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On Controllability Probabilities of Stochastic non – linear Control Systems

A Thesis

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By

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تكن تعلم وكان فضل الله عليك عظيماً ﴿113﴾

صدق الله العلي العظيم

SUPERVISOR CERTIFICATION

I certify that this thesis was prepared under my supervision at the Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University in partial fulfillment of the requirements for the degree of master of science in mathematics.

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الإهداء

الى ملهمي واستاذي الاول . . . .الى من فضله لا يجازى . . .الى من تعجز كلماتي عن

وصفه . . . . **ابي**

الى زهرة عمري . . . . .الى من حملتني وتحملت اعبائي حتى وصلت الى ما انا عليه . . . .الى

القلب الحنون . . . . **امي**

الى قطعة من قلبي فقدتها . . وآلمني فقدها . . . .**اخي المرحوم علي**

الى سندي وعضدي في الحياة . . . .الى املي المتجدد . . . . **حسوني**

الى من حبهم مغروس في قلبي . . . .الى سبب بهجتي وسعادتي . . . .**هوازن وثقيف**

اهدي هذا الجهد المتواضع

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Mohammed Amer

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Abstract

The main aim of this thesis is focused on studying some non-linear uncertain stochastic dynamical system.

The necessary background for stochastic process, stochastic integral for Brownian motion and fractional Brownian motion, stochastic dynamical system driven by Brownian motion and fractional Brownian motion are studied and discussed supported by useful comments and examples.

Some class of non-linear Itô stochastic ordinary control system driven by Brownian motion as well as fractional Brownian motion have been considered and discussed.

A necessary theorem of solvability and controllability of some class of non-linear Itô dynamical control system driven by Brownian motion are discussed and proved using Banach fixed point theorem and supported by useful concluding remark and illustration.

A theorem of solvability and controllability of some class of non-linear Itô dynamical system driven by fractional Brownian motion are also stated and proved supported by illustration.

Controllability concepts plays a vital role in deterministic control theory. It is well known that controllability of deterministic equation are widely used in many fields science and technology, say, physics and engineering. However, in many cases, some kind of randomness can appear in the problem, so that the system should be modeled by a stochastic form.

Controllability for linear and nonlinear deterministic systems has been well studied. For stochastic system the situation is less satisfactory. Only few papers deal with the stochastic control problem. Klamka and Socha [J. Klamka and L. Socha 1977] derived sufficient conditions for the stochastic controllability of linear and nonlinear systems using Lyapunov technique.

In [Y. Liu and S. Peng 2002] Liu and Peng has studied the exact controllability and exact terminal controllability of stochastic linear equation with control acting on the noise term.

Arapostathis in [Arapostathis 2001] established the controllability properties of the class of stochastic differential systems characterized by a linear controlled diffusion perturbed by a smooth, bounded, uniformly Lipschitz nonlinearity. Sirbu and Tessitore [M. Sirbu and G. Tessitore 2001] studied null controllability of an infinite dimensional stochastic differential equations with state and control dependent noise using Riccati approach.

In [A.E. Bashirov and K.R. Kerimov 1997] Bashirov and Kerimov proved that the approximate and complete controllability conditions for the partially observed linear control system to attain an arbitrarily small neighborhood of each point in the state space with probability arbitrarily close to one.

Controllability of nonlinear infinite dimensional stochastic system is studied in [J.P. Dauer and N.I. Mahmudov 2003] and the controllability of finite dimensional stochastic systems is studied in [Mahmudov 2000]. Mahmudov in [Mahmudov 2003] derived a set of sufficient conditions for the controllability of stochastic nonlinear systems.

In the past few years, fractional Brownian motion has been the subject of numerous investigations. Its potential applications to telecommunications and mathematical finance are the practical reasons for which this process is so much studied. On the theoretical point of view, it is an interesting process because it is neither a Markov process nor a semi-martingale so that stochastic calculus with respect to it is challenging. In particular, several attempts have been made to define a good stochastic integral with respect to fractional Brownian motion.

Y. Hu and B. Oksendal in [Hu 2003] showed that real inputs exhibit long-range dependence : the behavior of a real process after a given time t does not only depend on the situation at t but also of the whole history of the process up to time t. Moreover, it turns out that this property is far from being negligible because of the effects it induces on the expected behavior of the global system [Sakthivel 2003].

In this thesis, from the above literatures, our aim have been focused on studying and developing some result on controllability and solvability of [Mahmudov 2004].

The controllability of stochastic dynamic control system driven by Brownian motion and fractional Brownian motion have been considered.

The thesis is divided into three chapters. Chapter one is divided into five sections. Section 1 presents some basic concepts of Probability theory. Section 2 deals with Random Variable. Section 3 deals with Stochastic Processes. Section 4 deals with classes of stochastic processes. In Section 5 Brownian motion was studied and discussed.

Chapter two is divided into three sections. Section 1 presents some basic concepts of Brownian motion calculus. Section 2 deals with Control System Theory. In section 3 the complete controllability of a nonlinear stochastic dynamic system (Standard Brownian motion) are discussed and proved by using the contraction mapping principle.

Chapter three is divided into three sections. Section 1 presents some basic concepts of Fractional Brownian motion. In section 2 the complete controllability of a nonlinear stochastic dynamic system (fractional Brownian motion) are discussed and proved by using the contraction mapping principle. In section three we apply our results in example.

This chapter is divided into five sections. Section 1 presents some basic concepts of Probability theory. Section 2 deals with Random Variable. Section 3 deals with Stochastic Processes. Section 4 deals with classes of stochastic processes. In Section 5 Brownian motion was studied and discussed.

1.1 BASIC CONCEPTS OF PROBABILITY THEORY

The set of all possible outcomes of an experiment is called the sample space and denoted by Ω, that is $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ denotes the set of all finite outcomes, $\Omega = \{\omega_1, \omega_2, ...\}$ denotes the set of all countably infinite outcomes and $\Omega = \{\omega_t : 0 < t < T\}$ denotes the set of uncountably infinite outcomes.

Definition(1.1.1) Set Operations [Krishnan 1984]

Let A and B be two subsets of the sample space Ω, we define the following operation:

1. The Complement of A, denoted by A^c, represents the set of all ω-points not contained in A:

$$A^c = \{\omega : \omega \notin A\}. \quad (1.1.1)$$

The complement of Ω is the empty set $\varnothing$.

2. We said that A and B are equal iff A contained B and B contained A:

$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A \quad (1.1.2)$$

3. The Union of sets A and B, denoted by $A \bigcup B$ or $A + B$, represents the occurrence of $\omega-$points in either A or B:

$$A \bigcup B = \{\omega : \omega \in A \text{ or } \omega \in B\} \quad (1.1.3)$$

The union of an arbitrary collection of sets is defined by:

$$A_1 \bigcup A_2 \bigcup ... \bigcup A_n = \bigcup_{i=1}^{n\in N} A_i = \{\omega : \omega \in A_i \text{ for some } i\}, i = 1,2,...n$$

4. The intersection of sets A and B, denoted by $A \bigcap B$ or AB, represents the occurrence of $\omega -$points in A and B:

$$A \bigcap B = \{\omega : \omega \in A \text{ and } \omega \in B\} \qquad (1.1.4)$$

Clearly, if there is no commonality of $\omega -$points in A and B, then $A \bigcap B$ is the empty set $\varnothing$.

The intersection of an arbitrary collection of sets is defined by:

$$A_1 \bigcap A_2 \bigcap ... \bigcap A_n = \bigcap_{i=1}^{n\in N} A_i = \{\omega : \omega \in A_i \text{ for all } i\}, i = 1,2,...n$$

5. De Morgan's law

$(A \bigcap B)^c = A^c \bigcup B^c$ and $(A \bigcup B)^c = A^c \bigcap B^c$

In general

$$(\bigcup_{i=1}^{n} A_i)^c = \bigcap_{i=1}^{n} A_i^c \text{ and } (\bigcap_{i=1}^{n} A_i)^c = \bigcup_{i=1}^{n} A_i^c \qquad (1.1.5)$$

Example (1.1.1)

Let Ω be the $\omega -$points on the real line R.

$$\Omega = \{\omega : -\infty < \omega < \infty\}$$

Define

$$A = \{\omega : \omega \in (-\infty, a)\} = \{\omega < a\}$$
$$B = \{\omega : \omega \in (b, c)\} = \{b < \omega < c\}$$

Then the set operation yield

$$A^c = \{a \le \omega < \infty\}$$

$$B^c = \{-\infty < \omega \le b\} \bigcup \{c \le \omega < \infty\}$$

$$A \bigcup B = \begin{cases} \{\omega < a\} & c < a \\ \{\omega < c\} & b < a < c \\ \{\omega < a\} \bigcup \{b < \omega < c\} & a < b \end{cases}$$

$$A \bigcap B = \begin{cases} \{b < \omega < c\} & c < a \\ \{b < \omega < a\} & b < a < c \\ \varnothing & a < b \end{cases}$$

The union and intersection of an arbitrary collection of sets are defined by:

$$\bigcup_{n \in N} A_n = \{\omega : \omega \in A_n \text{ for some } n \in N\}$$

$$\bigcap_{n \in N} A_n = \{\omega : \omega \in A_n \text{ for all } n \in N\}$$

Definition (1.1.2) Sequences

A sequence of sets A_n, $n \in N$, is increasing if $A_{n+1} \supset A_n$ and decreasing if $A_{n+1} \subset A_n$ for every $n \in N$.

Remarks (1.1.1) [Krishnan 1984]

1. A sequence which is either increasing or decreasing is called a monotone sequence.
2. We can write the limits (N countably infinite) of monotone sequences as:

$$\lim_{n \to \infty} A_n = \lim_n A_n = \bigcup_{n=1}^{\infty} A_n = A \text{ (increasing).}$$

$$\lim_{n \to \infty} A_n = \lim_n A_n = \bigcap_{n=1}^{\infty} A_n = A \text{ (decreasing).}$$

3. The limit of monotone sequence $\{A_n\}$ is written as $A_n \uparrow A$ when it is increasing, and $A_n \downarrow A$ when it is decreasing.

Example (1.1.2)

Let $A_n = \{\omega : 0 < \omega \le 1 + \frac{1}{n}\}$ $n = 1, 2, ...$

Then:

$$A_1 = \{\omega : 0 < \omega \le 2\}$$
$$A_2 = \{\omega : 0 < \omega \le \frac{3}{2}\}$$
$$A_3 = \{\omega : 0 < \omega \le \frac{4}{3}\}$$
$$\vdots$$

We see that $\{A_n\}$ is decreasing, then

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n = \{\omega : 0 < \omega \le 1\}$$

Example (1.1.3)

Let $A_n = \{\omega : \frac{1}{k} < \omega < 2\}$ $n = 1, 2, ...$

then:

$$A_1 = \{\omega : 1 < \omega < 2\}$$
$$A_2 = \{\omega : \frac{1}{2} < \omega < 2\}$$
$$A_3 = \{\omega : \frac{1}{3} < \omega < 2\}$$
$$\vdots$$

We see that $\{A_n\}$ is increasing, then

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n = \{\omega : 0 < \omega \le 2\}.$$

Remarks (1.1.2) [Krishnan 1984]

1. A superior limit for any sequence $\{A_n\}$ not necessarily monotone is a sequence $\{B_n\}$ derived from $\{A_n\}$ as follow:

$$B_n = \operatorname{Sup}_{k\ge n} A_k = \bigcup_{k=1}^{\infty} A_k$$
$$= \{\omega : \omega \text{ belongs to at least } A_k \text{ except } A_1, A_2, \ldots, A_{n+1}\}$$

2. A_n inferior limit for any sequence $\{A_n\}$ not necessarily monotone is a sequence $\{C_n\}$ derived as follow:

$$C_n = \inf_{k\ge n} A_k = \bigcap_{k=n}^{\infty} A_k$$
$$= \{\omega : \omega \text{ belong to all } A_k \text{ except } A_1, A_2, \ldots, A_{n-1}\}$$

3. The sequences $\{B_n\}$ and $\{C_n\}$ are monotone and decreasing and increasing respectively.
4. We can define a limit for the sequences $\{B_n\}$ and $\{C_n\}$:

$$B = \lim_{n\to\infty} B_n = \lim_n B_n = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right)$$
$$= \lim_{n\to\infty} \operatorname{Sup} A_n = \lim \operatorname{Sup}_n A_n$$
$$= \{\omega : \omega \text{ belongs to inf initly many } A_n\}$$

$$C = \lim_{n\to\infty} C_n = \lim_n C_n = \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right)$$
$$= \lim_{n\to\infty} \inf A_n = \lim \inf_n A_n$$
$$= \{\omega : \omega \text{ belongs to all but a finite number of } A_n\}$$

5. If $\lim \operatorname{Sup}_n A_n = \lim \inf_n A_n$, then $\{A_n\}$ is convergent sequence and $\lim_n A_n = A$, say exists that is:

$$\lim \operatorname{Sup}_n A_n = \lim \inf_n A_n = \lim_n A_n = A.$$

Definition (1.1.3) Field (algebra) [Krishnan 1984]

Define $\mathcal{C}$ as a nonempty class of subsets drawn from the sample space Ω, and let A_i be a subset of Ω (i=1,2,…), we say that the class $\mathcal{C}$ is a field or an algebra of sets in Ω if it satisfies the following conditions

1. If $A_i \in \mathcal{C}$, then $A_i^c \in \mathcal{C}$.
2. If $\{A_i,\ \ i=1,2,3,...,n\} \in \mathcal{C}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{C}$.

Remarks (1.1.3) [Krishnan 1984]

1. De Morgan's law ensures that finite intersections also belong to the field.
2. A class of subsets is field if and only if it is closed under all finite set operations like union, intersection and complementation.
3. Every Boolean algebra of sets is isomorphic to an algebra of sets of Ω, we can also call the field a Boolean field or Boolean algebra.
4. Every field contains as elements the sample space Ω and the empty set $\varnothing$.
5. The smallest field containing $A \subset \Omega$ is:

$$\mathcal{C} = \{A, A^c, \Omega, \varnothing\}. \tag{1.1.6}$$

6. If a class of subsets is closed under finite set operations, it does not necessarily mean that it is also closed under countably infinite set operations.

Example (1.1.4)

Let Ω = R and consider a class $\mathcal{C}$ of all intervals of the form (a,b], that is $\{x \in R : a < x \le b\}$:

$$(a,b]\bigcap(c,d]=\begin{cases}\varnothing & ,a<b<c<d\\ =(c,b] & ,a<c<b<d\\ =(a,d] & ,c<a<d<b\\ =(a,b] & ,c<a<b<d\end{cases}$$

But $(a,b]^c=(-\infty,a]\bigcup(b,\infty)\notin\mathcal{C}$

$(a,b]\bigcup(c,d]\notin\mathcal{C}$ if $a<b<c<d$

The class $\mathcal{C}$ is not field.

Definition (1.1.4) σ-Field (σ-Algebra): [Krishnan2006]

A class of a countably infinite collection of subsets $A_j\subset\Omega$ denoted by $\mathcal{F}$ is a σ-field when the following conditions are satisfied:

1. If $A_i\in\mathcal{F}$, then $A_i^c\in\mathcal{F}$.
2. If $\{A_i,\ \ i=1,2,3,...\}\in\mathcal{F}$, then $\bigcup_{i=1}^{\infty}A_i\in\mathcal{F}$

Clearly, a σ-field is a field, but a field may not be a σ-field.

Example (1.1.5) [Krishnan1984]

Let Ω = R and $\mathcal{C}$ be the class of all intervals of the form $(-\infty,a], (a,c],$ and (d,∞):

$$\begin{aligned}(b,c]^c &= (-\infty,b]\bigcup(c,\infty)\in\mathcal{C}\\ (d,\infty)^c &= (-\infty,d]\in\mathcal{C}\\ (-\infty,a]^c &= (a,\infty)\in\mathcal{C}\end{aligned}$$

Clearly, the class $\mathcal{C}$ is closed under finite intersections. Similarly the example can be shown the $\mathcal{C}$ is closed under finite unions. Hence the class $\mathcal{C}$ is a field. However, for infinite intersections of the form:

$$\bigcap_{n=1}^{\infty}(b-\frac{1}{n},c)=[b,c)\notin\mathcal{C}.$$

The class $\mathcal{C}$ is not a σ-field.

Remarks (1.1.4) [Krishnan 2006]

1. The intersection of a non empty arbitrary collection of σ-field in Ω is a σ-field in Ω.
2. The arbitrary union of a collection of σ-fields may not be a σ-field.
3. We can always construct the smallest σ-field over $\mathcal{C}$ which will contain $\mathcal{C}$ and will be denoted by $\sigma(\mathcal{C})=\mathcal{F}$. This will always exist since $\sigma(\mathcal{C})$ can be defined as the intersections of all σ-fields containing $\mathcal{C}$, that is if $\sigma_1(\mathcal{C}),\sigma_2(\mathcal{C}),...$ are all σ-fields containing $\mathcal{C}$, then:

$$\sigma(\mathcal{C})=\bigcap_{i=1}^{\infty}\sigma_i(\mathcal{C}). \qquad (1.1.7)$$

Example (1.1.6)

Let the sample space Ω contain $\omega-\text{points}$ of the toss of a die. Ω is the set $\{1,2,3,4,5,6\}$. We shall now define a class of sets:

$$\mathcal{C}=\{\varnothing,\Omega,\{1,3,5\},\{2,4,6\},\{2,4\}\}$$

Clearly, $\mathcal{C}$ is not a field since $\{1,3,5\}\bigcup\{2,4\}=\{1,2,3,4,5\}$ is not in $\mathcal{C}$. However, we can generate the field containing $\mathcal{C}$ by:

$\sigma(\mathcal{C})=\mathcal{F}=\{\mathcal{C},\{1,3,5,6\},\{6\},\{1,2,3,4,5\}\}$ which is indeed a σ-field, and we can show that it is the minimum σ-field generated by $\mathcal{C}$.

Definition(1.1.5) Borel σ-field [Krishnan 2006]

The minimum σ-field generated by the collection of open sets of a topological space Ω is called the Borel σ-field or Borel field. Members of this σ-field are called Borel sets.

Remarks (1.1.5) [Krishnan 2006]

1. The Borel σ-field is a σ-field, and hence each closed set is also a Borel set.
2. The important topological space with which we will be concerned is the real line R. The collection of Borel sets on the real line is denoted by $\mathfrak{R}$. Each open interval is a member of $\mathfrak{R}$.
3. From the relationships:

$$(a,b] = \bigcap_{n=1}^{\infty}(a, b+\frac{1}{n})$$
$$[a,b) = \bigcap_{n=1}^{\infty}(a-\frac{1}{n}, b) \qquad (1.1.8)$$
$$[a,b] = \bigcap_{n=1}^{\infty}(a-\frac{1}{n}, b+\frac{1}{n})$$

we find that the intervals (a,b], [a,b) and [a,b] are Borel sets. Hence the Borel field $\mathfrak{R}$ contains all subsets of the form given above and their complements, countable unions, and intersections.

Definition (1.1.6) Measurable space [Krishnan 1984]

A suitable model of the random experiment is therefore a sample space Ω and a σ-field $\mathfrak{F}$ of subsets of Ω. The space $(\Omega, \mathfrak{F})$ thus created is called a measurable space.

Remarks (1.1.7) [Krishnan 1984]

1. Subsets of Ω which are elements in the σ-field are called events.
2. Element of Ω are points.
3. If $\{A_i, i=1,2,...,n\}$ is class of disjoint sets of Ω such that $\bigcup_{i=1}^{n} A_i = \Omega$, then the $\{A_i\}$ collectively exhaust Ω. The class is called a partition of Ω.

Definition(1.1.8) Probability measure [Krishnan2006]

A probability measure is a set function P defined on a σ-field $\mathfrak{F}$ of subsets of a sample space Ω such that it satisfies the following axioms of Kolmogorov for any $A \in \mathfrak{F}$

1. $P(A) \geq 0$ (non negativity).
2. $P(\Omega) = 1$ (normalization).
3. $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$ (σ-additivity).

With $A_n \in \mathfrak{F}$, and A_i and A_j being pairwise disjoint.

Remarks (1.1.8) [Krishnan 1984]

1. Any set function μ defined on a measurable space $(\Omega, \mathfrak{F})$ satisfying axioms (1) and (3) is called a measure.
2. A probability measure is normed or scaled measure because of axiom (2).
3. Any bounded measure with suitable normalization can be convert--ed into a probability measure.
4. If μ(A) is finite for each $A \in \mathfrak{F}$, then μ is a finite measure.
5. If μ(A) =∞ but if there exist a seq. $\{A_n\}$ of members of $\mathfrak{F}$ such that $A \subset \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n)$ is a finite for each n, then μ is a σ-finite measure.
6. The triplet $(\Omega, \mathfrak{F}, \mu)$ is a measure space.

Definition (1.1.9) Probability space [Krishnan 2006]

The measure space $(\Omega, \mathfrak{F}, P)$ is called a probability space, which serves to describe any random experiment where:

a. Ω is a nonempty set called the sample space, whose elements are the elementary outcomes of a random experiment.
b. $\mathfrak{F}$ is a σ-field of subsets of Ω.
c. P is a probability measure defined on the measurable space $(\Omega, \mathfrak{F})$.

Remark (1.1.9)

A signed measure is a σ-additive set function μ defined on a measurable space $(\Omega, \mathfrak{F})$ taking positive and negative values such that $\mu(\varnothing)=0$ and assuming at most one of two values, $(+\infty \text{ or } -\infty)$.

1.2 RANDOM VARIABLE

What is a random variable? An outlandish definition would be that it is neither random nor a variable!

An important class of functions are measurable functions which are different from the measure function μ, Whereas measure functions are set functions, measurable functions are invariably point functions.

