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Integro-Differential Inequalities

A Thesis

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بِسْمِ اللّه الْرَّحْمنِ الرَّحِيمِ

(قَالُواْ سُبْحَانَكَ لَا عِلْمَ لَنَا إِلَّا مَا عَلَّمْتَنَآ إِنَّكَ أَنتَ الْعَلِيمُ الْحَكِيمُ) .

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الى كل الاساتذة الذين استفدت من علمهم

الى كل من ساعدني في انجاز هذا العمل المتواضع

اهدي لهم جميعا ثمرة هذا الجهد المتواضع................

أسماء

Supervisor Certification

I certify that this thesis was prepared under my supervision at the Department of Mathematics and Computer Applications / College of Science / Al-Nahrain University as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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Abstract

The main purpose of this work can be classified into four objects. These are summarized as follows:

The first objective, is to classify the one-dimensional integral inequalities.

The second objective, is to find explicit bounds for the unknown function that appeared in special types of the one-dimensional Volterra linear and non-linear integral inequalities.

The third objective, is to classify the one-dimensional integro-differential inequalities.

The fourth objective, is to give explicit bounds for the unknown function that appeared in special types of the one-dimensional Volterra first order and second order linear and non-linear integro-differential inequalities.

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It is well known truth that the inequalities have always been of great importance for the development of many branches of mathematics, [Pachpatte, 2006, p.1].

The mathematical foundations of the theory of inequalities were established in part during the 18^{th} and 19^{th} century by mathematicians such as Gauss (1777-1855), Cauchy (1789-1857) and Chebyshev (1812-1894). In the years thereafter the influence of inequalities has been immense, and the subject has attracted many distinguished mathematicians, including Poincaré (1854-1912), Lyapunov (1857-1918), O. Hölder (1859-1937) and J. Hadamard (1865-1963). [Pachpatte, 1998, p.1].

Nowadays the theory of inequalities may be regarded as an independent branch of mathematics. This field is dynamic and experiencing an explosive growth in both theory and applications. A particular feature that makes the study of this interesting topic so fascinating arises from the numerous fields of applications. As a response to the needs of divers applications, a large variety of inequalities have been proposed and studied in the literature. This theory added some techniques which are instrumental in solving many important problems, [Pachpatte, 2006, p.1].

The integral inequalities that give explicit bounds on unknown functions provide a very useful and important device in the study of various properties of solutions of differential and integral equations. One of the best known and widely used inequalities in the study of nonlinear differential equations was given by Gronwall T., that in 1919 Gronwall found this integral inequality while investigating the dependence of systems of differential equations with respect to a parameter. In fact the roots of such an inequality can be found in the work of Peano (1885-1886), [Pachpatte, 1998, p.4-5].

During the period (1919-1975) a large number of papers appeared in the literature which were partly inspired by Gronwall inequality and its applications. An extensive survey of integral inequalities of the Gronwall type which are adequate in many applications in the theory of differential and integral equations may found in [Beesack P., 1975].

Pachpatte B. (1994-1997) has proved a number of integral inequalities which can be used as handy tools in the study of certain new classes of differential, integral and integro-differential equations, [Pachpatte B., 1998, p.6].

Integro-differential inequalities are integral inequalities involving functions and their derivatives.

The integro-differential inequalities have played a significant role in the developments of various branches of analysis. Pachpatte B. in (1977, 1978, 1982) gave some integro-differential inequalities which are useful in certain applications in the theory of differential and integro-differential equations.

The purpose of this work, is to classify the one-dimensional integral and integro-differential inequalities and finding explicit bounds for the unknown function in special types of the one-dimensional linear and nonlinear Volterra integral and integro-differential inequalities.

This thesis consists of two chapters.

In chapter one, we illustrate some concepts about the one-dimensional integral inequalities. Also it contains some of the famous integral inequalities, like Gronwall, Bellman and Bihari inequalities and their generalizations.

In chapter two, we give a simple classification of the one-dimensional first order and second order integro-differential inequalities. Moreover finding explicit bounds for special types of the one-dimensional integro-differential inequalities is presented.

CHAPTER ONE

THE ONE-DIMENSIONAL VOLTERRA INTEGRAL INEQUALITIES

Introduction

Integral inequalities that give explicit bounds on the unknown functions provide a very useful and important tool in the study of many qualitative as well as quantitative properties of solutions of nonlinear differential equations, [Pachpatte B., 1998, p.4].

The history of integral inequalities goes back to Gronwall T., in 1919, he discovered the integral inequalities and their use in studying problems of the theory of ordinary differential equations. This justifies the intensive investigations on integral inequalities, and the appearance of hundreds of publications on them, [Bainov D. and Simeonov P., 1992].

There are many types of integral inequalities of importance. One type is known as Gronwall type inequalities. Gronwall type inequalities are important, and have applications to questions of stability, uniqueness of solutions, asymptotic behaviour, etc., [Bainov D. and Simeonov P., 1992, p.vi].

The main purpose of this chapter is to classify the one-dimensional integral inequalities. Moreover explicit bounds for the unknown function that appeared in some famous Volterra integral inequalities, like Gronwall's inequality, Bellman's inequality and Bihari's inequality, and their generalizations are obtained.

This chapter consists of three sections.

In section one, we give the definition of the integral inequalities and the classification of special types of the one-dimensional integral inequalities.

In section two, we give explicit bounds for the unknown function in the one-dimensional Volterra linear integral inequalities and some theorems related with them.

In section three, we give explicit bounds for the unknown function in special types of the one-dimensional Volterra non-linear integral inequalities.

1.1 Classification of the One-Dimensional Integral Inequalities:-

The integral inequalities are inequalities involving functions of one or more than one independent variable appeared under an integral sign which provide explicit bounds.

The one-dimensional integral inequalities are integral inequalities in which the unknown function depends only on one-independent variable. On the other hand, the m-dimensional integral inequalities are integral inequalities in which the unknown function depends only on m-independent variables, [Bainov D. and Simeonov P., 1992].

In this section we classify special types of the one-dimensional Volterra linear and nonlinear integral inequalities.

The simplest form of the one-dimensional linear integral inequality that contains only one integral operator is:

$$h(x)y(x) \leq a(x) + \int_{\alpha}^{\beta(x)} k(x,t)y(t)dt, \qquad (1.1)$$

where a is a known function of x, k is a known function of x and t, β and h are known functions of x, α is a known constant and y is the unknown function that must determine its explicit bounds.

The integral inequality (1.1) is called Fredholm linear integral inequality when $\beta(x)=\beta$, where β, is a known constant. On the other hand, the integral inequality (1.1) is called the one-dimensional Volterra linear integral inequality in case $\beta(x)=x$.

If $h(x)=0$ in inequality (1.1), then this integral inequality is of the first kind, and if $h(x)=1$ then this integral inequality is of the second kind.

Also, if $a(x)=0$, then the integral inequality (1.1) is homogeneous, otherwise, it is nonhomogeneous, [Al-Azawi S., 2007, p.22].

On the other hand, the simplest form of the one-dimensional linear integral inequality that contains more than one integral operator is:

$$h(x)y(x)\leq a(x)+\int_{\alpha}^{\beta_1(x)} k_1(x,t)y(t)dt+\int_{\alpha}^{\beta_2(x)} k_2(x,t)\left[\int_{\alpha}^{\beta_3(t)} k_3(t,s)y(s)ds\right]dt, \qquad (1.2)$$

where a is a known function of x, k_1, k_2 are known functions of x and t, k_3 is a known function of t and s, β_1, β_2 are known functions of x, β_3 is a known function of t, α is a given constant and y is the unknown function that must determine its explicit bounds.

The integral inequality (1.2) is called Fredholm linear integral inequality when $\beta_1(x)=\beta_1$, $\beta_2(x)=\beta_2$ and $\beta_3(x)=\beta_3$ where β_1, β_2, β_3 are known constants. On the other hand, the integral inequality (1.2) is called Volterra linear integral inequality when $\beta_1(x)=\beta_2(x)=x$ and $\beta_3(t)=t$.

If $h(x)=0$ in inequality (1.2) then this integral inequality is of the first kind and if $h(x)=1$then this integral inequality is of the second kind.

Also, if $a(x)=0$ then the integral inequality (1.2) is said to be homogeneous, otherwise it is nonhomogeneous.

Moreover the simplest form of the one-dimensional non-linear integral inequality that contains only one integral operator is:

$$h(x)y(x)\leq a(x)+\int_{\alpha}^{\beta(x)} k(x,t,y(t))dt$$

where h, y, a, α, β are defined similar to the previous and k is a known function of x, t and y(t), in which it is non-linear with respect to y(t), [Al-Azawi S., 2007, p.23].

On the other hand, the simplest form of the non-linear integral inequality that contains only two integral operators is:

$$h(x)y(x)\leq a(x)+\int_{\alpha}^{\beta_1(x)} k_1(x,t,y(t))dt+\int_{\alpha}^{\beta_2(x)} k_2(x,t,y(t))dt$$

where h, y, a, α, β_1, β_2 are defined similar to the previous and k_1, k_2 are known functions of x, t and y(t), in which they are non-linear with respect to y(t).

Finally, similar to the linear case, one can recognize the one-dimensional Fredholm and Volterra non-linear integral inequalities of the first and second kinds.

1.2 Solutions of the One-Dimensional Volterra Linear Integral Inequalities:-

This section consists of some theorems that determine explicit bounds for the unknown function in the one-dimensional linear integral inequalities of Volterra type.

First, we give the following lemma which will be needed later.

Lemma (1.1), [Bainov D. and Simeonov P., 1992, p.2]:-

Let a and c be continuous functions for $x>\alpha$ and let y be a differentiable function for $x\geq\alpha$, and suppose that

$y'(x)\leq a(x)+c(x)y(x),\ x\geq\alpha$

and $y(\alpha)\leq y_o$,

then

$$y(x)\leq y_o e^{\int_\alpha^x c(s)ds}+\int_\alpha^x a(t)e^{\int_t^x c(s)ds}dt,\ x\geq\alpha.$$

Proof:-

Suppose that $y'(x)-c(x)y(x)\leq a(x)$.

By setting x=t in the above inequality and multiplying the resulting inequality by the integrating factor $e^{\int_t^x c(s)ds}$, one can get:

$$[y'(t)-c(t)y(t)]e^{\int_t^x c(s)ds}\leq a(t)e^{\int_t^x c(s)ds}.$$

Then the above inequality can be rewritten as:

$$\frac{d}{dt}\left[y(t)e^{\int_t^x c(s)ds}\right]\leq a(t)e^{\int_t^x c(s)ds}.$$

By integrating both sides of the above inequality from α to x, one can have:

$$y(x)-y_o\, e^{\int_\alpha^x c(s)ds} \le \int_\alpha^x a(t)e^{\int_t^x c(s)ds}dt\,.$$

Hence

$$y(x)\le y_o\, e^{\int_\alpha^x c(s)ds} + \int_\alpha^x a(t)e^{\int_t^x c(s)ds}dt\,,\ x\ge\alpha.$$

Gronwall T. in 1919 gave explicit bounds for the unknown function that appeared in the integral inequality given in the following theorem.

Theorem (1.1), [Gronwall T., 1919]:-

Let y be a continuous function defined on the interval $J=[\alpha,\alpha+h]$ and

$$0\le y(x)\le\int_\alpha^x [cy(t)+a]dt\,,\ x\in J,$$

where a and c are nonnegative constants. Then

$$0\le y(x)\le ahe^{ch},\ x\in J.$$

Proof:-

Suppose that $y(x)\le a(x-\alpha)+c\int_\alpha^x y(t)dt$ then $y(x)\le ah+c\int_\alpha^x y(t)dt$.

Let $z(x)=ah+c\int_{\alpha}^{x} y(t)dt$ then $y(x)\le z(x)$ and $z(\alpha)=ah$. But $z'(x)=cy(x)\le cz(x)$.

From lemma (1.1), one can have:

$$z(x)\le ahe^{\int_{\alpha}^{x} cdt}.$$

Since $\alpha\le x\le\alpha+h$, $a\ge 0$ and $c\ge 0$ then

$z(x)\le ahe^{c(x-\alpha)}\le ahe^{ch}$, $x\in[\alpha,\alpha+h]$.

Hence

$y(x)\le ahe^{ch}$, $x\in[\alpha,\alpha+h]$.

Bellman R. in 1943 gave explicit bounds for the unknown function that appeared in the integral inequality given in the following theorem.

Theorem (1.2), [Bellman R., 1943]:-

Let y and c be continuous and nonnegative functions defined on $J=[\alpha,\beta]$, and let a be a nonnegative constant. Then the inequality:

$$y(x)\le a+\int_{\alpha}^{x} c(t)y(t)dt,\ x\in J$$

implies that

$$y(x)\le ae^{\int_{\alpha}^{x} c(t)dt},\ x\in J.$$

Proof:-

Let $z(x)=a+\int_{\alpha}^{x} c(t)y(t)dt$, then $z(\alpha)=a$ and $y(x)\leq z(x)$. Also

$z'(x)=c(x)y(x)$

$\leq c(x)z(x)$, $x\in J$.

From lemma (1.1), one can get:

$$z(x)\leq ae^{\int_{\alpha}^{x} c(t)dt}, \quad x\in J.$$

Hence

$$y(x)\leq ae^{\int_{\alpha}^{x} c(t)dt}, \quad x\in J.$$

Bellman R. in 1958 gave the following generalization of theorem (1.2).

Crollary (1.1), [Bellman R., 1958]:-

Let y and c be continuous and nonnegative functions defined on $J=[\alpha,\beta]$, and let a be a continuous, positive and nondecreasing function defined on J, then

$$y(x)\leq a(x)+\int_{\alpha}^{x} c(t)y(t)dt, \quad x\in J,$$

implies that

$$y(x)\le a(x)e^{\int_{\alpha}^{x} c(t)dt}, \quad x\in J.$$

Proof:-

Let $w(x)=\frac{y(x)}{a(x)}$. Then $w(x)\le 1+\int_{\alpha}^{x} c(t)\frac{y(t)}{a(x)}dt$

Thus

$$w(x)\le 1+\int_{\alpha}^{x} c(t)\frac{a(t)y(t)}{a(x)a(t)}dt$$

$$=1+\int_{\alpha}^{x} c(t)\frac{a(t)w(t)}{a(x)}dt$$

$$\le 1+\int_{\alpha}^{x} c(t)\frac{a(x)w(t)}{a(x)}dt$$

$$\le 1+\int_{\alpha}^{x} c(t)w(t)dt, \quad x\in J.$$

Now an application of theorem (1.2) yields:

$$w(x)\le e^{\int_{\alpha}^{x} c(t)dt}, \quad x\in J.$$

Hence

$$y(x)\le a(x)e^{\int_{\alpha}^{x} c(t)dt}, \quad x\in J.$$

Next, the following theorem appeared in [Pachpatte B., 1975] without proof. Here we give its proof.

Theorem (1.3):-

Let y, a, b, c and g be nonnegative continuous functions defined on $J=[\alpha,\beta]$, and

$$y(x)\leq a(x)+b(x)\int_{\alpha}^{x}[c(t)y(t)+g(t)]dt,\ x\in J.$$

Then

$$y(x)\leq a(x)+b(x)\int_{\alpha}^{x}[a(t)c(t)+g(t)]e^{\int_{t}^{x}b(s)c(s)ds}dt,\ x\in J.$$

Proof:-

Let $z(x)=\int_{\alpha}^{x}[c(t)y(t)+g(t)]dt$, then $z(\alpha)=0$ and $y(x)\leq a(x)+b(x)z(x)$.

