Republic of Iraq
Ministry of Higher Education
and Scientific Research,
Al-Nahrain University,
College of Science,
Department of Mathematics and
Computer Applications

Solution of Stochastic Linear Ordinary Delay Differential Equations

A Thesis
Submitted to the College of Science of Al-Nahrain University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By
Hussna Ahmed Jassim
(B.Sc., Al-Nahrain University, 2006)

Supervised by
Asst. Prof. Dr. Fadhel Subhi Fadhel

May 2009 *Jamadi Al-Awal 1430*

بِسْمِ الله الرَّحمنِ الرّحيم

(يَرفَعُ ٱللَّــهَ ٱلذِينَ آمَنُوا مِنكُم وَٱلَّذِينَ
أوتُوا ٱلعِلمَ درجٰتٍ وٱللّــهُ بِما تَعمَلُونَ خَبِيرٌ)

صَدَقَ الله العَظيم

(المجادلة / ١١)

الإهداء

إلى نبي الرحمة وهادي الامة. إلى الرسول الاعظم

محمد (ص)

إلى الغالية التي عجزت الكلمات عن وصفها. إلى منبع الحنان

أمي الغالية

إلى من غرس هذه البذرة فكان من ثمارها هذا الجهد المتواضع

والدي العزيز

إلى الشموع المضيئة إلى قرة العين

أخوتي الاعزاء

إلى رفيق حياتي. إلى من كان لي خير سند وعون طوال هذه المسيرة.

زوجي الغالي

إلى اللواتي رافقنني طيلة مسيرتي الدراسية

صديقاتي العزيزات

إلى كل من شجعني وآزرني لإكمال هذه المسيرة الغراء حباً وإحتراماً أهدي جهدي المتواضع

حسنى

A cknowledgements

Praise to Allah for providing me the willingness, energy and patience to accomplish this work.

I wish to express sincere gratitude and grateful admiration to my supervisor Asst. Prof. Dr. Fadhel Subhi Fadhel for his over seeing, guidance, interests and suggestions that were very helpful for the completion of this thesis.

I would like also to express my appreciation to the staff members of the Department of Mathematics and Computer Applications, namely Dr. Akram M. Al-Abood, Dr. Alaudeen N. Ahmed, Dr. Ahlam J. Khaleel, Dr. Radhi A. Zboon, Dr. Osama H. Mohammed, Dr. Zainab A. Salman and Dr. Shatha A. Aziz.

I'm grateful to the College of Science of Al-Nahrain University, which gave me the chance to be one of their students.

Heartfelt thanks goes to my beloved family, in particular my father, mother, brothers and sister, for their love and giving me the facilities during my work and I'm truly grateful to my husband Asst. Lecturer Ammar Jaffer.

My deepest thanks to my dearest friends Zahraa, Roaa, Sora, Rona, Noor and Mohammed, for their support and encouragement.

Finally, I'd like to thank all those who hand participated in any way or another during the work.

***Hussna Ahmed Jassim* ?**

May, 2009

Supervisor Certification

I certify that this thesis was prepared under my supervision at the department of Mathematics, College of Science, Al-Nahrain University, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

Signature:

Name: Asst. Prof. Dr. Fadhel Subhi Fadhel

Data: / / 2009

In view of the available recommendations, I forward this thesis for debate by the examining committee.

Signature:

Name: Asst. Prof. Dr. Akram M. Al-Abood

Head of the Department of Mathematics and Computer Applications

Data: / / 2009

Examining Committee Certification

We certify that we have read this thesis entitled "*Solution of Stochastic Linear Ordinary Delay Differential Equations*" and as examining committee examined the student *(Hussna Ahmed Jassim)* in its contents and in what it connected with, and that, in our opinion, it meets the standards of a thesis for the degree of Master of Science in Mathematics.

(Chairman)

Signature:

Name: Dr. Akram M. Al-Abood

Asst. Prof.

Date: / / 2009

(Member)

Signature:

Name: Dr. Radhi A. Zboon

Asst. Prof.

Date: / / 2009

(Member)

Signature:

Name: Dr. Ahmed M. Abdulhadi

Asst. Prof.

Date: / / 2009

(Member and Supervisor)

Signature:

Name: Dr. Fadhel S. Fadhel

Asst. Prof.

Date: / / 2009

Approved by the Collage of Science

Signature

Name: Asst. Prof. Dr. Laith Abdul Aziz Al-Ani

Dean of the Collage of Science

Data: / / 2009

A bstract

This thesis have three main objectives. The first objective is to give a study of stochastic calculus, including the basic definitions and fundamental concepts related to this topic including the proof of some results, and among such results is the proof of Hölder's inequality of expectation, the existence and uniqueness theorem of stochastic differential equations and the Euler's method for solving stochastic differential equations. The second objective is to study the analytical and numerical methods for solving stochastic differential equations. The third objective is to modify the methods of solution to solve delay stochastic differential equations

Contents

I ntroduction

Stochastic differential equations (SDE's for short) are differential equations in which one or more of its terms are stochastic processes, and therefore will give solutions which are itself stochastic process, [Arnold, 1974]. SDE's are used in wide range of applications in environmental modeling, engineering and biological modeling, [Higham, 2001], and SDE's are a fundamental tool for mathematics and its applications, [Geiss, 2007].

The type of SDE's incorporated into the systems are also very important; therefore, various authors have made extensive work on the analytic solution of SDE's, [Smith, 1999], [ϕksendal, 2000], [ϕksendal, 2003], [Muszta, 2005], and the numerical solution of SDE's, [Han, 2005], [Mahony, 2006]. Since sometimes SDE's rarely have explicit solutions and hence in some cases accurate numerical methods are vital in order to make their implementation viable. Due to features of the stochastic calculus, the numerical analysis for solving SDE's differs in some key areas from the already well-developed area of the numerical analysis of ordinary differential equations, [Mahony, 2006].

There are two basic type of tasks connected with the simulation of solutions of SDE's. The first occurs in situation where a good path wise approximation is required, for instance in direct simulations, filtering or testing statistical estimators. The second interest focuses on approximating expectations of functional of the Itô process, such as its probability distribution and its moments, [Han, 2005]. As more realistic mathematical models become required to take into account random effects and influences in real world systems SDE's have become essential in the accurate description of such situations, [Mahony, 2006].

In recent years, stochastic processes and stochastic calculus have been applied to a wide range of financial problems. ApplicationS of stochastic processes and stochastic calculus may be found in many disciplines, such as physics, engineering, and finance. Stochastic calculus concerns with a specific class of stochastic processes that are stochastically integrable and are often expressed as solutions to the stochastic differential equations, [Lin, 2006]. They are typically describing the time dynamics of the evolution of a state vector, based on the (approximate) physics of the real system, together with a deriving noise process. The noise process can be though of in several ways. It is often represent processes not included in the model, but presented in real system, [Archambeau et al., 2007]. In the physical and engineering sciences, on the other hand, SDE's arise in a quite natural manner in the description of systems on which so-called "white noise" acts, [Lin, 2006], many physical systems are modeled by SDE's, where random effect are being modeled by a Wiener process (for more details, see for example [Soheili, 2008]). A natural extension is given by systems of SDE's, where system noise is modeled by including a diffusion on term of some suitable form in the driving equations, [Ditlevsen, 2006]. Statistical inference for diffusion type processes satisfying SDE's driven by Wiener process has been studied earlier and a comprehensive survey of various methods is given in, [Rao, 2003]. Recent years have witnessed that the most efficient and widely applicable approach in solving SDE's seems to be the simulation of sample paths of time discrete approximations on digital computers. This is based on a finite discretization of time interval [0, T] under consideration and generates an approximate values of sample paths step by step at the discretization times, [Han, 2005].

Starting from the well known Itô formula (as stochastic counter part of deterministic chain rule and as the link between continuous and discrete time stochastic dynamical systems). As in deterministic analysis, the latter formulas are essential for the systematic construction of stochastic numerical methods and the investigation of local behavior of their approximating trajectories, [Schurz, 2002]. The history of stochastic integration and the modeling of risky asset prices both begin with Brownian motion, so let us begin there too, and arrived at the notion of a SDE governing the paths of a Markov process that may be formulated in terms of the differential of a single differential process, but in the late of 1960 and 1970 which leads to even greater interest of Markov processes as solutions of SDE's, [Jarrow, 2003]. If these random functions have certain regular properties, one can consider the above mentioned problems simply as a family of classical problems for the individual samples functions, and treat them with the classical methods of the theory of differential equations, [Arnold, 1974]. The theory of SDE's was originally developed by mathematicians as a tool for explicit construction of the trajectories of diffusion processes for given coefficient of drift and diffusion, as a result of this variety in the motivations, existing detailed studies of the subject, as a rule, either are not written from a stand point applications or are inaccessible to the person intending to apply them, [Higham, 2001].

Stochastic delay differential equations (SDDE's for short) are a generalization of both deterministic delay differential equations (DDE's for short) and stochastic ordinary differential equations (SODE's for short). In many areas of science, such as population problems and the study of materials or systems with memory, there has been an increasing interest in the investigation of functional differential equations incorporating memory or after-effect, [Baker, 2000]. In such cases, stochastic delay differential equations or stochastic functional differential equations (SFDE's for short) provide important tools to describe and analyze these systems. SDDE's and SFDE's arising in many applications cannot be solved explicitly. Hence, one needs to develop effective numerical techniques for such systems, [Hu, 2004]. In general, there is no analytical closed form solution of the problems considered here and we usually require numerical techniques to investigate the models quantitatively, [Baker, 2000]. The analysis of numerical methods for SDDE's is based on the numerical analysis of DDE's and the numerical analysis of ODE's. There are few articles on numerical analysis of SDDE's to date, see (Tudor, 1992), (Küchler, 1999), and for the theory of SDDE's, see (Mohammed, 1984), (Kolmanovskii, 1992), (Mao, 1997).

This thesis consists of three chapters. In chapter one, some general concepts and definitions related to the subject of stochastic calculus and delay differential equations are given for completeness. In chapter two, the statement and the details of the proof of SDE's is given as well as with some additional theoretical results which are needed in the proof of the existence and uniqueness theorem in which some of them are given in literatures either without details. In chapter three, some analytical methods for solving SDE's are studied and explained with examples then these methods are modified for solving SDDE's. In addition, numerical method, namely Euler's method, is considered for solving SDE's and SDDE's.

Finally, the numerical results are obtained using computer programs written in Mathcad 2001i computer software and the results are given in a tabulated form.

Chapter One

General Concepts

Chapter One
General Concepts

Introduction

This chapter is of introductory nature, which consists of some the most common concepts related to this thesis. Therefore, this chapter consists of seven sections. In section 1.1, the main aspect of delay-differential equations are introduced, including its classification and basic properties. In section 1.2, we give the main concepts in probability theory including some definitions of fields (σ-fields) and probability space. In section 1.3, the concept of discrete random variables have been discussed in short while in section 1.4 we discuss in details the continuous random variables because of its strong relationship with the topics of this thesis. Section 1.5 presents an introduction to stochastic process which give the rise for stochastic differential equations. Also, as a second part in the introduction of stochastic differential equations is the Brownian motion which is introduced with some details in section 1.6, as well as, some results are given for completeness. Finally, section 1.7 presents the definition of stochastic differential equations in terms of the Itô process.

1.1 Delay Differential Equations

Delay differential equations are a large and important class of dynamical systems. They often arise in either natural or technological control problems, [Roussel, 2004], DDE's are of sufficient importance in many applications, say in mixing of liquids, population growth and automatic control system, [Driver, 1977], and DDE's are used to describe many phenomena of physical interests, [Shampine, 2000], DDE's are in which their time lags are constant (Sometimes called scalar, or point delays), [Abdulkadir, 2008].

The general form of the n-th order differential-difference equation with multiple delays is given by:

$$F(t; x(t), x(t-\tau_{01}), x(t-\tau_{02}), \ldots, x(t-\tau_{0m}), x'(t), x'(t-\tau_{11}),$$
$$x'(t-\tau_{12}), \ldots, x'(t-\tau_{1m}), \ldots, x^{(n)}(t), x^{(n)}(t-\tau_{n1}), x^{(n)}(t-\tau_{n2}), \ldots,$$
$$x^{(n)}(t-\tau_{nm})) = g(t) \quad \ldots\ldots(1.1)$$

where F is a given function and τ_{ij} (for i = 0, 1, …, n and j = 1, 2, …, m) are constants called delays, where i refers to the order of the derivative with respect to the dependent variable for each, j = 1, 2, …, m.

The first order linear differential-difference equation may be classified into three types. The first type, which is the simplest type of differential-difference equations is that; in which the delay terms is through the state variable and not through the derivative of the state variable and is called retarded differential-difference equations (RDDE’s, for short).

These types of equations occurred in a number of applications such as, in the physical applications, for example:

$$x'(t) = F(t; x(t), x(t-\tau)),$$

where $x(t) \in \mathbb{R}^n$, $F : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and $\tau > 0$ is a single constant fixed time delay, and in control problems, for example:

$$x'(t) = K(x(t) - x(t-\tau)),$$

where K is the feedback gain function and τ is the time delay, and also in the study of distribution of primes, for example:

$$x'(t) = -\alpha x(t-1)[1 + x(t)].$$

The second type of differential-difference equations is that; in which the delay terms is through the derivative of the state variable and not through the state variable itself and is called neutral differential-difference equations (NDDE's, for short), [Hale, 1993], for example:

$$x''(t) = -x'(t) - x'(t-1) - 3\sin x(t) + \cos(t).$$

Also variants of NDDE's have also been used as a model in the history of growth of single species, for example:

$$x'(t) = -\alpha\left\{\int_{-1}^{0} x'(s-\tau)\, ds\right\}(1+x(t));$$

and in the describing the spread of disease taking into account age dependence, for example:

$$x'(t) = -\int_{t-\tau}^{t} a(t-u)g(x'(t-u))\,du.$$

The third type is a combination between the two obvious types and is called the advanced differential-difference equations (ADDE's, for short), [Hale, 1993]. These types of equations occur in the theory of epidemics and models in the biomedical science, for example:

$$x''(t) = f(x(t-\tau))\, x'(t-\tau) - \alpha x'(t) - x(t).$$

Many theoretical and numerical solution are presented in literatures for solving DDE's, and among the most common used methods; the method of steps (or the method of successive integrations) and the Laplace transformation method for linear delay ODE's with constant coefficients. The method of steps; is to reduce the problem directly into an ordinary differential equation using the initial condition. This method has been used for many years in solving delay differential equations, [Gllsinn, 2006]. Although, Laplace transformation method is extremely useful in obtaining the solution of linear DDE's with constant coefficients. As it is known, Laplace transformation method may be used to solve ODE's and we can also use the same approach to solve DDE's. For this approach, suppose that f is a function of t defined on $[0, \infty)$ with $|f(t)| < Me^{-\beta t}$ then the Laplace transform of f(t) (denoted by F(s)) is defined by:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad s > 0$$

It is clear that this integral depends on f and on the number s, where f satisfies certain conditions. The function F(s) is called the Laplace transform of f(t), [Brauer, 1973]. Two approaches may be used in Laplace transformation method for solving DDE’s. The first approach is to solve the DDE’s by using Laplace transformation method directly without using the method of steps, [Bellman, 1963]. While the second approach depends on the method of steps firstly to transform the DDE, to an equivalent ODE and then apply the Laplace transformation method to solve the resulting equation, [Brauer, 1973].

Moreover, some numerical and approximate methods may be used to solve DDE's, such as the linear multistep methods [Al-Kubeisy, 2004], the collocation method, [Al-Saady, 2000], variational approach, [Abdukadir, 2008].

1.2 Basic Concepts of Probability Theory

Probability theory is that branch of mathematics which is concerned with random (or chance) phenomena. It has attracted people to its study both, because of its intrinsic interest and its successful applications to many areas within the physical, biological and social sciences, engineering, and in the business world, [Hoel, 1971], randomness and probability are not easy to define precisely, but we certainly recognize random events when we meet them. For example, randomness is in effect when we flip a coin a lottery ticket, run a horse race, [Krishnan, 1984]. The terminology given in the next remarks are necessary for the topics of this thesis:

Remarks (1.1), [Krishnan, 1984]:

1. Random experiment is an experiment satisfy the following conditions:

   (i) The outcome can be predicted with certainty.

   (ii) The outcome can be described prior to its performance.

   (iii) It can be repeated under the same conditions.

2. The collection of all possible outcomes of a random experiment is called sample space and is denoted by Ω. In set terminology, the sample space is termed as the universal set, thus, the sample space Ω is a set consisting of mutually exclusive, collectively exhaustive listing of all possible outcomes of a random experiment. That is,:

   $$\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$$

   denotes the set of all finite outcomes, while

   $$\Omega = \{\omega_1, \omega_2, \ldots\}$$

   denotes the set of all countably infinite outcomes, and

   $$\Omega = \{\omega_t : 0 \le t \le T\}$$

   denotes the set of uncountably outcomes.

