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Numerical Solution of Fractional Order Differential Equations Using Wavelets Methods

A Thesis
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By
Haneen Abdulkareem Ameen
(B.Sc.Math/College of Science/ Al-Nahrain University 2011)

Supervised by
Asst. Prof. Dr. Osama H. Mohammed

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بسم الله الرحمن الرحيم

إِنَّ اللَّهَ وَمَلَائِكَتَهُ يُصَلُّونَ عَلَى النَّبِيِّ يَا أَيُّهَا
الَّذِينَ آمَنُوا صَلُّوا عَلَيْهِ وَسَلِّمُوا تَسْلِيماً

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الإهداء

الى معلم البشرية ومنبع العلم...نبينا محمد (صلى الله عليه وسلم)

الى من سكنت روحه روحي الى رفيق دربي ... زوجي الغالي

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Supervisor Certification

I certify that this thesis, entitled "***Numerical Solution of Fractional Order Differential Equations Using Wavelets Methods***" was prepared under my supervision at the Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University as a partial fulfillment of the requirements for the degree of master in Mathematics.

Signature:

Name: Asst.Prof.Dr.Osama H.Mohammed

Date: /   /2014

In view of the available recommendations, I forward this thesis for debate by the examining Committee.

Signature:

Name: Asst. Prof. Dr. Fadhel S. Fadhel

Head of the Department of Mathematics and Computer Applications

Date: /   /2014

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We certifiy that we have read this thesis entitled " **Numerical Solution of Fractional Order Differential Equations Using Wavelets Methods''** and,as examining committee, examined the student " **Haneen Abdulkareem Ameen** " in its contents ,it is adequate as a thesis for the degree of Master of Science in Mathematics.

Signature:
Name: **Dr. Saad Naji**
Title: Professor
Date: / /2014
(Chairman)

Signature:
Name: **Dr. Fadhel S. Fadhel**
Title: Assistant Professor
Date: / / 2014
(Member)

Signature:
Name: **Dr. Jameel A. Ali**
Title: Assistant Professor
Date: / / 2014
(Member)

Signature:
Name: **Dr.Osama H. Mohammed**
Title: Assistant Professor
Date: / /2014
(Member and supervisior)

I, hereby certify upon the decision of the examining committee

Signature:
Name: **Dr.Hadi M.A.Abood**
Title: Assistant Professor
Dean of College of Science.
Date: / / 2014

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ABSTRACT

The main theme of this thesis is oriented about three objects:

The first one is to study the fundamental concepts of fractional calculus which are needed for finding the numerical solution of the differential equations (ordinary and partial) of fractional order.

The second objective is about finding the numerical solution of the non-linear ordinary differential equations of fractional order using wavelets methods which are Haar wavelets method, Chebyshev wavelets method and Legendre wavelets method. The main idea of these methods is to reduce the ordinary differential equation of fractional order into a system of algebraic equations then solved the obtained system. The solution of this system will give us the values of the coefficients of the desired solution which is expressed in an infinite series thus greatly simplifying such equations.

The third objective is to find the numerical solution of the linear partial differential equations of fractional order using three numerical methods which are: Chebyshev wavelets method, Haar-Chebyshev wavelets method and Chebyshev-Legendre wavelets method. The last two numerical methods (Haar-Chebyshev and Chebyshev-Legendre) are two modified numerical methods suggested in this thesis. The main characteristic of these methods is to express the solution of the partial differential equation as an infinite series in which its coefficients can be evaluated by converting the partial differential equations of fractional order into a system of algebraic equations which is named as Lyapunov type matrix and then solving this system of equations using MATLAB software which gives us the values of the coefficients and hence the desired solution of the partial differential equation of fractional order.

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INTRODUCTION

The subject of fractional calculus (that is, calculus of integral and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and wide spread fields of science and engineering. It does provide several potentially useful tool for solving differential and integral equations, and various other problems involving special functions of mathematical physics, as well as, their extensions and generalizations in one and more variables [Kilbas, 2006].

Most authors on this topic will cite a particular date of so called "fractional calculus" in a letter dated in September 30th, 1695 L'Hospital wrote to Leibaniz asking him about a particular notation, he had been used in his publication for the nth –derivative of the linear function $f(x) = \frac{D^n}{Dx^n}$. L'Hospital posed the question to Leibniz, what would the result be if $n = {}^1\!/_2$.

Leibniz response "An apparent paradox, from which one day useful consequences will be drawn". In these words fractional calculus was born.

Following L'Hosptial and Leibniz's first inquisition, fractional calculus was primary a study reserved for the best minds in mathematics, where Fourier, Euler and Laplace are among the many authors that dabbled with fractional calculus and the mathematical consequences [Nishimoto, 1983].

Many authors found, using their own notation and methodology, definitions that fit the concept of noninteger order integral or derivative. The most famous of these definitions that have been popularized in the word of fractional calculus are the Riemann-Liouville and Grünwald-Letnokov definition. Most of the mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the 20th century. However, it is the past 100 years that the most intriguing leaps in engineering and scientific application have been found.

The mathematics has in some cases to change to meet the requirements of physical reality, Caputo[Caputo,1997] reformulated the more 'classic' definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions to solve his fractional order differential equations [Podlubny, 1999]. However, during the last ten years fractional calculus starts to attract much more attention of physicists and mathematicians. It was found that various; especially interdisciplinary applications can be elegantly modeled with the help of the fractional derivatives. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [He, 1998] and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow, [He, 1999].

Fractional differential equations are generalized from classical integer-order ones, which are obtained by replacing integer-order derivatives by fractional ones.

Their advantages comparing with integer-order differential equations are the capability of simulating natural physical process and dynamic system more accurately [Chen, 2007].

partial differential equations involving derivatives with non-integer orders have shown to be adequate models for various physical phenomena in areas, such as damping laws, diffusion processes, etc. Other applications include electromagnetic, electrochemistry, arterial science, and the theory of ultra-slow processes and finance, [Wu, 2009].

However, several numbers of algorithms for solving fractional order partial differential equations have been investigated. Suarez [Suarez, 1997] used the eigenvector expansion method to find the solution of motion containing fractional derivative. Podlubny [Podlubny, 1999] used the Laplace transform method to solve fractional differential equations numerically with

Riemann-Liouville derivatives definition as well as the fractional partial differential equations with constant coefficients, Meerscharet and Tadjeran [Meerscharet, 2006] proposed the finite difference method to find the numerical solution of two-sided space- fractional partial differential equations. Momani [Momani, 2007] used a numerical algorithm to solve the fractional convection-diffusion equation with nonlinear source term. Odibat and Momani [Odibat, 2009] used the variation iteration method to handle fractional partial differential equations in fluid mechanics. Jafari and Seifi [Jafari, 2009] solved a system of nonlinear fractional partial differential equations using homotopy analysis method. Wu [Wu, 2009] derived a wavelet operational method to solve fractional partial differential equations numerically .Chen and Wu [Chen, 2010] used wavelet method to find the numerical solution for a class of fractional convection-diffusion equation with variable coefficients. Geng [Geng, 2011] suggested a wavelet method for solving nonlinear partial differential equations of fractional order. Guo and et.al [Guo, 2013] used the fractional variationl homotopy perturbation iteration method to solve a fractional diffusion equation.

In this thesis, numerical solution of partial differential equation of fractional order will be presented using the same approach given in [Wu, 2009] but with the aid of Chebyshev wavelets method, Haar-Chebyshev wavelets method and Chebyshev-Legendre wavelets . Wavelets analysis as a new approach of mathematics is widely applied in signal analysis, image manipulation, and numerical analysis, etc. It mainly studies the expression of functions, that is functions are decomposed into summation of “basic functions” and every “basic functions” is obtained by compression and translation of a mother wavelet function with good properties of locality and smoothness, which makes people able to analyze the properties of locality and integer in process of expressing functions [Li, 2005]. Beside their conventional applications in signal and image processing, wavelet basis had received attention dealing with numerical solutions of integer order as well as fractional order differential equations. Wavelet basis can be used to reduce the underlying problem to a system of algebraic equations by estimating the integrals using operational matrices [Chen, 2007], [Kilicman, 2007] and [Saadatmandi, 2010].

Recently the operational matrices of fractional order integration for the Haar wavelets, the Chebyshev wavelets and the Legendre wavelet have been developed in [Chen, 1997], [Yuanlu, 2010a], [Yuanlu, 2010b] and [Rehman, 2011] to solve the fractional order differential equations. This work consists of three chapters as well as this introduction. In chapter one, the fundamental concepts of fractional calculus are given. While in chapter two the numerical solution of ordinary differential equations using Haar wavelets method, Chebyshev wavelets method and Legendre wavelets method is presented. Finally the numerical solution of linear partial differential equations of fractional order by using Chebyshev wavelets method, Haar -Chebyshev wavelets method and Chebyshev-Legendre wavelets method are given in chapter three.

It is important to mention that, the calculation in chapter two and three are simplified using MATLAB R2013a computer software. The results are presented in figures or in a tabulated form.

Chapter One

Basic Concepts of Fractional Calculus

Chapter One

Basic Concepts of Fractional Calculus

1.1 Introduction:

This chapter consists of five sections, in section 1.2 the Beta and Gamma function were given, in section 1.3 we present some definitions of fractional order integration while in section 1.4 some definitions of fractional order derivatives are presented, finally in section 1.5 some analytical methods are used to find the solution of differential equations of fractional order.

1.2 The Gamma and Beta Functions,[Oldham,1974]:

The complete gamma function $\Gamma(x)$ plays an important role in the theory of fractional calculus. A comprehensive definition of $\Gamma(x)$ is that provided by Euler limit:

$$\Gamma(x) = \lim_{N \to \infty} \left(\frac{N!\, N^{x}}{x(x+1)(x+2)...(x+N)} \right), x > 0 \qquad \ldots(1.1)$$

but the integral transform definition is given by:

$$\Gamma(x) = \int_{0}^{\infty} y^{x-1} e^{-y} dy\ , x > 0 \qquad \ldots(1.2)$$

is often more useful, although it is restricted to positive value of x. An integration by parts applied to eq. (1.2) leads to the recurrence relationship:

$$\Gamma(x+1) = x\Gamma(x) \qquad \ldots(1.3)$$

This is the most important property of gamma function. The same result is a simple consequence of eq. (1.1), since Γ(1) = 1, this recurrence shows that for positive integer n:

$$\Gamma(n+1) = n\Gamma(n)$$

$$= n! \qquad \ldots(1.4)$$

The following are the most important properties of the gamma function:

1. $\Gamma\left(\frac{1}{2}-n\right) = \frac{(-4)^n n!\sqrt{\pi}}{(2n)!}$

2. $\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$

3. $\Gamma(-x) = \frac{-\pi csc(\pi x)}{\Gamma(x+1)}$

4. $\Gamma(nx) = \sqrt{\frac{2\pi}{n}}\left[\frac{nx}{\sqrt{2\pi}}\right]^n \prod_{k=0}^{n-1}\Gamma\left(x+\frac{k}{n}\right), n \in \mathbb{N}^+$

A function that is closely related to the gamma function is the complete beta function $\beta(p,q)$. For positive value of the two parameters *p* and *q*; the function is defined by the beta integral:

$$\beta(p,q) = \int_0^1 y^{p-1}(1-y)^{q-1}\,dy \,, p, q > 0 \qquad \ldots(1.5)$$

which is also known as the Euler's integral of the second kind. If either *p* or *q* is nonpositive, the integral diverges otherwise $\beta(p,q)$ is defined by the relationship:

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad \ldots(1.6)$$

where p and $q > 0$.

Both beta and gamma functions have "incomplete" analogues. The incomplete beta function of argument x is defined by the integral:

$$\beta_x(p,q) = \int_0^x y^{p-1}(1-y)^{q-1}\,dy \qquad \ldots(1.7)$$

and the incomplete gamma function of argument x is defined by:

$$\gamma^*(c,x) = \frac{c^{-x}}{\Gamma(x)}\int_0^c y^{x-1}e^{-y}\,dy$$

$$= e^{-x}\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+c+1)} \qquad \ldots(1.8)$$

$\gamma^*(c,x)$ is a finite single-valued analytic function of x and c.

1.3 Fractional Integration:

There are many literatures introduce different definitions of fractional integrations, such as:

1. Riemann-Liouville integral, [Oldham, 1974]:

The generalization to non-integer α of Riemann-Liouville integral can be written for suitable function $f(x)$, $x \in \mathbb{R}$; as:

$$I^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}f(s)\,ds,\ \alpha > 0 \qquad \ldots(1.9)$$

and $I^0 f(x) = f(x)$ is the identity operator.

The properties of the operator I^{α} can be founded in [Podlbuny, 1999] for $\beta \geq 0$, $\alpha > 0$, we have:

1. $I^{\alpha}I^{\beta} f(x) = I^{\alpha+\beta}f(x)$.
2. $I^{\alpha}I^{\beta}f(x) = I^{\beta}I^{\alpha}f(x)$. $\qquad \ldots(1.10)$

2. Weyl fractional integral, [Oldham, 1974]:

The left hand fractional order integral of order $\alpha > 0$ of a given function f is defined as:

$$ {}_{-\infty}I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{1-\alpha}} dy \qquad \ldots(1.11)$$

and the right fractional order integral of order $\alpha > 0$ of a given function f is given by:

$$ {}_{\infty}I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(y)}{(y-x)^{1-\alpha}} dy $$

3. Abel-Riemann fractional integral, [Mittal, 2008]:

The Abel-Riemann (A-R) fractional integral of any order $\alpha > 0$, for a function $f(x)$ with $x \in \mathbb{R}^+$ is defined as:

$$ I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-\tau)^{\alpha-1} f(\tau)\, d\tau,\ x > 0,\ \alpha > 0 \qquad \ldots(1.12)$$

$I^0 = I$ (identity operator)

The A-R integral posses the semigroup property:

$$ I^{\alpha} I^{\beta} = I^{\alpha+\beta}, \text{ for all } \alpha, \beta \geq 0 \qquad \ldots(1.13)$$

1.4 Fractional Derivatives:

Many literatures discussed and presented fractional derivatives of certain function, therefore in this section, some definitions of fractional derivatives are presented:

1. Riemann-Liouville fractional derivatives, [Oldham, 1974], [Nishimoto, 1983]:

Among the most important formulae used in fractional calculus is the Riemann-Liouville formula. For a given function $f(x)$, $\forall x \in [a,b]$; the left and right hand Riemann-Liouville fractional derivatives of order $\alpha > 0$ and m is a natural number, are given by:

$$ {}_x D^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x \frac{f(t)}{(x-t)^{\alpha-m+1}} dt \qquad \ldots(1.14)$$

$$ {}_x D^{\alpha}_{b-} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b \frac{f(t)}{(x-t)^{\alpha-m+1}} dt \qquad \ldots(1.15)$$

where $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$.

2. The A-R fractional derivative, [Mittal, 2008]:

The A-R fractional derivative of order $\alpha > 0$ is defined as the inverse of the corresponding A-R fractional integral, i.e.,

$$D^{\alpha} I^{\alpha} = I \qquad \ldots(1.16)$$

for positive integer m, such that $m - 1 < \alpha \leq m$,

$$(D^m I^{m-\alpha}) I^{\alpha} = D^m (I^{m-\alpha} I^{\alpha}) = D^m I^m = I$$

i.e.,

$$D^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dx^m} \displaystyle\int_a^x \frac{f(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m \\ \dfrac{d^m}{dx^m} f(x), & \alpha = m \end{cases} \qquad \ldots(1.17)$$

3. Caputo fractional derivative, [Caputo, 1967], [Minadri, 1997]:

In the late sixties of the last century, an alternative definition of fractional derivatives was introduced by Caputo. Caputo and Minadri used this definition in their work on the theory of viscoelasticity. According to Caputo's definition:

$$ {}^{c}D_{x}^{\alpha} = I^{m-\alpha}D^{m}, \text{ for } m-1 < \alpha \le m $$

which means that:

$$ {}^{c}D_{x}^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_{0}^{x} \dfrac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}}\, d\tau, & m-1<\alpha<m \\ \dfrac{d^{m}}{dx^{m}} f(x), & \alpha = m \end{cases} $$

The basic properties of the Caputo fractional derivative are:

1. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

2. $I^{\alpha}\, {}^{c}D_{x}^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^{+}) \dfrac{x^{k}}{k!}$.