But $z'(x)=c(x)y(x)+g(x)$

$\leq a(x)c(x)+b(x)c(x)z(x)+g(x)$.

From lemma (1.1), one can have:

$$z(x)\leq\int_{\alpha}^{x}[a(t)c(t)+g(t)]e^{\int_{t}^{x}b(s)c(s)ds}dt,\ x\in J.$$

Hence

$$y(x)\le a(x)+b(x)\int_{\alpha}^{x}[a(t)c(t)+g(t)]e^{\int_{t}^{x}b(s)c(s)ds}dt,\ x\in J.$$

Pachpatte B. in 1998 gave the following generalization of theorem (1.3).

Theorem (1.4), [Pachpatte B., 1998, p.20]:-

Let y, a, b, c and g be nonnegative continuous functions defined on J=[α,β]. Let k and its partial derivative $\frac{\partial k}{\partial x}$ be nonnegative continuous functions for α≤t≤x≤β. If

$$y(x)\le a(x)+b(x)\int_{\alpha}^{x}k(x,t)[c(t)y(t)+g(t)]dt,\ x\in J.$$

Then

$$y(x)\le a(x)+b(x)\int_{\alpha}^{x}B(t)e^{\int_{t}^{x}A(s)ds}dt,\ x\in J,$$

where

$$A(x)=k(x,x)b(x)c(x)+\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)b(t)c(t)dt,\ x\in J$$

and

$$B(x)=k(x,x)[a(x)c(x)+g(x)]+\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)[a(t)c(t)+g(t)]dt,\ x\in J.$$

Proof:-

Let $z(x)=\int_{\alpha}^{x} k(x,t)[c(t)y(t)+g(t)]dt$, then $z(\alpha)=0$ and

$y(x)\le a(x)+b(x)z(x)$.

By differentiating z and using the above inequality, one can get:

$$z'(x)=k(x,x)[c(x)y(x)+g(x)]+\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)[c(t)y(t)+g(t)]dt.$$

$$\le k(x,x)\{c(x)[a(x)+b(x)z(x)]+g(x)\}+$$

$$\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)[c(t)\{a(t)+b(t)z(t)\}+g(t)]dt.$$

By using the fact that z is nondecreasing in x, one can have

$$z'(x)\le z(x)\left[k(x,x)b(x)c(x)+\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)b(t)c(t)dt\right]+$$

$$k(x,x)[a(x)c(x)+g(x)]+\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)[a(t)c(t)+g(t)]dt$$

$$=A(x)z(x)+B(x),\quad x\in J$$

where A(x) and B(x) are defined previously.

From lemma (1.1), one can obtain:

$$z(x)\le\int_{\alpha}^{x}B(t)\,e^{\int_{t}^{x}A(s)ds}dt,\quad x\in J.$$

Hence

$$y(x)\leq a(x)+b(x)\int_{\alpha}^{x} B(t)\, e^{\int_{t}^{x} A(s)ds}\, dt,\ x\in J.$$

Now, explicit bounds for the unknown functions that appeared in another type of integral inequalities is given in the following theorem.

Theorem (1.5), [Pachpatte B., 1973a]:-

Let y, b and c be nonnegative continuous functions defined on J=[α,∞). If

$$y(x)\leq a+\int_{\alpha}^{x} c(t)y(t)dt+\int_{\alpha}^{x} c(t)\left[\int_{\alpha}^{t} b(s)y(s)ds\right]dt,\ x\in J,$$

where a is a nonnegative constant.

Then

$$y(x)\leq a\, e^{\int_{\alpha}^{x} c(t)\left[1+\int_{\alpha}^{t} b(s)ds\right]dt},\ x\in J.$$

Proof:-

Let $z(x)=a+\int_{\alpha}^{x} c(t)y(t)dt+\int_{\alpha}^{x} c(t)\left[\int_{\alpha}^{t} b(s)y(s)ds\right]dt$ then $z(\alpha)=a$ and $y(x)\leq z(x)$.

On the other hand

$$z'(x)=c(x)y(x)+c(x)\int_{\alpha}^{x} b(t)y(t)dt$$

Therefore

$$z'(x)\leq c(x)\left[z(x) + \int_{\alpha}^{x} b(t)z(t)dt\right], \ x\in J.$$

By using the fact that z is a nondecreasing function in x, one can get:

$$z'(x)\leq c(x)\left[1 + \int_{\alpha}^{x} b(t)dt\right]z(x), \ x\in J.$$

From lemma (1.1), one can have:

$$z(x)\leq a\, e^{\int_{\alpha}^{x} c(t)\left[1+\int_{\alpha}^{t} b(s)ds\right]dt}, \ x\in J.$$

Hence

$$y(x)\leq a\, e^{\int_{\alpha}^{x} c(t)\left[1+\int_{\alpha}^{t} b(s)ds\right]dt}, \ x\in J.$$

Next, a modification of theorem (1.5) can be given in the following corollary.

Corollary (1.2):-

Let y, b and c be nonnegative continuous functions defined on J=[α,∞), a is a positive nondecreasing function defined on J. If

$$y(x)\le a(x)+\int_{\alpha}^{x} c(t)y(t)dt+\int_{\alpha}^{x} c(t)\left[\int_{\alpha}^{t} b(s)y(s)ds\right]dt,\ x\in J$$

then

$$y(x)\le a(x)\, e^{\int_{\alpha}^{x} c(t)\left[1+\int_{\alpha}^{t} b(s)ds\right]dt},\ x\in J.$$

Proof:-

Let $w(x)=\dfrac{y(x)}{a(x)}$.

Then

$$w(x)\le 1+\int_{\alpha}^{x} c(t)\frac{a(t)w(t)}{a(x)}dt+\int_{\alpha}^{x} c(t)\left[\int_{\alpha}^{t} b(s)\frac{a(s)w(s)}{a(x)}ds\right]dt.$$

By using the fact that a is a nondecreasing function in x, one can get:

$$w(x)\le 1+\int_{\alpha}^{x} c(t)w(t)dt+\int_{\alpha}^{x} c(t)\left[\int_{\alpha}^{t} b(s)w(s)ds\right]dt,\ x\in J.$$

From theorem (1.5), one can have:

$$w(x)\le e^{\int_{\alpha}^{x} c(t)\left[1+\int_{\alpha}^{t} b(s)ds\right]dt},\ x\in J.$$

Hence

$$y(x)\le a(x)\, e^{\int_{\alpha}^{x} c(t)\left[1+\int_{\alpha}^{t} b(s)ds\right]dt},\ x\in J.$$

Next, another extension of the previous theorem can be seen below.

Theorem (1.6):-

Let y and b be nonnegative continuous functions defined on J=[α,∞). Let k and its partial derivative $\frac{\partial k}{\partial x}$ be nonnegative continuous functions defined on J=[α,∞). If

$$y(x)\leq a+\int_{\alpha}^{x} k_1(x,t)[y(t)+g(t)]dt+\int_{\alpha}^{x}\int_{\alpha}^{t} k_2(t,s)[y(s)+g(s)]dsdt,\ x\in J.$$

Then

$$y(x)\leq a\, e^{\int_{\alpha}^{x} A(t)dt}+\int_{\alpha}^{x} B(t)e^{\int_{t}^{x} A(s)ds}dt,\ x\in J,$$

where

$$A(x)=k_1(x,x)+\int_{\alpha}^{x}\frac{\partial}{\partial x}k_1(x,t)dt+c(x)\int_{\alpha}^{x}k_2(x,t)dt.$$

and

$$B(x)=k_1(x,x)g(x)+\int_{\alpha}^{x}\frac{\partial}{\partial x}k_1(x,t)g(t)dt+\int_{\alpha}^{x}k_2(x,t)g(t)dt.$$

Proof:-

Let

$$z(x) = a + \int_\alpha^x k_1(x,t)[y(t)+g(t)]dt + \int_\alpha^x \int_\alpha^t k_2(t,s)[y(s)+g(s)]dsdt,$$

then $z(\alpha)=a$ and $y(x) \le z(x)$, $x \in J$.

But $$z'(x)=k_1(x,x)[y(x)+g(x)] + \int_\alpha^x \frac{\partial}{\partial x} k_1(x,t)[y(t)+g(t)]dt$$

$$+ \int_\alpha^x k_2(x,t)[y(t)+g(t)]dt.$$

$$\le k_1(x,x)[z(x)+g(x)] + \int_\alpha^x \frac{\partial}{\partial x} k_1(x,t)[z(t)+g(t)]dt +$$

$$\int_\alpha^x k_2(x,t)[z(t)+g(t)]dt, \ x \in J.$$

By using the fact that z is a nondecreasing function in x, one can get:

$z'(x) \le A(x)z(x)+B(x)$,

where A(x) and B(x) are defined previously.

From lemma (1.1), one can have:

$$z(x) \le a e^{\int_\alpha^x A(t)dt} + \int_\alpha^x B(t) e^{\int_t^x A(s)ds} dt, \ x \in J.$$

Hence

$$y(x) \le a e^{\int_\alpha^x A(t)dt} + \int_\alpha^x B(t) e^{\int_t^x A(s)ds} dt, \ x \in J.$$

1.3 Solutions of the One-Dimensional Volterra Non-linear Integral Inequalities:-

As seen before in the previous section, all the inequalities that are discussed are of the linear type. On the other hand, the one-dimensional Volterra non-linear integral inequalities has many applications in the theory of integral, differential and integro-differential equations, [Pachpatte B., 1998].

So this section concerns with finding solutions of special types of the one-dimensional Volterra non-linear integral inequalities.

Bihari I. in 1956 gave the explicit bounds for the unknown function that appeared in the following non-linear integral inequality.

Theorem (1.7), [Bihari I., 1956]:-

Let y and c be nonnegative continuous functions defined on $J=[\alpha,\infty)$. Let w be a positive continuous nondecreasing function defined on J. If

$$y(x)\leq a+\int_{\alpha}^{x} c(t)w(y(t))dt,\ x\in J,$$

where a is a nonnegative constant, then for $\alpha\leq x\leq x_1$,

$$y(x)\leq G^{-1}\left[G(a)+\int_{\alpha}^{x} c(t)dt\right],$$

where

$$G(u)=\int_{u_0}^{u}\frac{dt}{w(t)},\quad u>u_0>0 \tag{1.3}$$

and G^{-1} is the inverse function of G and $x_1 \in J$ is chosen such that G(a)+ $\int_{\alpha}^{x} c(t)dt$ is in the domain of G^{-1} for all $x \in J$ lying in the interval $[\alpha, x_1]$.

Proof:-

Let $z(x)=a+\int_{\alpha}^{x} c(t)w(y(t))dt$, then $z(\alpha)=a$, $y(x) \leq z(x)$ and

$z'(x) \leq c(x)w(y(x))$

$\leq c(x)w(z(x))$

Then from equation (1.3), the above inequality can be rewritten as:

$$\frac{d}{dx}G(z(x)) = \frac{z'(x)}{w(z(x))} \leq c(x).$$

By taking x=t in the above inequality and integrating the resulting inequality over t from α to u, one can obtain:

$$G(z(u)) \leq G(a)+\int_{\alpha}^{x} c(t)dt, \quad \alpha \leq x \leq x_1.$$

Then

$$z(x) \leq G^{-1}\left[G(a)+\int_{\alpha}^{x} c(t)dt\right], \quad \alpha \leq x \leq x_1.$$

Hence

$$y(x) \leq G^{-1}\left[G(a)+\int_{\alpha}^{x} c(t)dt\right], \quad \alpha \leq x \leq x_1.$$

Now, the following theorem is a generalization of the previous theorem.

Theorem (1.8):-

Let y, a, a′ and c be nonnegative continuous functions defined on $J=[\alpha,\infty)$. Let w be a positive continuous nondecreasing function defined on J. If

$$y(x)\leq a(x)+\int_{\alpha}^{x} c(t)w(y(t))dt, \quad x\in J,$$

then for $\alpha\leq x\leq x_1$

$$y(x)\leq G^{-1}\left[G(a(\alpha))+\int_{\alpha}^{x}[a'(x)+c(t)]dt\right],$$

where

G and G^{-1} are defined previously and $x_1\in J$ is chosen such that $G(a(\alpha))+\int_{\alpha}^{x}[a'(x)+c(t)]dt$ is in the domain of G^{-1} for all $x\in J$ lying in the interval $[\alpha,x_1]$.

Proof:-

Let $z(x)=a(x)+\int_{\alpha}^{x} c(t)w(y(t))dt$, then $z(\alpha)=a(\alpha)$, $y(x)\leq z(x)$ and

$z'(x)\leq a'(x)+c(x)w(y(x))$

Therefore

$z'(x) \leq a'(x)+c(x)w(z(x))$.

Let $u \in R_+$ be any arbitrary number, then for $\alpha \leq x \leq u$, one can get:

$z(x) \leq a'(u)+ c(x)w(z(x))$

$\leq [a'(u)+c(x)]w(z(x))$.

Then from equation (1.3) the above inequality can be rewritten as:

$$\frac{d}{dx}G(z(x)) = \frac{z'(x)}{w(z(x))} \leq [a'(x)+c(x)].$$

By taking x=t in the above inequality and integrating the resulting inequality over t from α to u, one can obtain:

$$G(z(u)) \leq G(a(\alpha)) + \int_{\alpha}^{u} [a'(t)+c(t)]dt, \ \alpha \leq u \leq x_1.$$

Since u is an arbitrary number, then

$$z(x) \leq G^{-1}\left[G(a(\alpha)) + \int_{\alpha}^{x} [a'(t)+c(t)]dt\right], \ \alpha \leq x \leq x_1$$

Hence

$$y(x) \leq G^{-1}\left[G(a(\alpha)) + \int_{\alpha}^{x} [a'(t)+c(t)]dt\right], \ \alpha \leq x \leq x_1.$$

Next, we give explicit bounds for the unknown function that appeared in another type of Volterra nonlinear integral inequality.

Theorem (1.9), [Pachpatte B., 1975]:-

Let y, b, c and g be nonnegative continuous functions defined on $J=[\alpha,\infty)$. Let w be a positive continuous nondecreasing and submultiplicative function defined on J. If

$$y(x)\leq a+b(x)\int_{\alpha}^{x} g(t)y(t)dt+\int_{\alpha}^{x} c(t)w(y(t))dt,\ x\in J,$$

where a is a positive constant, then

$$y(x)\leq p(x)\left[a+G^{-1}\left\{G(A(x))+\int_{\alpha}^{x} c(t)w(p(t))dt\right\}\right],\ \alpha\leq x\leq x_1$$

where

$$p(x)=1+b(x)\int_{\alpha}^{x} g(t)e^{\int_{t}^{x} b(s)g(s)ds}dt,\ x\in J$$

and

$$A(x)=w(a)\int_{\alpha}^{x} c(t)w(p(t))dt,$$

G and G^{-1} are as defined in theorem (1.7) and $x_1\in J$ is chosen such that $G(a)+\int_{\alpha}^{x} c(t)w(p(t))dt$ is in the domain of G^{-1} for all $x\in J$ lying in the interval $[\alpha,x_1]$.

Proof:-

Let $v(x)=a+\int_{\alpha}^{x} c(t)w(y(t))dt$ then $y(x)\leq v(x)+b(x)\int_{\alpha}^{x} g(t)y(t)dt$.

Suppose that $z(x)=\int_{\alpha}^{x} g(t)y(t)dt$ then $y(x)\le v(x)+b(x)z(x)$.