Definition(1.2.1) Measurable function [Krishnan2006]

Let $(\Omega_1, \mathfrak{F}_1)$ and $(\Omega_2, \mathfrak{F}_2)$ be two measurable spaces. Let g be a function with domain $E_1 \subset \Omega_1$ and range $E_2 \subset \Omega_2$:

$$g: E_1 \to E_2$$

g is called an $\mathfrak{F}_1$-measurable function or an $\mathfrak{F}_1$-measurable mapping if for every $E_2 \in \mathfrak{F}_2$:

$$g^{-1}(E_2) = \{w : g(w) \in E_2\} = E_1 \qquad (1.2.1)$$

is in the σ-field $\mathfrak{F}_1$.

Definition (1.2.2) Random variable [Krishnan 2006]

A random variable is that it is a function X that assigns a rule of correspondence for every point ξ in the sample space Ω called the domain, a unique value $X(\xi)$ on the real line R called the range. Let $\mathfrak{F}$ be the field associated with the sample space and $\mathfrak{F}_X$ be the field associated with the real line. The random variable X induces a probability measure P_X in R and hence X is a mapping of the probability space $(\Omega, \mathfrak{F}, P)$ to the probability space $(\Omega_X, \mathfrak{F}_X, P_X)$ as shown below:

$$X:(\Omega,\mathfrak{F},P)\rightarrow(\Omega_X,\mathfrak{F}_X,P_X) \qquad (1.2.2)$$

Remarks (1.2.1) [Krishnan 2006]

1. Consider an event $A \subset \Omega \in \mathfrak{F}$. The random variable X maps every point x_i in the event A to points ξ_i in the event I_X, called the image of A under X, where $I_X \subset \Omega \in \mathfrak{F}_X$. X may be a random variable only if the inverse image $X^{-1}(I_X)$ belongs to the field $\mathfrak{F}$ of subsets of Ω, and hence it must be an event. The mapping and the inverse mapping are shown in Fig. 1.2.1. With this restriction we should be able to find the induced probability measure P_X in terms of the probability measure P as follows:

$$P_X(I_x) = P\{X^{-1}(I_X)\} = P(A) = P\{\xi : X(\xi) \in I_X\}. \qquad (1.2.3)$$

2. Since I_X belongs to the field $\mathfrak{F}_X$, on the real line it may consist of sets of the form $\{-\infty < \eta \le x\}, \{x < \eta < \infty\}, \{x_1 < \eta \le x_2\}, \{\eta = x\}$. Out of these sets we define I_X by

$$I_X = \{-\infty < \eta \le x\}$$

All the other sets $\{x < \eta < \infty\}, \{x_1 < \eta \le x_2\}, \{\eta = x\}$ can be expressed in terms of the defined I_X as follows:

$$\overline{I}_X = \{x < \eta \le \infty\}$$

$$I_{X_2} - I_{X_1} = \{-\infty < \eta \le x_2\} - \{-\infty < \eta \le x_1\} = \{x_1 < \eta \le x_2\}$$

$$\lim_{\Delta x \to 0^+} \{I_{X+\Delta X} - I_X\} = \{\eta = x\}$$

With the definition of I_X, we can write equation (1.2.3) as follows:

$$P_X(I_x) = P\{\xi : X(\xi) \in I_X\} = P\{\xi : X(\xi) \le x\} \qquad (1.2.4)$$

Define the quantity $P\{\xi : X(\xi) \le x\}$ as the cumulative distribution function $F_X(x)$, and rewrite equation (1.2.4) in abbreviated notation as

$$P\{\xi : X(\xi) \in I_X\} = P\{X \le x\} = F_X(x) \qquad (1.2.5)$$

Equation (1.2.5) converts from a cumbersome set function $P\{X \le x\}$ into a convenient point function $F_X(x)$. The value x scans the entire real line, that is, $-\infty < x < \infty$, and $F_X(x)$ must be defined for all x on the real line. Note that $\{X \le x\}$ in equation (1.2.5) is meaningless unless it is defined carefully, because X is a function and a function without an argument cannot be less than a number! For example, we cannot have an exponential less than 2, unless we specify the argument of the exponential. The abbreviated notation in equation (1.2.5) is to be defined as the probability of the set of all points $\xi \in \Omega$ such that the number $X(\xi)$ is less than or equal to the number x. The random variable X should not confused with the value x that the random variable takes. The random variable X can take any value, for example، $F_X(y) = P\{X \le y\}$.

Having defined the cumulative distribution function, we now define the probability density function (pdf) as the derivative of the cumulative distribution function $F_X(x)$ with respect to x:

$$f_X(x) = \lim_{\Delta x \to \infty} \frac{P(X \le x + \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to \infty} \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x} = \frac{d}{dx} F_X(x) \qquad (1.2.6)$$

The cdf $F_X(x)$ can be given in terms of the pdf $f_X(x)$ as follows:

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi) d\xi \qquad (1.2.7)$$

Whereas the distribution function $F_X(x)$ is a probability measure, the density function $f_X(x)$ is not a probability measure unless it is multiplied by the infinitesimal Δx to yield

$f_X(x)\Delta x = P(x < X \le x + \Delta x)$.

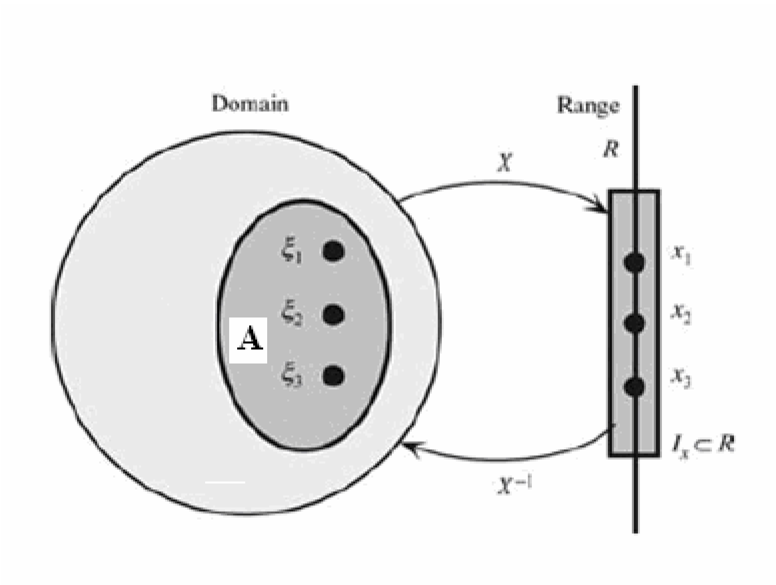


Figure (1.2.1) representation of random variable

Example (1.2.1)

A die is tossed, and the random variable X is defined by the amount won (+) or lost (−) on the face of the die as shown in Table 1.2.1.

We have to find the cdf $F_X(x)$ and the pdf $f_X(x)$. We will follow the steps to find the probability functions.

Step 1. The mapping diagram (Fig. 1.2.2) is drawn with positive numbers indicating win and negative numbers indicating loss.

Step 2. From the mapping diagram the regions of x are (a) $x \le -9$, (b)$-9 < x \le -4$, (c)$-4 < x \le 5$, (d)$5 < x \le 8$, and (e)$x > 8$. The corresponding set I_X on the real line for all the 5 regions is $I_X = \{-\infty < \eta < x\}$.

Step 3. We will find the all the points in the sample space that map into I_X for every region of x:

a. $x \le -9$: Since no points in Ω map into I_X (figure 1.2.2a), we have $F_X(x) = 0$.

b. $-9 < x \le -4$: In this region (Figure 1.2.2b), only one point {3} maps into I_X. Since $P(3) = \frac{1}{6}$, $F_X(3) = \frac{1}{6}$.

c. $-4 < x \le 5$: Here the points {3,2,6} from Ω into I_X(Figure 1.2.2c). Hence $F_X(x) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$.

d. $5 < x \le 8$: In this region (Figure 1.2.2d) the points {3,2,6,5} in Ω map in to I_X. Hence $F_X(x) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$.

e. $x > 8$: In this region (Figure 1.2.2e) all six points, that is, the entire sample space Ω, map into I_X. Hence $F_X(x) = 1$.

In terms of unit step functions we can write the cdf $F_X(x)$ as follows:

$$F_X(x)\frac{1}{6}u(x+9) + \frac{1}{3}u(x+4) + \frac{1}{6}u(x-5) + \frac{1}{3}u(x-8).$$

Step 4. The cdf $F_X(x)$ can be graphed as shown in Fig. 1.2.3.

Table (1.2.1)

Pips on the Die	Win or Loss $
1,4	+8
2,6	−4
3	−9
5	+5

Figure (1.2.2)

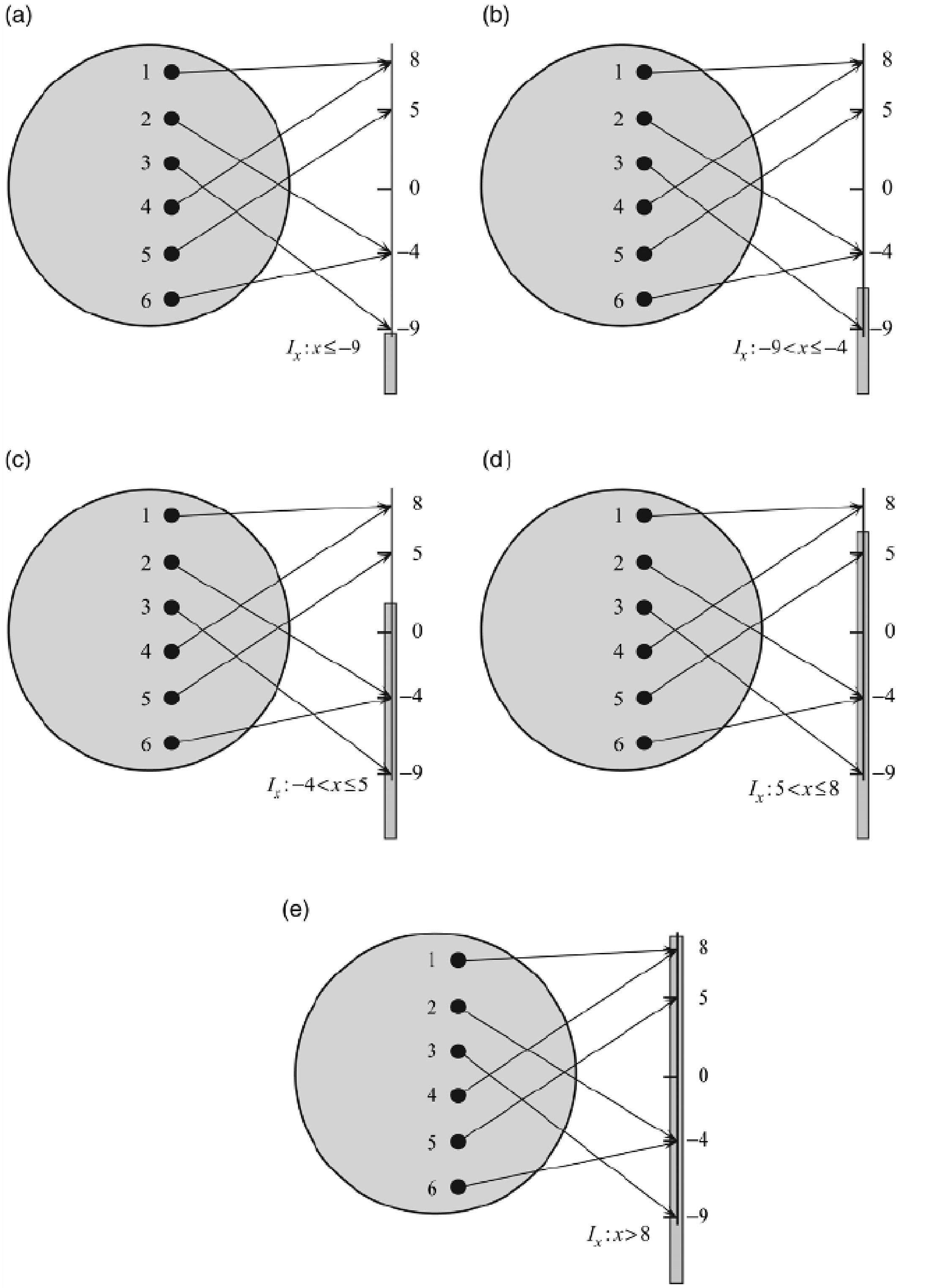
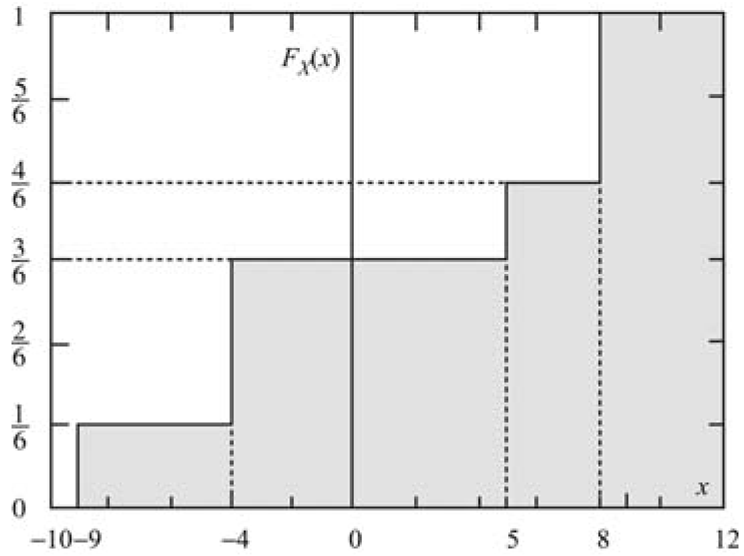


Figure (1.2.3)



Step 5. We can now find the density function $f_X(x)$ by differentiating the distribution function $F_X(x)$, bearing in mind that the differentiation of the unit step is the Dirac delta function. Performing the indicated differentiation, we obtain

$$f_X(x)\frac{1}{6}\delta(x+9)+\frac{1}{3}\delta(x+4)+\frac{1}{6}\delta(x-5)+\frac{1}{3}\delta(x-8)$$

Definition(1.2.3) Indicator functions [Krishnan 1984]

An indicator function is a real valued function defined on the sample space Ω taking either of the two values 0 or 1, depending upon whether or not the $\omega-\text{point}$ is in the event A:

$$I_A(\omega)=\begin{cases}1 & \omega\in A\\ 0 & \omega\notin A\end{cases} \qquad (1.2.3)$$

Remarks (1.2.2) [Krishnan 1984]

1. The indicator function is a random variable since it is a measurable mapping of the sample space into the real line.

2. As the indicator function is the simplest nontrivial function, we shall enumerate some of the properties. We shall use the quantities cup($\vee$) and cap($\wedge$) defined by:

$$a \vee b = \max(a,b), \qquad a \wedge b = \min(a,b) \tag{1.2.4}$$

a) $A \subset B \Leftrightarrow I_A \le I_B$.

b) $I_A = I_A^2 = \ldots = I_A^n$.

c) $I_\Omega = 1$.

d) $I_{A \cap B} = I_A \cdot I_B = I_A \wedge I_B$.

e) $I_{A \cup B} = I_A + I_B - I_A \cdot I_B = I_A + I_B - I_A \wedge I_B = I_A \vee I_B$.

f) $I_{A^c} = 1 - I_A$.

Definition (1.2.4) Simple function [Krishnan 2006]

Let $\{A_i, i = 1,2,\ldots,n\}$ be a partition of the sample space Ω. A simple function $g(\omega)$ can be written in the form:

$$g(\omega) = \sum_{k=1}^{n} g_k I_{A_k}(\omega) \tag{1.2.5}$$

where the g_k are distinct real numbers, and $I_{A_k}(\omega)$ is the indicator function of the set A_k.

Definition(1.2.5) Distribution function [Krishnan2006]

A function $F: R \to R$ is called a distribution function if it is increasing and right continuous. It is a probability distribution if in addition:

$$\lim_{x \to -\infty} F(x) = 0 \qquad \text{and} \qquad \lim_{x \to \infty} F(x) = 1$$

Remarks (1.2.3) [Krishnan 2006]

1. Right continuous function are those functions for which:

   $F(x) = \lim_{\varepsilon \downarrow 0} F(x + \varepsilon)$.

2. Left continuous function are those functions for which:

   $F(x) = \lim_{\varepsilon \uparrow 0} F(x - \varepsilon)$.

Definition (1.2.6) Expectation [Krishnan 2006]

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, and let X be a real random variable. The expectation of X is defined by:

$$EX = \int_{\Omega} X(\omega) dp(\omega) \quad \text{or} \quad \int_{\Omega} x dp. \tag{1.2.6}$$

Remark (1.2.4) Properties of expectation operator

We take a simple random variable of the form $X = \sum_{k=1}^{n} X_k I_{A_k}$ and define $EX = \sum_{k=1}^{n} X_k P(A_k)$:

1. Linearity: $E(aX + bY) = aEX + bEY$, a and b are constant.
2. Homogeneity: $E(cX) = cEX$ for constant c.
3. Order preservation: $X \geq Y \rightarrow EX \geq EY$.

Definition (1.2.7) Independence [Krishnan 2006]

1. Let $(\Omega, \mathfrak{F}, P)$ be a probability space and let subsets A and $B \in \mathfrak{F}$. The events A and B are independent if:

   $$P(A \cap B) = P(A) \cdot P(B) \tag{1.2.7}$$

2. n events $A_1, A_2, ..., A_n$ are independent if for any subset $\{k_1, k_2, ..., k_r\}$, where r=1,2,…,n,

$$P(\bigcap_{i=1}^{r} A_{k_i}) = \prod_{i=1}^{r} P(A_{k_i}) \qquad (1.2.8)$$

Definition (1.2.8) Independence of σ-fields

Let $(\Omega, \mathfrak{F}, P)$ be a probability space and let $\{\mathfrak{F}_i, i = 1,2,...,n\}$ be a sub σ-fields of $\mathfrak{F}$. These sub σ-fields are independent if for all $A_1 \in \mathfrak{F}_1, A_2 \in \mathfrak{F}_2, ..., A_n \in \mathfrak{F}_n$ the events $A_1, A_2, ..., A_n$ are independent.

Definition (1.2.9) Independence of random variables

Let $(\Omega, \mathfrak{F}, P)$ be a probability space and let X and Y be two integrable random variables with values in $(R, \mathfrak{F})$. If the random variables X and Y are independent, then EXY=EX EY. (1.2.9)

Definition(1.2.9) Conditional probability [Krishnan2006]

Let $(\Omega, \mathfrak{F}, P)$ be a probability space and let A and B be events in the probability space with $P(A) \neq 0$. The conditional probability of the event B given that A has occurred is defined by:

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \qquad (1.2.10)$$

Remarks (1.2.5) [Krishnan 2006]

1. If B_i are disjoint, then:

$$P(\bigcup_{i=1}^{n} B_i | A) = \frac{P[(\bigcup_{i=1}^{n} B_i) \cap A]}{P(A)} = \frac{P[\bigcup_{i=1}^{n} (B_i \cap A)]}{P(A)} = \sum_{i=1}^{n} \frac{P(B_i \cap A)}{P(A)}$$

2. If events A and B are independent, then:

   $P(B|A) = P(B)$.

Definition (1.2.10) Conditional Expectation

Let $(\Omega,\mathfrak{F},P)$ be a probability space. Let $\mathfrak{F}_1$ be a sub – σ – field of $\mathfrak{F}$, and let X be an integrable real – valued random variable. The conditional expectation of X relative to $\mathfrak{F}_1$ is an integrable $\mathfrak{F}_1$-measurable random variable $E(X|\mathfrak{F}_1)$ or $E^{\mathfrak{F}_1}X$, such that for every $A \in \mathfrak{F}_1$,

$$\int_A E(X|\mathfrak{F}_1)dP = \int_A E^{\mathfrak{F}_1}XdP = \int_A XdP.$$

1.3 STOCHASTIC PROCESS [Krishnan 2006]

In the earlier sections a random variable X was defined as a function that maps every outcome ξ_i of points in the sample space Ω to a number $X(t,\xi_i)$ on the real line R. A random process X(t) is a mapping that assigns a time function $X(t,\xi_i)$ to every outcome ξ_i of points in the sample space Ω. Alternate names for random processes are stochastic processes and time series. More formally, a random process is a time function assigned for every outcome $\xi \in \Omega$ according to some rule $X(t,\xi)$, $t \in T, \xi \in \Omega$, where T is an index set of time. As in the case of a random variable, we suppress ξ and define a random process by X(t). If the index set T is countably infinite, the random process is called a discrete-time process and is denoted by X_n.

Definition (1.3.1) Stochastic Process [Krishnan 2006]

Let $(\Omega,\mathfrak{F},P)$ be a complete probability space and let T be any time set. Let $(R,\mathfrak{R})$ be a measurable space, where R is the real line and $\mathfrak{R}$ is the σ-field of Borel sets on the real line. A stochastic process $\{X_t, t \in T\}$ is a family of random variables define on the probability space $(\Omega,\mathfrak{F},P)$ and taking values in the measurable space $(R,\mathfrak{R})$.

Some interpretations of Stochastic Processes [Klebaner 2005]

Referring to Fig.1.3.1, a random process has the following interpretations:

1. $X(\xi, t_1)$ is random variable for fixed time t_1.
2. $X(\xi_i, t)$ is a sample realization for any point ξ_i in the sample space Ω.
3. $X(\xi_i, t_1)$ is a number.
4. $X(\xi, t)$ is a collection or ensemble of realizations and is called a random process.