3. Caputo's fractional differentiation is linear operator, similar to integer order differentiation:

$$ {}^{c}D_{x}^{\alpha} [\lambda f(x) + \mu g(x)] = \lambda\, {}^{c}D_{x}^{\alpha} f(x) + \mu\, {}^{c}D_{x}^{\alpha} g(x) $$

4. Grünwald fractional derivatives, [Oldham, 1974]:

The Grünwald derivatives of any integer order to any function, can take the form:

$$\mathrm{D}^{\alpha} f(x) = \lim_{N \to \infty} \left\{ \frac{\left(\frac{x}{N}\right)^{-\alpha}}{\Gamma(-\alpha)} \sum_{j=0}^{N-1} \frac{\Gamma(j-\alpha)}{\Gamma(j+1)} f\left(x - j\frac{x}{N}\right) \right\} \qquad \ldots(1.18)$$

1.5 Analytic Methods for Solving Fractional Order Differential Equations, [Oldham, 1974]:

In the present section, some analytical methods are presented for solving fractional order differential equations, and among such method:

1.5.1. The Inverse Operator Method:

Consider the fractional order differential equation:

$$\frac{d^{\alpha} f}{dx^{\alpha}} = F \qquad \ldots(1.19)$$

where f is an unknown function and $\frac{d^{\alpha}}{dx^{\alpha}}$ is a fractional order derivative of Riemann-Liouville sense, hence upon taking the inverse operator $\frac{d^{-\alpha}}{dx^{-\alpha}}$ to the both sides of eq.(1.19) gives:

$$f = \frac{d^{-\alpha} F}{dx^{-\alpha}} \qquad \ldots(1.20)$$

additional terms must be added to eq. (1.20), which are:

$c_1 x^{\alpha-1}, c_2 x^{\alpha-2}, \ldots, c_m x^{\alpha-m}$

and hence:

$$f - \frac{d^{-\alpha}}{dx^{-\alpha}} \frac{d^{\alpha}}{dx^{\alpha}} f = c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + \ldots + c_m x^{\alpha-m}$$

where $c_1, c_2, \ldots, c_m$ are an arbitrary constants to be determined from the initial conditions and $m - 1 < \alpha \leq m$.

Thus:

$$f - c_1 x^{\alpha-1} - c_2 x^{\alpha-2} - \ldots - c_m x^{\alpha-m} = \frac{d^{-\alpha}}{dx^{-\alpha}} \frac{d^{\alpha}}{dx^{\alpha}} f$$

$$= \frac{d^{-\alpha}}{dx^{-\alpha}} F$$

Hence, the most general solution of eq. (1.19) is given by:

$$f = \frac{d^{-\alpha}}{dx^{-\alpha}} F + c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + \ldots + c_m x^{\alpha-m}$$

where $m - 1 < \alpha \leq m$.

As an illustration, we shall consider the following example:

<u>Example (1.1):</u>

Consider the fractional order differential equation:

$$\frac{d^{3/2}}{dx^{3/2}} f(x) = x^5 \qquad \ldots(1.21)$$

Applying $\frac{d^{-3/2}}{dx^{-3/2}}$ to the both sides of eq. (1.21), we get:

$$f(x) = \frac{d^{-3/2} x^5}{dx^{-3/2}} + c_1 x^{1/2} + c_2 x^{-1/2}$$

1.5.2 Laplace Transform Method:

In this section, we shall seek a transform of $d^m f / dx^m$ for all m and differintegrable f, i.e., we wish to relate:

$$\mathcal{L}\left\{\frac{d^m f}{dx^m}\right\} = \int_0^\infty exp(-sx)\frac{d^m f}{dx^m}dx$$

to the Laplace transform $\mathcal{L}\{f\}$ of the differintegrable function. Let us first recall the well-known transforms of integer-order derivatives:

$$\mathcal{L}\left\{\frac{d^m f}{dx^m}\right\} = s^m \mathcal{L}\{f\} - \sum_{k=0}^{m-1} s^{m-1-k}\frac{d^k f}{dx^k}(0) \quad m = 1, 2, 3, ...$$

and multiple integrals:

$$\mathcal{L}\left\{\frac{d^m f}{dx^m}\right\} = s^m \mathcal{L}\{f\},\ m = 0, -1, -2, ... \qquad ...(1.22)$$

and note that both formulae are embraced by:

$$\mathcal{L}\left\{\frac{d^m f}{dx^m}\right\} = s^m \mathcal{L}\{f\} - \sum_{k=0}^{m-1} s^k \frac{d^{m-1-k} f(0)}{dx^{m-1-k}},\ m = 0, \pm 1, \pm 2, ... \qquad ...(1.23)$$

Also, formula (1.23), can be generalized to include non integer *m* by the simple extension:

$$\mathcal{L}\left\{\frac{d^m f}{dx^m}\right\} = s^m \mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^k \frac{d^{m-1-k} f(0)}{dx^{m-1-k}},\ \text{for all } m \qquad ...(1.24)$$

where n is the integer such that $n - 1 < m \leq n$. The sum is empty vanishes when $m \leq 0$.

In proving (1.24), we first consider $m < 0$, so that the Riemann-Liouville definition:

$$\frac{d^m f}{dx^m} = \frac{1}{\Gamma(-m)} \int_0^x \frac{f(y)}{[x-y]^{m+1}} dy , m < 0$$

may be adopted and upon direct application of the convolution theorem [Churchill,1948]:

$$L \left\{ \int_0^x f_1(x-y) f_2(y) dy \right\} = L \{ f_1 \} L \{ f_2 \}$$

Then gives:

$$L\left\{ \frac{d^m f}{dx^m} \right\} = \frac{1}{\Gamma(-m)} L\{ x^{-1-m} \} L\{ f \} = s^m L\{ f \} , m < 0 \qquad \ldots(1.25)$$

So that eq.(1.22) generalized unchanged for negative m.

For noninteger positive *m*, we use the result, [Oldham, 1974]:

$$\left[\frac{d^m f}{dx^m} \right] = \frac{d^n}{dx^n} \left[\frac{d^{m-n} f}{dx^{m-n}} \right]$$

where *n* is the integer such that $n - 1 < m \le n$.

Now, on application of the formula (1.23), we find that :

$$L \left\{ \frac{d^m f}{dx^m} \right\} = L \left\{ \frac{d^n}{dx^n} \left[\frac{d^{m-n} f}{dx^{m-n}} \right] \right\}$$

$$= s^n L \left\{ \frac{d^{m-n} f}{dx^{m-n}} \right\} - \sum_{k=0}^{n-1} s^k \frac{d^{n-1-k}}{dx^{n-1-k}} \left[\frac{d^{m-n} f}{dx^{m-n}} \right] (0).$$

The difference $m - n$ being negative, the first right-hand term may be evaluated by use of (1.25).since $m - n < 0$,the composition rule may be applied to the terms within the summation. The result:

$$L\left\{ \frac{d^m f}{dx^m} \right\} = s^m L\{ f \} - \sum_{k=0}^{n-1} s^k \frac{d^{m-1-k} f(0)}{dx^{m-1-k}} , 0 < m \ne 1, 2, \ldots$$

Follows from these two operations and is seen to be incorporated in (1.24).

The transformation (1.24) is a very simple generalization of the classical formula for the Laplace transform of the derivative or integral of *f*. No similar generalization exists, however, for the classical formulae, [Oldham, 1974]:

$$\mathrm{L}\left\{\frac{-f}{x}\right\}=\frac{d^{-1}\mathrm{L}\{f\}}{ds^{-1}}(s)-\frac{d^{-1}\mathrm{L}\{f\}}{ds^{-1}}(\infty)$$

$$\mathrm{L}\{-xf\}=\frac{d\mathrm{L}\{f\}}{ds}$$

$$\mathrm{L}\left\{[-x]^{n}f\right\}=\frac{d^{n}\mathrm{L}\{f\}}{ds^{n}},\ n=1,2,...$$

As a final result of this section we shall establish the useful formula:

$$\mathrm{L}\left\{exp(-kx)\frac{d^{m}}{dx^{m}}[fe^{kx}]\right\}=[s+k]^{m}\ \{f\}$$

As an illustration, we consider the following example:

Example (1.2), [Abdulkhalik, 2008]:

Consider the integro differential equation:

$$\frac{d^{1/2}f(x)}{dx^{1/2}}+\frac{d^{-1/2}f(x)}{dx^{-1/2}}+2f(x)=\frac{2}{\sqrt{\pi x}}+6\sqrt{\frac{x}{\pi}}+\frac{4x^{3/2}}{3\sqrt{\pi}}+2x+4 \quad \ldots(1.27)$$

and in order to solve this equation using Laplace transformation method, first we take the Laplace transformation to the both sides of equation (1.27):

$$\mathrm{L}\left\{\frac{d^{1/2}f(x)}{dx^{1/2}}\right\}+\mathrm{L}\left\{\frac{d^{-1/2}f(x)}{dx^{-1/2}}\right\}+2\mathrm{L}\{f(x)\}=\frac{2}{\sqrt{\pi}}\mathrm{L}\left\{\frac{1}{\sqrt{x}}\right\}+$$

$$\frac{6}{\sqrt{\pi}}\mathrm{L}\{\sqrt{x}\}+\frac{4}{3\sqrt{\pi}}\mathrm{L}\{x^{3/2}\}+2\mathrm{L}\{x\}+\mathrm{L}\{4\}$$

Using the definition of the Laplace transformation for the non-integer order given by eq.(1.24) thus we get after simple simplification:

$$L(f) = \frac{2s^2 + 3s + 1 + 2\sqrt{s} + 4s\sqrt{s}}{s^2(s + 1 + 2\sqrt{s})}$$

$$= \frac{(2s + 1) + (s + 1 + 2\sqrt{s})}{s^2(s + 1 + 2\sqrt{s})}$$

Then upon using the inverse Laplace transform, we have:

$f(x) = 2 + x$

as the solution of the integro differential equation.

Chapter Two

Wavelets Methods for Solving Ordinary Differential Equations of Fractional Order

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Wavelets Methods for Solving Ordinary Differential Equations of Fractional Order

2.1 Introduction:

Wavelet analysis is relatively new area in mathematics research. It has been applied widely in signal analysis, time frequency analysis and numerical analysis. Wavelet analysis included the expression of functions. Which are expanded to summation of “basic function” and every “basic function” is achieved by dilation and translation locality. This chapter consists of five sections, in section 2.2 Haar wavelets method is presented, in section 2.3 Chebyshev wavelets method was given, while in section2.4 we present the Legendre wavelets method and finally in section 2.5 two illustrative examples are solved via the Haar wavelets, Chebyshev wavelets and Legendre wavelets methods and the results are documented either in figure or in tabulated form.

2.2 Haar Wavelets:

Haar functions have been used since 1910, when they were introduced by Hungarian mathematician Alferd Haar, [Haar, 1910].

The orthogonal set of Haar functions is defined as shown in Figures (2.1-2.8) that is a square waves with magnitude of ±1 in some interval and zero elsewhere. The first curve of Figure (2.1) is that $h_0(x) = 1$

during the whole interval [0,1]. It is called the scaling function. The second curve $h_1(x)$ is the fundamental square wave, or mother wavelet which also spans the whole interval [0,1]. All the other subsequent curve are generated from $h_1(x)$ with two operations translation and dilation, $h_2(x)$ is obtained from $h_1(x)$ with dilation, i.e., $h_1(x)$ is compressed from the whole interval [0,1] to half interval [0,1/2] to generate $h_2(x)$, $h_3(x)$ is the same as $h_2(x)$ but shifted (translated) to the right by 1/2. Similarly, $h_2(x)$ is compressed from the half interval to a quarter interval to generate $h_4(x)$. The function $h_4(x)$ is translated to the right by 1/4, 2/4, 3/4 to generate $h_5(x)$, $h_6(x)$ and $h_7(x)$; respectively.

In general:

$h_n(x) = h_1(2^j x - k/2^j)$, $n = 2^j + k$, $j \geq 0$, $0 \leq k \leq 2^j$

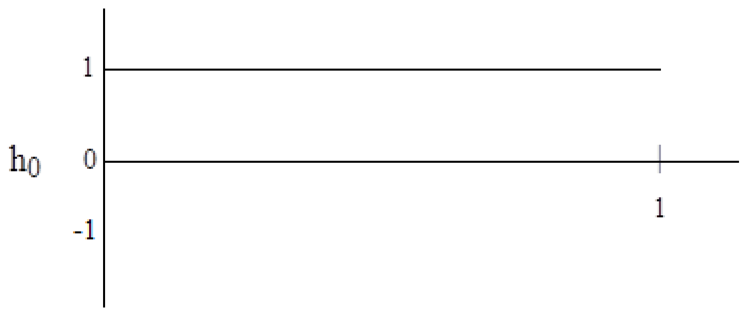

Figure (2.1) First Haar function.

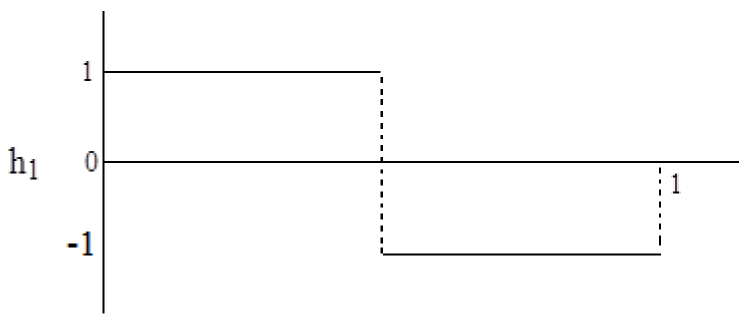

Figure (2.2) Second Haar function.

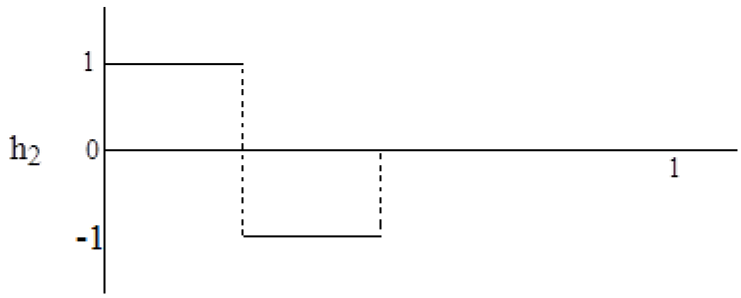

Figure (2.3) Third Haar function.

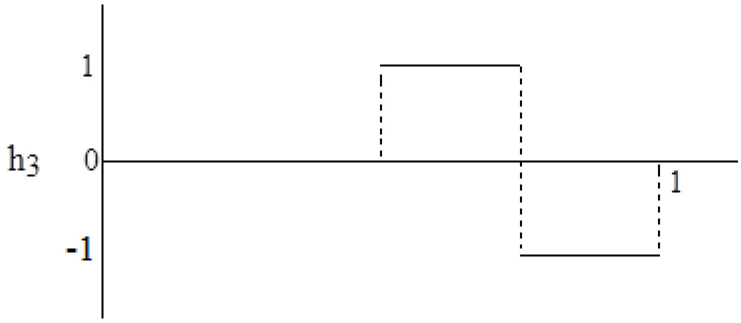

Figure (2.4) Fourth Haar function.