But $z'(x)=g(x)y(x)$

$$\le g(x)v(x)+g(x)b(x)z(x).$$

From lemma (1.1), one can have:

$$z(x)\le\int_{\alpha}^{x} g(t)v(t)e^{\int_{t}^{x} g(s)b(s)ds}\,dt.$$

By using the fact that v is a nondecreasing function in x, one can get:

$y(x)\le p(x)v(x)$,

where p(x) is defined previously.

Therefore

$$v(x)=a+\int_{\alpha}^{x} c(t)w(y(t))dt$$

$$\le a+\int_{\alpha}^{x} c(t)w(p(t)v(t))dt$$

$$\le a+\int_{\alpha}^{x} c(t)w(p(t))w(v(t))dt$$

By applying theorem (1.7) to the above inequality, one can obtain:

$$v(x)\le a+G^{-1}\left[G(A(x))+\int_{\alpha}^{x} c(t)w(p(t))dt\right],$$

where A(x) is defined previously.

Hence

$$y(x)\le p(x)\left[a+G^{-1}\left\{G(A(x))+\int_{\alpha}^{x}c(t)w(p(t))dt\right\}\right],\ x\in J.$$

Next, an extension of the previous theorem can be seen below.

<u>Theorem (1.10)</u>:-

Let y, b, c and g be nonnegative continuous functions defined on J=[α,∞). Let w be a positive continuous nondecreasing and submultiplicative function defined on J. Let a be a positive continuous and nondecreasing function defined on J. If

$$y(x)\le a(x)+b(x)\int_{\alpha}^{x}g(t)y(t)dt+\int_{\alpha}^{x}c(t)w(y(t))dt,\ x\in J.$$

Then

$$y(x)\le p(x)\left[a(x)+G^{-1}\left(G(A(x))+\int_{\alpha}^{x}c(t)w(p(t))dt\right)\right],$$

where G, G^{-1} and p(x) are defined previously, $x_1\in J$ is chosen such that $G(A(x))+\int_{\alpha}^{x}c(t)w(p(t))dt$ is in the domain of G^{-1} for all $x\in J$ lying in the interval $[\alpha,x_1]$ and

$$A(x)=\int_{\alpha}^{x}c(t)w(p(t))w(a(t))dt.$$

Proof:-

Let $v(x)=a(x)+\int_{\alpha}^{x} c(t)w(y(t))dt$ then $y(x)\le v(x)+b(x)\int_{\alpha}^{x} g(t)y(t)dt$.

Suppose that $z(x)=\int_{\alpha}^{x} g(t)y(t)dt$ then $y(x)\le v(x)+b(x)z(x)$ and

$z'(x)=g(x)y(x)$.

$\le g(x)v(x)+g(x)b(x)z(x)$.

From lemma (1.1), one can have:

$$z(x)\le\int_{\alpha}^{x} g(t)v(t)e^{\int_{t}^{x} g(s)b(s)ds}dt .$$

Then

$y(x)\le p(x)v(x)$,

where

p(x) is defined previously.

Therefore

$$v(x)=a(x)+\int_{\alpha}^{x} c(t)w(y(t))dt .$$

$$\le a(x)+\int_{\alpha}^{x} c(t)w(p(t)v(t))dt .$$

$$\le a(x)+\int_{\alpha}^{x} c(t)w(p(t))w(v(t))dt .$$

By applying theorem (1.8) to the above inequality, one can obtain:

$$v(x)\le a(x)+ G^{-1}\left(G(A(x))+\int_{\alpha}^{x} c(t)w(p(t))dt\right).$$

where A(x) is defined previously.

Then

$$y(x) \leq p(x)\left[a(x) + G^{-1}\left(G(A(x)) + \int_{\alpha}^{x} c(t)w(p(t))dt\right)\right].$$

CHAPTER TWO

THE ONE-DIMENSIONAL INTEGRO-DIFFERENTIAL INEQUALITIES

Introduction

The integro-differential inequalities are inequalities involving one (or more) unknown function, together with both differential and integral operations on the unknown functions which provide explicit bounds.

The one-dimensional integro-differential inequalities are integro-differential inequalities in which the unknown function depends only on one-independent variable. On the other hand the m-dimensional integro-differential inequalities in which the unknown function depends only on m-independent variables. We restrict the discussion here to the simplest types of one-dimensional integro-differential inequalities, which form a natural generalization of Volterra and Fredholm integral inequalities, [Pachpatte B., 1998].

The main purpose of this chapter is to classify the one-dimensional integro-differential inequalities. Moreover explicit bounds for the unknown function that appeared in some types of the one-dimensional Volterra integro-differential inequalities, like Gronwall-Bellman integro-differential inequality and their generalizations are obtained.

This chapter consists of three sections.

In section one, a simple classification of special types of the one-dimensional integro-differential inequalities is given. To the best of our knowledge, this classification seems to be new.

In section two, we give some theorems for finding solutions of special types of the one-dimensional Volterra linear integro-differential inequalities of the first and second order.

In section three, we give some theorems for finding solutions of special types of the one-dimensional Volterra non-linear integro-differential inequalities of the first and second order.

2.1 Classification of the One-Dimensional Integro-Differential Inequalities:-

The integro-differential inequalities are inequalities involving functions of one or more than one independent variable appeared under an integral sign and derivatives sign which provide explicit bounds.

The one-dimensional integro-differential inequalities are integro-differential inequalities in which the unknown function depends only on one-independent variable, [Pachpatte B., 1998].

In this section we classify special types of the one-dimensional Volterra linear and non-linear integro-differential inequalities of the first and second order.

The simplest form of the one-dimensional first order linear integro-differential inequality that contains only one integral operator is:

$$h_1(x)y'(x)+h_2(x)y(x)\leq a(x)+\int_{\alpha}^{\beta(x)}[k_1(x,t)y'(t)+k_2(x,t)y(t)]dt \qquad (2.1)$$

where h_1, h_2, a, β are known functions of x, k_1 and k_2 are known functions of x and t, α is a given constant and y is the unknown function that must determine its explicit bounds.

The integro-differential inequality (2.1) is called Fredholm if $\beta(x)=\beta$ where β is a given constant and is called Volterra if $\beta(x)=x$.

If $h_1(x)=h_2(x)=0$ then the integro-differential inequality (2.1) is said to be of the first kind and if $h_1(x)=1$ then the integro-differential (2.1) is said to be of the second kind.

Also, if $a(x)=0$ then the integro-differential inequality (2.1) is homogeneous, otherwise it is nonhomogeneous.

On the other hand, the simplest form of the one-dimensional first order linear integro-differential inequality that contains more than one integral operator is:

$h_1(x)y'(x)+h_2(x)y'(x)\leq a(x)+$

$$\int_{\alpha}^{\beta_1(x)}\left[k_1(x,t)y'(t)+k_2(x,t)y(t)+\int_{\alpha}^{\beta_2(t)}[k_3(t,s)y'(s)+k_4(t,s)y(s)]ds\right]dt \qquad (2.2)$$

where h_1, h_2, a, α, k_1, k_2, y are defined similar to the previous, k_3, k_4 are known functions of t, s, β_1 is a known function of x and β_2 is a known function of t.

The integro-differential inequality is said to be of Fredholm type if $\beta_1(x)=\beta_1$ and $\beta_2(t)=\beta_2$ where β_1 and β_2 are known constants and it is said to be of Volterra type if $\beta_1(x)=x$ and $\beta_2(t)=t$.

Next, the simplest form of the one-dimensional second order linear integro-differential inequality that contains only one integral operator is:

$h_1(x)y''(x)+h_2(x)y'(x)+h_3(x)y(x)\leq a(x)+$

$$\int_{\alpha}^{\beta(x)}[k_1(x,t)y''(t)+k_2(x,t)y'(t)+k_3(x,t)y(t)]dt$$

where h_1, h_2, α, β, k_1, k_2, y are defined similar to the previous, h_3 is a known function of x, k_3 is a known function of x and t.

In a similar manner one can easily define the one-dimensional second order linear Volterra and Fredholm integro-differential inequalities of the first and second kinds.

Moreover, the simplest form of the one-dimensional first order non-linear integro-differential inequality that contains only one integral operator is:

$$h_1(x)y'(x)+h_2(x)y(x)\leq a(x)+\int_{\alpha}^{\beta(x)}k(x,t,y(t),y'(t))dt$$

where h_1, h_2, a, α, y are defined similar to the previous and k is a known function of x, t, y(t), in which it is non-linear with respect to y(t).

In a similar manner one can easily define the one-dimensional second order non-linear integro-differential inequalities. Also, it is easy to recognize the one-dimensional first and second order non-linear Volterra and Fredholm integro-differential inequalities of the first and second kinds.

2.2 Solutions of the One-Dimensional Volterra Linear Integro-Differential Inequalities:-

In this section, we present some theorems about the one-dimensional linear integro-differential inequalities of Volterra type which provide explicit bounds for the unknown function that appeared in them.

Theorem (2.1), [Pachpatte B., 1977]:-

Let y, y′, a, b and c be nonnegative continuous functions defined on J=[α,∞).

(i) If $b(x)\geq 1$ for each $x\in J$ and

$$y'(x)\leq a(x)+b(x)\int_{\alpha}^{x} c(t)[y(t)+y'(t)]dt,\ x\in J$$

then

$$y(x)\leq y(\alpha)+\int_{\alpha}^{x}\left[a(t)+\int_{\alpha}^{t}\left\{b(s)c(s)[A(s)+B(s)]+b'(s)\int_{\alpha}^{s}c(\tau)[A(\tau)+B(\tau)]d\tau\right\}ds\right]dt,\ x\in J.$$

where

$$A(x)=y(\alpha)+a(x)+\int_{\alpha}^{x}a(t)dt,\ x\in J$$

and

$$B(x)=\int_{\alpha}^{x}\left[b(t)c(t)A(t)+b'(t)\int_{\alpha}^{t}c(s)A(s)ds\right]e^{\int_{t}^{x}\left[1+b(s)c(s)+b'(s)\int_{\alpha}^{s}c(\tau)d\tau\right]ds}dt,\ x\in J.$$

(ii) If

$$y'(x)\leq a(x)+b(x)\left[y(x)+\int_{\alpha}^{x}c(t)[y(t)+y'(t)]dt\right],\ x\in J,$$

then

$$y(x) \le y(\alpha) + \int_\alpha^x \left[a(t) + b(t) \left\{ y(\alpha) e^{\int_\alpha^t [b(s)+c(s)+b(s)c(s)]ds} + \int_\alpha^t a(s)[1+c(s)] e^{\int_t^s [b(\tau)+c(\tau)+b(\tau)c(\tau)]d\tau} ds \right\} \right] dt .$$

Proof:-

(i) Let $z(x)=b(x)\int_\alpha^x c(t)[y(t)+y'(t)]dt$ then $z(\alpha)=0$,

$z'(x)=b(x)c(x)[y(x)+y'(x)]+b'(x)\int_\alpha^x c(t)[y(t)+y'(t)]dt$ and

$y'(x) \le a(x)+z(x),\ x \in J.$

By setting x=t in the above inequality and integrating the resulting inequality over t from α to x, one can get:

$y(x) \le y(\alpha) + \int_\alpha^x a(t)dt + \int_\alpha^x z(t)dt,\ x \in J.$

Therefore

$$z'(x) \le b(x)c(x)\left[y(\alpha) + a(x) + \int_\alpha^x a(t)dt + z(x) + \int_\alpha^x z(t)dt \right] +$$

$$b'(x)\int_\alpha^x c(t)\left[y(\alpha) + a(t) + \int_\alpha^t a(s)ds + z(t) + \int_\alpha^t z(s)ds \right] dt,\ x \in J.$$

$$= b(x)c(x)\left[A(x) + z(x) + \int_\alpha^x z(t)dt \right] + b'(x)\int_\alpha^x c(t)\left[A(t) + z(t) + \int_\alpha^t z(s)ds \right] dt,\ x \in J.$$

where A(x) is defined previously.

Let $v(x)=z(x)+\int_\alpha^x z(t)dt,\ x \in J$, then $v(\alpha)=0$,

$z'(x) \le b(x)c(x)[A(x)+v(x)] + b'(x)\int_\alpha^x c(t)[A(t)+v(t)]dt,$

and

$v'(x)=z'(x)+z(x),\ x \in J.$

By using the fact that v is a nondecreasing function in x, one can have:

$$v'(x)\le b(x)c(x)A(x)+b'(x)\int_\alpha^x c(t)A(t)dt+\left[1+b(x)c(x)+b'(x)\int_\alpha^x c(t)dt\right]v(x).$$

From lemma (1.1), one can have:

$v(x)\le B(x),\ x\in J$

where B(x) is defined previously.

Then

$$z'(x)\le b(x)c(x)[A(x)+B(x)]+b'(x)\int_\alpha^x c(t)[A(t)+B(t)]dt\,.$$

By taking x=t in the above inequality and integrating the resulting inequality over t from α to x, one can have:

$$z(x)\le\int_\alpha^x\left\{b(t)c(t)[A(t)+B(t)]+b'(t)\int_\alpha^t c(s)[A(s)+B(s)]ds\right\}dt,\ x\in J.$$

Hence

$$y(x)\le y(\alpha)+\int_\alpha^x\left[a(t)+\int_\alpha^t\left\{b(s)c(s)[A(s)+B(s)]+b'(s)\int_\alpha^s c(\tau)[A(\tau)+B(\tau)]d\tau\right\}ds\right]dt,\ x\in J.$$

(ii) Let $z(x)=y(x)+\int_\alpha^x c(t)[y(t)+y'(t)]dt$, $x\in J$, then $z(\alpha)=y(\alpha)$ and

$y'(x)\le a(x)+b(x)z(x),\ x\in J.$

By using the above inequality and the fact that $y(x)\le z(x)$, one can have:

$$\begin{aligned}z'(x)&=y'(x)+c(x)[y(x)+y'(x)]\\&=[1+c(x)]y'(x)+c(x)y(x)\\&\le[1+c(x)][a(x)+b(x)z(x)]+c(x)z(x)\\&=a(x)[1+c(x)]+[b(x)+c(x)+b(x)c(x)]z(x),\ x\in J.\end{aligned}$$

From lemma (1.1), one can get:

$$z(x)\le y(\alpha)e^{\int_\alpha^x[b(t)+c(t)+b(t)c(t)]dt}+\int_\alpha^x a(t)[1+c(t)]e^{\int_t^x[b(s)+c(s)+b(s)c(s)]ds}dt,\ x\in J.$$

Hence

$y(x)\le y(\alpha)+$

$$\int_{\alpha}^{x}\left[a(t)+b(t)\left\{y(\alpha)e^{\int_{\alpha}^{t}[b(s)+c(s)+b(s)c(s)]ds}+\int_{\alpha}^{t}a(s)[1+c(s)]e^{\int_{t}^{s}[b(\tau)+c(\tau)+b(\tau)c(\tau)]d\tau}ds\right\}\right]dt.$$

Next, the following theorem is a modification of the first part of the previous theorem.