1.2.1 Fields, s-Fields, [Krishnan, 1984]:

Let R be the nonempty class of subsets drawn from the sample space Ω. We say that the class R is a field or an algebra of sets in Ω if it satisfies the following definition:

Definition (1.2) (Filed or Algebra), [Krishnan, 1984]:

A class of subsets $A_j \subset \Omega$, $\forall$ j = 1, 2, …, n denoted by R is a field when the following conditions are satisfied:

1. If $A_i \in R$, then $A_i^c \in R$.
2. If $\{A_i, \forall\, i = 1, 2, \ldots, n\} \in R$, then $\bigcup_{i=1}^{n} A_i \in R$.

Example (1.1), [Krishnan, 1984]:

Let $\Omega =$ ¡ and consider a class R of all intervals of the form (a, b], such that:

$$(a, b] \cap (c, d] = \begin{cases} \varnothing, & a < b < c < d \\ (c, b], & a < c < b < d \\ (a, d], & c < a < d < b \\ (c, d], & a < c < d < b \\ (a, b], & c < a < b < d \end{cases}$$

Clearly the class R is closed under intersections. However:

$$(a, b]^c = (-\infty, a] \cup (b, \infty) \notin R$$

The class R is not a field.

Definition (1.3) (s-Field or s-Algebra), [Krishnan, 1984]:

A class of countably infinite collection of subsets $A_j \subset \Omega$, $\forall\, j = 1, 2, \ldots$ denoted by F is a σ-field when the following conditions are satisfied:

1. If $A_i \in F$, then $A_i^c \in F$.

2. If $\{A_i, i = 1, 2, \ldots\} \in F$, then $\bigcup_{i=1}^{\infty} A_i \in F$.

Remarks (1.2), [Krishnan, 1984]:

1. In general a σ-field is a field, but a field may not be a σ-field.

2. The intersection of any nonempty but arbitrary collection of σ-fields in Ω is a σ-field in Ω.

3. In general the arbitrary union of a collection of σ-fields may not be a σ-field.

We can always construct the smallest σ-field over R in which R will contain R and will be denoted by $\sigma(R) = F$.

This will always exist since $\sigma(R)$ can be defined as the intersection of all σ-fields containing R. If $\sigma_1(R)$, $\sigma_2(R)$, … are all σ-fields containing R, then:

$$\sigma(R) = \bigcap_{i=1}^{\infty} \sigma_i(R)$$

Further, the minimal σ-field thus generated is unique, we shall call $\sigma(R)$ the σ-field generated by R.

1.2.2 Probability Space:

Definition (1.4) (Probability Measure), [Al-Bayaty, 2008]:

A probability measure is a set function P defined on a σ-field F of subsets of a sample space Ω such that it satisfies the following axioms of Kolmogrov for any $A \in F$:

1. $p(A) \geq 0$.

2. $p(\Omega) = 1$.

3. $p\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} p(A_n)$.

with $A_n \in \mathcal{F}$ and A_i and A_j, $i \neq j$ being pairwise disjoint. Any set function η defined on a measurable space $(\Omega, \mathcal{F})$ satisfying axioms (1) and (3) is called a measure.

Definition (1.5) (Probability Space), [Stirzaker, 2005]

The measure space $(\Omega, \mathcal{F}, P)$ is called a probability space, which serves to describe any random experiment, where:

1. Ω is a nonempty set called the sample space, whose elements are the elementary outcomes of a random experiment.
2. $\mathcal{F}$ is a σ-field of subsets of Ω.
3. P is a probability measure defined on the measurable space $(\Omega, \mathcal{F})$.

Definition (1.6) (Measurable Function), [Krishnan, 1984]:

Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. Let g be a function with domain $E_1 \subset \Omega_1$ and $E_2 \subset \Omega_2$; $g : E_1 \longrightarrow E_2$. Then g is called a $\mathcal{F}_1$-measurable function or an $\mathcal{F}_1$-measurable mapping, if for every $E_2 \in \mathcal{F}_2$:

$$g^{-1}(E_2) = \{\omega \in \Omega : g(\omega) \in E_2\} = E_1$$

is in the σ-field $\mathcal{F}_1$.

Remarks (1.3), [Krishnan, 1984]:

1. The set E_1 given by $g^{-1}(E_2)$ is called the inverse image or inverse mapping of E_2, and it is measurable set.
2. Let g be a measurable mapping from $(\Omega_1, \mathcal{F}_1) \longrightarrow (\Omega_2, \mathcal{F}_2)$. If $\mathcal{R}$ is a nonempty class of subsets of Ω_2, then:

   $$\sigma(g^{-1}(\mathcal{R})) = g^{-1}(\sigma(\mathcal{R})).$$

Definition (1.7) (Random Variable), [Poularikas, 1999]:

To every outcome η of any experiment we assign one and only one number $X(\eta) = x$. The function X, whose domain in the space Ω of all outcomes and its range is a set of numbers, is called a random variable.

Definition (1.8) (Distribution Function), [Stirzaker, 2005]:

The distribution function $F_X(x)$ of X is denoted by $F_X(x)=p(X\leq x)$, and is defined by:

$$F_X(x) = p(B_x)$$

where:

$$B_x = \{\omega : X(\omega) \leq x\}$$

It follows that we have for all x, a, and b > a:

$$P(X > x) = 1 - F_X(x)$$

and

$$P(a < X \leq b) = F(b) - F(a).$$

All random variables have a distribution function. There are two principle types of random variables; namely the discrete and the continuous.

1.3 Discrete Random Variables, [Strizaker, 2005]

A discrete random variable takes values only in some countable subset D of ¡ (very commonly this subset D is a subset of the integers). Then the probability that X takes some given values x in D is denoted by:

$$f(x) = p(X = x) = p(V_x)$$

where $V_x = \{\omega : X(\omega) = x\}$ is the event that $X = x$.

The function f(x) may be called the probability function of x, or the probability mass function of x, or the probability distribution of x. It may also be denoted by $f_X(x)$ to avoid ambiguity. Here are some familiar and important examples of discrete random variables and their distributions:

1. Binomial distribution.
2. Poisson distribution.
3. Geometric distribution.

1.4 Continuous Random Variables, [Strirzaker, 2005]

A random variable that is not discrete is said to be continuous if its distribution function F(x) is written in the form:

$$F_X(x) = \int_{-\infty}^{\infty} f_X(u)\, du$$

for some nonnegative integrable function $f_X(x)$ defined for all x in $(-\infty, \infty)$. Then $f_X(x)$ is called the density function (or simply, density) of x. We may denoted it by f(x) if there is no risk of ambiguity. It is analogous to the probability mass function f(x) of a discrete random variable.

Here are some important examples of continuous distributions:

1. Uniform distribution.
2. Exponential distribution.
3. Normal distribution.

Definition (1.9) (Pointwise Convergence), [Krishnan, 1984]:

A sequence $\{X_n\}$ converges to a limit X if and only if for any $\varepsilon > 0$, however small, we can find positive integer n_0, such that:

$$|X_n - X| < \varepsilon, \text{ for every } n > n_0$$

Remark (1.4), [Krishnan, 1984]:

If we consider a sequence of random variables $\{X_1, X_2, \ldots, X_n, \ldots\}$ and define a pointwise convergence to another random variable X as in definition (1.9), then we must have for every ω-point in Ω the sequence of numbers $X_1(\omega)$, $X_2(\omega)$, …, $X_n(\omega)$, converging to $X(\omega)$. This type of convergence is called everywhere convergence.

Definition (1.10) (Almost Sure Convergence), [Evans, 2006]:

A sequence of random variables $\{X_n\}$ converges almost surly (abbreviated by a.s.) or almost certainly or strongly to X if for every ω-point not belonging to the null event A,

$$\lim_{n\to\infty} |X_n(\omega) - X(\omega)| = 0$$

This type of convergence is known as convergence with probability 1 and is denoted by:

$$X_n(\omega) \xrightarrow[n\to\infty]{a.s.} X(\omega) = \lim_{n\to\infty} X_n(\omega) \text{ (a.s.)}$$

Remark (1.5), [Evans, 2006]:

If the limit X is not known a priori, then we can define a mutual convergence almost surely. The sequence X_n converges mutually almost surely if:

$$\sup_{m\geq n} |X_m - X_n| \xrightarrow[n\to\infty]{a.s.} 0$$

In which both definitions are equivalent.

Remark (1.6), [Evans, 2006]:

Let $A_1, A_2, \ldots, A_n, \ldots$, be events in a probability space. Then the event:

$$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \{\omega \in \Omega \mid \omega \text{ belongs to infinitely many of the } A_n\}$$

is called "An infinitely often" and abbreviated by i.o.

Definition (1.11) (Convergence in Probability), [Strizaker, 2005]:

A sequence of random variables $\{X_n\}$ converges in probability to X if and only if for every $\varepsilon > 0$, however small, $\lim_{n\to\infty} p(|X_n - X| \geq \varepsilon) = 0$, or equivalently $\lim_{n\to\infty} p(|X_n - X| < \varepsilon) = 1$, and it is denoted by:

$$X_n(\omega) \xrightarrow[n\to\infty]{\text{L.i.p.}} X(\omega) \quad \text{or} \quad X(\omega) = \underset{n\to\infty}{\text{L.i.p.}} X_n(\omega)$$

Remark (1.7), [Krishnan, 1984]:

1. We can define mutual convergence in probability as:

$$\lim_{n\to\infty} \underset{m\geq n}{\text{Sup}}\, p(|X_m - X_n| \geq \varepsilon) \longrightarrow 0$$

2. If a sequence of random variables $\{X_n\}$ converges almost surely to X, then it converges in probability to the same limit. The converse is not true.

3. If $\{X_n\}$ converges in probability to X, then there exists a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ which converges almost surely to the same limit.

4. $\{X_n\}$ converges in probability if and only if it is converges mutually in probability.

1.5 Introduction to Stochastic Processes

Differential equations for random functions (stochastic processes) arise in the investigation of numerous physical and engineering problems, [Arnold, 1974]. We have looked at single random variables

(X_1, X_2, …, X_n), which we termed random vectors. However, many practical application of probability are concerned with random processes evolving in time, or space, or both, without any limit on the time (or space), [Al-Bayaty, 2008].

Definition (1.12) (Stochastic Process), [Stirzaker, 2005]:

A stochastic process is a collection of random variables $\{X(t) : t \in T\}$, where t is a parameter that runs over an index set T. In general, we call t the time-parameter (or simply the time), and $T \subseteq$ ¡ . Each X(t) takes values in some set $S \subseteq$ ¡ called the state space, then X(t) is the state of the process at time t.

Example (1.2), [Stirzaker, 2005]:

X(t) may be a number of a time t, or the number of heads shown by t flips of some coin. There is also of course, some underlying probability space Ω and probability set function P; since we are not concerned here with a general theory of random processes, we also not need to stress this part of the structure.

Remark (1.8), [Stirzaker, 2005]:

1. If the index set T is a countable set, we call X a discrete-time stochastic process, and if T is a continuum, we call it a continuous-time stochastic process.
2. A continuous-time stochastic process $\{X(t) : t \in T\}$ is said to have an independent increment if for all $t_0 < t_1 < t_2 < \ldots < t_n$, the random

variables $X(t_1)-X(t_0)$, $X(t_2)-X(t_1)$, …, $X(t_n)-X(t_{n-1})$ are independent. The process stationary increments if $X(t + s) - X(t)$ has the same distribution for all t. That is, it posses independent increments if the changes in the processes values over nonoverlapping time intervals are independent, and it process stationary increments if the distribution of the change in the value between any two points depends only on the distance between those points.

Definition (1.13) (Stationary), [Al-Bayaty, 2008]:

A stochastic process X(t) is stationary if:

$$p\{X_1(t) \le x_1, X_2(t) \le x_2, \ldots, X_m(t) \le x_m\} = p\{X(t_1 + \theta) \le x_1, X(t_2 + \theta) \le x_2, \ldots, X(t_m + \theta) \le x_m\}$$

for all $t_1, t_2, \ldots, t_m > 0$ and real values $x_1, x_2, \ldots, x_m$. For every natural number m and for all θ.

1.6 Introduction to Brownian Motion

Brownian motion was introduced by Robert Brown in 1827, when he observed the motion of a pollen grain as it moved randomly in a glass of water. Because the water molecules collide with the pollen grain in a random fashion, the pollen grain moves about randomly. The motion of the pollen grain is stochastic, because its position from one point in time to the next can only be defined in terms of a probability density function, [Higham, 2001]. In 1900, L. Bachelier used the Brownian motion as a model for movement of stock prices in his mathematical theory of speculation. The mathematical foundation for Brownian motion as stochastic process was introduced by N. Wiener in 1931, and this process is also called the Wiener process, [Klebaner, 2005].

Definition (1.14) (Brownian Motion), [Friedman, 1975]:

A Brownian motion or a Wiener process is a stochastic process $W(t)$, $t \geq 0$, satisfying:

1. $W(0) = 0$.
2. For any $0 \leq t_0 < t_1 < \ldots < t_n$, the random variables:

   $$\Delta W_n = W(t_{n+1}) - W(t_n),\ 1 \leq n \leq k$$

   are independent.
3. If $0 \leq s < t$, $W(t) - W(s)$ is normally distributed with mean μ_t and variance σ_t^2, then:

   $$E[W(t) - W(s)] = (t - s)\mu_t$$

   $$E[W(t) - W(s)]^2 = (t - s)\sigma_t^2$$

   where μ_t and σ_t are real constants, $\sigma_t > 0$.

Remark (1.9), [Stirzaker, 2005]:

1. If $\sigma^2 = 1$, then $W(t)$ is said to be the standard Brownian motion, we always make this assumption unless stated otherwise.
2. In fact, the assumption (2) is not strictly necessary, in that case one can construct (by a limiting procedure) a random process $W(t)$ that obeys (1) and (3) and is almost surely continuous.

Definition (1.15) (Brownian Motion in n-Dimension), [Friedman, 1975]:

An n-dimensional process $W(t) = (W_1(t), W_2(t), \ldots, W_n(t))$ is called an n-dimensional Brownian motion if each process $W_i(t)$ is a Brownian motion and if the σ-field $\mathcal{F}(W_i(t), t \geq 0)$, $1 \leq i \leq n$, are independent.

The next theorem is given in literatures without details of the proof. Here, we give the details of the proof for completeness.

Theorem (1.1), [Friedman, 1975]:

If X is a Brownian motion with normal distribution $N(0, \sigma_t^2)$, then:

$$E(X^{2n}(t)) = \frac{(2n)!\sigma^{2n}}{2^n n!}$$

where E refers to the mathematical expectation.

Proof:

From the definition of the moment generating function, defined by:

$$M(t) = e^{t^2/2}, -\infty < t < \infty$$

and recall the Maclurian series of $e^{t^2/2}$ which is given by:

$$M(t) = e^{t^2/2} = 1 + \frac{t^2}{2!} + \frac{3\times1}{4!}t^4 + \ldots + \frac{(2k-1)\times\ldots\times3\times1}{(2k)!}t^{2k} + \ldots \qquad \ldots(1.2)$$

Also, the Maclurian's series for M(t) is:

$$M(t) = M(0) + \frac{M'(0)}{1!}t + \frac{M''(0)}{2!}t^2 + \ldots + \frac{M^{(m)}(0)}{m!}t^m + \ldots$$

$$= 1 + \frac{E(X(t))}{1!}t + \frac{E(X^2(t))}{2!}t^2 + \ldots + \frac{E(X^m(t))}{m!}t^m + \ldots \qquad \ldots(1.3)$$

Comparing the coefficients of $\frac{t^m}{m!}$ in the Maclurian's series representation of M(t) given in eqs.(1.2) and (1.3), gives:

$$E(X^{2n}(t)) = (2n-1)(2n-3)\times\ldots\times 3\times 1$$

$$= \frac{2n(2n-1)(2n-2)\ldots 3\times 2\times 1}{2n(2n-2)(2n-4)\ldots 4\times 2}$$

$$= \frac{(2n)!}{2n\times 2(n-1)\times 2(n-2)\times\ldots\times 2(2\times 2)}$$

$$= \frac{(2n)!}{2^n n(n-1)(n-2)\ldots 3\times 2\times 1} = \frac{(2n)!}{2^n n!}, \; n = 1, 2, \ldots$$

If $X(t) \sim N(0, 1)$, then $E(X^{2n}(t)) = \frac{(2n)!}{2^n n!}$ while if $Y(t) \sim N(0, \sigma_t^2)$, then we may consider the transformation $X(t) = \frac{Y(t)}{\sigma} \sim N(0, 1)$, or equivalently $Y(t) = X(t)\sigma_t \sim N(0, \sigma_t^2)$. Hence:

$$E(X^{2n}(t)) = E\left(\frac{Y^{2n}(t)}{\sigma_t^{2n}}\right)$$

$$\frac{1}{\sigma_t^{2n}} E(Y^{2n}(t)) = \frac{(2n)!}{2^n n!}$$

Hence:

$$E(Y^{2n}(t)) = \frac{(2n)!}{2^n n!}\sigma_t^{2n}.$$ **n**

Theorem (1.2):

If W(t) is a Brownian motion, then:

$$E|W(t) - W(s)|^{2n} = C_n|t - s|^n$$

where C_n is a constant.