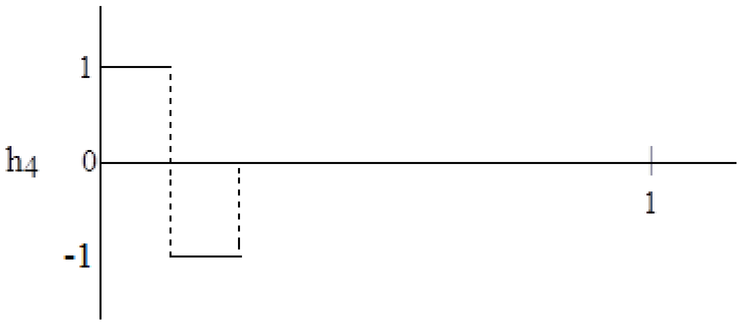

Figure (2.5) Fifth Haar function.

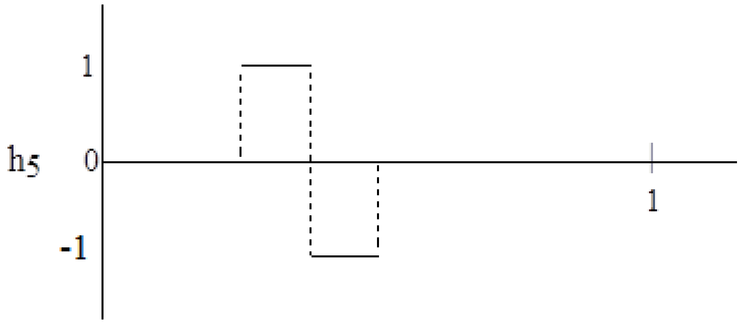

Figure (2.6) Sixth Haar function.

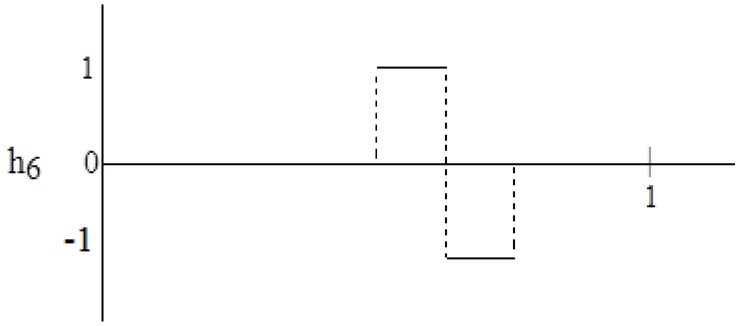

Figure (2.7) Seventh Haar function.

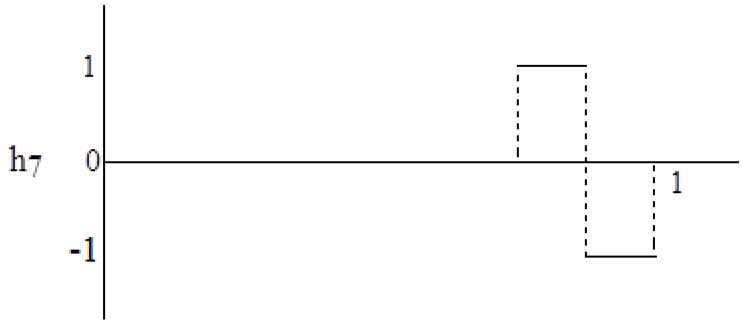

Figure (2.8) Eighth Haar function.

This orthogonal basis is a reminiscent of the Walsh basis, in which each Walsh function contains many wavelets to fill the interval [0,1] completely, and to form a global basis. While each Haar function contains just one wavelet during some subinterval of time, and remains zero elsewhere the Haar set form a local basis.

All the Haar wavelets are orthogonal to each other:

$$\int_0^1 h_i(x)h_\ell(x)\,dx = 2^{-j}\delta_i$$

$$= \begin{cases} 2^{-j}, & i = \ell = 2^{j+k} \\ 0, & i \neq \ell \end{cases}$$

Therefore, they form a very good transform basis.

2.2.1 Haar Wavelets Operational Matrix:

In this section we shall begin with the more convenient way for representing Haar wavelets in computer and for $x \in [A,B]$ which was given by [Lepik, 2009] and for this purpose we define the quantity $M=2^J$ where J is the maximal level of resolution and divide the interval $[A,B]$ into $2M$ subintervals of equal length; each subinterval has the length

$\Delta x = (B - A)/2M.$

Two parameters are introduced the dilation parameter j for which $j=0,1,\ldots, J$ and the translation parameter $k = 0,1,\ldots,m-1$ where $m = 2^j$. The wavelets number i is identified as $i = m + k + 1$ the i^{th} Haar wavelet is defined as:

$$h_i(x) = \begin{cases} 1, & \text{for } x \in [\xi_1(i), \xi_2(i)], \\ -1, & \text{for } x \in [\xi_2(i), \xi_3(i)], \\ 0, & elsewhere. \end{cases} \qquad \text{...(2.1)}$$

Where:

$$\xi_1(i) = A + 2k\mu\Delta x \;, \quad \xi_2(i) = A + (2k+1)\mu\Delta x$$

$$\xi_3(i) = A + 2(k+1)\mu\Delta x \;, \quad \mu = M/m$$

The case $i = 1$ corresponding to the scaling function

$$h_1(x) = \begin{cases} 1, \text{for } x \in [A,B] \\ 0, \; elsewhere \end{cases} \qquad \text{...(2.2)}$$

The following notations are introduced:

$$p_{i,1}(x) = \int_0^x h_i(\tau)d\tau$$

$$p_{i,v+1}(x)=\int_0^x p_{i,v}(\tau)d\tau \quad v=1,2,...$$

These integrals can be evaluated by using def. (2.1) and the first two of them are given by:

$$p_{i.1}(x)=\begin{cases} x-\xi_1(i), & x\in[\xi_1(i),\xi_2(i)) \\ \xi_3(i)-x, & x\in[\xi_2(i),\xi_3(i)) \\ 0, & Otherwise. \end{cases} \qquad \text{…(2.3)}$$

$$p_{i,2}(x)=\begin{cases} \frac{1}{2}(x-\xi_1(i))^2, & x\in[\xi_1(i),\xi_2(i)) \\ \frac{1}{4m^2}-\frac{1}{2}(\xi_3(i)-x)^2, & x\in[\xi_2(i),\xi_3(i)) \\ \frac{1}{4m^2}, & x\in[\xi_3(i),1) \\ 0, & Otherwise. \end{cases} \qquad \text{…(2.4)}$$

In general:

$$p_{i,n}(x)=\begin{cases} 0, & x<\xi_1(i) \\ \frac{1}{n!}(x-\xi_1(i))^n, & x\in[\xi_1(i),\xi_2(i)] \\ \frac{1}{n!}\left[(x-\xi_1(i))^n-2(x-\xi_2(i))^n\right], & x\in[\xi_2(i),\xi_3(i)] \\ \frac{1}{n!}\left[(x-\xi_1(i))^n-2(x-\xi_2(i))^n+2(x-\xi_3(i))^n\right], & x>\xi_3(i) \end{cases}$$

…(2.5)

For example, if $J=2$, then:

$$P_{4,1}=\frac{1}{16}\begin{bmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

And if $J=3$, then:

$$P_{8,1} = \frac{1}{64}\begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & -4 & 0 & 0 & 0 & 0 & -4 & 4 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Following figures (2.9-2.16) represent the first integral of $h_i(x)$, for all $i = 1, 2, \ldots, 8$.

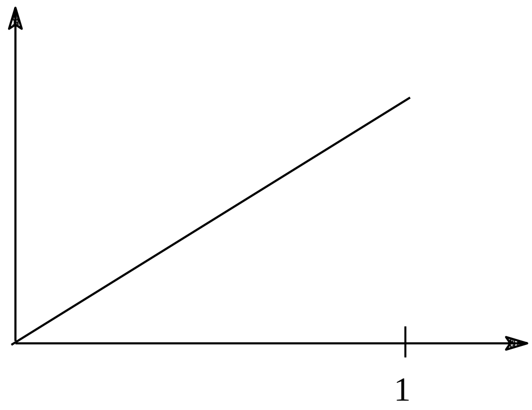

Figure (2.9) Integration of the first Haar wavelet.

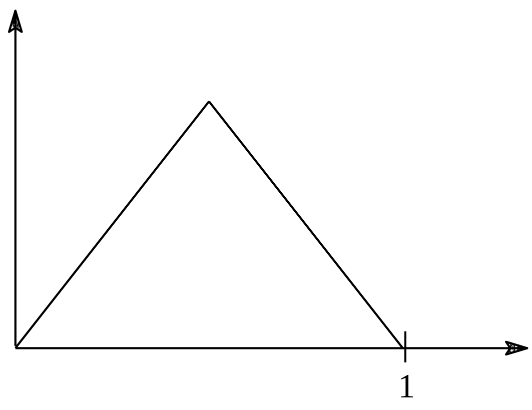

Figure (2.10) Integration of the second Haar wavelet.

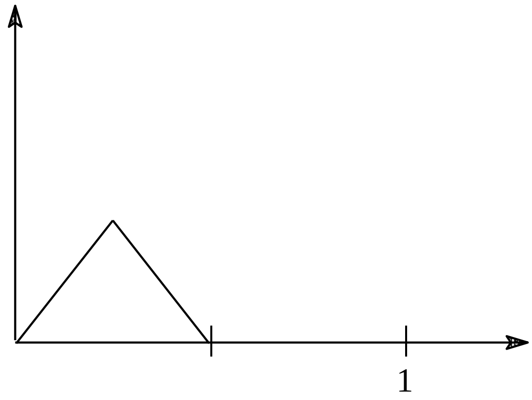

Figure (2.11) Integration of the third Haar wavelet.

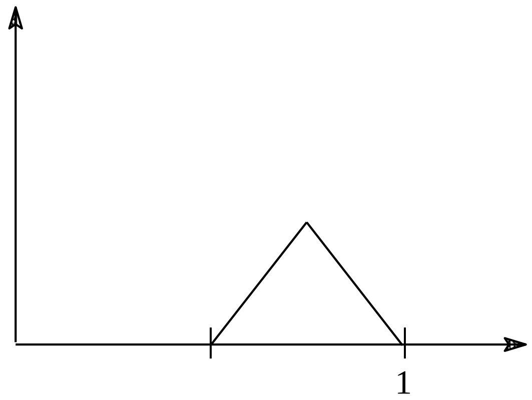

Figure (2.12) Integration of the forth Haar wavelet.

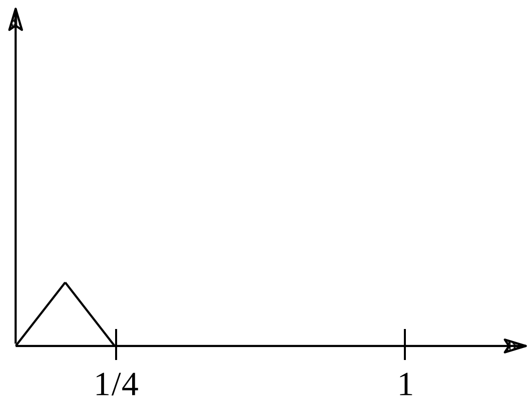

Figure (2.13) Integration of the fifth Haar wavelet.

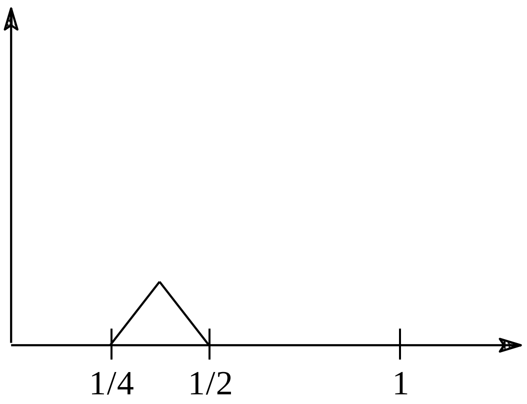

Figure (2.14) Integration of the sixth Haar wavelet.

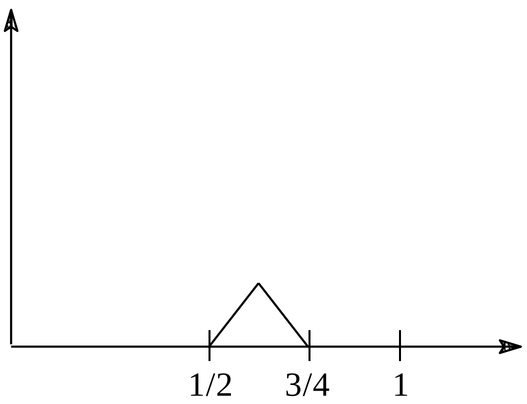

Figure (2.15) Integration of the seventh Haar wavelet.

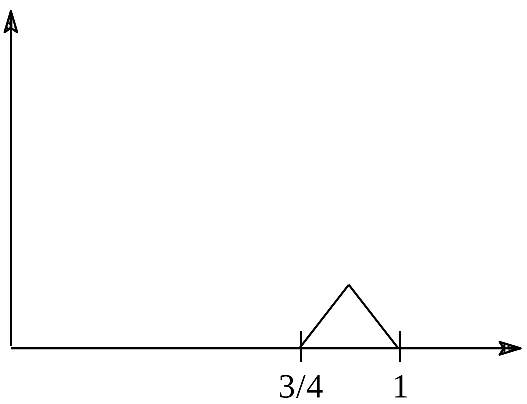

Figure (2.16) Integration of the eighth Haar wavelet.

Any function $f(x) \in L^2[0,1]$ can be expanded in terms of Haar series as:

$$f(x) = \sum_{i=1}^{\infty} c_i h_i(x) \qquad ...(2.6)$$

Where the coefficients c_i are determined by:

$$c_i = 2^j \int_0^1 f(x) h_i(x)$$

If $f(x)$ is piecewise constant or may be approximated as piecewise constant, then the sum in eq.(2.6) may be terminated after *2M* terms, that is:

$$f(x) = \sum_{i=1}^{2M} c_i h_i(x) = C_{2M}^T H_{2M}(x) = \hat{f}(x) \qquad ...(2.7)$$

$\hat{f}$ denotes the truncated sum, the Haar coefficients vector C_{2M} and Haar vector $H_{2M}(x)$ are defined as:

$$C_{2M} = [c_1, c_2, \dots, c_{2M}]^T$$

$$H_{2M}(x) = [h_1(x), h_2(x), \dots, h_{2M}(x)]^T \qquad \dots(2.8)$$

Taking the collocation points as following

$$x_i = A + (i - 0.5)\Delta x, i = 1,2, \dots 2M \qquad \dots(2.9)$$

By letting *A = 0, B = 1* and hence $\Delta x = \frac{1}{2M}$ in eq.(2.9) We define the 2M-square Haar matrix $\Phi_{2M\times 2M}$ as:

$$\Phi_{2M\times 2M} = \left[H_{2M}\left(\frac{1}{4M}\right) H_{2M}\left(\frac{3}{4M}\right) \dots H_{2M}\left(\frac{4M-1}{4M}\right)\right] \qquad \dots(2.10)$$

Correspondingly, we have:

$$\hat{f}_{2M} = \left[\hat{f}\left(\frac{1}{4M}\right) \hat{f}\left(\frac{3}{4M}\right) \dots \hat{f}\left(\frac{4M-1}{4M}\right)\right] = C_{2M}^T \Phi_{2M\times 2M} \qquad \dots(2.11)$$

Because the *2M* – square Haar wavelets matrix $\Phi_{2M\times 2M}$ is an invertible matrix, the Haar coefficients vector C_{2M}^T can be gotten by:

$$C_{2M}^T = \hat{f}_{2M} \ \Phi_{2M\times 2M}^{-1} \qquad \dots(2.12)$$

2.2.2 Block Pulse Function (BPF):

Defines a 2M – Set of Block Pulse Function (BPF) as:

$$b_i(x) = \begin{cases} 1, & \frac{i}{2M} \le x < \frac{i+1}{2M} \\ 0, & Otherwise \end{cases} \qquad \ldots(2.13)$$

where $i = 0,1,2 \ldots, 2M - 1$.