<u>Theorem (2.2)</u>:-

Let y, y′, a, b and c be nonnegative continuous functions defined on J=[α,∞). Let k:J×J→J, α≤t≤x be a nonnegative continuous function. If $y'(x)\le a(x)+\int_{\alpha}^{x}k(x,t)[y(t)+y'(t)]dt$, $x\in J$.

then

$$y(x)\le y(\alpha)+\int_{\alpha}^{x}\left[a(t)+\int_{\alpha}^{t}\left\{k(s,s)\left[A(s)+\int_{\alpha}^{s}B(\tau)e^{\int_{\tau}^{s}D(\vartheta)d\vartheta}d\tau\right]+\int_{\alpha}^{s}\frac{\partial}{\partial s}k(s,\tau)\left[A(\tau)+\int_{\alpha}^{\tau}B(\vartheta)e^{\int_{\vartheta}^{\tau}D(q)dq}d\vartheta\right]d\tau\right\}ds\right]dt,\ x\in J.$$

where

$A(x)=y(\alpha)+a(x)+\int_{\alpha}^{x}a(t)dt$, $x\in J$,

$B(x)=k(x,x)A(x)+\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)A(t)dt$, $x\in J$,

and

$D(x)=1+k(x,x)+\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)dt$, $x\in J$.

<u>Proof</u>:-

Let $z(x)=\int_{\alpha}^{x}k(x,t)[y(t)+y'(t)]dt$ then $z(\alpha)=0$ and

$y'(x)\le a(x)+z(x)$, $x\in J$.

By setting x=t in the above inequality and integrating the resulting inequality over t from α to x, one can obtain:

$y(x) \le y(\alpha) + \int_\alpha^x a(t)dt + \int_\alpha^x z(t)dt, \ x \in J.$

Therefore

$$z'(x) = k(x,x)[y(x)+y'(x)] + \int_\alpha^x \frac{\partial}{\partial x} k(x,t)[y(t)+y'(t)]dt,$$

$$\le k(x,x)\left[y(\alpha)+a(x)+\int_\alpha^x a(t)dt + z(x) + \int_\alpha^x z(t)dt\right] +$$

$$\int_\alpha^x \frac{\partial}{\partial x} k(x,t)\left[y(\alpha)+a(t)+\int_\alpha^t a(s)ds + z(t) + \int_\alpha^t z(s)ds\right]dt$$

$$= k(x,x)\left[A(x)+z(x)+\int_\alpha^x z(t)dt\right] + \int_\alpha^x \frac{\partial}{\partial x} k(x,t)\left[A(t)+z(t)+\int_\alpha^t z(s)ds\right]dt.$$

where A(x) is defined previously.

Let $v(x) = z(x) + \int_\alpha^x z(t)dt$ then $v(\alpha) = 0$, $v'(x) = z'(x) + z(x)$ and

$z'(x) \le k(x,x)[A(x)+v(x)] + \int_\alpha^x \frac{\partial}{\partial x} k(x,t)[A(t)+v(t)]dt, \ x \in J.$

By using the above inequality and the fact that $z(x) \le v(x)$, one can get:

$v'(x) \le k(x,x)[A(x)+v(x)] + \int_\alpha^x \frac{\partial}{\partial x} k(x,t)[A(t)+v(t)]dt + v(x), \ x \in J.$

By using the fact that v is a nondecreasing function in x, one can have:

$$v'(x) \le k(x,x)A(x) + \int_\alpha^x \frac{\partial}{\partial x} k(x,t)A(t)dt + \left[1 + k(x,x) + \int_\alpha^x \frac{\partial}{\partial x} k(x,t)dt\right] v(x).$$

$$\le B(x) + D(x)v(x), \ x \in J.$$

where B(x) and D(x) are defined previously.

From lemma (1.1), one can obtain:

$v(x) \le \int_\alpha^x B(t) e^{\int_t^x D(s)ds} dt, \ x \in J.$

Then

$$z'(x) \le k(x,x)\left[A(x) + \int_\alpha^x B(t) e^{\int_t^x D(s)ds} dt\right] +$$

$$\int_\alpha^x \frac{\partial}{\partial x} k(x,t) \left[A(t) + \int_\alpha^t B(s) e^{\int_s^t D(\tau)d\tau} ds\right] dt, \ x \in J.$$

By setting x=t in the above inequality and integrating the resulting inequality over t from α to x, one can have:

$$z(x)\le\int_\alpha^x\left\{k(t,t)\left[A(t)+\int_\alpha^t B(s)e^{\int_s^t D(\tau)d\tau}ds\right]+\int_\alpha^t \frac{\partial}{\partial t}k(t,s)\left[A(s)+\int_\alpha^s B(\tau)e^{\int_\tau^s D(\vartheta)d\vartheta}d\tau\right]ds\right\}dt,\ x\in J.$$

Hence

$$y(x)\le y(\alpha)+\int_\alpha^x\left[a(t)+\int_\alpha^t\left\{k(s,s)\left[A(s)+\int_\alpha^s B(\tau)e^{\int_\tau^s D(\vartheta)d\vartheta}d\tau\right]+\int_\alpha^s\frac{\partial}{\partial s}k(s,\tau)\left[A(\tau)+\int_\alpha^\tau B(\vartheta)e^{\int_\vartheta^\tau D(q)dq}d\vartheta\right]d\tau\right\}ds\right]dt,\ x\in J.$$

The following theorem appeared in [Pachpatte B., 1978] without proof. Here we give its proof.

Theorem (2.3):-

Let y, y′, b and c be nonnegative continuous functions defined on J=[α,∞). If

$$y'(x)\le a+\int_\alpha^x b(t)\left[y(t)+y'(t)+\int_\alpha^t c(s)\{y(s)+y'(s)\}ds\right]dt,\ x\in J.$$

Then

$$y(x)\le y(\alpha)+a(x-\alpha)+\int_\alpha^x\int_\alpha^t\left[A(s)+b(s)B(s)+b(s)\int_\alpha^s c(\tau)B(\tau)d\tau\right]ds\,dt,\ x\in J.$$

where

$$A(x)=b(x)\left[y(\alpha)+a(1+x-\alpha)+\int_\alpha^x c(t)\{y(\alpha)+a(1+t-\alpha)\}dt\right].$$

and

$$B(x)=\int_\alpha^x A(t)e^{\int_t^x\left[1+b(s)+b(s)\int_\alpha^s c(\tau)d\tau\right]ds}dt,\ x\in J.$$

Proof:

Let $z(x)=\int_\alpha^x b(t)\left[y(t)+y'(t)+\int_\alpha^t c(s)\{y(s)+y'(s)\}ds\right]dt$,

Then $z(\alpha)=0$ and $y'(x)\le a+z(x)$, $x\in J$.

By setting x=t in the above inequality and integrating the resulting inequality over t from α to x, one can get:

$y(x)\le y(\alpha)+a(x-\alpha)+\int_\alpha^x z(t)dt$, $x\in J$.

Therefore

$$z'(x)\le b(x)\Big[y(\alpha)+a(x-\alpha+1)+z(x)+\int_\alpha^x z(t)dt+\int_\alpha^x c(t)\Big\{y(\alpha)+a(t-\alpha+1)+z(t)+\int_\alpha^t z(s)ds\Big\}dt\Big].$$

Let $v(x)=z(x)+\int_\alpha^x z(t)dt$, then $v(\alpha)=0$, $z(x)\le v(x)$ and

$z'(x)\le A(x)+b(x)v(x)+b(x)\int_\alpha^x c(t)v(t)dt$

where A(x) is defined previously.

Thus

$$v'(x)=z'(x)+z(x)$$
$$\le A(x)+[1+b(x)]v(x)+b(x)\int_\alpha^x c(t)v(t)dt$$

By using the fact that v is a nondecreasing function in x, one can get:

$v'(x)\le A(x)+\left[1+b(x)+b(x)\int_\alpha^x c(t)dt\right]v(x)$.

From lemma (1.1), one can have:

$v(x)\le B(x)$,

where B(x) is defined previously.

Then

$z(x)\le\int_\alpha^x\left[A(t)+b(t)B(t)+b(t)\int_\alpha^t c(s)B(s)ds\right]dt$, $x\in J$.

Hence

$$y(x)\le y(\alpha)+a(x-\alpha)+\int_{\alpha}^{x}\int_{\alpha}^{t}\left[A(s)+b(s)B(s)+b(s)\int_{\alpha}^{s}c(\tau)B(\tau)d\tau\right]dsdt,\ x\in J.$$

Next, a modification of theorem (2.3) can be given in the second part of the following theorem.

Theorem (2.4):-

Let y, y′, a, b and c be nonnegative continuous functions defined on J=[α,∞).

(i) If

$$y'(x)\le a(x)+\int_{\alpha}^{x}b(t)\left[y(t)+y'(t)+\int_{\alpha}^{t}c(s)y'(s)ds\right]dt,\ x\in J.$$

Then

$$y(x)\le y(\alpha)+$$

$$\int_{\alpha}^{x}a(t)dt+\int_{\alpha}^{x}\int_{\alpha}^{t}\left[A(s)+b(s)\int_{\alpha}^{s}A(\tau)e^{\int_{\tau}^{s}\left[1+b(\vartheta)+b(\vartheta)\int_{\alpha}^{\vartheta}c(q)dq\right]d\vartheta}d\tau\right]e^{\int_{s}^{t}b(\tau)\int_{\alpha}^{\tau}c(\vartheta)d\vartheta d\tau}dsdt,$$

where

$$A(x)=b(x)\left[y(\alpha)+a(x)+\int_{\alpha}^{x}a(t)dt+\int_{\alpha}^{x}c(t)a(t)dt\right].$$

(ii) If

$$y'(x)\le a(x)+\int_{\alpha}^{x}b(t)\left[y(t)+y'(t)+\int_{\alpha}^{t}c(s)\{y(s)+y'(s)\}ds\right]dt,\ x\in J,$$

then

$$y(x)\le y(\alpha)+\int_{\alpha}^{x}a(t)dt+\int_{\alpha}^{x}\int_{\alpha}^{t}\left[A(s)+b(s)B(s)+b(s)\int_{\alpha}^{s}c(\tau)B(\tau)d\tau\right]dsdt,\ x\in J$$

where

$$A(x)=b(x)\left[y(\alpha)+a(x)+\int_{\alpha}^{x}a(t)dt+\int_{\alpha}^{x}c(t)\left\{y(\alpha)+a(t)+\int_{\alpha}^{t}a(s)ds\right\}dt\right],$$

and

$$B(x)=\int_{\alpha}^{x} A(t)e^{\int_{t}^{x}\left[1+b(s)+b(s)\int_{\alpha}^{s} c(\tau)d\tau\right]ds}dt,\ x\in J.$$

Proof:-

(i) Let $z(x)=\int_{\alpha}^{x} b(t)\left[y(t)+y'(t)+\int_{\alpha}^{t} c(s)y'(s)ds\right]dt$

then $z(\alpha)=0$ and $y'(x)\le a(x)+z(x)$.

By setting x=t in the above inequality and integrating the resulting inequality over t from α to x, one can get:

$y(x)\le y(\alpha)+\int_{\alpha}^{x}[a(t)+z(t)]dt.$

Therefore

$z'(x)\le b(x)\left[y(\alpha)+a(x)+z(x)+\int_{\alpha}^{x}\{a(t)+c(t)a(t)+z(t)+c(t)z(t)\}dt\right].$

Let $v(x)=z(x)+\int_{\alpha}^{x} z(t)dt$ then

$$v'(x)\le A(x)+[1+b(x)]v(x)+b(x)\int_{\alpha}^{x} c(t)v(t)dt,\ x\in J.$$

where A(x) is defined previously.

By using the fact that v is a nondecreasing function in x, one can get:

$$v'(x)\le A(x)+\left[1+b(x)+b(x)\int_{\alpha}^{x} c(t)dt\right]v(x),\ x\in J.$$

From lemma (1.1), one can obtain:

$$v(x)\le\int_{\alpha}^{x} A(t)e^{\int_{t}^{x}\left[1+b(s)+b(s)\int_{\alpha}^{s} c(\tau)d\tau\right]ds}dt,\ x\in J.$$

Then

$$z'(x)\le A(x)+b(x)\int_{\alpha}^{x} A(t)e^{\int_{t}^{x}\left[1+b(s)+b(s)\int_{\alpha}^{s} c(\tau)d\tau\right]ds}dt+b(x)\int_{\alpha}^{x} c(t)z(t)dt,\ x\in J.$$

By using the fact that z is a nondecreasing function in x, one can have:

$$z(x)\le A(x)+b(x)\int_{\alpha}^{x} A(t)e^{\int_t^x \left[1+b(s)+b(s)\int_{\alpha}^{s} c(\tau)d\tau\right]ds} dt + b(x)z(x)\int_{\alpha}^{x} c(t)dt,\ x\in J.$$

From lemma (1.1), one can get:

$$z(x)\le \int_{\alpha}^{x}\left[A(t)+b(t)\int_{\alpha}^{t} A(s)e^{\int_s^t \left[1+b(\tau)+b(\tau)\int_{\alpha}^{\tau} c(\vartheta)d\vartheta\right]d\tau}\right] e^{\int_t^x b(s)\int_{\alpha}^{s} c(\tau)d\tau ds} dt,\ x\in J.$$

Hence

$y(x)\le y(\alpha)+$

$$\int_{\alpha}^{x} a(t)dt+\int_{\alpha}^{x}\int_{\alpha}^{t}\left[A(s)+b(s)\int_{\alpha}^{s} A(\tau)e^{\int_\tau^s \left[1+b(\vartheta)+b(\vartheta)\int_{\alpha}^{\vartheta} c(q)dq\right]d\vartheta} d\tau\right] e^{\int_s^t b(\tau)\int_{\alpha}^{\tau} c(\vartheta)d\vartheta d\tau} dsdt.$$

(ii) Let $z(x)=\int_{\alpha}^{x} b(t)\left[y(t)+y'(t)+\int_{\alpha}^{t} c(s)\{y(s)+y'(s)\}ds\right]dt$,

then

$y(x)\le y(\alpha)+\int_{\alpha}^{x} a(t)dt+\int_{\alpha}^{x} z(t)dt,\ x\in J.$

Therefore

$$z'(x)\le b(x)\Big[y(\alpha)+a(x)+z(x)+\int_{\alpha}^{x}[a(t)+z(t)]dt+\int_{\alpha}^{x} c(t)\Big\{y(\alpha)+a(t)+ z(t)+\int_{\alpha}^{t}[a(s)+z(s)]ds\Big\}dt\Big],\ x\in J.$$

Let $v(x)=z(x)+\int_{\alpha}^{x} z(t)dt$, then $z(x)\le v(x)$ and

$$z'(x)\le A(x)+b(x)v(x)+b(x)\int_{\alpha}^{x} c(t)v(t)dt,$$

where A(x) is defined previously.

Thus

$$v'(x)\le A(x)+[1+b(x)]v(x)+b(x)\int_{\alpha}^{x} c(t)v(t)dt,\ x\in J.$$

By using the fact that v is a nondecreasing function in x, one can get:

$$v'(x)\le A(x)+\left[1+b(x)+b(x)\int_{\alpha}^{x} c(t)dt\right]v(x).$$

From lemma (1.1), one can have:

$v(x)\le B(x)$

where B(x) is defined previously.

Then

$$z(x)\le\int_{\alpha}^{x}\left[A(t)+b(t)B(t)+b(t)\int_{\alpha}^{t} c(s)B(s)ds\right]dt.$$

Hence

$$y(x)\le y(\alpha)+\int_{\alpha}^{x} a(t)dt+\int_{\alpha}^{x}\int_{\alpha}^{t}\left[A(s)+b(s)B(s)+b(s)\int_{\alpha}^{s} c(\tau)B(\tau)d\tau\right]dsdt,\ x\in J.$$

Now, another modification of theorem (2.3) can be given in the second part of the following theorem.

