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Approximate Solutions For Delay Differential Equations Of Fractional Order

A Thesis

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By

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بسم الله الرحمن الرحيم

يَٰٓأَيُّهَا ٱلَّذِينَ
ءَامَنُوٓاْ إِذَا قِيلَ لَكُمۡ تَفَسَّحُواْ فِي ٱلۡمَجَٰلِسِ فَٱفۡسَحُواْ يَفۡسَحِ
ٱللَّهُ لَكُمۡۖ وَإِذَا قِيلَ ٱنشُزُواْ فَٱنشُزُواْ يَرۡفَعِ ٱللَّهُ ٱلَّذِينَ ءَامَنُواْ
مِنكُمۡ وَٱلَّذِينَ أُوتُواْ ٱلۡعِلۡمَ دَرَجَٰتٖۚ وَٱللَّهُ بِمَا تَعۡمَلُونَ خَبِيرٞ ١١

صدق الله العظيم

سورة المجادلة
الآية (11)

Dedication

To My Family

With all Love and Respects

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My deepest thanks to Allah, for his generous by providing me the strength to accomplish this work.

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Abbas Ibrahim Khlaif

September, 2014.

SUMMERY

The main theme of this thesis is oriented about two objects:

The first objective is to find the approximate solution of delay differential equations of fractional order using Adomian decomposition method.

The second objective is to find the approximate solution of delay differential equations of fractional order using homotopy analysis method.

In both methods, the solutions are found in the form of a convergent power series with easily computed components.

Some numerical examples are presented and the results of these examples are compared with the exact solution in order to illustrate the accuracy and ability of the proposed methods.

CONTENTS

INTRODUCTION

Delay differential equations were initially introduced in the 18^{th} century by Laplace and Condorect, [Ulsoy, 2003]. However, the rapid development of the theory and applications of those equations did not come until after the Second World War, and continues till today. The basic theory concerning the stability of systems described by equations of this type was developed by Pontryagin in 1942. Important works have been written by Smith in 1957, Pinney in 1958, Bellman and Cooke in 1963, Halanay in 1966, Myshkis in 1972, Hale 1977, Yanusherski in 1978 and Marshal in 1979, [Ulsoy, 2003].

On the other hand, many complicated physical problems described in terms of partial differential equations can be approximated by much simpler problems described in terms of delay differential equations, [Pinney, 1958].

The impetus has mainly been due to the developments in many fields, such as the control theory, mathematical biology, and mathematical economics, etc. Minorsky, [Hale, 1977] was one of the first investigators of modern times to study the delay differential equation:

$$y'(t) = f(t, y(t), y(t-\tau))$$

and its effect on simple feed-back control systems in which the communication time cannot be neglected.

The abundance of applications is stimulating a rapid development of the theory of differential equations with deviating argument and, at present, this theory is one of the most rapidly developing branches of mathematical analysis.

Equations with a deviating argument describe many processes with an effect; such equations appear, for example, any time when in physics or technology we consider a problem of a force, acting on a material point, that depends on the velocity and position of the point not only at the given moment but at some moment preceding the given moment, [El'sgolt'c, 1973].

Fractional calculus is a field of mathematical study that grows out of the traditional definitions of the calculus integral and derivative operators in the same way fractional exponents is an outgrowth of exponents with integer value, [Loverro, 2004].

Many found, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative. The most famous of these definitions that have been popularized in the world of fractional calculus are the Riemann-Liouville and Grünwald-Letnikov definition. Also, Caputo, [Podlubny, 1999] reformulated the more "classic" definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions to solve his fractional order differential equations. Recently, [Kolowankar, 1996] reformulated again, the Riemann-Liouville fractional order derivative, in order to, differentiate no-where differentiable fractal functions.

In recent years, considerable interest in fractional calculus have been simulated by the applications that this subject finds in numerical analysis, differential equations and different areas of applied sciences, especially in physics and engineering, possibly including fractal phenomena.

This subject, devoted exclusively to the subject of fractional calculus in the book by Oldham and Spanier [Oldham, 1974] published in 1974. One of the most important works on the subject of fractional calculus is the book of Podlubny [Podlubny, 1999], published in 1999 which deals principally with fractional order differential equations, and today there exist at least two international journals which are devoted almost entirely to the subject of fractional calculus; (i) Journal of fractional calculus and (ii) Fractional calculus and Applied Analysis.

The purpose of this work is to find the approximate solution of delay differential equations of fractional order using two different approximate methods.

Fractional delay differential equations (FDDEs) are a very recent topic. Although it seems natural to model certain processes and systems in engineering and other science (with memory and heritage properties) with this kind of equations, only in the last few years has the attention of the scientific community been devoted to them [Moragdo, 2013].

Concerning the existence of solutions of (FDDEs) we refer [Lakshmikantham, 2008], [Ye, 2007], [Liao, 2009]. In [Lakshmikantham, 2008] Lakshmikantham provides sufficient conditions for the existence of solutions to initial value problems to single term nonlinear delay fractional differential equations, with the fractional derivative defined in the Riemann-Liouville sense. In [Ye, 2007], Ye, et al. investigate the existence of positive solutions for a class of single term delay fractional differential equations. Later in [Liao, 2009], for the same class of equations, sufficient condition for the uniqueness of the solution are reported [Moragdo, 2013].

This thesis consists of three chapters, as well as, this introduction. In chapter one, Basic concepts of delay differential equations and fractional calculus are given, while in chapter two, the Adomian decomposition method for solving delay differential equations of fractional order, as well as, some illustrative examples is presented. Finally in chapter three homotopy analysis method for solving delay differential equations of fractional order, with illustrative examples have been given .

It is remarkable that all the calculations are made by using Mathcad 2014.

CHAPTER ONE

BASIC CONCEPTS

1.1 Introduction

In this chapter we shall present the basic concepts of two main subjects which are so – called delay differential equations and fractional calculus which are necessary for the construction of this thesis .

This chapter consists of four sections. In section 1.2 the basic concepts of delay differential equations were given. In section 1.3 we shall give a brief introduction to the subject of fractional calculus including the beta and gamma functions, the fractional integration and fractional derivatives. Finally, we shall mention our main result in section 1.4.

1.2 Delay Differential Equations, [Bellman, 1963]

Delay differential equation "DDE" is defined as an unknown function y(t) and some of its derivatives, evaluated at arguments that differ by any of fixed number of values $\tau_1, \tau_2, \ldots, \tau_k$. The general form of the n-th order DDE is given by

$$F(t, y(t), y(t-\tau_1), \ldots, y(t-\tau_k), y'(t), y'(t-\tau_1), \ldots, y'(t-\tau_k), \ldots, y^{(n)}(t),$$
$$y^{(n)}(t-\tau_1), \ldots, y^{(n)}(t-\tau_k)) = 0 \qquad \ldots(1.1)$$

where F is a given functional and τ_i, $\forall$ i = 1, 2, …, k; are given fixed positive number called the "time delay".

In some literature equation (1.1) is called a difference-differential equation or functional differential equation, [Bellman, 1963], or an equation with time lag [Halanay, 1966], or a differential equation with deviating arguments, [Driver, 1977].

The emphasis will be, in general, on the linear equations with constant coefficients of the first order and with one delay (because as in "ODE" any differential equation with higher order than one may be transformed into a linear system of differential equations of the first order)

$$a_0 y'(t) + a_1 y'(t-\tau) + b_0 y(t) + b_1 y(t-\tau) = f(t) \qquad \ldots(1.2)$$

where f(t) is a given continuous function and τ is a positive constant and a_0, a_1, b_0, and b_1 are constants (if f(t) = 0, then equation (1.2) is said to be homogenous; otherwise it is nonhomogenous).

The kind of initial conditions that should be used in DDE's differ from ODE's so that one should specify in DDE's an initial function on some interval of length τ, say $[t_0 - \tau, t_0]$ and then try to find the solution of equation (1.2) for all $t \geq t_0$. Thus, we set $y(t) = \varphi_0(t)$, for $t_0 - \tau \leq t \leq t_0$ where $\varphi_0(t)$ is some given continuous function. Therefore the solution of DDE consist of finding a continuous extension of $\varphi_0(t)$ into a function y(t) which satisfies (1.2) for all $t \geq t_0$, [Halanay, 1966].

Delay differential equation given by equation (1.2) can be classified into three types which are retarded, neutral and mixed. The first type means an equation where the rate of change of state variable y is determined by the present and past states of the equation (1.2) where the coefficient of $y'(t-\tau)$ is zero, i.e., ($a_0 \neq 0$, $a_1 = 0$). If the rate of change of state depends on its own past values as well on its derivatives, the equation is then of neutral type, equation (1.2) where the coefficient of $y(t-\tau)$ is zero, i.e.,($a_0 \neq 0$, $a_1 \neq 0$ and $b_1 = 0$), while the third type is a combination of the previous two types, i.e., ($a_0 \neq 0$, $a_1 \neq 0$, $b_0 \neq 0$ and $b_1 \neq 0$).

1.2.1 Solution of the First Order Delay Differential Equations, [Driver, 1977]:

Because of the initial condition which is given for a time step interval with length equals to τ, we must find this solution for $t \geq t_0$ divided into steps with length τ also.

1- The Method of Successive Integrations:

The best well known method for solving DDE's is the method of steps or the method of successive integrations which is used to solve a DDE of the form:

$$y'(t) = f(t, y(t), y(t-\tau), y'(t-\tau)), \; t \geq t_0 \qquad \text{...(1.3)}$$

with initial condition $y(t) = \varphi_0(t)$, for $t_0 - \tau \leq t \leq t_0$. For such equations the solution is constructed step by step as follows:

Given that a function $\varphi_0(t)$ continuous on $[t_0 - \tau, t_0]$, therefore one can obtain the solution in the next step interval $[t_0, t_0 + \tau]$ by solving the following equation:

$$y'(t) = f(t, y(t), \varphi_0(t-\tau), \varphi'_0(t-\tau)), \text{ for } t_0 \leq t \leq t_0+\tau$$

with the initial condition $y(t_0) = \varphi_0(t_0)$. If we consider that $\varphi_1(t)$ is the desired first step solution, which exists by virtue of continuity hypotheses.