Remarks (1.3.1) [Klebaner 2005]

1. The probability space $(\Omega, \mathfrak{F}, P)$ is called the base space and the measurable space $(R, \Re)$ the state space.
2. If the random variables (vectors) X_t are discrete, we say that the stochastic process has a discrete state space.
3. the random variables (vectors) X_t are continuous, the process is said to have a continuous state space.
4. For each $t \in T$, the $\mathfrak{F}-$measurable random variable X_t is called the state of the process at time t.
5. For each $\omega \in \Omega$ the map $t \to X_t(\omega)$ defined on T and taking values in R is called a sample function.
6. If the time set T is N, then the stochastic process $\{X_t, t \in T\}$ becomes $\{X_n, n \in N\}$ and is called a discrete stochastic process.
7. If the time set T is R or R^+, then the stochastic process $\{X_t, t \in T\}$ becomes $\{X_n, n \in R\}$ and is called a continuous stochastic process.

8. An important point to emphasize is that a random process is a finite or an infinite ensemble of time functions and is not a single time function.

Example (1.3.1) [Klebaner 2005]

A fair coin is tossed. If heads come up, a sine wave $x_1(t) = \sin(5\pi t)$ is sent. If tails come up, then a ramp $x_2(t) = t$ is sent. The resulting random process X(t) is an ensemble of two realizations, a sine wave and a ramp, and is shown in Fig.1.3.2. The sample space S is discrete.

Example (1.3.2) [Klebaner 2005]

In this example a sine wave is in the form $X(t) = A\sin(\omega t + \Phi)$, where Φ is a random variable uniformly distributed in the interval $(0, 2\pi)$. Here the sample space is continuous, and the sequence of sine functions is shown in Fig. 1.3.3.

Figure (1.3.1)

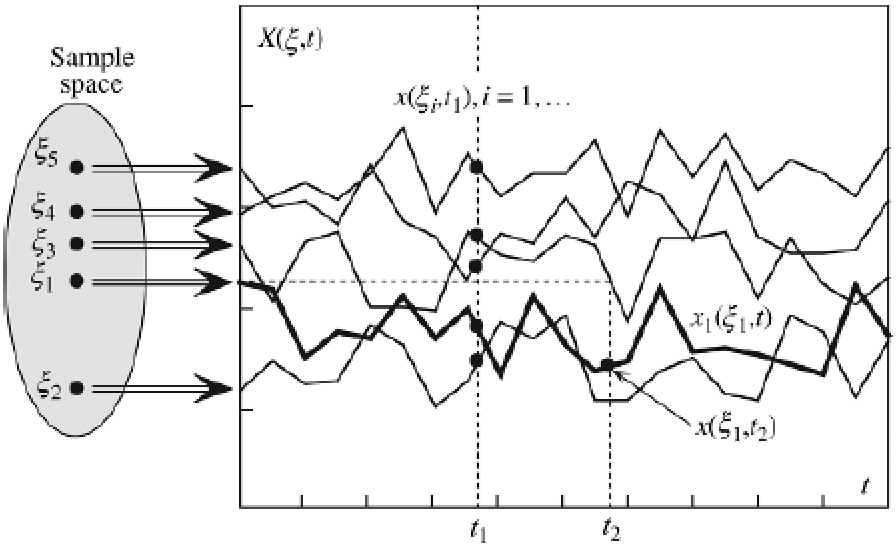


Figure (1.3.2)

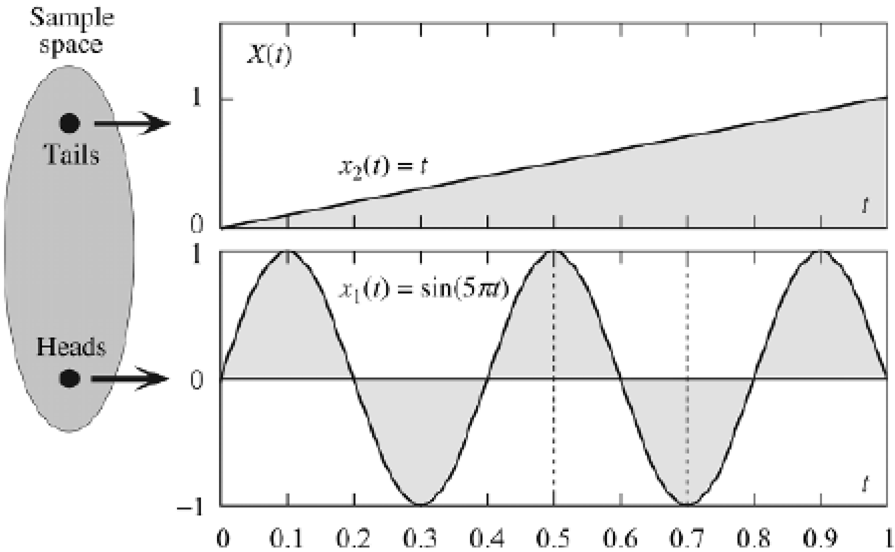
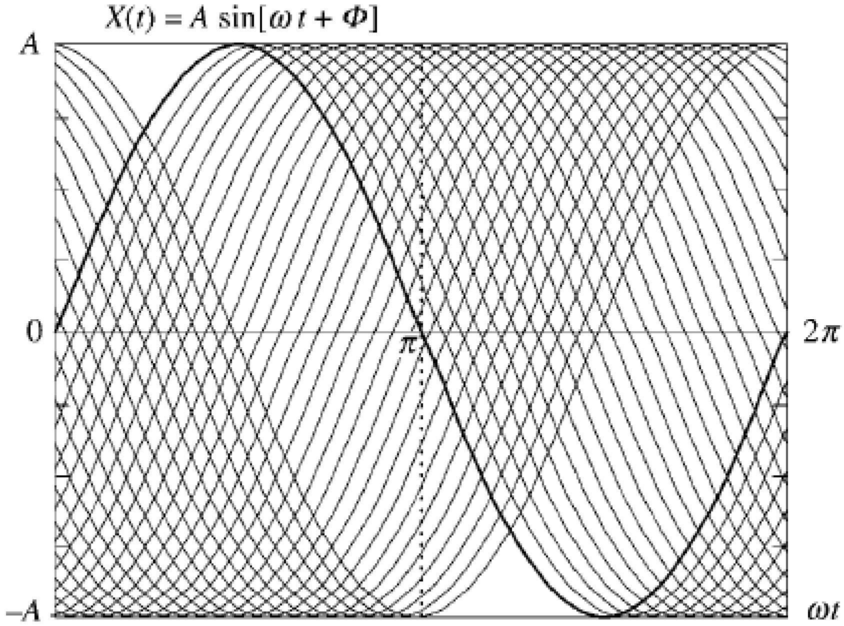


Figure (1.3.3)



Definition (1.3.2) Distribution and Density Functions

Since a random process is a random variable for any fixed time t, we can define a probability distribution and density functions as

$$F_x(x;t) = P\left(\xi,\ t : X(\xi;t) \le t\right) \text{ for a fixed t} \qquad (1.3.1)$$

and

$$f_x(x;t) = \frac{\partial}{\partial x} F_X(x;t) = \lim_{\Delta x \to 0} \frac{F_X\left(x+\Delta x;t\right) - F_X\left(x;t\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} P\left(x < X(t) \le x + \Delta x\right) \qquad (1.3.2)$$

These are also called first-order distribution and density functions, and in general, they are functions of time.

Definition (1.3.3) Means And Variances

Analogous to random variables, we can define the mean of a random process as

$$\mu_X(t) = E\left[X(t)\right] = \int_{-\infty}^{\infty} x f_x(x;t)dx \qquad (1.3.3)$$

and variance as

$$\sigma_X^2(t) = E\left[X(t) - \mu_X(t)\right]^2 = E\left[X^2(t)\right] - \mu_X^2(t)$$

$$= \int_{-\infty}^{\infty} \left[x - \mu_X(t)\right]^2 f_x(x;t)dx \qquad (1.3.4)$$

where

$$E\left[X^2(t)\right] = \int_{-\infty}^{\infty} x^2 f_X(x;t)dx$$

Since the density is a function of time, the means and variances of random processes are also functions of time.

Example (1.3.2) [Krishnan 2006]

We shall now find the distribution and density functions along with the mean and variance for the random process of Example 1.3.1 for times $t = 0, \frac{1}{2}, \frac{7}{10}$:

$$t = 0,\ x_1(0) = 0,\ x_2(0) = 0$$

At $t = 0$ the mapping diagram from the sample space to the real line is shown in Fig.(1.3.4a) along with the corresponding distribution and density functions.

The mean value is given by $\mu_X(0) = 0;\ 0 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = 0$. The variance is given by $\sigma_X^2(0) = (0-0)^2 \frac{1}{2}(0-0)^2 \frac{1}{2} = 0$:

$$t = \frac{1}{2},\ x_1\left(\frac{1}{2}\right) = 1,\ x_2\left(\frac{1}{2}\right) = \frac{1}{2}$$

At $t=\frac{1}{2}$ the mapping diagram from the sample space to the real line is shown in Fig.(1.3.4b) along with the corresponding distribution and density functions.

The mean value is given by $\mu_X\left(\frac{1}{2}\right)=\frac{1}{2}\cdot\frac{1}{2}+1\cdot\frac{1}{2}=\frac{3}{4}$. The variance is given by $\sigma_X^2\left(\frac{1}{2}\right)=\left(\frac{1}{2}-\frac{3}{4}\right)^2\frac{1}{2}+\left(1-\frac{3}{4}\right)^2\frac{1}{2}=\frac{1}{16}$:

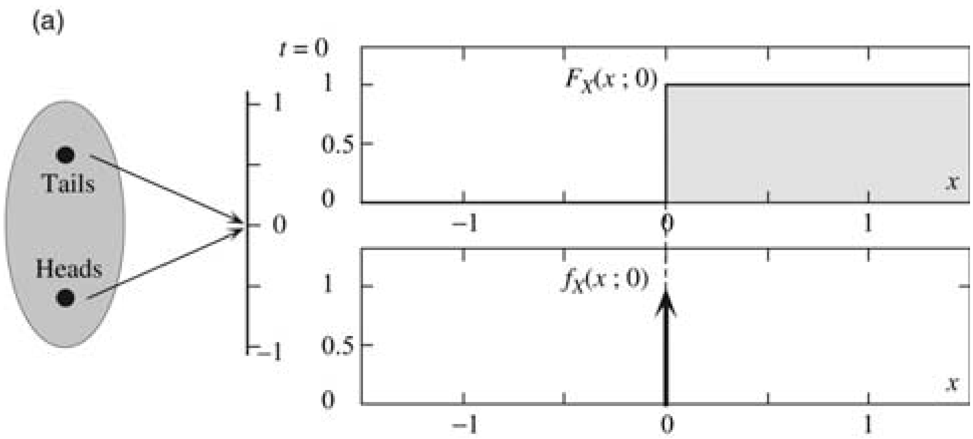
$$t=\frac{7}{10},\ x_1\left(\frac{7}{10}\right),\ x_2\left(\frac{7}{10}\right)=\frac{7}{10}$$

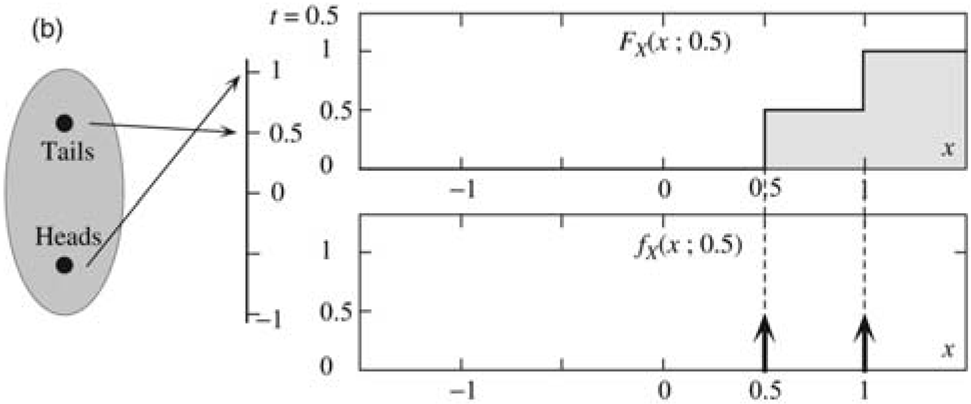
At $t=\frac{7}{10}$ the mapping diagram from the sample space to the real line is shown in Fig. 19.1.4c along with the corresponding distribution and density functions.

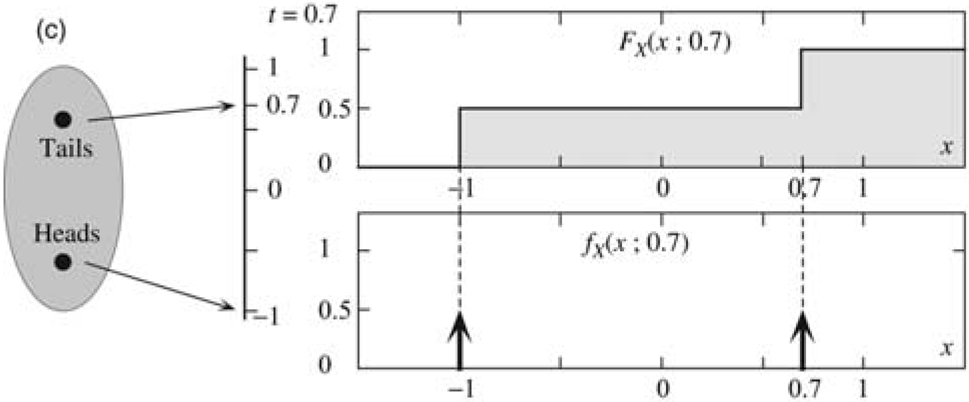
The mean value is given by $\mu_X\left(\frac{7}{10}\right)=\frac{7}{10}\cdot\frac{1}{2}-1\cdot\frac{1}{2}=-\frac{3}{20}$.

The variance is given by $\sigma_X^2\left(\frac{7}{10}\right)=\left(\frac{7}{10}+\frac{3}{20}\right)^2\frac{1}{2}+\left(-1+\frac{3}{20}\right)^2\frac{1}{2}=\frac{289}{400}$.

Figure (1.3.4)







Definition (1.3.4) Modes of convergence [Krishnan 2006]

In the mathematics of calculus, the convergence of a series in some measure is the basis for the concepts such as continuity, differentiation and integration of functions. When we study the calculus of random processes such as differentiation, integration and convergence of a series, we need to define the notions of convergence.

1. Convergence with probability one, also known as almost sure convergence, is defined as follows. Let X_n be a sequence of random variables, and

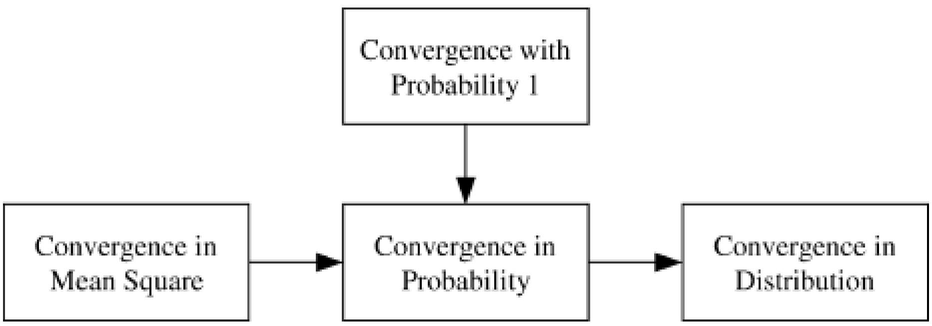
   $P\left(\lim_{n\to\infty} X_n = X\right) = 1$.

We say that X_n converges to X with probability 1. Here, $P(\cdot)$ is the probability measure of the random event $\lim_{n\to\infty} X_n = X$.

2. Convergence in probability is defined as
$$\lim_{n\to\infty} P\left(|X_n - X| > \varepsilon\right) = 0,\ \forall \varepsilon > 0.$$
3. Convergence in distribution is defined as
$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$
4. Convergence in mean square is defined as
$$\lim_{n\to\infty} E\left[|X_n - X|^2\right] = 0, \text{ or } \underset{n\to\infty}{\text{l.i.m.}} X_n = X.$$

The convergence in mean square is the strongest among the modes of convergence defined above. It implies the convergence in probability, which in turn implies the convergence in distribution. Convergence with probability 1 also implies the convergence in probability. The weakest mode of convergence is the convergence in distribution. This discussion is illustrated in Figure 1.3.5.

Figure (1.3.5)



Definition (1.3.5) Stochastic differentiation

Consider a stochastic process X(t) defined on the sample space Ω. A derivative process $\dot{X}(t)$ can be defined in terms of the following limit

$$\dot{X}(t) = \underset{\Delta t \to 0}{\text{l.i.m.}} \frac{X(t+\Delta t) - X(t)}{\Delta t}.$$

Definition (1.3.2) Separable Process [Krishnan 2006]

Let $\{X_t, t \in T\}$ be a stochastic process defined on $(\Omega, \mathfrak{F}, P)$ with time set T and values in R. Let K be any closed subset in R, and let I be an open interval in T. Then the process $\{X_t, t \in T\}$ is separable, relative to the class of all closed sets K in R, if there exist a countable subset $S \subset T$ and an $\omega -$set Λ of probability 0 such that the two $\omega -$sets

$\{\omega: \quad X_t(\omega) \in K, t \in I \cap T\}$, $\{\omega: \quad X_t(\omega) \in K, t \in I \cap S\}$ differ by Λ.

The countable set $S \subset T$ is called separating set or separant. What the definition implies is that if $\{X_t, t \in T\}$ is separable, then every set of the form $\{\omega: \quad X_t(\omega) \in K, t \in I \cap T\}$ differs from the event $\{\omega: \quad X_t(\omega) \in K, t \in I \cap S\}$ by the null set Λ and can be made an event by competing the underlying probability space.

Definition (1.3.3) Increasing σ-field or Filtration σ-field [Krishnan 2006]

Let $(\Omega, \mathfrak{F})$ be a complete measurable space and let $\{\mathfrak{F}_t, t \in T, T = R^+\}$ be a family of sub-σ-fields of $\mathfrak{F}$ such that for $s \le t$, $\mathfrak{F}_s \subset \mathfrak{F}_t$. Then $\{\mathfrak{F}_t\}$ is called an increasing family of sub-σ-fields on $(\Omega, \mathfrak{F})$ or the filtration σ-field of $(\Omega, \mathfrak{F})$.

$\mathfrak{F}_t$ is called the σ-field of events prior to t. If $\{X_t, t \in T\}$ is a stochastic process defined on $(\Omega, \mathfrak{F}, P)$ then clearly $\mathfrak{F}_t$ given by

$$\mathfrak{F}_t = \sigma\{X_s, s \le t, t \in T\}$$

is increasing.

Remark (1.3.2) [Krishnan 2006]

Since the probability space $(\Omega,\mathfrak{F},P)$ is complete, the σ-field $\mathfrak{F}$ contains all subsets of Ω having probability measure zero. We shall assume here that the filtration σ-field $\{\mathfrak{F}_t,\ t\in T\}$ also contains all the sets from $\mathfrak{F}$ having probability measure zero.

Definition (1.3.4) Continuity for the filtration σ-field

The filtration σ-field $\{\mathfrak{F}_t,\ t\in T,\ T=R^+\}$ is right continuous if

$$\mathfrak{F}_t=\mathfrak{F}_{t+}=\bigcap_{\tau>t}\mathfrak{F}_\tau \quad \text{for all } t\in T$$

And left continuous if

$$\mathfrak{F}_t=\mathfrak{F}_{t-}=\sigma\left\{\bigcup_{\tau<t}\mathfrak{F}_\tau\right\}=\bigvee_{\tau<t}\mathfrak{F}_\tau \quad \text{for all } t\in T.$$

Remark (1.3.3)

If the time set T is a set of positive integer N, then $\mathfrak{F}_{n+}$ and $\mathfrak{F}_{n-}$ are interpreted as $\mathfrak{F}_{n+1}$ and $\mathfrak{F}_{n-1}$ for all $n\in N$.

Definition (1.3.5) Adaptation of $\{X_t\}$ [Krishnan 1984]

Let $\{X_t,\ t\in T,\ T=R^+\}$ be a stochastic process defined on a probability space $(\Omega,\mathfrak{F},P)$ and let $\{\mathfrak{F}_t,\ t\in T,\ T=R^+\}$ be a filtration σ-field. The process $\{X_t\}$ is adapted to the family $\{\mathfrak{F}_t\}$ if X_t is $\mathfrak{F}_t-$measurable for every $t\in T$.

$\mathfrak{F}_t$-adapted random process are also $\mathfrak{F}_t$-measurable and nonanticipative with respect to the σ-field $\mathfrak{F}_t$.

Remark (1.3.4)

If $\mathfrak{F}_t$ is the σ-field generated by $\{X_s,\ s \le t\}$, then clearly the process $\{X_t,\ t \in T\}$ is adapted to the family $\{\mathfrak{F}_t,\ t \in T\}$, which is called the natural family or natural filtration of the process $\{X_t\}$.

Definition (1.3.6) Second – order process [Klebaner2005]

A second – order stochastic process $\{X_t,\ t \in T,\}$ is a process which satisfies $E|X_t|^2 < \infty$ for every $t \in T$. It is also called an L^2 – process.

Remark (1.3.5) Klebaner 2005]

The mean, autocovariance ant autocorrelation functions of a second – order process are defined by

$$\mu(t) = EX_t$$

$$C_x(t,s) = E[X_t - \mu(t)][X_s - \mu(s)]^* \qquad (1.3.1)$$

$$R_x(t,s) = EX_t X_s^*$$

where the asterisk denotes the complex conjugate.