Theorem (2.5):-

Let y, y′, b and c be nonnegative continuous functions defined on $J=[\alpha,\infty)$. Let k_1, k_2 and their partial derivatives $\frac{\partial k_1}{\partial x}$, $\frac{\partial k_2}{\partial x}$ be nonnegative continuous functions for $\alpha\le t\le x$.

(i) If

$$y'(x)\le a+\int_{\alpha}^{x} k_1(x,t)[y(t)+y'(t)]dt+\int_{\alpha}^{x}\int_{\alpha}^{t} k_2(t,s)y'(s)ds\,dt,\ x\in J.$$

Then

$$y(x)\le y(\alpha)+a(x-\alpha)+\int_{\alpha}^{x}\int_{\alpha}^{t}\left[A(s)+k_1(s,s)B(s)+\int_{\alpha}^{s}\frac{\partial}{\partial s}k_1(s,\tau)B(\tau)d\tau\right]e^{\int_{s}^{t}\int_{\alpha}^{\tau}k_2(\tau,\vartheta)d\vartheta d\tau}dsdt,$$

where

$$A(x)=k_1(x,x)\left[y(\alpha)+a(x-\alpha+1)\right]+\int_\alpha^x \frac{\partial}{\partial x}k_1(x,t)\left[y(\alpha)+a(x-\alpha+1)\right]dt+a\int_\alpha^x k_2(x,t)dt$$

and

$$B(x)=\int_\alpha^x A(t)e^{\int_t^x\left[1+k_1(s,s)+\int_\alpha^s \frac{\partial}{\partial s}k_1(s,\tau)d\tau+\int_\alpha^s k_2(s,\tau)d\tau\right]ds}dt\,.$$

(ii) If

$y'(x)\le a+\int_\alpha^x k_1(x,t)[y(t)+y'(t)]dt+\int_\alpha^x\int_\alpha^t k_2(t,s)[y(s)+y'(s)]ds\,dt,\ x\in J.$

Then

$$y(x)\le y(\alpha)+a(x-\alpha)+\int_\alpha^x\int_\alpha^t\left[A(s)+k_1(s,s)B(s)+\int_\alpha^s \frac{\partial}{\partial s}k_1(s,\tau)B(\tau)d\tau+\int_\alpha^s k_2(s,\tau)B(\tau)d\tau\right]ds\,dt\,,$$

where

$$A(x)=k_1(x,x)\left[y(\alpha)+a(x-\alpha+1)\right]+\left[y(\alpha)+a(t-\alpha+1)\right]\int_\alpha^x\left[\frac{\partial}{\partial x}k_1(x,t)+k_2(x,t)\right]dt$$

and

$$B(x)=\int_\alpha^x A(t)e^{\int_t^x\left[1+k_1(s,s)+\int_\alpha^s \frac{\partial}{\partial s}k_1(s,\tau)d\tau+\int_\alpha^s k_2(s,\tau)d\tau\right]ds}dt\,.$$

<u>Proof</u>:-

(i) Let

$z(x)=\int_\alpha^x k_1(x,t)[y(t)+y'(t)]dt+\int_\alpha^x\int_\alpha^t k_2(t,s)y'(s)ds\,dt,$

then $z(\alpha)=0$, $y'(x)\le a+z(x)$ and $y(x)\le y(\alpha)+a(x-\alpha)+\int_\alpha^x z(t)dt$.

Therefore

$z'(x)=k_1(x,x)[y(x)+y'(x)]+\int_\alpha^x \frac{\partial}{\partial x}k_1(x,t)[y(t)+y'(t)]dt+\int_\alpha^x k_2(x,t)y'(t)dt,$

and

$$z'(x)\le k_1(x,x)\left[y(\alpha)+a(x-\alpha+1)+z(x)+\int_\alpha^x z(t)dt\right]$$

$$+\int_\alpha^x \frac{\partial}{\partial x}k_1(x,t)\left[y(\alpha)+a(x-\alpha+1)+z(t)+\int_\alpha^t z(s)ds\right]dt$$

$$\int_\alpha^x k_2(x,t)[a+z(t)]dt,\ x\in J.$$

Thus

$$z'(x)=A(x)+\ k_1(x,x)\left[z(x)+\int_\alpha^x z(t)dt\right]+\int_\alpha^x \frac{\partial}{\partial x}k_1(x,t)\left[z(t)+\int_\alpha^t z(s)ds\right]dt$$

$$+\int_\alpha^x k_2(x,t)z(t)dt,$$

where A(x) is defined previously.

Let $v(x)=z(x)+\int_\alpha^x z(t)dt$, then $v(\alpha)=0$, $v'(x)=z'(x)+z(x)$ and

$$z'(x)\le A(x)+k_1(x,x)v(x)+\int_\alpha^x \frac{\partial}{\partial x}k_1(x,t)v(t)dt+\int_\alpha^x k_2(x,t)z(t)dt,\ \ x\in J.$$

Therefore

$$v'(x)\le A(x)+[1+k_1(x,x)]v(x)+\int_\alpha^x \frac{\partial}{\partial x}k_1(x,t)v(t)dt+b(x)\int_\alpha^x k_2(x,t)v(t)dt\,.$$

By using the fact that v is a nondecreasing function in x, one can get:

$$v'(x)\le A(x)+\left[1+k_1(x,x)+\int_\alpha^x \frac{\partial}{\partial x}k_1(x,t)dt+\int_\alpha^x k_2(x,t)dt\right]v(x).$$

From lemma (1.1), one can have:

$v(x)\le B(x),\ x\in J,$

where B(x) is defined previously.

Then

$$z'(x)\le A(x)+\left[k_1(x,x)B(x)+\int_\alpha^x \frac{\partial}{\partial x}k_1(x,t)B(t)dt+z(x)\int_\alpha^x k_2(x,t)dt\right].$$

From lemma (1.1), one can obtain:

$$z(x)\le\int_{\alpha}^{x}\left[A(t)+k_1(t,t)B(t)+\int_{\alpha}^{t}\frac{\partial}{\partial t}k_1(t,s)B(s)ds\right]e^{\int_t^x\int_\alpha^s k_2(s,\tau)d\tau ds}dt\,.$$

Hence

$$y(x)\le y(\alpha)+a(x-\alpha)+\int_{\alpha}^{x}\int_{\alpha}^{t}\left[A(s)+k_1(s,s)B(s)+\int_{\alpha}^{s}\frac{\partial}{\partial s}k_1(s,\tau)B(\tau)d\tau\right]e^{\int_s^t\int_\alpha^\tau k_2(\tau,\vartheta)d\vartheta d\tau}dsdt\,.$$

(ii) Let $z(x)=\int_\alpha^x k_1(x,t)[y(t)+y'(t)]dt+\int_\alpha^x\int_\alpha^t k_2(t,s)[y(s)+y'(s)]ds\,dt$, then $z(\alpha)=0$ and $y'(x)\le a+z(x)$.

By setting x=t in the above inequality and integrating the resulting inequality over t from α to x, one can get:

$$y(x)\le y(\alpha)+a(x-\alpha)+\int_{\alpha}^{x}z(t)dt\,.$$

Therefore

$$z'(x)\le k_1(x,x)\left[y(\alpha)+a(x-\alpha+1)+z(x)+\int_{\alpha}^{x}z(t)dt\right]+$$

$$\int_{\alpha}^{x}\frac{\partial}{\partial x}k_1(x,t)\left[y(\alpha)+a(t-\alpha+1)+z(t)+\int_{\alpha}^{t}z(s)ds\right]dt+$$

$$\int_{\alpha}^{x}k_2(x,t)\left[y(\alpha)+a(t-\alpha+1)+z(t)+\int_{\alpha}^{t}z(s)ds\right]dt\,.$$

Let $v(x)=z(x)+\int_{\alpha}^{x}z(t)dt$ then $v(\alpha)=0$, $z(x)\le v(x)$ and

$$z'(x)\le A(x)+k_1(x,x)v(x)+\int_{\alpha}^{x}\frac{\partial}{\partial x}k_1(x,t)v(t)dt+\int_{\alpha}^{x}k_2(x,t)v(t)dt\,,$$

where A(x) is defined previously.

Therefore

$v'(x)=z'(x)+z(x)$

$$v'(x) \le A(x) + [1 + k_1(x,x)]v(x) + \int_\alpha^x \frac{\partial}{\partial x} k_1(x,t)v(t)dt + \int_\alpha^x k_2(x,t)v(t)dt.$$

By using the fact that v is a nondecreasing function in x, one can get:

$$v'(x) \le A(x) + \left[1 + k_1(x,x) + \int_\alpha^x \frac{\partial}{\partial x} k_1(x,t)dt + \int_\alpha^x k_2(x,t)dt\right]v(x).$$

From lemma (1.1), one can have:

$v(x) \le B(x)$.

where B(x) is defined previously.

Then

$$z'(x) \le A(x) + k_1(x,x)B(x) + \int_\alpha^x \frac{\partial}{\partial x} k_1(x,t)B(t)dt + \int_\alpha^x k_2(x,t)B(t)dt.$$

and

$$z(x) \le \int_\alpha^x \left[A(t) + k_1(t,t)B(t) + \int_\alpha^t \frac{\partial}{\partial t} k_1(t,s)B(s)ds + \int_\alpha^t k_2(t,s)B(s)ds\right]dt.$$

Hence

$$y(x) \le y(\alpha) + a(x-\alpha) + \int_\alpha^x \int_\alpha^t \left[A(s) + k_1(s,s)B(s) + \int_\alpha^s \frac{\partial}{\partial s} k_1(s,\tau)B(\tau)d\tau + \int_\alpha^s k_2(s,\tau)B(\tau)d\tau\right]dsdt.$$

Now, explicit bounds for the unknown function that appeared in the one-dimensional second order Volterra linear integro-differential inequality can be seen below.

Theorem (2.6), [Pachpatte B., 1980]:

Let y, y', y'' and c be nonnegative continuous functions defined on $J=[\alpha,\infty)$. If

$$y''(x) \le a + \int_\alpha^x c(t)[y(t) + y'(t) + y''(t)]dt, \quad x \in J,$$

where a is a nonnegative constant.

Then

$$y(x)\le y(\alpha)+y'(\alpha)[x-\alpha]+\frac{a(x-\alpha)^2}{2}+$$

$$\int_\alpha^x\int_\alpha^t\int_\alpha^s\left\{A(\tau)+c(\tau)\int_\alpha^\tau\left[A(\vartheta)+\int_\alpha^\vartheta A(q)e^{\int_q^\vartheta[2+c(\gamma)]d\gamma}dq\right]e^{\int_\vartheta^\tau c(q)dq}d\vartheta\right\}d\tau ds dt,\ x\in J$$

where

$$A(x)=c(x)\left[y(\alpha)+y'(\alpha)\{x-\alpha+1\}+\frac{a[(x-\alpha)^2+2(x-\alpha)+2]}{2}\right].$$

Proof:

Let $z(x)=\int_\alpha^x c(t)[y(t)+y'(t)+y''(t)]dt$, then $y''(x)\le a+z(x)$,

$y'(x)\le y'(\alpha)+a(x-\alpha)+\int_\alpha^x z(t)dt$ and

$$y(x)\le y(\alpha)+y'(\alpha)[x-\alpha]+\frac{a(x-\alpha)^2}{2}\int_\alpha^x\int_\alpha^t z(s)ds\,dt.$$

Therefore

$$z'(x)\le c(x)\Bigg[y(\alpha)+y'(\alpha)\{x-\alpha+$$

$$1\}+\frac{a[(x-\alpha)^2+2(x-\alpha)+2]}{2}+z(x)+\int_\alpha^x z(t)dt+\int_\alpha^x\int_\alpha^t z(s)ds\,dt\Bigg].$$

$$=A(x)+c(x)\left[z(x)+\int_\alpha^x z(t)dt+\int_\alpha^x\int_\alpha^t z(s)dsdt\right].$$

where A(x) is defined previously.

Let $u(x)=z(x)+\int_\alpha^x z(t)dt+\int_\alpha^x\int_\alpha^t z(s)ds\,dt$,

then it is easy to check that

$$u'(x)\le A(x)+u(x)+c(x)u(x)+\int_\alpha^x u(t)dt,\ \ x\in J.$$

By setting $v(x)=u(x)+\int_{\alpha}^{x}u(t)dt$, then $u(x)\le v(x)$ and $u'(x)\le A(x)+c(x)u(x)+v(x)$.

Therefore

$v'(x)\le A(x)+[2+c(x)]v(x),\ x\in J.$

From lemma (1.1), one can obtain:

$$v(x)\le\int_{\alpha}^{x}A(t)e^{\int_{t}^{x}[2+c(s)]ds}dt,\ x\in J.$$

and this implies that

$$u'(x)\le A(x)+\int_{\alpha}^{x}A(t)e^{\int_{t}^{x}[2+c(s)]ds}dt+c(x)u(x).$$

From lemma (1.1), one can have:

$$u(x)\le\int_{\alpha}^{x}\left[A(t)+\int_{\alpha}^{t}A(s)e^{\int_{s}^{t}[2+c(\tau)]d\tau}ds\right]e^{\int_{t}^{x}c(s)ds}dt.$$

Then

$$z'(x)\le A(x)+c(x)\int_{\alpha}^{x}\left[A(t)+\int_{\alpha}^{t}A(s)e^{\int_{s}^{t}[2+c(\tau)]d\tau}ds\right]e^{\int_{t}^{x}c(s)ds}dt,$$

and this implies that

$$z(x)\le\int_{\alpha}^{x}\left\{A(t)+c(t)\int_{\alpha}^{t}\left[A(s)+\int_{\alpha}^{s}A(\tau)e^{\int_{\tau}^{s}[2+c(\vartheta)]d\vartheta}d\tau\right]e^{\int_{s}^{t}c(\tau)d\tau}ds\right\}dt.$$

Hence

$$y(x)\le y(\alpha)+y'(\alpha)[x-\alpha]+\frac{a(x-\alpha)^2}{2}+$$

$$\int_{\alpha}^{x}\int_{\alpha}^{t}\int_{\alpha}^{s}\left\{A(\tau)+c(\tau)\int_{\alpha}^{\tau}\left[A(\vartheta)+\int_{\alpha}^{\vartheta}A(q)e^{\int_{q}^{\vartheta}[2+c(\gamma)]d\gamma}dq\right]e^{\int_{\vartheta}^{\tau}c(q)dq}d\vartheta\right\}d\tau ds dt,\ x\in J.$$

Next, the following theorem is a generalization of the previous theorem.

