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Generating Random Variates for Estimating the Parameters of Logistic Distribution by Monte Carlo Simulation

A Thesis
Submitted to the College of Science of Al-Nahrain University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By
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بِسْــــمِ اللهِ الرَّحْمَنِ الرَّحِيْـــمِ

إقرَأ بِأسمِ رَبِكَ الَّذِي خَلَقَ ﴿١﴾ خَلَقَ
الإِنسَانَ مِن عَلَقٍ ﴿٢﴾ إقرَأ وَرَبُّكَ
الأَكرَمُ ﴿٣﴾ الَّذي عَلَّمَ بِالقَلَمِ ﴿٤﴾ عَلَّمَ
الإِنسَانَ مَا لَم يَعلَمْ ﴿٥﴾

صَدَقَ اللهُ العَليُ العَظيم

سورة العلق

الإهداء

إلى سيد الخلق وأشرف المرسلين

إلى من أشرقت الدنيا بنور وجهه

نبي الرحمة محمد (صلى الله عليه واله وسلم)

إلى الشمس التي أحاطتني بدفئها

إلى القلب الذي غمرني بحبها

والدتي العزيزة

إلى النور الذي أنار طريقي

إلى البحر الذي سقاني من فيض علمه

والدي العزيز

إلى أعظم أحبة ساندوني

إلى من شدوا من أزري في الحياة

أخوتي الاعزاء

إلى أبتسامة الحياة وعطر الورد

إلى من رافقوني في مسيرتي

صديقاتي العزيزات

إلى من نوروا لي الطريق

إلى من أبجلهم من كل أعماقي

أساتذتي الكرام

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Supervisor Certification

I certify that this thesis was prepared under my supervision at the department of Mathematics, College of Science, Al-Nahrain University, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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Data: / / 2009

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We certify that we have read this thesis entitled "*Generating Random Variates for Estimating the Parameters of Logistic Distribution by Monte Carlo Simulation*" and as examining committee examined the student *(Zahraa Ammory Ali)* in its contents and in what it connected with, and that, in our opinion, it meets the standards of a thesis for the degree of Master of Science in Mathematics.

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Notations and Abbreviations

p.d.f.	Probability Density Function
c.d.f.	Cumulative Distribution Function
m.g.f	Moment Generating Function
r.v.	Random Variable
r.v.'s	Random Variables
r.s.	Random Sample
distn.	Distribution
distn's.	Distributions
M.L.E.	Maximum Likelihood Estimator
eq.	Equation
eq.'s	Equations
m.l.e	Maximum Likelihood Estimate
$L(\alpha, b)$	Logistic Distribution With Parameters α, b
L.S.M.	Least Square Method
M.M.	Moments Method
M.M.M.	Modified Moments Method
M.L.M.	Maximum Likelihood Method

IT	Inverse Transform
m_r	r^{th} moment about the mean
$m_r^{/}$	r^{th} moment about the origin
d^2	Variance
m.s.e	Mean Square Error
Exp(1)	Exponential Distribution With Parameter 1
R.H.S	Right Hand Side

Abstract

In this work, we consider the Logistic distribution of two parameters for its importance in statistics. Mathematical and statistical properties of Logistic distribution are considered, moments and higher moments are illustrated to the distribution parameters, namely, moments methods, maximum likelihood method, modified moments method, least squares method are discussed theoretically and assessed practically by utilizing two procedures of Monte Carlo simulation for generating random variates from the Logistic distribution. Properties of the estimators, such as Bias, variance, skewness, kurtosis and mean square error measurement are tabulated.

Contents

Introduction

The Logistic growth function was first proposed as a tool for use in demographic studies by Verhulst (1838-1845), [7]. Logistic function used in the present work had been discovered by Pearl and Reed in (1920). It was firstly used in (1924) as a model of growth field of biology. In (1929) this function was given its present name by Reed and Berkson, [7]. It was used by Schultz (1930) as a function of estimating the growth of human population. Persk (1932) proposed a very general class of distributions which includes the Logistic and used it in the graduation of mortality statistics. Pearl et al. (1940) used Logistic function for estimating the growth of human population, while Pearl (1940), Emmens (1941), Wilson and Worcester (1943), Berkson (1944, 1951, 1953) and Finney (1947, 1952) gave studies for some applications of the Logistic function in bioassay problems. Plackett (1959) used the Logistic function in the analysis of survival data. Fisk (1961) used it in the study of income distributions. Furthermore, the Logistic function was used by Oliver (1964) as a model for agricultural production data, [7], and in the studies of physiochemical phenomenon, geological studies and psychological studies by Sanathanan (1974) and Formann (1983), [6].

Balakrishnan (1985) [28] established some recurrence relations for the moment and product moments of order statistics for half Logistic distribution.

Balakrishnan and Leung (1988) [27] show the probability density function of a random variable X that has type I generalized logistic distn. which is given by:

$$f(x; b) = \frac{be^{-x}}{\left(1+e^{-x}\right)^{b+1}}, -\infty < x < \infty, b > 0$$

The log-Logistic distn. is obtained by applying the logarithmic transformation to the Logistic distn. in much the same way the log-Normal distn. is obtained from Normal distn..

Singh et al. (1993) derived a new method of parameter estimation for 3-parameter log-Logistic distn. and used Monte Carlo simulation to evaluate the parameter estimates and compare it with the methods of moments, probability weighted moments and maximum Likelihood estimators, [24]. In (1994) Rassol et al. estimated the probability weighted moments of the generalized Logistic distn. [22].

In (1996) Scerri and Farrugia compared between the Logistic and Weibull distn. for modeling wind speed data, [29]. Al-Yousef (1999) [5] discuss the problem of estimating the parameters of the doubly truncated Logistic distn. when truncation points are unknown and he estimated the parameters of the distn. and the truncation points by using the method of maximum likelihood estimation where he utilize an iterative techniques for approximating the unknown parameters. Wujong-Wuu et al. (2000) made an extension to the usual four-parameter generalized Logistic distn. to the density function involve five-parameters. Olapade (2000) stated some properties of the Logistic distn. in relation to other probability distn['s]., [27].

Rasool et al. (2002) made an applications for generalized Logistic distn. and they estimated by probability weighted moments on 24 hour maximum rainfall events recorded for different cities of Pakistan, [22]. In (2004) [21] Ojo and Olapad define a suitable r.v. to represent the six-parameters generalized Logistic distn. and gave an approximation to the distn. c.d.f. and proved some theorems related to the Logistic distn.. Jones (2006) [25] discovered that the Logistic distn. is a special case of the class formula:

$$f(x) = f(x; \alpha, \beta) = F^{\alpha}(x)(1 - F(x))^{\beta}, \text{ when } \alpha = \beta = 1.$$

The density function of the Logistic distn. is symmetric and uni-model. It is similar in appearance to the normal distn. and in practical applications, [29]. The Logistic gives a nice looking S-shaped curve with a relatively simple mathematical formula. The S-shaped curve is used in what so called Logistic regression model, which uses input variables to make predictions about how Likelihood of certain outcomes.

The S-shaped curve of the Logistic c.d.f. is though to be a substantively useful description of how the probability of an "event" or other outcome rises as a function of some input variables. The Logistic distn. was used instead "as an approximation to other symmetrical distn's. due to the mathematical tractability of its c.d.f", [29]. The simplicity of the Logistic distn. and its importance as a growth curve have made it one of the important statistical distn's., [5].

It was also attracted interesting applications in the modeling of the dependence of chronic obstructive respiratory disease prevalence on smoking and age, degrees of pneumoconiosis in coal miners, geological issues hemolytic uremic syndrome data for children, physiochemical phenomenon, psychological issues, survival time of diagnosed leukemia patients, and weighted gain data, [20].

Literaturey published show that there is a little work dealing with parameters estimate of Logistic distn. by using the implemented four methods of estimation and procedures of generating random variates from Logistic distn. by using Monte Carlo simulation. Hence, this work is an attempt to study the possibility of parameters estimation of Logistic distn. by using Monte Carlo simulation.

This thesis involve three chapters. In chapter one, we introduce some properties of Logistic distn. and moment properties of the distn. are illustrated and modified. Four methods of estimation for the distn. parameters are discussed theoretically. Finally, we proved a theorem related to Logistic distn.

In chapter two, we introduce some concepts of the history of stochastic simulation. Procedures for generating random numbers and random variates from different distn. is discussed theoretically and supported by various examples. Two procedures for generating random variates from Logistic distn. are considered and their algorithms are illustrated.

In chapter three, we found practically moments properties of the estimators such as bias, variance, skewness, kurtosis and mean square error measurement by using four methods of estimation. The professional Mathcad, 13 computer software is used to make the programs of thesis.

Chapter One

On Logistic Distribution

1 On Logistic Distribution

1.1 Introduction

The aim of this chapter is to find the estimators to the parameters of Logistic distn. by using four methods of estimation. In section (1.2), we introduce some important mathematical and statistical properties of Logistic distn., in section (1.3) we represent the genesis of the Logistic distn., in section (1.4) moment properties of the distn. are illustrated and unified, section (1.5) we illustrated point estimation and some important definitions about the estimators, and four methods of estimation are discussed theoretically. Finally, in section (1.6) we introduced some related theorem about the Logistic distn. .

1.2 Some Basic Concepts of Logistic Distribution

In this section, we shall give some mathematical and statistical properties of Logistic distribution.

Definition (1.1), [30]:

A continuous r.v. X is said to have Logistic distn., denoted by X ~ L(α, β) if X has p.d.f:

$$f(x;\alpha,\beta) = \frac{e^{-\left(\frac{x-\alpha}{\beta}\right)}}{\beta\left[1+e^{-\left(\frac{x-\alpha}{\beta}\right)}\right]^2}, -\infty < x < \infty, -\infty < \alpha < \infty, \beta > 0 \quad ..(1.1)$$

$$= 0, \qquad \text{e.w.}$$

where α and β are respectively known as the shape and scale parameters. When $\alpha = 0$, $\beta = 1$ the distn. is called the standard Logistic distn. that is $X \sim L(0, 1)$.

To verify that $f(x; \alpha, \beta)$ of eq.(1.1) is valid p.d.f., we have to show that:

(i) $f(x; \alpha, \beta) > 0, \forall\, x \in (-\infty, \infty)$, obvious.

(ii) The integral side of eq.(1.1) is unity viz.

$$\int_{-\infty}^{\infty} f(x;\alpha,\beta)\,dx = \int_{-\infty}^{\infty} \frac{e^{-\left(\frac{x-\alpha}{\beta}\right)}}{\beta\left[1+e^{-\left(\frac{x-\alpha}{\beta}\right)}\right]^2}\,dx$$

$$= \left.\frac{1}{1+e^{-\left(\frac{x-\alpha}{\beta}\right)}}\right|_{-\infty}^{\infty} = 1$$

The Logistic distn. depends on two parameters α and β, whose graph is bell shape extended indefinitely in both directions.

Figure (1.1) show a graphical representation of the p.d.f. of eq.(1.1) for any values of α and β.

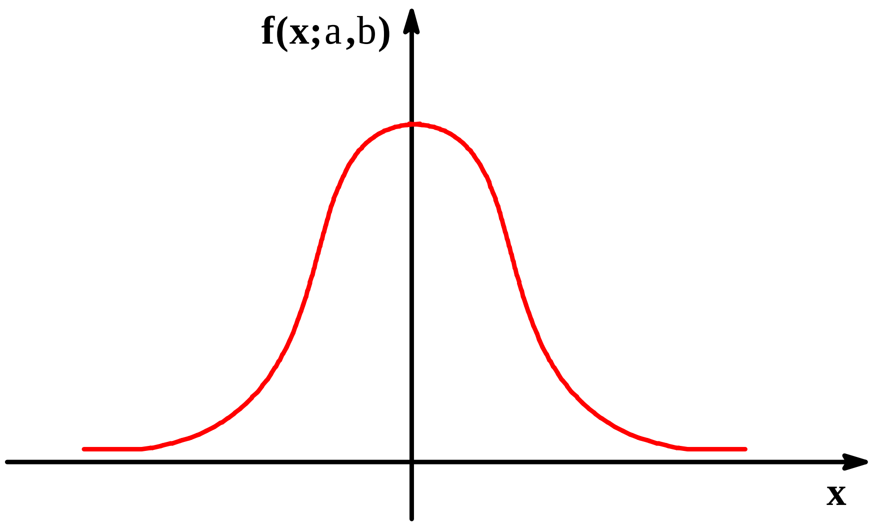

Fig.(1.1) Graph of Logistic distn.

The graph of the Logistic distn. as shown in Fig.(1.1) have generally the following properties:

1. Symmetric about $x = \alpha$.
2. Have the x- axis as a horizontal asymptote.
3. Increasing for $-\infty < x < \alpha$ and decreasing for $\alpha < x < \infty$.
4. Have maximum point at $x = \alpha$.
5. Have two inflection points at $x = \alpha - \beta \ln(2 \pm \sqrt{3})$.
6. Concave up for $-\infty<x<\alpha-\beta\ln(2+\sqrt{3})$ and for $\alpha-\beta\ln(2-\sqrt{3})<x<\infty$ and concave downward for $\alpha-\beta\ln(2+\sqrt{3})<x<\alpha-\beta\ln(2-\sqrt{3})$.