The functions $b_i(x)$ are disjoint and orthogonal, that is:

$$b_i(x)b_l(x) = \begin{cases} 0 & , i \neq l \\ b_i(x) & , i = l \end{cases} \qquad \ldots(2.14)$$

Kilicman and Zhour [Kilicman, 2007] had given the block pulse operational matrix of fractional order integration F^{α} as following:

$$I^{\alpha} B_{2M}(x) = F^{\alpha} B_{2M}(x) \qquad \ldots(2.15)$$

where:

$$F^{\alpha} = \frac{1}{(2M)^{\alpha}} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{2M-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{2M-2} \\ 0 & 0 & 1 & \cdots & \xi_{2M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots(2.16)$$

where:

$$\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1} \text{ ,k=1,2,…,2M-1}$$

2.2.3 Operational matrix of the fractional order integration of Haar Wavelet Functions:

The integration of $H_{2M}(x)$ defined in Eq.(2.8) can be approximated by Haar series with Haar coefficient matrix P_{ha} as:

$$\int_0^x H_{2M}(\tau) d\,\tau \approx P_{ha\,2M\times 2M}\ H_{2M}(x) \qquad \ldots(2.17)$$

where a *2M*-square matrix P_{ha} is called the Haar wavelets operational matrix of integration [Chen, 1997].

Zhao, [Zhao, 2010] derive the Haar wavelets operational matrix of the fractional order integration.

He introduced the Riemann-Liouville fractional order integration, as given in chapter one as:

$$(I^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)}\int_{o}^{x}(x-\tau)^{\alpha-1}f(\tau)d\tau = \frac{1}{\Gamma(\alpha)}x^{\alpha-1} * f(x) \qquad \ldots(2.18)$$

where $\alpha \in \mathbb{R}$ is the order of integration, $\Gamma(\alpha)$ is the Gamma function and $x^{\alpha-1} * f(x)$ is the convolution product of $x^{\alpha-1}$ and $f(x)$.

Now if $f(x)$ is expanded in Haar function, the Riemann- Liouville fractional order integration is solved via the Haar function, because the Haar functions are piecewise constant, it may be expanded into *2M*- term Block Pulse Function (BPF) as:

$$H_{2M}(x) = \Phi_{2M\times 2M}B_{2M}(x) \qquad \ldots(2.19)$$

where $B_{2M}(x) \triangleq [b_1(x)b_2(x) \ldots b_{2M}(x)]$

Next, the Haar wavelets operational matrix of the fractional order integration is derived by letting

$$(I^{\alpha}H_{2M})(x) = P^{\alpha}_{ha\ 2M\times 2M}H_{2M}(x) \qquad \ldots(2.20)$$

Where the *2M* – square matrix $P^{\alpha}_{ha\ 2M\times 2M}$ is called the Haar wavelets operational matrix of the integration.

Using Eqs.(2.15) and (2.19) we have

$$(I^{\alpha}H_{2M})(x) \approx (I^{\alpha}\Phi_{2M\times 2M}B_{2M})(x) = \Phi_{2M\times 2M}(I^{\alpha}B_{2M})(x)$$

$$\approx \Phi_{2M\times 2M}F^{\alpha}B_{2M}(x) \qquad \ldots(2.21)$$

From eqs.(2.20) and (2.21), we get:

$$P^{\alpha}_{ha\;2M\times 2M}\;H_{2M}(x) = P^{\alpha}_{ha\;2M\times 2M}\;\Phi_{2M\times 2M}\;\;B_{2M}(x)$$

$$= \Phi_{2M\times 2M}\;\;F^{\alpha}\;B_{\;2M}(x) \qquad \ldots(2.22)$$

Then the Haar wavelet operational matrix of the fractional order of integration $P^{\alpha}_{ha\;2Mx2M}$ is given by

$$P^{\alpha}_{ha\;2M\times 2M} = \;\Phi_{2M\times 2M}\;\;F^{\alpha}\;\;\Phi^{-1}_{2M\times 2M} \qquad \ldots(2.23)$$

For example, let $\alpha = 0.5$, $J = 2$ hence $2M = 8$, the operational matrix $P^{\alpha}_{ha\;2M\times 2M}$ is computed below as:

$$P^{0.5}_{ha\;8\times 8} = \begin{bmatrix} 0.7523 & -0.2203 & -0.1558 & -0.0820 & -0.1102 & -0.0580 & -0.0447 & -0.0377 \\ 0.2203 & 0.3116 & -0.1558 & 0.2296 & -0.1102 & -0.0580 & 0.1756 & 0.0782 \\ 0.0410 & 0.1148 & 0.2203 & -0.0350 & -0.1102 & 0.1623 & -0.0389 & -0.0063 \\ 0.0779 & -0.0779 & 0 & 0.2203 & 0 & 0 & -0.1102 & 0.1623 \\ 0.0094 & 0.0196 & 0.0812 & -0.0032 & 0.1558 & -0.0247 & -0.0026 & -0.0009 \\ 0.0112 & 0.0439 & -0.0551 & -0.0194 & 0 & 0.1558 & -0.0247 & -0.0026 \\ 0.0145 & -0.0145 & 0 & 0.0812 & 0 & 0 & 0.1558 & -0.0247 \\ 0.0275 & -0.0275 & 0 & -0.0551 & 0 & 0 & 0 & 0.1558 \end{bmatrix}$$

2.3 Chebyshev Wavelets:

Wavelets are a family of function constructed from dilation and translation of a single function called mother wavelet.

In this section we will present another type of wavelets which is so called the second kind Chebyshev wavelets as follows:

When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets as [Fan, 2008]:

$$\Psi_{a,b}(x) = |a|^{-1/2}\,\Psi\left(\frac{x-b}{a}\right), a, b \in \mathbb{R}, a \neq 0 \qquad \ldots(2.24)$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 1$, $b_0 > 0$, where n and k are positive integers, the family of discrete wavelets are defined as:

$$\psi_{k,n}(x) = |a_0|^{k/2}\psi(a_0^k x - nb_0) \qquad \ldots(2.25)$$

where $\psi_{k,n}$ forms a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$ and $b_0 = 1$, $\psi_{k,n}(x)$ forms an orthogonal basis.

The second kind Chebyshev wavelets $\psi_{n,m}(x) = \psi(k,n,m,x)$ involve four arguments $n = 1, 2, \ldots, 2^{k-1}$; k is assumed any positive integer, m is the degree of the second kind Chebyshev polynomial and x is the normalized time. They are defined on the interval [0,1] as [Fan, 2008]:

$$\psi_{n,m}(x)=\begin{cases} 2^{k/2}\tilde{U}_m(2^k x-2n+1), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}} \\ 0, & Otherwise \end{cases} \qquad \ldots(2.26)$$

where:

$$\tilde{U}_m(x) = \sqrt{\frac{2}{\pi}}U_m(x) \qquad \ldots(2.27)$$

and $m = 0, 1, \ldots, M - 1$. Here are the second kind Chebyshev polynomial of degree m with respect to the weight function $\omega(x)=\sqrt{1-x^2}$ on the interval [−1,1] and satisfy the following recursive formula:

$$U_0(x) = 1$$

$$U_1(x) = 2x$$

$$\vdots$$

$$U_{m+1}(x) = 2xU_m(x) - U_{m-1}(x),\ m = 1, 2, \ldots .$$

We should note that in dealing with the second kind Chebyshev wavelet, the weight function $\tilde{\omega}(x)=\omega(2x-1)$ have to be dilated and translated as:

$$\omega_n(x) = \omega(2kx - 2n + 1)$$

2.3.1 Function Approximation and Operational Matrix:

A function $f(x)$ defined over [0,1] may be expanded as:

$$f(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\, \Psi_{nm}(x) = C^T\, \Psi(x) = \hat{f}\,(x) \qquad \ldots(2.28)$$

where C and $\Psi(x)$ are $2^{k-1}M \times 1$ matrice given by:

$$C \triangleq \left[c_{10}, c_{11}, \ldots, c_{1(M-1)}, c_{20}, c_{21}, \ldots, c_{2(M-1)}, \ldots, c_{2^{k-1}0}, c_{2^{k-1}1}, \ldots, c_{2^{k-1}(M-1)}\right]^T \qquad \ldots(2.29)$$

and

$$\Psi(x) \triangleq \left[\psi_{10}, \psi_{11}, \ldots, \psi_{1(M-1)}, \psi_{20}, \psi_{21}, \ldots, \psi_{2(M-1)}, \ldots, \psi_{2^{k-1}0}, \psi_{2^{k-1}1}, \ldots, \psi_{2^{k-1}(M-1)}\right]^T \qquad \ldots(2.30)$$

From now we define $m = 2^{k-1}M$

Taking the collocation points as following:

$$x_i = \frac{2i-1}{2^k M}, i = 1\ 2, \ldots, m$$

The second kind Chebyshev wavelets matrix $\Omega_{m\times m}$ is given by:

$$\Omega_{m\times m} = \left[\Psi\left(\frac{1}{2m}\right), \Psi\left(\frac{3}{2m}\right), \ldots, \Psi\left(\frac{2m-1}{2m}\right)\right] \qquad \ldots(2.31)$$

For example, when $M = 3$ and $K = 2$, the second kind Chebyshev wavelets is expressed as

$$\Omega_{6\times 6} = \begin{bmatrix} 1.5959 & 1.5958 & 1.5958 & 0 & 0 & 0 \\ -2.1278 & 0 & 2.1278 & 0 & 0 & 0 \\ 1.2415 & -1.5958 & 1.2415 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.5959 & 1.5958 & 1.5958 \\ 0 & 0 & 0 & -2.1278 & 0 & 2.1278 \\ 0 & 0 & 0 & 1.2415 & -1.5958 & 1.2415 \end{bmatrix}$$

Correspondingly, we have:

$$\hat{f}_m = \left[\hat{f}\left(\frac{1}{2m}\right), \hat{f}\left(\frac{3}{2m}\right), \dots, \hat{f}\left(\frac{2m-1}{2m}\right)\right] \qquad \dots(2.32)$$

Because the second kind wavelets matrix $\Omega_{m\times m}$ is an invertible matrix, the Chebyshev wavelets coefficients vector C^T can be determined by:

$$C^T = \hat{f}_m \, \Omega^{-1}{}_{m\times m} \qquad \dots(2.33)$$

The convergence of the second kind Chebyshev wavelet bases was given by Wang [Wang, 2011].

2.3.2 Operational Matrix of the Fractional Order Integration of Chebyshev Wavelet Functions:

The integration of $\Psi(x)$ defined in eq.(2.30) can be approximated by Chebyshev series with Chebyshev coefficient matrix P_{Ch} as:

$$\int_0^x \Psi(\tau)d\,\tau \approx P_{Ch\, m\times m}\ \Psi(x) \qquad \dots(2.34)$$

where a $m \times m$ square matrix P_{Ch} is called the Chebyshev wavelets operational matrix of integration.

Next, we shall present the derivation of the second kind Chebyshev wavelets operational matrix of the fractional order integration.

Now, if $f(x)$ is expanded in a second kind Chebyshev wavelets, as given in Eq.(2.28). The Riemann- Liouville fractional integration becomes

$$(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} * f(x)$$

$$\approx C^T \frac{1}{\Gamma(\alpha)} \{x^{\alpha-1} * \Psi(x)\} \qquad \dots(2.35)$$

Thus if $x^{\alpha-1} * f(x)$ can be integrated then expanded in the second kind Chebyshev wavelets, the Riemann-Liouville fractional integration is solved via the second kind Chebyshev wavelets.

Because the Chebyshev wavelets are piecewise constant, it may be expanded into m – term Block Pulse Function (BPF) as:

$$\Psi_m(x) = \Omega_{m\times m}B_m(x) \qquad \ldots(2.36)$$

where:

$$B_m(x) \triangleq [b_0(x)b_1(x) \ldots b_i(x) \ldots b_{m-1}(x)]^T$$

Next, we shall derive the Chebyshev wavelets operational matrix of the fractional order integration by letting:

$$(I^\alpha \Psi_m)(x) = P^\alpha_{Ch\, m\times m}\Psi_m(x) \qquad \ldots(2.37)$$

where the matrix $P^\alpha_{Ch\, m\times m}$ is called the Chebyshev wavelets operational matrix of the fractional integration

Using eqs.(2.36) and (2.15), we have:

$$(I^\alpha \Psi_m)(x) = (I^\alpha \Omega_{m\times m}B_m)(x) = \Omega_{m\times m}(I^\alpha B_m)(x)$$

$$\approx \Omega_{m\times m}F^\alpha B_m(x) \qquad \ldots(2.38)$$

where F^αdefined in eq. (2.16)with 2M= m.

From eqs.(2.37) and (2.38), we get:

$$P^\alpha_{Ch\, m\times m}\, \Psi_m(x) = P^\alpha_{Ch\, m\times m}\, \Omega_{m\times m}B_m(x)$$

$$= \Omega_{m\times m}\ F^\alpha\ B_m(x) \qquad \ldots(2.39)$$

Then the second kind Chebyshev wavelet operational matrix of the fractional integration $P^\alpha_{Ch\, m\times m}$ is given by

$$P^\alpha_{Ch\, m\times m} = \Omega_{m\times m}\ F^\alpha\ \Omega^{-1}_{m\times m} \qquad \ldots(2.40)$$

2.4 Legendre Wavelets:

The Legendre wavelets constitute a family of functions constructed from dilation and translation of single function called the mother wavelet $\psi(x)$. They are defined by:

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right). \quad a,b \in \mathbb{R},\ a \neq 0,$$

where *a* is dilation parameter and *b* is translation parameter.

By restricting *a*, *b* to discrete values as:

$a = a_0^j, b = kb_0a_0^j$ where $a_0 > 1$, $b_0 > 0$ and $n,k \in \mathbb{N}$,

we get following family of discrete wavelets as

$$\psi_{j,k}(x) = |a_0|^{\frac{j}{2}}\psi(a_0^j x - kb_0)$$

The set of wavelets forms an orthogonal basis of $L^2(\mathbb{R})$.In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{j,k}(x)$ forms an orthonormal basis. The Legendre polynomial of order m, denoted by $L_m(x)$ are defined on the interval [–1,1] and can be determined with the aid of the following recurrence formulae:

$$L_0(x) = 1, \quad L_1(x) = x\ ,$$

$$L_{m+1}(x) = \frac{2m+1}{m+1}xL_m(x) - \frac{m}{m+1}L_{m-1}(x),\ \ m = 1,2,3,\dots .$$

The Legendre wavelets are defined on interval [0,1), by:

$$\psi_{n,m}(x) = \begin{cases} (2m+1)^{\frac{1}{2}}2^{\frac{k}{2}}L_m(2^k x - \hat{n}), & \frac{\hat{n}-1}{2^k} \le x < \frac{\hat{n}+1}{2^k} \\ 0, & elsewhere, \end{cases}$$

where $k=2,3,\dots,\hat{n} = 2n-1, n=1,2,3,\dots,2^{k-1}, m=0,1,2,\dots,M-1$ is the order of the Legendre polynomials and *M* is fixed positive integer .