Theorem (2.7):-

Let y, y′, y″, a and c be nonnegative continuous functions defined on $J=[\alpha,\infty)$. If

$y''(x)\le a(x)+\int_\alpha^x c(t)[y(t)+y'(t)+y''(t)]dt,\ x\in J.$

Then

$$y(x)\le y(\alpha)+y'(\alpha)[x-\alpha]+\int_\alpha^x\int_\alpha^t a(s)ds+$$

$$\int_\alpha^x\int_\alpha^t\int_\alpha^s\left\{A(\tau)+c(\tau)\int_\alpha^\tau\left[A(\vartheta)+\int_\alpha^\vartheta A(q)e^{\int_q^\vartheta[2+c(\gamma)]d\gamma}dq\right]e^{\int_\vartheta^\tau c(q)dq}d\vartheta\right\}d\tau ds dt,\ x\in J.$$

where

$$A(x)=c(x)\left[y(\alpha)+y'(\alpha)\{x-\alpha+1\}+a(x)+\int_\alpha^x a(t)dt+\int_\alpha^x\int_\alpha^t a(s)dsdt\right].$$

Proof:-

Let $z(x)=\int_\alpha^x c(t)[y(t)+y'(t)+y''(t)]dt$ then $z(\alpha)=0$ and

$y''(x)\le a(x)+z(x),\ \ y'(x)\le y'(\alpha)+\int_\alpha^x[a(t)+z(t)]dt,$ and

$y(x)\le y(\alpha)+y'(\alpha)[x-\alpha]+\int_\alpha^x\int_\alpha^t a(s)ds\,dt+\int_\alpha^x\int_\alpha^t z(s)dsdt,\ x\in J.$

Thus

$$z'(x)\le c(x)\Big[y(\alpha)+y'(\alpha)\{x-\alpha+1\}+a(x)+\int_\alpha^x a(t)dt+\int_\alpha^x\int_\alpha^t a(s)ds\,dt+$$
$$z(x)+\int_\alpha^x z(t)dt+\int_\alpha^x\int_\alpha^t z(s)ds\,dt\Big].$$
$$\le A(x)+c(x)\Big[z(x)+\int_\alpha^x z(t)dt+\int_\alpha^x\int_\alpha^t z(s)ds\,dt\Big],$$

where A(x) is defined previously.

Let $u(x)=z(x)+\int_\alpha^x z(t)dt+\int_\alpha^x\int_\alpha^t z(s)ds\,dt.$

Then it is easy to check that

$u'(x) \leq A(x) + u(x) + c(x)u(x) + \int_{\alpha}^{x} u(t)dt, \ x \in J.$

Let $v(x) = u(x) + \int_{\alpha}^{x} u(t)dt$ then $v(\alpha) = 0$, $u(x) \leq v(x)$ and

$u'(x) \leq A(x) + c(x)u(x) + v(x).$

Therefore

$v'(x) \leq A(x) + [2 + c(x)]v(x).$

From lemma (1.1), one can have:

$$v(x) \leq \int_{\alpha}^{x} A(t) e^{\int_t^x [2+c(s)]ds} dt, \ x \in J.$$

and this implies that:

$$u'(x) \leq A(x) + \int_{\alpha}^{x} A(t) e^{\int_t^x [2+c(s)]ds} dt + c(x)u(x), \ x \in J.$$

From lemma (1.1), one can get:

$$u(x) \leq \int_{\alpha}^{x} \left[A(t) + \int_{\alpha}^{t} A(s) e^{\int_s^t [2+c(\tau)]d\tau} ds\right] e^{\int_t^x c(s)ds} dt, \ x \in J.$$

Then

$$z'(x) \leq A(x) + c(x) \int_{\alpha}^{x} \left\{A(t) + \int_{\alpha}^{t} A(s) e^{\int_s^t [2+c(\tau)]d\tau} ds\right\} e^{\int_t^x c(s)ds} dt$$

and this implies that

$$z(x) \leq \int_{\alpha}^{x} \left\{ A(t) + c(t) \int_{\alpha}^{t} \left[A(s) + \int_{\alpha}^{s} A(\tau) e^{\int_{\tau}^{s} [2+c(\vartheta)]d\vartheta} d\tau \right] e^{\int_{s}^{t} c(\tau)d\tau} ds \right\} dt.$$

Hence

$$y(x) \leq y(\alpha) + y'(\alpha)[x-\alpha] + \int_{\alpha}^{x}\int_{\alpha}^{t} a(s)ds +$$

$$\int_{\alpha}^{x}\int_{\alpha}^{t}\int_{\alpha}^{s} \left\{ A(\tau) + c(\tau) \int_{\alpha}^{\tau} \left[A(\vartheta) + \int_{\alpha}^{\vartheta} A(q) e^{\int_{q}^{\vartheta} [2+c(\gamma)]d\gamma} dq \right] e^{\int_{\vartheta}^{\tau} c(q)dq} d\vartheta \right\} d\tau ds dt, \ x \in J.$$

Now, another extension of theorem (2.6) can be given below.

Theorem (2.8):-

Let y, y′ and y″ be nonnegative continuous functions defined on J=[α,∞). Let k and its partial derivative $\frac{\partial k}{\partial x}$ be nonnegative continuous functions for α≤t≤x. If

$y''(x) \le a + \int_\alpha^x k(x,t)[y(t) + y'(t) + y''(x)]\,dt, \; x \in J,$

where a is a nonnegative constant.

Then

$$y(x) \le y(\alpha) + y'(\alpha)[x-\alpha] + \frac{a[x-\alpha]^2}{2}$$

$$\int_\alpha^x \int_\alpha^t \int_\alpha^s \left[A(\tau) + k(\tau,\tau)B(\tau) + \int_\alpha^\tau \frac{\partial}{\partial \tau} k(\tau,\vartheta)B(\vartheta)\,d\vartheta \right] d\tau\,ds\,dt, \; x \in J,$$

where

$$A(x) = k(x,x)\left[y(\alpha) + y'(\alpha)\{x-\alpha+1\} + \frac{a[(x-\alpha)^2 + 2(x-\alpha) + 2]}{2} \right] +$$

$$\int_\alpha^x \frac{\partial}{\partial x} k(x,t)\left[y(\alpha) + y'(\alpha)\{t-\alpha+1\} + \frac{a[(t-\alpha)^2 + 2(t-\alpha) + 2]}{2} \right] dt$$

and

$$B(x) = \int_\alpha^x \left[A(t) + \int_\alpha^t A(s) e^{\int_s^t \left[2 + k(\tau,\tau) + \int_\alpha^\tau \frac{\partial}{\partial \tau} k(\tau,\vartheta)\,d\vartheta \right] d\tau} ds \right] e^{\int_t^x \left[k(s,s) + \int_\alpha^s \frac{\partial}{\partial s} k(s,\tau)\,d\tau \right] ds} dt.$$

Proof:-

Let $z(x) = \int_\alpha^x k(x,t)[y(t) + y'(t) + y''(t)]\,dt$ then z(α)=0, y″(x)≤a+z(x),

$y'(x) \le y'(\alpha) + a(x-\alpha) + \int_\alpha^x z(t)\,dt$ and

$$y(x) \le y(\alpha) + y'(\alpha)[x-\alpha] + \frac{a(x-\alpha)^2}{2} + \int_\alpha^x \int_\alpha^t z(s)\,ds\,dt.$$

Therefore

$$z'(x)\le k(x,x)\left[y(\alpha)+y'(\alpha)\{x-\alpha+1\}+\frac{a[(x-\alpha)^2+2(x-\alpha)+2]}{2}+z(x)+\int_\alpha^x z(t)dt+\int_\alpha^x\int_\alpha^t z(s)ds\,dt\right]$$

$$+\int_\alpha^x\frac{\partial}{\partial x}k(x,t)\left[y(\alpha)+y'(\alpha)\{t-\alpha+1\}+\frac{a[(t-\alpha)^2+2(t-\alpha)+2]}{2}+z(t)+\int_\alpha^t z(s)ds+\int_\alpha^t\int_\alpha^s z(\tau)d\tau\,ds\right]dt.$$

$$z'(x)\le A(x)+k(x,x)\left[z(x)+\int_\alpha^x z(t)dt+\int_\alpha^x\int_\alpha^t z(s)dsdt\right]+\int_\alpha^x\frac{\partial}{\partial x}k(x,t)\left[z(t)+\int_\alpha^t z(s)dt+\int_\alpha^t\int_\alpha^s z(\tau)d\tau ds\right]dt,\ x\in J.$$

Let $u(x)=z(x)+\int_\alpha^x z(t)dt+\int_\alpha^x\int_\alpha^t z(s)ds\,dt$.

Then it is easy to check that

$u'(x)\le A(x)+u(x)+k(x,x)u(x)+\int_\alpha^x\frac{\partial}{\partial x}k(x,t)u(t)dt+\int_\alpha^x u(t)dt.$

By setting $v(x)=u(x)+\int_\alpha^x u(t)dt$, one can have:

$v(\alpha)=0$, $u(x)\le v(x)$ and

$u'(x)\le A(x)+k(x,x)u(x)+\int_\alpha^x\frac{\partial}{\partial x}k(x,t)u(t)dt+v(x).$

Therefore

$v'(x)=u'(x)+u(x),$

$\le A(x)+[2+k(x,x)]v(x)+\int_\alpha^x\frac{\partial}{\partial x}k(x,t)v(t)dt,\ x\in J.$

By using the fact that v is a nondecreasing function in x, then one can obtain:

$v'(x)\le A(x)+\left[2+k(x,x)+\int_\alpha^x\frac{\partial}{\partial x}k(x,t)dt\right]v(x),\ x\in J.$

From lemma (1.1), one can get:

$$v(x)\le\int_{\alpha}^{x}A(t)e^{\int_t^x\left[2+k(s,s)+\int_\alpha^s\frac{\partial}{\partial s}k(s,\tau)d\tau\right]ds}dt,\ x\in J.$$

Thus

$$u'(x)\le A(x)+k(x,x)u(x)+\int_\alpha^x\frac{\partial}{\partial x}k(x,t)u(t)dt+\int_\alpha^x A(t)e^{\int_t^x\left[2+k(s,s)+\int_\alpha^s\frac{\partial}{\partial s}k(s,\tau)d\tau\right]ds}dt.$$

By using the fact that u is a nondecreasing function in x, one can have:

$$u'(x)\le A(x)+\int_\alpha^x A(t)e^{\int_t^x\left[2+k(s,s)+\int_\alpha^s\frac{\partial}{\partial s}k(s,\tau)d\tau\right]ds}dt+\left[k(x,x)+\int_\alpha^x\frac{\partial}{\partial x}k(x,t)dt\right]u(x)$$

From lemma (1.1), one can obtain:

$u(x)\le B(x),\ x\in J$

where B(x) is defined previously.

Then

$$z'(x)\le A(x)+k(x,x)B(x)+\int_\alpha^x\frac{\partial}{\partial x}k(x,t)B(t)dt,\ x\in J.$$

and

$$z(x)\le\int_\alpha^x\left[A(t)+k(t,t)B(t)+\int_\alpha^t\frac{\partial}{\partial t}k(t,s)B(s)ds\right]dt.$$

Hence

$$y(x)\le y(\alpha)+y'(\alpha)[x-\alpha]+\frac{a[x-\alpha]^2}{2}$$

$$\int_\alpha^x\int_\alpha^t\int_\alpha^s\left[A(\tau)+k(\tau,\tau)B(\tau)+\int_\alpha^\tau\frac{\partial}{\partial\tau}k(\tau,\vartheta)B(\vartheta)d\vartheta\right]d\tau ds dt,\ x\in J,$$

Next, explicit bounds for the function that appeared in another type of the one-dimensional second order Volterra linear integro-differential inequality can be given below.

Theorem (2.9):-

Let y, y′, y″ and a be nonnegative continuous functions defined on J=[α,∞). Let k and its partial derivative $\frac{\partial k}{\partial x}$ be nonnegative continuous functions for α≤t≤x. If

$$y''(x)\le a(x)+\int_\alpha^x k(x,t)[y(t)+y'(t)]dt,\quad x\in J.$$

Then

$$y(x)\le y(\alpha)+y'(\alpha)[x-\alpha]+\int_\alpha^x\int_\alpha^t\Big[a(s)+\int_\alpha^s\Big\{A(\tau)+k(\tau,\tau)\int_\alpha^\tau B(q)dq+\int_\alpha^\tau\frac{\partial}{\partial\tau}k(\tau,q)\Big[\int_\alpha^q B(\vartheta)d\vartheta\Big]dq\Big\}d\tau\Big]dsdt,$$

where

$$A(x)=k(x,x)\Big[y(\alpha)+y'(\alpha)\{x-\alpha+1\}+\int_\alpha^x a(t)dt+\int_\alpha^x\int_\alpha^t a(s)dsdt\Big]+\int_\alpha^x\frac{\partial}{\partial x}k(x,t)\Big[y(\alpha)+y'(\alpha)\{t-\alpha+1\}+\int_\alpha^t a(s)ds+\int_\alpha^t\int_\alpha^s a(\tau)d\tau ds\Big]dt$$

and

$$B(x)=\int_\alpha^x A(t)e^{\int_t^x\left[1+(s-\alpha)k(s,s)+\int_\alpha^s\frac{\partial}{\partial s}k(s,\tau)(\tau-\alpha)d\tau\right]ds}dt.$$

Proof:-

Let $z(x)=\int_\alpha^x k(x,t)[y(t)+y'(t)]dt$, then z(α)=0, y″(x)≤a(x)+z(x).

Therefore

$$y'(x)\le y'(\alpha)+\int_\alpha^x a(t)dt+\int_\alpha^x z(t)dt$$

and

$$y(x)\le y(\alpha)+y'(\alpha)\{x-\alpha\}+\int_\alpha^x\int_\alpha^t[a(s)+z(s)]dsdt,\quad x\in J.$$

Since z is a nondecreasing function in x for x∈J then

$z'(x) \le A(x)+$

$$k(x,x)\left[\int_\alpha^x\left[z(t)+\int_\alpha^t z(s)ds\right]dt+\int_\alpha^x \frac{\partial}{\partial x}k(x,t)\left[\int_\alpha^t\{z(s)+\int_\alpha^s z(\tau)d\tau\}ds\right]dt\right],$$

where A(x) is defined previously.

Let $v(x)=z(x)+\int_\alpha^x z(t)dt$,

then

$$z'(x) \le A(x)+k(x,x)\int_\alpha^x v(t)dt+\int_\alpha^x \frac{\partial}{\partial x}k(x,t)\left[\int_\alpha^t v(s)ds\right]dt,\ z(x) \le v(x)$$

and

$v'(x)=z'(x)+z(x)$,

Therefore

$$v'(x) \le A(x)+k(x,x)\int_\alpha^x v(t)dt+\int_\alpha^x \frac{\partial}{\partial x}k(x,t)\left[\int_\alpha^t v(s)ds\right]dt+v(x).$$

By using the fact that v is a nondecreasing function in x, one can have:

$$v'(x) \le A(x)+\left[1+(x-\alpha)k(x,x)+\int_\alpha^x \frac{\partial}{\partial x}k(x,t)(t-\alpha)dt\right]v(x).$$

From lemma (1.1), one can obtain:

$v(x) \le B(x)$,

where

$$B(x)=\int_\alpha^x A(t)e^{\int_t^x\left[1+(s-\alpha)k(s,s)+\int_\alpha^s \frac{\partial}{\partial s}k(s,\tau)(\tau-\alpha)d\tau\right]ds}dt.$$

Then

$$z(x) \le \int_\alpha^x\left[A(t)+k(t,t)\int_\alpha^t B(s)ds+\int_\alpha^t \frac{\partial}{\partial t}k(t,s)\left[\int_\alpha^s B(\tau)d\tau\right]ds\right]dt,\ x\in J.$$

Therefore by substituting this result in the estimate of y(x) one can get the desired result.

2.3 Solutions of the One-Dimensional Volterra Non-linear Integro-Differential Inequalities:

As seen before in the previous section, all the previous integro-differential inequalities are of the linear type. Here we give some theorems which determine explicit bounds for the unknown function in special types of the one-dimensional Volterra non-linear integro-differential inequalities of the first and second order.

First we give the following lemma which will be needed later.

Lemma (2.1), [Bainov D. and Simeonov P., 1992, p.38]:-

Let y be a positive differentiable function satisfying the inequality $y'(x)\leq a(x)y(x)+b(x)y^{p}(x),\ x\in J=[\alpha,\beta]$

and

$y(\alpha)\leq y_o$,

where a and b are continuous functions defined on J and $p\neq 1$ is a nonnegative integer.