1.2.1 The Cumulative Distribution Function:

The c.d.f. of Logistic distn. is known by the following integral:

$$F(x;\alpha,\beta) = \int_{-\infty}^{x} f(t;\alpha,\beta)\,dt$$

$$= \int_{-\infty}^{x} \frac{e^{-\left(\frac{t-\alpha}{\beta}\right)}}{\beta\left[1+e^{-\left(\frac{t-\alpha}{\beta}\right)}\right]^2}\,dt$$

$$F(x;\alpha,\beta) = pr(X \le x) = \begin{cases} 0, & x \to -\infty \\ \dfrac{1}{1+e^{-\left(\frac{x-\alpha}{\beta}\right)}}, & -\infty < x < \infty \\ 1, & x \to \infty \end{cases} \qquad ....(1.2)$$

1.3 Genesis of the Logistic Distn.

The Belgian scientist Pierre Francois VERHVLST [2] (1804-1849) proposed in (1838) a "demographic growth curve" which was called later as the Logistic function of the form:

$$Y = \frac{A}{B+e^{a+bx}} \qquad ....(1.3)$$

where $x \ge 0$, a, b > 0, e being the Euler's number ($e \approx 2.71828$). The usual form used in econometric studies [17]:

$$Y = \frac{A}{B + e^{-Cx}},\ A, B, C > 0,\ x \geq 0 \quad \text{...(1.4)}$$

The Vechulst's function given by eq.(1.8) is obtained from a differential equation by taking:

$$u(y) = y - y^2 \quad \text{.. (1.5)}$$

Therefore:

$$\frac{dy}{dx} = y(1-y) \ \text{ or } \ \frac{dy}{y(1-y)} = dx \quad \text{...(1.6)}$$

which provides the reduced form of the c.d.f. of the Logistic distn.

$$y = \frac{1}{1+e^{-x}},\ x \in \mathbb{R} \quad \text{..(1.7)}$$

where $y(x)$ is considered as a c.d.f. of a given r.v. $X : y = F(x) = pr(X \leq x)$.

If we differentiate eq.(1.7), we obtain the p.d.f. of standardized Logistic distn. of the form:

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2},\ -\infty < x < \infty \quad \text{..(1.8)}$$

The form of eq.(1.8) generates the well-known BURR-HATKE family of distributions (1942 [8] and 1949 [10]).

1.4 Moments and Higher Moments Properties of Logistic Distribution, [19]

Moments are set of constants used for measuring a distn. properties and under certain circumstances they specify the distn.. The moments of r.v. X (or distn.) are defined in terms of the mathematical expectation of a certain power of X, when they exist. For instance $\mu_r' = E(X^r)$ is called the rth moment of X about the origin and $\mu_r = E[(x-\mu)^r]$ is called the rth central moments of X. That is:

$$\mu_r' = E(X^r) = \begin{cases} \sum_x x^r f(x), & \text{x is discrete r.v.} \\ \int_x x^r f(x)\, dx, & \text{x is continuous r.v.} \end{cases}$$

and

$$\mu_r = E[(x-\mu)^r] = \begin{cases} \sum_x (x-\mu)^r f(x), & \text{x is discrete r.v.} \\ \int_x (x-\mu)^r f(x)\, dx, & \text{x is continuous r.v.} \end{cases}$$

provided that the sum or integral converges absolutely. The generating function reflector certain properties of the distn., they could be used to generate moments. Sometimes they are defining some specific distn$^{'s}$., and also have a particular usefulness in connection with sums of independent r.v$^{'s}$.

First, we shall consider a function of a real t called the moment generating function, denoted by $\mu(t)$ which can be used to generate moments of r.v. X. For continuous r.v. X, the m.g.f is defined by:

$$M(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

provided the integral converge absolutely.

To find the m.g.f of Logistic distn.:

$$M(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{e^{-\left(\frac{x-\alpha}{\beta}\right)}}{\beta\left[1+e^{-\left(\frac{x-\alpha}{\beta}\right)}\right]^2} dx$$

Set $y = \frac{x-\alpha}{\beta} \Rightarrow x = \alpha + \beta y \Rightarrow dx = \beta dy$

$$M(t) = \int_{-\infty}^{\infty} e^{t(\alpha+\beta y)} \frac{e^{-y}}{\beta\left[1+e^{-y}\right]^2} \beta\, dy$$

$$= e^{\alpha t} \int_{-\infty}^{\infty} \frac{e^{\beta t y} e^{-y}}{(1+e^{-y})^2} dy$$

$$= -\, e^{\alpha t} \int_{-\infty}^{\infty} \frac{(e^{-y})^{-\beta t}(-e^{-y})}{(1+e^{-y})^2} dy$$

Set $u = e^{-y} \Rightarrow du = -e^{-y}\, dy$

$$M(t) = -e^{\alpha t} \int_{\infty}^{0} \frac{u^{-\beta t}}{(1+u)^2} du$$

$$= e^{\alpha t} \int_{0}^{\infty} \frac{u^{-\beta t}}{(1+u)^2} du$$

Let $x = \frac{1}{1+u} \Rightarrow u = \frac{1-x}{x} \Rightarrow du = -\frac{1}{x^2}dx$. Therefore:

$$M(t) = e^{\alpha t}\int_1^0 \left(\frac{1-x}{x}\right)^{-\beta t} x^2\left(-\frac{1}{x^2}\right)dx$$

$$= e^{\alpha t}\int_0^1 x^{(1+\beta t)-1}(1-x)^{(1-\beta t)-1}\,dx$$

$$= e^{\alpha t}\Gamma(1+\beta t)\Gamma(1-\beta t),\quad t > -\frac{1}{\beta} \quad\text{.......................................(1.9)}$$

The R.H.S of eq. (1.9) could be simplified as [30]:

$$\Gamma(1+\beta t)\Gamma(1-\beta t) = \pi\beta t\csc(\pi\beta t)\quad\text{.......................................(1.10)}$$

Then the m.g.f of eq.(1.9) becomes:

$$M(t) = \pi\beta t e^{\alpha t}\csc(\pi\beta t)\quad\text{...(1.11)}$$

Many methods could be used to find the moments and higher moments of Logistic distn. such as direct expectation approach, differentiation of m.g.f, or representing the Maclurian series expansion of the m.g.f.

We shall write the Maclurian series expansion for finding the moments of the Logistic distn. as follow:

For simplicity, set $\pi\beta = \theta$, and expansion of the m.g.f given by eq.(1.11) is:

$$\mu(t) = \theta t e^{\alpha t}\csc(\theta t)$$

$$= \frac{\theta t e^{\alpha t}}{\sin(\theta t)}$$

$$= \theta t \frac{1 + \frac{\alpha t}{1!} + \frac{\alpha^2 t^2}{2!} + \frac{\alpha^3 t^3}{3!} + \frac{\alpha^4 t^4}{4!} + o(t^5)}{\theta t - \frac{\theta^3 t^3}{3!} + \frac{\theta^5 t^5}{5!} + o(t^7)}$$

$$= \frac{1 + \frac{\alpha t}{1!} + \frac{\alpha^2 t^2}{2!} + \frac{\alpha^3 t^3}{3!} + \frac{\alpha^4 t^4}{4!} + o(t^5)}{1 - \frac{\theta^2 t^2}{3!} + \frac{\theta^4 t^4}{5!} + o(t^6)}$$

$$= \left[1 + \frac{\alpha t}{1!} + \frac{\alpha^2 t^2}{2!} + \frac{\alpha^3 t^3}{3!} + \frac{\alpha^4 t^4}{4!} + o(t^5)\right]$$

$$\left[1 - \frac{\theta^2 t^2}{3!} + \frac{\theta^4 t^4}{5!} + o(t^6)\right]^{-1}, \text{ where } \left|\frac{\theta^2 t^2}{3!} + \frac{\theta^4 t^4}{5!} + o(t^6)\right| < 1$$

$$= \left[1 + \frac{\alpha t}{1!} + \frac{\alpha^2 t^2}{2!} + \frac{\alpha^3 t^3}{3!} + \frac{\alpha^4 t^4}{4!} + o(t^5)\right]$$

$$\left[1 + \left(\frac{\theta^2 t^2}{3!} - \frac{\theta^4 t^4}{5!} + o(t^6)\right) + \left(\frac{\theta^2 t^2}{3!} - \frac{\theta^4 t^4}{5!} + o(t^6)\right)^2 + \ldots\right]$$

$$= 1 + \frac{\alpha t}{1!} + \frac{\alpha^2 t^2}{2!} + \frac{\alpha^3 t^3}{3!} + \frac{\alpha^4 t^4}{4!} + \frac{\theta^2 t^2}{3!} + \frac{\alpha\theta^2 t^3}{3!} + \frac{\alpha^2\theta^2 t^4}{2!3!} -$$

$$\frac{\theta^4 t^4}{5!} + \frac{\theta^4 t^4}{36} + o(t^5)$$

$$= 1 + \alpha t + (\alpha^2 + \frac{\theta^2}{3})\frac{t^2}{2!} + (\alpha^3 + \alpha\theta^2)\frac{t^3}{3!} + (\alpha^4 + 2\alpha^2\theta^2 - \frac{\theta^4}{5} +$$

$$\frac{2\theta^4}{3})\frac{t^4}{4!} + o(t^5) \quad \ldots\ldots(1.12)$$

The r^{th} moment $E(X^r)$ is the coefficient of $\frac{t^r}{r!}$, $r = 1, 2, \ldots$

The first four moments are:

$$\left.\begin{aligned} &1. E(X) = \alpha \\ &2. E(X^2) = \alpha^2 + \frac{\pi^2\beta^2}{3} \\ &3. E(X^3) = \alpha^3 + \alpha\pi^2\beta^2 \\ &4. E(X^4) = \alpha^4 + 2\alpha^2\pi^2\beta^2 + \frac{7}{15}\pi^4\beta^4 \end{aligned}\right\} \ldots\ldots\ldots\ldots(1.13)$$

i) Mean:

$E(X) = \mu = \mu_1'$ is called the mean of the r.v. X. It is measure of central tendency. Use of eq.(1.13), we have:

$$E(X) = \mu = \alpha \ldots\ldots\ldots\ldots(1.14)$$

ii) Variance:

$Var(X) = \sigma^2 = E[(X - \mu)^2] = E(X^2) - \mu^2$ is called the variance of the r.v. X. It is a measure of dispersion. Use of eq.(1.13) we have:

$$\sigma^2 = \alpha^2 + \frac{\pi^2\beta^2}{3} - \alpha^2 = \frac{\pi^2\beta^2}{3} \ldots\ldots\ldots\ldots(1.15)$$

iii) Coefficient of Variation:

c.v. $= \frac{\sigma}{\mu}$ is called the variational coefficient of the r.v. X. It is a measure of dispersion. Use of eq.(1.13), we have:

$$\text{c.v.} = \frac{\sigma}{\mu} = \frac{\pi\beta}{\alpha\sqrt{3}} \ldots\ldots\ldots\ldots(1.16)$$

iv) Coefficient of Skewness:

$\gamma_1 = \dfrac{\mu_3}{\mu_2^{3/2}} = E[(X - \mu)^3]/(\sigma^2)^{3/2}$ is called the coefficient of skewness. It is a measure of the departure of the frequency curve from symmetry.

If $\gamma_1 = 0$, the curve is not skewed, $\gamma_1 > 0$, the curve is positively skewed, and $\gamma_1 < 0$, the curve is negatively skewed. Use of eq.(1.13) with

$$\mu_3 = E[(X - \mu)^3] = E(X^3) - 3\mu E(X^2) + 2\mu^3$$

$$= \alpha^3 + \alpha\pi^2\beta^2 - 3\alpha\left(\alpha^2 + \frac{\pi^2\beta^2}{3}\right) + 2\alpha^3$$

$$= \alpha^3 + \alpha\pi^2\beta^2 - 3\alpha^3 - \alpha\pi^2\beta^2 + 2\alpha^3 = 0$$

Thus:

$$\gamma_1 = 0 \quad \text{...}(1.17)$$

v) Coefficient of Kurtosis

$\gamma_2 = \dfrac{\mu_4}{\mu_2^2} - 3 = \dfrac{E[(X-\mu)^4]}{(\sigma^2)^2} - 3$ is called the coefficient of kurtosis.

It is a measure of the degree of flatting of the frequency curve. If $\gamma_2 = 0$, the curve is called mesokurtic, if $\gamma > 0$, the curve is called leptokurtic, and if $\gamma_2 < 0$, the curve is called platykurtic. Use eq.(1.13) with

$$\mu_4 = E[(X - \mu)^4]$$

$$= E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4$$

$$= \alpha^4 + 2\alpha^2\pi^2\beta^2 + \frac{7}{15}\pi^4\beta^4 - 4\alpha(\alpha^3 + \alpha\pi^2\beta^2) +$$

$$6\alpha^2\left(\alpha^2 + \frac{\pi^2\beta^2}{3}\right) - 3\alpha^4 = \frac{7}{15}\pi^4\beta^4$$

Thus:

$$\gamma_2 = \frac{\frac{7}{15}\pi^4\beta^4}{\left(\frac{\pi^2\beta^2}{3}\right)^2} - 3$$

$$= \frac{21}{5} - 3 = \frac{6}{5} \quad \text{...(1.18)}$$

vi) Mode:

A mode of distn. is the value x of a r.v. X that maximize the p.d.f f(x). For continuous distn's., the mode x is a solution of:

$$\frac{df(x)}{dx} = 0 \quad \text{and} \quad \frac{d^2f(x)}{dx^2} < 0$$

A mode is a measure of location. Also, we note that the mode may not exist or may have more than one mode.