Proof:

If $W(t) = \dfrac{Z(t)}{\sigma_t}$, then from theorem (1.1)

$$E(Z^{2n}(t)) = \frac{(2n)!}{2^n n!}\sigma_t^{2n} \qquad \ldots(1.4)$$

Then from condition (3) of the Brownian motion, we have:

$$E(W(t) - W(s))^2 = (t - s)\sigma_t^2 \qquad \ldots(1.5)$$

To prove $E|W(t) - W(s)|^{2n} = C_n|t - s|^n$, where $C_n = \dfrac{(2n)!}{2^n n!}$

Let in eq.(1.4), $Z = |W(t) - W(s)|$, hence:

$$E(Z^{2n}(t)) = E|W(t) - W(s)|^{2n}$$

$$= E(|W(t) - W(s)|^2)^n$$

$$= \underbrace{E|W(t) - W(s)|^2 E|W(t) - W(s)|^2 \ldots E|W(t) - W(s)|^2}_{\text{multiplied n-times}}$$

$$= \underbrace{|t-s|\sigma_t^2 \, |t-s|\sigma_t^2 \cdots |t-s|\sigma_t^2}_{\text{n-times}}$$

$$= |t-s|^n (\sigma_t^2)^n$$

$$= |t-s|^n \sigma_t^{2n} = C_n |t-s|^n. \quad \blacksquare$$

Definition (1.16) (White Noise), [Klebaner, 2005]:

The white noise process Y(t) is formally defined as the derivative of the Brownian motion:

$$Y(t) = \frac{dW(t)}{dt} = W'(t) \qquad \ldots(1.6)$$

It does not exist as a function of t in the usual sense, since a Brownian motion is nowhere differentiable function.

Remark (1.10), [Al-Bayaty, 2008]:

A special case that is of considerable interest occurs when the processes X(t) from which the white noise derives is the Brownian motion. The white noise process then obtained is often referred to as Gaussian white noise.

1.7 Stochastic Differential Equations

Stochastic differential equations incorporate white noise which can be thought of as the derivative of Brownian motion. However, it should be mentioned that other types of random fluctuations are possible, [Arnold, 1974]. Solution of SDE's from every large class of stochastic processes. This class include Brownian motion, and many other stochastic processes used in stochastic modeling, [Lin, 2006]. A system of SDE's which arise when a random noise is introduced into ordinary differential equations, [Klebaner, 2005].

Definition (1.17) (Stochastic Differential Equations), [Haugh, 2005]:

An n-dimensional Itô process, X_t, is a process that can be represented as:

$$X_t = X_0 + \int_0^t A(s, X_s)\, ds + \int_0^t B(s, X_s)\, dW_s \qquad \ldots(1.7)$$

where W is an m-dimensional standard Brownian motion, and A and B are n-dimensional and n×m-dimensional F_t-adapted processes, respectively which is defined later in chapter two. We often use the notation:

$$dX_t = A(t, X_t)\, dt + B(t, X_t)\, dW_t,\ X(t_0) = X_0 \qquad \ldots(1.8)$$

Chapter Two

The Existence and Uniqueness Theorem of Stochastic Differential Equations

Chapter Two
The Existence and Uniqueness Theorem of Stochastic Differential Equations

Introduction

Solution of stochastic differential equations is of great difficulty in applications because of the existence of random processes, and hence the existence of a unique solution for such type of equations seems also to be very difficult, since such type of equations needs for some additional conditions on their solutions without explicitly evaluating the solution. Therefore, this chapter presents some basic and necessary and basic preliminaries in the theory of stochastic differential equations, and followed by the statement and the proof of the existence and uniqueness theorem of stochastic differential equations.

2.1 Preliminaries

Following are some fundamental and necessary concepts in the theory of stochastic differential equations, which are needed later on in the proof of the existence and uniqueness theorem.

Definition (2.1), [Friedman, 1975]:

A stochastic process f(t) defined on $[\alpha, \beta]$ is called a step function if there exists a partition $\alpha = t_0<t_1<\ldots<t_r=\beta$ of $[\alpha, \beta]$, such that:

$$f(t) = f(t_i) \text{ if } t_i < t \leq t_{i+1},\ i = 0, 1, \ldots, r-1$$

Lemma (2.1), [Friedman, 1975]:

Let $f \in L^2_\omega[\alpha, \beta]$, then:

1. If $L^2_\omega[\alpha, \beta]$ is the space of all functions f such that $\int_\alpha^\beta |f(t)|^2\, dt < \infty$, and there exists a sequence of continuous functions g_n in $L^2_\omega[\alpha, \beta]$, such that:

$$\lim_{n\to\infty} \int_\alpha^\beta |f(t) - g_n(t)|^2\, dt = 0 \text{ a.s.} \qquad \ldots(2.1)$$

2. There exists a sequence of step functions f_n in $L^2_\omega[\alpha, \beta]$, such that:

$$\lim_{n\to\infty} \int_\alpha^\beta |f(t) - f_n(t)|^2\, dt = 0 \text{ a.s.} \qquad \ldots(2.2)$$

The next theorem is given in [Krishnan, 1984] without details of the proof, we give here the complete details of the proof for completeness.

Theorem (2.1) (Hölder's Inequality of Expectation), [Krishnan, 1984]:

If p and q are real numbers greater than 1, with $\frac{1}{p} + \frac{1}{q} = 1$ and if the random variables X, Y and $|X|^p$, $|Y|^p$ are integrable, then:

$$E|XY| \le [E|X|^p]^{1/p}[E|Y|^q]^{1/q}$$

Proof:

Let X be a positive number and consider the function:

$$\phi(X) = \frac{X^p}{p} + \frac{X^{-q}}{q}$$

This function has a minimum value at X = 1, since:

$$\phi'(X) = X^{p-1} - X^{-1-q} = 0$$

and multiplying by X, yields:

$$X^p - X^{-q} = 0 \qquad \ldots(2.3)$$

Also, p = pq − q. Hence, substituting in eq.(2.3) give:

$$X^{pq-q} - X^{-q} = 0$$

i.e.,

$$X^{-q}(X^{pq} - 1) = 0$$

Therefore $X^{pq} - 1 = 0$ which implies that $X^{pq} = 1$, i.e., X = 1 is the critical point which give a minimum value of ϕ which is $\phi(1) = 1$.

Let us now substitute $X = b^{1/q}/a^{1/p}$, with a, b > 0 in $\phi(X)$, then:

$$\phi(X) = \frac{\left[\frac{b^{1/q}}{a^{1/p}}\right]^p}{p} + \frac{\left[\frac{b^{1/q}}{a^{1/p}}\right]^{-q}}{q}$$

$$= \frac{b^{p/q}}{ap} + \frac{a^{q/p}}{bq}$$

$$= \frac{p^{-1}b^{p/q}}{a} + \frac{q^{-1}a^{q/p}}{b} \geq 1$$

Hence:

$$bp^{-1}b^{p/q} + aq^{-1}a^{q/p} \geq ab$$

i.e.,

$$b^{\frac{p}{q}+1} p^{-1} + a^{\frac{q}{p}+1} q^{-1} \geq ab$$

Therefore:

$$b^{\frac{p+q}{q}} p^{-1} + a^{\frac{q+p}{p}} q^{-1} \geq ab$$

since $\frac{p+q}{pq} = 1$, we have $\frac{p+q}{q} = p$ and $\frac{p+q}{p} = q$, and therefore:

$$ab \leq \frac{b^p}{p} + \frac{a^q}{q} \qquad \ldots(2.4)$$

Suppose that:

$$b = \frac{|X|}{\left[E|X|^p\right]^{1/p}} \text{ and } a = \frac{|Y|}{\left[E|Y|^q\right]^{1/q}} \qquad \ldots(2.5)$$

Substitute (2.4) in (2.5), yields:

$$\frac{|Y|}{\left[E|Y|^q\right]^{1/q}} \frac{|X|}{\left[E|X|^p\right]^{1/p}} \leq \frac{\left(\frac{|X|}{\left[E|X|^p\right]^{1/p}}\right)^p}{p} + \frac{\left(\frac{|Y|}{\left[E|Y|^q\right]^{1/q}}\right)^q}{q}$$

and taking the expectation to the both sides of the last inequality, yields:

$$\frac{E|XY|}{\left[E|X|^p\right]^{1/p}\left[E|Y|^q\right]^{1/q}} \leq \frac{\frac{E|X|^p}{E|X|^p}}{p} + \frac{\frac{E|Y|^q}{E|Y|^q}}{q}$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

Hence:

$$E|XY| \le [E|X|^p]^{1/p}\,[E|Y|^q]^{1/q}$$

and Hölder's inequality of expectation follows. **n**

Remark (2.1), [Krishnan, 1984]:

Cauchy-Schwarz's inequality of expectation follows directly from Hölder's inequality, if we substitute p = q = 2, then we obtain that:

$$E|XY| \le \sqrt{E\,|\,X\,|^2\;E\,|\,Y\,|^2}$$

if X, Y and $|X|^2$, $|Y|^2$ are integrable.

Definition (2.2) (Stochastic Integral), [Friedman, 1975]:

Let f(t) be a step function $L^2_\omega[\alpha, \beta]$, say:

$$f(t) = f_i,\ \text{if } t_i < t \le t_{i+1},\ 0 \le i \le r-1$$

where $\alpha = t_0 < t_1 < t_2 < \ldots < t_r = \beta$. The random variable:

$$\sum_{k=0}^{r-1} f(t_k)[W(t_{k+1}) - W(t_k)]$$

is denoted by:

$$\int_{\alpha}^{\beta} f(t)\,dW(t)$$

and is called the stochastic integral of f with respect to Brownian motion W, it is also called the Itô integral.

Theorem (2.2), [Friedman, 1975]:

Let f, f_n be in $L^2_\omega[\alpha, \beta]$ and suppose that:

$$\int_\alpha^\beta |f_n(t) - f(t)|^2 \, dt \xrightarrow{p.} 0 \text{ as } n \longrightarrow \infty \qquad \ldots(2.6)$$

Then:

$$\int_\alpha^\beta f_n(t) \, dW(t) \xrightarrow{p.} \int_\alpha^\beta f(t) \, dW(t) \text{ as } n \longrightarrow \infty \qquad \ldots(2.7)$$

where $\xrightarrow{p.}$ refers that the converge is in probability.

Lemma (2.2), [Friedman, 1975]:

If $f \in L^2_\omega[\alpha, \beta]$ and f is continuous, then for any sequence π_n of partitions $\alpha = t_{n,0} < t_{n,1} < \ldots < t_{n,m_n} = \beta$ of $[\alpha, \beta]$ with mesh $|\pi_n| \longrightarrow 0$,

$$\sum_{k=0}^{m_n-1} f(t_{n,k})[W(t_{n,k+1}) - W(t_{n,k})] \xrightarrow{p.} \int_\alpha^\beta f(t) \, dW(t) \text{ as } n \longrightarrow \infty$$

$$\ldots(2.8)$$

Proof:

Introduce the step function g_n:

$g_n(t) = f(t_{n,k})$ if $t_{n,k} \le t \le t_{n,k+1}$, $0 \le k \le m_n - 1$

for $g_n(t) \longrightarrow f(t)$ uniformly in $t \in [\alpha, \beta)$ as $n \longrightarrow \infty$. Hence:

$$\int_\alpha^\beta |g_n(t) - f(t)|^2 \, dt \longrightarrow 0 \text{ a.s.}$$

By theorem (2.2), we then have:

$$\int_{\alpha}^{\beta} g_n(t)\, dW(t) \xrightarrow{p.} \int_{\alpha}^{\beta} f(t)\, dW(t)$$

Since $\int_{\alpha}^{\beta} g_n(t)\, dW(t) = \sum_{k=0}^{m_n-1} f(t_{n,k})[W(t_{n,k+1}) - W(t_{n,k})]$, then assertion (2.8) follows. ■

Example (2.1), [Friedman, 1975]:

If $0 \le t_1 < t_2$ and $\pi_n = \{t_1 = t_{n,1}, t_{n,2}, \ldots, t_{n,n} = t_n\}$ is a sequence of partitions of $[t_1, t_2]$ with mesh $|\pi_n| \longrightarrow 0$, then from lemma (2.2)

$$\int_{t_1}^{t_2} W(t)\, dW(t) = \lim_{n\to\infty} \sum_{k=1}^{n-1} W(t_{n,k})[W(t_{n,k+1}) - W(t_{n,k})]$$

$$= \frac{1}{2} \lim_{n\to\infty} \sum_{k=1}^{n-1} [(W(t_{n,k+1}))^2 - (W(t_{n,k}))^2] - [W(t_{n,k+1}) - W(t_{n,k})]^2$$

$$= \frac{1}{2}(W(t_2))^2 - \frac{1}{2}(W(t_1))^2 - \frac{1}{2} \lim_{n\to\infty} \sum_{k=1}^{n-1} [W(t_{n,k+1}) - W(t_{n,k})]^2$$

where $\lim_{n\to\infty}$ is taken as the limit in probability. The last limit in probability is equal to $t_2 - t_1$. Hence:

$$\int_{t_1}^{t_2} W(t)\, dW(t) = \frac{1}{2}(W(t_2))^2 - \frac{1}{2}(W(t_1))^2 - \frac{1}{2}(t_2 - t_1) \qquad \ldots(2.9)$$

Lemma (2.3), [Friedman, 1975]:

Let f_1, f_2 be two step functions in $L^2_\omega[\alpha, \beta]$ and let λ_1, λ_2 be two real numbers. Then $\lambda_1 f_1 + \lambda_2 f_2$ is in $L^2_\omega[\alpha, \beta]$ and

$$\int_\alpha^\beta [\lambda_1 f_1(t) + \lambda_2 f_2(t)]\, dW(t) = \lambda_1 \int_\alpha^\beta f_1(t)\, dW(t) + \lambda_2 \int_\alpha^\beta f_2(t)\, dW(t) \quad \ldots(2.10)$$

Theorem (2.3), [Friedman, 1975]:

If f is a step function in $\mu^2_\omega[\alpha, \beta]$ (where $\mu^2_\omega[\alpha, \beta]$ will be defined later in this chapter) and W(t) is Brownian motion, then:

$$E\int_\alpha^\beta f(t)\, dW(t) = 0 \quad \ldots(2.11)$$

$$E\left|\int_\alpha^\beta f(t)\, dW(t)\right|^2 = E\int_\alpha^\beta f^2(t)\, dt \quad \ldots(2.12)$$

Proof:

Since:

$$E\int_\alpha^\beta f^2(t)\, dt = \sum_{i=0}^{r-1} Ef^2(t_i)(t_{i+1} - t_i) \quad \ldots(2.13)$$

is finite, by assumption, we deduce that $Ef^2(t_i) < \infty$. In particular, $E|f(t_i)| < \infty$. Also, $E|W(t_{i+1}) - W(t_i)| < \infty$. But since $f(t_i)$ is $\mathcal{F}_{t_i}$ measurable, where as $W(t_{i+1}) - W(t_i)$ is independent of $\mathcal{F}_{t_i}$.

Since, from the properties of the mathematical expectation of independent events, then:

$$Ef(t_i)(W(t_{i+1}) - W(t_i)) = Ef(t_i)E(W(t_{i+1}) - W(t_i)) = 0$$

Summing over i, (2.11) follows.

Next, since $f^2(t_i)$ and $[W(t_{i+1}) - W(t_i)]^2$ are independent and have finite expectation, also $f^2(t_i)[W(t_{i+1}) - W(t_i)]^2$ has finite expectation.

By Schawrz's inequality it follows that:

$$E|f(t_k)f(t_i)(W(t_{i+1})-W(t_i))| \le \sqrt{E\,|\,f(t_k)f(t_i)\,|^2\;E\,|\,W(t_{i+1})-W(t_i)\,|^2} < \infty$$

i.e., finite.