Then

$$y(x)\leq e^{\int_{\alpha}^{x} a(t)dt}\left[y_o^q+q\int_{\alpha}^{x} b(t)e^{-q\int_{\alpha}^{t} a(s)ds}dt\right]^{\frac{1}{q}},\ x\in[\alpha,\beta_1),$$

where $q=1-p$ and β_1 is chosen such that the expression between [...] is positive in the subinterval $[\alpha,\beta_1)$.

Now the following theorem determined explicit bounds for the unknown function for special types of the one-dimensional Volterra nonlinear integro-differential inequalities of the first order.

Theorem (2.10), [Pachpatte B., 1977b]:-

Let y, y′ and b be nonnegative continuous functions defined on $J=[\alpha,\infty)$.

If

$$y'(x)\le a+\int_{\alpha}^{x} b(t)y'(t)[y(t)+y'(t)]dt, \text{ for } x\in J,$$

where a is a positive constant.

Then

$$y(x)\le y(\alpha)+a\int_{\alpha}^{x} e^{[a+y(\alpha)]\int_{\alpha}^{t} b(s)[E(s)]^{-1}e^{s}ds}dt,\ x\in[\alpha,\infty),$$

where

$$E(x)=e^{\alpha}-[a+y(\alpha)]\int_{\alpha}^{x} b(t)e^{t}dt,$$

Proof:-

Let $z(x)=a+\int_{\alpha}^{x} b(t)y'(t)[y(t)+y'(t)]dt$ then $z(\alpha)=a$ and $y'(x)\le z(x)$.

By setting x=t in the above inequality and integrating the resulting inequality over t from α to x, one can get:

$$y(x)\le y(\alpha)+\int_{\alpha}^{x} z(t)dt.$$

Therefore

$$z'(x)=b(x)y'(x)[y(x)+y'(x)],$$

$$\le b(x)z(x)\left[y(\alpha)+z(x)+\int_{\alpha}^{x} z(t)dt\right].$$

Let $v(x)=y(\alpha)+z(x)+\int_{\alpha}^{x} z(t)dt$ then $z(x)\le v(x)$ and $z'(x)\le b(x)z(x)v(x)$.

Therefore

$$v'(x)=z'(x)+z(x),$$

$$\le b(x)v^{2}(x)+v(x).$$

From lemma (2.1) one can have:

$v(x)\le e^{x}[a+y(\alpha)][E(x)]^{-1}$, $x\in[\alpha,\infty)$,

where E(x) is defined previously.

Then

$z'(x)\le b(x)e^{x}[a+y(\alpha)][E(x)]^{-1}z(x)$, $x\in[\alpha,\infty)$.

From lemma (1.1), one can obtain:

$z(x)\le a e^{[a+y(\alpha)]\int_{\alpha}^{x} e^{t}b(t)[E(t)]^{-1}dt}$, $x\in[\alpha,\infty)$

Hence

$y(x)\le y(\alpha)+a\int_{\alpha}^{x} e^{[a+y(\alpha)]\int_{\alpha}^{t} e^{s}b(s)[E(s)]^{-1}ds}dt$, $x\in[\alpha,\infty)$.

Next, the following theorem is a generalization of the previous theorem.

Theorem (2.11):-

Let y and y' be nonnegative continuous functions defined on $J=[\alpha,\infty)$ and $a>0$ is a constant. Let k and its partial derivative $\frac{\partial k}{\partial x}$ be nonnegative continuous functions for $\alpha\le t\le x$. If

$y'(x)\le a+\int_{\alpha}^{x} k(x,t)y'(t)[y(t)+y'(t)]dt$, $x\in J$,

Then

$y(x)\le y(\alpha)+a\int_{\alpha}^{x} e^{[a+y(\alpha)]\int_{\alpha}^{t}\left[k(s,s)e^{s}[E(s)]^{-1}+\int_{\alpha}^{s}\frac{\partial}{\partial s}k(s,\vartheta)e^{\vartheta}[E(\vartheta)]^{-1}d\vartheta\right]ds}dt$, $x\in[\alpha,\infty)$,

where

$E(x)=e^{\alpha}-[a+y(\alpha)]\int_{\alpha}^{x} e^{t}\left[k(t,t)+\int_{\alpha}^{t}\frac{\partial}{\partial t}k(t,s)ds\right]dt$.

Proof:-

Let $z(x)=a+\int_{\alpha}^{x} k(x,t)y'(t)[y(t)+y'(t)]dt$ then $z(\alpha)=a$ and $y(x)\le y(\alpha)+\int_{\alpha}^{x} z(t)dt$.

Therefore

$z'(x)=k(x,x)y'(x)[y(x)+y'(x)]+\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)y'(t)[y(t)+y'(t)]dt$,

$z'(x) \le k(x,x)z(x)\left[y(\alpha) + z(x) + \int_\alpha^x z(t)dt\right] + \int_\alpha^x \frac{\partial}{\partial x} k(x,t)z(t)\left[y(\alpha) + z(t) + \int_\alpha^t z(s)ds\right]dt.$

Let $v(x)=y(\alpha)+z(x)+\int_\alpha^x z(t)dt$ then $v(\alpha)=a+y(\alpha)$, $z(x)\le v(x)$ and

$z'(x) \le k(x,x)z(x)v(x)+\int_\alpha^x \frac{\partial}{\partial x} k(x,t)z(t)v(t)dt.$

Thus

$$v'(x)=z'(x)+z(x),$$
$$\le k(x,x)v^2(x)+\int_\alpha^x \frac{\partial}{\partial x} k(x,t)v^2(t)dt + v(x).$$

By using the fact that v is a nondecreasing function in x, one can get: $v'(x)\le v(x)+\left[k(x,x) + \int_\alpha^x \frac{\partial}{\partial x} k(x,t)dt\right] v^2(x).$

From lemma (2.1) one can have:

$v(x)\le[a + y(\alpha)]e^x[E(x)]^{-1}$, $x\in[\alpha,\beta)$,

then

$z'(x)\le k(x,x)[a + y(\alpha)]e^x[E(x)]^{-1}z(x) + \int_\alpha^x \frac{\partial}{\partial x} k(x,t)e^t[E(t)]^{-1} z(t)dt.$

By using the fact that z is a nondecreasing function in x, one can have: $z'(x)\le\left[k(x,x)\{a + y(\alpha)\}e^x[E(x)]^{-1} + \int_\alpha^x \frac{\partial}{\partial x} k(x,t)e^t[E(t)]^{-1} dt\right]z(x).$

From lemma (1.1), one can obtain:

$z(x)\le a e^{\int_\alpha^x\left[k(t,t)[a+y(\alpha)]e^t[E(t)]^{-1}+\int_\alpha^t\frac{\partial}{\partial t}k(t,s)e^s[E(s)]^{-1}ds\right]dt}$, $x\in[\alpha,\infty)$,

Hence

$y(x)\le y(\alpha)+\int_\alpha^x a\, e^{\int_\alpha^t\left[k(s,s)[a+y(\alpha)]e^s[E(s)]^{-1}+\int_\alpha^s\frac{\partial}{\partial s}k(s,\tau)e^\tau[E(\tau)]^{-1}d\tau\right]ds}dt$, $x\in[\alpha,\infty)$.

Now, solutions of another type of the one-dimensional first order linear Volterra non-linear integro-differential inequality are given below.

Theorem (2.12), [Pachpatte B., 1977]:-

Let y, y′ and b be nonnegative continuous functions defined on $J=[\alpha,\infty)$ and $b(x)\geq 1$ for each $x\in J$. Let w be a positive continuous nondecreasing function defined on J and $a\geq 0$ be a constant. If

$$y'(x)\leq a+\int_{\alpha}^{x} b(t)w[y(t)+y'(t)]dt,\ x\in J.$$

Then for $\alpha\leq x\leq x_1$

$$y(x)\leq y(\alpha)+a[x-\alpha]+\int_{\alpha}^{x}\int_{\alpha}^{t} b(s)w\{G^{-1}[G(A(s))+\int_{\alpha}^{s} b(\tau)d\tau]\}dsdt,$$

where

$$A(x)=y(\alpha)+a[x-\alpha+1]$$

and

$$G(u)=\int_{u_0}^{u}\frac{dt}{t+w(t)},\ u>0,\ u_0>0 \qquad (2.3)$$

and G^{-1} is the inverse of G, and $x_1\in J$ is chosen such that $G(A(x))+\int_{\alpha}^{x} b(t)dt$ is in the domain of G^{-1} for $x\in J$ lying in the interval $[\alpha,x_1]$.

Proof:-

Let $z(x)=\int_{\alpha}^{x} b(t)w[y(t)+y'(t)]dt$ then $z(\alpha)=0$, $y'(x)\leq a+z(x)$ and $y(x)\leq y(\alpha)+a[x-\alpha]+\int_{\alpha}^{x} z(t)dt$.

Therefore

$$z'(x)=b(x)w[y(x)+y'(x)]$$
$$\leq b(x)w[y(\alpha)+a[x-\alpha+1]+z(x)+\int_{\alpha}^{x} z(t)dt]$$
$$=b(x)w[A(x)+z(x)+\int_{\alpha}^{x} z(t)dt],$$

where A(x) is defined previously.

Let $u\in R_+$ be any arbitrary number then for $\alpha\leq x\leq u$, one can obtain:

$$z'(x)\leq b(x)w[A(u)+z(x)+\int_{\alpha}^{x} z(t)dt].$$

Let $v(x)=A(u)+z(x)+\int_{\alpha}^{x} z(t)dt$ then $v(\alpha)=A(u)$ and

$z'(x)\leq b(x)w(v(x))$.

By using the above inequality and the fact that $z(x)\leq v(x)$ in the equation

$v'(x)=z'(x)+z(x)$,

$\leq b(x)w(v(x))+v(x)$,

$\leq b(x)[v(x)+w(v(x))]$

By dividing the above inequality by $[v(x)+w(v(x))]$ and integrating the resulting inequality from α to x and using equation (2.3), one can get:

$G(v(x))\leq G(A(u))+\int_{\alpha}^{x} b(t)dt$.

Then

$z'(x)\leq b(x)w\{G^{-1}[G(A(u))+\int_{\alpha}^{x} b(t)dt]\}$.

By integrating the above inequality from α to x, one can have:

$z(x)\leq\int_{\alpha}^{x} b(t)w\left\{G^{-1}\left[G(A(u))+\int_{\alpha}^{t} b(s)ds\right]\right\}dt$.

Since u is an arbitrary number then

$z(x)\leq\int_{\alpha}^{x} b(t)w\left\{G^{-1}\left[G(A(x))+\int_{\alpha}^{t} b(s)ds\right]\right\}dt$.

Hence

$y(x)\leq y(\alpha)+a[x-\alpha]+\int_{\alpha}^{x}\int_{\alpha}^{t} b(s)w\{G^{-1}[G(A(s))+\int_{\alpha}^{s} b(\tau)d\tau]\}dsdt$.

Now, an extension of theorem (2.12) can be seen below.

Theorem (2.13):-

Let y, y′, a and b be nonnegative continuous functions defined on $J=[\alpha,\infty)$ and $b(x)\geq 1$. Let w be a positive continuous nondecreasing function defined on J. If

$y'(x)\leq a(x)+\int_{\alpha}^{x} b(t)w[y(t)+y'(t)]dt,\ x\in J$.

Then

$y(x)\le y(\alpha)+\int_\alpha^x \left\{a(t)+\int_\alpha^t b(s)w\left[G^{-1}\left\{G\left(A(s)\right)+\int_\alpha^s b(\tau)d\tau\right\}\right]ds\right\}dt,$

where

$A(x)=y(\alpha)+a(x)+\int_\alpha^x a(t)dt,$

G and G^{-1} are defined previously and $x_1\in J$ is chosen such that $G(A(x))+\int_\alpha^x b(t)dt$ is in the domain of G^{-1} for all $x\in J$ lying in the interval $[\alpha,x_1]$.

<u>Proof</u>:-

Let $z(x)=\int_\alpha^x b(t)w[y(t)+y'(t)]dt$ then $z(\alpha)=0$ and

$y(x)\le y(\alpha)+\int_\alpha^x a(t)dt+\int_\alpha^x z(t)dt.$

Therefore

$z'(x)=b(x)w[y(x)+y'(x)],$

and

$z'(x)\le b(x)w\left[y(\alpha)+a(x)+\int_\alpha^x a(t)dt+z(x)+\int_\alpha^x z(t)dt\right].$

$\le b(x)w\left[A(x)+z(x)+\int_\alpha^x z(t)dt\right].$

where A(x) is defined previously.

Let $u\in R_+$ be any arbitrary number then for $\alpha\le x\le u$, one can obtain:

$z'(x)\le b(x)w\left[A(u)+z(x)+\int_\alpha^x z(t)dt\right].$

Let $v(x)=A(u)+z(x)+\int_\alpha^x z(t)dt$ then $v(\alpha)=A(u)$ and

$z'(x)\le b(x)w(v(x)).$

By using the above inequality and the fact that $z(x)\le v(x)$ one can have: $v'(x)=z'(x)+z(x),$

$\le b(x)w(v(x))+v(x),$

$\le b(x)[v(x)+w(v(x))]$

By dividing the above inequality by $[v(x)+w(v(x))]$ then integrating the resulting inequality from α to x and using equation (2.3), one can get:

$G(v(x))\le G(A(u))+\int_\alpha^x b(t)dt.$

Then

$z'(x)\le b(x)w\{G^{-1}[G(A(u))+\int_\alpha^x b(t)dt]\}.$

By integrating the above inequality from α to x, one can have:

$z(x)\le\int_\alpha^x b(t)w\left\{G^{-1}\left[G(A(u))+\int_\alpha^t b(s)ds\right]\right\}dt.$

Since u is an arbitrary number then

$z(x)\le\int_\alpha^x b(t)w\left\{G^{-1}\left[G(A(x))+\int_\alpha^t b(s)ds\right]\right\}dt.$

Hence

$y(x)\le y(\alpha)+\int_\alpha^x \left\{a(t)+\int_\alpha^t b(s)w\left[G^{-1}\left(G(A(s)+\int_\alpha^s b(\tau)d\tau\right)\right]ds\right\}dt.$

Now, the following theorem appeared in [Pachpatte B.,1998, p.176] without proof. Here we give its proof.

Theorem (2.14):-

Let y, y′ and c be nonnegative continuous functions on J=[α,∞). Let w be a positive continuous nondecreasing function defined on J. Let a≥0 and b≥0 be constants. If

$y'(x)\le a+b\left[y(x)+\int_\alpha^x c(t)w[y(t)+y'(t)]dt\right],\ x\in J.$

then

$$y(x)\le y(\alpha)+a[x-\alpha]+\int_\alpha^x \left[by(\alpha)e^{\int_\alpha^t b ds}+\int_\alpha^t\left(A(s)+bc(s)w\left[G^{-1}\left\{G(by(\alpha))+\int_\alpha^s[A(\tau)+(1+b)c(\tau)]d\tau\right\}\right]\right)e^{\int_s^t bd\tau}ds\right]dt,\ \alpha\le x\le x_1$$

where

$A(x)=b[a+c(x)w\{y(\alpha)+a(x-\alpha+1)\}],$

G and G^{-1} are defined previously and $x_1 \in J$ is chosen such that $G[a + (1+b)y(\alpha)] + (1+b)\int_\alpha^x c(t)dt$ is in the domain of G^{-1} for all $x \in J$ lying in the interval $[\alpha, x_1]$.