For Logistic case we take the logarithm of the p.d.f given by eq.(1.1), are have

$$\text{Ln } f(x;\ \alpha,\beta) = -\text{Ln}\beta - \left(\frac{x-\alpha}{\beta}\right) - 2\text{Ln}\left[1 + e^{-\left(\frac{x-\alpha}{\beta}\right)}\right]$$

$$\frac{dLnf(x,\alpha,\beta)}{dx}=-\frac{1}{\beta}+\frac{2e^{-\left(\frac{x-\alpha}{\beta}\right)}}{\beta\left[1+e^{-\left(\frac{x-\alpha}{\beta}\right)}\right]},\ -\infty<x<\infty,\ -\infty<\alpha<\infty,\ \beta>0$$

............................(1.19)

$$\frac{dLnf(x,\alpha,\beta)}{dx}=0\Rightarrow 1+e^{-\left(\frac{x-\alpha}{\beta}\right)}=2e^{-\left(\frac{x-\alpha}{\beta}\right)}$$

$$\Rightarrow\ 1=e^{-\left(\frac{x-\alpha}{\beta}\right)}$$

$$\Rightarrow Ln(1)=-\left(\frac{x-\alpha}{\beta}\right)$$

$$\Rightarrow x=\alpha$$

$$\frac{d^2Lnf(x,\alpha,\beta)}{dx^2}=\frac{2}{\beta}\left[0-\frac{e^{-\left(\frac{x-\alpha}{\beta}\right)}}{\beta\left[1+e^{-\left(\frac{x-\alpha}{\beta}\right)}\right]^2}\right]$$

$$=\frac{-2e^{-\left(\frac{x-\alpha}{\beta}\right)}}{\beta^2\left[1+e^{-\left(\frac{x-\alpha}{\beta}\right)}\right]^2}$$

$$\left.\frac{d^2Lnf(x,\alpha,\beta)}{dx^2}\right|_{x=\alpha}=\frac{-2}{\beta^2(1+1)^2}=\frac{-1}{2\beta^2}<0$$

vii) Median:

A median of a distn. is defined to be the value of x of r.v. X such that $F(x) = pr(X \leq x) = \frac{1}{2}$. The median is a measure of location.

For Logistic case, the c.d.f. given by eq.(1.2), we have:

$$\frac{1}{2} = \frac{1}{1+e^{-\left(\frac{x-\alpha}{\beta}\right)}}$$

$$\Rightarrow 2 = 1 + e^{-\left(\frac{x-\alpha}{\beta}\right)} \Rightarrow 1 = e^{-\left(\frac{x-\alpha}{\beta}\right)}$$

$$\Rightarrow Ln(1) = -\left(\frac{x-\alpha}{\beta}\right)$$

$$\Rightarrow x = \alpha$$

1.5 Point Estimation

The point estimation concerned with inference about the unknown parameters of a distn. from a sample. It provides a single value for each unknown parameter. The following definitions are needed for the interest of this work.

Definition (1.2) (Statistic), [16]:

A statistic is a function of one or more r.v's. which does not depends on any unknown parameters.

Definition (1.3) (Estimator), [16]:

Any statistic whose value used to estimate the unknown parameter θ for some function of θ say $\tau(\theta)$ is called point estimator

Definition (1.4) (Unbiased Estimator), [19]:

An estimator $\hat{\theta} = u(X_1, X_2, \ldots, X_n)$ is defined to be an unbiased estimator of θ if and only if $E(\hat{\theta}) = \theta$ for all $\theta \in \Omega$, where Ω is a parameter space. The term $E(\hat{\theta}) - \theta$ is called the bias of the estimator $\hat{\theta}$.

Definition (1.5) (Asymptotically Unbiased Estimator), [18]:

An estimator $\hat{\theta} = u(X_1, X_2, \ldots, X_n)$ is defined to be asymptotically unbiased estimator for θ if $\lim_{n\to\infty} E(\hat{\theta}) = \theta$.

Definition (1.6) (Consistent Estimator), [19]:

An estimator $\hat{\theta}$ is called consistent estimator for θ if $\hat{\theta}$ converge stochastically to θ.

Remark (1.1) [16]:

Point estimation admits two problems:

First, developing methods for obtaining a statistic, to represent or estimate the unknown parameters in the p.d.f. such statistic is called point estimator.

Second, selecting criteria and technique to define and find the best estimator among many possible estimators.

1.5.1 Methods of Finding Estimators, [1]:

Many techniques have been proposed in the literatures of finding estimators for the distn. parameters, such as Moments, Maximum Likelihood, Minimum Chi-Square, Minimum Distance, Least Square, and Bayesian method. These methods provide a single value for each unknown parameter of the distn. .

For Logistic case, we shall consider four methods for finding estimators of distn. parameters.

(i) Moments method (M.M).

(ii) Maximum Likelihood method (M.L.M).

(iii) Modified Moments method (M.M.M).

(iv) Least-Square method (L.S.M).

1.5.1.1 Estimation of Parameters by Moments Method, [1]:

Let $X_1, X_2, \ldots, X_n$, be a r.s. of size n from distn. whose p.d.f $f(x; \underset{\sim}{\theta})$, $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ is a vector of unknown parameters. Let $\mu'_r = E(X^r)$ be the rth moment of the distn. about the origin and $M_r = \frac{1}{n}\sum_{i=1}^{n} X_i^r$ be the rth moment of the sample about origin. The method of Moments can be describe as follows:

Since, we have k-unknown parameters, equate μ'_r to M_r at $\underset{\sim}{\theta} = \underset{\sim}{\hat{\theta}}$. That is:

$$\mu'_r = M_r \text{ at } \underset{\sim}{\theta} = \underset{\sim}{\hat{\theta}}, r = 1, 2, \ldots, k.$$

For these k equations, we find a unique solution for $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k$ and we say that $\hat{\theta}_r$ (r = 1, 2,…, k) is an estimator of θ_r obtained by method of moments and the corresponding statistic $\hat{\Theta}_r$ is an estimator of θ_r. For Logistic distn. case:

Let $X_1, X_2, \ldots, X_n$, be a r.s. of size n from L(α, β) is taken. Since L(α, β) distn. involve two unknown parameters, we set:

$$\mu'_r = M_r \text{ at } \alpha = \hat{\alpha}, \beta = \hat{\beta}, r = 1, 2$$

For r = 1, we have $\mu'_1 = E(X) = \alpha$ and $M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$, then:

$$\hat{\alpha} = \bar{X} \quad \text{...(1.20)}$$

where $\hat{\alpha}$ is the M.M estimator for α.

For r = 2, we have $\mu'_2 = E(X^2) = \alpha^2 + \frac{\pi^2\beta^2}{3}$ and

$$M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{n-1}{n}S^2 + \bar{X}^2$$

which implies that $\bar{X}^2 + \frac{\pi^2\hat{\beta}^2}{3} = \frac{n-1}{n}S^2 + \bar{X}^2$, and hence:

$$\hat{\beta} = \frac{S}{\pi}\left[\frac{3(n-1)}{n}\right]^{1/2} \quad \text{...(1.21)}$$

where $S^2 = \frac{1}{n-1}\sum_{i=0}^{n}(X_i - \bar{X})^2$ and $\hat{\beta}$ is the M.M. estimator for β.

The estimators $\hat{\alpha}$ and $\hat{\beta}$ given by eq's.(1.20) and (1.21) have the following properties:

(i) $\hat{\alpha} = \bar{X}$ is an unbiased estimator for α, since

$$E(\hat{\alpha}) = E(\bar{X}) = \mu = \alpha \quad \text{.........(1.22)}$$

with

$$\text{Var}(\hat{\alpha}) = \text{var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{\pi^2\beta^2}{3n} \quad \text{.........(1.23)}$$

(ii) $\hat{\beta} = \frac{S}{\pi}\left(\frac{3(n-1)}{n}\right)^{1/2}$ Is asymptotically unbiased estimator for β, since:

$$E(\hat{\beta}) = E\left[\frac{S}{\pi}\left(\frac{3(n-1)}{n}\right)^{1/2}\right] = \frac{1}{\pi}\left(\frac{3(n-1)}{n}\right)^{1/2} E(S)$$

Since S^2 converge stochastically to σ^2, [11].

Implies S converge stochastically to σ

So:

$$E(\hat{\beta}) \approx \frac{1}{\pi}\left(\frac{3(n-1)}{n}\right)^{1/2}\sigma$$

$$= \frac{1}{\pi}\left(\frac{3(n-1)}{n}\right)^{1/2}\frac{\pi\beta}{\sqrt{3}}$$

$$= \left(\frac{n-1}{n}\right)^{1/2}\beta$$

$$\lim_{n\to\infty} E(\hat{\beta}) = \beta \lim_{n\to\infty} \sqrt{1-\frac{1}{n}} = \beta\sqrt{(1-0)} = \beta \qquad \text{.........................(1.24)}$$

With:

$$Var(\hat{\beta}) = Var\left[\frac{1}{\pi}\left(\frac{3(n-1)}{n}\right)^{1/2} S\right]$$

$$= \frac{1}{\pi^2}\left(\frac{3(n-1)}{n}\right)Var(S)$$

Since:

$$Var(S^2) = \frac{1}{n}\left[\mu_4 - \frac{n-3}{n-1}\sigma^4\right]$$

where $\mu_4 = E\left[(X-\mu)^4\right] =$

$$Var(S) = \left[\frac{1}{n}\left[\mu_4 - \frac{n-3}{n-1}\sigma^4\right]\right]^{1/2}$$

Therefore:

$$Var(\hat{\beta}) \approx \frac{1}{\pi^2}\left(\frac{3(n-1)}{n-1}\right)\left[\frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\sigma^4\right)\right]^{1/2} \qquad \text{......................(1.25)}$$

Definition (1.7) (Likelihood Function), [1]:

The likelihood function of r.v's. X_1, X2,… X_n of size n from a distn. having $f(x;\, \underset{\sim}{\theta})$, where $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ is a vector of unknown parameters, is defined to be the joint p.d.f. of the r.v's $X_1, X_2, \ldots, X_n$ which is considered as a function of $\underset{\sim}{\theta}$ and denoted by $L(\underset{\sim}{\theta},\, \underset{\sim}{x})$ is:

$$L = L(\underset{\sim}{\theta},\, \underset{\sim}{x}) = f(\underset{\sim}{x};\, \underset{\sim}{\theta}) = \prod_{i=1}^{n} f(x_i; \underset{\sim}{\theta})$$

1.5.1.2 Estimation of Parameters by Maximum Likelihood Method:

Maximum likelihood method (together with some of its variants) is the most widely used method of estimation, and a list of its applications would cover practically the whole field of statistics, [9].

Let $L(\underset{\sim}{\theta}, \underset{\sim}{x})$ be the likelihood function of a r.s. $X_1, X_2, \ldots, X_n$ of size n from a distn. whose p.d.f $f(x; \underset{\sim}{\theta})$, $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ is a vector of unknown parameters. Let $\hat{\underset{\sim}{\theta}} = \underset{\sim}{u}(\underset{\sim}{x}) = (u_1(\underset{\sim}{x}), u_2(\underset{\sim}{x}), \ldots, u_k(\underset{\sim}{x}))$ be a vector of statistics of observations $\underset{\sim}{x} = (x_1, x_2, \ldots, x_k)$. If $\hat{\underset{\sim}{\theta}}$ have the value of $\underset{\sim}{\theta}$ which maximize $L(\hat{\underset{\sim}{\theta}}, \underset{\sim}{x})$, then $\hat{\underset{\sim}{\theta}}$ is the m.l.e of $\underset{\sim}{\theta}$ and the corresponding statistic $\hat{\underset{\sim}{\Theta}} = \underset{\sim}{u}(\underset{\sim}{x})$ is the m.l.e of $\underset{\sim}{\theta}$. We note that:

(i) Many likelihood functions satisfy the condition that the m.l.e is a solution of the likelihood eq's.

$$\frac{\partial L(\underset{\sim}{\theta}, \underset{\sim}{x})}{\partial \theta_r} = 0, \text{ at } \underset{\sim}{\theta} = \hat{\underset{\sim}{\theta}},\ r = 1, 2, \ldots, k.$$

(ii) Since $L(\underset{\sim}{\theta}, \underset{\sim}{x})$ and $\text{Ln } L(\underset{\sim}{\theta}, \underset{\sim}{x})$ have their maximum at the same value of $\underset{\sim}{\theta}$ so sometimes it is easier to find the maximum of the logarithm of the likelihood. In such case, the M.L.E $\hat{\underset{\sim}{\theta}}$ of $\underset{\sim}{\theta}$ which maximizes $L(\underset{\sim}{\theta}, \underset{\sim}{x})$ may be give the solution of the likelihood eq's.

$$\frac{\partial \text{Ln } L(\underset{\sim}{\theta}, \underset{\sim}{x})}{\partial \theta_r} = 0, \text{ at } \underset{\sim}{\theta} = \hat{\underset{\sim}{\theta}},\ r = 1, 2, \ldots, k, [8].$$

For Logistic distn. case:

Let $X_1, X_2, \ldots, X_n$ be a r.s. of size n from $L(\alpha, \beta)$, where the distn. p.d.f is given by eq.(1.1). The likelihood function is:

$$L(\alpha, \beta, \underset{\sim}{x}) = f(\underset{\sim}{x}, \alpha, \beta)$$

$$= \prod_{i=1}^{n} f(x_i, \alpha, \beta)$$

$$= \prod_{i=1}^{n} \frac{e^{-\left(\frac{x_i-\alpha}{\beta}\right)}}{\beta\left[1+e^{-\left(\frac{x_i-\alpha}{\beta}\right)}\right]^2}$$

$$= \frac{e^{-\frac{1}{\beta}\sum_{i=1}^{n}(x_i-\alpha)}}{\beta^n \prod_{i=1}^{n}\left[1+e^{-\left(\frac{x_i-\alpha}{\beta}\right)}\right]^2}$$

$$\text{Ln } L = -\frac{1}{\beta}\sum_{i=1}^{n}(x_i-\alpha) - n\text{Ln }\beta - 2\sum_{i=1}^{n}\text{Ln}\left[1+e^{-\left(\frac{x_i-\alpha}{\beta}\right)}\right]$$

$$\frac{\partial \text{Ln } L}{\partial \alpha} = \frac{n}{\beta} - \frac{2}{\beta}\sum_{i=1}^{n}\frac{e^{-\left(\frac{x_i-\alpha}{\beta}\right)}}{1+e^{-\left(\frac{x_i-\alpha}{\beta}\right)}}$$

$$= \frac{n}{\beta} - \frac{2}{\beta}\sum_{i=1}^{n}\left[1-\frac{1}{1+e^{-\left(\frac{x_i-\alpha}{\beta}\right)}}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.26)$$

and

$$\frac{\partial \text{Ln } L}{\partial \beta} = \frac{1}{\beta^2}\sum_{i=1}^{n}(x_i - \alpha) - \frac{n}{\beta} - \frac{2}{\beta^2}\sum_{i=1}^{n}\frac{(x_i - \alpha)e^{-\left(\frac{x_i-\alpha}{\beta}\right)}}{1+e^{-\left(\frac{x_i-\alpha}{\beta}\right)}}$$

$$= \frac{1}{\beta^2}\sum_{i=1}^{n}(x_i - \alpha) - \frac{n}{\beta} - \frac{2}{\beta^2}\sum_{i=1}^{n}(x_i - \alpha)\left[1 - \frac{1}{1+e^{-\left(\frac{x_i-\alpha}{\beta}\right)}}\right]$$

............................(1.27)

Set:

$$\frac{\partial \text{Ln } L}{\partial \alpha} = 0 \text{ and } \frac{\partial \text{Ln } L}{\partial \beta} = 0 \text{ at } \alpha = \hat{\alpha},\ \beta = \hat{\beta}$$

We have:

$$\frac{n}{\hat{\beta}} - \frac{2}{\hat{\beta}}\sum_{i=1}^{n}\left[1 - \frac{1}{1+e^{-\left(\frac{x_i-\hat{\alpha}}{\hat{\beta}}\right)}}\right] = 0 \quad \text{...(1.28)}$$

and

$$\frac{1}{\hat{\beta}^2}\sum_{i=1}^{n}(x_i - \hat{\alpha}) - \frac{n}{\hat{\beta}} - \frac{2}{\hat{\beta}^2}\sum_{i=1}^{n}(x_i - \hat{\alpha})\left[1 - \frac{1}{1+e^{-\left(\frac{x_i-\hat{\alpha}}{\hat{\beta}}\right)}}\right] = 0 \quad \text{....(1.29)}$$

Solution for $\hat{\alpha}$ and $\hat{\beta}$ can not be found analytically from the nonlinear eq's. (1.28) and (1.29).

An approximate solution for $\hat{\alpha}$ and $\hat{\beta}$ from eq's. (1.28) and (1.29) can be made iteratively by using Newton-Raphson method for solving a non-linear eq's. as follows:

Let:

$$f_1 = f_1(\hat{\alpha}, \hat{\beta}) = \frac{n}{\hat{\beta}} - \frac{2}{\hat{\beta}} \sum_{i=1}^{n} \left[1 - \frac{1}{1 + e^{-\left(\frac{x_i - \hat{\alpha}}{\hat{\beta}}\right)}}\right] = 0$$

and

$$f_2 = f_2(\hat{\alpha}, \hat{\beta}) = \frac{1}{\hat{\beta}^2} \sum_{i=1}^{n} (x_i - \hat{\alpha}) - \frac{n}{\hat{\beta}} - \frac{2}{\hat{\beta}^2} \sum_{i=1}^{n} (x_i - \hat{\alpha}) \left[1 - \frac{1}{1 + e^{-\left(\frac{x_i - \hat{\alpha}}{\hat{\beta}}\right)}}\right] = 0$$

Suppose that ($\hat{\alpha}_{(s)}$, $\hat{\beta}_{(s)}$) represent the approximate solution of $(\hat{\alpha}, \hat{\beta})$ at stage (s). Then approximate solution at stage (s + 1) for $(\hat{\alpha}, \hat{\beta})$ is:

$$\hat{\alpha}_{(s+1)} = \hat{\alpha}_{(s)} + \delta_1 \qquad \text{.......... (1.30)}$$

$$\hat{\beta}_{(s+1)} = \hat{\beta}_{(s)} + \delta_2 \qquad \text{.......... (1.31)}$$

In matrix form, we may write:

$$\underline{\delta} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = -\begin{bmatrix} \frac{\partial f_1}{\partial \hat{\alpha}_{(s)}} & \frac{\partial f_1}{\partial \hat{\beta}_{(s)}} \\ \frac{\partial f_2}{\partial \hat{\alpha}_{(s)}} & \frac{\partial f_2}{\partial \hat{\beta}_{(s)}} \end{bmatrix}^{-1} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad \text{.......... (1.32)}$$

Provided that:

$$\begin{vmatrix} \dfrac{\partial f_1}{\partial \hat{\alpha}_{(s)}} & \dfrac{\partial f_1}{\partial \hat{\beta}_{(s)}} \\ \dfrac{\partial f_2}{\partial \hat{\alpha}_{(s)}} & \dfrac{\partial f_2}{\partial \hat{\beta}_{(s)}} \end{vmatrix} \neq 0$$

Set:

$$a = \frac{\partial f_1}{\partial \hat{\alpha}_{(s)}} = -\frac{2}{\hat{\beta}^2} \sum_{i=1}^{n} \frac{e^{-\left(\frac{x_i - \hat{\alpha}}{\hat{\beta}}\right)}}{\left[1 + e^{-\left(\frac{x_i - \hat{\alpha}}{\hat{\beta}}\right)}\right]^2}$$

$$b = \frac{\partial f_1}{\partial \hat{\beta}_{(s)}} = \frac{\partial f_2}{\partial \hat{\alpha}_{(s)}} = -\frac{n}{\hat{\beta}^2} - \frac{2}{\hat{\beta}^3} \sum_{i=1}^{n} \frac{(x_i - \hat{\alpha}) e^{-\left(\frac{x_i - \hat{\alpha}}{\hat{\beta}}\right)}}{\left[1 + e^{-\left(\frac{x_i - \hat{\alpha}}{\hat{\beta}}\right)}\right]^2} +$$

$$\frac{2}{\hat{\beta}^2} \sum_{i=1}^{n} \left[1 - \frac{1}{1 + e^{-\left(\frac{x_i - \hat{\alpha}}{\hat{\beta}}\right)}}\right]$$

$$c = \frac{\partial f_2}{\partial \hat{\beta}_{(s)}} = -\frac{2}{\hat{\beta}^3} \sum_{i=1}^{n} (x_i - \hat{\alpha}) + \frac{n}{\hat{\beta}^2} - \frac{2}{\hat{\beta}^4} \sum_{i=1}^{n} \frac{(x_i - \hat{\alpha})^2 e^{-\left(\frac{x_i - \hat{\alpha}}{\hat{\beta}}\right)}}{\left[1 + e^{-\left(\frac{x_i - \hat{\alpha}}{\hat{\beta}}\right)}\right]^2} +$$

$$\frac{4}{\hat{\beta}^3} \sum_{i=1}^{n} (x_i - \hat{\alpha}) \left[1 - \frac{1}{1 + e^{-\left(\frac{x_i - \hat{\alpha}}{\hat{\beta}}\right)}}\right]$$

We have:

$$\begin{bmatrix}\delta_1\\ \delta_2\end{bmatrix} = -\begin{bmatrix}a & b\\ b & c\end{bmatrix}^{-1}\begin{bmatrix}f_1\\ f_2\end{bmatrix}$$

$$= -\frac{1}{ac-b^2}\begin{bmatrix}c & -b\\ -b & a\end{bmatrix}\begin{bmatrix}f_1\\ f_2\end{bmatrix}, \text{ for } ac-b^2 \neq 0$$

Then:

$$\delta_1 = -\frac{1}{ac-b^2}(cf_1 - bf_2)$$

$$\delta_2 = -\frac{1}{ac-b^2}(-bf_1 + af_2)$$

and according eq$^{'s}$. (1.30) and (1.31), we have:

$$\hat{\alpha}_{(s+1)} = \hat{\alpha}_{(s)} + -\frac{1}{ac-b^2}(cf_1 - bf_2) \qquad (1.33)$$

$$\hat{\beta}_{(s+1)} = \hat{\beta}_{(s)} + -\frac{1}{ac-b^2}(-bf_1 + af_2) \qquad (1.34)$$

1.5.1.3 Estimation of Parameters by Modified Moments Method, [18]:

This method can be described as follows:

Let $X_1, X_2, \ldots, X_n$ be a r.s. of size n from distn. whose p.d.f $f(x, \underset{\sim}{\theta})$ where $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$, is a vector of k-unknown parameters. Let $Y_1 < Y_2 < \ldots < Y_n$ represent the arrangement of the sample set $\{X_i\}$ in ascending order of magnitude. Let $\mu'_r = E(X^r)$ be the rth distn. moment about the origin and $M_r = \frac{1}{n}\sum_{i=0}^{n} X_i^r$ is the r^{th} sample moment about the origin, $r = 1, 2, \ldots k$

In this method, we equate $\mu'_r = M_r$ at $\underset{\sim}{\theta} = \hat{\theta}_i$,$i = 1, 2, \ldots, k$ with $r = 1$ and ranking $E(Y_i) = Y_i$ beginning with $i = 1$ until $i = k - 1$ this process will gives k eq's. to provide a unique solution for θ_i at $\hat{\theta}_i$, $i = 1, 2, \ldots, k$.

For Logistic distn. case:

We have two unknown parameters α and β and if we take a r.s. of size n from $L(\alpha, \beta)$, we let Y_1 represent the first order statistic of the sample.

From the statistic theory the p.d.f of Y_1 is

$g_1(y_1) = n[1 - F(y_1)]^{n-1} f(y_1)$, and hence:

$$g_1(y_1) = n\left[1 - \frac{1}{1+e^{-\left(\frac{y_1-\alpha}{\beta}\right)}}\right]^{n-1} \frac{e^{-\left(\frac{y_1-\alpha}{\beta}\right)}}{\beta\left[1+e^{-\left(\frac{y_1-\alpha}{\beta}\right)}\right]^2}$$

$$= \frac{n\left[e^{-\left(\frac{y_1-\alpha}{\beta}\right)}\right]^n}{\beta\left[1+e^{-\left(\frac{y_1-\alpha}{\beta}\right)}\right]^{n+1}}$$

To find $E(Y_1)$,we shall consider the m.g.f of Y_1

$$M_{y1}(t) = E(e^{ty_1}) = \int_{-\infty}^{\infty} e^{ty_1} \frac{n\left[e^{-\left(\frac{y_1-\alpha}{\beta}\right)}\right]^n}{\beta\left[1+e^{-\left(\frac{y_1-\alpha}{\beta}\right)}\right]^{n+1}} dy_1$$

Let $x = \dfrac{y_1 - \alpha}{\beta}$, then $\beta dx = dy_1$

$$= \int_{-\infty}^{\infty} e^{t(\alpha+\beta x)} \frac{n\left(e^{-x}\right)^n}{\beta\left[1+e^{-x}\right]^{n+1}} \beta\, dx$$

$$= ne^{\alpha t} \int_{-\infty}^{\infty} \frac{e^{\beta tx}\left(e^{-x}\right)^n}{\left(1+e^{-x}\right)^{n+1}} dx$$

$$= -ne^{\alpha t} \int_{-\infty}^{\infty} \frac{(e^{-x})^{-\beta t+n-1}(-e^{-x})}{(1+e^{-x})^{n+1}} dx$$

Let $u = e^{-x} \Rightarrow du = -e^{-x}\, dx$

$$M_y(t) = -ne^{\alpha t} \int_{\infty}^{0} \frac{u^{-\beta t+n-1}}{(1+u)^{n+1}} du$$

$$M_y(t) = ne^{\alpha t} \int_{0}^{\infty} \frac{u^{-\beta t+n-1}}{(1+u)^{n+1}} du$$

Let $x = \dfrac{1}{1+u} \Rightarrow u = \dfrac{1-x}{x} \Rightarrow du = \dfrac{-1}{x^2}\, dx$

$$M_y(t) = ne^{\alpha t}\int_1^0 \left(\frac{1-x}{x}\right)^{-\beta t+n-1} x^{n+1}\left(\frac{-1}{x^2}\right)dx$$

$$= ne^{\alpha t}\int_0^1 x^{(1+\beta t)-1}(1-x)^{(n-\beta t)-1}\, dx$$

$$= ne^{\alpha t}\frac{\Gamma(1+\beta t)\Gamma(n-\beta t)}{\Gamma(n+1)}$$

Set $\varphi(t) = \text{Ln } M_y(t)$

$$= \text{Ln}(n) + \alpha t + \text{Ln }\Gamma(1+\beta t) + \text{Ln }\Gamma(n-\beta t) - \text{Ln }\Gamma(n+1)$$

$$\frac{d\varphi(t)}{dt} = 0 + \alpha + \beta\psi(1+\beta t) - \beta\psi(n-\beta t) \qquad (1.35)$$

Where $\psi(n) = \frac{d}{dx}\text{Ln}\Gamma(x)$ is known as diagamma function.

$$\left.\frac{d\varphi(t)}{dt}\right|_{t=0} = \alpha + \beta\psi(1) - \beta\psi(n) = E(Y_1)$$

Now, we apply the modified moments method by setting:

$\mu'_1 = \bar{X}$ and $E(Y_1) = Y_1$ at $\alpha = \hat{\alpha}$, $\beta = \hat{\beta}$, which leads to:

$$\hat{\alpha} = \bar{X} \qquad (1.36)$$

$$\hat{\alpha} + \hat{\beta}\psi(1) - \hat{\beta}\psi(n) = Y_1 \qquad (1.37)$$

From eq's. (1.36) and (1.37) the estimators of α and β are respectively:

$$\hat{\alpha} = \bar{X} \qquad (1.38)$$

$$\hat{\beta} = \frac{Y_1 - \bar{X}}{\psi(1) - \psi(n)} \qquad (1.39)$$

where $\psi(1) = -0.577$ and $\psi(n)$ is approximated by:

$$\psi(n) = \text{Ln}(n) - \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \frac{1}{240n^8}$$, [3]

The estimators $\hat{\alpha}$ and $\hat{\beta}$ given by eq.(1.38) and (1.39), have the following properties:

(i) $\hat{\alpha} = \bar{X}$ is an unbiased estimator for α, since :

$$E(\hat{\alpha}) = E(\bar{X}) = \mu = \alpha \qquad (1.40)$$

with:

$$\text{Var}(\hat{\alpha}) = \text{Var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{\pi^2\beta^2}{3n} \qquad (1.41)$$

(ii) $\hat{\beta} = \dfrac{Y_1 - \bar{X}}{\psi(1) - \psi(n)}$ is an unbiased estimator for β, since:

$$E(\hat{\beta}) = E\left[\frac{Y_1 - \bar{X}}{\psi(1) - \psi(n)}\right] = \frac{1}{\psi(1) - \psi(n)} E\left[Y_1 - \bar{X}\right]$$

$$= \frac{1}{\psi(1) - \psi(n)}[(\alpha + \beta\psi(1) - \beta\psi(n)) - \alpha]$$

Hence:

$$E(\hat{\beta}) = \frac{1}{\psi(1) - \psi(n)}(\beta(\psi(1) - \psi(n))) = \beta \qquad (1.42)$$

With:

$$\text{Var}(\hat{\beta}) = \frac{1}{(\psi(1) - \psi(n))^2} \text{Var}(Y_1 - \bar{X})$$

$$\text{Var}(Y_1 - \bar{X}) = \text{Var}(Y_1) + \text{Var}(\bar{X}) - 2\text{Cov}(Y_1, \bar{X})$$

From eq.(1.35), we have:

$$\frac{d^2\varphi(t)}{dt^2} = \beta^2\psi'(1+\beta t) + \beta^2\psi'(n-\beta t)$$

Where $\psi'(x)= \frac{d^2}{dx^2}\mathrm{Ln}\Gamma(x)$ is known as trigamma function.

$$\left.\frac{d^2\varphi(t)}{dt^2}\right|_{t=0} = \beta^2\psi'(1) + \beta^2\psi'(n)$$

$$=\beta^2\,[\psi'(1) + \psi'(n)] = \mathrm{Var}(Y_1)$$

Where $\psi'(1) = 1.645$ and $\psi'(n)$ is approximated by:

$$\psi'(n)= \frac{1}{n}+\frac{1}{2n^2}+\frac{1}{6n^3}-\frac{1}{30n^5}+\frac{1}{24n^7}-\frac{1}{30n^9}\text{ , [3]}$$

$$\mathrm{Cov}(Y_1, \bar{X}) = E(Y_1\bar{X}) - E(Y_1)E(\bar{X})$$

$$E(Y_1\bar{X}) = E[\min(X_i)\bar{X}] = \frac{1}{n}E\left[\min(X_i)\left(\sum_{i=1}^{n}X_i\right)\right]$$

$$= \frac{1}{n}E[\min(X_i)X_1 + \min(X_i)X_2 + \ldots + \min(X_i)X_n]$$

$$= \frac{1}{n}E\left[X_i^2 + \sum_{\substack{j=1\\ i\neq j}}^{n}X_iX_j\right]$$

$$= \frac{1}{n}E(X_i^2) + \sum_{\substack{j=1\\ i\neq j}}^{n}E(X_iX_j)$$

$$= \frac{1}{n}E(X_i^2) + \sum_{\substack{j=1\\ i\neq j}}^{n}E(X_i)E(X_j)$$

$$= \frac{1}{n}[\mathrm{Var}(X_i) + [E(X_i)]^2] + \sum_{\substack{j=1 \\ i \neq j}}^{n} E(X_i)E(X_j)$$

$$= \frac{1}{n}\left[\frac{\pi^2\beta^2}{3} + \alpha^2 + \sum_{\substack{j=1 \\ i \neq j}}^{n} \alpha^2\right]$$

$$= \frac{1}{n}\left[\frac{\pi^2\beta^2}{3} + \alpha^2 + (n-1)\alpha^2\right]$$

$$= \frac{1}{n}\left[\frac{\pi^2\beta^2}{3} + n\alpha^2\right]$$

Hence:

$$\mathrm{Var}(\hat{\beta}) = \frac{1}{(\psi(1)-\psi(n))^2}\left[\beta^2(\psi'(1)+\psi'(n)) + \frac{\pi^2\beta^2}{3n}\right] - 2\left[\frac{1}{n}\left(\frac{\pi^2\beta^2}{3} + n\alpha^2\right) - \left(\alpha^2 + \alpha\beta\psi(1) - \alpha\beta\psi(n)\right)\right] \quad ....(1.43)$$

1.5.1.4 Estimation of Parameters by Least Squares Method, [18]:

The Least squares method is a general technique for estimating parameters in fitting a set of points to generate a curve whose trend might be linear, quadratic, or of higher order. In order to utilize this method, the error terms due to experiment must satisfy the following conditions:

(i) They have zero mean.

(ii) They have the same variance.

(iii) They must be uncorrelated.

For good results of fitting curve to the data set, the error must be minimized as small as possible.

Let us assume that we have a set of n data points (x_i, t_i) through which we desire to pass a straight line. This line is representing the best fit in the least square sense.

Suppose that the best fitting straight line to the data (x_i, t_i) is $x = \lambda_0 + \lambda_1 t$, where λ_0 and λ_1 are two unknown parameters representing respectively the vertical intercept and the slop. To assist in visualizing the process, assume that the data points as well as the line to be fitted, unless the data fall in a straight line, usually the general curve will not pass through all of the data points. For convenience, let us consider the i^{th} point where ordinate of the point is given as x_i.

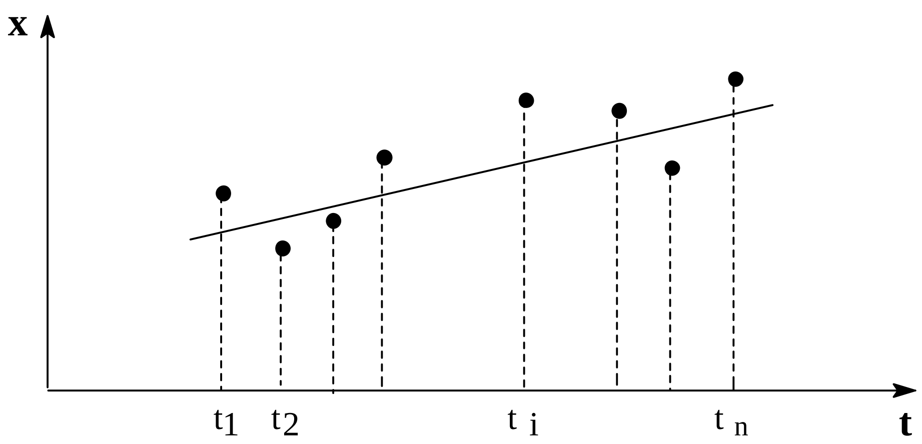

Fig.(1.2).

Figure (1.2) show the best fitted line to the data (t_i ,x_i,). The ordinate x_i as given by the general line is $\lambda_0 + \lambda_1 t_i$ .The difference between these two values is the error of fit at the ith point $e_i = x_i - (\lambda_0 + \lambda_1 t_i)$. Let the sum of squares of all errors at the data points be:

$$\Omega = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (x_i - \lambda_0 - \lambda_1 t_i)^2$$

For minimum, we set:

$$\frac{\partial\Omega}{\partial\lambda_0} = 0 \text{ and } \frac{\partial\Omega}{\partial\lambda_1} = 0, \text{ at } \lambda_0 = \hat{\lambda}_0, \lambda_1 = \hat{\lambda}_1$$

$$\left.\frac{\partial\Omega}{\partial\lambda_0}\right|_{\substack{\lambda_0=\hat{\lambda}_0 \\ \lambda_1=\hat{\lambda}_1}} = -2\sum_{i=1}^{n}(x_i - \hat{\lambda}_0 - \hat{\lambda}_1 t_i) = 0 \qquad (1.44)$$

$$\left.\frac{\partial\Omega}{\partial\lambda_1}\right|_{\substack{\lambda_0=\hat{\lambda}_0 \\ \lambda_1=\hat{\lambda}_1}} = -2\sum_{i=1}^{n}(x_i - \hat{\lambda}_0 - \hat{\lambda}_1 t_i) t_i = 0 \qquad (1.45)$$

From (1.44) and (1.45), we can get two eq$^{'s}$. as:

$$n\hat{\lambda}_0 + \hat{\lambda}_1 \sum_{i=1}^{n} t_i = \sum_{i=1}^{n} x_i \qquad (1.46)$$

$$\hat{\lambda}_0 \sum_{i=1}^{n} t_i + \hat{\lambda}_1 \sum_{i=1}^{n} t_i^2 = \sum_{i=1}^{n} t_i x_i \qquad (1.47)$$

Equations (1.46) and (1.47) are simultaneous algebraic eq$^{'s}$. for the two parameters λ_0 and λ_1.

In matrix notation (1.46) and (1.47) may be written as:

$$\underset{\sim}{A}\,\underset{\sim}{\hat{\lambda}} = (b) \qquad (1.48)$$

where:

$$\underset{\sim}{A} = \begin{bmatrix} n & \sum_{i=1}^{n} t_i \\ \sum_{i=1}^{n} t_i & \sum_{i=1}^{n} t_i^2 \end{bmatrix}, \ \underset{\sim}{\hat{\lambda}} = \begin{bmatrix} \hat{\lambda}_0 \\ \hat{\lambda}_1 \end{bmatrix}, \ \underset{\sim}{b} = \begin{bmatrix} \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} t_i x_i \end{bmatrix}$$

The solution of eq.(1.48) is:

$$\underset{\sim}{\hat{\lambda}} = \underset{\sim}{A}^{-1} \underset{\sim}{b} \text{ if and only if } |\underset{\sim}{A}| \text{ exists.}$$

Thus, whenever the data points t_i, $\forall$ i are given, then the two matrices $\underset{\sim}{A}$ and $\underset{\sim}{b}$ may be computed and hence $\underset{\sim}{\hat{\lambda}}$ is determined as follows:

$$\hat{\lambda}_0 = \frac{\bar{X}\sum_{i=1}^{n} t_i^2 - \bar{t}\sum_{i=1}^{n} t_i x_i}{\sum_{i=1}^{n} t_i^2 - \bar{t}\sum_{i=1}^{n} t_i} \quad \text{...(1.49)}$$

$$\hat{\lambda}_1 = \frac{\sum_{i=1}^{n} t_i x_i - \bar{t}\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} t_i^2 - \bar{t}\sum_{i=1}^{n} t_i} \quad \text{..(1.50)}$$

provided that $\left(\sum_{i=1}^{n} t_i^2 - \bar{t}\sum_{i=1}^{n} t_i\right) \neq 0$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\bar{t} = \frac{1}{n}\sum_{i=1}^{n} t_i$.

For the Logistic distn. case, suppose that X_1, X_2, …, X_n be a random sample of size n from Logistic distn. having cumulative function:

$$F(x) = \Pr(X \le x) = \begin{cases} 0, & x \to -\infty \\ \dfrac{1}{1+e^{-\left(\frac{x-\alpha}{\beta}\right)}}, & -\infty < x < \infty \\ 1, & x \to \infty \end{cases}$$

We set $u_i = F(x_i)$, then $u_i = \dfrac{1}{1+e^{-\left(\frac{x_i-\alpha}{\beta}\right)}}$, which implies:

$$x_i = \alpha + \beta \operatorname{Ln}\left(\frac{u_i}{1-u_i}\right),\ i = 1, 2, \ldots, n \qquad (1.51)$$

Set $y_i = x_i$, $t_i = \operatorname{Ln}\left(\frac{u_i}{1-u_i}\right)$, $i = 1, 2, \ldots, n$ and $\hat{\lambda}_0 = \alpha$, $\hat{\lambda}_1 = \beta$. Then $y_i = \lambda_0 + \lambda_1 t_i$, $i = 1, 2, \ldots, n$.

Utilizing eq.(1.51) for obtaining the estimator $\hat{\lambda}_0$ and $\hat{\lambda}_1$, therefore; the least squares estimators $\hat{\alpha}$ and $\hat{\beta}$ can be obtained from the (1.51):

$$\hat{\alpha} = \hat{\lambda}_0 \qquad (1.52)$$

$$\hat{\beta} = \hat{\lambda}_1 \qquad (1.53)$$

The estimators $\hat{\alpha}$ and $\hat{\beta}$ given by (1.52) and (1.53) have the following properties:

(i) $\hat{\beta} = \dfrac{\sum_{i=1}^{n} t_i x_i - \bar{t}\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} t_i^2 - \bar{t}\sum_{i=1}^{n} t_i}$ is an unbiased estimator.

