\ldots,n .$$

Conclutions and Recommendations

Conclusions and Recommendations

From the present study, we can conclude the following:

1. When the problem is defined in a finite range $a \le x \le b$, then by using the linear transformations $x = \frac{1}{2}(b-a)t + \frac{1}{2}(b+a)$ and $x = (b-a)t + a$, the range $a \le x \le b$ can be transformated to the range $-1 \le t \le 1$ and $0 \le t \le 1$ respectively, which are the domain of T_r and T_r^* respectively.
2. The Chebyshev polynomials of the second kind defined on [0,1], denoted by $U_r^*(x)$, can be defined by:

   $U_r^*(x) = U_r(2x-1),\ 0 \le x \le 1,\ r = 0,1,\dots.$
3. Proposition (1.16) does not hold in case r is an odd positive integer since the constant of integration is taken to be nonzero. For example:

   $$\int U_1(x)dx = \int 2x\,dx = x^2 + c_1 = \frac{1}{2}T_2(x) + \frac{1}{2}T_0(x) + c_1 = T_2(x) + c\,.$$
4. From remark (1.5) one can see that the set of the Chebyshev polynomials of the first kind does not form an orthogonal sequence.
5. Remark (1.6) can be also include the sine Fourier series which is the infinite Chebyshev series of the second kind.
6. Chebyshev series method can be also used to solve the linear Fredholm integral equations of the first kind in case K^{-1} exists.
7. Chebyshev-matrix method fails to be applied in case $a_r \ne 0$ for $r > N$.

For future work the following problems could be recommended:

1. Modify the Chebyshev series method to solve the linear Volterra integral and integro-differential equations.

2. Extend the Chebyshev-matrix method to solve nonlinear differential equations.
3. Use Chebyshev series method for solving nonlinear problems.
4. Devote the Chebyshev polynomials of the second kind defined on [-1,1] and [0,1] as a method to solve non-linear problems.

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المستخلص

الغرض الرئيسي من هذا العمل يمكن تقسيمه إلى المحاور التالية:

١.دراسة متعددة حدود تشيبيتشيف من النوع الأول والثاني والمعرفة على الفترات [0,1] و[-1,1] وتطوير بعض خواصهم.

٢.أستعمال طريقتين لحل المعادلات التفاضلية الأعتيادية الخطية ذات المعاملات الغير ثابتة وهما طريقة مصفوفة تشيبيتشيف وطريقة متسلسلة تشيبيتشيف.

٣.تبني طريقة متسلسلة تشيبيتشيف لحل أنظمة من معادلات فريدهولم التكاملية و التكاملية-التفاضلية الخطية.

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