Similarly, if $\varphi_1(t)$ is defined on the whole segment $[t_0, t_0+\tau]$ then, one can find the solution $\varphi_2(t)$ to the equation:

$$y'(t) = f(t, y(t), \varphi_1(t-\tau), \varphi'_1(t-\tau)), \text{ for } t_0+\tau \leq t \leq t_0 + 2\tau$$

with the initial condition $y(t_0 + \tau) = \varphi_1(t_0 + \tau)$.

In general, by assuming that $\varphi_{k-1}(t)$, $\forall(k=1,2,\ldots)$ is defined on the interval $[t_0+(k-2)\tau,\ t_0+(k-1)\tau]$, then, one can find the solution $\varphi_k(t)$ to the equation:

$$y'(t) = f(t, y(t), \varphi_{k-1}(t-\tau), \varphi'_{k-1}(t-\tau)), \text{ for } t_0+(k-1)\tau \le t \le t_0+k\tau$$

with the initial condition $y(t_0 + (k-1)\tau) = \varphi_{k-1}(t_0 + (k-1)\tau)$.

Now, we shall consider some illustrative examples for all types of DDE:

<u>Example (1.1):</u>

Consider the retarded first order DDE:

$$y'(t) = y(t-1),\ t \ge 0$$

with the initial condition:

$$y(t) = \varphi_0(t) = t, \text{ for } -1 \le t \le 0$$

To find the solution in the first step interval [0, 1] we have to solve the following equation:

$$\begin{aligned} y'(t) &= \varphi_0(t-1) \\ &= t-1, \text{ for } 0 \le t \le 1 \end{aligned}$$

Integrating both sides from 0 to t where $0 \le t \le 1$, we have:

$$\int_0^t y'(s)ds = \int_0^t (s-1)ds$$

and hence after carrying some calculations we get the first time step solution:

$$y(t) = \frac{t^2}{2} - t, \text{ for } 0 \le t \le 1$$

In order to find the solution in the second step interval, suppose that:

$$\varphi_1(t) = y_1\,(t) = \frac{t^2}{2} - t\,,\ 0 \le t \le 1$$

Since $\varphi_1(t)$ is defined on the whole segment [0, 1].

Hence by forming the new equation:

$$y'(t) = \varphi_1(t-1),\ \text{for } 0 \le t \le 1 \qquad \ldots(1.4)$$

with the initial condition $\varphi_1(t) = \frac{t^2}{2} - t$, for $0 \le t \le 1$.

One can find the solution in the next step interval [1, 2], and we shall solve equation (1.4)

$$y'(t) = \varphi_1(t-1),\ \text{for } 1 \le t \le 2$$

$$= \frac{(t-1)^2}{2} - (t-1)$$

$$= \frac{t^2}{2} - t + \frac{1}{2} - t + 1$$

$$= \frac{t^2}{2} - 2t + \frac{3}{2},\ \text{for } 1 \le t \le 2$$

Integrating both sides from 1 to t, where $t \in [1, 2]$, we get:

$$y(t) = -\frac{7}{6} + \frac{t^3}{6} - t^2 + \frac{3}{2}t,\ \text{for } 1 \le t \le 2$$

Similarly, let:

$$y_2\,(t) = \frac{-7}{6} + \frac{t^3}{6} - t^2 + \frac{3}{2}t$$

and suppose $\varphi_2(t)$ is the desired second step solution, i.e.,

$$\varphi_2(t) = y_2(t) = \frac{-7}{6} + \frac{t^3}{6} - t^2 + \frac{3}{2}t$$

Since $\varphi_2(t)$ is defined on the whole segment [1, 2] hence by forming the new equation:

$$y'(t) = \varphi_2(t - 1), \text{ for } 2 \le t \le 3$$

with the initial condition:

$$\varphi_2(t) = \frac{-7}{6} + \frac{t^3}{6} - t^2 + \frac{3}{2}t$$

Similarly, one can find $y_3(t)$, $y_4(t)$ and so on.

Example (1.2):

Consider the neutral first order DDE:

$$y'(t) = y'(t-1) + t, \; t \ge 0$$

with initial condition

$$\varphi_0(t) = t + 1, \text{ for } -1 \le t \le 0$$

To find the solution of the first interval [0, 1] . We solve the following:

$$y'(t) = \varphi_0'(t-1) + t, \text{ for } -1 \le t \le 0$$

$$= 1 + t, \text{ for } -1 \le t \le 0$$

Integrating both sides from 0 to t where $0 \le t \le 1$, we have:

$$\int_0^t y'(s)\,ds = \int_0^t (1+s)\,ds$$

and hence:

$$y_1(t) = t + \frac{t^2}{2} + 1, \text{ for } 0 \le t \le 1$$

In order to find the solution in the second step interval suppose that:

$$\varphi_1(t) = y_1(t) = t + \frac{t^2}{2} + 1$$

is the initial condition. Since $\varphi_1(t)$ is defined on the whole segment [0, 1]. Hence by forming the new equation:

$$y'(t) = \varphi_1'(t-1) + t, \text{ for } 1 \le t \le 2 \qquad \ldots(1.5)$$

where $\varphi_1(t) = t + \frac{t^2}{2} + 1$, for $0 \le t \le 1$.

One can find the solution in the next step interval [1, 2], and solving equation (1.5) for y (t), we have:

$$y'(t) = \varphi_1'(t - 1) + t$$

$$= 2t, \text{ for } 1 \le t \le 2$$

Integrating both sides from 1 to t where $1 \le t \le 2$, we get:

$$y(t) = t^2 + \frac{3}{2}, \text{ for } 1 \le t \le 2$$

Therefore, y(t) is the desired second step solution which is denoted by:

$$y(t) = \varphi_2(t) = t^2 + \frac{3}{2}, \text{ for } 1 \le t \le 2$$

Similarly, we proceed to the next intervals.

Example (1.3):

Consider the mixed DDE:

$$y'(t) = y(t-1) + 2y'(t-1), \ t \geq 1$$

with initial condition:

$$\varphi_0(t) = 1, \text{ for } 0 \leq t \leq 1$$

To find the solution in the first step interval [1,2], we will solve the following equation:

$$y'(t) = \varphi_0(t-1) + 2\varphi_0'(t-1), \text{ for } 1 \leq t \leq 2$$

$$y'(t) = 1$$

By integrating from 1 to t, where $1 \leq t \leq 2$, we have:

$$y(t) = t, \text{ for } 1 \leq t \leq 2$$

and suppose that $\varphi_1(t)$ is the desired first step solution

$$y_1(t) = \varphi_1(t) = t, \text{ for } 1 \leq t \leq 2$$

Since $\varphi_1(t)$ is defined on the whole segment [1,2], hence by forming the new equation:

$$y'(t) = \varphi_1(t-1) + 2\varphi_1'(t-1), \text{ for } 2 \leq t \leq 3$$

with initial condition:

$$y_1(t) = \varphi_1(t) = t, \text{ for } 1 \leq t \leq 2$$

and so on, we proceed to the next intervals.

The next example considers the solution of DDE with variable delay which can be solved by successive integration method.

Example (1.4):

Consider the retarded first order DDE:

$$y'(t) = -y(t - e^t), \text{ for } 0 \le t \le 1$$

with initial condition:

$$y(t) = \varphi_0(t) = 1, \text{ for } -1 \le t \le 0$$

To find the solution in the first step interval [0,1] we have to solve the following equation:

$$y'(t) = -\varphi_0(t - e^t)$$

$$= -1, \text{ for } 0 \le t \le 1$$

Integrating both sides from 0 to t where $0 \le t \le 1$, we have:

$$\int_0^t y'(s)ds = \int_0^t -ds$$

Hence:

$$y(t) = 1 - t, \text{ for } 0 \le t \le 1$$

In order to find the solution in the second step interval suppose that:

$$\varphi_1(t) = y_1(t) = 1 - t$$

Therefore:

$$y_1(t) = 1 - t, \text{ for } 0 \le t \le 1$$

Since $\varphi_1(t)$ is defined on the whole segment [0,1].

Hence by forming the new equation:

$$y'(t) = -\varphi_1(t - e^t)$$
$$= -1 + (t - e^t)$$

Integrating both sides from 1 to t, where $t \in [1,2]$, yields:

$$y(t) = 3.2 - t + \frac{t^2}{2} - e^t, \text{ for } 1 \le t \le 2$$

Similarly, let:

$$y_2(t) = 3.2 - t + \frac{t^2}{2} - e^t, \text{ for } 1 \le t \le 2$$

and suppose $\varphi_2(t)$ is the desired second step solution, i.e.,

$$\varphi_2(t) = y_2(t)$$
$$= 3.2 - t + \frac{t^2}{2} - e^t, \text{ for } 1 \le t \le 2$$

Since $\varphi_2(t)$ is defined on the whole segment [1, 2], hence by forming the new equation:

$$y'(t) = -\varphi_2(t - e^t), \text{ for } 2 \le t \le 3$$

with initial condition

$$\varphi_2(t) = 3.2 - t + \frac{t^2}{2} - e^t, \text{ for } 1 \le t \le 2$$

similarly, one can find $y_3(t)$, $y_4(t)$ and so on.

2- Laplace Transformation Method, [Ross, 1984]:

Laplace transformation method is also, one of the most widely use methods for solving DDE's. It is important here to review the Laplace transformation of a given function.

Suppose that f is a real-valued function of the real variable defined for $x > 0$. Let s be a parameter that we shall assume to be real, and consider the function F defined by

$$\mathcal{L}\{f\} = F(s) = \int_0^\infty e^{-sx} f(x) dx \qquad \ldots(1.6)$$

For all values of s for which this integral exists. The function $\mathcal{L}\{f\}$ defined by the integral (1.6) is called the Laplace transformation of the function f and we shall denote the Laplace transform $\mathcal{L}\{f\}$ of f by F(s).