Set:

$$S_{tt} = \sum_{i=1}^{n}(t_i - \bar{t})^2 = \sum_{i=1}^{n}(t_i - \bar{t})t_i = \sum_{i=1}^{n} t_i^2 - \frac{\left(\sum_{i=1}^{n} t_i\right)^2}{n}$$

$$S_{xx} = \sum_{i=1}^{n}(x_i - \bar{X})^2 = \sum_{i=1}^{n}(x_i - \bar{X})x_i = \sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}$$

$$S_{tx} = \sum_{i=1}^{n}(t_i - \bar{t})(x_i - \bar{X}) = \sum_{i=1}^{n}(t_i - \bar{t})x_i = \sum_{i=1}^{n} t_i x_i - \frac{\left(\sum_{i=1}^{n} t_i\right)\left(\sum_{i=1}^{n} x_i\right)}{n}$$

So $\hat{\beta}$ may be written as:

$$\hat{\beta} = \frac{S_{tx}}{S_{tt}}$$

$$E(\hat{\beta}) = E\left(\frac{S_{tx}}{S_{tt}}\right) = \frac{1}{S_{tt}} E(S_{tx}) = E\frac{1}{S_{tt}}\left[\sum_{i=1}^{n}(t_i - \bar{t})x_i\right]$$

$$= \frac{1}{S_{tt}} \sum_{i=1}^{n}(t_i - \bar{t})E(x_i) = \frac{1}{S_{tt}} \sum_{i=1}^{n}(t_i - \bar{t})(\alpha + \beta t_i)$$

$$= \frac{1}{S_{tt}}\left[\alpha \sum_{i=1}^{n}(t_i - \bar{t}) + \beta \sum_{i=1}^{n}(t_i - \bar{t})t_i\right]$$

Hence:

$$E(\hat{\beta}) = \frac{1}{S_{tt}}(\beta S_{tt}) = \beta \qquad \text{.......(1.54)}$$

with:

$$Var(\hat{\beta}) = Var\left(\frac{S_{tx}}{S_{tt}}\right) = \frac{1}{S_{tt}^2} Var(S_{tx})$$

$$= \frac{1}{S_{tt}^2} Var\left[\sum_{i=1}^{n} (t_i - \bar{t}) x_i\right]$$

$$= \frac{1}{S_{tt}^2} \sum_{i=1}^{n} Var\left[(t_i - \bar{t}) x_i\right]$$

$$= \frac{1}{S_{tt}^2} \sum_{i=1}^{n} (t_i - \bar{t})^2 Var(x_i) = \frac{1}{S_{tt}^2} \sum_{i=1}^{n} (t_i - \bar{t})^2 \sigma^2$$

Hence:

$$Var(\hat{\beta}) = \frac{1}{S_{tt}^2} \sigma^2 S_{tt} = \frac{\sigma^2}{S_{tt}} = \frac{\pi^2 \beta^2}{3\sum_{i=1}^{n} (t_i - \bar{t})^2} \quad \text{.............................(1.55)}$$

(ii) $\hat{\alpha}$ is an unbiased estimator. From eq.(1.46), we have:

$$\hat{\alpha} = \bar{X} - \hat{\beta}\,\bar{t}$$

$$E(\hat{\alpha}) = E(\bar{X} - \hat{\beta}\,\bar{t}) = E(\bar{X}) - \bar{t}\,E(\hat{\beta})$$

Since $x_i = \alpha + \beta t_i + e_i$, then:

$$\sum_{i=1}^{n} x_i = n\alpha + \beta \sum_{i=1}^{n} t_i + \sum_{i=1}^{n} e_i$$

Which implies to:

$$\bar{X} = \alpha + \beta\,\bar{t} + \bar{e}$$

$$E(\bar{X}) = \alpha + \beta\,\bar{t} + 0$$

Hence:

$$E(\hat{\alpha}) = \alpha + \beta\,\bar{t} - \beta\,\bar{t} = \alpha \quad \text{...}(1.56)$$

$$Var(\hat{\alpha}) = Var(\bar{X} - \hat{\beta}\,\bar{t}) = Var(\bar{X}) + \bar{t}^{\,2}Var(\hat{\beta}) - 2\bar{t}\,Cov(\bar{X}, \hat{\beta})$$

$$= \frac{\pi^2\beta^2}{3n} + \bar{t}^{\,2}\frac{\pi^2\beta^2}{3\sum_{i=1}^{n}(t_i - \bar{t})^2} - 2\bar{t}\;Cov(\bar{X}, \hat{\beta})$$

$$Cov(\bar{X}, \hat{\beta}) = Cov\left(\frac{1}{n}\sum_{i=1}^{n} x_i, \frac{S_{tx}}{S_{tt}}\right)$$

$$= \frac{1}{nS_{tt}} Cov\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n}(t_i - \bar{t})x_i\right)$$

$$= \frac{1}{nS_{tt}}\sum_{i=1}^{n} Cov(x_i, (t_i - \bar{t})x_i)$$

$$= \frac{1}{nS_{tt}}\sum_{i=1}^{n}\left[E\left[(t_i - \bar{t})x_i^2\right] - E(x_i)E\left[(t_i - \bar{t})x_i\right]\right]$$

$$= \frac{1}{nS_{tt}}\sum_{i=1}^{n}(t_i - \bar{t})\left[E(x_i^2) - (E(x_i))^2\right]$$

$$= \frac{1}{nS_{tt}}\sum_{i=1}^{n}(t_i - \bar{t})Var(x_i)$$

$$= \frac{1}{nS_{tt}}\sigma^2\sum_{i=1}^{n}(t_i - \bar{t}) = \frac{\sigma^2.0}{nS_{tt}} = 0$$

Hence:

$$Var(\hat{\alpha}) = \frac{\pi^2\beta^2}{3}\left[\frac{1}{n} + \frac{\overline{t^2}}{\sum_{i=1}^{n}(t_i - \bar{t})^2}\right] \quad \text{...}(1.57)$$

1.6 Theorem Related to the Logistic Distribution

In this section, we illustrate theorem related with Logistic distn. given in the [13] without proof.

1.6.1 Theorem (1.1) :

If the r.v. $X \sim Exp(1)$, then the r.v. $Y = -Ln\left[\frac{e^{-x}}{1-e^{-x}}\right] \sim L(0, 1)$.

Proof:

Given $X \sim Exp(1)$, then the p.d.f of X is:

$$f(x) = \begin{cases} e^{-x}, & 0 < x < \infty \\ 0, & e.w. \end{cases}$$

The function $y = -Ln\left[\frac{e^{-x}}{1-e^{-x}}\right]$ define a one-to-one transformation that maps the space $A = \{x : 0<x<\infty\}$ onto the space $B = \{y : 0<y<\infty\}$ with inverse:

$$e^{-y} = \frac{e^{-x}}{1-e^{-x}} \Rightarrow e^{-x} = \frac{e^{-y}}{1+e^{-y}}$$

Hence:

$$x = y + Ln(1 + e^{-y}), \text{ and } J = \frac{dx}{dy} = 1 - \frac{e^{-y}}{1+e^{-y}}$$

Then the p.d.f of Y is:

$$g(y) = f(y + \text{Ln}(1 + e^{-y}))\ |J|$$

$$= \frac{e^{-y}}{1+e^{-y}}\left(1-\frac{e^{-y}}{1+e^{-y}}\right)$$

$$= \frac{e^{-y}}{(1+e^{-y})^2}$$, which is the p.d.f of L(0, 1).

Chapter Two

Sampling Techniques for Logistic Distribution by Monte-Carlo Methods

2 Sampling Techniques for Logistic Distribution by Monte-Carlo Methods

2.1 Introduction

After constructing mathematical model for the problem under consideration, the next step is to derive a solution. There are analytic and numerical solution methods. The analytic solution is usually obtained directly from its mathematical representation in the form of a formula, while the numerical solution is generally an approximate solution obtained as a result of substitution of numerical values for the variable and parameters of the model. Many numerical methods are iterative, that is, each successive step in the solution uses the results from the previous step, such as Newton's method for approximating the root of non-linear equation. Two special types of numerical methods simulation and Monte Carlo are designed for a solution of deterministic and stochastic problem. Simulation in a wide sense is defined as a numerical technique for conducting experiments on a digital computer which involve certain types of mathematical and logical models that describes the behavior of a system over extended periods of real time, for example, simulating a football game, supersonic jet flight, a telephone communication system, wind tunnel, a large scale military battle (to evaluate defensive or offensive weapon system), or a maintenance operation (to determine the optimal size of repair crews) and alive application of real equipment in work combat scenarios or firing range, these allow pilots, tank drivers and other soldiers to practice the physical activities of war with their real equipment, etc. .

Whereas, simulation in a narrow sense (also called stochastic simulation) is defined as experimenting with the model over time, it includes sampling stochastic variates from probability distn. Often simulation is viewed as "Method of Last resort" to be used when every things else has failed. Software building and technical development have made simulation one of the most widely used and accepted tools for designers in the system analysis and operational research, [23]. Application areas for simulation are numerous and diverse, we can point out some particular kinds of problems for which simulation has been found to be a useful and powerful tool, [14]:

- Designing and analyzing manufacturing systems.
- Evaluating military weapons systems or their logistics requirements.
- Designing and operating transportation systems such as airports, freeways, ports, and subways.
- Reengineering of business process.
- Determining ordinary policies for an inventory system.
- Analyzing financial or economic system.

The goal of this chapter is to generate random variates from Logistic distn. by using inverse transformation method given by theorem (2.1) and by using theorem (1.1). This chapter involves five sections, in section (2.2) we introduce the historic genesis of Monte Carlo simulation and the uses of its methods. In section (2.3), we illustrate the congruential method and its fundamental relationship model for random number generation. Sections (2.4) show the random variates generation form continuous distn's. which consists of inverse transform method (IT). In section (2.5) we consider two procedures for generating random variates from Logistic distn.

2.2 Monte Carlo Simulation

Historically, the Monte Carlo method was considered as a technique using random or pseudorandom numbers, for solution of a model. The term "Monte Carlo" was introduced by Von Neumann and Ulam during World War II, as a code war for the secret work at Los Alamos; it was suggested by the gambling casinos at the city of Monte Carlo in Monaco, [23].

The idea behind Monte Carlo simulation gained its work to develop the first major use in 1944 in the research work to develop the first atomic bomb, [26].

The general accepted birth date of the Monte Carlo methods is 1949 when the first article entitled "The Monte Carlo Method" by N. Metropolis and S. Ulam appeared in the Journal of the American Statistical Association. The Monte Carlo method is a method of approximately solving mathematical and real life problems "in physics or engineering" by simulation of random quantities, [9].

In the beginning of the 20^{th} century the Monte Carlo method was used to examine the Boltzman equation. In 1908, the famous statistician W. S. Gooset (student) used the Monte Carlo method (experimental sampling) for estimating the correlation coefficient in his distn., [23].

Kolmogorov (1931) showed the relationship between Markov stochastic processes and certain integro-differential equations. About 1948, Fermi, Metropolis and Ulam obtained Monte Carlo estimates the eigenvalues of Schrodinger equation. Shortly thereafter Monte Carlo methods used to evaluate complex multidimensional integrals, stochastic problems, and deterministic problems if they have the same formal expression as some stochastic process. Also, Monte Carlo method is used for solution of certain integrals and differential eq's, sampling of random variates from probability distn's., and for analyzing complex problem (such as radiation transport to rivers), [4]. Monte Carlo results are not efficient for small samples, since the error is of order $N^{-1/2}$, where N is the total number of observations, [9].

2.3 Random Number Generation

Many techniques for generating random numbers on digital computers by Monte Carlo method and simulation have been suggested, tested and used in recent years. Some of these methods are based on random phenomena, others on deterministic recurrence procedures, [23].

Initially manual methods were used to generate a sequence of numbers such as coin flipping, dice rolling, card shuffling, and roulette wheels, but these methods were to slow for general use, and moreover the generated sequence by such methods could not be reproduced. Within the computer aid it becomes possible to obtain random numbers. In (1951) Von Neumann suggested the mid-square method using the arithmetic operations of computer. His idea was to take the square of the preceding number and extract the middle digits. For instance, suppose we wish to generate 4-digits numbers

1. Choose any 4-digits to generate 4-digits numbers, say 4103.
2. Square it to have 16834609.
3. The next 4-digits numbers is the middle 4-digits in step (2), that is 8346.
4. Repeat the process.

The method prove slow and not suitable for statistical analysis, furthermore the sequence tend to cyclicity, and once a zero is encountered the sequence terminates, [23]. One method for generating random numbers on digital computers was published by RAND Corporation (1955); consists of preparing a table of million random digits stored in the computer memory, [4]. The advantage of this method is reproducibility and its disadvantage, was its slow and the risk of exhausting the table. It mentioned in the literature that the random numbers generated by any method is a "good one" if the random numbers are uniformly distributed, statistically independent and reproducible; moreover the method is necessarily fast and requires minimum capacity in the computer memory. The congruential methods of generating pseudorandom numbers are designed specifically to satisfy as many of the above requirements as possible.

These methods produce a nonrandom sequence of numbers according to some recursive formula based on calculating the residues module of some integer m of a linear transformation. The congruential methods are based on a fundamental congruence relationship which may be formulated as:

$$x_{i+1} = (ax_i + c)(\text{mod } m),\ 0 \le x \le m,\ i = 1, 2, \ldots, m. \qquad \ldots(2.1)$$

where a is the multiplier, c is the increment and m is the modulus (a, c, m are nonnegative integers), (mod m) mean that eq.(2.1) can be written as:

$$x_{i+1} = ax_i + c - m[z]\ , \qquad \ldots(2.2)$$

where $[z] = \left[\frac{ax_i + c}{m}\right]$ is the largest integer in z. Given an initial starting value x_1 with fixed values of a, c and m; then eq.(2.2) yields congruence relationship (modulo m) for any values i of the sequence $\{x_i\}$. The sequence $\{x_i\}$ will repeat itself in at most m steps and will be therefore periodic.