If k > i, then $W(t_{k+1}) - W(t_k)$ is independent of $f(t_k)f(t_i)(W(t_{i+1}) - W(t_i))$

In view of the last inequality and the finiteness of $E|W(t_{k+1}) - W(t_k)|$, we deduce that:

$$E[f(t_i)(W(t_{i+1}) - W(t_i))] = Ef(t_i)E[W(t_{i+1}) - W(t_i)] = 0$$

Replace i by k, yields:

$$E[f(t_k)(W(t_{k+1}) - W(t_k))] = Ef(t_k)E[W(t_{k+1}) - W(t_k)] = 0$$

Hence:

$$E[f(t_i)f(t_k)(W(t_{i+1}) - W(t_i))\,(W(t_{k+1}) - W(t_k))]\;E[f(t_i)(W(t_{i+1}) - W(t_i))\;f(t_k)(W(t_{k+1}) - W(t_k))] = 0$$

Therefore:

$$E\left|\int_{\alpha}^{\beta} f(t)\,dW(t)\right|^2 = \sum_{i=0}^{r-1} E\left[f^2(t_i)\big(W(t_{i+1})-W(t_i)\big)^2\right]$$

$$= \sum_{i=0}^{r-1} Ef^2(t_i)E\big(W(t_{i+1})-W(t_i)\big)^2$$

and from theorem (1.2) with n = 1, $C_n = 1$, we have:

$$E(W(t_{i+1}) - W(t_i))^2 = |t_{i+1} - t_i|$$

Hence:

$$E\left|\int_{\alpha}^{\beta} f(t)\, dW(t)\right|^2 = \sum_{i=0}^{r-1} Ef^2(t_i)\,|t_{i+1} - t_i|$$

$$= E\int_{\alpha}^{\beta} f^2(t)\, dt. \quad \blacksquare$$

Theorem (2.4), [Friedman, 1975]:

Let $f \in \mu_{\omega}^2[0, T]$, then:

$$\sup_{0\le t\le T}\left|\int_0^t f(s)\, dW(s)\right|^2 \le 4E\left|\int_0^T f(t)\, dW(t)\right|^2$$

$$= 4E\int_0^T |f(t)|^2\, dt$$

Definition (2.3), [Krishnan, 1984]:

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let X be a real random variable. The expectation of X is defined by:

$$E(X) = \int_{\Omega} X(\omega)\, dp(\omega) \text{ or } \int_{\Omega} X\, dp \qquad \ldots(2.14)$$

We now have to define the integral, which we will do by stages. First, we take a simple random variable of the form:

$$X = \sum_{k=1}^{n} x_k I_{A_k} \qquad \ldots(2.15)$$

and define:

$$E(X) = \sum_{k=1}^{n} x_k p(A_k) \qquad \ldots(2.16)$$

Proposition (2.1) (Fatou's Lemma), [Krishnan, 1984]:

Let $\{X_n\}$ be a sequence of random variables and X be an integrable random variable, such that $X_n(\omega) \geq X(\omega)$, for all n and ω is bounded below, then:

$$E(\liminf_{n\to\infty} X_n) \leq \liminf_{n\to\infty} E(X_n)$$

and if $X(\omega)$ is such that $X_n(\omega) < X$, for all n and ω is bounded above, then:

$$\limsup_{n\to\infty} E(X_n) \leq E(\limsup_{n\to\infty} X_n)$$

Definition (2.4) (The Itô Process], [Friedman, 1975]:

A stochastic process $X(t)$, $0 \leq t \leq T$ is called an Itô process with respect to $\{W(t), P, \mathcal{F}_t\}$, (where $\mathcal{F}_t$ is adapted to $W(t)$) relative to $B(t)$, $A(t)$ if:

$$X(t) = X(0) + \int_0^t A(s)\, ds + \int_0^t B(s)\, dW(s), \; 0 \leq t \leq T \qquad \ldots(2.17)$$

Definition (2.5) (Increasing s-Field or Filtration s-Field),[Krishnan,1984]:

Let (Ω, F) be a complete measurable space and let $\{F_t, t \in T, T = ¡^{+}\}$ be a family of sub-σ-fields of F, such that for $s \leq t$, $F_s \subset F_t$. Then $\{F_t\}$ is called an increasing family of sub-σ-fields on (Ω, F) or the filtration σ-filed of (Ω, F).

F_t is called the σ-field of events prior to t. If $\{X_t, t \in T\}$ is a stochastic process defined on (Ω, F, P), then clearly F_t given by:

$$F_t = \sigma \{X_s, s \leq t, t \in T\} \qquad \ldots(2.18)$$

is increasing.

Remark (2.2), [Krishnan, 1984]:

Since the probability space (Ω, F, P) is complete, the σ-field F contains all subsets of Ω having probability measure zero. We shall assume here that the filtration σ-field $\{F_t, t \in T\}$ also contains all the sets from F having probability measure zero.

Definition (2.6), (Adaptation of $\{X_t\}$), [Krishnan, 1984]:

Let $\{X_t, t \in T, T = ¡^{+}\}$ be a stochastic process defined on a probability space (Ω, F, P) and let $\{F_t, t \in T, T = ¡^{+}\}$be a filtration σ-field. The process $\{X_t\}$ is adapted to the family $\{F_t\}$ if X_t is F_t-measurable for every $t \in T$, or:

$$E^{F_t} X_t = X_t, \ t \in T$$

F_t-adapted random processes are also F_t-measurable.

2.2 Martingles

Martingales play a central role in the modern theory of stochastic processes and stochastic calculus. Martingales converge almost surely. Stochastic integrals are martingales. These are most important properties of martingales which hold under some condition, [Klebaner, 2005].

Definition (2.7), [Klebaner, 2005]:

Adapted to filtration F = (F_t) is a martingales if for any t, μ(t) is integrable, that is, E|μ(t)| < ∞ and for any t and s with $0 \le s < t \le T$

$$E(\mu(t) \mid F_s) = \mu(s) \quad \text{a.s.}$$

where μ(t) is a martingale on [0, ∞) if it is integrable and the martingale property holds for any $0 \le s < t < \infty$.

Definition (2.8), [Klebaner, 2005]:

A stochastic process X(t), t ≥ 0 adapted to filtration F is a super martingale (submartingale) if it is integrable, and for any t and s, $0 \le s < t \le T$

$$E(X(t) \mid F_s) \le X(s) \quad \text{a.s.}$$

$$E(X(t) \mid F_s) \ge X(s) \quad \text{a.s.}$$

If X(t) is a supermartingale, then X(t) is submartingale.

Theorem (2.5) (Martingale Inequalities), [Evans, 2006]:

Let X(t) be a stochastic process with continuous sample paths a.s.

1. If X(.) is a submartingale, then:

$$p(\max_{0\le s\le t} X(s) \ge \lambda) \le \frac{1}{\lambda}E(X(t)^+), \text{ for all } \lambda > 0,\ t \ge 0$$

2. If X(.) is a martingale and $1 < p < \infty$, then:

$$E(\max_{0\le s\le t} |X(s)|^p) \le \left(\frac{p}{p-1}\right)^p E(|X(t)|^p).$$

__Example (2.2), [Evans, 2006]:__

Let W(.) be a 1-dimensional standard Wiener process, then W(.) is a martingale. To see this, write:

$\mathcal{W}(t) = \mathcal{F}_t(W(s) \mid 0 \le s \le t)$, and let $t \ge s$, then:

$$E(W(t) \mid \mathcal{W}(s)) = E(W(t) - W(s) \mid \mathcal{W}(s)) + E(W(s) \mid \mathcal{W}(s))$$

$$= E(W(t) - W(s)) + W(s)$$

$$= W(s) \text{ a.s.}$$

__Remark (2.3):__

We may denote by $\mu_\omega^2[0, T]$ to be the set of all finite square integrable martingales over [0, T].

2.3 The Existence and Uniqueness Theorem of Stochastic Differential Equations, [Friedman, 1975]

Consider the stochastic differential equation:

$$dX(t) = A(X(t), t)\, dt + B(X(t), t)\, dW(t) \qquad \ldots(2.19)$$

with initial condition:

$$X(0) = X_0 \qquad \ldots(2.20)$$

where A(X(t), t) and B(X(t), t) are measurable functions.

Hence, to find the equivalent stochastic integral equation, integrate both sides of eq.(2.19) and use the initial condition (2.20)

$$\int_0^t dX(s)\, ds = \int_0^t A(X(s), s)\, ds + \int_0^t B(X(s), s)\, dW(s)$$

therefore:

$$X(t) = X_0 + \int_0^t A(X(s), s)\, ds + \int_0^t B(X(s), s)\, dW(s)$$

and hence an iterated sequence of solutions of the resulting integral equation may be evaluated as follows:

$$\left.\begin{aligned} X_1(t) &= X_0 + \int_0^t A(X_0(s),s)\, ds + \int_0^t B(X_0(s),s)\, dW(s) \\ X_2(t) &= X_0 + \int_0^t A(X_1(s),s)\, ds + \int_0^t B(X_1(s),s)\, dW(s) \\ &\vdots \\ X_{m+1}(t) &= X_0 + \int_0^t A(X_m(s),s)\, ds + \int_0^t B(X_m(s),s)\, dW(s) \end{aligned}\right\} \qquad \ldots(2.21)$$

Theorem (2.6) (The Existence Theorem):

Suppose that A(X(t), t), B(X(t), t) are measurable functions in $(X(t), t) \in \mathbb{R}^n \times [0, T]$ and A(x(t), t), B(X(t), t) satisfies:

$$\left.\begin{aligned} &|A(X, t) - A(\bar{X}, t)| \le K_*|X - \bar{X}| \\ &|B(X, t) - B(\bar{X}, t)| \le K_*|X - \bar{X}| \\ &|A(X, t)| \le K(1 + |X|) \\ &|B(X, t)| \le K(1 + |X|) \end{aligned}\right\} \quad \ldots(2.22)$$

Where K_*, K are constants. Let x_0 be any n-dimensional random vector independent of $F(W(t))$, $0 \le t \le T$, such that $E|x_0|^2 < \infty$. Then there exist a solution of (2.19) with condition (2.20) in $\mu^2_\omega[0, T]$.

Proof:

Since the iterated sequence of solutions of the integral equation may be given as:

$$X_{m+1}(t) = X_0 + \int_0^t A(X_m(s), s)\, ds + \int_0^t B(X_m(s), s)\, dW(s) \quad \ldots(2.23)$$

for all m = 0, 1, … . The proof will proceed by induction on the sequence of solutions $X_m(t) \in \mu^2_\omega[0, T]$.

If m = 0, then:

$$|X_1(t) - X_0|^2 = |X_0 + \int_0^t A(X_0(s),s)\, ds + \int_0^t B(X_0(s),s)\, dW(s) - X_0|^2$$

$$= \left| \int_0^t A(X_0(s),s)\,ds + \int_0^t B(X_0(s),s)\,dW(s) \right|^2$$

$$\leq \left| \int_0^t A(X_0(s),s)\,ds \right|^2 + \left| \int_0^t B(X_0(s),s)\,dW(s) \right|^2$$

Taking the expectation on both sides and using (2.22), give:

$$E|X_1(t) - X_0|^2 \leq E\left| \int_0^t A(X_0(s),s)\,ds \right|^2 + E\left| \int_0^t B(X_0(s),s)\,dW(s) \right|^2$$

Using the following inequality $(a + b)^2 \leq 2a^2 + 2b^2$, then:

$$E|X_1(t) - X_0|^2 \leq 2E\left| \int_0^t A(X_0(s),s)\,ds \right|^2 + 2E\left| \int_0^t B(X_0(s),s)\,dW(s) \right|^2$$

Now, from theorem (2.3), we have:

$$E|X_1(t) - X_0|^2 \leq 2Et\int_0^t |A(X_0(s),s)|^2\,ds + 2E\int_0^t |B(X_0(s),s)|^2\,ds$$

$$\leq (2K^2t + 2K^2)(1 + E|X_0|^2)t$$

$$\leq Mt = \frac{Mt}{1!}$$

If m = 1, then:

$$|X_2(t) - X_1(t)|^2 = |X_0 + \int_0^t A(X_1(s),s)\,ds + \int_0^t B(X_1(s),s)\,dW(s) -$$

$$X_0 - \int_0^t A(X_0(s),s)\,ds + \int_0^t B(X_0(s),s)\,dW(s)|^2$$

$$\leq \left|\int_0^t \big(A(X_1(s),s) - A(X_0(s),s)\big)ds\right|^2 +$$

$$\left|\int_0^t \big(B(X_1(s),s) - B(X_0(s),s)\big)dW(s)\right|^2$$

Taking the expectation on both sides and using eq.(2.22), yields:

$$E|X_2(t) - X_1(t)|^2 = E\left|\int_0^t \big(A(X_1(s),s) - A(X_0(s),s)\big)ds\right|^2 +$$

$$E\left|\int_0^t \big(B(X_1(s),s) - B(X_0(s),s)\big)dW(s)\right|^2$$

$$\leq 2E\left|\int_0^t \big(A(X_1(s),s) - A(X_0(s),s)\big)ds\right|^2 +$$

$$2E\left|\int_0^t \big(B(X_1(s),s) - B(X_0(s),s)\big)dW(s)\right|^2$$

$$\leq 2Et\int_0^t \left|A(X_1(s),s) - A(X_0(s),s)\right|^2 ds +$$

$$2E\int_0^t \left|B(X_1(s),s) - B(X_0(s),s)\right|^2 ds$$

$$\leq 2K_*^2 t\int_0^t E\left|X_1(s) - X_0(s)\right|^2 ds +$$

$$2K_*^2\int_0^t E\left|X_1(s) - X_0(s)\right|^2 ds$$

$$\leq (2K_*^2 t + 2K_*^2)E|X_1(s) - X_0(s)|^2 t \leq \frac{(Mt)^2}{2!}$$

If the inequality is satisfied for m = k and to prove it is true for m = k + 1

$$|X_{k+1}(t) - X_k(t)|^2 = |X_0 + \int_0^t A(X_k(s),s)\,ds + \int_0^t B(X_k(s),s)\,dW(s) -$$

$$X_0 - \int_0^t A(X_{k-1}(s),s)\,ds + \int_0^t B(X_{k-1}(s),s)\,dW(s)|^2$$

$$\leq \left|\int_0^t \big(A(X_k(s),s) - A(X_{k-1}(s),s)\big)ds\right|^2 +$$

$$\left|\int_0^t \big(B(X_k(s),s) - B(X_{k-1}(s),s)\big)dW(s)\right|^2$$

Taking the expectation on both sides and using eq.(2.22), yields:

$$E|X_{k+1}(t) - X_k(t)|^2 = E\left|\int_0^t \big(A(X_k(s),s) - A(X_{k-1}(s),s)\big)ds\right|^2 +$$

$$E\left|\int_0^t \big(B(X_k(s),s) - B(X_{k-1}(s),s)\big)dW(s)\right|^2$$

$$\leq 2E\left|\int_0^t \big(A(X_k(s),s) - A(X_{k-1}(s),s)\big)ds\right|^2 +$$

$$2E\left|\int_0^t \big(B(X_k(s),s) - B(X_{k-1}(s),s)\big)dW(s)\right|^2 \qquad \ldots(2.24)$$

$$\leq 2Et\int_0^t |A(X_k(s),s) - A(X_{k-1}(s),s)|^2\,ds +$$

$$2E\int_0^t |B(X_k(s),s) - B(X_{k-1}(s),s)|^2\,ds$$

$$\leq 2K_*^2 t\int_0^t E\left|X_k(s)-X_{k-1}(s)\right|^2 ds \ +$$

$$2K_*^2\int_0^t E\left|X_k(s)-X_{k-1}(s)\right|^2 ds$$

$$\leq (2K_*^2 t + 2K_*^2)E|X_k(s) - X_{k-1}(s)|^2 t$$

$$\leq \frac{(Mt)^{k+1}}{(k+1)!} \qquad \ldots(2.25)$$

Since this implies that $X_{k+1} \in \mu_\omega^2[0, T]$, the proof of inductive assumption for k + 1 is complete. From (2.24), we have also:

$$E|X_{k+1}(t) - X_k(t)|^2 \leq 2E\left|\int_0^t \left(A(X_k(s),s)-A(X_{k-1}(s),s)\right)ds\right|^2 +$$

$$2E\left|\int_0^t \left(B(X_k(s),s)-B(X_{k-1}(s),s)\right)dW(s)\right|^2$$

Hence:

$$\sup_{0\leq t\leq T} |X_{k+1}(t) - X_k(t)|^2 \leq 2TK_*^2\int_0^T \left|X_k(s)-X_{k-1}(s)\right|^2 ds \ +$$

$$2\sup_{0\leq t\leq T}\left|\int_0^t \left(B(X_k(s),s)-B(X_{k+1}(s),s)\right)\ dW(s)\right|^2$$

using theorem (2.4)

$$\sup_{0\le t\le T} E|X_{k+1}(t) - X_k(t)|^2 \le 2TK_*^2 \int_0^T E|X_k(s) - X_{k-1}(s)|^2\,ds +$$

$$8K_*^2 \int_0^t E|X_k(s) - X_{k-1}(s)|^2\,ds$$

$$\le (2K_*^2 T + 8K_*^2)T$$

$$E|X_{k+1}(t) - X_k(t)|^2 \le C\frac{(MT)^k}{k!}$$

where $C = (2K_*^2 T + 8K_*^2)T$.

Now, to prove the convergence of the sequence $\{X_m\}_{m=1}^{\infty}$ uniformly in $t \in [0, T]$.