Also, as it is known, Laplace transformation method may be used to solve linear ODE's and we can use it also to solve DDE by two approaches. The first approach is by mixing between method of steps and Laplace transform method and the other approach is by applying directly the Laplace transform method to the original DDE.

- ***The First Approach, [Brauer, 1973]:***

This approach depends mainly on applying first the method of steps to transform the DDE into ODE and then applying Laplace transformation method to solve the resulting equation. This approach can be explained in the following examples:

Example (1.5):

Consider the following neutral DDE:

$$y'(t) = y'(t-1) + t \text{, for } 0 \le t \le 1$$

with initial condition:

$$y(t) = \varphi_0(t) = t+1 \text{, for } -1 \le t \le 0$$

To find the solution in the first step interval [0, 1], we apply the method of steps, to get:

$$y'(t) = \varphi_0'(t-1) + t$$

$$= 1 + t \text{, for } 0 \le t \le 1$$

which is an ODE of the first order.

Now, taking the Laplace transformation approach:

$$\mathcal{L}\{\, y'(t) \,\} = \mathcal{L}\,\{1\} + \mathcal{L}\{t\}$$

$$sY(s) - y\,(0) = \frac{1}{s} + \frac{1}{s^2}$$

and so the Laplace transform of the solution y(t) into Y(s) is given by:

$$Y(s) = \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s}$$

Taking inverse Laplace transform, we have:

$$y(t) = \mathcal{L}^{-1}\left[\frac{1!}{s^2}\right] + \frac{1}{2!}\,\mathcal{L}^{-1}\left[\frac{2!}{s^3}\right] + \mathcal{L}^{-1}\left[\frac{1}{s}\right]$$

$$y(t) = t + \frac{t^2}{2} + 1 \text{, for } 0 \le t \le 1$$

Hence, the solution in the first step interval is given by:

$$y(t) = \varphi_1(t) = t + \frac{t^2}{2} + 1, \text{ for } 0 \leq t \leq 1$$

In order to find the solution in the second step interval [1,2], we proceed similarly as in the first step with initial condition:

$$\varphi_1(t) = t + \frac{t^2}{2} + 1, \text{ for } 0 \leq t \leq 1$$

and hence:

$$y'(t) = \varphi_1'(t-1) + t, \text{ for } 0 \leq t \leq 1$$

with the equivalent ODE $y'(t) = 2t$, for $1 \leq t \leq 2$ with initial condition, $y(1) = \frac{5}{2}$

By making changing independent variable $w = t - 1$ then $w \in [0,1]$, so that

$$y'(w+1) = 2(w+1), \; y(1-1) = \frac{5}{2}$$

and by considering:

$$z(w) = y(w + 1)$$

Implies that:

$$z'(w) - 2(w+1) = 0, \text{ with } z(0) = \frac{5}{2}, \; w \in [0,1]$$

Taking the Laplace transform of both sides, we have:

$$sZ(s) - z(0) = \frac{2}{s^2} + \frac{2}{s}$$

where Z(s) is the Laplace transform of z(w) hence:

$$Z(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{5}{2s}$$

Taking inverse Laplace, we have:

$$z(w) = w^2 + 2w + \frac{5}{2}$$

Hence the solution in the second step interval [1, 2] is given by:

$$z(w) = y(t) = (t-1)^2 + 2(t-1) + \frac{5}{2}$$

Similarly, we proceed to the next intervals.

Similarly, as in the method of successive integration we can use Laplace transformation method to solve DDE with variable delay:

Example (1.6):

Consider the following DDE:

$$y'(t) = y'(t - e^t) + t, \text{ for } 0 \le t \le 1$$

with initial condition

$$y\,(t) = \varphi_0(t) = t+1, \text{ for } -1 \le t \le 0$$

To find the solution in the first step interval [0, 1], we apply the method of steps, to get:

$$y'(t) = \varphi_0'(t - e^t) + t$$

$$= 1 + t - e^t, \text{ for } 0 \le t \le 1$$

and this is an ODE of the first order.

Now, taking the Laplace transform produces:

$$\mathcal{L}\{\, y'(t)\,\} = \mathcal{L}\{1\} + \mathcal{L}\{t\} - \mathcal{L}\{e^t\}$$

$$sY(s) - y\,(0) = \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s-1}$$

and so the Laplace transform of the solution y(t) into Y(s) is given by:

$$Y(s) = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s(s-1)}$$

Taking inverse Laplace transform, we have:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1!}{s^2}\right\} + \frac{1}{2!}\,\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\}$$

$$y(t) = 2 + t + \frac{t^2}{2} - e^t, \text{ for } 0 \le t \le 1$$

- ***Second Approach, [Brauer, 1973]:***

This approach is to solve DDE's by using Laplace transform method directly without using the method of steps. Laplace transformation method is extremely useful in obtaining the solution of the linear DDE's with constant coefficients. Let us illustrate this method by considering the following example:

<u>Example (1.7):</u>

Consider the following DDE:

$$y'(t) = y(t-1)$$

with initial condition:

$$y(t) = \varphi_0(t) = t, \text{ for } -1 \le t \le 0$$

such that $y\,(0) = 0$, $y'(0) = 1$.

Applying the Laplace transform method to both sides of the equation, we get:

$$sY(s) = \int_0^\infty y(t-1)e^{-st}dt$$

Using the transform z = t – 1, yields:

$$\int_0^\infty y(t-1)e^{-st}dt = \int_{-1}^\infty y(z)e^{-s(z+1)}dz$$

$$= e^{-s}\int_{-1}^{0} y(z)e^{-sz}dz + e^{-s}\int_{0}^{\infty} y(z)e^{-sz}dz$$

$$= e^{-s}\int_{-1}^0 (z)e^{-sz}dz + e^{-s}\int_0^\infty y(z)e^{-sz}dz$$

Since y(z) = z, for $-1 \le z \le 0$.

Finally:

$$Y(s) = \left[\frac{-1}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]\left[\frac{1}{s-e^{-s}}\right] \qquad \ldots(1.7)$$

From equation (1.7), it follows that:

$$Y(s) = \left[\frac{-1}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]\left[\frac{1}{s-e^{-s}}\right]$$

and upon taking the inverse Laplace transform one can find the solution y(t), where it is so difficult to obtain, which in force us to prefer using the numerical methods.

Now, after we stated the definition of the delay differential equations and its analytical methods of solution we shall start the next section, with another important concept which is so called fractional calculus.

1.3 Fractional Calculus

In this section we shall give some basic concepts about fractional calculus, which are needed in this thesis and in order to make this thesis self-contained as soon as possible.

1.3.1 The Gamma and Beta Functions, [Oldham, 1974]:

The complete gamma function *Γ(x)* plays an important role in the theory of fractional calculus. A comprehensive definition of *Γ(x)* is that provided by Euler limit:

$$\Gamma(x) = \lim_{N\to\infty}\left(\frac{N!N^x}{x(x+1)(x+2)...(x+N)}\right), x > 0 \qquad \text{...(1.8)}$$

but the integral transform definition is given by:

$$\Gamma(x) = \int_0^\infty y^{x-1}e^{-y}dy\,, x > 0 \qquad \text{...(1.9)}$$

is often more useful, although it is restricted to positive value of *x*. An integration by parts applied to eq. (1.9) leads to the recurrence relationship:

$$\Gamma(x + 1) = x\Gamma(x) \qquad \text{...(1.10)}$$

This is the most important property of gamma function. The same result is a simple consequence of eq. (1.8), since $\Gamma(1) = 1$, this recurrence shows that for positive integer n:

$$\begin{aligned}\Gamma(n + 1) &= n\Gamma(n)\\ &= n!\end{aligned} \qquad \text{...(1.11)}$$

The following are the most important properties of the gamma function:

1. $\Gamma\left(\frac{1}{2}-n\right)=\frac{(-4)^n n!\sqrt{\pi}}{(2n)!}$

2. $\Gamma\left(\frac{1}{2}+n\right)=\frac{(2n)!\sqrt{\pi}}{4^n n!}$

3. $\Gamma(-x)=\frac{-\pi\csc(\pi x)}{\Gamma(x+1)}$

4. $\Gamma(nx)=\sqrt{\frac{2\pi}{n}}\left[\frac{nx}{\sqrt{2\pi}}\right]^n \prod_{k=0}^{n-1}\Gamma\left(x+\frac{k}{n}\right), n\in N$.

A function that is closely related to the gamma function is the complete beta function β(p,q). For positive value of the two parameters p and q; the function is defined by the beta integral:

$$\beta(p,q)=\int_0^1 y^{p-1}(1-y)^{q-1}\,dy\,,\ p,\,q>0 \qquad \ldots(1.12)$$

which is also known as the Euler's integral of the second kind. If either p or q is nonpositive, the integral diverges otherwise β(p,q) is defined by the relationship:

$$\beta(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\,,\ p,\,q>0 \qquad \ldots(1.13)$$

Both beta and gamma functions have "incomplete" analogues. The incomplete beta function of argument *x* is defined by the integral:

$$\beta_x(p,q)=\int_0^x y^{p-1}(1-y)^{q-1}\,dy \qquad \ldots(1.14)$$

and the incomplete gamma function of argument *x* is defined by:

$$\gamma^*(c,x) = \frac{c^{-x}}{\Gamma(x)}\int_0^c y^{x-1}e^{-y}\,dy$$

$$= e^{-x}\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+c+1)} \qquad \text{...(1.15)}$$

$\gamma^*(c,x)$ is a finite single-valued analytic function of x and c.