For example:

Let $a = c = x_1 = 5$, and $m = 9$, then the sequence obtained from the recursive formula:

$$x_{i+1} = (5x_i + 5)(\text{mod } 9)$$

is:

$$x_i = 5, 3, 2, 6, 8, 0, 5, \ldots$$

The random number on the unit interval [0, 1] can be obtained by:

$$U_i = \frac{x_i}{m},\ i = 1, 2, \ldots, m \qquad \ldots(2.3)$$

It follows from eq.(2.3) that $x_i \le m$, $\forall$ i, this inequality means that the period of the generator cannot exceed m, that is, the sequence {xi} contains at most m distinct numbers. So, we should choose m as large as possible to ensure, a sufficiently large sequence of distinct numbers in the cycle. It is noted in the literature, [14] that good statistical results can be achieved from computers by choosing $a = 2^7+1$, $c = 1$ and $m = 2^{35}$.

2.4 Random Variates Generation from Continuous Distribution

Many methods and procedures are proposed in the literatures for generating random numbers from different distributions. We shall utilize the inverse transform method, (IT).

2.4.1 Inverse Transform Method:

One of the more useful ways of generating random variates is through the inverse transformation techniques which are based on the following theorem [12]:

Theorem (2.1), [26]:

The random variable $U = F(X) \sim U(0, 1)$ if and only if the random variable $X = F^{-1}(U)$ has c.d.f $pr(X \le x) = F(x)$.

Proof:

Let the random variable $U = F(X) \sim U(0, 1)$ then U has c.d.f

$$G(u) = pr(U \le u) = \begin{cases} 0, & u \le 0 \\ u, & 0 < u < 1 \\ 1, & u \ge 1 \end{cases}$$

Now:

$$pr(X \le x) = pr[F^{-1}(U) \le x] = pr[U \le F(x)] = F(x)$$

Conversely, let the random variable X has c.d.f pr $(X \le x) = F(x)$ and let g(u) be the c.d.f of U ,then

$$G(u) = pr(U \le u) = pr[F(X) \le u] = pr[X \le F^{-1}(u)]$$

$$= F[F^{-1}(u)] = u.$$

The algorithm of generating random variates by inverse transform method can be described by the steps of IT-algorithm:

IT-Algorithm:

1. Generate U from U(0, 1).
2. Set $X = F^{-1}(U)$.
3. Deliver X as a random variable generated from the p.d.f f(x).
4. Stop.

As an application of IT-Algorithm, we shall consider the following examples:

Example (2.1):[26]

Consider, we wish to generate a r.v. X from C(0, 1), where the distn. p.d.f:

$$f(x) = \frac{1}{\pi(1+x^2)}, \ -\infty < x < \infty$$

then, the c.d.f of this p.d.f

$$F(x) = pr(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \frac{1}{\pi}\int_{-\infty}^{x} \frac{1}{1+t^2}\,dt$$

$$= \frac{\tan^{-1}(x)}{\pi} + \frac{1}{2}$$

Set u = F(x), implies:

$$x = \tan\left[\pi\left(u - \frac{1}{2}\right)\right]$$

Apply IT-Algorithm:

1. Generate U from U(0, 1).
2. Set $X = \tan\left[\pi\left(U - \frac{1}{2}\right)\right]$.
3. Deliver X as a random variable generated from $f(x) = \frac{1}{\pi(1+x^2)}$.
4. Stop.

Example (2.2), [26]:

If a r.v. X required from the distn. whose distn. p.d.f:

$$f(x) = \frac{1}{2}e^{-|x|} = \begin{cases} \frac{1}{2}e^{x}, & -\infty < x \le 0 \\ \frac{1}{2}e^{-x}, & 0 < x < \infty \end{cases}$$

then, the c.d.f of this p.d.f is:

$$F(x) = pr(X \le x) = \begin{cases} 0, & x \longrightarrow -\infty \\ \int_{-\infty}^{x} f(t)dt, & -\infty < x \le 0 \\ \int_{-\infty}^{0} f(t)dt + \int_{0}^{x} f(t)dt, & 0 \le x < \infty \\ 1, & x \longrightarrow \infty \end{cases}$$

So:

$$F(x) = \begin{cases} 0, & x \longrightarrow -\infty \\ \frac{1}{2}e^{x}, & -\infty < x \le 0 \\ 1 - \frac{1}{2}e^{-x}, & 0 \le x < \infty \\ 1, & x \longrightarrow \infty \end{cases}$$

For $-\infty < x \le 0$, set $u = F(x) \Rightarrow u = \frac{1}{2}e^{x}$, implies:

$$x = Ln(2u), \text{ for } 0 < u < \frac{1}{2}$$

For $0 \le x < \infty$, set $u = F(x) \Rightarrow u = 1 - \frac{1}{2}e^{-x}$, implies:

$$x = -Ln(2u), \text{ for } \frac{1}{2} \le u < 1$$

Apply IT-Algorithm:

1. Generate U from U(0, 1).
2. If $0 < U < \frac{1}{2}$ set $X = Ln(2U)$: go to step (4).
3. Else, set $X = -Ln(2U)$.

4. Deliver X as a random variable generated from $f(x)=\frac{1}{2}e^{-|x|}$, $-\infty<x<\infty$.

5. Stop.

To apply the inverse transform method, the c.d.f F(x) must exist in a form for which the corresponding inverse transform can be solve analytically.

Some probability distn., it's either impossible or difficult to find the inverse transform , that is, to solve, $u = F(x) = \int_{-\infty}^{x} f(t)\,dt$.

For example:

1. $X \sim Exp(\lambda)$, where $f(x) = \frac{1}{\lambda}e^{-x/\lambda}$, $0 < x < \infty$ (possible).
2. $X \sim G(2, 1)$, where $f(x) = xe^{-x}$, $0 < x < \infty$ (difficult).
3. $X \sim N(0, 1)$, where $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $-\infty < x < \infty$ (impossible).

2.5 Procedures for generating Random Variates of Logistic Distribution

In this section, we shall consider the procedures for generating random variates from Logistic distn. by utilizing theorems (1.1) and theorem (2.1).

2.5.1 Procedure (L-1):

This procedure is based on Inverse Transform method given by theorem (2.1):

$$f(x) = \begin{cases} \dfrac{e^{-\left(\frac{x-\alpha}{\beta}\right)}}{\beta\left[1+e^{-\left(\frac{x-\alpha}{\beta}\right)}\right]^2}, & -\infty < x < \infty, -\infty < \alpha < \infty, \beta > 0 \\ 0, & \text{e.w.} \end{cases}$$

The distn. c.d.f is:

$$F(x) = pr(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \frac{1}{\beta}\int_{-\infty}^{x} \frac{e^{-\left(\frac{t-\alpha}{\beta}\right)}}{\left[1+e^{-\left(\frac{t-\alpha}{\beta}\right)}\right]^2}\,dt$$

Implies:

$$F(x) = \begin{cases} 0, & x \longrightarrow -\infty \\ \dfrac{1}{1+e^{-\left(\frac{x-\alpha}{\beta}\right)}}, & -\infty < x < \infty, -\infty < \alpha < \infty, \beta > 0 \\ 1, & x \longrightarrow \infty \end{cases}$$

Setting $u = F(x)$ implies $u = \dfrac{1}{1+e^{-\left(\frac{x-\alpha}{\beta}\right)}}$, implies that:

$$x = \alpha - \beta \mathrm{Ln}\left(\frac{1-u}{u}\right)$$

The (L-1) algorithm describes the necessary steps for generating random variates by the inverse transform method.

Algorithm (L-1):

1. Read α, β.
2. Generate U form U(0, 1).
3. Set $X = \alpha - \beta \mathrm{Ln}\left(\frac{1-U}{U}\right)$.
4. Deliver X as a r.v. generated from L(α, β).
5. Stop.

2.5.2 Procedure (L-2):

This procedure is based on theorem (1.1) and the L-2 algorithm describe the steps of generation.

Algorithm (L-2):

1. Read α, β.
2. Generate U form U(0, 1).
3. Set $X = -\mathrm{Ln}(1 - U)$.
4. Set $Y = -\mathrm{Ln}\left(\frac{e^{-X}}{1-e^{-X}}\right) = X + \mathrm{Ln}(1 - e^{-X})$.
5. Set $Z = \alpha + \beta Y$.
6. Deliver Z as a r.v. generated from L(α, β).
7. Stop.

Chapter Three

Monte Carlo Results

3 Monte Carlo Results

3.1 Introduction

In this chapter, a large scale of samples generated from the Logistic distn. by procedures (L-1) and (L-2) of chapter two where the sample sizes n = 5(1) 10(2) 20(5) 30 and the run size m = 500 is used.

These samples are used to estimate the parameters of the Logistic distn. by the four methods of estimation namely, Moments method, Maximum likelihood method, Modified moments method and Least squares method. Moments properties of the estimators, such as bias, variance, skewness, and kurtosis are tabulated, efficiency of the methods are illustrated and discussed by using mean square error measurements.

3.2 The Estimates of the Parameters Using Procedure (L-1)

To access the results obtained by the four methods of estimation, we generate samples of different sizes from Logistic distn. by the inverse transform method.

A computer program (5) of Appendix (B) uses procedure (L-1) of section (2.5.1) is made which utilize the Inverse Transform Method, which generate samples from L(0, 1).

The samples of program (5) of Appendix (B) is used in Appendix (A) in programs (1), (2), (3), and (4) for estimating the unknown parameters of L(0, 1)

The estimators by the four methods of estimations are displayed in table (3.1).

Table (3.1)

Parameters estimation.

Sample size n	***Estimation of*** $(\hat{a}, \hat{b})$			
	M.M	***M.L.M***	***M.M.M***	***L.S.M***
5	(0.054, 1.018)	(0.1, 1.626)	(0.054, 0.97)	(0.045, 0.963)
6	(–0.026, 1.01)	(0.09, 1.654)	(–0.026, 1.106)	(0.012, 0.967)
7	(–0.037, 1.009)	(0.0941, 1.528)	(–0.037, 1.02)	(0.032, 0.968)
8	(–0.023, 1.012)	(–0.071, 1.434)	(–0.023, 0.96)	(0.028, 0.999)
9	(–0.016, 1.001)	(–0.032, 1.428)	(–0.016, 1.03)	(8.204×10^{-3},1.003)
10	(9.794×10^{-3},1.107)	(0.044, 1.415)	(9.794×10^{-3},1.061)	(9.719×10^{-3},1.003)
12	(–0.023, 0.998)	(–0.014, 1.301)	(–0.023, 1.035)	(–0.34, 0.986)
14	(-3.405×10^{-3},1.032)	(–0.038, 1.34)	(3.405×10^{-3}, 1.017)	(4.667×10^{-3},0.99)
16	(0.035, 1.011)	(0.017, 1.271)	(0.035, 1.055)	(–0.015, 0.996)
18	(0.036, 1.032)	(–0.031, 1.236)	(0.036, 0.99)	(–0.013, 0.97)
20	(–0.023, 1.005)	(0.018, 1.201)	(–0.023, 0.968)	(-6.648×10^{-3},1.003)
25	(–0.01, 0.997)	(0.028, 1.167)	(–0.01, 1.046)	(5.313×10^{-4},0.974)
30	(2.032×10^{-3},1.011)	(0.02, 1.112)	(2.032×10^{-3},1.019)	(0.047, 0.971)

Table(3.1) show that methods M.M, M.M.M ,and L.S.M give a good agreement between the true values of the parameters α=0 and β=1 with the estimators $(\hat{\alpha})$ and $(\hat{\beta})$ for all sample sizes, while the M.L.M give higher values for small and moderate sample sizes and become ad gate for large sample sizes.

3.3 The Bias of Estimators Using Procedure (L-1)

The biases of estimators $\hat{\alpha}$ and $\hat{\beta}$ which can be obtained by:

$$\text{Bias}(\hat{\alpha}) = \hat{\alpha} - \alpha$$

$$\text{Bias}(\hat{\beta}) = \hat{\beta} - \beta$$

Tables (3.2) and (3.3) show the biases of estimators ($\hat{\alpha}$) and ($\hat{\beta}$) obtained by the four methods of estimation:

Table (3.2)
Bias of Estimator ($\hat{a}$).

Sample size n	Bias of Estimation ($\hat{a}$)			
	M.M	M.L.M	M.M.M	L.S.M
5	0.054	0.1	0.054	0.045
6	–0.026	0.09	–0.026	0.12
7	–0.037	0.094	–0.037	0.032
8	–0.023	–0.071	–0.023	0.028
9	–0.016	–0.032	–0.016	8.204×10^{-3}
10	9.794×10^{-3}	0.044	9.794×10^{-3}	9.719×10^{-3}
12	–0.023	–0.014	–0.023	–0.034
14	-3.405×10^{-3}	–0.038	-3.405×10^{-3}	5.667×10^{-3}
16	0.035	0.017	0.035	–0.015
18	0.036	–0.031	0.036	–0.013
20	–0.023	0.018	–0.023	-6.648×10^{-3}
25	–0.01	0.028	–0.01	5.313×10^{-4}
30	2.023×10^{-3}	0.02	2.023×10^{-3}	0.047

Table (3.3)
Bias of Estimator ($\hat{b}$).

Sample size n	Bias of Estimation ($\hat{b}$)			
	M.M	M.L.M	M.M.M	L.S.M
5	0.018	0.626	–0.03	–0.064
6	0.01	0.654	0.106	–0.033
7	0.009	0.528	0.02	–0.032
8	0.012	0.434	–0.04	–0.001
9	0.001	0.428	0.03	0.003
10	0.017	0.415	0.061	0.003
12	–0.002	0.301	0.035	–0.04
14	0.032	0.34	0.017	–0.01
16	0.011	0.271	0.055	–0.004
18	0.032	0.236	–0.001	–0.03
20	0.005	0.201	–0.032	0.003
25	–0.003	0.197	0.046	–0.026
30	0.011	0.152	0.019	–0.029

3.4 The Variance of Estimators Using Procedure (L-1)

The variances of estimator ($\hat{\alpha}$) are shown in table (3.4), where the true values of variances are given:

1- Equation (1.23) by moments method.