It follows that the sequence of partial sums:

$$X_k(t) = X_0 + \sum_{m=0}^{k-1} (X_{m+1}(t) - X_m(t)),\ \forall\ k = 1, 2, \ldots$$

$$= X_0 + (X_1(t) - X_0(t)) + \ldots + (X_m(t) - X_{m-1}(t))$$

Which must be converge uniformly. Now, $\{X_m(t)\}_{m=1}^{\infty}$ is converge if, the series:

$$X_k(t) = X_0 + \sum_{m=0}^{k-1} (X_{m+1}(t) - X_m(t)),\ \forall\ k = 1, 2, \ldots$$

is converge.

From (2.22), we have:

$$E|X_{m+1}(t)|^2 = E|X_0 + \int_0^t A(X_m(s),s)\, ds + \int_0^t B(X_m(s),s)\, dW(s)|^2$$

$$\leq E|X_0|^2 + K^2 t\int_0^t (1+E|X_m(t)|^2)\, ds + K^2\int_0^t (1+E|X_m(t)|^2)\, ds$$

$$\leq E|X_0|^2 + K^2 t\int_0^t E|X_m(t)|^2\, ds + K^2\int_0^t E|X_m(t)|^2\, ds$$

$$\leq E|X_0|^2 + (K^2 t + K^2)\int_0^t E|X_m(t)|^2\, ds$$

$$\leq C(1 + E|X_0|^2) + C\int_0^t E|X_m(t)|^2\, ds$$

Now, carrying the last inequality recursively

$$E|X_{m+1}(t)|^2 \leq C(1 + E|X_0|^2) + Ct[C(1 + E|X_0|^2 + C\int_0^t E|X_m(t)|^2\, ds]$$

$$\leq C(1 + E|X_0|^2 + Ct[C(1 + E|X_0|^2 + Ct[C(1 + E|X_0|^2) + C\int_0^t E|X_{m-1}(t)|^2\, ds]]$$

So, carrying this inequality m-times will produce:

$$E|X_{m+1}(t)|^2 \leq (1 + C + C^2 t + \ldots + C^m \frac{t^m}{m!} + \ldots)(1 + E|X_0|^2)$$

$$= e^{Ct}(1 + E|X_0|^2)$$

Therefore:

$$E|X_{m+1}(t)|^2 \leq C(1 + E|X_0|^2)e^{Ct}$$

To prove $X_k(t)$ is converge to $X(t)$ as $k \longrightarrow \infty$, i.e., $\lim_{k\to\infty} X_k(t) = X(t)$,

i.e., given any $\varepsilon > 0$, there exist $N \in ¥$, such that:

$$|X_k(t) - X(t)| < \varepsilon, \forall k \geq N$$

$$|X_0 + \sum_{m=0}^{k-1} (X_{m+1}(t) - X_m(t)) - X(t)| = |X_0 + (X_1(t) - X_0(t)) + (X_2(t) - X_1(t)) + \ldots + (X_m(t) - X_{m-1}(t))| < \varepsilon$$

which implies

$$|X_{k-1}(t) - X(t)| < \varepsilon$$

Since:

$$X_k(t) = X_0 + \sum_{m=0}^{k-1} (X_{m+1}(t) - X_m(t))$$

Converges on a compact interval [0, T], then the sequence is converge uniformly.

Now, to prove that X(t) is continuous, i.e., to prove:

$$\lim_{h\to 0} X(t + h) = X(t)$$

Hence:

$$|X(t+h)-X(t)| = |X(t+h)-X_k(t+h)+X_k(t+h)-X_k(t)+X_k(t)-X(t)|$$

$$\leq |X(t+h)-X_k(t+h)|+|X_k(t+h)-X_k(t)|+|X_k(t)-X(t)|$$

$$= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

i.e., $\lim_{h\to 0} X(t + h) = X(t)$

Hence, X(t) is continuous.

To prove X(t) is a solution, i.e., to prove X(t) satisfy the stochastic integral equation:

$$X(t) = X_0 + \int_0^t A(X(s), s)\, ds + \int_0^t B(X(s), s)\, dW(s)$$

and since:

$$X_{m+1}(t) = X_0 + \int_0^t A(X_m(s), s)\, ds + \int_0^t B(X_m(s), s)\, dW(s)$$

and, as $m \longrightarrow \infty$, then:

$$X(t) = X_0 + \lim_{m\to\infty} \int_0^t A(X_m(s), s)\, ds + \lim_{m\to\infty} \int_0^t B(X_m(s), s)\, dW(s)$$

Therefore, it is enough to prove that:

$$\lim_{m\to\infty} \int_0^t A(X_m(s), s)\, ds = \int_0^t A(X(s), s)\, ds$$

and

$$\lim_{m\to\infty} \int_0^t B(X_m(s), s)\, dW(s) = \int_0^t B(X(s), s)\, dW(s)$$

Now:

$$E\left|\int_0^t \left(A(X(s),s) - A(X_m(s),s)\right) ds\right|^2 + E\left|\int_0^t \left(B(X(s),s) - B(X_m(s),s)\right) dW(s)\right|^2$$

$$\leq 2K_*^2 tE\int_0^t |X(s) - X_m(s)|^2\, ds + 2K_*^2 E\int_0^t |X(s) - X_m(s)|^2\, ds$$

and taking $m \longrightarrow \infty$ and using Fatou's lemma, we conclude that:

$$E|X(t)|^2 \leq C(1 + E|X_0|^2)e^{Ct}$$

Thus X(t) is a solution of (2.19) and (2.20). ∎

Theorem (2.7) (The Uniqueness Theorem):

Under the same conditions of theorem (2.6), there exists a unique solution of (2.19) with condition (2.20).

Proof:

Suppose that $X_1(t)$ and $X_2(t)$ are any two solutions belonging to $\mu^2_\omega[0, T]$ of eqs.(2.19) and (2.20), hence:

$$X_1(t) = X_0 + \int_0^t A(X_1(s), s)\, ds + \int_0^t B(X_1(s), s)\, dW(s) \quad \text{a.s.}$$

$$X_2(t) = X_0 + \int_0^t A(X_2(s), s)\, ds + \int_0^t B(X_2(s), s)\, dW(s) \quad \text{a.s.}$$

Therefore:

$$X_1(t) - X_2(t) = \int_0^t [A(X_1(s), s) - A(X_2(s), s)]\, ds + \int_0^t [B(X_1(s), s) - B(X_2(s), s)]\, dW(s) \quad \text{a.s.}$$

and hence

$$|X_1(t) - X_2(t)|^2 = \left|\int_0^t [A(X_1(s),s) - A(X_2(s),s)]\, ds\right|^2 + \left|\int_0^t [B(X_1(s),s) - B(X_2(s), s)]\, dW(s)\right|^2$$

Taking the expectation and using (2.22), we get:

$$E|X_1(t) - X_2(t)|^2 \le E|\int_0^t [A(X_1(s), s) - A(X_2(s), s)]\, ds|^2 +$$

$$E|\int_0^t [B(X_1(s),s) - B(X_2(s), s)]\, dW(s)|^2$$

Hence:

$$E|X_1(t) - X_2(t)|^2 \le 2E|\int_0^t [A(X_1(s), s) - A(X_2(s), s)]\, ds|^2 +$$

$$2E|\int_0^t [B(X_1(s),s) - B(X_2(s), s)]\, dW(s)|^2$$

$$\le 2K_*^2 t \int_0^t E\left|X_1(s) - X_2(s)\right|^2 ds +$$

$$2K_*^2 \int_0^t E\left|X_1(s) - X_2(s)\right|^2 ds$$

By using Gronwall inequality, thus the function

$$\phi(t) = E|X_1(t) - X_2(t)|^2$$

Satisfies:

$$\phi(t) \le (2K_*^2 t + 2K_*^2) \int_0^t \phi(s)\, ds$$

i.e.,

$$\phi(t) \le 0 \times \int_0^t \phi(s)\, ds$$

which implies $\phi(0) \le 0$, i.e., $\phi(0) = 0$

Therefore, $\phi(t) = 0$, $\forall\, t \in [0, T]$

Hence $E|X_1(t) - X_2(t)|^2 = 0$, which means that $X_1(t) - X_2(t) = 0$

This means that $X_1(t) = X_2(t)$, $\forall\ t \in [0, T]$. ■

Remark (2.4), [Friedman, 1975]:

The assertion of the uniqueness theorem means that if $X_1(t)$ and $X_2(t)$ are two solutions of (2.19) with condition (2.20) and if they belong to $\mu_{\omega}^{2}[0, T]$, then:

$$P\{X_1(t) = X_2(t) \text{ for all } 0 \leq t \leq T\} = 1$$

Chapter Three

Stochastic Differential Equations

Chapter Three
Stochastic Differential Equations

Introduction

As it is defined previously, stochastic differential equations SDE's are differential equations in which one or more of its terms are stochastic processes, and therefore will give solutions which are itself stochastic process, and because of the importance of its solution, we discuss in this chapter some of its analytical methods of solution and its numerical solution using Euler's method. Then an improvement of these methods have been made to solve stochastic delay differential equations, which have many applications in different branches of applied mathematics. Therefore, this chapter consists of four sections. Ins section 3.1, the Itô formula have been discussed which has its importance in defining stochastic integrals and SDE's. In section 3.2, we discuss the analytical method for solving SDE's, namely the reduction method and total differentiation method, and explained with some illustrative examples. Section 3.3 presents the numerical solution of SDE's using explicit and implicit Euler's method, while section 3.4 presents an improvement to the SDE's to stochastic delay differential equations, as well as, the methods of solution with some illustrative examples.

3.1 Itô's Formula, [Han, 2005], [Stirzaker, 2005]

Itô formula is the analog of integration by parts in the stochastic calculus. In stochastic calculus this is not possible, the useful range of techniques is practically restricted to those that deal with integral equations. Of these by far the most important is that known as Itô's formula, where may be seen as a stochastic chain rule. Let us recall some elementary non-random chain rule; as usual primes may denote differentiation.

1. One-variable chain rule: If y(t) = f(g(t)), then:

$$y'(t) = \frac{dy}{dt} = f'(g(t))g'(t)$$

Assuming that the derivatives f' and g' exists. We may express this in differential notion as:

$$dy = f'(g)g'(t)\, dt = f'(g)\, dg$$

2. Two-variables chain rule: If:

$$y(t) = f(x(t), w(t))$$

Then:

$$\frac{dy}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial w}\frac{dw}{dt}$$

where differentiation may be denoted by suffices in an obvious way. In particular, if x = t, we obtain, for y = f(t, w(t))

$$dy = f_t\, dt + f_w\, dw$$

Itô formula are extremely useful in many topics, particularly in evaluating stochastic integrals.

Theorem (3.1) (Itô Formula), [Evans, 2006]:

Suppose that x(.) has a stochastic differential equation:

$$dx(t) = A(t, x(t))\, dt + B(t, x(t))\, dW(t) \qquad \ldots(3.1)$$

for $A \in L^1(0, T)$, $B \in L^2(0, T)$. Assume $u : R\times[0, T] \longrightarrow ¡$ is continuous and that $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$ exist and are continuous. Set:

$$y(t) = u(x(t), t) \qquad \ldots(3.2)$$

Then Y has the stochastic differential:

$$dy = \frac{\partial u}{\partial t}\, dt + \frac{\partial u}{\partial x}\, dx + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} B^2 dt$$

$$= \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} A + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} B^2\right) dt + \frac{\partial u}{\partial x} B\, dW \qquad \ldots(3.3)$$

is called the Itô's formula or Itô chain rule.

Remark (3.1), [Evans, 2006]:

i. In view of our definitions, the expression (3.3) means that for all $0 \le s \le r \le T$

$$y(r) - y(s) = u(x(r), r) - u(x(s), r)$$

$$= \int_s^r \left(\frac{\partial u}{\partial t}(x,t) + \frac{\partial u}{\partial x}(x,t)A + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x,t)B^2\right) dt +$$

$$\int_s^r \left(\frac{\partial u}{\partial x}(x,t)B\right) dW \quad \text{a.s.} \qquad \ldots(3.4)$$

ii. Since:

$$x(t) = x(0) + \int_0^t A(s, x(s))\, ds + \int_0^t B(s, x(s))\, dW(s)$$

where x(.) has continuous sample paths almost surely. Thus for almost every W, the functions $t \mapsto \frac{\partial u}{\partial t}(x(t),t), \frac{\partial u}{\partial x}(x(t),t), \frac{\partial^2 u}{\partial x^2}(x(t),t)$ are continuous and so the integrals in (3.4) are defined.

__Theorem (3.2) (The General Itô Formula), [Øksendal, 2003]:__

Consider the SDE:

$$dx(t) = A(t, x(t))\, dt + B(t, x(t))\, dW(t)$$

let:

$$g(t, x) = (g_1(t, x), g_2(t, x), \ldots, g_p(t, x))$$

be a C^2 map from $[0, \infty)\times \mathbb{R}^n$. Then the process:

$$y(t) = g(t, x(t))$$

is again a process, whose component number k, y_k, is given by:

$$dy_k = \frac{\partial g_k}{\partial t}(t, x)\, dt + \sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i}(t, x)\, dx_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t, x)\, dx_i dx_j$$

where $dW_i dW_j = \sigma_{ij}\, dt$, $dW_i dt = dt dW_i = 0$.

__Example (3.1), [Øksendal, 2000]:__

Using the Itô formula to solve the SDE:

$$\left.\begin{array}{l} dx(t) = x(t)\,dW(t),\ t \in [0,1] \\ x(0) = 1 \end{array}\right\} \qquad \ldots(3.5)$$

Hence from the stochastic differential equation, we have:

$$\frac{dx(t)}{x(t)} = dW(t)$$

and therefore:

$$\int_0^t \frac{dx(s)}{x(s)} = \int_0^t dW(s)$$

i.e.,

$$\int_0^t \frac{dx(s)}{x(s)} = W(t) \qquad \ldots(3.6)$$

Using the Itô formula for the function:

$$g(t, x) = \ln x,\ x > 0$$

and obtain that from eq.(3.5)

$$d(\ln x(t)) = \frac{1}{x(t)} dx(t) + \frac{1}{2}\left(-\frac{1}{x^2(t)}\right)(dx(t))^2$$

$$= \frac{dx(t)}{x(t)} + \frac{1}{2}\left(-\frac{1}{x^2(t)}\right)(x(t)\,dW(t))^2$$

and since $dW(t) \simeq (dt)^{1/2}$. Hence:

$$d(\ln x(t)) = \frac{dx(t)}{x(t)} + \frac{1}{2}\left(-\frac{1}{x^2(t)}\right)x^2(t)\,dt$$

$$= \frac{dx(t)}{x(t)} - \frac{1}{2}\,dt$$

or equivalently:

$$\frac{dx(t)}{x(t)} = d(\ln x(t)) + \frac{1}{2}\,dt$$

So, from eq.(3.6), one can conclude that:

$$\int_0^t \frac{dx(s)}{x(s)} = \int_0^t d(\ln x(s))\,ds + \int_0^t \frac{1}{2}\,ds$$

$$W(t) = \ln\frac{x(t)}{x(0)} + \frac{1}{2}t$$

Therefore, the solution is given by:

$$x(t) = \exp(W(t) - \frac{1}{2}t)$$

Example (3.2), [Evans, 2006]:

Consider the linear stochastic differential equation:

$$dx(t) = \eta x(t)\,dt + \sigma x(t)\,dW(t),\ t \in [0, 1] \qquad \ldots(3.7)$$

where η and σ are constants.