1.3.2 Fractional Integration:

There are many literatures introduce different definitions of fractional integrations, such as:

1. Riemann-Liouville integral, [Oldham, 1974]:

The generalization to non-integer α of Riemann-Liouville integral can be written for suitable function f(x), $x \in R^+$; as:

$$I^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}f(s)\,ds\,,\ \alpha > 0 \qquad \text{...(1.16)}$$

and $I^0f(x) = f(x)$ is the identity operator.

The properties of the operator I^{α} can be founded in [Podlbuny, 1999] for $\beta \geq 0$, $\alpha > 0$, we have:

1. $I^{\alpha}I^{\beta}f(x) = I^{\alpha+\beta}f(x)$.
2. $I^{\alpha}I^{\beta}f(x) = I^{\beta}I^{\alpha}f(x)$. ...(1.17)

2. Weyl fractional integral, [Oldham, 1974]:

The left hand fractional order integral of order $\alpha > 0$ of a given function f is defined as:

$$_{-\infty}I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{1-\alpha}} dy \qquad \ldots(1.18)$$

and the right fractional order integral of order $\alpha > 0$ of a given function f is given by:

$$_{\infty}I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(y)}{(y-x)^{1-\alpha}} dy$$

3. Abel-Riemann fractional integral, [Mittal, 2008]:

The Abel-Riemann (A-R) fractional integral of any order $\alpha > 0$, for a function f(x) with $x \in R^+$ is defined as:

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-\tau)^{\alpha-1} f(\tau) d\tau, \; x > 0, \; \alpha > 0 \qquad \ldots(1.19)$$

$I^0 = I$ (identity operator)

The A-R integral posses the semigroup property:

$I^{\alpha} I^{\beta} = I^{\alpha+\beta}$, for all $\alpha, \beta \geq 0$ …(1.20)

1.3.3 Fractional Derivatives:

Many literatures discussed and presented fractional derivatives of certain function, therefore in this section, some definitions of fractional derivatives are presented:

1. Riemann-Liouville fractional derivatives, [Oldham, 1974], [Nishimoto, 1983]:

Among the most important formulae used in fractional calculus is the Riemann-Liouville formula. For a given function f(x), $\forall\, x \in [a,b]$; the left and right hand Riemann-Liouville fractional derivatives of order $\alpha > 0$ and m is a natural number, are given by:

$$ {}_xD^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_a^x \frac{f(t)}{(x-t)^{\alpha-m+1}}dt \quad \ldots(1.21)$$

$$ {}_xD^{\alpha}_{b-}f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_x^b \frac{f(t)}{(x-t)^{\alpha-m+1}}dt \quad \ldots(1.22)$$

where $m - 1 < \alpha \leq m$, $m \in R$

2. The A-R fractional derivative, [Mittal, 2008]:

The A-R fractional derivative of order $\alpha > 0$ is defined as the inverse of the corresponding A-R fractional integral, i.e.,

$$D^{\alpha}I^{\alpha} = I \quad \ldots(1.23)$$

for positive integer m, such that $m - 1 < \alpha \leq m$,

$$(D^m I^{m-\alpha})I^{\alpha} = D^m(I^{m-\alpha}I^{\alpha}) = D^m I^m = I$$

i.e.,

$$D^{\alpha}f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\dfrac{d^m}{dx^m}\displaystyle\int_a^x \frac{f(\tau)}{(x-\tau)^{\alpha+1-m}}d\tau, & m-1<\alpha<m \\ \dfrac{d^m}{dx^m}f(x), & \alpha = m \end{cases} \quad \ldots(1.24)$$

3. Caputo fractional derivative, [Caputo, 1967]:

In the late sixties of the last century, an alternative definition of fractional derivatives was introduced by Caputo. Caputo and Minardi used this definition in their work on the theory of viscoelasticity. According to Caputo's definition:

${}^cD^{\alpha}_x = I^{m-\alpha}D^m$, for $m - 1 < \alpha \leq m$

which means that:

$$ {}^{c}D_{x}^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_{0}^{x} \dfrac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m \\ \dfrac{d^{m}}{dx^{m}} f(x), & \alpha = m \end{cases} $$

The basic properties of the Caputo fractional derivative are:

1. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

2. $I^{\alpha}\, {}^{c}D_{x}^{\alpha} f(x) = f(x) - \sum\limits_{k=0}^{m-1} f^{(k)}(0^{+}) \dfrac{x^{k}}{k!}$.

3. Caputo's fractional differentiation is linear operator, similar to integer order differentiation:

   $$ {}^{c}D_{x}^{\alpha} [\lambda f(x) + \mu g(x)] = \lambda\, {}^{c}D_{x}^{\alpha} f(x) + \mu\, {}^{c}D_{x}^{\alpha} g(x) $$

4. Grünwald fractional derivatives, [Oldham, 1974]:

The Grünwald derivatives of any integer order to any function, can take the form:

$$ D^{\alpha} f(x) = \lim_{N\to\infty} \left\{ \frac{\left(\dfrac{x}{N}\right)^{-\alpha}}{\Gamma(-\alpha)} \sum_{j=0}^{N-1} \frac{\Gamma(j-\alpha)}{\Gamma(j+1)} f\left(x - j\frac{x}{N}\right) \right\} \qquad \ldots(1.25) $$

CHAPTER TWO

ADOMIAN DECOMPOSITION METHOD FOR SOLVING DELAY DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

2.1 Introduction

In this chapter the adomian decomposition method will be presented in order to find the approximate solution of fractional delay differential equations .

This chapter consists of six sections, in section 2.2, the literature review of ADM was presented. While in section 2.3, the Adomian Decomposition Method will be given. In section 2.4, we discussed the Adomian Decomposition Method for solving delay differential equations. In section 2.5 , we will focused on the approximate solution of the delay differential equation of fractional order using ADM . Finally in section 2.6 numerical examples are presented in order to illustrate the ability and efficiency of the proposed method .

2.2Literature Review of ADM

Many of the phenomena that arise in real world are described by nonlinear differential and integral equations. However, most of the Methods developed in mathematics are usually used in solving linear differential and integral equations The recently developed decomposition method proposed by American mathematician, Georg Adomian has been receiving much attention in recent years in applied mathematics. The ADM emerged as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, integro-differential equations, and partial differential equations. Thus yields rapidly convergent series solutions for both linear and nonlinear deterministic and stochastic equations; it has many advantages over the classical techniques, namely, it avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculations and avoidance of physically unrealistic assumptions, the theoretical treatment of convergence of the decomposition method has been considered in [Seng, 1996] and the obtained results about the speed of convergence of this method. The solution of the fractional differential equation has been obtained through the Adomian decomposition by [Ray, 2004].

However, El-Sayed and Kaya proposed ADM to approximate the numerical and analytical solution of system of two-dimensional Burger's equations with initial conditions in [El-Sayed, 2004], and the advantages of this work is that the decomposition method reduces the computational work and improves with regards to its accuracy and rapid convergence. The convergence of decomposition method is proved as [Inc, 2005], in [Celik, 2006] applied ADM to obtain the approximate solution for the DAEs system and the result obtained by this method indicate a high degree of accuracy through the comparison with the analytic solutions. In [Hosseini, 2006 a], [Hosseini, 2006 b] standard and modified ADMs are applied to solve non-linear DAEs. While, the error analysis of Adomian series solution to a class of nonlinear differential equation, whereas numerical experiments show that Adomian solution using this formula converges faster is discussed in [El-Kala, 2007]. Also, a new discrete ADM to approximate the theoretical solution of discrete nonlinear Schrodinger equations is presented in [Bratsos, 2007] where this examined for plane waves and single solution waves in case of continuous, semi discrete and fully discrete Schrodinger equations. Momani and Jafari, [Momani, 2008] presented numerical study of system of fractional differential equation by ADM. Also, A review of the Adomian decomposition method and its applications to fractional differential equations [Jun, 2012]. Yinwei and Cha'o [Yinwei, 2014] presented Modified Adomian Decomposition method for Double singular boundary value problems. Also [Behman, 2014] presented Adomian Decomposition method for solving Fractional Bratu- type Equations.

2.3 The Adomian Decomposition Method (ADM), [Moragdo,2013]

To introduce the basic idea of the ADM, we consider the operator equation Fy = G, where F represents a general nonlinear ordinary differential operator and G is a given function. Then F can be decomposed as:

$$Ly + Ry + Ny = G \qquad \ldots(2.1)$$

where, N is a nonlinear operator, L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of order less than L and G is the nonhomogeneous term.

The method is based by applying the operator L^{-1}formally to the expression:

$$Ly = G - Ry - Ny \qquad \ldots(2.2)$$

so by using the given conditions, we obtain:

$$y = h + L^{-1} G - L^{-1} Ry - L^{-1} Ny \qquad \ldots(2.3)$$

where, h is the solution of the homogeneous equation Ly = 0, with the initial-boundary conditions. The problem now is the decomposition of the nonlinear term Ny. To do this, Adomian developed a very elegant technique as follows:

Define the decomposition parameter λ as:

$$y = \sum_{n=0}^{\infty} \lambda^n y_n$$

then N(y) will be a function of λ, y_0, y_1 …. Next expanding N(y) in Maclurian series with respect to λ we obtain $N(y) = \sum_{n=0}^{\infty} \lambda^n A_n$, where:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N\left(\sum_{k=0}^{n} \lambda^k y_k \right) \right]_{\lambda=0} \qquad \ldots(2.4)$$

where, the components of A_n are the so called Adomian polynomials they are generated for each nonlinearity, for example, for N(y) = f(y) the Adomian polynomials, are given as:

$$A_0 = f(y_0)$$

$$A_1 = y_1 f'(y_0)$$

$$A_2 = y_2 f'(y_0) + \frac{y_1^2}{2} f''(y_0)$$

$$A_3 = y_3 f'(y_0) + y_1 y_2 f''(y_0) + \frac{y_1^3}{3!} f'''(y_0)$$

$$\vdots$$

Now, we parameterize eq.(2.3) in the form:

$$y = h + L^{-1} G - \lambda L^{-1} Ry - \lambda L^{-1} Ny \qquad \ldots(2.5)$$

where, λ is just an identifier for collecting terms in a suitable way such that A_n depends on $y_0, y_1, \ldots, y_n$ and we will later set $\lambda = 1$

$$\sum_{n=0}^{\infty}\lambda^n y_n = h + L^{-1}G - \lambda L^{-1}R\sum_{n=0}^{\infty}\lambda^n y_n - \lambda L^{-1}\sum_{n=0}^{\infty}\lambda^n A_n \qquad \ldots(2.6)$$