2.4.1 Function Approximation and Operational Matrix:

A function *f(x)* defined over [0,1] may be approximated as:

$$f(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\, \psi_{nm}(x) = C^T\, \Psi(x) = \hat{f}\,(x) \qquad \ldots(2.41)$$

where C and $\Psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$C \triangleq \left[c_{1,0}, c_{1,1}, \ldots, c_{1,(M-1)}, c_{2,0}, c_{2,1}, \ldots, c_{2,(M-1)}, \ldots,\ c_{2^{k-1},0}, c_{2^{k-1},1}, \ldots, c_{2^{k-1},(M-1)}\right]^T$$

and

$$\Psi(x) = [\psi_{1,0}, \psi_{1,1},\ \ldots, \psi_{1,(M-1)}, \psi_{2,0}, \psi_{2,1}, \ldots, \psi_{2,(M-1)}, \ldots,\ \psi_{2^{k-1},0},\ \psi_{2^{k-1},1}, \ldots, \psi_{2^{k-1},(M-1)}]^T \qquad \ldots(2.42)$$

From now we will define $m = 2^{k-1}M$

Talking the collocation point as following:

$$x_i = \frac{2i-1}{2^k M}, i = 1, 2, \ldots, m$$

The Legendre wavelets matrix $\Theta_{m\times m}$, is given by:

$$\Theta_{m\times m} = \left[\Psi\left(\frac{1}{2m}\right), \Psi\left(\frac{3}{2m}\right), \ldots, \Psi\left(\frac{2m-1}{2m}\right)\right] \qquad \ldots(2.43)$$

Correspondingly, we have:

$$\hat{\mathrm{f}}_m = \left[\hat{\mathrm{f}}\left(\frac{1}{2m}\right), \hat{\mathrm{f}}\left(\frac{3}{2m}\right), \ldots, \hat{\mathrm{f}}\left(\frac{2m-1}{2m}\right)\right] \qquad \ldots(2.44)$$

2.4.2 Operational Matrix of the Fractional Order Integration of the Legendre Wavelets Functions:

The integration of $\Psi(x)$ defined in Eq.(2.42) can be approximated by Legendre series with Legendre coefficient matrix $P_{L\, m\times m}$ as:

$$\int_0^x \Psi(\tau) d\,\tau \approx\ P_{L\, m\times m}\ \Psi(x) \qquad \ldots(2.45)$$

where a $m \times m$ square matrix $P_{L\,m\times m}$ is called the Legendre wavelets operational matrix of integration.

Next, we shall present the derivation of the Legendre wavelets operational matrix of the fractional order integration.

Now if $f(x)$ is expanded in the Legendre wavelets. The Riemann-liouville fractional integration becomes:

$$(I^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} * f(x)$$

$$\approx C^{T} \frac{1}{\Gamma(\alpha)} \{x^{\alpha-1} * \Psi(x)\} \qquad \ldots(2.46)$$

Thus if $x^{\alpha-1} * f(x)$ can be integrated, then expanded in the Legendre wavelets, the Riemann-Liouville integration is solved via the Legendre wavelets.

Because the Legendre wavelets are piecewise constant, it may be expanded into *m*- term Block Pulse Function (BPF) as

$$\Psi_m(x) = \Theta_{m\times m} B_m(x) \qquad \ldots(2.47)$$

$$B_m(x) \triangleq [b_1(x) b_2(x) \ldots b_i(x) \ldots b_m(x)]$$

Next, we shall derive the Legendre wavelets operational matrix of the fractional order integration by letting

$$(I^{\alpha}\Psi_m)(x) = P^{\alpha}_{L\,m\times m} \Psi_m(x) \qquad \ldots(2.48)$$

where the matrix $\mathrm{P}^{\alpha}_{\mathrm{L}\,\mathrm{m}\times\mathrm{m}}$ is called the Legendre wavelets operational matrix of the fractional integration

Using eqs.(2.47) and (2.15), we have:

$$(I^{\alpha}\Psi)(x) = (I^{\alpha}\Theta_{m\times m} B_m)(x) = \Theta_{m\times m}(I^{\alpha} B_m)(x)$$

$$\approx \Theta_{m\times m} F^{\alpha} B_m(x) \qquad \ldots(2.49)$$

From eqs. (2.48) and (2.49), we get:

$$P_{L\,m\times m}^{\alpha}\ \Psi_m(x) = P_{L\,m\times m}^{\alpha}\ \Theta_{m\times m}B_m(x)$$

$$= \Theta_{m\times m}\ F^{\alpha}\ B_m(x) \qquad \ldots(2.50)$$

Then the Legendre wavelets operational matrix of the fractional order of integration $P_{L\,m\times m}^{\alpha}$ is given by

$$P_{L\,m\times m}^{\alpha} = \Theta_{m\times m}\ F^{\alpha}\ \Theta_{m\times m}^{-1}$$

In particular, if $k = 3$, $M = 2$ and $\alpha = 0.5$ the Legendre wavelets operational matrix of fractional integration is given by:

$$p_{L\,m\times m}^{0.5} = \begin{bmatrix} 0.3761 & 0.1272 & 0.3116 & -0.0602 & 0.2028 & -0.0153 & 0.1640 & -0.0080 \\ -0.0954 & 0.1558 & 0.0452 & -0.0247 & 0.0115 & -0.0026 & 0.0060 & -0.0009 \\ 0 & 0 & 0.3761 & 0.1272 & 0.3116 & -0.0602 & 0.2028 & -0.0153 \\ 0 & 0 & -0.0954 & 0.1558 & 0.0452 & -0.0247 & 0.0115 & -0.0026 \\ 0 & 0 & 0 & 0 & 0.3761 & 0.1272 & 0.3116 & -0.0602 \\ 0 & 0 & 0 & 0 & -0.0954 & 0.1558 & 0.0452 & -0.0247 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.3761 & 0.1272 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.0954 & 0.1558 \end{bmatrix}$$

2.5 Numerical Examples:

Next, we will use the Haar wavelets, Chebyshev wavelets and Legendre wavelets operational matrices of the fractional order integration in order to solve the fractional order differential equations for the sake of demonstrating the effectiveness of these schemes. The results obtained using Haar, Chebyshev and Legendre wavelets operational matrices of fractional order integration are compared with the analytical solution or with the solution obtained using the existing methods.

Example (2.1):

Consider the Bagley – Torvik equation

$$aD^2y(x) + bD^{1.5}y(x) + cy(x) = f(x), 0 \le x \le 1 \qquad \ldots(2.51)$$

We will consider the case:

$f(x) = c(x+1), a = 1, b = 1, c = 1$

Subject to the following initial condition:

$y(0) = 1, y'(0) = 1$

The exact solution of this problem [Zhao, 2010] is $y(x) = x + 1$.

In order to find the solution of eq. (2.51), we let:

$$D^2 y(x) = C_{2M}^T H_{2M}(x) \quad \ldots(2.52)$$

Together with the initial states, we have:

$$D^{1.5} y(x) = (I^{0.5} D^2 y)(x) = C_{2M}^T P_{ha\,2M\times 2M}^{0.5} H_{2M}(x) \quad \ldots(2.53)$$

and

$$Dy(x) = C_{2M}^T P_{ha\,2M\times 2M} H_{2M}(x) + y'(0)$$

$$= C_{2M}^T P_{ha\,2M\times 2M} H_{2M}(x) + 1 \quad \ldots(2.54)$$

Therefore:

$$y(x) = C_{2M}^T P_{ha\,2M\times 2M}^2 H_{2M}(x) + [1, 1, \ldots, 1]\Phi_{2M\times 2M}^{-1} P_{ha\,2M\times 2M} H_{2M}(x) + y(0) \quad \ldots(2.55)$$

Similarly, the input signal $f(x)$ can be expanded by the Haar functions as follows:

$$f(x) = f_{2M}^T H_{2M}(x) \quad \ldots(2.56)$$

where f_{2M}^T is known constant vector substituting eqs.(2.52), (2.53), (2.55) and (2.56) into eq.(2.51), then we get:

$$C_{2M}^T H_{2M}(x) + C_{2M}^T P_{ha\,2M\times 2M}^{0.5} H_{2M}(x) + (C_{2M}^T P_{ha\,2M\times 2M}^2 H_{2M}(x) + [1,1,\ldots,1]\Phi_{2M\times 2M}^{-1} P_{ha\,2M\times 2M} H_{2M}(x) + [1,1,\ldots,1]) = f_{2M}^T H_{2M}(x) \quad \ldots(2.57)$$

Thus eq.(2.51) has been transformed into a system of algebraic equations. Solving the system (2.57) of algebraic equations, we obtain the coefficients C_{2M}^T and hence by using eq.(2.55), we get our desired solution of eq. (2.51).

Following table (2.1) represent a comparison between the numerical solution using Haar, Chebyshev and Legendre wavelets methods and the exact solution of example (2.1)

Table (2.1)

The numerical and the exact solution of example (2.1).

x	Y_{Haar} *J=2,2M=8*	$Y_{Legendre}$ *J=2,M=2,K=3*	$Y_{Chebyshev}$ *J=2,M=2,K=3*	*Exact solution*
0.0625	1.0653	1.0545	1.0630	1.0625
0.1875	1.2072	1.0933	1.1880	1.1875
0.3125	1.3277	1.1978	1.4175	1.3125
0.4375	1.4587	1.2998	1.4625	1.4375
0.5625	1.5721	1.5425	1.5125	1.5625
0.6875	1.78	1.5979	1.6625	1.6875
0.8125	2.0042	1.7984	1.8375	1.8125
0.9375	2.0068	1.9856	1.9875	1.9375

For more accurate solution one can use larger values of J and hence M.

<u>Example (2.2):</u>

Consider the nonlinear fractional order differential equation

$$D^{1.5}y(x) = [y(x)]^2 + 1 \quad 0 \leq x < 1 \qquad \ldots(2.58)$$

Subject to the initial condition $y(0) = 0$.

Also, in order to find the approximate solution of Eq.(2.58), we let:

$$D^{1.5}y(x) = C_{2M}^{T}H_{2M}(x) \qquad \ldots(2.59)$$

Together with the initial states, then we have

$$y(x) = C_{2M}^{T}\, P_{ha\;2M\times 2M}^{1.5} H_{2M}(x) \qquad \ldots(2.60)$$

Hence:

$$y(x) = C_{2M}^{T}\, P_{ha\;2M\times 2M}^{1.5}\; \Phi_{2M\times 2M} B_{2M}(x) \qquad \ldots(2.61)$$

Suppose that:

$$C_{2M}^{T}\, P_{ha\;2M\times 2M}^{1.5}\; \Phi_{2M\times 2M} = [a_{1,} a_2, \ldots, a_m] \qquad \ldots(2.62)$$

and using Eq.(2.61), we have:

$$[y(x)]^2 = [a_1 b_1(x) + a_2 b_2(x) + \cdots + a_{2M} b_{2M}(x)]^2$$

$$= a_1^2 b_1(x) + a_2^2 b_1(x) + \cdots + a_{2M}^2 b_{2M}(x)$$

$$= [a_1^2, a_2^2, \ldots a_{2M}^2] B_{2M}(x) \qquad \ldots(2.63)$$

Substituting eqs.(2.59) and (2.63) into eq.(2.58), we have

$$\mathrm{C}_{2\mathrm{M}}^{\mathrm{T}}\, \Phi_{2\mathrm{M}\times 2\mathrm{M}}\, \mathrm{B}_{2\mathrm{M}}(\mathrm{x}) - [\mathrm{a}_1^2, \mathrm{a}_2^2, \ldots\ldots \mathrm{a}_\mathrm{m}^2]\, \mathrm{B}_{2\mathrm{M}}(\mathrm{x}) - [\,1, 1, \ldots, 1]\mathrm{B}_{2\mathrm{M}}(\mathrm{x}) = 0 \qquad \ldots(2.64)$$

This is a nonlinear system of algebraic equations which can be solved easily using MATLAB.

The solution of Eq.(2.58) for J = 2 is presents by the following figure

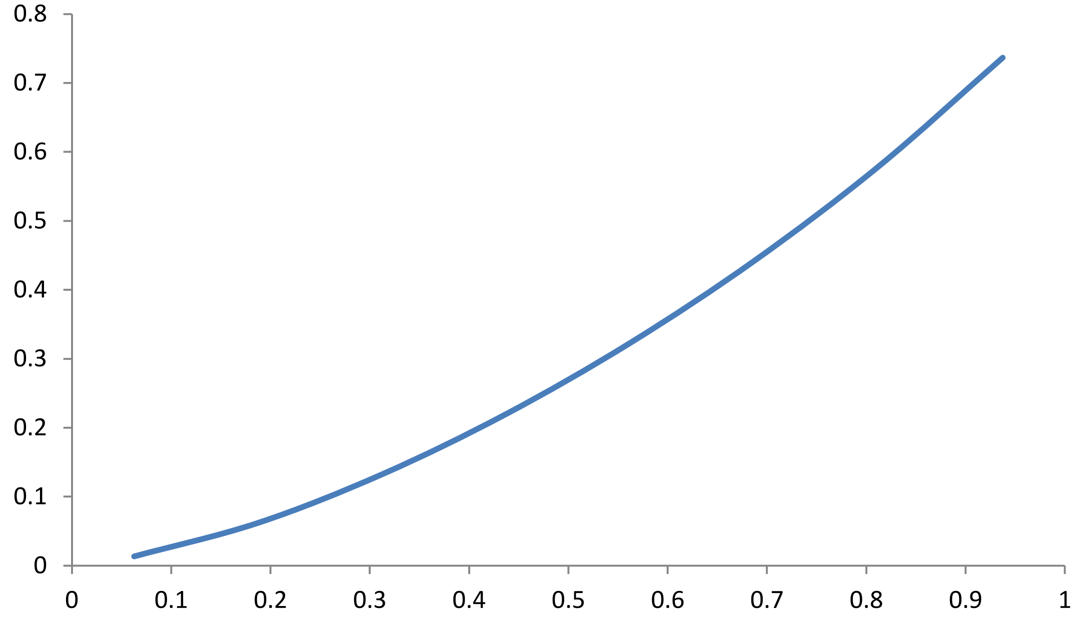

Figure (2.1)

Numerical solution of example (2.2).

It seems from Figure (2.1) that our results are coincides with the results that have been found in [Arikoglu, 2007].

Chapter Three

Wavelets Methods for Solving Partial Differential Equations of Fractional Order

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Wavelets Method for Solving Partial Differential Equations of Fractional Order

3.1 Introduction:

In this chapter, we shall present the application of Chebyshev wavelets, Haar-Chebyshev and Chebyshev-Legendre wavelets methods for solving linear partial differential equations of fractional order.

This chapter consists of four sections, in section 3.2 Chebyshev wavelets method for solving partial differential equations of fractional order is presented, while in section 3.3 the Haar-Chebyshev wavelets method will be given for solving partial differential equations of fractional order Finally the Chebyshev-Legendre wavelets method for solving partial differential equations of fractional order will be presented in section 3.4.

3.2 Chebyshev Wavelets Method for Solving Partial Differential Equations of Fractional Order:

In this section, we shall use the second kind Chebyshev wavelet operational matrix of fractional integration for solving linear partial differential equations of fractional order.

By using this method the fractional order linear partial differential equation is translated into Lyapunov type matrix equation which can be solved easily using MATLAB.