To solve this stochastic differential equation, divide eq.(3.7) on x(t) yields:

$$\frac{dx(t)}{x(t)} = \eta\,dt + \sigma\,dW(t)$$

and hence:

$$\int_0^t \frac{dx(s)}{x(s)} = \int_0^t \eta\,ds + \int_0^t \sigma\,dW(s)$$

$$\int_0^t \frac{dx(s)}{x(s)} = \eta t + \sigma W(t) \qquad \ldots(3.8)$$

to evaluate the integral on the left hand side use the Itô formula (3.1) for the function:

$$g(x, t) = \ln x; \; x > 0$$

and obtain:

$$d(\ln x(t)) = \frac{1}{x(t)} dx(t) + \frac{1}{2}\left(-\frac{1}{x^2(t)}\right)(dx(t))^2$$

$$= \frac{1}{x(t)}(\eta x(t)\, dt + \sigma x(t)\, dW(t)) + \frac{1}{2}\left(-\frac{1}{x^2(t)}\right)(\eta x(t)\, dt + \sigma\, dW(t))^2$$

$$= \frac{dx(t)}{x(t)} - \frac{1}{2x^2(t)}[\eta^2 x^2(t)\, d^2t + 2\sigma\eta x^2(t)\, dt dW(t) + \sigma^2 x^2(t) d^2W(t)]$$

and since $dW(t) \approx (dt)^{1/2}$. Hence:

$$d(\ln x(t)) = \frac{dx(t)}{x(t)} - \frac{1}{2x^2(t)}\sigma^2 x^2(t)\, dt$$

Hence:

$$d(\ln x(t)) = \frac{dx(t)}{x(t)} - \frac{1}{2}\sigma^2\, dt$$

and therefore:

$$\frac{dx(t)}{x(t)} = d(\ln x(t)) + \frac{1}{2}\sigma^2\, dt$$

so from (3.8), we get:

$$\int_0^t \frac{dx(s)}{x(s)} = \int_0^t d(\ln x(s))\, ds + \int_0^t \frac{1}{2}\sigma^2\, ds$$

i.e.,

$$\eta t + \sigma W(t) = \ln\frac{x(s)}{x(0)} + \frac{1}{2}\sigma^2 t$$

$$x(t) = \exp[(\eta - \frac{1}{2}\sigma^2)t + \sigma W(t)]$$

3.2 Analytical Methods for Solving Stochastic Differential Equations, [Goldys, 2008]

After the publication in 1973 of the groundbreaking paper of Blackan schools on the arbitrary pricing of European call options, it became clear that stochastic analysis is an indispensable tool for the theory of financial markets, derivation of prices of standard and exotic options and other derivative securities, hedging related to financial risk, as well as managing the interestrate risk. Because of the difficulties encountered in solving SDE's, several approaches for solving such type of equations are proposed by several authors, but still with so many difficulties. Therefore, in this section, these methods of solution are discussed in details and more calculations, and give the proposed method which is termed as (in this work) as total differences method for solving SDE's, namely:

1. Reduction method.
2. Total differentiation method.

Each of such methods will be discussed in details with some illustrative examples.

3.2.1 The Reduction Method, [Smith, 1999]:

This method was proposed by Smith in 1999 as a method for evaluating the analytic solution of certain SDE's which is based on transformation method of the solution and then evaluating the inverse transformation after the reduction of the solution. First, start with the following fundamental example in this method which is given without details in literature and we give the details of the solution.

Example (3.3):

Consider the SDE:

$$dx(t) = \left(\sqrt{1+x^2} + \frac{x}{2}\right)dt + \sqrt{1+x^2}\, dW(t),\ t \in [0, 1]$$

subject to the initial condition $x_0 = 0$.

In order to solve this SDE, consider the transformation y = g(x) with a monotonic function g to set by using the Itô formula:

$$dy \; ; \; g_x dx + \frac{1}{2} g_{xx}(dx)^2 + \ldots$$

and since $dW(t) \; ; \; (dt)^{1/2}$, then consequently if we compute dy and keep all terms of order dt or $(dt)^{1/2}$ we obtain that:

$$dy \approx g_x\left[\left(\sqrt{1+x^2}+\frac{x}{2}\right)dt+\sqrt{1+x^2}\,dW\right]+$$

$$\frac{1}{2}g_{xx}\left[\left(\sqrt{1+x^2}+\frac{x}{2}\right)dt+\sqrt{1+x^2}\,dW\right]^2$$

$$= g_x\left(\sqrt{1+x^2}+\frac{x}{2}\right)dt + g_x\sqrt{1+x^2}\,dW +$$

$$\frac{1}{2}g_{xx}\left[\left(\sqrt{1+x^2}+\frac{x}{2}\right)^2 dt^2 + 2\left(\sqrt{1+x^2}+\frac{x}{2}\right)\sqrt{1+x^2}\,dtdW +\right.$$

$$\left.(1+x^2)\,dW^2\right]$$

$$= g_x\left(\sqrt{1+x^2}+\frac{x}{2}\right)dt + g_x\sqrt{1+x^2}\,dW +$$

$$\frac{1}{2}g_{xx}\left(\sqrt{1+x^2}+\frac{x}{2}\right)^2 dt^2 + \frac{1}{2}g_{xx}2\left(\sqrt{1+x^2}+\frac{x}{2}\right)\sqrt{1+x^2}$$

$$dtdW + \frac{1}{2}g_{xx}(1+x^2)\,dW^2$$

and since $dW(t) \approx (dt)^{1/2}$, then:

$$dy \approx \left[\ g_x\left(\sqrt{1+x^2}+\frac{x}{2}\right) + \frac{1}{2}g_{xx}(1+x^2)\ \right]dt + g_x\sqrt{1+x^2}\ dW$$

so look for a function g with constant factor of the form $g_x\sqrt{1+x^2} = 1$, which mean's that this process occurs with probability 1, (for more details see [Arnold, 1974]).

Now, $g(x_0) = g(0) = 0$, and $g_x = \dfrac{1}{\sqrt{1+x^2}}$ and integrate both sides,

yields:

$$g(x) = \int_0^x \frac{1}{\sqrt{1+s^2}} \frac{s+\sqrt{1+s^2}}{s+\sqrt{1+s^2}}\, ds$$

$$= \int_0^x \frac{\dfrac{s}{\sqrt{1+s^2}} + \dfrac{\sqrt{1+s^2}}{\sqrt{1+s^2}}}{s+\sqrt{1+s^2}}\, ds$$

$$= \int_0^x \frac{1}{s+\sqrt{1+s^2}} \left[1 + \frac{2s}{2\sqrt{1+s^2}}\right] ds$$

$$= \ln\left[s+\sqrt{1+s^2}\right]\Big|_0^x = \ln\left[x+\sqrt{1+x^2}\right]$$

Hence $g(x) = \ln\left[x+\sqrt{1+x^2}\right]$.

But, we may find that the factor of dt is also:

$$g_x\left(\sqrt{1+x^2} + \frac{x}{2}\right) + \frac{1}{2} g_{xx}(1 + x^2) = 1$$

then:

$$dy(t) = dt + dW(t)$$

and since $y(0) = g(x_0) = 0$ and $y = g(x)$, hence $y = \ln\left[x+\sqrt{1+x^2}\right]$, i.e.,

$$e^y = x + \sqrt{1+x^2}$$

and squaring both sides, yields to:

$$e^{2y} - 1 = 2xe^{y}$$

Hence:

$$x(t) = \frac{1}{2}(e^{y} - e^{-y})$$

$$= \sinh(y(t))$$

3.2.2 The Total Differentiation Method,[Rainville,1989],[Muszta,2005]:

Suppose that a function x(t) can be found such that has for its total differential given by the form:

$$dx(t) = a(t, x(t))\, dt + b(t, x(t))\, dW(t) \qquad \ldots(3.9)$$

where a(t, x(t)) and b(t, x(t)) are constant functions. Then certainly:

$$x(t) = f(t, W(t)) \qquad \ldots(3.10)$$

and define implicitly a set of solutions of (3.9). For this, from eq.(3.10) it follows that:

$$dx(t) = 0$$

or, in view of eq.(3.9)

$$a(t, x(t))\, dt + b(t, x(t))\, dW(t) = 0$$

as desired. Two things, then, are needed; the first one is to find out under what conditions on a(t, x(t)) and b(t, x(t)) a function x(t) exists such that its total differential is exactly the same of a(t, x(t))dt + b(t, x(t))dW(t); and second, if those conditions are satisfied actually to determine the function x(t). If a function x(t) exists, such that:

$$a(t, x(t))\, dt + b(t, x(t))\, dW(t)$$

is exactly the total differential of x(t). Equation (3.9) is called an exact equation. If the equation

$$a(t, x(t))\,dt + b(t, x(t))\,dW(t) \qquad \ldots(3.11)$$

is exact, then by definition x(t) exists, such that:

$$dx(t) = a(t, x(t))\,dt + b(t, x(t))\,dW(t)$$

is obtained by applying the Itô formula to f(t, W(t)). This gives:

$$f(t, W(t)) = f(0, 0) + \int_0^t \left[\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, W(s)) + \frac{\partial f}{\partial s}(s, W(s))\right] ds \; +$$

$$\int_0^t \frac{\partial f}{\partial x}(s, W(s))\,dW(s) \qquad \ldots(3.12)$$

compare eq.(3.12) with x(t), gives:

$$x(t) = x_0 + \int_0^t a(s, x(s))\,ds + \int_0^t b(s, x(s))\,dW(s) \qquad (3.13)$$

choosing a function f so that it satisfies the following system of partial differential equations, then will have a candidate for the solution of the SDE (3.9):

$$\left.\begin{aligned} &\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, X(s)) + \frac{\partial f}{\partial s}(s, X(s)) = a(s, f(s, x)); \\ &\frac{\partial f}{\partial x}(s, X(s)) = b(s, f(s, x)); \\ &f(0,0) = x_0 \end{aligned}\right\} \qquad \ldots(3.14)$$

This technique is useful mostly for SDE's with linear coefficients a(s, x) and b(s, x).

The next examples illustrate the above method of solution:

Example (3.4), [Muszta, 2005]:

Consider the SDE:

$dx(t) = dt + dW(t),\ t \in [0, 1]$

$x(t_0) = x_0$

It is known that the solution to this equation is:

$x(t) = x_0 + t + W(t)$

Let us see what the coefficient matching technique gives. The system need to solve in this case is by using the Itô formula to f(t, W(t))

$$f(t, W(t)) = f(0, 0) + \int_0^t \left[\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, W(s)) + \frac{\partial f}{\partial s}(s, W(s))\right] ds + \int_0^t \frac{\partial f}{\partial x}(s, W(s))\ dW(s)$$

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, x) + \frac{\partial f}{\partial s}(s, x) = a(s, f(s, x))$$

$$\frac{\partial f}{\partial x}(s, x) = b(s, f(s, x))$$

$$f(0, 0) = x_0$$

such that:

$$\left.\begin{array}{l} \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, x) + \frac{\partial f}{\partial s}(s, x) = 1 \\ \frac{\partial f}{\partial x}(s, x) = 1 \\ f(0,0) = x_0 \end{array}\right\} \qquad \ldots(3.15)$$

The solution is computed as follows:

$$\frac{\partial f}{\partial x}(s,x) = 1$$

Hence integrating with respect to x, yields:

$$f(s, x) = x + g(s)$$

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,x) + \frac{\partial f}{\partial s}(s,x) = 1$$

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}(x+g(s)) + \frac{\partial}{\partial s}(x+g(s)) = 1$$

which implies $g'(s) = 1$, and hence $g(s) = s + c$; and from the initial condition $f(0, 0) = x_0$, implies that $c = x_0$

Thus, we have obtained $f(s, x) = x_0 + s + x$ and the candidate solution to the SDE is $f(t, W(t)) = x_0 + t + W(t)$, which in this case actually is the solution to eq.(3.15).

Example (3.5), [Muszta, 2005]:

Consider the SDE:

$$dx(t) = -rx(t)\,dt + \sigma\,dW(t);$$

where $r, \sigma \in \mathbb{R}$. If we apply the coefficient, we get the system:

$$\left.\begin{aligned} &\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,x) + \frac{\partial f}{\partial s}(s,x) = -rx(s) \\ &\frac{\partial f}{\partial x}(s,x) = \sigma \\ &f(0,0) = x_0 \end{aligned}\right\} \qquad \ldots(3.16)$$

Since $\frac{\partial f}{\partial x}(s,x) = \sigma$, then $f(s, x) = \sigma x + g(s)$

and $\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,x) + \frac{\partial f}{\partial s}(s,x) = -rx(s)$, then:

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma x + g(s)) + \frac{\partial}{\partial s}(\sigma x + g(s)) = -rf(s,x)$$

Therefore:

$$g'(s) = -rf(s, x)$$

$$= -r\sigma x - rg(s)$$

$$\frac{dg}{ds} + rg = -r\sigma x$$

Solving this linear equation, yields:

$$g(s) = -\sigma x + ce^{-rs}$$

and since $f(s, x) = \sigma x - \sigma x + ce^{-rs}$, then

$$f(s, x) = x_0 e^{-rs}$$

The candidate solution to the SDE is $f(t, W(t)) = x_0 e^{-rt}$, which is in this case the solution of eq.(3.16).

3.3 Numerical Methods for Solving Stochastic Differential Equations

Unfortunately explicitly solvable SDE's are rare in practical applications. However, there are an increasing number of numerical methods for the solution of SDE's. In SDE's, Euler's method is one of the simplest time discrete approximation of SDE's, [Han, 2005].

In the next theorem, we will derive the general for of Euler's method for solving numerically SDE's which is given in [Han, 2005] without derivation, we give the proof for completeness.

Theorem (3.3) (Explicit Euler's Method):

Consider the stochastic differential equation:

$$dx(t) = A(t, x(t))\, dt + B(t, x(t))\, dW(t) \qquad \ldots(3.17)$$

where $t \in [a, b]$ with initial condition $x(t_0) = x_0$. Descritize the interval $[a, b]$ as, $a = t_0 < t_1 < \ldots < t_N = b$, or $t_n = a + nh$, for fixed $N \in$ ¥ and $n \in$ ¥, $h = \frac{b-a}{N}$, $\Delta W_n = W(t_{n+1}) - W(t_n)$ is a random increment, $n = 0, 1, \ldots, N - 1$; then the sequence of numerical solutions are given by:

$$x_{n+1} = x_n + A(t_n, x_n)h + B(t_n, x_n)\Delta W_n,\ n = 0, 1, \ldots, N - 1$$

which is called the explicit Euler's method.

Proof:

Consider the stochastic differential equation given by eq.(3.17), then the method is based on the formal integration of eq.(3.17) over a time step $[t_n, t_{n+1}]$, which gives:

$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} A(s, x(s))\, ds + \int_{t_n}^{t_{n+1}} B(s, x(s))\, dW(s) \quad \ldots(3.18)$$

Letting $h = t_{n+1} - t_n$ and replacing the integration time s in the functions $A(s, x(s))$ and $B(s, x(s))$ by the lower limit of integration, t_n, give:

$$x(t_{n+1}) \ ; \ x(t_n) + A(t_n, x(t_n))h + B(t_n, x(t_n))\Delta W_n \qquad \ldots(3.19)$$

where the random increment ΔW_n is defined as:

$$\Delta W_n = \int_{t_n}^{t_{n+1}} W(s)\,dW(s) = W(t_{n+1}) - W(t_n) \qquad \ldots(3.20)$$

Hence we obtain the following time discrete approximation:

$$x_{n+1} = x_n + A(t_n, x_n)h + B(t_n, x_n)\Delta W_n \qquad \ldots(3.21)$$

where $x_0 = x(t_0)$.

From the definition of a Brownian motion, it follows that these increments are independent Gaussain random variables with mean 0 and Variance h, if $0 < t_n < t_{n+1} < T$, then the random increment t, $\Delta W_n = W(t_{n+1}) - W(t_n)$ and $h = t_{n+1} - t_n$. Hence:

$$\Delta W_n = W_{n+1} - W_n \sim N(0, t_{n+1} - t_n)$$

or, $\dfrac{\Delta W_n}{\sqrt{t_{n+1} - t_n}} \sim N(0, 1)$

and hence $\Delta W_n \sim \sqrt{t_{n+1} - t_n}\, N(0, 1)$. **n**

Remark (3.2):

In applications, the Euler's method (3.21) requires a generation of independent Gaussian random variables with mean 0 and variance $h = t_{n+1} - t_n$, which are generated using the following algorithm:

Algorithm (3.1):

1. Input: n, a, b.
2. h = (b – a)/n.
3. For i = 1 to n/2.

4. Generate U_1 and U_2 from U(0, 1) (uniform distribution over [0, 1]).
5. Set $x_1 = (-\ln U_1)^{1/2}\cos(2\pi U_2)$ and $x_2 = (-\ln U_1)^{1/2}\sin(2\pi U_2)$.
6. $W_1 = (h)^{1/2}x_1$ and $W_2 = (h)^{1/2}x_2$.
7. $z_i = W_1$ and $z_{i+1} = W_2$.
8. Output: deliver z as a vector of independent r.v's. from N(0, 1).

Proposition (3.1) (Implicit Euler's Method):

Consider the stochastic differential equation:

$$dx(t) = A(t, x(t))\, dt + B(t, x(t))\, dW(t) \qquad \ldots(3.22)$$

where $t \in [a, b]$ with initial condition $x(t_0) = x_0$. Descritize the interval $[a, b]$ as, $a = t_0 < t_1 < \ldots < t_N = b$, or $t_n = a + nh$, for fixed $N \in$ ¥ and $n \in$ ¥, $h = \frac{b-a}{N}$, $\Delta W_n = W(t_{n+1}) - W(t_n)$ is a random increment, n = 0, 1, …, N – 1; then the sequence of numerical solutions are given by:

$$x_{n+1} = x_n + A(t_{n+1}, x_{n+1})h + B(t_{n+1}, x_{n+1})\Delta W_n, \quad n = 0, 1, \ldots, N-1$$

which is called the implicit Euler's method.