Proof:-

Let $z(x)=b\left[y(x) + \int_\alpha^x c(t)w[y(t) + y'(t)]dt\right]$ then $z(\alpha)=by(\alpha)$ and $y(x) \le y(\alpha)+a[x-\alpha]+ \int_\alpha^x z(t)dt$.

Therefore

$$z'(x)=b\left[y'(x) + c(x)w[y(x) + y'(x)]\right].$$

$$\le b\left[a + z(x) + c(x)w\left\{y(\alpha) + a\{x - \alpha + 1\} + z(x) + \int_\alpha^x z(t)dt\right\}\right].$$

$$\le b[a + c(x)w\{y(\alpha) + a[x - \alpha + 1]\}]+b\left[z(x) + c(x)w\left\{z(x) + \int_\alpha^x z(t)dt\right\}\right].$$

$$=A(x)+b\left[z(x) + c(x)w\left\{z(x) + \int_\alpha^x z(t)dt\right\}\right],$$

where A(x) is defined previously.

Let $v(x)=z(x)+\int_\alpha^x z(t)dt$ then $v(\alpha)=by(\alpha)$, $z(x) \le v(x)$ and

$$z'(x) \le A(x)+b[z(x) + c(x)w(v(x))].$$

Thus

$$v'(x)=z'(x)+z(x),$$

$$\le A(x)+(1+b)v(x)+bc(x)w(v(x)).$$

$$\le A(x)+(1+b)c(x)[v(x) + w(v(x))].$$

By dividing the above inequality by $[v(x) + w(v(x))]$, one can have:

$$\frac{v'(x)}{[v(x) + w(v(x))]} \le \frac{A(x)}{[v(x) + w(v(x))]} + (1+b)c(x),$$

$$\le A(x)+(1+b)c(x).$$

Then

$$v(x) \le G^{-1}\left\{G(by(\alpha) + \int_\alpha^x [A(t) + (1+b)c(t)]dt\right\},$$

and

$z'(x) \leq A(x) + bc(x)w\left[G^{-1}\left\{G\left(by(\alpha)\right) + \int_{\alpha}^{x}[A(t) + (1+b)c(t)]dt\right\}\right] + bz(x).$

From lemma (1.1), one can get:

$$z(x) \leq by(\alpha)e^{\int_{\alpha}^{x} bdt} + \int_{\alpha}^{x}\left(A(t) + bc(t)w\left[G^{-1}\left\{G\left(by(\alpha)\right) + \int_{\alpha}^{t}[A(s) + (1+b)c(s)]ds\right\}\right]\right)e^{\int_{t}^{x} bds}dt.$$

Hence

$$y(x) \leq y(\alpha) + a[x-\alpha] + \int_{\alpha}^{x}\left[by(\alpha)e^{\int_{\alpha}^{t} bds} + \int_{\alpha}^{t}\left(A(s) + bc(s)w\left[G^{-1}\left\{G\left(by(\alpha)\right) + \int_{\alpha}^{s}[A(\tau) + (1+b)c(\tau)]d\tau\right\}\right]\right)e^{\int_{s}^{t} bd\tau}ds\right]dt.$$

Next, the following two theorems are generalizations of the previous theorem.

Theorem (2.15):-

Let y, y′, a and c be nonnegative continuous functions defined on $J=[\alpha,\infty)$. Let w be a positive continuous nondecreasing function defined on J and $b \geq 0$ be a constant. If

$y'(x) \leq a(x) + b\left[y(x) + \int_{\alpha}^{x} c(t)w[y(t) + y'(t)]dt\right]$, $x \in J$.

Then

$$y(x) \leq y(\alpha) + \int_{\alpha}^{x}\left[a(t) + by(\alpha)e^{\int_{\alpha}^{t} bds} + \int_{\alpha}^{t}\left(A(s) + bc(s)w\left[G^{-1}\left\{G\left(by(\alpha)\right) + \int_{\alpha}^{s}[A(\tau) + (1+b)c(\tau)]d\tau\right\}\right]\right)e^{\int_{s}^{t} bd\tau}ds\right]dt, \quad \alpha \leq x \leq x_1$$

where

$A(x) = b\left[a(x) + c(x)w\left(y(\alpha) + a(x) + \int_{\alpha}^{x} a(t)dt\right)\right]$,

where G and G^{-1} are as defined in theorem (2.12) and $x_1 \in J$ is chosen such that $G\left(by(\alpha)\right) + \int_{\alpha}^{x}[A(t) + (1+b)c(t)]dt$ is in the domain of G^{-1} for all $x \in J$ lying in the interval $\alpha \leq x \leq x_1$.

Proof:-

Let $z(x)=b\left[y(x)+\int_{\alpha}^{x} c(t)w[y(t)+y'(t)]dt\right]$ then $z(\alpha)=by(\alpha)$

and

$y(x)\leq y(\alpha)+\int_{\alpha}^{x} a(t)dt+\int_{\alpha}^{x} z(t)dt.$

Therefore

$z'(x)=b\left[y'(x)+c(x)w[y(x)+y'(x)]\right].$

$\leq b\left[a(x)+z(x)+c(x)w\left(y(\alpha)+a(x)+\int_{\alpha}^{x} a(t)dt+z(x)+\int_{\alpha}^{x} z(t)dt\right)\right].$

$\leq b\left[a(x)+c(x)w\left\{y(\alpha)+a(x)+\int_{\alpha}^{x} a(t)dt\right\}\right]+b\left[z(x)+c(x)w\left\{z(x)+\int_{\alpha}^{x} z(t)dt\right\}\right].$

Thus

$z'(x)\leq A(x)+b\left[z(x)+c(x)w\left\{z(x)+\int_{\alpha}^{x} z(t)dt\right\}\right].$

where A(x) is defined previously.

Let $v(x)=z(x)+\int_{\alpha}^{x} z(t)dt$ then $v(\alpha)=by(\alpha)$, $z(x)\leq v(x)$ and

$z'(x)\leq A(x)+b[z(x)+c(x)w(v(x))].$

Thus

$v'(x)=z'(x)+z(x),$

$\leq A(x)+(1+b)v(x)+bc(x)w(v(x)).$

$\leq A(x)+(1+b)c(x)[v(x)+w(v(x))].$

By dividing the above inequality by $[v(x)+w(v(x))]$, one can have:

$$\frac{v'(x)}{[v(x)+w(v(x))]}\leq \frac{A(x)}{[v(x)+w(v(x))]}+(1+b)c(x),$$

$$\leq A(x)+(1+b)c(x).$$

Then

$v(x)\leq G^{-1}\left\{G(by(\alpha)+\int_{\alpha}^{x}[A(t)+(1+b)c(t)]dt\right\},$

and

$z'(x)\leq A(x)+bc(x)w\left[G^{-1}\left\{G\left(by(\alpha)\right)+\int_{\alpha}^{x}[A(t)+(1+b)c(t)]dt\right\}\right]+bz(x).$

From lemma (1.1), one can get:

$$z(x)\le by(\alpha)e^{\int_\alpha^x bdt}+\int_\alpha^x\Big(A(t)+bc(t)w\Big[G^{-1}\Big\{G\big(by(\alpha)\big)+\int_\alpha^t[A(s)+(1+b)c(s)]ds\Big\}\Big]\Big)e^{\int_t^x bds}dt.$$

Hence

$$y(x)\le y(\alpha)+\int_\alpha^x\Big[a(t)+by(\alpha)e^{\int_\alpha^t bds}+\int_\alpha^t\big(A(s)+bc(s)w\big[G^{-1}\big\{G\big(by(\alpha)\big)+\int_\alpha^s[A(\tau)+(1+b)c(\tau)]d\tau\big\}\big]\big)e^{\int_s^t bd\tau}ds\Big]dt.$$

<u>Theorem (2.16)</u>:-

Let y and y′ be nonnegative continuous functions defined on J=[α,∞). Let w be a positive continuous nondecreasing function defined on J. Let k and its partial derivative $\frac{\partial k}{\partial x}$ be nonnegative continuous functions defined on J.

If

$$y'(x)\le a+b\Big[y(x)+\int_\alpha^x k(x,t)w[y(t)+y'(t)]dt\Big],\ x\in J.$$

Then

$$y(x)\le y(\alpha)+a[x-\alpha]+\int_\alpha^x\left[by(\alpha)e^{\int_\alpha^t bds}+\int_\alpha^t\left[A(s)+b\left\{k(s,s)+\int_\alpha^s\frac{\partial}{\partial s}k(s,\tau)d\tau\right\}w(B(s))\right]e^{\int_s^t bd\tau}ds\right]dt,$$

where

$$A(x)=b\left[a+k(x,x)w\{y(\alpha)+a(x-\alpha+1)\}+\int_\alpha^x\frac{\partial}{\partial x}k(x,t)w\{y(\alpha)+a(x-\alpha+1)\}dt\right],$$

$$B(x)=G^{-1}\left[G(by(\alpha))+\int_\alpha^x\left\{A(x)+(b+1)\left[k(t,t)+\int_\alpha^t\frac{\partial}{\partial t}k(t,s)ds\right]\right\}dt\right],$$

G and G^{-1} are defined previously and $x_1\in J$ is chosen such that $G\big(A(u)+by(\alpha)\big)+\int_\alpha^x(b+1)\Big[1+k(t,t)+\int_\alpha^t\frac{\partial}{\partial t}k(t,s)ds\Big]$ is in the domain of G^{-1} for all $x\in J$ lying in the interval $[\alpha,x_1]$.

<u>Proof</u>:-

Let $z(x)=b\left[y(x)+\int_{\alpha}^{x} k(x,t)w[y(t)+y'(t)]dt\right]$ then $z(\alpha)=by(\alpha)$

and

$y(x)\leq y(\alpha)+a[x-\alpha]+\int_{\alpha}^{x} z(t)dt.$

Therefore

$$z'(x)=b\left[y'(x)+k(x,x)w[y(x)+y'(x)]+\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)w[y(t)+y'(t)]dt\right],$$

$$\leq b\left[a+z(x)+k(x,x)w\{y(\alpha)+a[x-\alpha+1]+z(x)+\int_{\alpha}^{x}z(t)dt\}+\right.$$

$$\left.\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)w\left[y(\alpha)+a[x-\alpha+1]+z(t)+\int_{\alpha}^{t}z(s)ds\right]\right]$$

$$=A(x)+b\left[z(x)+k(x,x)w\left\{z(x)+\int_{\alpha}^{x}z(t)dt\right\}+\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)w\left\{z(t)+\int_{\alpha}^{t}z(s)ds\right\}dt\right]$$

where A(x) is defined previously.

Let $v(x)=z(x)+\int_{\alpha}^{x}z(t)dt$ then $v(\alpha)=by(\alpha)$, $z(x)\leq v(x)$ and

$$z'(x)\leq A(x)+b[z(x)+k(x,x)w(v(x))]+\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)w\big(v(t)\big)dt.$$

By using the fact that w is a nondecreasing function, one can get:

$$z'(x)\leq A(x)+bz(x)+b\left[k(x,x)+\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)dt\right]w(v(x)).$$

Thus

$v'(x)=z'(x)+z(x),$

Therefore

$$v'(x)\leq A(x)+(b+1)v(x)+b\left[k(x,x)+\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)dt\right]w\big(v(x)\big).$$

$$\leq A(x)+(b+1)\left[k(x,x)+\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)dt\right][v(x)+w(v(x))].$$

Let $u\in R_+$ be any arbitrary number then for $\alpha\leq x\leq u$, one can obtain:

$$v'(x)\leq A(u)+(b+1)\left[k(x,x)+\int_{\alpha}^{x}\frac{\partial}{\partial x}k(x,t)dt\right][v(x)+w(v(x))].$$

By dividing the above inequality by $[v(x)+w(v(x))]$ and then integrating the resulting inequality from α to x and using equation (2.3), one can obtain:

$$v(x)\leq G^{-1}\left\{G(by(\alpha))+\int_{\alpha}^{x}\left[A(u)+(b+1)\left\{k(t,t)+\int_{\alpha}^{t}\frac{\partial}{\partial t}k(t,s)ds\right\}\right]dt\right\},\ \alpha\leq x\leq u.$$

Then for x=u, one can get:

$v(x) \leq B(x)$

where B(x) is defined previously.

then

$$z'(x) \leq A(x) + b\left[k(x,x) + \int_{\alpha}^{x} \frac{\partial}{\partial x} k(x,t)dt\right] w(B(x)) + bz(x).$$

From lemma (1.1), one can obtain:

$$z(x) \leq by(\alpha)\, e^{\int_{\alpha}^{x} bdt} + \int_{\alpha}^{x} \left\{ A(t) + b\left[k(t,t) + \int_{\alpha}^{t} \frac{\partial}{\partial t} k(t,s)ds \right] w(B(t)) \right\} e^{\int_{t}^{x} bds} dt\,.$$

Hence

$y(x) \leq y(\alpha) + a[x-\alpha] +$

$$\int_{\alpha}^{x} \left[by(\alpha)e^{\int_{\alpha}^{t} bds} + \int_{\alpha}^{t} \left[A(s) + b\left\{ k(s,s) + \int_{\alpha}^{s} \frac{\partial}{\partial s} k(s,\tau)d\tau \right\} w(B(s)) \right] e^{\int_{s}^{t} bd\tau} ds \right] dt\,.$$

Conclusions and Recommendations

From the present study, we can conclude the following :

(1) The classification of the one-dimensional integral and integro-differential inequalities can be extended to include the multi-dimensional integral and integro-differential inequalities.

(2) Defining the one-dimensional n-th order integro-differential inequalities is similar to the definition of the one dimensional first order integro-differential inequalities.

(3) The upper bound for the unknown function in the previous integral and integro-differential inequalities may not be unique.

(4) The existence of the upper bound for the unknown function in all of the integral and integro-differential inequalities is based on the existence of the lower bound of it.

(5) Finding explicit bounds for the unknown function of the integral and integro-differential inequalities that contains the sign $\geq$ instead of the sign $\leq$ can be easily discussed.

(6) Discussing the existence of the solutions of integral and integro-differential inequalities depends on the existence of the solutions of special types of differential inequalities.

Also, for further work, we can recommend the introduction of the following open problems:

(1) Use the integro-differential inequalities to ensure the existence and uniqueness of the solutions for the integro-differential equations.

(2) Find explicit bounds for the unknown functions in the integro-differential inequalities of Fredholm type.

(3) Determine explicit bounds for the unknown function in the multi-dimensional integro-differential inequalities.

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المستخلص

الهدف الرئيسي من هذا العمل يمكن تصنيفه الى أربعة محاور. هذه المحاور يمكن تلخيصها كالاتي:

الهدف الاول: هو اعطاء تصنيف للمتراجحات التكاملية ذات البعد الواحد.

الهدف الثاني: هو ايجاد قيود صريحة للدوال المجهولة الموجودة في انواع خاصة من المتراجحات التكاملية الخطية و اللاخطية من نوع فولتيرا ذات البعد الواحد.

الهدف الثالث: هو تصنيف المتراجحات التكاملية-التفاضلية ذات البعد الواحد.

الهدف الرابع: هو اعطاء قيود صريحة للدوال المجهولة الموجودة في انواع خاصة من المتراجحات التكاملية-التفاضلية الخطية واللاخطية من نوع فولتيرا ذات البعد الواحد من الرتبة الاولى والثانية.

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