Equating the coefficients of equal powers of λ, we obtain:

$$\left.\begin{aligned} y_0 &= h + L^{-1}G \\ y_1 &= -L^{-1}(Ry_0) - L^{-1}(A_0) \\ y_2 &= -L^{-1}(Ry_1) - L^{-1}(A_1) \\ &\ \ \vdots \end{aligned}\right\} \qquad \ldots(2.7)$$

and in general:

$$y_n = -L^{-1}(Ry_{n-1}) - L^{-1}(A_{n-1}), n \ \geq 1$$

Finally, an N-terms that approximate the solution is given by:

$$\Phi_N(t) = \sum_{n=0}^{N-1} y_n(t) \ , \quad N \geq 1$$

and the exact solution is $y(t) = \lim_{N\to\infty}\Phi_N(t)$

2.4 ADM for Solving Delay Differential Equations, [Evans,2004]

In this section the approximate solution of the following DDEs will be given:

$$Ly(t) = \ N\Big(t, y(t), y\big(g(t)\big)\Big), 0 \ \leq \ t \ \leq \ 1 \qquad \ldots(2.8)$$

$$y^{(i)}(0) = \ y_0^i \quad , i = \ 0, 1, \ldots, N \ - \ 1$$

$$y(t) = \ \Phi(t), t \ \leq \ 0$$

where the differential operator L is given by:

$$L(.) = \frac{d^N(.)}{dx^N} \qquad \ldots(2.9)$$

the inverse operator $L^{-1}$is therefore considered a N-fold integral operator defined by:

$$L^{-1}(\#) = \int_0^t (\#)_{N-\text{times}} dt \qquad \ldots(2.10)$$

operating with L^{-1} on Eq. (2.8), it then follows:

$$y(t) = \sum_{j=0}^{N-1} \frac{\alpha_j}{j!} t^j + L^{-1}\left(N\left(t, y(t), y\big(g(t)\big)\right)\right) \qquad \ldots(2.11)$$

where the α_j are constants that describe the boundary conditions. The Adomian decomposition method assumes that the unknown function y(t) can be expressed by an infinite series of the form:

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \qquad \ldots(2.12)$$

so that the components $y_n(t)$will be determined recursively. Moreover, the method defines the nonlinear term $N\left(t, y(t), y\big(g(t)\big)\right)$ by the Adomian polynomials

$$N\left(t, y(t), y\big(g(t)\big)\right) = \sum_{n=0}^{\infty} A_n \qquad \ldots(2.13)$$

where A_n are Adomian polynomials that can be generated for all forms of nonlinearity [Wazwaz, 2000] as:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\big(t, \sum_{j=0}^{\infty} \lambda^j y_j(t), \sum_{j=0}^{\infty} \lambda^j y_j(g(t))\big)\right]_{\lambda=0}$$

Substituting Eqs. (2.12) and (2.13) into Eq. (2.11) gives

$$\sum_{n=0}^{\infty} y_n(t) = \sum_{j=0}^{N-1} \frac{\alpha_j}{j!} t^j + L^{-1}\Big(\sum_{n=0}^{\infty} A_n\Big) \qquad \ldots(2.14)$$

to determine the components $y_n(t)$, $n \geq 0$. First, we identify the zero component $y_0(t)$ by all terms that arise from the boundary conditions at $t = 0$ and from integrating the source term if it exists. Second, the remaining components of $y(t)$

can be determined in a way such that each component is determined by using the preceding components. In other words, the method introduces the recursive relation:

$$y_0(t) = \sum_{j=0}^{N-1} \frac{\alpha_j}{j!} t^j \,, \quad y_{n+1}(t) = L^{-1}(A_n) \qquad \ldots(2.15)$$

for the determination of the components $y_n(t)$, $n \geq 0$ of y(t) the series solution of y(t) follows immediately with the constants α_j, $j = 0,1,\ldots,N-1$ are as yet undetermined.

2.5 ADM for Solving Delay Differential Equations of Fractional Order

In this section we shall approximate the solution of the following FDDE$_s$:

$${}^{c}D_t^{\alpha} y(t) = N(t, y(t), y(\Phi(t)), n-1 < \alpha \leq n \qquad \ldots(2.16)$$

$$\begin{aligned} &y(t)=\psi(t). \qquad -\tau\leq t\leq 0 \\ &y^{(i)}(0)=y_0^i \quad i=0,1,..,n-1 \end{aligned} \qquad \ldots(2.17)$$

where ${}^{c}D_t^{\alpha}$ is the Caputo fractional derivative of order α, N is a nonlinear operator, t is the independent variable, $\Phi(t)$ is the delay function , y(t) is the unknown function , $\psi(t)$ is a given continuous function and $y^{(i)}(0)$ are given constants .

And in order to solve the problem (2.16 - 2.17) by using the ADM operating I_t^{α} to the both sides of equation (2.16), we have:

$$y(t) = I_t^{\alpha} N\left(t, y(t), y\left(\Phi(t)\right)\right) + \sum_{k=0}^{n-1} y^k(0^+) \frac{t^k}{k!} \qquad (2.18)$$

Adomian's method defined the solution y(t) by the series:

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \qquad \ldots(2.19)$$

So that the Components y_n will be determined recursively. Moreover, the method defines the nonlinear term $N(t, y(t), y(\phi(t))$ by the Adomian polynomials:

$$N\Big(t, y(t), y\big(\phi(t)\big)\Big) = \sum_{n=0}^{\infty} A_n \qquad \ldots(2.20)$$

where A_n are the Adomian polynomials that can be generated for all forms of nonlinearity as:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\Big[N\big(t, \sum_{j=0}^{\infty} \lambda^j y_j(t), \sum_{j=0}^{\infty} \lambda^j y_j(\phi(t))\big)\Big]_{\lambda=0} \qquad \ldots(2.21)$$

Substituting equations (2.19) and (2.20) into equation (2.18) gives:

$$\sum_{n=0}^{\infty} y_n(t) = \sum_{k=0}^{n-1} y^{(k)}(0^+)\frac{t^k}{k!} + I_t^{\alpha}\Big(\sum_{n=0}^{\infty} A_n\Big) \qquad \ldots(2.22)$$

To determine the components $y_n(t)$, $n \geq 0$. First we identify the zero component $y_0(t)$ by the terms $\sum_{k=0}^{n-1} y^{(k)}(0^+)\frac{t^k}{k!}$, $\psi(t)$ and $I_t^{\alpha} f(t)$ where f(t) represent the nonhomogeneous part in $N\Big(t, y(t), y\big(\phi(t)\big)\Big)$. Secondly , the remaining components of y(t) can be determined in a way such that each component is determined by using the preceding components. In other words, the method introduces the recursive relation:

$$y_0(t) = \psi(t) + \sum_{k=0}^{n-1} y^{(k)}(0^+)\frac{t^k}{k!} + I_t^{\alpha} f(t) \qquad \ldots(2.23)$$

$$y_{n+1}(t) = I_t^{\alpha} A_n \qquad n \geq 0 \qquad \ldots(2.24)$$

And in order to represent this approach let us take some illustrative examples as given in the next section.

2.6 Numerical Examples

In this section we shall use the ADM to solve the non-linear fractional differential equations with variable delay and the results obtained using this scheme will be compare with the analytical solution.

Example (2.1):

Consider the FDDE$_S$

$$ {}^{c}D_t^{\alpha} y(t) = \frac{1}{2} e^{\frac{t}{2}} y\left(\frac{t}{2}\right) + \frac{1}{2} y(t) \,,\ 0 \le t \le 1 \,,\ 0 < \alpha \le 1 \qquad \ldots(2.25) $$

$$ y(0) = 1 $$

The exact solution of equation (2.25) when $\alpha = 1$ was given in [Evans,2004] as $y(t) = e^t$. According to equations (2.23) and (2.24) thus we have:

$$ y_0(t) = 1 $$

$$ y_{n+1}(t) = I_t^{\alpha}\left(\frac{1}{2} e^{\frac{t}{2}} y_n\left(\frac{t}{2}\right) + \frac{1}{2} y_n(t)\right) \qquad \ldots (2.26) $$

And upon taking the Maclurian series expansion of $e^{\frac{t}{2}}$ up to three terms. One can get $y_1, y_2, y_3, \ldots$, respectively.

Following table (2.1) represent the approximate solution of example (2.1) using ADM up to three terms for different values of α with a comparison with the exact solution when $\alpha = 1$.

Table (2.1)

The approximate solution of example (2.1) using different values of α with a comparison with the exact solution when $\alpha = 1$.

t	*ADM* $\alpha = 0.5$	*ADM* $\alpha = 0.75$	*ADM* $\alpha = 1$	*Exact* $\alpha = 1$
0	1	1	1	1
0.1	1.45	1.216	1.105	1.105
0.2	1.701	1.391	1.221	1.221
0.3	1.926	1.565	1.347	1.35
0.4	2.142	1.745	1.485	1.492
0.5	2.356	1.932	1.636	1.649
0.6	2.573	2.128	1.799	1.822
0.7	2.794	2.334	1.976	2.014
0.8	3.022	2.553	2.167	2.226
0.9	3.26	2.784	2.374	2.46
1	3.508	3.029	2.598	2.718

Example (2.2):

Consider the $FDDE_s$:

$$ {}^{c}D_t^{\alpha} y(t) = \frac{3}{4} y(t) + y\left(\frac{t}{2}\right) - t^2 + 2\,,\ \ 0 \le t \le 1,\ \ 1 < \alpha \le 2 \qquad \ldots(2.27)$$

$$y(0) = 0, \quad y'(0) = 0$$

The exact solution of equation (2.27) when $\alpha = 2$ was given in [Evans,2004] as $y(t) = t^2$ According to equations (2.23) and (2.24) thus we have:

$$y_0(t) = \frac{2}{\Gamma(\alpha+1)} t^{\alpha} - \frac{2}{\Gamma(\alpha+3)} t^{\alpha+2}$$

$$y_{n+1}(t) = I_t^{\alpha}\left(\frac{3}{4}y_n(t) + y_n\left(\frac{t}{2}\right)\right) \qquad \ldots(2.28)$$

Following table (2.2) represent the approximate solution of example (2.2) using ADM up to three terms for different values of α with a comparison with the exact solution when $\alpha = 2$.