Proof:

Consider the stochastic differential equation given by eq.(3.22), then the formal integration of eq.(3.22) over a time step $[t_n, t_{n+1}]$, which gives:

$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} A(s, x(s))\, ds + \int_{t_n}^{t_{n+1}} B(s, x(s))\, dW(s) \qquad \ldots(3.23)$$

Letting $h = t_{n+1} - t_n$ and replacing the integration time s in the functions A(s, x(s)) and B(s, x(s)) by the upper limit of integration, t_{n+1}, give:

$$x(t_{n+1}) \approx x(t_n) + A(t_{n+1}, x(t_{n+1}))h + B(t_{n+1}, x(t_{n+1}))\Delta W_n \quad \ldots(3.24)$$

where the random increment ΔW_n is defined as:

$$\Delta W_n = \int_{t_n}^{t_{n+1}} W(s)\, dW(s) = W(t_{n+1}) - W(t_n) \quad \ldots(3.25)$$

Thus, the implicit Euler's method is given by:

$$x_{n+1} = x_n + A(t_{n+1}, x_{n+1})h + B(t_{n+1}, x_{n+1})\Delta W_n \quad \ldots(3.26)$$

where $x_0 = x(t_0)$. n

Now, consider some illustrative examples for solving SDE's numerically using the explicit method:

Example (3.6):

Consider the stochastic differential equation:

$$dx(t) = \left(\sqrt{1+x^2} + \frac{x}{2}\right)dt + \sqrt{1+x^2}\, dW(t),\ t \in [0, 1]$$

with initial condition x(0) = 0 and suppose that we want to find the numerical solution by using Euler method (3.21) with step lengths h = 0.1 and h = 0.01. The results obtained upon using the explicit Euler's method and its comparison with the exact solution given in example (3.3) are presented in table (3.1) in which the results are obtained by using the computer programs written in Mathcad 2001i. In addition the Gaussian random numbers with mean 0 and variances 0.1 and 0.01, respectively, are presented in this table for completeness (which will be used in evaluating the exact solution also).

Table (3.1)

Numerical and exact results of example (3.6).

t	h = 0.1			h = 0.01		
	DW_n	Numerical solution	Absolute error	DW_n	Numerical solution	Absolute error
0.1	−0.068	0.076	0.12	9.851×10^{-3}	0.117	0.016
0.2	−0.047	0.201	0.162	-7.231×10^{-3}	0.209	0.029
0.3	−0.095	0.264	0.111	−0.029	0.302	0.051
0.4	−0.169	0.305	0.1	-4.091×10^{-3}	0.455	0.069
0.5	4.353×10^{-3}	0.605	0.374	4.285×10^{-3}	0.603	0.103
0.6	−0.012	0.733	0.229	-1.932×10^{-4}	0.75	0.167
0.7	0.056	0.978	0.39	1.437×10^{-3}	0.922	0.26
0.8	0.219	1.395	0.64	5.391×10^{-3}	1.12	0.33
0.9	0.081	1.399	0.38	7.911×10^{-3}	1.34	0.445
1.0	0.099	1.672	0.691	4.377×10^{-3}	1.581	0.587

Example (3.7):

Consider the SDE:

$dx(t) = dt + dW(t),\ t \in [0, 1]$

with initial condition x(0) = 0 and suppose that we want to find the numerical solution by using Euler's method (3.21) with step lengths h = 0.1 and h = 0.01. The results obtained upon using the explicit Euler's method and its comparison with the exact solution given in example (3.4) are presented in table (3.2) in which the results are obtained by using computer programs written in Mathcad 2001i. In addition the Gaussian random numbers with mean 0 and variances 0.1 and 0.01, respectively, are presented in this table for completeness (which will be used in evaluating the exact solution also).

Table (3.2)

Numerical and exact results of example (3.7).

t	*h = 0.1*			*h = 0.01*		
	DW_n	**Numerical solution**	**Absolute error**	DW_n	**Numerical solution**	**Absolute error**
0.1	–0.068	0.076	0.12	9.851×10^{-3}	0.114	0.013
0.2	–0.047	0.196	0.165	-7.231×10^{-3}	0.197	0.016
0.3	–0.095	0.249	0.096	–0.029	0.275	0.022
0.4	–0.169	0.275	0.07	-4.091×10^{-3}	0.4	0.012
0.5	4.353×10^{-3}	0.548	0.317	4.285×10^{-3}	0.509	5.353×10^{-3}
0.6	–0.012	0.632	0.127	-1.932×10^{-4}	0.604	0.015
0.7	0.056	0.8	0.212	1.437×10^{-3}	0.706	0.03
0.8	0.219	1.063	0.307	5.391×10^{-3}	0.81	0.011
0.9	0.081	1.025	5.52×10^{-3}	7.911×10^{-3}	0.912	7.908×10^{-3}
1.0	0.099	1.142	0.162	4.377×10^{-3}	1.009	0.015

Example (3.8):

Consider the SDE:

$$dx(t) = \psi x(t)\, dt + \sigma x(t)\, dW(t)$$

with initial condition x(0) = 0.1 and suppose that we want to find the numerical solution by using Euler's method (3.21) with step lengths h = 0.1 and h = 0.01. The obtained results upon using the explicit Euler method and its comparison with the exact solution given in example (3.5) are presented in table (3.3) in which the results are obtained by using computer programs written in Mathcad 2001i. In addition the Gaussian random numbers with mean 0 and variances 0.1 and 0.01, respectively, are presented in this table for completeness (which will be used in evaluating the exact solution also).

Table (3.3)

Numerical and exact results of example (3.8).

t	*h* = 0.1			*h* = 0.01		
	DW_n	**Numerical solution**	**Absolute error**	DW_n	**Numerical solution**	**Absolute error**
0.1	–0.068	0.115	0.015	9.851×10^{-3}	0.116	1.616×10^{-3}
0.2	–0.047	0.132	0.016	-7.231×10^{-3}	0.135	1.73×10^{-3}
0.3	–0.095	0.152	0.017	–0.029	0.156	1.842×10^{-3}
0.4	–0.169	0.175	0.018	-4.091×10^{-3}	0.181	1.918×10^{-3}
0.5	4.353×10^{-3}	0.201	0.019	4.285×10^{-3}	0.211	1.978×10^{-3}
0.6	–0.012	0.231	0.02	-1.932×10^{-4}	0.244	2.046×10^{-3}
0.7	0.056	0.266	0.02	1.437×10^{-3}	0.284	2.1×10^{-3}
0.8	0.219	0.307	0.021	5.391×10^{-3}	0.329	2.007×10^{-3}
0.9	0.081	0.352	0.019	7.911×10^{-3}	0.382	1.89×10^{-3}
1.0	0.099	0.405	0.019	4.377×10^{-3}	0.443	1.751×10^{-3}

3.4 Stochastic Delay Differential Equations

Stochastic delay differential equations appears in the description of many random time varying processes in applications. Moreover, for an overview, they extend naturally the classical time series models to continuous time. Among the huge variety of equations affine SDDE's form a fundamental class, [Reiss, 2000]. In the last few decades, statistical inference for SDDE's has been studied from various view points, [Sϕrensen, 2007], when modeling a system which do not noticeably affect their environment, stochastic variables are often used to model the environmental fluctuations, thus leading to a SDDE's, [Guillouzic, 1999], which evolves according to the following SDDE:

$$dx(t) = A(t, x(t), x(t-\tau))\,dt + B(t, x(t), x(t-\tau))\,dW(t) \quad \ldots(3.27)$$

where $A(x_0, x_\tau)$ and $B(x_0, x_\tau)$ are known function, τ is the delay which is considered to be constant, and B is a parameter which scales the noise amplitude. Also, x_0 and x_τ are used as dummy variables, and do not necessarily refer to $x(t)$ and $x(t-\tau)$, not to the initial conditions.

The quantity W(t) in eq.(3.27) is a Brownian motion whose initial condition is 0 at time t = 0, [Guillouzic, 1999]. The related integral equation of stochastic delay is:

$$x(t) = x_0 + \int_0^t A(s, x(s), x(s-\tau))\,ds + \int_0^t B(s, x(s), x(s-\tau))\,dW(s) \quad \ldots(3.28)$$

All those generalizations and modifications have certain common features, but need to be scrutinized in details in order to build an analogous theory as we do for SDDE's in the sequel, [Reib, 2003].

3.4.1 Analytical Method for Solving Stochastic Delay Differential Equations:

In this section, the solution of SDDE's is studied using the total differential method that is discussed previously in solving SDE's with the corporation of the method of successive integrations for solving ordinary differential equations. The method of solution will be explained more accurately in the next examples which are solved with details.

Example (3.9):

Consider the following SDDE:

$$dx(t) = x(t-1)\,dt + dW(t),\ t \in [0, 1]$$

with initial condition:

$$x(t) = \phi_0(t) = t, \text{ for } -1 \le t < 0$$

Therefore, in order to find the solution, we consider the first time step interval [0, 1], i.e., consider:

$$dx(t) = \phi_0(t-1)\,dt + dW(t)$$

$$= (t-1)\,dt + dW(t)$$

which is a SDE with initial condition $x(0) = x_0 = 0$.

In order to solve this SDE by using the total differential method, the solution to this equation is:

$$x(t) = x_0 + \frac{t^2}{2} - t + W(t)$$

we let $x(t) = f(t, W(t))$, and by using the Itô formula to $f(t, W(t))$, we get:

$$f(t, W(t)) = f(0, 0) + \int_0^t \left[\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, W(s)) + \frac{\partial f}{\partial s}(s, W(s))\right] ds \ +$$

$$\int_0^t \frac{\partial f}{\partial x}(s, W(s))\ dW(s)$$

such that:

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, x) + \frac{\partial f}{\partial s}(s, x) = s - 1$$

$$\frac{\partial f}{\partial x}(s, x) = 1$$

$$f(0, 0) = x_0 = c$$

Therefore, $\frac{\partial f}{\partial x}(s, x) = 1$, will implies that:

$$f(s, x) = x + g(s)$$

Hence, from:

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, x) + \frac{\partial f}{\partial s}(s, x) = s - 1$$

we get $g'(s) = s - 1$, and therefore $g(s) = \frac{s^2}{2} - s + c$.

Now, since $f(0, 0) = x_0 = c$, therefore we obtain that $f(s, x) = x + \frac{s^2}{2} - s + x_0$, and the candidate solution to the SDDE is:

$$x(t) = f(t, W(t)) = x_0 + \frac{t^2}{2} - t + W(t)$$

$x(t)$ is a solution of all $t \geq t_0$.

Example (3.10):

Consider the following SDDE:

$$dx(t) = t\, dt + x(t - 1)\, dW(t),\ t \in [0, 1]$$

with initial condition:

$$x(t) = \phi_0(t) = t + 1, \text{ for } -1 \leq t < 0$$

Hence to find the solution, consider the first time step interval [0, 1], i.e., consider:

$$dx(t) = t\,dt + \phi_0(t-1)\,dW(t)$$

which is reduced to:

$$dx(t) = t\,dt + t\,dW(t)$$

which is a SDE with initial condition $x(0) = x_0 = 0$.

and to solve this SDE by using the total differential method, let $x(t) = f(t, W(t))$, then by using the Itô formula to f(t, W(t)), we get:

$$f(t, W(t)) = f(0, 0) + \int_0^t \left[\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, W(s)) + \frac{\partial f}{\partial s}(s, W(s))\right] ds + \int_0^t \frac{\partial f}{\partial x}(s, W(s))\, dW(s)$$

such that:

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, x) + \frac{\partial f}{\partial s}(s, x) = s$$

$$\frac{\partial f}{\partial x}(s, x) = s$$

$$f(0, 0) = x_0$$

Hence $\frac{\partial f}{\partial x}(s, x) = s$, will implies that:

$$f(s, x) = sx + g(s) \qquad \ldots(3.29)$$

Therefore:

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, x) + \frac{\partial f}{\partial s}(s, x) = s$$

Which implies $g'(s) = s - x$, and therefore $g(s) = \frac{s^2}{2} - sx + c$.

Now, substituting in eq.(3.29), yields to:

$$f(s, x) = \frac{s^2}{2} + c$$

where:

$$f(0, 0) = x_0 = c$$

Hence:

$$f(t, x) = \frac{t^2}{2} + x_0$$

and the candidate solution is:

$$x(t) = \frac{t^2}{2} + x_0$$

Similarly, to find the solution for the second time step interval [1, 2], let:

$$x(t) = \phi_1(t) = \frac{t^2}{2}$$

Therefore:

$$dx(t) = t\,dt + \phi(\frac{t^2}{2} - 1)\,dW(t)$$

$$= t\,dt + \frac{t^2}{2} - 1\,dW(t)$$

and by using the total differential method in SDE's and the Itô formula for f(t, W(t)), yields to:

$$f(t, W(t)) = f(0, 0) + \int_1^t \left[\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, W(s)) + \frac{\partial f}{\partial s}(s, W(s))\right] ds \; +$$

$$\int_1^t \frac{\partial f}{\partial x}(s, W(s)) \; dW(s)$$

such that:

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, x) + \frac{\partial f}{\partial s}(s, x) \; = s$$

$$\frac{\partial f}{\partial x}(s, x) \; = \frac{s^2}{2} - 1$$

Therefore:

$$f(s, x) = \frac{s^2}{2} x - x + g(s) \qquad \ldots(3.30)$$

therefore $g'(s) = s - sx$, and so:

$$g(s) = \frac{s^2}{2} - \frac{x}{2} s^2 + c$$

and substituting g(s) in equation (3.30), yields:

$$f(s, x) = \frac{s^2}{2} - x + c$$

thus, we have obtained

$$f(s, x) = x_0 + \frac{s^2}{2} - x$$

and the candidate solution to the SDE is:

$$f(t, W(t)) = x_0 + \frac{t^2}{2} - W(t)$$

Example (3.11):

Consider the following SDDE:

$$dx(t) = x(t-1)\,dt + x(t-2)\,dW(t),\ t \in [0, 1]$$

with initial condition:

$$x(t) = \phi_0(t) = t, \text{ for } -2 \leq t \leq 0$$

Hence to find the solution, consider the first time step interval [0, 1], i.e., consider:

$$dx(t) = \phi_0(t-1)\,dt + \phi_0(t-2)\,dW(t)$$

and hence:

$$dx(t) = (t-1)\,dt + (t-2)\,dW(t)$$

and to solve this SDE by using the total differential method, let x(t) = f(t, W(t)), then by using the Itô formula to f(t, W(t)), we get:

$$f(t, W(t)) = f(0, 0) + \int_0^t \left[\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, W(s)) + \frac{\partial f}{\partial s}(s, W(s))\right] ds + \int_0^t \frac{\partial f}{\partial x}(s, W(s))\, dW(s)$$

such that:

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, x) + \frac{\partial f}{\partial s}(s, x) = s - 1$$

$$\frac{\partial f}{\partial x}(s, x) = s - 2$$

$$f(0, 0) = x_0$$

Hence $\frac{\partial f}{\partial x}(s,x) = s - 2$, which implies that:

$$f(s, x) = sx - 2x + g(s) \qquad \ldots(3.31)$$

Therefore, substituting in the partial differential equation:

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,x) + \frac{\partial f}{\partial s}(s,x) = s - 1$$

Which will implies to $g'(s) = s - 1 - x$, and therefore $g(s) = \frac{s^2}{2} - s - sx + c$.

Now, substituting g(s) in eq.(3.31), yields:

$$f(s, x) = \frac{s^2}{2} - s - 2x + c$$

and since $f(0, 0) = x_0 = c$, then we have:

$$f(t, x) = x_0 + \frac{t^2}{2} - t - 2x$$

and the candidate solution to the SDE is:

$$x(t) = f(t, W(t)) = x_0 + \frac{t^2}{2} - t - 2W(t)$$

In this example, it is so difficult to find the solution for the next time interval, unless when a new method is proposed or numerical methods are used to solve for further time intervals.

3.4.2 Euler's Method for Solving Stochastic Delay Differential Equations:

The present section consists of using Euler's method for solving SDDE's. For this purpose, we consider, for simplicity and without loose of generality, the first order retarded SDDE's, which has the form as:

$$dx(t) = A(t, x(t), x(t-\tau))\, dt + B(t, x(t)), x(t-\tau))\, dW(t) \quad \ldots(3.32)$$

with initial condition:

$$x(t) = \phi_0(t), \text{ for } t_0 - \tau \leq t \leq t_0$$

where τ is a fixed number. The next theorem introduces the derivation of Euler's method for solving SDDE's.