3.2.1 Function Approximation and Operational matrix:

A function $y(x,t) \in L^2(\mathbb{R})$ may be expanded as

$$y(x,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,j}\Psi_j(x)\Psi_i(t) \qquad \ldots(3.1)$$

Where

$$c_{i,j} = \int_0^1 y(x,t)\Psi_i(x)dx \int_0^1 y(x,t)\Psi_j(t)dt \qquad \ldots(3.2)$$

Eq. (3.1) can be written into the discrete form as:

$$Y(x,t) = \Omega^T(x)C\Omega(t) \qquad \ldots(3.3)$$

Taking the collocation points by following

$$x_i = \frac{2i-1}{2^k M} \ ,i=1,2,\ldots,m$$

$$t_j = \frac{2j-1}{2^k M} \ ,j=1,2,\ldots,m$$

Here $Y(x,t)$ is the discrete form of $y(x,t)$,and the matrices Ω and C are given by:

$$\Omega = \begin{bmatrix} \Psi_{1,1} & \Psi_{1,2} & \cdots & \Psi_{1,m} \\ \Psi_{2,1} & \Psi_{2,2} & \cdots & \Psi_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{m,1} & \Psi_{m,2} & \cdots & \Psi_{m,m} \end{bmatrix} ,C = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,m} \\ c_{2,1} & c_{2,2} & c_{2,m} \\ & & \\ c_{m,1} & c_{m,2} & c_{m,m} \end{bmatrix}$$

Where $m = 2^{k-1}M$ and C is the coefficient matrix of Y, which can be obtained by the following formula

$$C = (\Omega^T)^{-1} Y \ \Omega^{-1} \qquad \ldots(3.4)$$

3.2.2 The Numerical Approach for solving linear partial differential equations of fractional order:

In this section we shall use the numerical approach given by [Wu, 2009] to find the numerical solution for the linear partial differential equations of fractional order but by using the second kind Chebyshev wavelet.

consider the following first-order PDE of fractional order

$$\frac{\partial^{\alpha} y}{\partial x^{\alpha}}+\frac{\partial^{\alpha} y}{\partial t^{\alpha}}=k \qquad \ldots(3.5)$$

The fractional integration of order α with respect to the variable t of $Y(x,t)=\Omega^T(x)C\,\Omega(t)$

It yields:

$$I_t^{\alpha}Y=I_t^{\alpha}\left(\Omega^T(x)C\,\Omega(t)\right)=\Omega^T(x)C\left[I_t^{\alpha}\Omega(t)\right]=\Omega^T(x)CP^{\alpha}_{Ch\,m\times m}\Omega(t) \qquad \ldots(3.6)$$

Similarly, the fractional integration of order α where $0<\alpha\leq 1$of $Y(x,t)$ with respect to the variable x can be expressed as:

$$I_x^{\alpha}Y=I_x^{\alpha}\left(\Omega^T(x)C\Omega(t)\right)$$

$$=\left[I_x^{\alpha}\Omega(x)\right]^T C\Omega(t)$$

$$=\left[P^{\alpha}_{Ch\,m\times m}\Omega(x)\right]^T C\Omega(t)$$

$$=\Omega^T(x)(P^{\alpha}_{Ch\,m\times m})^T C\Omega(t) \qquad \ldots(3.7)$$

In general, performing the double integration to the function $Y(x,t)$ with fractional order α to the variable t and to the variable x, we obtain:

$$I_x^{\alpha}I_t^{\alpha}Y=\Omega^T(x)(P^{\alpha}_{Ch\,m\times m})^T CP^{\alpha}_{Ch\,m\times m}\Omega(t) \qquad \ldots(3.8)$$

eqs. (3.6), (3.7) and (3.8) are the main formulae for solving a fractional order partial differential equation (3.5) numerically via the second kind Chebyshev wavelet operational method.

Next we will give two illustrative examples in order to illustrate the above scheme and the results obtained using this scheme will be compared with the analytical solution or the solution obtained by using other methods or approaches.

Example (3.1):

Solve the following partial differential equation:

$$\frac{\partial y}{\partial x}+\frac{\partial y}{\partial t}=1, \quad x,t\geq 0 \qquad \ldots (3.9)$$

with the initial conditions y(0,t) = y(x,0) = 0.

First we shall integrate Eq. (3.9) with respect to t, yields to:

$$\int_0^t \frac{\partial y}{\partial x}dt+\left(y(x,t)-y(x,0)\right)=\int_0^t dt \qquad \ldots (3.10)$$

then integrating (3.10) with respect to *x*, we obtain:

$$\int_0^x\int_0^t \frac{\partial y}{\partial x}dtdx+\int_0^x ydx=\int_0^x\int_0^t dtdx \qquad \ldots (3.11)$$

or:

$$\int_0^t \left(y(x,t)-y(0,t)\right)dt+\int_0^x y\,dx=\int_0^x\int_0^t dtdx \qquad \ldots (3.12)$$

$$\int_0^t y\,dt+\int_0^x y\,dx=\int_0^x\int_0^t dtdx \qquad \ldots (3.13)$$

For solving the partial differential equation (3.9) by the proposed method, we shall let

$Y(x,t) = \Omega^T(x) C \Omega(t)$ and substitute (3.6) and (3.7) using $\alpha=1$ into (3.13), it gives:

$$\Omega^T C P_{Ch\,m\times m} \Omega + \Omega^T (P_{Ch\,m\times m})^T C \Omega = \Omega^T (P_{Ch\,m\times m})^T J P_{Ch\,m\times m} \Omega \quad \ldots(3.14)$$

where J is the matrix

$$J = \left(\Omega^T\right)^{-1} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{m\times m} \Omega^{-1}$$

Multiplying Eq.(3.14) from the left by $\left(\Omega^T\right)^{-1}(x)$ and from the right by $\Omega^{-1}(t)$, it yields:

$$C P_{Ch\,m\times m} + (P_{Ch\,m\times m})^T C = (P_{Ch\,m\times m})^T J P_{Ch\,m\times m} \quad \ldots(3.15)$$

Which is a Lyapanov equation and if m = 8 (k = 3, M = 2), then Eq.(3.15) becomes

$$\begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} & c_{1,5} & c_{1,6} & c_{1,7} & c_{1,8} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} & c_{2,5} & c_{2,6} & c_{2,6} & c_{2,6} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} & c_{3,5} & c_{3,6} & c_{3,6} & c_{3,6} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} & c_{4,5} & c_{4,6} & c_{4,6} & c_{4,6} \\ c_{5,1} & c_{5,2} & c_{5,3} & c_{5,4} & c_{5,5} & c_{5,6} & c_{5,6} & c_{5,6} \\ c_{6,1} & c_{6,2} & c_{6,3} & c_{6,4} & c_{6,5} & c_{6,6} & c_{6,6} & c_{6,6} \\ c_{7,1} & c_{7,2} & c_{7,3} & c_{7,4} & c_{7,5} & c_{7,6} & c_{7,7} & c_{7,8} \\ c_{8,1} & c_{8,2} & c_{8,3} & c_{8,4} & c_{8,5} & c_{8,6} & c_{8,7} & c_{8,8} \end{bmatrix} \begin{bmatrix} 0.1250 & 0.1250 & 0.2500 & 0 & 0.2500 & 0 & 0.2500 & 0 \\ -0.0313 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1250 & 0.1250 & 0.2500 & 0 & 0.2500 & 0 \\ 0 & 0 & -0.0313 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1250 & 0.1250 & 0.2500 & 0 \\ 0 & 0 & 0 & 0 & -0.0313 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1250 & 0.1250 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.0313 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0.1250 & 0.1250 & 0.2500 & 0 & 0.2500 & 0 & 0.2500 & 0 \\ -0.0313 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1250 & 0.1250 & 0.2500 & 0 & 0.2500 & 0 \\ 0 & 0 & -0.0313 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1250 & 0.1250 & 0.2500 & 0 \\ 0 & 0 & 0 & 0 & -0.0313 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1250 & 0.1250 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.0313 & 0 \end{bmatrix}^T \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} & c_{1,5} & c_{1,6} & c_{1,7} & c_{1,8} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} & c_{2,5} & c_{2,6} & c_{2,6} & c_{2,6} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} & c_{3,5} & c_{3,6} & c_{3,6} & c_{3,6} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} & c_{4,5} & c_{4,6} & c_{4,6} & c_{4,6} \\ c_{5,1} & c_{5,2} & c_{5,3} & c_{5,4} & c_{5,5} & c_{5,6} & c_{5,6} & c_{5,6} \\ c_{6,1} & c_{6,2} & c_{6,3} & c_{6,4} & c_{6,5} & c_{6,6} & c_{6,6} & c_{6,6} \\ c_{7,1} & c_{7,2} & c_{7,3} & c_{7,4} & c_{7,5} & c_{7,6} & c_{7,7} & c_{7,8} \\ c_{8,1} & c_{8,2} & c_{8,3} & c_{8,4} & c_{8,5} & c_{8,6} & c_{8,7} & c_{8,8} \end{bmatrix} =$$

$$\begin{bmatrix} 0.1250 & 0.1250 & 0.2500 & 0 & 0.2500 & 0 & 0.2500 & 0 \\ -0.0313 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1250 & 0.1250 & 0.2500 & 0 & 0.2500 & 0 \\ 0 & 0 & -0.0313 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1250 & 0.1250 & 0.2500 & 0 \\ 0 & 0 & 0 & 0 & -0.0313 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1250 & 0.1250 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.0313 & 0 \end{bmatrix}^T \begin{bmatrix} 0.1963 & 0 & 0.1963 & 0 & 0.1963 & 0 & 0.1963 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1963 & 0 & 0.1963 & 0 & 0.1963 & 0 & 0.1963 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1963 & 0 & 0.1963 & 0 & 0.1963 & 0 & 0.1963 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1963 & 0 & 0.1963 & 0 & 0.1963 & 0 & 0.1963 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.1250 & 0.1250 & 0.2500 & 0 & 0.2500 & 0 & 0.2500 & 0 \\ -0.0313 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1250 & 0.1250 & 0.2500 & 0 & 0.2500 & 0 \\ 0 & 0 & -0.0313 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1250 & 0.1250 & 0.2500 & 0 \\ 0 & 0 & 0 & 0 & -0.0313 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1250 & 0.1250 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.0313 & 0 \end{bmatrix}$$

solving the above equation with respect to the matrix C yields:

$$C = \begin{bmatrix} 0.0153 & 0.0123 & 0.0245 & 0 & 0.0245 & 0 & 0.0245 & 0 \\ 0.0123 & 0.0123 & 0.0245 & 0 & 0.0245 & 0 & 0.0245 & 0 \\ 0.0245 & 0.0245 & 0.0644 & 0.0123 & 0.0736 & 0 & 0.0736 & 0 \\ 0 & 0 & 0.0123 & 0.0123 & 0.0245 & 0 & 0.0245 & 0 \\ 0.0245 & 0.0245 & 0.0736 & 0.0245 & 0.1135 & 0.0123 & 0.1227 & 0 \\ 0 & 0 & 0 & 0 & 0.0123 & 0.0123 & 0.0245 & 0 \\ 0.0245 & 0.0245 & 0.0736 & 0.0245 & 0.1227 & 0.0245 & 0.1626 & 0.0123 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0123 & 0.0123 \end{bmatrix}$$

Hence, the matrix form of the approximate solution (see Appendix A program1) given by equation (3.3) at the point

$$x = [0.0625 \quad 0.1875 \quad 0.3125 \quad 0.4375 \quad 0.5625 \quad 0.6875 \quad 0.8125 \quad 0.9375]$$

$$t = [0.0625 \quad 0.1875 \quad 0.3125 \quad 0.4375 \quad 0.5625 \quad 0.6875 \quad 0.8125 \quad 0.9375]$$

becomes:

$$Y_{Chebyshev} = \begin{bmatrix} 0.0312 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 \\ 0.0625 & 0.1563 & 0.1875 & 0.1875 & 0.1875 & 0.1875 & 0.1875 & 0.1875 \\ 0.0625 & 0.1875 & 0.2812 & 0.3125 & 0.3125 & 0.3125 & 0.3125 & 0.3125 \\ 0.0625 & 0.1875 & 0.3125 & 0.4062 & 0.4375 & 0.4375 & 0.4375 & 0.4375 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5312 & 0.5625 & 0.5625 & 0.5625 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6563 & 0.6875 & 0.6875 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6875 & 0.7813 & 0.8125 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6875 & 0.8125 & 0.9062 \end{bmatrix}$$

The exact solution of the example (3.1)[Wu,2009] is given by:

$$y(x,t) = \begin{cases} t, & x \ge t \\ x, & t < x \end{cases}$$

Hence the matrix form of the exact solution is given by:

$$y_{exact} = \begin{bmatrix} 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 \\ 0.0625 & 0.1875 & 0.1875 & 0.1875 & 0.1875 & 0.1875 & 0.1875 & 0.1875 \\ 0.0625 & 0.1875 & 0.3125 & 0.3125 & 0.3125 & 0.3125 & 0.3125 & 0.3125 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.4375 & 0.4375 & 0.4375 & 0.4375 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.5625 & 0.5625 & 0.5625 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6875 & 0.6875 & 0.6875 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6875 & 0.8125 & 0.8125 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6875 & 0.8125 & 0.9375 \end{bmatrix}$$

and therefore, the error matrix will be

$$error = \begin{bmatrix} 0.0313 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0313 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0313 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0313 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0313 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0313 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0313 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0313 \end{bmatrix}$$

Example 3.2:

Consider the linear partial differential equation of fractional order:

$$\frac{\partial^{\frac{1}{2}} y}{\partial x^{\frac{1}{2}}} + \frac{\partial^{\frac{1}{2}} y}{\partial t^{\frac{1}{2}}} = 1, \quad x, t \geq 0 \qquad \ldots(3.16)$$

with zero initial conditions.

Applying the fractional order integration of order $\frac{1}{2}$ twice with respect to *x* and *t* respectively, thus we get

$$I_t^{\frac{1}{2}} I_X^{\frac{1}{2}} \frac{\partial^{\frac{1}{2}} y}{\partial x^{\frac{1}{2}}} + I_t^{\frac{1}{2}} I_X^{\frac{1}{2}} \frac{\partial^{\frac{1}{2}} y}{\partial t^{\frac{1}{2}}} = I_t^{\frac{1}{2}} I_X^{\frac{1}{2}} 1, \quad x,t \geq 0 \qquad \ldots(3.17)$$

Also, we shall let the approximate solution of eq.(3.16) given by

$$Y(x,t) = \Omega^T(x) C \Omega(t)$$

Then eq.(3.16) becomes:

$$\Omega^T \left(P_{Ch\, m\times m}^{\frac{1}{2}} \right)^T C\Omega + \Omega^T C P_{Ch\, m\times m}^{\frac{1}{2}} \Omega = \Omega^T \left(P_{Ch\, m\times m}^{\frac{1}{2}} \right)^T J P_{Ch\, m\times m}^{\frac{1}{2}} \Omega, \qquad \ldots(3.18)$$

Multiplying Eq.(3.18) from the left by $\left(\Omega^T\right)^{-1}(x)$ and from the right by $\Omega^{-1}(t)$, it yields:

$$\left(P_{Ch\, m\times m}^{\frac{1}{2}} \right)^T C + C P_{Ch\, m\times m}^{\frac{1}{2}} = \left(P_{Ch\, m\times m}^{\frac{1}{2}} \right)^T J P_{Ch\, m\times m}^{\frac{1}{2}}, \qquad \ldots(3.19)$$

where J is the matrix

$$J = \left(\Omega^T\right)^{-1} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{m\times m} \Omega^{-1}$$

In the case of $m = 8$ $(k = 3, M = 2)$, solving the Lyapunov eq.(3.19) with respect to the matrix C therefore we get

$$C = \begin{bmatrix} 0.0392 & 0.0153 & 0.0528 & 0.0029 & 0.0569 & 0.0015 & 0.0593 & 0.0009 \\ 0.0153 & 0.0090 & 0.0249 & 0.0023 & 0.0283 & 0.0012 & 0.0303 & 0.0008 \\ 0.0528 & 0.0249 & 0.0790 & 0.0066 & 0.0888 & 0.0036 & 0.0947 & 0.0024 \\ 0.0029 & 0.0023 & 0.0066 & 0.0014 & 0.0089 & 0.0009 & 0.0104 & 0.0006 \\ 0.0569 & 0.0283 & 0.0888 & 0.0089 & 0.1024 & 0.0051 & 0.1108 & 0.0035 \\ 0.0015 & 0.0012 & 0.0036 & 0.0009 & 0.0051 & 0.0006 & 0.0062 & 0.0005 \\ 0.0593 & 0.0303 & 0.0947 & 0.0104 & 0.1108 & 0.0062 & 0.1213 & 0.0043 \\ 0.0009 & 0.0008 & 0.0024 & 0.0006 & 0.0035 & 0.0005 & 0.0043 & 0.0004 \end{bmatrix}$$

And the numerical solution of example 3.2 using Chebyshev wavelet operational matrix will be similar to the Haar wavelet matrix given by [Wu, 2009] as below(see Appendix A program1):

$$Y_{Chebyshev} = \begin{bmatrix} 0.1330 & 0.1881 & 0.2011 & 0.2098 & 0.2157 & 0.2201 & 0.2235 & 0.2262 \\ 0.1881 & 0.2888 & 0.3221 & 0.3425 & 0.3568 & 0.3675 & 0.3759 & 0.3827 \\ 0.2011 & 0.3221 & 0.3702 & 0.4003 & 0.4217 & 0.4379 & 0.4508 & 0.4614 \\ 0.2098 & 0.3425 & 0.4003 & 0.4376 & 0.4645 & 0.4853 & 0.5019 & 0.5157 \\ 0.2157 & 0.3568 & 0.4217 & 0.4645 & 0.4960 & 0.5205 & 0.5404 & 0.5569 \\ 0.2201 & 0.3675 & 0.4379 & 0.4853 & 0.5205 & 0.5482 & 0.5709 & 0.5899 \\ 0.2235 & 0.3759 & 0.4508 & 0.5019 & 0.5404 & 0.5709 & 0.5960 & 0.6171 \\ 0.2262 & 0.3827 & 0.4614 & 0.5157 & 0.5569 & 0.5899 & 0.6171 & 0.6401 \end{bmatrix}$$

the error was given by the matrix :

$$\text{error} = 1 \times 10^{-14}\begin{bmatrix} 0.0139 & 0.0083 & 0.0611 & 0 & 0.1860 & 0.0638 & 0.0444 & 0.1499 \\ 0.0139 & 0.0111 & 0.0500 & 0.0167 & 0.3053 & 0.0167 & 0.1166 & 0.2831 \\ 0.0278 & 0.0111 & 0.0999 & 0.0333 & 0.3275 & 0.0500 & 0.1055 & 0.3830 \\ 0.0416 & 0.0111 & 0.0999 & 0.0111 & 0.3608 & 0.0722 & 0.1443 & 0.4108 \\ 0.0472 & 0.0056 & 0.0444 & 0.0555 & 0.3275 & 0.1332 & 0.2109 & 0.4330 \\ 0.0444 & 0.0111 & 0.0500 & 0.0611 & 0.3775 & 0.1554 & 0.1665 & 0.4330 \\ 0.0111 & 0.0444 & 0.0777 & 0.0333 & 0.3886 & 0.0777 & 0.1332 & 0.4219 \\ 0.0167 & 0.0278 & 0.0666 & 0.0333 & 0.3775 & 0.1110 & 0.1665 & 0.4552 \end{bmatrix}$$

3.3 Haar- Chebyshev Wavelets Method for Solving Linear Partial Differential Equations of Fractional Order:

In this section, we shall suggest a new approach for solving linear partial equations of fractional order by mixing the Chebyshev wavelets method with Haar wavelet method by expanding the required approximate solution as the elements of Chebyshev basis functions of the second kind in time and the Haar basis function in space.