Table (2.2)

The approximate solution of example (2.2) using different values of α with a comparison with the exact solution when $\alpha = 2$.

t	*ADM* $\alpha = 1.5$	*ADM* $\alpha = 1.75$	*ADM* $\alpha = 2$	*Exact* $\alpha = 2$
0	0	0	0	0
0.1	0.048	0.022	0.01	0.01
0.2	0.137	0.075	0.04	0.04
0.3	0.255	0.153	0.09	0.09
0.4	0.398	0.254	0.16	0.16
0.5	0.564	0.377	0.25	0.25
0.6	0.753	0.521	0.36	0.36
0.7	0.963	0.687	0.49	0.49
0.8	1.195	0.873	0.64	0.64
0.9	1.1448	1.08	0.809	0.81
1	1.723	1.307	0.999	1

Example (2.3):

Consider the FDDE_S

$${}^{c}D_t^{\alpha}y(t) = 1 - 2y^2\left(\frac{t}{2}\right), \quad 0 \le t \le 1, \ 0 < \alpha \le 1 \qquad \ldots(2.29)$$

$y(0) = 0$

The exact solution of equation (2.29), when $\alpha = 1$ was given in [Evans,2004] as $y(t) = \sin t$.

According to equations (2.23) and (2.24), thus we have:

$$y_0(t) = \frac{1}{\Gamma(\alpha+1)} t^{\alpha}$$

$$y_{n+1}(t) = -2I_t^{\alpha} A_n \qquad \ldots(2.30)$$

where A_n, $n \geq 0$ are the Adomian polynomials that represent the nonlinear term. We list the set of Adomian polynomial as:

$$A_0(t) = y_0^2\left(\frac{t}{2}\right),\ A_1(t) = 2y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right)$$

$$A_2(t) = y_1^2\left(\frac{t}{2}\right) + 2y_0\left(\frac{t}{2}\right)y_2\left(\frac{t}{2}\right)$$

$$A_3(t) = 2y_1\left(\frac{t}{2}\right)y_2\left(\frac{t}{2}\right) + 2y_0\left(\frac{t}{2}\right)y_3\left(\frac{t}{2}\right)$$

$$\vdots$$

Following table (2.3) represent the approximate solution of example (2.3) using ADM up to three terms for different values of α with a comparison with the exact solution when $\alpha = 1$.

Table (2.3)

The approximate solution of example (2.3) using different values of α with a comparison with the exact solution when $\alpha=1$.

t	*ADM* $\alpha=0.5$	*ADM* $\alpha=0.75$	*ADM* $\alpha=1$	*Exact* $\alpha=1$
0	0	0	0	0
0.1	0.329	0.191	0.1	0.1
0.2	0.431	0.314	0.199	0.199
0.3	0.493	0.413	0.296	0.296
0.4	0.537	0.494	0.389	0.389
0.5	0.574	0.562	0.479	0.479
0.6	0.61	0.616	0.565	0.565
0.7	0.65	0.66	0.644	0.644
0.8	0.696	0.693	0.717	0.717
0.9	0.752	0.719	0.783	0.783
1	0.821	0.737	0.842	0.841

CHAPTER THREE
HOMOTOPY ANALYSIS METHOD FOR SOLVING DELAY DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

3.1 Introduction

The homotopy analysis method will be given in this chapter in order to approximate the solution of the fractional delay differential equations .

This chapter consists of four sections, in section 3.2, the Homotopy Analysis Method (HAM) was given, while in section 3.3HAM for solving delay differential equation of fractional order is presented. Finally, in section 3.4, some numerical examples are given for illustration purpose.

3.2 Homotopy Analysis Method (HAM), [Liao, 2003a]

In this section we shall give a brief introduction to the HAM which will be used later on in this chapter in order to solve the delay differential equations of fractional order.

Liao [Liao, 2003a] developed an analytic method such that it satisfy the following conditions:

1. It is valid for strongly nonlinear problems even if a given nonlinear problem does not contain any small/large parameters.

2. Provide us with a convenient way to adjust the convergence region and rate of approximation series.
3. Provide us with freedom to use different base functions to approximate a nonlinear problem.

A kind of analytic technique, namely the homotopy analysis method [Liao,1992 a], [Liao,1992 b], [Liao,1997], [Liao,1999a], [Liao,2003a], was proposed by means of homotopy, a fundamental concept of topology [Sen,1983]. The idea of the homotopy is very simple and straightforward. For example, consider a differential equation

$$A[y(t)] = 0 \qquad \ldots(3.1)$$

where A is a nonlinear operator, t denotes the time, and y(t) is an unknown function. Let $y_0(t)$ denote an initial approximation of y(t) and L denote an auxiliary linear operator with the property:

$$Lf = 0\,,\ \text{when } f = 0 \qquad \ldots(3.2)$$

We then construct the so-called homotopy:

$$H[\phi(t;\ q);\ q] = (1-q)\, L[\phi\ (t;q) - y_0(t)] + q\, A[\phi\ (t;q)] \qquad \ldots(3.3)$$

Where $q \in [0,1]$ is an embedding parameter and $\phi(t;q)$ is a function of t and q.

When q = 0 and q = 1, we have:

$$H[\phi(t;q);q]|_{q=0} = L[\phi\ (t;\ 0) - y_0(t)]$$

and

$$H[\phi(t;q);q]|_{q=1} = A[\phi\ (t;\ 1)]$$

respectively. Using (3.2), it is clear that

$$\phi\ (t;0) = y_0(t)$$

is the solution of the equation:

$$H[\phi\ (t;q);q]|_{q=0} = 0$$

and

$$\phi\ (t;1) = y(t)$$

is therefore obviously the solution of the equation

$$H[\phi\ (t;q);q]|_{q=1} = 0$$

as the embedding parameter q increases from 0 to 1, the solution $\phi(t;q)$ of the equation

$$H[\phi\ (t;q);q] = 0$$

depends upon the embedding parameter q and varies from the initial approximation $y_0(t)$ to the solution y(t) of Equation (3.1). In topology, such a kind of continuous variation is called deformation. Based on the idea of homotopy, some numerical techniques such as the continuation method [Grigolyuk,1991] and the homotopy continuation method [Alexander,1978] were developed. In fact, the artificial small parameter method and the δ-expansion method can be described by the homotopy if we replace the artificial parameter ε or δ by the embedding parameter q. However, although the above-mentioned traditional way to construct the homotopy (3.3) might be enough from viewpoints of numerical techniques, it is not good enough from viewpoints of analytic ones. This is mainly because we have great freedom to choose the so-called auxiliary operator L and the initial approximations but lack any rules to direct their choice. More importantly, the traditional way to construct a homotopy cannot provide a convenient way to adjust convergence region and rate of approximation series. However, instead of using the traditional homotopy (3.3), we introduce a nonzero auxiliary parameter ħ and a nonzero auxiliary function H(t) to construct such a new kind of homotopy:

$$\widetilde{H}[\phi; q, \hbar, H] = (1-q)L[\phi(t, q, \hbar, H) - y_0(t)] - q\hbar\, H(t)A[\phi(t, q, \hbar, H)] \quad \ldots(3.4)$$

which is more general than (3.3) because (3.3) is only a special case of (3.4). When $\hbar = -1$ and $H(t) = 1$, i.e.,

$$H(\phi\,;\,q) \;=\; \widetilde{H}\,(\phi;\ q, -1, 1) \quad \ldots(3.5)$$

Similarly, as q increases from 0 to 1, $\phi(t;\, q, \hbar, H)$ varies from the initial approximation $y_0(t)$ to the exact solution y(t) of the original nonlinear problem. However, the solution $\phi(t;\ q, \hbar, H)$ of the equation:

$$\widetilde{H}\,[\phi\,(t;\ q, \hbar, H)] = 0 \quad \ldots(3.6)$$

depends not only on the embedding parameter q but also on the auxiliary parameter ħ and the auxiliary function H(t). So, at q = 1, the solution still depends upon the auxiliary parameter ħ and the auxiliary function H(t). Thus, different from the traditional homotopy (3.3), the generalized homotopy (3.4) can provide us with a family of approximation series whose convergence region depends upon the auxiliary parameter ħ and the auxiliary function H(t). More importantly, this provides us with a simple way to adjust and control the convergence regions and rates of approximation series. The homotopy analysis method is rather general and valid for nonlinear ordinary and partial differential equations in many different types. It has been successfully applied to many nonlinear problems such as nonlinear oscillations [Liao,1992c], [Liao,1995], [Liao,1998], [Liao,2003c], [Liao,2004] boundary layer flows [Liao,1997], [Liao,1999a], [Liao,1999b], [Liao,2002a], heat transfer [Liao and ,2002b],[Wang,2003] viscous flows in porous medium [Ayub2003], viscous flows of Oldroyd 6-constant fluids, magnetohydrodynamic flows of Non-Newtonian fluids [Liao,2003d], nonlinear water waves [Liao,1992d], Thomas-Fermi equation [Liao,2003d], and so on.