The matrix Γ formed by setting $\Gamma_{ij} = C_x(t_i, t_j)$ defined on the square $T \times T$ is a nonnegative definite function.

<u>Definition (1.3.7) Continuous in Probability</u>

A stochastic process $\{X_t, t \in T\}$ is continuous in probability at a point $t \in T$ if for every $\varepsilon > 0$,

$$\lim_{s \to t} P\{|X_s - X_t| \ge \varepsilon \to 0.$$

If $\{X_t, t \in T\}$ is continuous in probability at every point $t \in T$, then we say, that it is continuous in probability on T or simply continuous in probability.

Example (1.3.1)

We assume that the stochastic process $\{X_t, t \in T\}$ is both separable and continuous in probability. We have to determine $P\{X_t \geq 0, t \in T, T = [0,1]\}$.

Clearly this probability is given by the probability of the uncountable intersections $\bigcap_{t \in T} \{\omega : X_t(\omega) \geq 0\}$. To calculate the probability, we assume that the countably dense set S is the partition of T given by:

$$S = \left\{ \frac{k}{2^n}, 0 \leq k \leq 2^n, n = 1,2,... \right\}$$

Then

$$P\{X_t \geq 0, t \in T\} = P\left(\bigcap_{t \in T} \omega \ : X_t(\omega) \geq 0 \right)$$

$$= P\left(\bigcap_{n=0}^{\infty} \omega : X_{k/2^n}(\omega) \geq 0, 0 \leq k \leq 2^n \right)$$

If we define $A_n = \{\omega : X_{k/2^n}(\omega) \geq 0, 0 \leq k \leq 2^n\}$, then the sequence $\{A_n\}$ is a decreasing sequence, and since the probability measure is sequentially continuous, the limit exists and we can write:

$$P\left\{X_t \geq 0, t \in T\right\} = \lim_{n \to \infty} P\left\{\omega : X_{k/2^n}(\omega) \geq 0, 0 \leq k \leq 2^n\right\}.$$

Definition (1.3.8) p^{th} Mean Continuity

A stochastic process $\{X_t, t \in T\}$ defined on a probability space $(\Omega, \mathfrak{F}, P)$ is continuous in the p^{th} mean at $t \in T$ if

$$\lim_{s \to t} E\left|X_s - X_t\right|^p \to 0 \qquad (1.3.2)$$

Remarks (1.3.6)

1. If $\{X_t\}$ is continuous in the p^{th} mean at every $t \in T$, then it is a p^{th} mean continuous process.
2. If p=2, we have a quadratic mean continuous process define by

$$\lim_{s \to t} E|X_s - X_t|^2 \to 0 \tag{1.3.3}$$

Definition (1.3.9) Almost Sure Continuity [Krishnan 1984]

Let $\{X_t, t \in T\}$ be a separable stochastic process defined on a complete probability space $(\Omega, \mathfrak{F}, P)$. The process is almost surely (a.s.) continuous at a point $t \in T$ if

$$P\left\{\omega : \lim_{s \to t} |X_s(\omega) - X_t(\omega)| = 0\right\} = 1 \tag{1.3.4}$$

Remarks (1.3.7) [Krishnan 1984]

1. If the process is almost surely continuous at every point $t \in T$, then it is an almost surely continuous process.
2. Almost sure continuity at a point $t \in T$ can be interpreted in a different way. If we now define an $\mathfrak{F}-$measurable set

   $$\Lambda_t = \left\{\text{set of all trajectories of } X_t \text{ discontinuous at t, } X_t \neq \lim_{s \to t} X_s\right\}.$$

   then almost sure continuity at a point t implies that Λ_t is a null set, that is, $P(\Lambda_t) = 0$. Almost sure continuity of a process implies only that the probability of countable union of Λ_t for $t \in T$ is zero. By its very definition almost sure continuity implies continuity in probability.

Even though almost sure continuity implies that the countable union of Λ_t is also a null set, it is not necessary that the uncountable union of Λ_t be a null set.

Example (1.3.2) [Krishnan 1984]

Let $(\Omega,\mathcal{F},P)$ be a probability space with $\Omega=[0,1]$. Let $\{X_t\}$ be the stochastic process defined on T=[0,1] and Ω by

$$X_{t_i}(\omega)=\begin{cases}0 & t_i<\omega\\ 1 & t_i\geq\omega\end{cases}$$

Clearly all the trajectories of this process are continuous except at the point $t_i=\omega$. Hence the set $N_{t_i}=\{t_i\}$ is a null set, and the countable union $\bigcup_{i=1}^{\infty}\{t_i\}$ is also a null set. However, the uncountable union $\bigcup_{t_i\in T}\{t_i\}$ is equal to Ω and is not a null set.

Definition (1.3.10) Almost surely sample continuous [Krishnan 1984]

Let $\{X_t, t\in T\}$ be a separable stochastic process defined on a complete probability space $(\Omega,\mathcal{F},P)$. The process is almost surely sample continuous if

$$P\left[\bigcup_{t\in T}\left\{\omega:\lim_{x\to\infty}X_s(\omega)-X_t(\omega)\neq 0\right\}\right]=0 \qquad (1.3.5)$$

Remarks (1.3.8) Krishnan 1984]

1. If a stochastic process $\{X_t, t\in T\}$ is almost surely sample continuous, then there is a representation of X_t whose sample function will be continuous at t with probability 1.

2. Almost sure continuity at every $t \in T$ does not imply almost sure sample continuity.
3. Almost sure sample continuity implies almost sure continuity at every $t \in T$, which in turn implies continuity in probability.
4. It is difficult to determine the sample continuity of a process from the definition. To this end the Kolmogorov condition is used to verify sample continuity.

Definition (1.3.11) Kolmogorov Condition [Krishnan 1984]

Let $\{X_t, t \in T\}$ be a separable stochastic process defined on a complete probability space $(\Omega, \mathfrak{F}, P)$. Let T be a closed interval in R^+. If there exist three strictly positive constant α, β, and C such that for all $t \in T$

$$E\left|X_{t+h} - X_t\right|^{\alpha} \leq Ch^{1+\beta} \qquad (1.3.6)$$

Then

$$\lim_{h \to 0} \sup_{\substack{s,t \in T \\ |s-t|<h}} \left|X_s - X_t\right| \xrightarrow{a.s.} 0$$

And almost every sample function is uniformly continuous on T.

1.4 CLASSES OF STOCHASTIC PROCESSES

In this section we shall consider several types of stochastic processes frequently encountered in this thesis and discuss their properties.

Definition(1.4.1) Stationary process [Krishnan 1984]

Let $\{X_t, t \in T\}$ be a stochastic process with time set T defined on a probability space $(\Omega, \mathfrak{F}, P)$ taking values in the state space $(R, \mathfrak{R})$.

Let $T_n = \{t_1, t_2, ... t_n\}$ be any finite set of values belonging to T. Then the process is strictly stationary or stationary if for any real Δt the joint distribution of the sequence $\{X(t_1), X(t_2), ..., X(t_n)\}$ is the same as the joint distribution of $\{X(t_1 + \Delta t), X(t_2 + \Delta t), ..., X(t_n + \Delta t)\}$ for any positive integer n.

Definition (1.4.2) Wide Sense Stationary [Krishnan 2006]

A real stochastic process X_t, $t \in T$, is wide sense stationary or covariance stationary if

1. $EX^2 < \infty$.
2. $\mu_x = EX_t$ a constant. (1.4.1)
3. $C_x(t-s) = E(X_t - \mu)(X_s - \mu)$ depends only on the time difference $t-s$ and not on either t or s.

Remark (1.4.1)

By the very nature of the definition (1.4.2), strict sense stationary implies wide sense stationary, but the converse is not true.

Definition(1.4.3) Dirac Delta Function [Krishnan 2006]

The Dirac delta or Dirac's delta is a mathematical construct introduced by the British theoretical physicist Paul Dirac. Informally, it is a function representing an infinitely sharp peak bounding unit area. a function $\delta(x)$ that has the value zero everywhere except at $x = 0$ where its value is infinitely large in such a way that its total integral is 1.

Definition (1.4.4) White Noise Process [Krishnan 2006]

A zero mean stationary random process X_t whose autocovariance or autocorrelation is given by

$$C_x(\tau) = R_x(\tau) = \sigma_x^2\delta(\tau)$$

where $\delta(\tau)$ is the Dirac delta function, is called a white-noise process. The energy of a white-noise process is infinite since $C_x(0) = R_x(0) = E\left[X^2(t)\right] = \infty$.

Definition (1.4.6) Gaussian Random Variable [Krishnan 1984]

Let X be a random variable with $EX^2 < \infty$, and let $\mu = EX$ and $\sigma^2 = E(X-\mu)^2$. Then the random variable X is Gaussian if the probability distribution function

$$F_x(a) = P\{X \le a\} = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(X-\mu)^2}{2\sigma^2}\right] dx \qquad (1.4.3)$$

Remarks (1.4.2) [Krishnan 2006]

1. σ^2 can have a value 0, in which case the random variable $X = \mu$ with probability 1 and we have a degenerate Gaussian random variable.
2. The probability density function of X in the nondegenerate case is given by

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(X-\mu)^2}{2\sigma^2}\right] \qquad 0 < \sigma^2 < \infty \qquad (1.4.4)$$

3. The characteristic function of a Gaussian random variable is given by

$$\varphi(u) = Ee^{juX} = \exp(ju\mu - \frac{1}{2}u^2\sigma^2) \qquad (1.4.5)$$

Definition (1.4.7) Gaussian Process [Krishnan 1984]

A second – order stochastic process $\{X_t, t \in T\}$ is a Gaussian process if for every finite collection $\{t_1, t_2, ..., t_n\} \subset T$ and every finite linear combination of random variables $X_{t_1}, X_{t_2}, ..., X_{t_n}$, the random variable X given by

$$X = \sum_{i=1}^{n} \alpha_i X_{t_i}, \ \alpha_i \text{ constant} \qquad (1.4.6)$$

is Gaussian for any n.

Definition (1.4.8) Markov Process [Krishnan 2006]

A stochastic process $\{X_t, t \in T\}$ defined on a probability space $(\Omega, \mathfrak{F}, P)$ is a Markov process if for any increasing collection $\{t_1, t_2, ... t_n\} \in T$.

$$P\left\{X_{t_n} \le x_n \middle| X_{t_1} = x_1, X_2 = x_2, ..., X_{t_{n-1}} = x_{n-1}\right\} = P\left\{X_{t_n} \le x_n \middle| X_{t_n - 1} = x_{n-1}\right\}$$

with probability 1.

Remark (1.4.5)

Let $(\Omega, \mathfrak{F}, P)$ be a probability space and let $\{X_t, t \in T\}$ be a stochastic process defined on it. Let $\mathfrak{F}_t$ be the σ-field generated by $\{X_s, s \le t\}$. Let $\mathfrak{F}_t^c$ be the σ-field generated by $\{X_t, s > t\}$, and let $A \in \mathfrak{F}_t$ and $B \in \mathfrak{F}_t$. The process $\{X_t, t \in T\}$ is a Markov process with respect to the family $\{\mathfrak{F}_t, t \in T\}$ if

$$P(A \cap B | \mathfrak{F}_t) = P(A | \mathfrak{F}_t) P(B | \mathfrak{F}_t) \qquad (1.4.7)$$

Definition (1.4.9) Independent Increment Process

A stochastic process $\{X_t, t \in T\}$ defined on the probability space $(\Omega, \mathfrak{F}, P)$ is an independent increment process if for any collection $\{t_1, t_2, ..., t_n\} \subset T$ satisfying $t_1 < t_2 < ... < t_n$ the increments of the process X_{t_1}, $(X_{t_2} - X_{t_1}), (X_{t_3} - X_{t_2}), ..., (X_{t_n} - X_{t_{n-1}})$ are the sequence of independent random variables.

Remarks (1.4.6)

1. If the time set is discrete, $T = N = (0,1,...)$, then an independent increment process reduced to a sequence of independent random variables $Y_0 = X_0$, $Y_i = X_i - X_{i-1}$, i=1,2… .
2. The independent increment process is a special case of a Markov process.
3. If the distribution of the increments $(X_t - X_s), t < s$, depends only on the difference time t – s, then the process is a stationary independent increment process.
4. A stationary independent increment is not a stationary process.

Definition (1.4.10) Poisson Process

The stochastic process $\left\{N_t, t \in T, T = R^+\right\}$ defined on probability space $(\Omega, \mathfrak{F}, P)$ is Poisson process if

1. $N_0 = 0$.
2. For almost all ω, the sample functions $N_t(\omega)$ are monotone functions increasing in isolated jumps of unit magnitude.
3. For every pair s<t, the increment $(N_t - N_s)$ is integral valued with distribution

$$P\left\{N_t - N_s = k\right\} = e^{-\lambda(t-s)\frac{\lambda^k (t-s)^k}{k!}} \tag{1.4.8}$$

Where $\lambda > 0$ is a parameter associated with the Poisson process and is called the density parameter.

4. For every collection $\left\{t_1 < t_2 < ... < t_n\right\} \subset T$ and Borel sets $A_1, A_2, ..., A_n$,

$$\begin{aligned} &P\left\{N_{t_1} \in A_1, N_{t_2} - N_{t_2} \in A_2, ..., N_{t_n} - N_{t_{n-1}} \in A_n\right\} \\ &\quad = P\left\{N_{t_1} \in A_1\right\} P\left\{N_{t_2} - N_{t_2} \in A_2\right\} ... P\left\{N_{t_n} - N_{t_{n-1}} \in A_n\right\} \end{aligned} \tag{1.4.9}$$

Remark (1.4.7)

In the definition (1.4.9) we conclude that the increments N_{t_1}, $(N_{t_2} - N_{t_1}), (N_{t_3} - N_{t_2}), ..., (N_{t_n} - N_{t_{n-1}})$ are independent random variables. If the density parameter $\lambda = 1$, then we have a standard Poisson process.

Example (2.1), [Krishnan, 1984]:

If $\mathcal{F}_t$ is the σ-field by $\left\{X_t, s \le t\right\}$, then clearly the process $\left\{X_t, t \in T\right\}$ is adapted to the family $\{\mathfrak{F}_t, t \in T\}$ which is called the *natural family* or *natural filtration* of the process $\{X_t\}$.

Definition (1.4.12) Step Process, [Friedman, 75]

A stochastic process $f(t)$ defined on $\left[\alpha, \beta\right]$ is called a step function if there exists a partition $\alpha = t_0 < t_1 < ... < t_k = \beta$ of $\left[\alpha, \beta\right]$ such that $f(t) = f(t_i)$ if $t_i \le t \le t_{i+1}$, $0 \le i \le k-1$.

(1.5) BROWNIAN MOTION

Robert Brown in 1826–27 observed the irregular motion of pollen particles suspended in water. He and others noted that

- The path of a given particle is very irregular, having a tangent at no point.
- The motions of two distinct particles appear to be independent.

Definition (1.5.1) Brownian Motion [Krishnan 2006]

The random motion of a particle in a fluid subject to collisions and influence of other particles is called Brownian motion. One of the mathematical models of this motion is the Wiener process.

If the following conditions are satisfied, W(t) is a Wiener process with parameter σ^2:

1. $W(0)=0$.
2. W(t) is a Gaussian process with sample continuous function.
3. $E[W(t)]=0$.
4. The autocovariance function

$$C_w(t,s)=R_w(t,s)=E[W(t),W(s)]=\sigma^2\min(t,s).$$

5. The variance of $W(t)$ from (4) is $\sigma_w^2(t)=\sigma^2 t$.

Remarks (1.5.1) [Krishnan 2006]

1. The probability density function of the wiener process is given by

$$f_w(w,t)=\frac{1}{\sqrt{2\pi\sigma t}}e^{-\frac{1}{2}(\frac{w^2}{\sigma^2 t})} \qquad (1.5.1).$$

2. The covariance matrix of the random variable vector $W^T=[W(t_1),W(t_2),...,W(t_n)]^T$ for times $0<t_1<t_2<...<t_n$ can be given from condition 4 as.

$$C_W = E[WW^T] = \sigma^2 \begin{bmatrix} t_1 & t_1 & t_1 & \dots & t_1 \\ t_1 & t_2 & t_2 & \dots & t_2 \\ t_1 & t_2 & t_3 & \dots & t_3 \\ \vdots & \vdots & \vdots & & \vdots \\ t_1 & t_2 & t_3 & \dots & t_n \end{bmatrix} \qquad (1.5.2)$$

and this matrix is positive definite. Since W(t) is Gaussian, the finite-dimensional density function using Eq. (1.5.2) is given by

$$f_W(\omega_1, t_1; \omega_2, t_2; \dots; \omega_n, t_n) = \frac{1}{(2\pi)^{n/2} |C_W|^{1/2}} e^{-(1/2) W^T C_W^{-1} W}$$

3. The determinant $|C_W| = t_1 (t_2 - t_1) \dots (t_{n-1} - t_n)$ and equation (1.5.2) can be written as

$$f_W(\omega_1, t_1; \omega_2, t_2; \dots; \omega_n, t_n) = \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi(t_k - t_{k-1})\sigma^2}} \exp\left\{ -\frac{1}{2} \left[\frac{(\omega_k - \omega_{k-1})^2}{\sigma^2 (t_k - t_{k-1})} \right] \right\}$$

(1.5.3) with $t_0 = 0 = \omega_0$.

4. The density function of equation (1.5.3) shows that the sequence $\{W_{t_1}, (W_{t_2} - W_{t_1}), \dots, (W_{t_n} - W_{t_{n-1}})\}$ is a collection of independent random variables for increasing $\{0 < t_1 < t_2 < \dots < t_n\}$ and the density depends only on the time difference, showing that the Wiener process is a process of stationary independent increments.
5. From the properties of the random walk from which W(t) was constructed, we conclude that for $(0 < t < s)$ the increments $W(t) - W(0)$ and $W(s) - W(t)$ are independent.
6. from equation (1.5.1), for $t_2 < t_1$ the density function of the increment $W(t_2) - W(t_1)$ is

$$f_W(\omega_1, t_1; \omega_2, t_2) = \frac{1}{\sqrt{2\pi(t_2 - t_1)\sigma^2}} e^{-(1/2)\left[(\omega_2 - \omega_1)^2 / \sigma^2 (t_2 - t_1)\right]^2} \qquad (1.5.4)$$

and since it depends only on the time difference $\left(t_2 - t_1\right)$, $W(t)$ is a stationary independent increment process.

This chapter is divided into three sections. Section 1 presents some basic concepts of Brownian motion calculus. Section 2 deals with Control System Theory. In section 3 the complete controllability of a nonlinear stochastic dynamic system (Standard Brownian motion) are discussed and proved by using the contraction mapping principle.

2.1 BROWNIAN MOTION CALCULUS

In this Section stochastic integrals with respect to Brownian motion are introduced and their properties are given. They are also called Itô integrals, and the corresponding calculus Itô calculus.

Definition (2.1.1) Stochastic Integral[Klebaner 2005]

Let $f(t)$ be a step function in $L^2_\omega[\alpha,\beta]$, say $f(t)=f$ if $t_i \le t \le t_{i+1}$, $0 \le i \le k-1$ where $\alpha = t_0 < t_1 < ... < t_k = \beta$ the random variable

$$\sum_{r=0}^{k-1} f(t_r)\left[W(t_{r+1}) - W(t_r)\right]$$

is denoted by

$$\int_\alpha^\beta f(t)dW(t)$$

and is called the stochastic integral of f with respect to the Brownian motion W; it is also called the Itô integral.

Itô Integral

Our goal is to define the stochastic integral $\int_0^T X(t)dB(t)$, also denoted $\int XdB$ or $X \cdot B$ This integral should have the property that if $X(t) = 1$ then $\int_0^T dB(t) = B(t) - B(0)$. Similarly, if $X(t)$ is a constant c, then the integral should be $c\left(B(t) - B(0)\right)$. In this way we can integrate constant processes with respect to B. The integral over (0,T] should be the sum of integrals over two subintervals (0,a] and (a,T]. Thus if X(t) takes two values c_1 on (0,a], and c_2 on (a,T], then the integral of X with respect to B is easily defined. In this way the integral is defined for simple processes, that is, processes which are constant on finitely many intervals. By the limiting procedure the integral is then defined for more general processes.

Itô Integral of Simple Process

Consider first integrals of a non-random simple process $X(t)$, which is a function of t and does not depend on B(t). By definition a simple non-random process X(t) is a process for which there exist times $0 = t_0 < t_1 < ... < t_n = T$ and constants $c_0, c_1, ..., c_{n-1}$, such that

$$X(t) = c_0 I_0(t) + \sum_{i=0}^{n-1} c_i I_{(t_i, t_{i+1}]}(t)$$

The Itô integral $\int_0^T X(t)dB(t)$ is defined as a sum

$$\int_0^T X(t)dB(t) = \sum_{i=0}^{n-1} c_i \left(B(t_{i+1}) - B(t_i)\right) \tag{2.1}$$

It is easy to see by using the independence property of Brownian increments that the integral, which is the sum in (2.1) is a Gaussian random variable with mean zero and variance

$$\begin{aligned}
Var\left(\int_0^T X(t)dB(t)\right) &= Var\left(\sum_{i=0}^{n-1} c_i \left(B(t_{i+1}) - B(t_i)\right)\right) \\
&= \sum_{i=0}^{n-1} Var\left(c_i \left(B(t_{i+1}) - B(t_i)\right)\right) \\
&= \sum_{i=0}^{n-1} c_i^2 \left(t_{i+1} - t_i\right)
\end{aligned}$$

Example (2.1.1) [Klebaner 2005]

Let $X(t) = -1$ for $0 \le t \le 1$, $X(t) = 1$ for $1 < t \le 2$, and $X(t) = 2$ for $2 < t \le 3$. Then (note that $t_i = 0,1,2,3$, $c_i = X(t_{i+1})$, $c_0 = -1$, $c_1 = 1$, $c_2 = 2$)

$$\begin{aligned}
\int_0^3 X(t)dB(t) &= c_0\left(B(1) - B(0)\right) + c_2\left(B(2) - B(1)\right) + c_3\left(B(3) - B(2)\right) \\
&= -B(1) + \left(B(2) - B(1)\right) + 2\left(B(3) - B(2)\right) \\
&= 2B(3) - B(2) - 2B(1)
\end{aligned}$$

Its distribution is $N(0,6)$, either directly as a sum of independent $N(0,1) + N(0,1) + N(0,4)$ or by using the result above.