By using this approach, the fractional order partial differential equation is translated also into Lyapunov type matrix equations which can be solved easily using MATLAB.

3.3.1 Function Approximation using Haar-Chebyshev wavelets method:

A function $y(x,t) \in L^2(\mathbb{R})$ may be also expanded as:

$$y(x,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,j} h_j(x) \Psi_i(t) \qquad \ldots(3.20)$$

Where $h_j(x)$ and $\Psi_i(t)$ are the Haar functions and the Chebyshev functions respectively, the coefficients appear in eq. (3.20) can be obtained as:

$$c_{i,j}=\int_0^1 y(x,t)h_i(x)dx.\int_0^1 y(x,t)\Psi_j(t)dt \qquad (3.21)$$

Eq.(3.20) can be written into the discrete form as follow:

$$Y(x,t)=H^T(x)C\Omega(t) \qquad \ldots(3.22)$$

The matrices Ω, C and H are given by

$$\Omega=\begin{bmatrix}\Psi_{1,1} & \Psi_{1,2} & \cdots & \Psi_{1,m}\\ \Psi_{2,1} & \Psi_{2,2} & \cdots & \Psi_{2,m}\\ \vdots & \vdots & \ddots & \vdots\\ \Psi_{m,1} & \Psi_{m,2} & \cdots & \Psi_{m,m}\end{bmatrix},C=\begin{bmatrix}c_{1,1} & c_{1,2} & c_{1,m}\\ c_{2,1} & c_{2,2} & c_{2,m}\\ & & \\ c_{m,1} & c_{m,2} & c_{m,m}\end{bmatrix}$$

$$H=\begin{bmatrix}h_{1,1} & h_{1,2} & \cdots & h_{1,m}\\ h_{2,1} & h_{2,2} & \cdots & h_{2,m}\\ \vdots & \vdots & \ddots & \\ h_{m,1} & h_{m,2} & \cdots & h_{m,m}\end{bmatrix}$$

The coefficients matrix C of Y can be obtained by the following formula

$$C=(H^T)^{-1}Y\Omega^{-1} \qquad \ldots(3.23)$$

3.3.2 The Numerical Approach:

For solving the linear partial differential equation of fractional order (3.5) using Haar-Chebyshev wavelets method the integration of $Y(x,t)=H^T(x)C\Omega(t)$ with respect to the variable t yields:

$$I_t^{\alpha}Y=I_t^{\alpha}\left(H^T(x)C\Omega(t)\right)=H^T(x)C\left[I_t^{\alpha}\Omega(t)\right]=H^T(x)CP^{\alpha}_{Chm\times m}\Omega(t)$$

or

$$I_t^{\alpha} Y = H^T C P^{\alpha}_{Ch\, m\times m} \Omega \qquad \ldots(3.24)$$

Similarly, the fractional integration of order *α* of *Y(x,t)* with respect to variable *x* can be expressed as:

$$I_x^{\beta} Y = I_x^{\alpha}\left(H^T(x) C\, \Omega(t)\right)$$

$$= \left[I_x^{\alpha} H(x)\right]^T C\, \Omega(t)$$

$$= \left[P^{\alpha}_{ha\, m\times m} H(x)\right]^T C\, \Omega(t)$$

$$= H^T(x)(P^{\alpha}_{ha\, m\times m})^T C\, \Omega(t)$$

$$= H^T(x)(P^{\alpha}_{ha\, m\times m})^T C\, \Omega(t) \qquad \ldots(3.25)$$

In general, performing the fractional order integration of order α twice with respect to the variables *x* and *t* respectively, we obtain:

$$I_t^{\alpha} I_x^{\alpha} Y = H^T(x)(P^{\alpha}_{ha\, m\times m})^T C P^{\alpha}_{Ch\, m\times m} \Omega(t) \qquad \ldots(3.26)$$

Eq.(3.24), (3.25) and (3.26) are the main formulae for solving a fractional partial differential equation numerically via the Haar- Chebyshev wavelet operational method.

The above procedure will be clear and illustrated by considering the following numerical examples.

<u>Example 3.3:</u>

In this example we will consider the problem given in example (3.1) we will follow the same approach considered in example (3.1) to solve this problem via Haar-Chebyshev wavelets method and therefore we let

$$Y(x,t) = H^T(x) C\, \Omega(t)$$

and substitute (3.25) and (3.26) with $\alpha = 1$ into (3.13), it gives:

$$H^T(x)CP_{Ch\,m\times m}\Omega(t)+H^T(x)(P_{ha\,m\times m})^T C\,\Omega(t)=H^T(x)(P_{ha\,m\times m})^T JP_{Ch\,m\times m}\Omega(t) \quad \ldots(3.27)$$

where J is given by:

$$J=\left(H^T(x)\right)^{-1}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{m\times m} \Omega(t)^{-1}$$

Multiplying Eq.(3.27) from the left by $(H^T(x))^{-1}$ and by $\Omega(t)^{-1}$ from the right, it yields:

$$CP_{Ch\,m\times m}+(P_{ha\,m\times m})^T C=(P_{ha\,m\times m})^T JP_{Ch\,m\times m}$$

If $m=8\ (k=3, M=2)$, solving the above equation yields:

$$C=\begin{bmatrix} 0.1420 & 0.1371 & 0.3769 & 0.0979 & 0.5336 & 0.0587 & 0.6119 & 0.0196 \\ -0.0147 & -0.0196 & -0.0930 & -0.0587 & -0.2203 & -0.0587 & -0.2986 & -0.0196 \\ -0.0208 & -0.0277 & -0.0900 & -0.0277 & -0.1108 & 0 & -0.1108 & 0 \\ 0 & 0 & 0 & 0 & -0.0208 & -0.0277 & -0.0900 & -0.0277 \\ -0.0196 & -0.0196 & -0.0392 & 0 & -0.0392 & 0 & -0.0392 & 0 \\ 0 & 0 & -0.0196 & -0.0196 & -0.0392 & 0 & -0.0392 & 0 \\ 0 & 0 & 0 & 0 & -0.0196 & -0.0196 & -0.0392 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.0196 & -0.0196 \end{bmatrix}$$

The matrix form of *Y(x,t)* (see Appendix A program1) is given by:

$$Y_{H,Ch}=\begin{bmatrix} 0.0313 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 \\ 0.0625 & 0.1563 & 0.1875 & 0.1875 & 0.1875 & 0.1875 & 0.1875 & 0.1875 \\ 0.0625 & 0.1875 & 0.2813 & 0.3126 & 0.3126 & 0.3126 & 0.3126 & 0.3126 \\ 0.0625 & 0.1875 & 0.3126 & 0.4064 & 0.4376 & 0.4376 & 0.4376 & 0.4376 \\ 0.0625 & 0.1875 & 0.3126 & 0.4376 & 0.5314 & 0.5626 & 0.5626 & 0.5626 \\ 0.0625 & 0.1875 & 0.3126 & 0.4376 & 0.5626 & 0.6564 & 0.6877 & 0.6877 \\ 0.0625 & 0.1875 & 0.3126 & 0.4376 & 0.5626 & 0.6877 & 0.7815 & 0.8127 \\ 0.0625 & 0.1875 & 0.3126 & 0.4376 & 0.5626 & 0.6877 & 0.8127 & 0.9065 \end{bmatrix}$$

And the matrix form of the exact solution is given by:

$$Y_{exact}=\begin{bmatrix} 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 \\ 0.0625 & 0.1875 & 0.1875 & 0.1875 & 0.1875 & 0.1875 & 0.1875 & 0.1875 \\ 0.0625 & 0.1875 & 0.3125 & 0.3125 & 0.3125 & 0.3125 & 0.3125 & 0.3125 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.4375 & 0.4375 & 0.4375 & 0.4375 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.5625 & 0.5625 & 0.5625 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6875 & 0.6875 & 0.6875 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6875 & 0.8125 & 0.8125 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6875 & 0.8125 & 0.9375 \end{bmatrix}$$

The error was given by:

$$\text{error}=\begin{bmatrix} 0.0312 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0312 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0312 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0311 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0311 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0311 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0310 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0310 \end{bmatrix}$$

<u>Example 3.4:</u>

The same example considered in example(3.2) will be considered and to find the solution via the Haar-Chebyshev wavelets method so we let

$$Y(x,t)=H^{T}(x)C\Omega(t)$$

and by substituting Eq.(3.24), (3.25) and (3.26) using $\alpha=\dfrac{1}{2}$ into(3.16)thus we have

$$\left(P_{ha\,m\times m}^{\frac{1}{2}}\right)^{T}C+CP_{Ch\,m\times m}^{\frac{1}{2}}=\left(P_{ha\,m\times m}^{\frac{1}{2}}\right)^{T}JP_{Ch\,m\times m}^{\frac{1}{2}},$$

Solving the above equation in terms of the matrix *C* *m=8 (k = 3, M=2)* thus we get:

$$C=\begin{bmatrix} 0.0392 & 0.0153 & 0.0528 & 0.0029 & 0.0569 & 0.0015 & 0.0593 & 0.0009 \\ 0.0153 & 0.0090 & 0.0249 & 0.0023 & 0.0283 & 0.0012 & 0.0303 & 0.0008 \\ 0.0528 & 0.0249 & 0.0790 & 0.0066 & 0.0888 & 0.0036 & 0.0947 & 0.0024 \\ 0.0029 & 0.0023 & 0.0066 & 0.0014 & 0.0089 & 0.0009 & 0.0104 & 0.0006 \\ 0.0569 & 0.0283 & 0.0888 & 0.0089 & 0.1024 & 0.0051 & 0.1108 & 0.0035 \\ 0.0015 & 0.0012 & 0.0036 & 0.0009 & 0.0051 & 0.0006 & 0.0062 & 0.0005 \\ 0.0593 & 0.0303 & 0.0947 & 0.0104 & 0.1108 & 0.0062 & 0.1213 & 0.0043 \\ 0.0009 & 0.0008 & 0.0024 & 0.0006 & 0.0035 & 0.0005 & 0.0043 & 0.0004 \end{bmatrix}$$

And the numerical solution of the above example using Haar-Chebyshev wavelet operational matrix will be given as below (see Appendix A program1):

$$Y_{H,Ch}=\begin{bmatrix} 0.1330 & 0.1881 & 0.2011 & 0.2098 & 0.2157 & 0.2201 & 0.2235 & 0.2262 \\ 0.1881 & 0.2888 & 0.3221 & 0.3425 & 0.3568 & 0.3675 & 0.3759 & 0.3827 \\ 0.2011 & 0.3221 & 0.3702 & 0.4003 & 0.4217 & 0.4379 & 0.4508 & 0.4614 \\ 0.2098 & 0.3425 & 0.4003 & 0.4376 & 0.4645 & 0.4853 & 0.5019 & 0.5157 \\ 0.2157 & 0.3568 & 0.4217 & 0.4645 & 0.4960 & 0.5205 & 0.5404 & 0.5569 \\ 0.2201 & 0.3675 & 0.4379 & 0.4853 & 0.5205 & 0.5482 & 0.5709 & 0.5899 \\ 0.2235 & 0.3759 & 0.4508 & 0.5019 & 0.5404 & 0.5709 & 0.5960 & 0.6171 \\ 0.2262 & 0.3827 & 0.4614 & 0.5157 & 0.5569 & 0.5899 & 0.6171 & 0.6401 \end{bmatrix} \text{ and}$$

hence the error matrix

$$error=1\times 10^{-14}\begin{bmatrix} 0.0111 & 0.0222 & 0.0722 & 0.0222 & 0.2137 & 0.0583 & 0.1082 & 0.1971 \\ 0.0056 & 0.0222 & 0.0666 & 0.0111 & 0.3109 & 0.0111 & 0.1388 & 0.3275 \\ 0.0083 & 0.0222 & 0.0555 & 0.0222 & 0.3553 & 0.0555 & 0.1665 & 0.3775 \\ 0.0305 & 0 & 0.0666 & 0.0111 & 0.3775 & 0.0666 & 0.1665 & 0.4108 \\ 0.0250 & 0.0056 & 0.0611 & 0.0278 & 0.3775 & 0.0888 & 0.1776 & 0.4219 \\ 0.0194 & 0.0056 & 0.0555 & 0.0333 & 0.4108 & 0.1221 & 0.1776 & 0.4219 \\ 0.0250 & 0.0056 & 0.0444 & 0.0444 & 0.4330 & 0.1443 & 0.1887 & 0.4552 \\ 0.0167 & 0.0056 & 0.0500 & 0.0333 & 0.4330 & 0.1443 & 0.1776 & 0.4663 \end{bmatrix}$$

3.4 Chebyshev - Legendre Wavelets Method for Solving Partial Differential Equations of Fractional Order

In this section, a similar approach that have been given in section 3.3 will be given to solve partial differential equations of fractional order but by mixing the Chebyshev wavelets method with Legendre wavelet method by expanding the required approximate solution as the elements of Chebyshev basis functions of the second kind in time and the Legendre basis function in space.

By using this method the fractional order partial differential equation is translated also into Lyapunov type matrix equation which can be solved easily using MATLAB.