3.3 HAM for Solving Delay Differential Equations of Fractional Order

In this section the basic ideas of the HAM are introduced in order to solve the following problem:

$$ {}^{c}D_t^{\alpha} y(t) = f(t, y(t), y(\Phi(t))) ,\ n-1 < \alpha \le n \qquad \ldots(3.7)$$

$$ y(t) = \psi(t),\ -\tau \le t \le 0 \qquad \ldots(3.8)$$

$$ y^{(i)}(0) = y_0^i \quad i = 0,1,..,n-1 \qquad \ldots(3.9)$$

where ${}^{c}D_t^{\alpha}$ is the fractional derivative in the Caputo sense and $\psi(t)$ is a continuous function, f is a nonlinear operator and $y_0^{(i)}$ are prescribed constants.

3.3.1 Zero – Order Deformation Equation:

In HAM equation (3.7) is first written in the form

$$ N\left[t, y(t), y\left(\Phi(t)\right)\right] = 0 \qquad \ldots(3.10)$$

where N is a nonlinear operator given by the form:

$$ N\left[t, y(t), y\left(\Phi(t)\right)\right] = {}^{c}D_t^{\alpha} y(t) - f(t, y(t), y(\Phi(t))) \qquad \ldots(3.11)$$

t denote the independent variable, y(t) is the unknown function and $\Phi(t)$ is the delay function. Let $y_0(t)$ denote the initial guess of the exact solution y(t), $h \neq 0$ an auxiliary parameter and let L to be an auxiliary operator defined by:

$$ L = {}^{c}D_t^{\alpha} $$

Then using $q \in [0,1]$ as an embedding parameter, in view of Liao [Liao,2003] we construct such a homotopy:

$$H[\tilde{y}(t,q), y_0, h, q] = (1-q)L[\tilde{y}(t,q) - y_0] - qhHN\big[t, \tilde{y}(t), \tilde{y}\big(\phi(t)\big), q\big] \quad \ldots(3.12)$$

for the FDDEs (3.7).

Enforcing the homotopy (3.12) to be zero, we have the so called zeroth-order deformation equation as

$$(1-q)L[\tilde{y}(t,q) - y_0] = qhHN\big[t, \tilde{y}(t), \tilde{y}\big(\phi(t)\big), q\big] \quad \ldots(3.13)$$

where $\tilde{y}(t,q)$ is the solution which depends on the initial guess $y_0(t)$, the auxiliary linear operator L, the nonzero auxiliary parameter h, the auxiliary function H and the embedding parameter $q \in [0,1]$.

Obviously, when q = 0 and q = 1, both

$$\tilde{y}(t,0) = y_0(t), \;\; \tilde{y}(t,1) = y(t) \quad \ldots(3.14)$$

respectively hold. Thus, according to above equation, as the embedding parameter q increases from 0 to 1, $\tilde{y}(t,q)$ varies continuously from the initial approximate $y_0(t)$ to the exact solution y(t) of the original equation (3.7).

The zero-order deformation equation (3.13) defines a family of homotopies between the initial approximation $y_0(t)$ and the exact solution y(t) via auxiliary parameter h.

The mapping to the exact solution is implemented through a successive approximation with the initial approximation as the first term.

To this end, the mapping function $\tilde{y}(t,q)$ are expanded in Taylor series about q = 0 as:

$$\tilde{y}(t,q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t)q^m \qquad \ldots(3.15)$$

where:

$$y_m = \frac{1}{m!}\frac{\partial^m \tilde{y}(t,q)}{\partial q^m}\bigg|_{q=0} \qquad \ldots(3.16)$$

Assume that the auxiliary parameter h, the auxiliary function H, the initial guess $y_0(t)$ and the auxiliary linear operator L are so properly chosen that the series (3.15) converges at q = 1. Then at q = 1, the series (3.15) becomes:

$$\tilde{y}(t,1) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) \qquad \ldots(3.17)$$

Therefore, using equation (3.14), we have

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) \qquad \ldots(3.18)$$

The above expression provides us with a relationship between the initial guess $y_0(t)$ and the exact solution y(t) by means of the terms $y_m(t)$ $(m = 1,2,3,\ldots)$ which are unknown up to now.

3.3.2 High – Order Deformation Equation:

Define the vector:

$$\vec{y}_m = \{y_0(t), y_1(t), \ldots, y_m(t)\}$$

According to the definition (3.16) the governing equation of $y_m(t)$ can be derived from the zeroth-order deformation equation (3.13).

Differentiating zeroth-order deformation equation (3.13) m times with respect to the embedding parameter q and then setting q = 0 and finally dividing by m! We have the so called m^{th} – order deformation equation

$$L[y_m(t) - X_m y_{m-1}(t)] = hH(t)R_{y_m}(\vec{y}_{m-1}) \qquad \ldots(3.19)$$

where:

$$R_{y_m}(\vec{y}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[t,\tilde{y}(t),\tilde{y}(\phi(t))]}{\partial q^{m-1}}\bigg|_{q=0} \qquad \ldots(3.20)$$

and

$$X_m = \begin{cases} 0 \ , & m \leq 1 \\ 1 \ , & m > 1 \end{cases} \qquad \ldots(3.21)$$

Notice that $R_{y_m}(\vec{y}_{m-1})$ given by the above expression is only dependent upon $y_0(t), y_1(t), y_2(t), \ldots, y_{m-1}(t)$, which are known when solving the m^{th} – order deformation equation (3.19). The m^{th} – order approximation of y(t) is given by:

$$y(t) \approx \sum_{n=0}^{m} y_n(t) \qquad \ldots(3.22)$$

It should be emphasized that the zero – order deformation equation (3.18) is determined by the auxiliary linear operator L the initial approximation $y_0(t)$, the auxiliary parameter h, and the auxiliary function H(t).

Theoretically speaking the solution y(t) given by the above approach is dependent of the auxiliary linear operator L, the initial approximation $y_0(t)$, the auxiliary operator h and the auxiliary function H(t). Thus the convergence region rate of solution series given by the above approach might not be uniquely determined.

3.4 Numerical Examples:

In this section we shall use the HAM to solve the non-linear delay differential equations of fractional order and the results obtained by using this scheme will be compare with the analytical solution.

Here the same examples given in chapter two have bee attacked for comparison purposes.

Example (3.1):

Consider the FDDEs

$${}^{c}D_t^{\alpha}y(t) = \frac{1}{2}e^{\frac{t}{2}}y\left(\frac{t}{2}\right) + \frac{1}{2}y(t)\,, \quad 0 \le t \le 1\,, 0 < \alpha \le 1 \qquad \ldots(3.23)$$

$y(0) = 1$

First we choose the initial approximation $y_0(t)$ to be:

$$y_0(t) = 1 \qquad \ldots(3.24)$$

and according to equation (3.11) then:

$$N\left[t, y(t), y\left(\frac{t}{2}\right)\right] = {}^{c}D_t^{\alpha}y(t) - \frac{1}{2}e^{\frac{t}{2}}y\left(\frac{t}{2}\right) - \frac{1}{2}y(t) \qquad \ldots(3.25)$$

Set $L = {}^{c}D_t^{\alpha}$, $h = -1$ and $H = 1$, hence according to equation (3.19) we have

$${}^{c}D_t^{\alpha}[y_m(t) - X_m y_{m-1}(t)] = -R_{y_m}(\vec{y}_{m-1}) \qquad \ldots(3.26)$$

where:

$$R_{y_m}(\vec{y}_{m-1}) = {}^{c}D_t^{\alpha}y_{m-1}(t) - \frac{1}{2}e^{\frac{t}{2}}y_{m-1}\left(\frac{t}{2}\right) - \frac{1}{2}y_{m-1}(t) \qquad \ldots(3.27)$$

Operating I_t^{α} to the both sides of equation (3.26) and using equation (3.24) therefore one can get the functions $y_1, y_2, \ldots$ one after one by solving the resulting equations with respect to these functions:

First at n = 1, we have:

$$y_1(t) = -I_t^{\alpha}\, N\left[t, y_0(t), y_0\left(\frac{t}{2}\right)\right]$$

$$y_1(t) = -I_t^{\alpha}\left(D^{\alpha}y_0(t) - \frac{1}{2}e^{\frac{t}{2}}y_0\left(\frac{t}{2}\right) - \frac{1}{2}y_0(t)\right)$$

and upon taking the Maclurian series expansion of $e^{\frac{t}{2}}$ up to three terms. One can get as :

$$y_1 = \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{4\Gamma(\alpha+2)} + \frac{t^{\alpha+2}}{8\Gamma(\alpha+3)}$$

at n = 2, we have:

$$y_2(t) = y_1(t) - I_t^{\alpha}N\left[t, y_1(t), y_1(\phi(t))\right]$$

Hence

$$y_2(t) = \frac{(1+2^{\alpha})\,t^{2\alpha}}{2^{\alpha+1}\Gamma(2\alpha+1)} + \frac{\Gamma(\alpha+3)\Gamma(\alpha+2)+\Gamma(\alpha+1)+2^{\alpha-1}\Gamma(\alpha+1)\Gamma(\alpha+2)\,t^{2\alpha+1}}{2^{\alpha+2}\Gamma(2\alpha+2)\Gamma(\alpha+1)\Gamma(\alpha+3)} +$$

$$\frac{2\Gamma(\alpha+1)\Gamma(\alpha+2)+2\Gamma(\alpha+1)\Gamma(\alpha+3)+\Gamma(\alpha+2)\Gamma(\alpha+3)+2^{\alpha-1}\Gamma(\alpha+1)\Gamma(\alpha+2)\,t^{2\alpha+2}}{2^{\alpha+4}\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(2\alpha+3)} +$$

$$\left(\frac{2\Gamma(\alpha+2)+\Gamma(\alpha+3)}{2^{\alpha+4}\Gamma(\alpha+2)\Gamma(\alpha+3)}\right)\frac{\Gamma(\alpha+4)\,t^{2\alpha+3}}{\Gamma(2\alpha+4)} + \frac{\Gamma(\alpha+5)\;\;t^{2\alpha+4}}{2^{\alpha+5}\Gamma(\alpha+3)\Gamma(2\alpha+5)}$$

Similarly according to eq.(3.26) and eq.(3.27) we can find $y_3, y_4, \ldots$.