Definition (2.1.3) Simple Adapted Process

A process $X = \{X(t), 0 \le t \le T\}$ X is called a simple adapted process if there exist times $0 = t_0 < t_1 < ... < t_n = T$ and random variables $\xi_0, \xi_1, ..., \xi_{n-1}$, such that ξ_0 is a constant, ξ_i is $\mathcal{F}_{t_i}$-measurable (depends on the values of B(t) for $t \le t_i$), but not on values of B(t) for $t > t_i$), and $E(\xi_i^2) < \infty$, $i = 0,1,...,n-1$; such that

$$X(t) = \xi_0 I_0(t) + \sum_{i=0}^{n-1} \xi_i I_{(t_i, t_{i+1}]}(t) \tag{2.2}$$

Remark (2.1.1)

For simple adapted processes Itô integral is defined as a sum

$$\int_0^T X(t) dB(t) = \sum_{i=0}^{n-1} \xi_i \left(B(t_{i+1}) - B(t_i) \right) \tag{2.3}$$

Note that when ξ_i's are random, the integral need not have a Normal distribution.

Theorem (2.1.1) Properties of the Itô Integral of simple Adapted processes [Klebaner 2005]

The following properties carry over to the Itô integral of general processes

1. Linearity. If X(t) and Y(t) are simple processes and α and β are constants then

$$\int_0^T (\alpha X(t)+\beta Y(t))dB(t)=\alpha\int_0^T X(t)dB(t)+\beta\int_0^T Y(t)dB(t).$$

2. For the indicator function of an interval $I_{(a,b]}(t)$ $\left\{I_{(a,b]}(t)=1 \text{ when } t\in(a,b],\text{and zero otherwise}\right\}$.

$$\int_0^T I_{(a,b]}(t)dB(t)=B(b)-B(a),\ \int_0^T I_{(a,b]}(t)X(t)dB(t)=\int_a^b X(t)dB(t).$$

3. Zero mean property. $E\left(\int_0^T X(t)dB(t)\right)=0$.

4. Isometry property . $E\left(\int_0^T X(t)dB(t)\right)^2=\int_0^T E\left(X^2(t)\right)dt$

Definition (2.1.4) Itô Integral of adapted processes

Let $X^n(t)$ be a sequence of simple processes convergent in probability to the process X(t). Then, under some conditions, the sequence of their integrals $\int_0^T X^n(t)dB(t)$ also converges in probability to a limit J. the random variable J is taken to be the integral $\int_0^T X(t)dB(t)$.

Definition (2.1.5) Quadratic Variation [Klebaner2005]

If g is the function of real variable, define its quadratic variation over the interval [0,t] as the limit (when it exists)

$$[g,g](t)=\lim_{\delta_n\to 0}\sum_{i=0}^{n}\left(g(t_i^n)-g(t_{i-1}^n)\right)^2, \qquad (2.4)$$

where the limit is taken over partitions: $0 = t_0^n < t_1^n < t_2^n < ... < t_n^n = t$, with

$\delta_n = \max_{1\le i\le n}\left(t_i^n - t_{i-1}^n\right)$.

Remark (2.1.2) [Klebaner 2005]

The stochastic calculus definition of quadratic variation is different to the classical one with p=2 (unlike for the first variation p=1, when they are the same). In stochastic calculus the limit in (2.4) is taken over shrinking partitions with $\delta_n = \max_{1\le i\le n}\left(t_i^n - t_{i-1}^n\right) \to 0$, and not over all possible partitions. We shall use only the stochastic definition.

Definition (2.1.6) Quadratic Variation of Brownian motion

The quadratic variation of Brownian motion [B,B](t) is defined as

$$[B,B](t) = [B,B]\left(\left[0,t\right]\right) = \lim \sum_{i=1}^{n} \left|B\left(t_i^n\right) - B(t_{i-1}^n)\right|^2,$$

Where the limit taken over all shrinking partition of [0,t], with $\delta_n = \max_{1\le i\le n}\left(t_i^n - t_{i-1}^n\right) \to 0$ as $n \to \infty$.

Theorem (2.1.2) [C Klebaner 2005]

Quadratic variation of a Brownian motion over [0,t] is t.

Example (2.1.2)

We find $\int_0^T X(t)dB(t)$.

Let $0 = t_0^n < t_1^n < t_2^n < ... < t_n^n = T$ be a partition of [0,T], and let

$$X^n(t) = \sum_{i=0}^{n-1} B(t_i^n) I_{(t_i^n, t_{i+1}^n]}(t).$$

Then for any n, $X^n(t)$ is a simple adapted process. $\left(\text{Here } \xi_i^n = B(t_i^n)\right)$. By the continuity of B(t), $\lim_{n\to\infty} X^n(t) = B(t)$ almost surly as $\max_i \left(t_{i+1}^n - t_i^n\right) \to 0$. The Itô integral of the simple function $X^n(t)$ is given by

$$\int_0^T X^n(t)dB(t) = \sum_{i=1}^{n-1} B(t_i^n)\left(B(t_{i+1}^n) - B(t_i^n)\right)$$

We show that this sequence of integrals converges in probability to $J = \frac{1}{2}B^2(t) - \frac{1}{2}T$.

Adding and subtracting $B^2(t_{i+1}^n)$, we obtain

$$B(t_i^n)\left(B(t_{i+1}^n) - B(t_i^n)\right) = \frac{1}{2}\left[B^2\left(t_{i+1}^n\right) - B^2\left(t_i^n\right) - \left(B\left(t_{i+1}^n\right) - B\left(t_i^n\right)\right)^2\right],$$

and

$$\int_0^T X^n(t)dB(t) = \frac{1}{2}\sum_{i=0}^{n-1}\left(B^2\left(t_{i+1}^n\right) - B^2\left(t_i^n\right)\right) - \frac{1}{2}\sum_{i=0}^{n-1}\left(B\left(t_{i+1}^n\right) - B\left(t_i^n\right)\right)^2$$

$$= \frac{1}{2}B^2(T) - \frac{1}{2}B^2(0) - \frac{1}{2}\sum_{i=0}^{n-1}\left(B\left(t_{i+1}^n\right) - B\left(t_i^n\right)\right)^2$$

By the definition of the quadratic variation of Brownian motion the second sum converges in probability to T. Therefore $\int_0^T B(t)dB(t)$ converges in probability to the limit J

$$\int_0^T B(t)dB(t) = J = \lim_{n\to\infty}\int_0^T X^n(t)dB(t) = \frac{1}{2}B^2(t) - \frac{1}{2}T.$$

Remarks (2.1.3)

1. If X(t) is a differentiable function (more generally, a function of finite variation), then the stochastic integral $\int_0^T X(t)dB(t)$ can be defined by formally using the integration by part:

   $$\int_0^T X(t)dB(t) = X(T)B(T) - X(0)B(0) - \int_0^T B(t)dX(t), \tag{2.5}$$

   But this approach fails when X(t) depends on B(t).
2. Brownian motion has no derivative, but it has a generalize derivative as a Schwartz distribution. It is defined by the following relation. For a smooth function g with a compact support (zero outside a finite interval)

   $$\int g(t)B'(t)dt = -\int B(t)g'(t)dt.$$

   But the approach fails when g(t) depends on B(t).
3. For simple processes the Itô integral is defined for each ω, path by path, but in general, this is not possible. For example,

$\int_0^1 B(\omega,t)dB(\omega,t)$ is not defined, whereas $\left(\int_0^1 B(t)dB(t)\right)(\omega) = J(\omega)$ is defined as a limit in probability of integrals (sums) of simple processes.

Theorem (2.1.3) [C Klebaner 2005]

Let X(t) be a regular adapted process such that with probability one $\int_0^T X^2(t)dt < \infty$. Then Itô integral $\int_0^T X(t)dB(t)$ is defined and has the following properties:

1. Linearity. If Itô integrals of X(t) and Y(t) are defined and α and β are some constants then

   $$\int_0^T \left(\alpha X(t) + \beta Y(t)\right)dB(t) = \alpha\int_0^T X(t)dB(t) + \beta\int_0^T Y(t)dB(t).$$

2. $\int_0^T X(t)I_{(a,b]}(t)dB(t) = \int_a^b X(t)dB(t)$.

The following two properties hold when the process satisfies an additional assumption

$$\int_0^T E\left(X^2(t)\right)dt < \infty. \qquad (2.6)$$

3. Zero mean property. If condition (2.5) holds then

   $$E\left(\int_0^T X(t)dB(t)\right) = 0.$$

4. Isometry property. If condition (2.6) holds then

$$E\left(\int_0^T X(t)dB(t)\right)^2 = \int_0^T E\left(X^2(t)\right)dt$$

Remark (2.1.4)

Note that the Itô integral does not have the monotonicity property: $X(t) \le Y(t)$ does not imply $\int_0^T X(t)dB(t) \le \int_0^T Y(t)dB(t)$. A simple counter – example is $\int_0^1 1\times dB(t) = B(1)$ with probability half this is smaller than zero, the Itô integral of zero.

Example (2.1.3)

Take $f(t) = e^t$. $\int_0^1 e^{B(t)}dB(t)$ is well defined as e^x is continuous on R.

Since $E\left(\int_0^1 e^{2B(t)}dt\right) = \int_0^1 E\left(e^{2B(t)}\right)dt = \int_0^1 e^{2t}dt = \frac{1}{2}\left[e^2 - 1\right] < \infty$,

$$E\left(\int_0^1 e^{B(t)}dB(t)\right) = 0, \text{ and } E\left(\int_0^1 e^{B(t)}dB(t)\right)^2 = \frac{1}{2}\left[e^2 - 1\right].$$

Example (2.1.4)

Take $f(t)=t$, that is, consider $\int_0^1 B(t)dB(t)$. Since $\int_0^1 E\left(B^2(t)\right)dt=\int_0^1 tdt=\frac{1}{2}<\infty$. Thus $\int_0^1 B(t)dB(t)$ has mean zero and variance $1/2$.

Example (2.1.5)

Take $f(t)=e^{t^2}$. That is, consider $\int_0^1 e^{B^2(t)}dB(t)$. Although this integral is well defined, the condition (2.15) fails, as $\int_0^T E\left(e^{2B^2(t)}\right)dt=\infty$, due to the fact that $E\left(e^{2B^2(t)}\right)=\int e^{2x^2}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}=\infty$ for $t\geq\frac{1}{4}$. Therefore we can not claim that this Itô integral has finite moments. Then the Expectation of the Itô integral does not exist.

Theorem (2.1.4) [C Klebaner 2005].

Let X(t) and Y(t) be regular adapted processes, such that $E\left(\int_0^T X^2(t)dt\right)<\infty$ and $E\left(\int_0^T Y^2(t)dt\right)<\infty$. Then

$$E\left(\int_0^T X(t)dB(t)\int_0^T Y(t)dB(t)\right)=\int_0^T E\left(X(t)Y(t)\right)dt. \qquad (2.7)$$

Proof

Denote the Itˆo integrals $I_1 = \int_0^T X(t)dB(t)$, $I_2 = \int_0^T Y(t)dB(t)$.

Write their product by using the identity $I_1 I_2 = {(I_1 + I_2)^2}/{2} - {I_1^2}/{2} - {I_2^2}/{2}$.

Then use the isometry property.

Itô's Formula for Brownian motion

Itô formula, also known as the change of variable and the chain rule, is one of the main tools of stochastic calculus. It gives rise to many others, such as Dynkin, Feynman-Kac, and integration by parts formulae.

Theorem (2.1.5)

If B(t) is a Brownian motion on [0,T] and f(x) is a twice continuously differentiable function on R, then for any $t \le T$

$$f(B(t) = f(0) + \int_0^t f'\left(B(s)\right)dB(s) + \frac{1}{2}\int_0^t f''\left(B(s)\right)ds \tag{2.8}$$

Proof in ***[C Klebaner 2005]***.

Theorem (2.1.6) [C Klebaner 2005]

If g is a bounded continuous function and $\{t_i^n\}$ represents partitions of [0,t], then for any $\theta_i^n \in \left(B(t_{i+1}^n) - B(t_i^n)\right)$, the limit in probability

$$\lim_{\delta_n \to \infty} \sum_{i=0}^{n-1} g(\theta_i^n)\left(B(t_{i+1}^n) - B(t_i^n)\right)^2 = \int_0^t g\left(B(s)\right)ds \tag{2.9}$$

Example (2.1.6)

Taking $f(x) = x^m, m \geq 2$, we have

$$B^m(t) = m\int_0^t B^{m-1}(s)dB(s) + \frac{m(m-1)}{2}\int_0^t B^{m-2}(s)ds$$

With m=2,

$$B^2(t) = 2\int_0^t B(s)dB(s) + t$$

Rearranging, we recover the result on the stochastic integral

$$\int_0^t B(s)dB(s) = \frac{1}{2}B^2(t) - \frac{1}{2}t.$$

Definition (2.1.7) Itô Process [C Klebaner 2005].

An Itô process has the form

$$Y(t) = Y(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dB(s),\ 0 \leq t \leq T. \tag{2.10}$$

where Y(0) is $\mathfrak{F}_0$ −measurable, processes $\mu(t)$ and $\sigma(t)$ are $\mathfrak{F}_t$ −adapted, such that $\int_0^T |\mu(t)|dt < \infty$ and $\int_0^T \sigma^2(t)dt < \infty$.

It is said that the process Y(t) has the stochastic differential on [0,T]

$$dY(t) = \mu(t)dt + \sigma(t)dB(t),\ \ 0 \leq t \leq T \tag{2.11}$$

Example (2.1.7)

Example (2.3.6) shows that

$$B^2(t) = t + 2\int_0^t B(s)dB(s)$$

In other words, with $Y(t) = B^2(t)$ we can write $Y(t) = \int_0^t ds + \int_0^t 2B(s)dB(s)$. Thus $\mu(s) = 1$ and $\sigma(s) = 2B(s)$. The stochastic differential of $B^2(t)$

$$d\left(B^2(t)\right) = 2B(t)dB(t) + dt$$

Example (2.1.8)

Taking $f(x) = e^x$, we have

$$e^{B(t)} = 1 + \int_0^t e^{B(s)}dB(s) + \frac{1}{2}\int_0^t e^{B(s)}ds$$

$\therefore Y(t) = e^{B(t)}$ has stochastic differential

$$de^{B(t)} = e^{B(t)}dB(t) + \frac{1}{2}e^{B(t)}dt,$$

or

$$dY(t) = Y(t)dB(t) + \frac{1}{2}Y(t)dt$$

Remark (2.1.5)

Itô's formula in theorem (2.3.5) in differential notation becomes: for C^2 function f

$$d\left(f(B(t))\right)=f'\left(B(t)\right)dB(t)+\frac{1}{2}f''\left(B(t)\right)dt. \qquad (2.12)$$

Example (2.1.9)

We find $d\left(\sin\left(B\left(t\right)\right)\right)$.

$f(x)=\sin\left(x\right)$, $f'(x)=\cos\left(x\right)$, $f''(x)=-\sin\left(x\right)$. Thus

$$d\left(\sin\left(B\left(t\right)\right)\right)=\cos\left(B\left(t\right)\right)dB(t)-\frac{1}{2}\sin\left(B(t)\right)dt.$$

Similarly,

$$d\left(\cos\left(B\left(t\right)\right)\right)=-\sin\left(B\left(t\right)\right)dB(t)-\frac{1}{2}\cos\left(B(t)\right)dt.$$

Example (2.1.10)

We find $d\left(e^{iB(t)}\right)$ with $i^2=-1$.

The application of Itô's formula to a complex-valued function means its application to the real and complex parts of the function. A formal application by treating i as another constant gives the same result. Using example (2.3.9) , we can calculate

$$d\left(e^{iB(t)}\right)=d\cos\left(B(t)\right)+id\sin\left(B(t)\right),$$

or directly by using Itô's formula with

$f(x)=e^{ix}$, we have $f'(x)=ie^{ix}$, $f''(x)=-e^{ix}$ and

$$d\left(e^{iB(t)}\right)=ie^{iB(t)}dB(t)-\frac{1}{2}e^{iB(t)}dt.$$

Thus $X(t)=e^{iB(t)}$ has stochastic differential

$$dX(t)=iX(t)dB(t)-\frac{1}{2}X(t)dt.$$

Remark (2.1.6)

If $Y(t)$ and X(t) have stochastic differentials with respect to the same Brownian motion B(t), then clearly process Y(t)+X(t) also has a stochastic differential with respect to the same Brownian motion. It follows that covariation of X and Y on [0, t] exists and is given by

$$\left[X,Y\right](t)=\frac{1}{2}\left(\left[X+Y,X+Y\right](t)-\left[X,X\right](t)-\left[Y,Y\right](t)\right). \qquad (2.13)$$

Theorem (2.1.7) [C Klebaner 2005].

If X and Y are Itô processes and X is of finite variation, then covariation [X,Y] (t) = 0.

Example (2.1.11)

Let $X(t)=\exp(t)$, $Y(t)=B(t)$, then $\left[X,Y\right](t)=\left[\exp,B\right](t)=0$.

Introduce a convention that allows a formal manipulation with stochastic differentials.

$$dY(t)dX(t)=d\left[X,Y\right](t),$$

And in particular

$$\left(dY(t)\right)^2 = d\left[Y, Y\right](t).$$

Since $X(t) = t$ is a continuous function of finite variation and $Y(t) = B(t)$ is continuous with quadratic variation t, the following rules follow

$$dB(t) = 0,\ \left(dt\right)^2 = 0$$

but

$$\left(dB\left(t\right)\right)^2 = d\left[B, B\right](t) = dt$$

Integrals with respect to Itô processes

It is necessary to extend integration with respect to processes obtained from Brownian motion. Let the Itô integral process $Y(t) = \int_0^t X(s)dB(s)$ be defined for all $t \le T$, where X(t) is an adapted process, such that $\int_0^T X^2(s)ds < \infty$ with probability one. Let an adapted process H(t) satisfy $\int_0^T H^2(s)X^2(s)d(s) < \infty$ with probability one. Then the Itô integral process $Z(t) = \int_0^t H(s)X(s)dB(s)$ is also defined for all $t \le T$.

In this case one can formally write by identifying dY(t) and X(t)dB(t)

$$Z(t) = \int_0^t H(s)dY(s) := \int_0^t H(s)X(s)dB(s) \qquad (2.14)$$

Theorem (2.1.8) Itô's formula for Itô process

Let X(t) have a stochastic differential for $0 \le t \le T$

$$dX(t) = \mu(t)dt + \sigma(t)dB(t)$$

If f(x) is twice continuously differentiable (C^2 function), then the stochastic differential of the process $Y(t) = f\left(X\left(t\right)\right)$ exists and is given by

$$df\left(X\left(t\right)\right) = f'\left(X\left(t\right)\right)dX(t) + \frac{1}{2}f''\left(X\left(t\right)\right)d\left[X,X\right](t)$$
$$= f'\left(X\left(t\right)\right) + \frac{1}{2}f''\left(X\left(t\right)\right)\sigma^2\left(t\right)dt$$
$$= \left(f'\left(X\left(t\right)\right)\mu\left(t\right) + \frac{1}{2}f''\left(X\left(t\right)\right)\sigma^2\left(t\right)\right)dt + f'\left(X\left(t\right)\right)\sigma(t)dB(t)$$

The meaning of the above is

$$f\left(X\left(t\right)\right) = f\left(X\left(0\right)\right) + \int_0^t f'\left(X\left(s\right)\right)dX\left(s\right) + \frac{1}{2}\int_0^t f''\left(X\left(s\right)\right)\sigma^2\left(s\right)ds$$

where the first integral is an Itˆo integral with respect to the stochastic differential.

Definition (2.1.8) Integration By Part

We give a representation of the quadratic covariation [X,Y](t) of two Itô processes X(t) and Y(t) in terms of Itô integrals. This representation gives rise to the integration by parts formula.