3.4.1 Function Approximation using Chebyshev-Legendre wavelets method:

A function $y(x,t) \in L^2(\mathbb{R})$ may be expanded as

$$y(x,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,j} L_j(x) \Psi_i(t) \qquad \ldots(3.28)$$

Where the coefficients $c_{i,j}$ are given by

$$c_{i,j} = \int_0^1 y(x,t) L_i(x)dx.\int_0^1 y(x,t)\Psi_j(t)dt \qquad \ldots(3.29)$$

Equation (3.28) can be written in discrete form as

$$Y(x,t) = \Psi^T(x) C \Omega(t) \qquad \ldots(3.30)$$

Where:

$$\Omega = \begin{bmatrix} \Psi_{1,1} & \Psi_{1,2} & \cdots & \Psi_{1,m} \\ \Psi_{2,1} & \Psi_{2,2} & \cdots & \Psi_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{m,1} & \Psi_{m,2} & \cdots & \Psi_{m,m} \end{bmatrix}, C = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,m} \\ c_{2,1} & c_{2,2} & c_{2,m} \\ & & \\ c_{m,1} & c_{m,2} & c_{m,m} \end{bmatrix}$$

$$\Psi = \begin{bmatrix} L_{1,1} & L_{1,2} & \cdots & L_{1,m'} \\ L_{2,1} & L_{2,2} & \cdots & L_{2,m'} \\ \vdots & \vdots & \ddots & \\ L_{m,1} & L_{m,2} & \cdots & L_{m,m} \end{bmatrix}$$

The matrix C is the coefficient matrix of the approximate solution *Y(x,t)* , and it can be obtained by the formula

$$C = (\Psi^T(x))^{-1} Y \Omega(t)^{-1} \qquad \ldots(3.31)$$

3.4.2 The Numerical Approach:

For solving the partial differential equation of fractional order (3.5) using the Chebyshev–Legendre wavelet method we integrate:

$$Y(x,t) = \Psi^T(x) C \Omega(t)$$

Fractionally of order α with respect to the variable t it yields:

$$I^{\alpha}Y = I^{\alpha}\left(\Psi^T(x) C \Omega(t)\right) = \Psi^T(x) C \left[I^{\alpha}\Omega(t)\right] = \Psi^T(x) C P_{Ch\, m\times m}\Omega(t) \qquad \ldots(3.32)$$

Similarly, the fractional integration order α of *Y(x,t)* with respect to variable *x* can be expressed as:

$$I_x^{\alpha}Y = I_x^{\alpha}\left(\Psi^T(x) C \Omega(t)\right)$$

$$= \left[I_x^{\alpha}\Psi(x)\right]^T C \Omega(t)$$

$$= \left[P_{L\, m\times m}^{\alpha}\Psi(x)\right]^T C \Omega(t)$$

$$= \Psi^T(x)(P_{L\, m\times m}^{\alpha})^T C \Omega(t)$$

$$= \Psi^T(x)(P_{L\, m\times m}^{\alpha})^T C \Omega(t) \qquad \ldots(3.33)$$

In general, performing the fractional order integration of order α twice with respect to the variables x and t respectively, we obtain:

$$I_t^{\alpha} I_x^{\alpha} Y = \Psi^T(x)(P_{L\, m\times m}^{\alpha})^T C P_{Ch\, m\times m}^{\alpha}\Omega(t) \qquad \ldots(3.34)$$

Eqs. (3.32),(3,33) and (3.34) are the main formulae for solving a fractional partial differential equation numerically via the Chebyshev-Legendre wavelets operational matrices method.

The above procedure will be clear and illustrated by the following numerical examples given in the next section

3.4.3 Numerical Examples:

In this section we will use the Chebyshev-Legendre wavelets operational matrices of the fractional integration to solve linear fractional order partial differential equations and the results obtained using this scheme will be compare with the analytical solution or the solution obtained using other method or approaches.

<u>Example 3.5:</u>

We will consider also in this example the same equation given in example 3.1 and in order to find the approximate solution of this equation using Chebyshev-Legendre wavelets method, we let:

$$Y(x,t) = \Psi^T(x) C \Omega(t)$$

and substitute (3.34) ,(3.35)and (3.36) into (3.9), we get::

$$\Psi^T C P_{Ch\, m\times m} \Omega + \Psi^T (P_{L\ m\times m})^T C\Omega = \Psi^T (P_{L\, m\times m})^T J P_{Ch\, m\times m} \Omega \quad \ldots(3.35)$$

where J is the matrix given by the following formula:

$$J = \left(\Psi^T\right)^{-1} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \Omega^{-1}$$

Multiply eq. (3.35) from the left by $\left(\Psi^{T}\right)^{-1}$ and from the right by Ω^{-1},we get

$$CP_{Ch\,m\times m}+(P_{L\,m\times m})^{T}C=(P_{L\,m\times m})^{T}JP_{Ch\,m\times m}$$

If $m=8$ $(k=3, M=2)$, then the coefficient matrix C becomes:

$$C=\begin{bmatrix} 0.0122 & 0.0057 & 0.0196 & 0 & 0.0196 & 0 & 0.0196 & 0 \\ 0.0098 & 0.0057 & 0.0196 & 0 & 0.0196 & 0 & 0.0196 & 0 \\ 0.0196 & 0.0113 & 0.0514 & 0.0057 & 0.0587 & 0 & 0.0587 & 0 \\ 0 & 0 & 0.0098 & 0.0057 & 0.0196 & 0 & 0.0196 & 0 \\ 0.0196 & 0.0113 & 0.0587 & 0.0113 & 0.0906 & 0.0057 & 0.0979 & 0 \\ 0 & 0 & 0 & 0 & 0.0098 & 0.0057 & 0.0196 & 0 \\ 0.0196 & 0.0113 & 0.0587 & 0.0113 & 0.0979 & 0.0113 & 0.1297 & 0.0057 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0098 & 0.0057 \end{bmatrix}$$

And hence the solution matrix Y(x,t) (see Appendix A program1) is given by:

$$Y_{Ch,L}=\begin{bmatrix} 0.0312 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 \\ 0.0625 & 0.1563 & 0.1875 & 0.1875 & 0.1875 & 0.1875 & 0.1875 & 0.1875 \\ 0.0625 & 0.1875 & 0.2813 & 0.3125 & 0.3125 & 0.3125 & 0.3125 & 0.3125 \\ 0.0625 & 0.1875 & 0.3125 & 0.4063 & 0.4375 & 0.4375 & 0.4375 & 0.4375 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5313 & 0.5625 & 0.5625 & 0.5625 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6563 & 0.6875 & 0.6875 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6875 & 0.7812 & 0.8125 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6875 & 0.8125 & 0.9062 \end{bmatrix}$$

Example3.5:

The same example considered in example3.2 will be considered and to find the solution via the Chebyshev-Legendre wavelets method so we let

$$Y(x,t)=\Psi^{T}(x)C\,\Omega(t)$$

and by substituting Eq.(3.32), (3.33) and (3.34) using $\alpha=\frac{1}{2}$ into(3.16)thus we have

$$\left(P_{L\,m\times m}^{\frac{1}{2}}\right)^{T} C + CP_{Ch\,m\times m}^{\frac{1}{2}} = \left(P_{L\,m\times m}^{\frac{1}{2}}\right)^{T} JP_{Ch\,m\times m}^{\frac{1}{2}},$$

Solving the above equation in terms of the matrix C $m=8$ $(k=3, M=2)$ thus we get

$$C = \begin{bmatrix} 0.0122 & 0.0057 & 0.0196 & 0 & 0.0196 & 0 & 0.0196 & 0 \\ 0.0098 & 0.0057 & 0.0196 & 0 & 0.0196 & 0 & 0.0196 & 0 \\ 0.0196 & 0.0113 & 0.0514 & 0.0057 & 0.0587 & 0 & 0.0587 & 0 \\ 0 & 0 & 0.0098 & 0.0057 & 0.0196 & 0 & 0.0196 & 0 \\ 0.0196 & 0.0113 & 0.0587 & 0.0113 & 0.0906 & 0.0057 & 0.0979 & 0 \\ 0 & 0 & 0 & 0 & 0.0098 & 0.0057 & 0.0196 & 0 \\ 0.0196 & 0.0113 & 0.0587 & 0.0113 & 0.0979 & 0.0113 & 0.1297 & 0.0057 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0098 & 0.0057 \end{bmatrix}$$

And the numerical solution of the above example using Chebyshev-Legendre wavelet operational matrix will be given as below:

$$Y_{Ch,L} = \begin{bmatrix} 0.0312 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 \\ 0.0625 & 0.1563 & 0.1875 & 0.1875 & 0.1875 & 0.1875 & 0.1875 & 0.1875 \\ 0.0625 & 0.1875 & 0.2813 & 0.3125 & 0.3125 & 0.3125 & 0.3125 & 0.3125 \\ 0.0625 & 0.1875 & 0.3125 & 0.4063 & 0.4375 & 0.4375 & 0.4375 & 0.4375 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5313 & 0.5625 & 0.5625 & 0.5625 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6563 & 0.6875 & 0.6875 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6875 & 0.7812 & 0.8125 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6875 & 0.8125 & 0.9062 \end{bmatrix}$$

Conclusions and Future Works

From the present study, we can conclude the following:

1. Wavelets methods have been proved to be powerful methods for solving non-linear differential equations of fractional order.
2. Chebyshev, Haar-Chebyshev and Chebyshev-Legendre wavelets methods gave reasonable results when they used to solve partial differential equations of fractional order.
3. It seems from the results that Haar-Chebyshev wavelets method gave more accurate results than the other methods (Chebyshev wavelets and Chebyshev-Legendre wavelets).

Also, we recommend the following problems as future work:

1. Wavelets methods for solving nonlinear fuzzy differential equations of fractional order.
2. Wavelets methods for solving fuzzy integral equations of fractional order.
3. Wavelets methods for solving differential algebraic equations and delay differential equations of fractional order.

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APPENDIX A

Program1.

```
clc
clear
k=3
M1=2
a1=input('enter the value of alpha    ')
J=2
M=2^J
mm=2^(k-1)*M1
for l=1:2*M
   x(l)=(l-0.5)/(2*M);
end
x1=x
t=x
for j=0:J
   m=2^j;
   for k=0:m-1
      i=k+m+1;
      z1(i)=k/m;
      z2(i)=(k+0.5)/m;
      z3(i)=(k+1)/m;
      for n=1:2*M
         h(1,n)=1/((2*M)^0.5);
         x=x1(n);
         if x>=z1(i)&& x<=z2(i)
      h(i,n)=(1/(2*M)^0.5)*(2^(j/2));
   elseif x>=z2(i) && x<=z3(i)
      h(i,n)=(1/(2*M)^0.5)*-2^(j/2);
   elseif x>=z3(i)&& x<=1
      h(i,n)=0;
   else
      h(i,n)=0;
   end
      end
   end
end
h
aa=[2.8284 2.8284 0 0 0 0 0 0;-2.4495 2.4495  0 0  0 0 0 0;0 0 2.8284  2.8284 0
0 0 0;0 0 -2.4495 2.4495 0 0 0 0;0 0 0 0 2.8284 2.8284  0 0;0 0 0 0 -2.4495
2.4495 0 0;0 0 0 0 0 0 2.8284 2.8284;0 0 0 0 0 0 -2.4495 2.4495]
ch=[4/pi^0.5 -2/pi^0.5 0 0 0 0 0 0 ;4/pi^0.5  2/pi^0.5 0 0 0 0 0 0;0 0 4/pi^0.5
-2/pi^0.5 0 0 0 0;0 0 4/pi^0.5  2/pi^0.5 0 0 0 0;0 0 0 0 4/pi^0.5 -2/pi^0.5 0 0
```

```
;0 0 0 0 4/pi^0.5 2/pi^0.5 0 0;0 0 0 0 0 0 4/pi^0.5 -2/pi^0.5; 0 0 0 0 0 0
4/pi^0.5 2/pi^0.5]'
for s1=1:8
   for s2=1:8
      if x1(s1)>=t(s2)
         yexact(s1,s2)=t(s2);
      else
         yexact(s1,s2)=x1(s1);
      end
   end
end
yexact
for i=1:mm
   z1='(i+1)^(a1+1)-2*((i)^(a1+1))+(i-1)^(a1+1)';
   zz1(i)=eval(z1);
end
z1
zz1
for jj=1:mm
   for ii=1:mm
      if ii==jj
         ppf1(ii,jj)=1;
      elseif ii>jj
         ppf1(ii,jj)=0;
      elseif ii<jj
         ppf1(ii,jj)=zz1(jj-ii);
      end
   end
end
ppf1
ppff1=(1/(mm)^a1)*(1/(gamma(a1+2)))*ppf1
ph=h*ppff1*inv(h)
pch=ch*ppff1*inv(ch)
pa=aa*ppff1*inv(aa)
 for u=1:8
    for uu=1:8
       J1(u,uu)=1;
    end
end
 J1
   Ja=inv(aa')*J1*inv(aa)
    Jh=h*J1*inv(h)
    Jch=inv(ch')*J1*inv(ch)
    Jhch=h*J1*inv(ch)
```

```
Jchl=inv(ch')*J1*inv(aa)
      Aa=pa'
       Ah=ph'
       Ach=pch'
        A2=ph'
        A3=pch'
      Ba=pa
         Bh=ph
         Bch=pch
         B2=pch
         B3=pa
         Qh=-1*(ph'*Jh*ph)
          Qch=-1*(pch'*Jch*pch)
      Qa=-1*(pa'*Ja*pa)
      Q2=-1*(ph'*Jhch*pch)
      Q3=-1*(pch'*Jchl*pa)
      C=lyap(Aa,Ba,Qa)
         Ch=lyap(Ah,Bh,Qh)
         Cch=lyap(Ach,Bch,Qch)
         C2=lyap(A2,B2,Q2)
         C3=lyap(A3,B3,Q3)
      Ya=aa'*C*aa
       Yh=h'*Ch*h
       Ych=ch'*Cch*ch
        Ymixhaarchebyshev=h'*C2*ch
        Ymixchebyshevlegendre=ch'*C3*aa
      yexact
```

المستخلص

الهدف الرئيسي لهذه الرسالة يتمحور حول ثلاثة أهداف :

الهدف الأول هو دراسة المبادئ الأساسية للحساب الكسري والتي سوف تطرأ لها الحاجة عند إيجاد الحل العددي للمعادلات التفاضلية (إعتيادية و وجزئية) ذات الرتب الكسريه.

الهدف الثاني هو إيجاد الحل العددي للمعادلات التفاضلية الإعتياديه خطيه وغير خطيه ذات الرتب الكسرية بإستخدام طرائق المويجات والتي هي طريقه مويجات هار وطريقة مويجات تشيبيشيف وطريقه مويجات ليجيندر.

الفكرة الرئيسية لهذه الطرائق هو إنها تخفض المعادلات التفاضلة الإعتيادية ذات الرتب الكسريه الى حل نظام جبري من المعادلات الحل لهذا النظام سوف يعطينا قيم المعاملات للحل والذي هو ممثل على شكل متسلسله لانهائيه وهكذا فهي تبسط الى حد كبير هكذا نوع من المعادلات .

الهدف الثالث هو إيجاد الحل العددي للمعادلات التفاضلية الجزئية الخطية ذات الرتب الكسرية بإستخدام ثلاث طرائق والتي هي طريقه مويجات Chebyshev وطريقة مويجات Haar-Chebyshev وطريقه Chebyshev-Legendre، الطريقتان العدديتان الاخيرتان (Haar- Chebyshev and Chebyshev-Legendre) هما طريقتان عدديتان جديدتان تم اقتراحهما في هذه الرسالة.

الفكره الرئيسية لهذه الطرائق هو تمثيل الحل على شكل متسلسله لانهائية بحيث معاملاتها يتم حسابها عن طريق تحويل المعادلات التفاضليه الجزئيه ذات الرتب الكسرية الى نظام جبري من المعادلات والذي يسمى مصفوفات نوع ليابانوف وعن طريق حل هذا النظام الجبري من المعادلات بإستخدام برنامج MATLAB سوف نحصل على المعاملات وعليه سوف نحصل على الحل المطلوب للمعادلات التفاضلية الجزئية ذات الرتب الكسرية.

جمهـورية العـراق
وزارة التعليم العالي والبحث العلمي
جــامعة النهــــريـن
كليــة العلوم
قسم الريـاضيات وتطبيقات الحاسوب

الحلول العددية للمعادلات التفاضليه ذات الرتب الكسرية بأستخدام طرائق المويجات

رسالة
مقدمة إلى مجلس كلية العلوم – جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل
حنين عبد الكريم أمين
بكالوريوس 2011 (جامعة النهرين)

إشراف
أ.م.د.أسامة حميد محمد

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