Following table (3.1) represent the approximate solution of example (3.1) using HAM up to three terms for different values of α with a comparison with the exact solution when $\alpha = 1$.

Table (3.1)

The approximate solution of example (3.1) using different values of α with a comparison with the exact solution when $\alpha=1$.

t	*HAM* $\alpha=0.5$	*HAM* $\alpha=0.75$	*HAM* $\alpha=1$	*Exact* $\alpha=1$
0	1	1	1	1
0.1	1.45	1.216	1.105	1.105
0.2	1.701	1.391	1.221	1.221
0.3	1.926	1.565	1.347	1.35
0.4	2.142	1.745	1.485	1.492
0.5	2.356	1.932	1.636	1.649
0.6	2.573	2.128	1.799	1.822
0.7	2.794	2.334	1.976	2.014
0.8	3.022	2.553	2.167	2.226
0.9	3.26	2.784	2.374	2.46
1	3.508	3.029	2.598	2.718

Example (3.2):

Consider the FDDE_S:

$$ {}^{c}D_{t}^{\alpha}y(t)=\frac{3}{4}y(t)+y\left(\frac{t}{2}\right)-t^{2}+2\,,\ 0\le t\le 1,\quad 1<\alpha\le 2 \qquad \text{…(3.28)}$$

$y(0)=0$

First we choose the initial approximation $y_0(t)$ to be:

$$y_0(t) = 0 \qquad \ldots(3.29)$$

and according to equation (3.11), then:

$$N\left[t, y(t), y\left(\frac{t}{2}\right)\right] = {}^{c}D_t^{\alpha} y(t) - \frac{3}{4} y(t) - y\left(\frac{t}{2}\right) + g(t; q) \qquad \ldots(3.30)$$

where $g(t; q) = 2 - t^2 q$.

Set $L = {}^{c}D_t^{\alpha}$, $h = -1$ and $H = 1$, hence according to equation (3.19), we have:

$${}^{c}D_t^{\alpha}[y_m(t) - X_m y_{m-1}(t)] = -R_{y_m}(\vec{y}_{m-1}) \qquad \ldots(3.31)$$

where:

$$R_{y_m}(\vec{y}_{m-1}) = {}^{c}D_t^{\alpha} y_{m-1}(t) - \frac{3}{4} y_{m-1}(t) - y_{m-1}\left(\frac{t}{2}\right) - W_n(t) \qquad \ldots(3.32)$$

where $W_1 = 2$ and $W_2 = -t^2$ and $W_n = 0$, $n = 3,4, \ldots$

Operating I_t^{α} to the both sides of equation (3.31) and using equation (3.29) therefore one can get the functions $y_1, y_2, \ldots$ one after one by solving the resulting equations with respect to these functions:

First at n = 1, we have:

$$y_1(t) = -I_t^{\alpha}\left\{N\left[t, y_0(t), y_0\left(\frac{t}{2}\right)\right] - W_1\right\}$$

$$y_1(t) = -I_t^{\alpha}\left({}^{c}D_t^{\alpha} y_0(t) - \frac{3}{4} y_0(t) - y_0\left(\frac{t}{2}\right) - 2\right)$$

Hence:

$$y_1(t) = \frac{2\, t^{\alpha}}{\Gamma(\alpha+1)}$$

n = 2, we have:

$$y_2(t) = y_1(t) - I_t^{\alpha}\left\{N\left[t, y_1(t), y_1\left(\frac{t}{2}\right)\right] + t^2\right\}$$

Therefore:

$$y_2(t) = \left(\frac{2^{\alpha-2}3+1}{2^{\alpha-1}\Gamma(2\alpha+1)}\right) t^{2\alpha} - \frac{2}{\Gamma(\alpha+3)} t^{\alpha+2}$$

and similarly according to eq.(3.30) and eq.(3.31) we can find $y_3, y_4, \ldots$.

Following table (3.2) represent the approximate solution of example (3.2) using HAM up to three terms for different values of α with a comparison with the exact solution when $\alpha = 2$.

Table (3.2)

The approximate solution of example (3.2) using different values of α with a comparison with the exact solution when $\alpha = 2$.

t	*HAM* $\alpha = 1.5$	*HAM* $\alpha = 1.75$	*HAM* $\alpha = 2$	*Exact* $\alpha = 2$
0	0	0	0	0
0.1	0.048	0.022	0.01	0.01
0.2	0.137	0.075	0.04	0.04
0.3	0.255	0.153	0.09	0.09
0.4	0.397	0.254	0.16	0.16
0.5	0.563	0.377	0.25	0.25
0.6	0.75	0.521	0.36	0.36
0.7	0.958	0.686	0.49	0.49
0.8	1.186	0.872	0.64	0.64
0.9	1.434	1.077	0.81	0.81
1	1.7	1.303	1	1

<u>Example (3.3):</u>

Consider the $FDDE_S$

$$ {}^{c}D_t^{\alpha} y(t) = 1 - 2y^2\left(\frac{t}{2}\right), \quad 0 \le t \le 1\,,\ 0 < \alpha \le 1 \qquad \ldots(3.33)$$

$$y(0) = 0$$

First we choose the initial approximation $y_0(t)$ to be:

$$y_0(t) = t \qquad \ldots(3.34)$$

and according to equation (3.11), then:

$$N\left[t, y(t), y\left(\frac{t}{2}\right)\right] = {}^{c}D_t^{\alpha} y(t) - 1 + 2y^2\left(\frac{t}{2}\right) \qquad \ldots(3.35)$$

Set $L = {}^{c}D_t^{\alpha}$, $h = -1$ and $H = 1$, hence according to equation (3.19), we have:

$${}^{c}D_t^{\alpha}[y_m(t) - X_m y_{m-1}(t)] = -R_{y_m}(\vec{y}_{m-1}) \qquad \ldots(3.36)$$

where:

$$R_{y_m}(\vec{y}_{m-1}) = {}^{c}D_t^{\alpha} y_{m-1}(t) + 2\sum_{i=0}^{m-1} y_i\left(\frac{t}{2}\right) y_{m-1-i}\left(\frac{t}{2}\right) - (1 - X_m) \qquad \ldots(3.37)$$

Operating I_t^{α} to the both sides of equation (3.36) and using equation (3.34) therefore one can get the functions $y_1, y_2, \ldots$ one after one by solving the resulting equations with respect to these functions:

First at n = 1, we have:

$$y_1(t) = -I_t^{\alpha}\left({}^{c}D_t^{\alpha} y_n(t) + 2y_0^2\left(\frac{t}{2}\right) - 1 \right)$$

Hence:

$$y_1(t) = -t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)}$$

Similarly for n = 2 we have :

$$y_2(t) = \frac{2\, t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{\Gamma(\alpha+4)\, t^{2\alpha+3}}{2^{\alpha+1}\Gamma(\alpha+3)\Gamma(2\alpha+4)} - \frac{\Gamma(\alpha+2)\, t^{2\alpha+1}}{2^{\alpha-1}\Gamma(\alpha+1)\Gamma(2\alpha+2)}$$

and similarly according to eq.(3.36) and eq. (3.37) we can find $y_3, y_4, \ldots$.

Following table (3.3) represent the approximate solution of example (3.3) using HAM up to three terms for different values of α with a comparison with the exact solution when $\alpha = 1$.

Table (3.3)

The approximate solution of example (3.3) using different values of α with a comparison with the exact solution when $\alpha = 1$.

t	*HAM* $\alpha = 0.5$	*HAM* $\alpha = 0.75$	*HAM* $\alpha = 1$	*Exact* $\alpha = 1$
0	0	0	0	0
0.1	0.099	0.1	0.1	0.1
0.2	0.195	0.197	0.199	0.199
0.3	0.286	0.292	0.296	0.296
0.4	0.371	0.382	0.389	0.389
0.5	0.45	0.467	0.479	0.479
0.6	0.523	0.547	0.565	0.565
0.7	0.589	0.619	0.644	0.644
0.8	0.649	0.685	0.717	0.717
0.9	0.703	0.744	0.783	0.783
1	0.751	0.795	0.842	0.841

CONCLUSIONS AND FUTURE WORKS

From the present study we conclude the following:

1- One can conclude from the results of the numerical examples that the ADM and HAM gave us a good agreements with the exact solutions.
2- According to example 2 HAM gave us an accurate results when solving the delay differential equations of fractional order than the ADM due to the adjust and control the convergence regions and rates of approximation series.
3- In the proposed methods that have been used to approximate the solution of the delay differential equations of fractional order which can be approximate as a convergent power series all the components of these series are easily computed and thus we can get an approximate solution in easily manner.

For future works we recommended the following:

1- Numerical solution of delay differential equations of fractional order using Haar wavelet method.
2- Numerical solution of Integro-delay differential equations of fractional order using homotopy analysis method, Adomian decomposition method and variational iteration method.
3- Solving system of delay differential equations of fractional order using Adomian decomposition method.

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المستخلص

الهدف الرئيسي لهذة الرسالة يتمحور حول هدفين هما :

الهدف الاول هو ايجاد الحلول التقريبية للمعادلات التفاضلية التباطؤية ذات الرتب الكسرية باستخدام طريقة ادومين للتجزئة .

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في كلتا الطريقتين تم تمثيل الحلول على شكل متسلسة اسية متقاربة والتي ممكن ايجاد حدودها بشكل بسيط.

تم عرض بعض الامثلة التوضيحية وقد تمت مقارنة نتائج هذه الامثلة مع الحل المظبوط وذلك من اجل بيان دقة وقابلية الطرائق المقترحة .

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رسالة

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إشراف
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ذي الحجه ١٤٣٥ **ايلول ٢٠١٤**