<u>Theorem (3.4):</u>

Consider the SDDE:

$$dx(t) = A(t, x(t), x(t-\tau))\, dt + B(t, x(t), x(t-\tau))\, dW(t)$$

where $t \in [a, b]$ with initial condition $x(t_0) = \phi_0(t)$, $t_0 - \tau \leq t \leq t_0$. Descritize the interval $[a, b]$ as, $a = t_0 < t_1 < \ldots < t_N = b$, or $t_n = a + nh$, for fixed $N \in$ ¥ and $n \in$ ¥, $h = \dfrac{b-a}{N}$, $\Delta W_n = W(t_{n+1}) - W(t_n)$ is a random increment, $n = 0, 1, \ldots, N-1$; then the sequence of numerical solutions are given by:

$$x_{n+1} = x_n + A(t_n, x_n, \phi_{n-\tau})h + B(t_n, x_n, \phi_{n-\tau})\Delta W_n, \; n = 0, 1, \ldots, N-1$$

Proof:

Consider the SDDE:

$$dx(t) = A(t, x(t), x(t-\tau))\, dt + B(t, x(t), x(t-\tau))\, dW(t) \quad \ldots(3.33)$$

The method is based on the formal integration of eq(3.33) over a time step which gives:

$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} A(s,x(s),x(s-\tau))\,ds + \int_{t_n}^{t_{n+1}} B(s,x(s),x(s-\tau))\,dW(s)$$

…(3.34)

where $h = t_{n+1} - t_n$. Replacing the integration time s in the functions $A(s, x(s), x(s - \tau))$ and $B(s, x(s), x(s - \tau))$ by the lower limit of integration t_n, we have:

$$x(t_{n+1}) \simeq x(t_n) + A(t_n, x(t_n), x(t_n - \tau))h + B(t_n, x(t_n), x(t_n - \tau))\Delta W_n$$

…(3.35)

where the random increment ΔW_n is defined as:

$$\Delta W_n = \int_{t_n}^{t_{n+1}} W(s)\,dW(s) = W(t_{n+1}) - W(t_n)$$

Hence, w e obtain the following time discrete approximation:

$$x_{n+1} = x_n + A(t_n, x_n, \phi_{n-\tau})h + B(t_n, x_n, \phi_{n-\tau}))\Delta W_n \qquad \text{…(3.36)}$$

where $x(t) = \phi_0(t)$, $t_0 - \tau \le t \le t_0$. ∎

Example (3.12):

Consider the SDDE:

$$dx(t) = x(t - 1)\,dt + dW(t)$$

with initial condition $x(t)= \phi_0(t) = t$, $-1 \le t \le 0$, and suppose we want to find the numerical solution by using Euler's method (3.36) with step lengths $h = 0.1$ and $h = 0.01$. The obtained results upon using the Euler method and its comparison with the exact solution given in example (3.8) are presented in table (3.4) in which the results are obtained by using computer programs written in Mathcad 2001i. In addition the Gaussian random numbers with mean 0 and variances 0.1 and 0.01, respectively, are presented in this table for completeness (which will be used in evaluating the exact solution also).

Table (3.4)

Numerical and exact results of example (3.11).

t	$h = 0.1$			$h = 0.01$		
	DW_n	**Numerical solution**	**Absolute error**	DW_n	**Numerical solution**	**Absolute error**
0.1	–0.068	–0.124	0.124	9.851×10^{-3}	–0.081	0.096
0.2	–0.047	–0.193	0.198	-7.231×10^{-3}	–0.184	0.21
0.3	–0.095	–0.321	0.341	–0.029	–0.281	0.326
0.4	–0.169	–0.465	0.510	-4.091×10^{-3}	–0.322	0.41
0.5	4.353×10^{-3}	–0.352	0.432	4.285×10^{-3}	–0.369	0.508
0.6	–0.012	–0.418	0.543	-1.932×10^{-4}	–0.419	0.602
0.7	0.056	–0.390	0.570	1.437×10^{-3}	–0.453	0.685
0.8	0.219	–0.257	0.502	5.391×10^{-3}	–0.474	0.799
0.9	0.081	–0.415	0.735	7.911×10^{-3}	–0.487	0.9
1.0	0.099	–0.408	0.813	4.377×10^{-3}	–0496.	0.986

Example (3.13):

Consider the SDDE:

$$dx(t) = t\,dt + x(t-1)\,dW(t)$$

with initial condition $x(t)= \phi_0(t) = t + 1, -1 \le t \le 0$, and suppose we want to find the numerical solution by using Euler's method (3.36) with step lengths $h = 0.1$ and $h = 0.01$. The obtained results upon using the Euler method and its comparison with the exact solution given in example (3.9) are presented in table (3.5) for two time step intervals [0, 1] and [1, 2], in which the results are obtained by using computer programs written in Mathcad 2001i. In addition the Gaussian random numbers with mean 0 and variances 0.1 and 0.01, respectively, are presented in this table for completeness (which will be used in evaluating the exact solution also).

Table (3.5)

Numerical and exact results of example (3.11).

t	h = 0.1			h = 0.01		
	DW_n	Numerical solution	Absolute error	DW_n	Numerical solution	Absolute error
0.1	–0.068	0	0	9.851×10^{-3}	5.418×10^{-3}	1.295×10^{-3}
0.2	–0.047	0.012	0.007061	-7.231×10^{-3}	0.017	8.735×10^{-5}
0.3	–0.095	0.022	0.002498	–0.029	0.035	4.98×10^{-3}
0.4	–0.169	0.03	0.015	-4.091×10^{-3}	0077.	3.288×10^{-4}
0.5	4.353×10^{-3}	0.14	0.06	4.285×10^{-3}	0.125	2.406×10^{-4}
0.6	–0.012	0.181	0.056	-1.932×10^{-4}	0.177	3.467×10^{-3}
0.7	0.056	0.282	0.102	1.437×10^{-3}	0.243	0.016
0.8	0.219	0.467	0.222	5.391×10^{-3}	0.321	6.437×10^{-3}
0.9	0.081	0.436	0.116	7.911×10^{-3}	0.109	6.998×10^{-3}
1.0	0.099	0.542	0.137	4.377×10^{-3}	0.501	0.011
1.1	–0.068	0.076	0.424	9.851×10^{-3}	0.12	0.479
1.2	–0.047	0.209	0.396	-7.231×10^{-3}	0.215	0.491
1.3	–0.095	0.271	0.449	–0.029	0.31	0.512
1.4	–0.169	0.306	0.539	-4.091×10^{-3}	0.477	0.493
1.5	4.353×10^{-3}	0.688	0.292	4.285×10^{-3}	0.633	0.49
1.6	–0.012	0.813	0.312	-1.932×10^{-4}	0.781	0.482
1.7	0.056	1.082	0.198	1.437×10^{-3}	0.949	0.453
1.8	0.219	1.53	0.085	5.391×10^{-3}	1.131	0.477
1.9	0.081	1.461	0.159	7.911×10^{-3}	1.321	0.477
2.0	0.099	1.648	0.121	4.377×10^{-3}	1.51	0.47

Example (3.14):

Consider the SDDE:

$$dx(t) = x(t-1)\,dt + x(t-2)\,dW(t)$$

with initial condition:

$$x(t)= \phi_0(t) = t,\ -1 \le t \le 0$$

carrying out the computer program written in Mathcad 2001i to find the numerical solution in the first time step interval [0, 1] by using Euler's method with step length h = 0.1 and h = 0.01. The results of table (3.6) are obtained:

Table (3.6)

Numerical and exact results of example (3.11).

t	$h = 0.1$			$h = 0.01$		
	DW_n	**Numerical solution**	**Absolute error**	DW_n	**Numerical solution**	**Absolute error**
0.1	–0.068	–0.052	0.052	9.851×10^{-3}	–0.123	0.135
0.2	–0.047	–0.181	0.186	-7.231×10^{-3}	–0.177	0.209
0.3	–0.095	–0.175	0.195	–0.029	–0.216	0.281
0.4	–0.169	–0.12	0.165	-4.091×10^{-3}	–0.324	0.406
0.5	4.353×10^{-3}	–0.457	0.537	4.285×10^{-3}	–0.393	0.509
0.6	–0.012	–0.482	0.607	-1.932×10^{-4}	–0.431	0.616
0.7	0.056	–0.617	0.797	1.437×10^{-3}	–0.469	0.737
0.8	0.219	–0.86	1.105	5.391×10^{-3}	–0.499	0.817
0.9	0.081	–0.714	1.034	7.911×10^{-3}	–0.516	0.915
1.0	0.099	–0.743	1.148	4.377×10^{-3}	–0.517	1.007

Conclusions and Recommendations

From the present and study, we may conclude the following:

1. The solution of SDDE's either analytically or numerically may be carried in a similar manner that followed in solving SDE's without delay or may be solved directly without transforming to an ODE which is by using the Laplace transformation method.
2. In comparison of the numerical results, the residue error in some examples may be used, to check the accuracy of the results when the analytic solution is not available.
3. Implicit Euler's method have more accurate results than explicit Euler's method, which is due to the bounds of local truncation error for each method.

Also, from the present study, we may recommend the following problems for future work:

1. Numerical solution of SDE's using the Itô-Taylor's expansion method, linear multistep method and Runge-Kutta methods.
2. Studying the theory of stochastic partial differential equations and delay stochastic partial differential equations, as well as, the analytical and numerical methods of solution.
3. Studying the theory of stochastic integral equations and its analytical and numerical methods of solution.

4. Introducing fractional derivatives and fractional integrals in stochastic calculus and then study the resulting stochastic fractional differential equations and stochastic fractional integral equations.

References

References

1. Abdulkadir S., "Variational Formulations of Some Variable Delay Differential Systems", M.Sc. Thesis, Department of Mathematics, College of Science, Al-Nahrain University, Baghdad, Iraq, 2008.

2. Al-Bayaty N. A., "Stochastic Nonlinear Control Stabilizability Based on Inverse Optimality", M.Sc. Thesis, Department of Mathematics, College of Science, Al-Nahrain University, Baghdad, Iraq, 2008.

3. Al-Kubeisy S., "Numerical Solution of Delay Differential Equations Using Linear Multistep Methods", M.Sc. Thesis, Department of Mathematics, College of Science, Al-Nahrain University, Baghdad, Iraq, 2004.

4. Al-Saady A. S., "Cubic Spline Technique in Solving Delay Differential Equations", M.Sc. Thesis, Department of Mathematics, College of Science, Al-Nahrain University, Baghdad, Iraq, 2000.

5. Archambeau C., Cornford D., Opper M. and Taylor J. S., "Gaussian Process Approximations of Stochastic Differential Equations", JMLR: Workshop and Conference Proceedings, Vol.1, pp.1-16, 2007.

6. Arnold L., "Stochastic Differential Equations; Theory and Applications", John Wiley and Sons, Inc., 1974.

7. Baker C. T. H. and Buckwar E., "Numerical Analysis of Explicit One-Step Method for Stochastic Delay Differential Equations",

London, Mathematical Society, Compute. Math. 3, pp.135-335, 2000.

8. Bellman R. and Cook K., "Differential Difference Equations", Academic Press, Inc., New York, 1963.
9. Brauer F. and Nohel J. A., "Ordinary Differential Equations", W. A. Benjamin, Inc., 1973.
10. Burill C. W. and Knudsen J. R., "Real Variables", Holt, Rinehart and Winston, Inc., 1969.
11. Ditlevsen S. and Gaetano A., "Mixed Effects in Stochastic Differential Equation's Models", Revstat Statistical Journal, Vol.3, No.2, pp.137-153, 2006.
12. Driver R. D. E., “Ordinary and Delay Differential Equations”, Springer Verlag Inc., New York, 1977.
13. Evans L. C., "An Introduction to Stochastic Differential Equations", Version 12, Lecture Notes, Short Course at SIAM Meeting, July, 2006.
14. Fridman A., "Stochastic Differential Equations and Applications", Vol.1, Academic Press, Inc., 1975.
15. Geiss S., "Brownian Motion and Stochastic Differential Equations", At Department of Mathematics, Tampere University of Technology, 2007.
16. Gllsinn D. E. and Porta F. A., "Integral Operators and Delay Differential Equations", ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conferences, 2006.

17. Guillouzic S., L'Heurenx I. and Longtin A., "Small Delay Approximation of Stochastic Delay Differential Equations", The American Physical Society, Vol.59, No.4, pp.3970-3982, 1999.

18. Han S., "Numerical Solution of Stochastic Differential Equations", M.Sc. Thesis, University of Edinburgh and Heriot-Watt, 2005.

19. Haugh M., "Overview of Stochastic Calculus", Term Structure Models, 2005.

20. Higham D., "An Algorithmic Introduction o Numerical Simulation of Stochastic Differential Equations", SIAM Review, Vol.43, No.3, pp.525-546, 2001.

21. Hoel P. G., Port S. C. and Stone C. J., "Introduction to Probability Theory", Houghton Mifflin Company, 1971.

22. Hu Y., Mohammed S. E. and Yan F., "Discrete-Time Approximations of Stochastic Delay Differential Equations; Time Milstein Scheme", Institute of Mathematical Statistics, Vol.32, No.1A, pp.265-314, 2004.

23. Klebaner F. C., "Introduction to Stochastic Calculus with Application", Imperial College Press, 2005.

24. Krishnan V., "Nonlinear Filtering and Smoothing", John Wiley and Sons, Inc., 1984.

25. Lin X. S., "Introduction Stochastic Analysis for Finance and Insurance", John Wiley and Sons, Inc., 2006.

26. Mahony C. O., "The Numerical Analysis of Stochastic Differential Equations", Transactions on Circuits and Systems II, 2006.

27. Poularikas A. D., "Probability and Stochastic Processes", CRC Press, LIC, 1999.

28. Reiss M., "Estimating the Delay Time in Affine Stochastic Delay Differential Equations" Wias-Berlin, 2000.

29. Schurz H., "A Brief Introduction to Numerical Analysis of Ordinary Stochastic Differential Equations Without Tears", Marcel Dekker, Basel, pp.237-259, 2002.

30. Shampine L. F. and Thompson S., "Solving Delay Differential Equations with dde23", J. Comp. and Maths. With Appls., Vol.39, pp.43-54, 2000.

31. Soheili A. R., "Stochastic Runge-Kutta Method with Weak and Strong Convergencey", J. Contemp. Math. Sciences, Vol.3, No.9, pp.411-418, 2008.

32. Strizaker D., "Stochastic Processes and Models", Oxford University Press, Inc., New York, 2005.

33. Sϕrensen M. and Küchler U., "Statistical Inference for Discrete-Time Samples from Affine Stochastic Delay Differential Equations", University of AArhus-AArhus School of Business, 2007.

34. ϕksendal B., "Stochastic Differential Equations", Springer-Verlag Heidelberg, New York, 2000.

35. ϕksendal B., "Stochastic Differential Equations; An Introduction with Applications", Springer-Verlag, Berlin, Heidelberg, 2003.

Internet References:

Goldys B., "Introduction to Stochastic Analysis", 2008. Internet Reference (http://www.Ic.unsw.edu.au/plagiarism).

Jarrow R. and Protter P., "A Short History of Stochastic Integration and Mathematical Finance", 2003. Internet Reference (http://www.orie.cornell.edu/~protter/webpapers).

Muszta A., "Solving Stochastic Differential Equations", 2005. Internet Reference (http://www.math.chalmers.se/~eojan/SDE1/).

Rao B. L. S., "Parametric Estimation for Linear Stochastic Delay Differential Equations Driven by Fractional Brownian Motion", 2003. Internet Reference (http://www.isid.ac-in/~statmath/eprints).

Reib M., "Stochastic Delay Differential Equations", 2003. Internet Reference (http://www.mathematik.hu-berlindu/~reiss).

Roussel M. R., "Delay Differential Equations", 2004. Internet Reference (http://ww.cs.uleth.ca/~roussel).

Smith L., "Solving Stochastic Differential Equations by the Reduction Method", 1999. Internet Reference (http://www.personal. umich.edu/~lones/ftp/b10/sde99).

المستخلص

لهذه الإطروحة ثلاثة أهداف رئيسية. الهدف الأول هو إعطاء دراسة شاملة لموضوع التفاضل والتكامل متغير العشوائية، حيث تتضمن الدراسة التعاريف الأساسية والمفاهيم الأساسية المتعلقة بهذا الموضوع متضمنة برهان بعض النتائج، ومن بين هذه النتائج برهان متباينة هولدر للتوقع، نظرية ومبرهنة وجود ووحدانية حلول المعدلات التفاضلية متغيرة العشوائية وطريقة أويلر العددية لحل المعادلات التفاضلية متغيرة العشوائية. الهدف الثاني هو لدراسة الطرق التحليلية والعددية لحل المعادلات التفاضلية متغيرة العشوائية. بينما كان الهدف الثالث هو تطوير طرق الحل المتبعة للمعادلات التفاضلية متغيرة العشوائية وذلك لحل المعادلات التفاضلية متغيرة العشوائية التباطؤية.

جمهـورية العـراق
وزارة التعليم العالي والبحث العلمي
جـــامعة النهــــريــن
كليـــة العلوم
قسم الريـاضيات وتطبيقات الحاسوب

حلول المعادلات التفاضلية متغيرة العشوائية الخطية التباطؤية الاعتيادية

رسالة
مقدمة إلى كلية العلوم - جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل
حسنى أحمد جاسم
(بكالوريوس علوم، جامعة النهرين، ٢٠٠٦)

إشراف
أ.م.د. فاضل صبحي فاضل

آيار ٢٠٠٩ جمادي الاول ١٤٣٠