Quadratic covariation is a limit over decreasing partitions of [0, t],

$$\left[X,Y\right](t)=\lim_{\delta_n\to 0}\sum_{i=0}^{n-1}\left(X\left(t_{i+1}^n\right)-X\left(t_i^n\right)\right)\left(Y\left(t_{i+1}^n\right)-Y\left(t_i^n\right)\right). \qquad (2.16)$$

The sum on the right above can be written as

$$=\sum_{i=0}^{n-1}\left(X\left(t_{i+1}^n\right)Y\left(t_{i+1}^n\right)-X\left(t_i^n\right)Y\left(t_i^n\right)\right)-\sum_{i=0}^{n-1}\left(X\left(t_i^n\right)Y\left(t_{i+1}^n\right)-Y\left(t_i^n\right)\right)$$

$$-\sum_{i=0}^{n-1}\left(Y\left(t_i^n\right)X\left(t_{i+1}^n\right)-X\left(t_i^n\right)\right)$$

$$=X\left(t\right)Y\left(t\right)-X\left(0\right)Y\left(0\right)-\sum_{i=0}^{n-1}\left(X\left(t_i^n\right)Y\left(t_{i+1}^n\right)-Y\left(t_i^n\right)\right)$$

$$-\sum_{i=0}^{n-1}\left(Y\left(t_i^n\right)X\left(t_{i+1}^n\right)-X\left(t_i^n\right)\right)$$

The last two sums converge in probability to Itô integrals $\int_0^t X(s)dY(s)$ and $\int_0^t Y(s)dX(s)$. Thus the following expression is obtained

$$\left[X,Y\right](t)=X(t)Y(t)-X(0)Y(0)-\int_0^t X(s)dY(s)-\int_0^t Y(s)dX(s).$$

The formula for integration by parts (stochastic product rule) is given by

$$X(t)Y(t)-X(0)Y(0)=\int_0^t X(s)dY(s)+\int_0^t Y(s)dX(s)+\left[X,Y\right](t)$$

In differential notations this reads

$$d\left(X(t)Y(t)\right)=X(t)dY(t)+Y(t)dX(t)+d\left[X,Y\right](t) \qquad (2.17)$$

If

$$dX(t)=\mu_X(t)dt+\sigma_X(t)dB(t),$$
$$dY(t)=\mu_Y(t)dt+\sigma_Y(t)dB(t),$$

then, their quadratic covariation can be obtained formally by multiplication of dX and dY, namely

$$d[X,Y](t) = dX(t)dY(t)$$
$$= \sigma_X(t)\sigma_Y(t)\left(dB(t)\right)^2$$
$$= \sigma_X(t)\sigma_Y(t)dt,$$

Leading to the formula

$$d\left(X(t)Y(t)\right) = X(t)dY(t) + Y(t)dX(t) + \sigma_X(t)\sigma_Y(t)dt \qquad (2.18)$$

Note that if one of the processes is continuous and is of finite variation, then the covariation term is zero. Thus for such processes the stochastic product rule is the same as usual.

Example (2.1.12)

X(t) has stochastic differential

$$dX(t) = B(t)dt + tdB(t),\ X(0) = 0$$

We find X(t), give its distribution, its mean and covariance. X(t) = tB(t) satisfies the above equation, since the product rule for stochastic differentials is the same as usual, when one of the processes is continuous and of finite variation. Thus X(t) = tB(t) is Gaussian, with mean zero, and covariance function

$$\gamma(t,s) = \mathrm{Cov}\left(X(t),X(s)\right) = E\left(X(t)X(s)\right)$$
$$= E\left(B(t)B(s)\right)$$
$$= \mathrm{Cov}\left(B(t)B(s)\right)$$
$$= \min(t,s).$$

Example (2.1.13)

Let Y(t) have stochastic differential

$$dY(t) = \frac{1}{2}Y(t)dt + Y(t)dB(t),\ Y(0) = 1.$$

Let $X(t) = tB(t)$. We find $d\big(X(t)Y(t)\big)$.

Y(t) is a Geometric Brownian motion $e^{B(t)}$. For $d\big(X(t)Y(t)\big)$ use the product rule. We need the expression for d[X,Y](t).

$$\begin{aligned} d\big[X,Y\big](t) &= dX(t)dY(t) \\ &= \big(B(t)dt + tdB(t)\big)\left(\frac{1}{2}Y(t)dt + Y(t)dB(t)\right) \\ &= \frac{1}{2}B(t)Y(t)\big(dt\big)^2 + \left(B(t)Y(t) + \frac{1}{2}tY(t)\right)dB(t)dt + tY(t)\big(dB(t)\big)^2 \\ &= tY(t)dt, \end{aligned}$$

as $\big(dB(t)\big)^2 = dt$ and all the other terms are zero. Thus

$$\begin{aligned} d\big(X(t)Y(t)\big) &= X(t)dY(t) + Y(t)dX(t) + d[X,Y](t) \\ &= X(t)dY(t) + Y(t)dX(t) + tY(t)dt, \end{aligned}$$

and substitution the expression for X and Y the answer is obtained.

Itô's formula for Functions of Two Variables

If two processes X and Y both possess a stochastic differential with respect to B(t) and f(x, y) has continuous partial derivatives up to order two, then $f\big(X(t),Y(t)\big)$ also possesses a stochastic differential. To find its form consider formally the Taylor expansion of order two,

$$df(x,y)=\frac{\partial f(x,y)}{\partial x}dx+\frac{\partial f(x,y)}{\partial y}dy$$
$$+\frac{1}{2}\left(\frac{\partial^2 f(x,y)}{\partial x^2}(dx)^2+\frac{\partial^2 f(x,y)}{\partial y^2}(dy)^2+\frac{\partial^2 f(x,y)}{\partial x\partial y}dxdy\right). \quad (2.19)$$

Now, $(dX(t))^2=dX(t)dX(t)=d[X,X](t)=\sigma_X^2(X(t))dt$,

$(dY(t))^2=d[Y,Y](t)=\sigma_Y^2(Y(t))dt$,

and $dX(t)dY(t)=d[X,Y](t)=\sigma_X(X(t))\sigma_Y(Y(t))dt$, where $\sigma_X(t)$, and $\sigma_Y(t)$ are the diffusion coefficients of X and Y respectively.

Theorem (2.1.9) [C Klebaner 2005]

Let f(x,y) have continuous partial derivatives up to order two (a C^2 function) and X, Y be Itô processes, then

$$df(X(t),Y(t))=\frac{\partial f}{\partial x}(X(t),Y(t))dX(t)+\frac{\partial f}{\partial y}(X(t),Y(t))dY(t)$$
$$+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X(t),Y(t))\sigma_X^2(X(t))dt$$
$$+\frac{1}{2}\frac{\partial^2 f}{\partial y^2}(X(t),Y(t))\sigma_Y^2(Y(t))dt$$
$$+\frac{\partial^2 f}{\partial x\partial y}(X(t),Y(t))\sigma_X(X(t))\sigma_Y(Y(t))dt$$

Example (2.1.14)

If $f(x,y)=xy$, then we obtain a differential of a product (or product rule) which gives the integration by parts formula.

$$d\big(X(t)Y(t)\big)=X(t)dY(t)+Y(t)dX(t)+\sigma_X(t)\sigma_Y(t)dt \qquad (2.20)$$

Remark (2.1.7)

An important case of Itô's formula is for functions of the form $f\big(X(t),t\big)$.

Theorem (2.1.10) [C Klebaner 2005]

Let f(x,t) be twice continuously differentiable in x, and continuously differentiable in t (a $C^{2,1}$ function) and X be an Itô process, then

$$df\big(X(t),t\big)=\frac{\partial f}{\partial x}\big(X(t),t\big)dX(t)+\frac{\partial f}{\partial t}\big(X(t),t\big)d(t)+\frac{1}{2}\sigma_x^2\big(X(t),t\big)\frac{\partial^2 f}{\partial x^2}\big(X(t),t\big)dt$$
(2.21)

This formula can be obtained from Theorem 2.3.9 by taking Y(t) = t and observing that d[Y,Y] = 0 and d[X,Y] = 0.

Example (2.1.15)

We find the stochastic differential of $X(t)=e^{B(t)-t/2}$.

Use Itô's formula with $f(x,t)=e^{x-t/2}$. $X(t)=f\big(B(t),t\big)$ satisfies

$$dX(t) = df\left(B(t),t\right) = \frac{\partial f}{\partial x}dB(t) + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial^2 x}dt$$
$$= f\left(B(t),t\right)dB(t) - \frac{1}{2}f\left(B(t),t\right)dt + \frac{1}{2}f\left(B(t),t\right)dt$$
$$= f\left(B(t),t\right)dB(t) = X(t)dB(t).$$

So that

$$dX(t) = X(t)dB(t).$$

STOCHASTIC DIFFERENTIAL EQUATIONS

Differential equations are used to describe the evolution of a system. Stochastic Differential Equations (SDEs) arise when a random noise is introduced into ordinary differential equations (ODEs).

Definition (2.1.9) Stochastic Differential Equation

Let B(t), $t \geq 0$, be Brownian motion process. An equation of the form

$$dX(t) = \mu\left(X(t),t\right)dt + \sigma\left(X(t),t\right)dB(t)\,, \tag{2.22}$$

where functions $\mu(x,t)$ and $\sigma(x,t)$ are given and X(t) is the unknown process, is called a stochastic differential equation (SDE) driven by Brownian motion. The functions $\mu(x,t)$ and $\sigma(x,t)$ are called the coefficients.

Definition (2.1.10) Strong Solution of the Stochastic Differential Equation [C Klebaner 2005]

A process X(t) is called a strong solution of the SDE (2.22) if for all $t > 0$ the integrals $\int_0^t \mu(X(t),t)ds$ and $\int_0^t \sigma(X(s),s)dB(s)$ exist, with the second being an $\hat{Ito}$ integral, and

$$dX(t) = X(0) + \int_0^t \mu(X(s),s)ds + \int_0^t \sigma(X(s),s)dB(s). \qquad (2.23)$$

Remarks (2.1.8) [C Klebaner 2005]

1. A strong solution is some function (functional) $F(t,(B(s),s < t))$ of the given Brownian motion B(t).
2. When σ = 0, the SDE becomes an ordinary differential equation (ODE).
3. Equations of the form (2.22) are called diffusion-type SDEs. More general SDEs have the form
   $dX(t) = \mu(t)dt + \sigma(t)dB(t),$

   where $\mu(t)$ and $\sigma(t)$ can depend on t and the whole past of the processes X(t) and B(t) $(X(s), B(s), s \le t)$, that is, $\mu(t) = \mu(X(s), s \le t, t)$, $\sigma(t) = \sigma(X(s), s \le t, t)$. The only restriction on μ(t) and σ(t) is that they must be adapted processes, with respective integrals defined.

2.2 BASIC CONCEPTS OF CONTROL THEORY [Ogata, 1997]

The following definitions are needed for complete understanding of the subject:

Definition (2.2.1) Control

Control is a general term for the theory and techniques to change the dynamic performance of a system by imposed control action on the systems, so as to satisfy certain requirement to their best.

Definition (2.2.2) Plant and Control System

A physical object to be controlled is called plant. It may be a heating furnace, a chemical reactor, a spacecraft, etc. A combination of components that acts together and performs a certain objective is called a system. A control system is a system which consists of such components as a sensor, controller, actuator, plant, etc. A plant is usually a given fixed component of a control system.

Definition (2.2.3) Disturbance

A disturbance is signal that tends to a diversely affect the value of the output of system. If the disturbance is generated within the system, it is called internal while an external disturbance is generated outside the system and is an input.

Definition (2.2.4) Feedback Control

Feedback control refers to an operation that, in the presence of disturbances, tends to reduce the difference between the output of a system and some reference input and that does so on the basis of this difference. Here only unpredictable disturbances are so specified, since predictable or known disturbance can always be compensated for within the system.

Definition (2.2.5) Feedback Control System

A system that maintains a prescribed relationship between the output and some reference input by comparing them and using the difference as means of control is called a feedback control system.

Definition (2.2.6) Closed- Loop Control System

Feedback control systems are often referred to as closed loop control systems. In closed-loop control system the actuating error, which is the difference between the input signal and the feed back signal, which may be the output signal itself or a function of the output signal and its derivatives and/or (integrals), is fed to the controller so as to reduce the error and bring the output of the systems to a desired value.

Definition (2.2.7) Open- Loop Control System

Those systems in which the output has no effect on the control action are called open-loop control systems. In other words, in an open-loop control system the output is neither measured nor fed back for comparison with the input. Thus, to each reference input there corresponds a fixed operating condition; as a result, the accuracy of the system depends on calibration.

Definition (2.2.8) State

The state of dynamical control system is the smallest set of variables (called state variables) such that the knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$. Thus, the state of dynamic system at time t is uniquely determined by the state at time t_0 and the input for $t \geq t_0$, which is independent to the state and input before t_0. In this work, the state variables are symbolized by x_1, x_2, …, x_n.

Definition (2.2.9) State Space

The n-dimensional space whose coordinate axes consist of the x_1-axis, x_2-axis, ..., x_n- axis is called state space. Any state can be represented by a point in the state space.

Definition (2.2.10) Controllability [Ogata, 1997]

Consider the system (1.6)

$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$

This linear system is said to be controllable at time t_0 if it is possible by means of an unconstrained control vector $u(t)$ to transfer the system from any initial state $x(t_0)$ to any other state in a finite interval of time.

Definition (2.2.11) (Completely State Controllable) [C Klebaner 2005]

The system (1.6) is said to be completely state controllable if there exists some input $u(t)$ defining on $[t_0, t_1]$, which gives $x(t_1) = 0$, for all initial time t_0 and all initial state $x(t_0)$.

Theorem 2.2.1 [Arapostathis 2001]

The system described by equation (1.6)

$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$

is completely state controllable if and only if the composite $n \times nr$ matrix M where

$$\mathrm{M} = [\mathrm{B} \vdots \mathrm{AB} \vdots \mathrm{A}^2\mathrm{B} \vdots \ \dots \ \vdots \mathrm{A}^{n-1}\mathrm{B}]$$

is of rank n.

Theorem 2.2.2 [C Klebaner 2005]

The system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is controllable if and only if one of the following conditions is satisfied

1- All rows of $e^{-At}B$ (and consequently of $e^{At}B$) are linearly independent on $[0,\infty)$ over $\not{c}$, the filed of complex numbers;

2- For every eigenvalue λ of A (and consequently for every λ in $\not{c}$), the $n \times nr$ complex matrix $[\lambda I - A \vdots B]$ has rank n;

3- If the rank of $[B \vdots AB \vdots A^2B \vdots \ \ldots \ \vdots A^{n-1}B]$ is n, which n is the dimension of the system.

(2.3) PROBLEM FORMULATION

Consider the nonlinear stochastic control system

$$\begin{aligned} dx(t) &= [Ax(t) + Bu(t) + f\big(t, x(t)\big) + N_1\big(t, x(t), N_2\big(t, x(t)\big)\big) \\ &\quad + \sigma\big(t, x(t)\big)dW(t) \qquad (*) \\ x(0) &= x_0 \in R^n, \quad t \in [0, b] \end{aligned}$$

Where A and B are matrices of dimension $n \times n$ and $n \times m$ respectively, $N_2 : [0,b] \times R^n \to R^n$, $N_1 : [0,b] \times R^n \times R^n \to R^n$, $f : [0,b] \times R^n \to R^{n \times n}$, $\sigma : [0,b] \times R^n \to R^{n \times n}$, u(t) is a feedback control and W is a n – dimensional standard Brownian motion.

Let (Ω, Γ, P) be a probability space with probability measure P on Ω and $\{\Gamma_t \mid t \in [0,b]\}$ be the filtration generated by $\{w(s), 0 \le s \le t\}$. Let $L_2(\Omega, \Gamma_b, R^n)$ be the Hilbert space of all Γ_b – measurable square integrable variables with values in R^n. Let $L_p^{\Gamma}([0,b],R^n)$ be the Banach space of all $p-$integrable and Γ_t-measurable processes with values in R^n for $p \geq 2$. Further H_2 is the Banach space of all square integrable and Γ_t-adapted processes $\vartheta(t)$ with norm $\|\vartheta\|^2 = \sup_{t\in[0,b]} E\|\vartheta(t)\|^2$. Denote $\Phi(t)=\exp(At)$ and by a control set $U = L_2^{\Gamma}([0,b],R^m)$.

Now let us introduce the following operators and sets.

The operator $L_0^b \in \mathscr{L}(L_2^{\Gamma}([0,b],R^m), L_2(\Omega,\Gamma_b,R^n))$ is defined by

$$L_0^b u = \int_0^b \Phi(b-s)Bu(s)ds$$

and set of all states attainable from x_0 in time $t>0$ is

$$\mathscr{R}_t(x_0) = \{x(t;x_0,u) : u(.) \in L_2^{\Gamma}([0,b],R^m)\},$$

where $x(t;x_0,u)$ is the solution of (*) corresponding to $x_0 \in R^n$, $u(.) \in L_2^{\Gamma}([0,b],R^m)$.

Clearly the adjoint $(L_0^b)^* : L_2([0,b],R^n) \to L_2^{\Gamma}([0,b],R^m)$ is defined by

$$(L_0^b)^* z = B^*\Phi^*(b-t)E\{z|\Gamma_t\}.$$

The controllability operator $\prod_0^b(.)$ is

$$\prod\nolimits_0^b(.) = \int_0^b \Phi(b-t)BB^*\Phi^*(b-t)E\{.|\Gamma_t\}dt$$

which belongs to $\mathscr{L}(L_2(\Omega,\Gamma_b,R^n), L_2(\Omega,\Gamma_b,R^n))$ and the controllability matrix $\Psi_s^b \in \mathscr{L}(R^n,R^n)$ is

$$\Psi_s^b = \int_s^b \Phi(b-t)BB^*\Phi^*(b-t)dt.$$

Definition (2.3.1) Complete controllability

The system (*) is said to be completely controllable on [a,b] if $\mathcal{R}_b(x_0) = L_2\left(\Omega, \Gamma_b, R^n\right)$, that is, all the points in $L_2\left(\Omega, \Gamma_b, R^n\right)$ can be reached from the point x_0 in time b.

Remark (2.3.1)

By the solution of the system (*)

$$\begin{aligned} x(t) &= \Phi(t)x_0 + \int_0^t \Phi(t-s)[Bu(s) + f(s,x(s))ds \\ &\quad + \int_0^t \Phi(t-s)N_1(s,x(s),N_2(s,x(s)))ds + \int_0^t \Phi(t-s)\sigma(s,x(s))dW(s) \end{aligned} \qquad (**)$$

To prove the complete controllability of the equation (*), it is enough to prove the complete controllability of (**).

Lemma (2.3.1)

1. $\ell_1 = \max\left\{\|\Phi(t)\|^2 : t \in [0,b]\right\}$.

2. For some $v > 0$ $E\left\langle \prod_0^b z, z \right\rangle \geq vE\|z\|^2$, for all $z \in L_2(\Omega, \Gamma_b, R^n)$, and consequently $E\left\|\left(\prod_0^b\right)^{-1}\right\|^2 \leq \frac{1}{v} = \ell_2$ **[Mahmudov 2003].**

3. $\ell_3 = E\|x_b\|^2$.

4. $M = \max\left\{\|\Psi_s^b\|^2 : s \in [0,b]\right\}$.

5. $E\left\|\prod_0^t z\right\|^2 \leq M\ E\|z\|^2 \quad \forall z \in L_2(\Omega, \Gamma_t, R^n)$ **[Mahmudov 2003].**

Definition (2.3.2) Lipchitz Condition

A function f is said to satisfy Lipchitz condition in a region D if it is satisfy the following inequality

$$|f(t,y_2)-f(t,y_1)|\le K|y_2-y_1|$$

Definition (2.3.3) Growth Condition

Linear growth condition also appears in the results on existence and uniqueness of solutions of differential equations. f(x) satisfies the linear growth condition if

$$\|f(x)\|\le K(1+\|x\|)$$

This condition describes the growth of a function for large values of x, and states that f is bounded for small values of x.

Example (3.3.1)

It can be shown that if f(0, t) is a bounded function of t, $\|f(0,t)\|\le C$ for all t, and f(x, t) satisfies the Lipschitz condition in x uniformly in t, $\|f(x,t)-f(y,t)\|\le K\|x-y\|$, then f(x, t) satisfies the linear growth condition in x, $\|f(x,t)\|\le K_1(1+\|x\|)$.

Remark (2.3.2)

The polynomial growth condition on f is the condition of the form

$$\|f(x)\|\le K(1+\|x\|^m),\ \text{for some}\ \ K,m>0.$$

Lemma (2.3.2) [Mahmudov 2003]

Assume that the operator $\prod_0^t$ is invertible. Then for arbitrary $x_T \in L_2(\Omega,\mathfrak{F}_t,R^n)$, $f(.)\in L_2^{\mathfrak{F}}\left([0,b],R^n\right)$, $\sigma(.)\in L_2^{\mathfrak{F}}\left([0,b],R^{n\times n}\right)$, the control

$$u(t)=B^*\Phi^*(t-s)E[\left(\prod\nolimits_0^b\right)^{-1}\{x_b-\Phi(b)x_0-\int_0^b\Phi(b-s)f(s,x(s))ds$$
$$-\int_0^b\Phi(b-s)N_1\left(s,X(s),N_2(s,x)\right)ds$$
$$-\int_0^b\Phi(b-s)\sigma\left(s,x(s)\right)dW(t)\}\left|\Gamma_t\right.]$$

transfers the system

$$x(t)=\Phi(t)x_0+\int_0^t\Phi(t-s)[Bu(s)+f\left(s,x(s)\right)$$
$$+N_1\left(s,X(s),N_2(s,x)\right)]ds+\int_0^t\Phi(t-s)\sigma\left(s,x(s)\right)dW(s)$$

From $x_0\in R^n$ to x_b at time b and

$$\therefore T_x(t)=\Phi(t)x_0+\prod\nolimits_0^t\Big(\Phi^*(t-s)\left(\prod\nolimits_0^b\right)^{-1}\{x_b-\Phi(b)x_0$$
$$-\int_0^b\Phi(b-s)f(s,x(s))ds-\int_0^b\Phi(b-s)N_1\left(s,X(s),N_2(s,x)\right)ds$$
$$-\int_0^b\Phi(b-s)\sigma\left(s,x(s)\right)dW(s)\}\Big)$$
$$+\int_0^t\Phi(t-s)[f\left(s,x(s)\right)+N_1\left(s,X(s),N_2(s,x)\right)]ds$$
$$+\int_0^t\Phi(t-s)\sigma\left(s,x(s)\right)dW(s)$$

Some Lipchitz and linear growth conditions

i. f, N_1, N_2 and σ satisfies the Lipchitz condition and there exist L_1, L_2, L_3, L_4 and k for every $t \in [0,b]$, $x_i, y_i \in R^n, i=1,2$ such that

$$\|N_1(t,x_1,y_1) - N_1(t,x_2,y_2)\|^2 \le L_1\left[\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2\right],$$

$$\|N_2(t,x) - N_2(t,y)\|^2 \le L_2\|x-y\|^2,$$

$$\|f(t,x) - f(t,y)\|^2 \le L_3\|x-y\|^2,$$

$$\|\sigma(t,x) - \sigma(t,y)\|^2 \le L_4\|x-y\|^2$$

and

$$\|N_2(t,x)\|^2 \le k\|x\|^2$$

ii. f, N_1 and σ are continuous and satisfies

$$\|f(t,x)\|^2 + \|N_1(t,x,y)\|^2 + \|\sigma(t,x)\|^2 \le L\left(\|x\|^2 + \|y\|^2 + 1\right),$$

where L is a positive constant.

iii. $E\left\|\prod_0^t z\right\|^2 \le M\, E\|z\|^2$ **[Mahmudov 2003].**

iv. Let $p = \left[4M\ell_1^2\ell_2 bL_1\left(bL_2+1\right) + 4M\ell_1^2\ell_2 L_3 + 4\ell_1 bL_1\left(bL_2+1\right) + 4\ell_1 L_3\right]b$ be such that $0 \le p < 1$.

Theorem(2.4.1)

If the hypotheses (i) - (iv) are satisfied, then the system (**) is completely controllable on [a,b].

Proof

Define the nonlinear operator $T: H_2 \to H_2$ by

$$T_x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-s)[Bu(s) + f(s,x(s)) + N_1(s,X(s),N_2(s,x))]ds + \int_0^t \Phi(t-s)\sigma(s,x(s))dW(s) \quad (2.1)$$

Choose a feed back control function

$$u(t) = B^*\Phi^*(t-s)E[\left(\prod_0^b\right)^{-1}\{x_b - \Phi(b)x_0 - \int_0^b \Phi(b-s)f(s,x(s))ds - \int_0^b \Phi(b-s)N_1(s,X(s),N_2(s,x))ds - \int_0^b \Phi(b-s)\sigma(s,x(s))dW(t)\}|\Gamma_t] \quad (2.2)$$

Put (2.2) in (2.1) to obtain

$$\begin{aligned} T_x(t) = \Phi(t)x_0 &+ \int_0^t \Phi(t-s)[BB^*\Phi^*(t-s)E[\left(\prod_0^b\right)^{-1}\{x_b - \Phi(b)x_0 \\ &- \int_0^b \Phi(b-s)f(s,x(s))ds - \int_0^b \Phi(b-s)N_1(s,X(s),N_2(s,x))ds \\ &- \int_0^b \Phi(b-s)\sigma(s,x(s))dW(s)\}|\Gamma_t]]ds \quad (2.3) \\ &+ \int_0^t \Phi(t-s)[f(s,x(s)) + N_1(s,X(s),N_2(s,x))]ds \\ &+ \int_0^t \Phi(t-s)\sigma(s,x(s))dW(s) \end{aligned}$$

$$\because \prod_0^b(.) = \int_0^b \Phi(b-t)BB^*\Phi^*(b-t)E\{.|\Gamma_t\}dt$$

$$\therefore T_x(t) = \Phi(t)x_0 + \prod_0^t \Big(\Phi^*(t-s)\Big(\prod_0^b\Big)^{-1} \{x_b - \Phi(b)x_0$$

$$-\int_0^b \Phi(b-s)f(s,x(s))ds - \int_0^b \Phi(b-s)N_1\big(s,X(s),N_2(s,x)\big)ds$$

$$-\int_0^b \Phi(b-s)\sigma\big(s,x(s)\big)dW(s)\} \Big)$$

$$+\int_0^t \Phi(t-s)[f\big(s,x(s)\big) + N_1\big(s,X(s),N_2(s,x)\big)]ds$$

$$+\int_0^t \Phi(t-s)\sigma\big(s,x(s)\big)dW(s)$$

Note that the control (2.2) transfers the system (*) from the initial state x_0 to the final state x_b provided that the operator T has a fixed point then the system (*) is completely controllable. To prove the complete controllability it is enough to show that the operator T has a fixed point in H_2. To prove this result, we use the contraction mapping principle, first we show that T maps H_2 into itself.

Now we have:

$$E\|T_x(t)\|^2 = \Big\|\Phi(t)x_0 + \prod_0^t\Big(\Phi^*(t-s)\Big(\prod_0^b\Big)^{-1}\ \{x_b - \Phi(b)x_0$$

$$-\int_0^b \Phi(b-s)f(s,x(s))ds - \int_0^b \Phi(b-s)N_1\big(s,X(s),N_2(s,x)\big)ds$$

$$-\int_0^b \Phi(b-s)\sigma\big(s,x(s)\big)dW(s)\}\Big)$$

$$+\int_0^t \Phi(t-s)[f\big(s,x(s)\big) + N_1\big(s,X(s),N_2(s,x)\big)]ds$$

$$+\int_0^t \Phi(t-s)\sigma\big(s,x(s)\big)dW(s)\Big\|^2$$

$$\le 4\|\Phi(t)x_0\|^2 + 4E\Big\|\prod_0^t\Big(\Phi^*(t-s)\Big(\prod_0^b\Big)^{-1}\ \{x_b - \Phi(b)x_0$$

$$-\int_0^b \Phi(b-s)f(s,x(s))ds - \int_0^b \Phi(b-s)N_1\big(s,X(s),N_2(s,x)\big)ds$$

$$-\int_0^b \Phi(b-s)\sigma\big(s,x(s)\big)dW(s)\}\Big)\Big\|^2$$

$$+4b\int_0^t \|\Phi(t-s)\|^2 E\|f\big(s,x(s)\big)\|^2 ds + 4b\int_0^t \|\Phi(t-s)\|^2 E\|N_1\big(s,X(s),N_2(s,x)\big)\|^2 ds$$

$$+4\int_0^t \|\Phi(t-s)\|^2 E\|\sigma\big(s,x(s)\big)\|^2 dW(s) \tag{2.4}$$

$\because E\left\|\prod_0^t z\right\|^2 \le M\ E\|z\|^2$

$$E\|T_x(t)\|^2 \le 4\|\Phi(t)\|^2\|x_0\|^2 + 4M\,E\Big\|\Phi^*(t-s)\left(\prod\nolimits_0^b\right)^{-1}\{x_b - \Phi(b)x_0$$

$$-\int_0^b \Phi(b-s)f(s,x(s))ds - \int_0^b \Phi(b-s)N_1\left(s,X(s),N_2(s,x)\right)ds$$

$$-\int_0^b \Phi(b-s)\sigma\left(s,x(s)\right)dW(s)\}\Big\|^2$$

$$+4\int_0^t \|\Phi(t-s)\|^2 E\|f\left(s,x(s)\right)\|^2 ds + 4\int_0^t \|\Phi(t-s)\|^2 E\|N_1\left(s,X(s),N_2(s,x)\right)\|^2 ds$$

$$+4\int_0^t \|\Phi(t-s)\|^2 E\|\sigma\left(s,x(s)\right)\|^2 dW(s)$$

$$\le 4\ell_1\|x_0\|^2 + 16M\ell_1\ell_2\left[\ell_3 + \ell_1\|x_0\|^2 + \int_0^t \|\Phi(t-s)\|^2 E\|f(s,x)\|^2 ds\right.$$

$$\left.+\int_0^t \|\Phi(t-s)\|^2 E\|N_1\left(s,X(s),N_2(s,x)\right)\|^2 ds + \int_0^t \|\Phi(t-s)\|^2 E\|\sigma\left(s,x(s)\right)\|^2 dW(s)\right]$$

$$+4\ell_1\left(\int_0^t\left(E\|f(s,x)\|^2\right)ds + \int_0^t E\|N_1\left(s,x(s),N_2(s,x)\right)\|^2 ds + \int_0^t E\|\sigma\left(s,x(s)\right)\|^2 dW(s)\right)$$

$$\le 4\ell_1\|x_0\|^2 + 16M\ell_1\ell_2\left[\ell_3 + \ell_1\|x_0\|^2 + \int_0^t \ell_1 LE\left(\|x(s)\|^2 + \|N_2(s,x(s))\|^2 + 1\right)ds\right]$$

$$+4\ell_1\left[\int_0^t LE\left(\|x(s)\|^2 + \|N_2(s,x(s))\|^2 + 1\right)ds\right]$$

$$\le\left(16M\ell_1^2\ell_2 + 4\ell_1\right)\|x_0\|^2 + 16M\ell_1\ell_2\ell_3 + \left(16M\ell_1^2\ell_2 + 4\ell_1\right)L\int_0^t\left[\left((k+1)E\|x(s)\|^2 + 1\right)\right]ds$$

$$\because\left(16M\ell_1^2\ell_2 + 4\ell_1\right)\|x_0\|^2 + 16M\ell_1\ell_2\ell_3 \ge 0$$

$$\therefore E\|T_x(t)\|^2 \le \left(16M\ell_1^2\ell_2 + 4\ell_1\right)L\int_0^t\left[\left((k+1)E\|x(s)\|^2 + 1\right)\right]ds$$

There exist a constant K_1 such that

$$E\|T_x(t)\|^2 \le K_1\left(1+(k+1)\int_0^t E\|x(s)\|^2 ds\right)$$

$$\le K_1\left(1+(k+1)\sup_{0\le s<1} E\|x(s)\|^2\right) \qquad \text{for all } t\in[0,b]$$

Thus T maps H_2 into itself.

Secondly, we prove that T is a contraction mapping

$$E\|T_x(t)-T_y(t)\|^2 = E\Bigg\|\prod_0^t\Big(\Phi^*(t-s)\Big(\prod_0^b\Big)^{-1}\times\Big[\int_0^b \Phi(b-s)\big(f(s,x(s))-f(s,y(s))\big)ds$$

$$+\int_0^b \Phi(b-s)\big(N_1(s,x(s),N_2(s,x))-N_1(s,y(s),N_2(s,x))\big)ds$$

$$-\int_0^b \Phi(b-s)\big(\sigma(s,x(s))-\sigma(s,y(s))\big)dW(s)\Big]\Big)$$

$$+\int_0^t \Phi(b-s)\big(f(s,x(s))-f(s,y(s))\big)ds$$

$$+\int_0^t \Phi(b-s)\big(N_1(s,x(s),N_2(s,x))-N_1(s,y(s),N_2(s,x))\big)ds$$

$$+\int_0^t \Phi(b-s)\big(\sigma(s,x(s))-\sigma(s,y(s))\big)dW(s)\Bigg\|^2$$

$$\leq 4M\ell_1\ell_2\left\{b\ell_1 L_3\int_0^t E\|x(s)-y(s)\|^2 ds\right.$$
$$+b\ell_1 L_1\left[\int_0^t E\|x(s)-y(s)\|^2 ds+\int_0^t E\left\|N_2\left(s,x(s)\right)-N_2\left(s,y(s)\right)\right\|^2 ds\right]$$
$$\left.+\ell_1 L_4\int_0^t E\|x(s)-y(s)\|^2 ds\right\}$$
$$+b\ell_1 L_3\int_0^t E\|x(s)-y(s)\|^2 ds$$
$$+b\ell_1 L_1\left[\int_0^t E\|x(s)-y(s)\|^2 ds+\int_0^t E\left\|N_2\left(s,x(s)\right)-N_2\left(s,y(s)\right)\right\|^2 ds\right]$$
$$+\ell_1 L_4\int_0^t E\|x(s)-y(s)\|^2 ds$$

$$\leq 4M\ell_1\ell_2\left\{b\ell_1 L_3\int_0^t E\|x(s)-y(s)\|^2 ds\right.$$
$$+b\ell_1 L_1\left[\int_0^t E\|x(s)-y(s)\|^2 ds+L_2\int_0^t E\|x(s)-y(s)\|^2 ds\right]$$
$$\left.+\ell_1 L_4\int_0^t E\|x(s)-y(s)\|^2 ds\right\}$$
$$+b\ell_1 L_3\int_0^t E\|x(s)-y(s)\|^2 ds$$
$$+b\ell_1 L_1\left[\int_0^t E\|x(s)-y(s)\|^2 ds+L_2\int_0^t E\|x(s)-y(s)\|^2 ds\right]$$
$$+\ell_1 L_4\int_0^t E\|x(s)-y(s)\|^2 ds$$

$$\le 4M\ell_1\ell_2\left\{b\ell_1L_3\int_0^t E\|x(s)-y(s)\|^2\,ds\right.$$
$$\left.+\,b\ell_1L_1\left(1+L_2\right)\int_0^t E\|x(s)-y(s)\|^2\,ds+\ell_1L_4\int_0^t E\|x(s)-y(s)\|^2\,ds\right\}$$
$$+\,b\ell_1L_3\int_0^t E\|x(s)-y(s)\|^2\,ds$$
$$+\,b\ell_1L_1\left(1+L_2\right)\int_0^t E\|x(s)-y(s)\|^2\,ds+\ell_1L_4\int_0^t E\|x(s)-y(s)\|^2\,ds$$
$$\le\left\{4M\ell_1^2\ell_2bL_3+4M\ell_1^2bL_1\left(1+L_2\right)+4M\ell_1^2L_4\right.$$
$$\left.+b\ell_1L_3+b\ell_1L_1\left(1+L_2\right)+\ell_1L_4\right\}\int_0^t E\|x(s)-y(s)\|^2\,ds$$

$$\le\left\{4M\ell_1^2\ell_2bL_3+4M\ell_1^2bL_1\left(1+L_2\right)+4M\ell_1^2L_4\right.$$
$$\left.+b\ell_1L_3+b\ell_1L_1\left(1+L_2\right)+\ell_1L_4\right\}\sup_{s\in[0,b]}E\|x(s)-y(s)\|^2$$

Therefore, T is a contraction mapping and hence there exists a unique fixed point $x(.)$ in H_2 which is the solution of the equation (**). Thus the system (**) is completely controllable on [0,b].

This chapter is divided into two sections. Section 1 presents some basic concepts of Fractional Brownian motion. In section 2 the complete controllability of a nonlinear stochastic dynamic system (fractional Brownian motion) are discussed and proved by using the contraction mapping principle

3.1 FRACTIONAL BROWNIAN MOTION [Biagini and Others 2008]

The fractional Brownian motion was first introduced within a Hilbert space framework by Kolmogorov in 1940, where it was called Wiener Helix. It was further studied by Yaglom in [25]. The name fractional Brownian motion is due to Mandelbrot and Van Ness, who in 1968 provided a stochastic integral representation of this process in terms of a standard Brownian motion.

Definition (3.1.1) Fractional Brownian Motion [Biagini and Others 2008]

Let H be a constant belonging to (0,1). A fractional Brownian motion (fBm) $(B^{(H)}(t))_{t\geq 0}$ of Hurst index H is a continuous and centered Gaussian process with covariance function

$$E\,[B^{(H)}(t)B^{(H)}(s)] = 1/2(t^{2H} + s^{2H} - |t - s|^{2H}).$$

Remark (3.1.1) Properties of Fractional Brownian Motion

For H = 1/2, the fBm is then a standard Brownian motion. By Definition 1.1.1 we obtain that a standard fBm $B^{(H)}$ has the following properties:

1. $B^{(H)}(0) = 0$ and $E[B^{(H)}(t)] = 0$ for all $t \geq 0$.
2. $B^{(H)}$ has homogeneous increments, i.e., $B^{(H)}(t + s) - B^{(H)}(s)$ has the same law of $B^{(H)}(t)$ for s, $t \geq 0$.
3. $B^{(H)}$ is a Gaussian process and $E\left[B^{(H)}(t)^2\right] = t^{2H}, t \geq 0, \text{for all } H \in (0,1)$.
4. $B^{(H)}$ has continuous trajectories.

Remark (3.1.2) Stochastic integral representation [Biagini and Others 2008]

Here we discuss some of the integral representations for the fBm. In [25], it is proved that the process

$$Z(t) = \frac{1}{\Gamma(H + 1/2)} \int_R \left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right) dB(s)$$

$$= \frac{1}{\Gamma(H + 1/2)} \left(\int_{-\infty}^{0} \left((t-s)^{H-1/2} - (-s)^{H-1/2} \right) dB(s) + \int_0^t (t-s)^{H-1/2} dB(s) \right) \quad \ldots(3.1)$$

where B(t) is a standard Brownian motion and Γ represents the gamma function , is a fBm with Hurst index $H \in (0,1)$.

If B(t) is replaced by a complex-valued Brownian motion, the integral (1.1) gives the complex fBm. First we notice that Z(t) is a continuous centered Gaussian process. Hence, we need only to compute the covariance functions. In the following computations we drop the constant 1/Γ (H+1/2) for the sake of simplicity. We obtain

$$\begin{aligned} E\left[Z^2(t)\right] &= \int_R \left[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right]^2 ds \\ &= t^{2H} \int_R \left[(1-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}\right]^2 du \\ &= C(H) t^{2H}, \end{aligned}$$

where we have used the change of variable s = tu. Analogously, we have that

$$\begin{aligned} E\left[|Z(t)-Z(s)|^2\right] &= \int_R \left[(t-u)_+^{H-\frac{1}{2}} - (s-u)_+^{H-\frac{1}{2}}\right]^2 ds \\ &= t^{2H} \int_R \left[(t-s-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}\right]^2 du \\ &= C(H) |t-s|^{2H} \end{aligned}$$

Now

$$\begin{aligned} E\left[Z(t)Z(s)\right] &= -\frac{1}{2}\left\{E\left[|Z(t)-Z(s)|^2\right] - E\left[Z(t)^2\right] - E\left[Z(s)^2\right]\right\} \\ &= \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \end{aligned}$$

Hence we can conclude that Z(t) is a fBm of Hurst index H.

Definition (3.1.2) Correlation between two increments [Biagini and Others 2008]

For H = 1/2, B(H) is a standard Brownian motion; hence, in this case the increments of the process are independent. On the contrary, for $H \neq 1/2$ the increments are not independent. More precisely, by Definition 1.1.1 we know that the covariance between $B^{(H)}(t+h)\ B^{(H)}(t)$ and $B^{(H)}(s+h)\ B^{(H)}(s)$ with s + h   t and t   s = nh is

$$\rho_H(n) = \frac{1}{2} h^{2H} \left[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H} \right].$$

Remark (3.1.3)

In particular, we obtain that two increments of the form $B^{(H)}(t+h)\ B^{(H)}(t)$ and $B^{(H)}(t + 2h)\quad B^{(H)}(t + h)$ are positively correlated for $H>1/2$, while they are negatively correlated for $H < 1/2$. In the first case the process presents an aggregation behavior and this property can be used in order to describe “cluster” phenomena (systems with memory and persistence). In the second case it can be used to model sequences with intermittency and anti-persistence.

Definition (3.1.3) Long-range dependence [Biagini and Others 2008]

A stationary sequence $\{X_n\}_{n\in N}$ exhibits long-range dependence if the autocovariance functions ρ(n) := cov(X_k,X_{k+n}) satisfy

$$\lim_{n\to\infty}\frac{\rho(n)}{cn^{-\alpha}}=1$$

for some constants c and α (0, 1).

Remark (3.1.4)

the dependence between X_k and X_{k+n} decays slowly as n tends to infinity and

$$\sum_{n=1}^{\infty}\rho(n)=\infty.$$

Hence, we obtain immediately that the increments $X_k := B^{(H)}(k)\, B^{(H)}(k\;\;1)$ of $B^{(H)}$ and $X_{k+n} := B^{(H)}(k + n)\;\; B^{(H)}(k + n\;\;1)$ of $B^{(H)}$ have the long-range dependence property for H >1/2 since

$$\rho_H(n) = \frac{1}{2}\;[(n + 1)^{2H} + (n\;\;1)^{2H}\;\;2n^{2H}]\;\;H(2H\;\;1)^{n2H-2}$$

as n goes to infinity. In particular,

$$\lim_{n\to\infty}\frac{\rho_H(n)}{H(2H-1)n^{2H-2}}=1$$

Summarizing, we obtain

1. For H >1/2, $\sum_{n=1}^{\infty} \rho_H(n) = \infty$

2. For H <1/2, $\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$.

There are alternative definitions of long-range dependence. We recall that a function L is slowly varying at zero (respectively, at infinity) if it is bounded on a finite interval and if, for all a > 0, L(ax)/L(x) tends to 1 as x tends to zero (respectively, to infinity).

We introduce now the spectral density of the autocovariance functions ρ(k)

$$f(\lambda) := \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-i\lambda k} \rho(k)$$

for λ [π, π].

Definition (3.1.4) [Biagini and Others 2008]

For stationary sequences $\{X_n\}_{n\in N}$ with finite variance, we say that $\{X_n\}_{n\in N}$ exhibits long-range dependence if one of the following holds:

1. $\lim_{n\to\infty} (\sum_{k=-n}^{n} \rho(k))/(cn^{\beta}L_1(n)) = 1$ for some constant c and $\beta \in (0, 1)$.
2. $\lim_{k\to\infty} \rho(k)/ck^{-\gamma}L_2(k) = 1$ for some constant c and $\gamma \in (0, 1)$.

3. $\lim_{\lambda \to 0} f(\lambda)/c|\lambda|^{-\delta} L_3(|\lambda|) = 1$ for some constant c and $\delta \in (0, 1)$.

Here L_1, L_2 are slowly varying functions at infinity, while L_3 is slowly varying at zero.

Lemma (1.1.1)

For fBm B(H) of Hurst index $H \in (1/2, 1)$, the three conditions s of long-range dependence of Definition (3.1.3) are equivalent. They hold with the following choice of parameters and slowly varying functions:

1. $\beta = 2H \;\; 1$, $L_1(x) = 2H$.

2. $\gamma = 2 \;\; 2H$, $L_2(x) = H(2H \;\; 1)$.

3. $\delta = 2H \;\; 1$, $L_3(x) = \pi^{-1} H\Gamma(2H) \sin \pi H$.

Definition (3.1.5) Self – similarity [Biagini and Others 2008]

We say that an R^n – valued random process $X = \{X_t\}_{t \geq 0}$ is self – similar or satisfies the property of self – similarity if for every $a > 0$ there exists $b > 0$ such that

$$\text{Law}\left(X_{at},\, t \geq 0\right) = \text{Law}\left(bX_t, t \geq 0\right) \qquad (3.2)$$

Note that (3.2) means that the two processes X_{at} and bX_t have the same finite dimensional distribution functions, i. e., for every choice $t_0, t_1, ..., t_n$ in R,

$$P\left(X_{at_0} \le x_0, ..., X_{at_n} \le x_n\right) = P\left(bX_{t_0} \le x_0, ..., bX_{t_n} \le x_n\right)$$

for every $x_0, ..., x_n$ in R.

Remark (3.1.5) [Biagini and Others 2008]

If $b = a^{-H}$ in definition (3.1.5), then we say that $X = \{X_t\}_{t \ge 0}$ is a self similar process with Hurst index H. the quantity D=1/H is called the statistical fractal dimension of X.

3.2 PROBLEM FORMULATION

Consider the nonlinear stochastic control system

$$\begin{aligned} dx(t) &= [Ax(t) + Bu(t) + f\left(t, x(t)\right) + N_1\left(t, x(t), N_2\left(t, x(t)\right)\right) \\ &\quad + \sigma\left(t, x(t)\right) dW^H(t) \qquad (*) \\ x(0) &= x_0 \in R^n, \quad t \in [0, b] \end{aligned}$$

where W^H is the fractional Brownian motion with $0 < H < 1$.
and

$$x(t)=\Phi(t)x_0+\int_0^t\Phi(t-s)[Bu(s)+f(s,x(s))ds \\ +\int_0^t\Phi(t-s)N_1(s,x(s),N_2(s,x(s)))ds+\int_0^t\Phi(t-s)\sigma(s,x(s))dW^H(s) \qquad (**)$$

Remarks (3.2.1)

1. As well known, the fractional Brownian motion is an extension of the Brownian motion and provides the useful mathematical model when the long range dependence is accounted importantly. In general the steps in the fractional Brownian motion are strongly correlated and have long memory.
2. Hence fractional Brownian motion has become a powerful mathematical model for studying correlated random motion with wide application in Physics and it has been ubiquitous model in Physics.

We show in this section the stochastic control system (*) is completely controllable on [0,b].

Theorem (3.2.1)

If the conditions

i. f, N_1, N_2 and σ satisfies the Lipchitz condition and there exist L_1, L_2, L_3, L_4 and k for every $t \in [0,b]$, $x_i, y_i \in R^n, i = 1,2$ such that

$$\|N_1(t,x_1,y_1) - N_1(t,x_2,y_2)\|^2 \le L_1\left[\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2\right],$$

$$\|N_2(t,x) - N_2(t,y)\|^2 \le L_2\|x - y\|^2,$$

$$\|f(t,x) - f(t,y)\|^2 \le L_3\|x - y\|^2,$$

$$\|\sigma(t,x) - \sigma(t,y)\|^2 \le L_4\|x - y\|^2$$

and

$$\|N_2(t,x)\|^2 \le k\|x\|^2$$

ii. f, N_1 and σ are continuous and satisfies

$$\|f(t,x)\|^2 + \|N_1(t,x,y)\|^2 + \|\sigma(t,x)\|^2 \le L\left(\|x\|^2 + \|y\|^2 + 1\right),$$

where L is a positive constant.

iii. $E\left\|\prod_0^t z\right\|^2 \le M\, E\|z\|^2$ **[Mahmudov 2003].**

iv. Let $p = \left[4M\ell_1^2\ell_2 bL_1\left(bL_2 + 1\right) + 4M\ell_1^2\ell_2 L_3 + 4\ell_1 bL_1\left(bL_2 + 1\right) + 4\ell_1 L_3\right]b$ be such that $0 \le p < 1$.

v. $\|D_s^\phi f\|^2 \le q\|f\|^2$.

are satisfying, then the system (**) is completely controllable

The difference is in the estimate (2.4) since the Itô that used in the Brownian motion case is not valid anymore in the fractional Brownian motion case.

Let $\mathscr{L}_{\phi}^{1,2}(R)$ denote the completion of the set of all $\Gamma_t^{(H)}$ −adapted processes $f(t) = f(t,\omega)$ such that

$$\|f\|^2_{\mathscr{L}_{\phi}^{1,2}(R)} = E_{\mu_\phi}\left[\int_R\int_R f(s)f(t)\phi(s,t)dsdt\right] + E_{\mu_\phi}\left[\left(\int_R D_s^{\phi} f(s)ds\right)^2\right] < \infty,$$

where D_s^{ϕ} is the ϕ derivative defined in [Imbrie 1988]. Let $F \in L^2(\mu_\phi)$, where μ_ϕ is the probability law of W^H. Let $D_s F$ denotes the usual Malliavin derivative then $D_s^{\phi} F = \int_R \phi(s,t) D_t F dt$. Then we have the following fractional Itô isometry (see [Y. Hu 2003])

$$E\left[\left(\int_R f(t,\omega)dW^H(t)\right)^2\right] = \|f\|^2_{\mathscr{L}_{\phi}^{1,2}(R)} \tag{3.5}$$

Here the function ϕ is chosen as

$$\phi(s,t) = H(2H-1)|s-t|^{2(H-1)}, \quad H > \frac{1}{2}.$$

Instead of using the Itô isometry we use the above fractional Itô isometry to prove the required result, Now then we have

$$
\begin{aligned}
E\left\|T_x(t)\right\|^2 = E\Bigg\| &\Phi(t)x_0 + \prod\nolimits_0^t \Big(\Phi^*(t-s)\Big(\prod\nolimits_0^b\Big)^{-1} \{x_b - \Phi(b)x_0 \\
&- \int_0^b \Phi(b-s)f(s,x(s))ds - \int_0^b \Phi(b-s)N_1\big(s,X(s),N_2(s,x)\big)ds \\
&- \int_0^b \Phi(b-s)\sigma\big(s,x(s)\big)dW^H(s)\} \Big) \\
&+ \int_0^t \Phi(t-s)[f\big(s,x(s)\big) + N_1\big(s,X(s),N_2(s,x)\big)]ds \\
&+ \int_0^t \Phi(t-s)\sigma\big(s,x(s)\big)dW^H(s)\Bigg\|^2 \\
\le 4&\left\|\Phi(t)x_0\right\|^2 + 4E\Bigg\|\prod\nolimits_0^t \Big(\Phi^*(t-s)\Big(\prod\nolimits_0^b\Big)^{-1} \{x_b - \Phi(b)x_0 \\
&- \int_0^b \Phi(b-s)f(s,x(s))ds - \int_0^b \Phi(b-s)N_1\big(s,X(s),N_2(s,x)\big)ds \\
&- \int_0^b \Phi(b-s)\sigma\big(s,x(s)\big)dW^H(s)\} \Big)\Bigg\|^2 \\
&+ 4b\int_0^t \left\|\Phi(t-s)\right\|^2 E\left\|f\big(s,x(s)\big)\right\|^2 ds + 4b\int_0^t \left\|\Phi(t-s)\right\|^2 E\left\|N_1\big(s,X(s),N_2(s,x)\big)\right\|^2 ds \\
&+ 4\int_0^t \left\|\Phi(t-s)\right\|^2 E\left\|\sigma\big(s,x(s)\big)\right\|^2 dW^H(s)
\end{aligned}
$$

From the conditions (i), (ii) an fractional Itô isometry (3.5) with the assumption $\left\|D_s^{\phi}f\right\|^2 \le q\left\|f\right\|^2$, q is constant, we have

$$E\|T_x(t)\|^2 \le 4\|\Phi(t)\|^2\|x_0\|^2 + 4M\,E\Big\|\Phi^*(t-s)\Big(\prod_0^b\ \Big)^{-1}\{x_b - \Phi(b)x_0$$
$$-\int_0^b \Phi(b-s)f(s,x(s))ds - \int_0^b \Phi(b-s)N_1\big(s,X(s),N_2(s,x)\big)ds$$
$$-\int_0^b \Phi(b-s)\sigma\big(s,x(s)\big)dW^H(s)\}\Big\|^2$$
$$+4\int_0^t \|\Phi(t-s)\|^2 E\|f\big(s,x(s)\big)\|^2 ds + 4\int_0^t \|\Phi(t-s)\|^2 E\|N_1\big(s,X(s),N_2(s,x)\big)\|^2 ds$$
$$+4\int_0^t \|\Phi(t-s)\|^2 E\|\sigma\big(s,x(s)\big)\|^2 dW^H(s)$$

$$\le 4\ell_1\|x_0\|^2 + 16M\ell_1\ell_2\Big[\ell_3 + \ell_1\|x_0\|^2 + b\int_0^t \|\Phi(t-s)\|^2 E\|f(s,x)\|^2 ds$$
$$+b\int_0^t \|\Phi(t-s)\|^2 E\|N_1\big(s,X(s),N_2(s,x)\big)\|^2 ds + \int_0^t \|\Phi(t-s)\|^2 E\|\sigma\big(s,x(s)\big)\|^2 dW^H(s)\Big]$$
$$+4\ell_1 b\Big(\int_0^t \big(E\|f(s,x)\|^2\big)ds + \int_0^t E\|N_1\big(s,x(s),N_2(s,x)\big)\|^2 ds\Big) + 4\ell_1\int_0^t E\|\sigma\big(s,x(s)\big)\|^2 dW^H(s)$$

$$\le 4\ell_1\|x_0\|^2 + 16M\ell_1\ell_2\Big[\ell_3 + \ell_1\|x_0\|^2 + b\int_0^t \|\Phi(t-s)\|^2 E\|f(s,x)\|^2 ds$$
$$+b\int_0^t \|\Phi(t-s)\|^2 E\|N_1\big(s,X(s),N_2(s,x)\big)\|^2 ds + q\int_0^t \|\Phi(t-s)\|^2 E\|\sigma\big(s,x(s)\big)\|^2 d(s)\Big]$$
$$+4\ell_1 b\Big(\int_0^t \big(E\|f(s,x)\|^2\big)ds + \int_0^t E\|N_1\big(s,x(s),N_2(s,x)\big)\|^2 ds\Big) + 4\ell_1 q\int_0^t E\|\sigma\big(s,x(s)\big)\|^2 d(s)$$

$$\leq 4\ell_1\|x_0\|^2 + 16M\ell_1\ell_2\left[\ell_3 + \ell_1\|x_0\|^2 + \int_0^t \ell_1 LE\left(q\|x(s)\|^2 + \|N_2(s,x(s))\|^2 + 1\right)ds\right]$$
$$+4\ell_1\left[\int_0^t LE\left(q\|x(s)\|^2 + \|N_2(s,x(s))\|^2 + 1\right)ds\right]$$

$$\leq \left(16M\ell_1^2\ell_2 + 4\ell_1\right)\|x_0\|^2 + 16M\ell_1\ell_2\ell_3$$
$$+\left(16M\ell_1^2\ell_2 + 4\ell_1\right)L\int_0^t\left[\left((q+k+1)E\|x(s)\|^2 + 1\right)\right]ds$$

$$E\|T_x(t)\|^2 \leq \left(16M\ell_1^2\ell_2 + 4\ell_1\right)L\int_0^t\left[\left((q+k+1)E\|x(s)\|^2 + 1\right)\right]ds$$

There exist a constant K_2 such that

$$E\|T_x(t)\|^2 \leq K_2\left(1 + (q+k+1)\int_0^t E\|x(s)\|^2 ds\right)$$
$$\leq K_2\left(1 + (q+k+1)b\sup_{0\leq s<1} E\|x(s)\|^2\right) \qquad \text{for all } t\in[0,b]$$

Thus T maps H_2 into itself.

Secondly, we prove that T is a contraction mapping

$$
\begin{aligned}
E\left\|T_x(t)-T_y(t)\right\|^2 = E\Bigg\| & \prod_0^t\Bigg(\Phi^*(t-s)\Big(\prod_0^b\ \Big)^{-1}\times\Bigg[\int_0^b \Phi(b-s)\big(f(s,x(s))-f(s,y(s))\big)ds \\
& +\int_0^b \Phi(b-s)\Big(N_1\big(s,x(s),N_2(s,x)\big)-N_1\big(s,y(s),N_2(s,x)\big)\Big)ds \\
& -\int_0^b \Phi(b-s)\Big(\sigma\big(s,x(s)\big)-\sigma\big(s,y(s)\big)\Big)dW^H(s)\Bigg]\Bigg) \\
& +\int_0^t \Phi(b-s)\big(f(s,x(s))-f(s,y(s))\big)ds \\
& +\int_0^t \Phi(b-s)\Big(N_1\big(s,x(s),N_2(s,x)\big)-N_1\big(s,y(s),N_2(s,x)\big)\Big)ds \\
& +\int_0^t \Phi(b-s)\Big(\sigma\big(s,x(s)\big)-\sigma\big(s,y(s)\big)\Big)dW^H(s)\Bigg\|^2 \\
\le 4E\Bigg\| & \prod_0^t\Bigg(\Phi^*(t-s)\Big(\prod_0^b\ \Big)^{-1}\times\Bigg[\int_0^b \Phi(b-s)\big(f(s,x(s))-f(s,y(s))\big)ds \\
& +\int_0^b \Phi(b-s)\Big(N_1\big(s,x(s),N_2(s,x)\big)-N_1\big(s,y(s),N_2(s,x)\big)\Big)ds \\
& -\int_0^b \Phi(b-s)\Big(\sigma\big(s,x(s)\big)-\sigma\big(s,y(s)\big)\Big)dW^H(s)\Bigg]\Bigg)\Bigg\|^2 \\
& +4E\Bigg\|\int_0^t \Phi(b-s)\big(f(s,x(s))-f(s,y(s))\big)ds \\
& +\int_0^t \Phi(b-s)\Big(N_1\big(s,x(s),N_2(s,x)\big)-N_1\big(s,y(s),N_2(s,x)\big)\Big)ds \\
& +\int_0^t \Phi(b-s)\Big(\sigma\big(s,x(s)\big)-\sigma\big(s,y(s)\big)\Big)dW^H(s)\Bigg\|^2
\end{aligned}
$$

$$\leq 4M\ell_1\ell_2\Big\{b\ell_1L_3\int_0^t E\|x(s)-y(s)\|^2 ds$$
$$+b\ell_1L_1\left[q\int_0^t E\|x(s)-y(s)\|^2 ds+\int_0^t E\|N_2(s,x(s))-N_2(s,y(s))\|^2 ds\right]$$
$$+\ell_1L_4\int_0^t E\|x(s)-y(s)\|^2 ds\Big\}$$
$$+b\ell_1L_3\int_0^t E\|x(s)-y(s)\|^2 ds$$
$$+b\ell_1L_1\left[q\int_0^t E\|x(s)-y(s)\|^2 ds+\int_0^t E\|N_2(s,x(s))-N_2(s,y(s))\|^2 ds\right]$$
$$+\ell_1L_4\int_0^t E\|x(s)-y(s)\|^2 ds$$

$$\leq 4M\ell_1\ell_2\Big\{b\ell_1L_3\int_0^t E\|x(s)-y(s)\|^2 ds$$
$$+b\ell_1L_1(q+L_2)\int_0^t E\|x(s)-y(s)\|^2 ds+\ell_1L_4\int_0^t E\|x(s)-y(s)\|^2 ds\Big\}$$
$$+b\ell_1L_3\int_0^t E\|x(s)-y(s)\|^2 ds$$
$$+b\ell_1L_1(q+L_2)\int_0^t E\|x(s)-y(s)\|^2 ds+\ell_1L_4\int_0^t E\|x(s)-y(s)\|^2 ds$$
$$\leq\Big\{4M\ell_1^2\ell_2bL_3+4M\ell_1^2bL_1(q+L_2)+4M\ell_1^2L_4$$
$$+b\ell_1L_3+b\ell_1L_1(q+L_2)+\ell_1L_4\Big\}\int_0^t E\|x(s)-y(s)\|^2 ds$$

$$\leq \left\{4M\ell_1^2\ell_2 bL_3 + 4M\ell_1^2 bL_1\left(q+L_2\right)+4M\ell_1^2 L_4 \right.$$
$$\left. + b\ell_1 L_3 + b\ell_1 L_1\left(q+L_2\right)+\ell_1 L_4\right\} \sup_{s\in[0,b]} E\left\|x(s)-y(s)\right\|^2$$

Therefore, T is a contraction mapping and hence there exists a unique fixed point $x(.)$ in H_2 which is the solution of the equation (**). Thus the system (**) is completely controllable on [0,b].

ILLUSTRATION 3.2.1

Consider a two dimensional nonlinear stochastic control equation

$$\begin{aligned} dx(t) &= [Ax(t) + Bu(t) + f\left(t, x(t)\right) + N_1\left(t, x(t), N_2\left(t, x(t)\right)\right) \\ &\quad + \sigma\left(t, x(t)\right) dW(t) \\ x(0) &= x_0 \in R^2, \quad t \in [0, b] \end{aligned} \tag{3.6}$$

where W(t) is a one-dimensional Brownian motion and

$$A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$f\left(t, x(t)\right) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$N_1\left(t, x(t), N_2\left(t, x(t)\right)\right) = \begin{bmatrix} -\left(3 + \sin x_2(t)\right) x_1(t) \\ -\left(3 - \cos x_1(t)\right) x_2(t) + 2x_1(t) \end{bmatrix},$$

$$\sigma(t) = \begin{bmatrix} \left(2t^2 + 1\right) e^{-2t} \\ \sin t \cos t \cdot e^{-t} \end{bmatrix}$$

for $x(t) = \left(x_1(t), x_2(t)\right) \in R^2$. Take the final point $x(b) \in R^2$. For this system, the controllability matrix

$$\Psi_0^b = \int_0^b X(0,t)BB^*X^*(0,t)dt$$
$$= \frac{1}{2}\left(e^{2b}-1\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

is nonsingular if b>0.

We need some remarks to prove the Lipschitz condition

Remarks(3.3.1) [C Klebaner 2005]

1. If f is continuously differentiable on a finite interval [a, b], then it is Lipschitz.
2. A Lipschitz function does not have to be differentiable, for example f(x)=|x| is Lipschitz but it is not differentiable at zero.
3. A Lipschitz function multiplied by a constant, and a sum of two Lipschitz functions are Lipschitz functions. The product of two bounded Lipschitz functions is again a Lipschitz function.
4. If f is Lipschitz on [0,N] for any N >0 but with the constant K depending on N, then it is called locally Lipschitz. For example, x^2 is Lipschitz on [0,N] for any finite N, but it is not Lipschitz on [0,+∞), since its derivative is unbounded.

5. If f is a function of two variables f(t,x) and it satisfies Lipschitz condition in x for all t, $0 \leq t \leq T$, with same constant K independent of t, it is said that f satisfies Lipschitz condition in x uniformly in t, $0 \leq t \leq T$.

From above remarks and the definition of Lipschitz condition and the definition of Growth condition, we see that (f, N_1, N_2, and σ) are differentiable and satisfying Lipschitz and Growth conditions.

Then the System (3.6) is completely controllable.

Conclusions

1. The controllability study of a nonlinear stochastic control system in the presence of system uncertainty driven by some Brownian motion or/and fractional Brownian motion is not an easy task as one can see, but due to its important applications, the present work have been adaptive.

2. the necessary and sufficient condition of controllability of this work are applicable, as one can see this fact from the illustration. This part makes the present work suitable for many real life applications.

3. the difficult of this work comes from the hard background of nonlinear stochastic dynamic system using Ito formula and some conditional expectations operator in the presence of adapted filtration .

Future work

1. study the feedback of nonlinear stochastic dynamic control system using the concept of controllability that discussed in this work and then go for the real life application.

2. one can try to develop the result of this work for delay stochastic control system in the presence of system uncertainty driven by Brownian motion or fractional Brownian motion.

3. Another case of Fractional Brownian motion where $\{0<H<0.5\}$ may also be considered and try to develop this case for the work of chapter

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الخلاصة

الهدف الرئيسي لهذه الرسالة هو التركيز على دراسة بعض أنظمة غير خطية غير عشوائية ديناميكية .

العمليات العشوائية ، التكاملات العشوائية للحركة البراونية و الحركة البراونية الكسرية العشوائية الديناميكية المشتقة من نظام الحركة البراونية والحركة البراونية الكسرية درست ونوقشت بدعم من التعليقات المفيدة والأمثلة على ذلك .

بعض الحالات من النظام العشوائي (Itô) غير الخطي المشتقة من الحركة البراونية كذلك الحركة البراونية الكسرية تم النظر فيها ومناقشتها .

النظرية الأساسية من قابلية الحل و التفسير والتحكم ببعض حالات من الانظمة غير الخطية (Itô) الدينامية المشتقة من الحركة البراونية نوقشت وأثبتت باستخدام نظرية النقطة الثابتة لـ(بناخ) مدعومة بملاحظات ختامية ومثال .

و النظرية لقابلية الحل و التفسير والتحكم لبعض حالات الأنظمة العشوائية غير الخطية (Itô) الدينامية يقودها نظام الحركة البراونية الكسرية أيضا ذكرت وأثبتت مدعومة بمثال .

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حول قابلية السيطرة الاحتمالية لأنظمة سيطرة غير خطية متغيرة العشوائية

رسالة

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في الرياضيات

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