2- Equation (1.41) by modified moments method.

3- Equation (1.57) by least squares method.

Table (3.4) show the variance of estimator ($\hat{\alpha}$) where the true value of variance ($\hat{\alpha}$) are shown in parenthesis.

Table (3.4)

Variance of Estimator ($\hat{a}$).

Sample size n	Variance of Estimation ($\hat{a}$)			
	M.M	M.L.M	M.M.M	L.S.M
5	0.715 (0.658)	0.141	0.715 (0.658)	0.024 (0.048)
6	0.604 (0.548)	0.146	0.604 (0.548)	0.0202 (0.033)
7	0.389 (0.4703)	0.214	0.389 (0.4703)	0.189 (0.036)
8	0.415 (0.4115)	0.202	0.415 (0.4115)	0.286 (0.034)
9	0.176 (0.365)	0.335	0.176 (0.365)	0.259 (0.014)
10	0.351 (0.3922)	0.272	0.351 (0.3922)	0.187 (0.011)
12	0.264 (0.274)	0.184	0.264 (0.274)	0.131 (7.536×10^{-3})
14	0.205 (0.235)	0.126	0.205 (0.235)	0.143 (5.719×10^{-3})
16	0.259 (0.205)	0.125	0.259 (0.205)	0.131 (4.179×10^{-3})
18	0.282 (0.1829)	0.212	0.282 (0.1829)	0.064 (3.308×10^{-3})
20	0.138 (0.1646)	0.227	0.138 (0.1646)	0.091 (2.633×10^{-3})
25	0.118 (0.1317)	0.091	0.118 (0.1317)	0.028 (1.676×10^{-3})
30	0.16 (0.1097)	0.071	0.16 (0.1097)	0.085 (1.201×10^{-3})

Table (3.4) show that the variances values of the estimator $\hat{\alpha}$ are close to the true variance values given by eq.(1.23) and (1.41),while the variances values given by L.S.M are not satisfactory

The variances of estimator ($\hat{\beta}$) are shown in table (3.5), where the true values of variances are given:

1- Equation (1.25) by moments method.

2- Equation (1.43) by modified moments method.

3- Equation (1.55) by least squares method.

Table (3.5) show the variance of estimator ($\hat{\beta}$) where the true value of variance ($\hat{\beta}$) are shown in parenthesis.

Table (3.5)

Variance of Estimator ($\hat{b}$).

Sample size n	***Variance of Estimation ($\hat{b}$)***			
	M.M	***M.L.M***	***M.M.M***	***L.S.M***
5	0.353 (0.688)	0.564	0.197 (0.521)	0.081 (0.036)
6	0.468 (0.645)	0.495	0.124 (0.42)	0.065 (0.075)
7	0.213 (0.609)	0.435	0.236 (0.356)	0.086 (7.86×10^{-3})
8	0.125 (0.578)	0.317	0.208 (0.31)	0.073 (0.015)
9	0.232 (0.55)	0.306	0.109 (0.277)	0.089 (0.018)
10	0.261 (0.526)	0.276	0.122 (0.252)	0.084 (0.011)
12	0.102 (0.487)	0.271	0.121 (0.215)	0.077 (0.015)
14	0.113 (0.454)	0.178	0.072 (0.19)	0.055 (3.008×10^{-3})
16	0.054 (0.428)	0.154	0.068 (0.172)	0.048 (1.535×10^{-3})
18	0.067 (0.405)	0.109	0.055 (0.158)	0.046 (1.001×10^{-3})
20	0.061 (0.386)	0.075	0.047 (0.147)	0.039 (1.499×10^{-3})
25	0.042 (0.348)	0.063	0.032 (0.127)	0.026 (9.356×10^{-4})
30	0.028 (0.319)	0.049	0.027 (0.114)	8.523×10^{-3} (2.779×10^{-4})

Table (3.5) show that the variances values of the estimator ($\hat{\beta}$) are less than true values given by eq.(1.25) and (1.43) while the L.S.M is higher than the values given by eq.(1.55).

3.5 The Skewness of Estimators Using Procedure (L-1)

The skewness of estimators ($\hat{\alpha}$) and ($\hat{\beta}$) which can be obtained by:

$$\text{Skewness }(\hat{\alpha}) = \frac{\frac{1}{n}\left[\sum_{i=1}^{n}(\alpha_i)^3 - 3\hat{\alpha}\sum_{i=1}^{n}(\alpha_i)^2 + 3\hat{\alpha}^2\sum_{i=1}^{n}(\alpha_i) - \hat{\alpha}^3\right]}{(\sigma^2)^{3/2}}$$

$$\text{Skewness }(\hat{\beta}) = \frac{\frac{1}{n}\left[\sum_{i=1}^{n}(\beta_i)^3 - 3\hat{\beta}\sum_{i=1}^{n}(\beta_i)^2 + 3\hat{\beta}^2\sum_{i=1}^{n}(\beta_i) - \hat{\beta}^3\right]}{(\sigma^2)^{3/2}}$$

Tables (3.6) and (3.7) show the skewness of estimators ($\hat{\alpha}$) and ($\hat{\beta}$) by the four methods of estimation.

Table (3.6)

Skewness of Estimator ($\hat{a}$).

Sample size n	Skewness of Estimation ($\hat{a}$)			
	M.M	M.L.M	M.M.M	L.S.M
5	0.419	1.284	0.419	0.655
6	0.462	1.279	0.462	1.099
7	–0.952	0.131	–0.952	0.507
8	1.262	–0.5	1.262	0.085
9	0.779	–0.477	0.779	–0.714
10	–0.836	0.149	–0.836	0.799
12	0.921	0.893	0.921	–1.165
14	–0.189	0.927	–0.189	–0.957
16	0.336	–0.85	0.336	–1.471
18	–0.202	–0.843	–0.202	–0.794
20	0.794	0.276	0.794	–0.927
25	–0.601	–0.895	–0.601	–0.197
30	0.338	–0.655	0.338	–0.248

Table (3.6) show that all methods give a very little skewness to left and to the right which indicate that the estimator ($\hat{\alpha}$) approach rapidly to normal distn.

Table (3.7)

Skewness of Estimator ($\hat{b}$).

Sample size n	Skewness of Estimation ($\hat{b}$)			
	M.M	**M.L.M**	**M.M.M**	**L.S.M**
5	4.406	7.265	8.872	27.69
6	3.587	9.898	25.923	44.039
7	10.158	9.706	8.996	30.644
8	21.124	13.469	8.812	44.189
9	9.113	14.26	27.18	33.417
10	8.265	16.438	25.123	36.578
12	28.471	19.256	24.758	42.743
14	27.072	28.659	51.121	69.608
16	76.928	30.753	62.471	88.056
18	62.002	47.952	73.752	88.823
20	64.683	79.379	86.649	125.656
25	110.405	104.137	195.319	210.945
30	213.813	134.567	233.612	1.125×10^3

Table (3.7) show that when the sample size increase the skewness loose centrality

3.6 The Kurtosis of Estimators Using Procedure (L-1)

The kurtosis of estimators ($\hat{\alpha}$) and ($\hat{\beta}$) which can be obtained by:

$$\text{Kurtosis }(\hat{\alpha})=\frac{\frac{1}{n}\left[\sum_{i=1}^{n}(\alpha_i)^4-4\hat{\alpha}\sum_{i=1}^{n}(\alpha_i)^3+6\hat{\alpha}^2\sum_{i=1}^{n}\alpha_i-4\hat{\alpha}^3\sum_{i=1}^{n}\alpha_i\right]}{(\sigma^2)^2}-3$$

$$\text{Kurtosis }(\hat{\beta})=\frac{\frac{1}{n}\left[\sum_{i=1}^{n}(\beta_i)^4-4\hat{\beta}\sum_{i=1}^{n}(\beta_i)^3+6\hat{\beta}^2\sum_{i=1}^{n}\beta_i-4\hat{\beta}^3\sum_{i=1}^{n}\beta_i\right]}{(\sigma^2)^2}-3$$

Tables (3.8) and (3.9) show the kurtosis of estimators ($\hat{\alpha}$) and ($\hat{\beta}$) by the four methods of estimation.

Table (3.8)

Kurtosis of Estimator ($\hat{a}$).

Sample size n	Kurtosis of Estimation ($\hat{a}$)			
	M.M	***M.L.M***	***M.M.M***	***L.S.M***
5	–1.722	–0.78	–1.722	–1.992
6	–1.82	–0.793	–1.82	–1.008
7	–0.093	–1.95	–0.093	–0.728
8	–0.746	–1.683	–0.746	–0.99
9	–1.335	–1.505	–1.335	–1.737
10	–0.81	–1.883	–0.81	–1.758
12	–0.862	–1.414	–0.862	–0.307
14	–0.638	–0.5	–0.638	–0.443
16	–1.274	–1.357	–1.274	–0.215
18	–1.109	–0.641	–1.109	–0.899
20	–0.678	–0.892	–0.678	–0.51
25	–0.14	–0.578	–0.14	–0.409
30	–0.687	–0.441	–0.687	–0.193

Table (3.8) show that all methods give a little kurtosis which make the maximum point of the distn. osculate up and down on the y-axis which indicate that the estimator ($\hat{\alpha}$) have limiting N (0,1) distn.

Table (3.9)

Kurtosis of Estimator ($\hat{b}$).

Sample size n	Kurtosis of Estimation ($\hat{b}$)			
	M.M	M.L.M	M.M.M	L.S.M
5	–8.74	–19.632	–20211	–95.705
6	–4.567	–27.333	–85.24	–171.762
7	–20.225	–26.59	–17.09	–104.117
8	–60.346	–38.671	–18.476	–165.637
9	–17.393	–41.203	–84.912	–115.33
10	–13.639	–48.762	–77.398	–131.452
12	–86.414	–57.92	–72.376	–143.444
14	–48.016	–95.9	–19.449	–289.346
16	–330.577	–104.754	–252.579	–402.27
18	–239.972	–184.851	–308.5	–395.159
20	–258.073	–355.288	–383.713	–643.913
25	–533.601	–505.262	-1.147×10^{3}	-1.27×10^{3}
30	-1.292×10^{3}	–707.416	-1.459×10^{3}	-1.184×10^{4}

Table (3.9) show that all methods give higher kurtosis which make osculate the y-axis which indicate that maximum value of ($\hat{\beta}$) unstable.

3.7 Mean Square Error of Estimators Using Procedure (L-1)

The mean square error of estimators ($\hat{\alpha}$) and ($\hat{\beta}$) which can be obtained by:

$$\text{m.s.e}\,(\hat{\alpha}) = \text{Variance}(\hat{\alpha}) + [\text{bias}(\hat{\alpha})]^2$$

$$\text{m.s.e}\,(\hat{\beta}) = \text{Variance}(\hat{\beta}) + [\text{bias}(\hat{\beta})]^2$$

Tables (3.10) and (3.11) show the mean square error of estimators ($\hat{\alpha}$) and ($\hat{\beta}$) by the four methods of estimation.

Table (3.10)

Mean square error of Estimator ($\hat{a}$).

Sample size n	Mean square error of Estimation ($\hat{a}$)			
	M.M	M.L.M	M.M.M	L.S.M
5	0.717	0.151	0.717	0.026
6	0.605	0.154	0.605	0.202
7	0.39	0.222	0.39	0.19
8	0.416	0.207	0.416	0.287
9	0.177	0.336	0.177	0.259
10	0.351	0.273	0.351	0.17
12	0.262	0.189	0.262	0.132
14	0.205	0.127	0.205	0.143
16	0.26	0.125	0.26	0.131
18	0.283	0.212	0.283	0.064
20	0.139	0.227	0.139	0.091
25	0.118	0.091	0.118	0.082
30	0.16	0.071	0.16	0.087

Table (3.10) show that the m.s.e of ($\hat{\alpha}$) is very good by using L.S.M in comparison with other methods.

Table (3.11)

Mean square error of Estimator ($\hat{b}$).

Sample size n	Mean square error of Estimation ($\hat{b}$)			
	M.M	**M.L.M**	**M.M.M**	**L.S.M**
5	0.353	0.956	0.198	0.085
6	0.468	0.923	0.136	0.066
7	0.213	0.714	0.236	0.087
8	0.125	0.505	0.209	0.073
9	0.232	0.484	0.11	0.089
10	0.261	0.448	0.126	0.084
12	0.102	0.307	0.122	0.077
14	0.114	0.293	0.073	0.056
16	0.054	0.227	0.071	0.048
18	0.068	0.165	0.055	0.047
20	0.061	0.115	0.048	0.039
25	0.042	0.101	0.034	0.027
30	0.028	0.072	0.019	09349×10^{3}

Table (3.11) show that the m.s.e of ($\hat{\beta}$) is very good by using L.S.M in comparison with other methods.

3.8 Parameters Estimation Using Procedure (L-2)

To access the results obtained by the four methods of estimation, we generate samples of different sizes from Logistic distn. by procedure (L-2). A computer program (6) of Appendix (B) uses procedure (L-2) of section (2.5.2) is made which utilize the procedure (L-2), which generate samples from L(0, 1).

The samples of program (6) of Appendix (B) is used in Appendix (A) in programs (1), (2), (3), and (4) for estimating the unknown parameters of L(0, 1)

The estimators by the four methods of estimations are displayed in table (3.12).

Table (3.12)
Parameters estimation

Sample size n	***Estimation*** $(\hat{a}, \hat{b})$			
	M.M	***M.L.M***	***M.M.M***	***L.S.M***
5	$(-2.263\times10^{-3}$,1.048)	(0.31,1.802)	$(-2.263\times10^{-3}$,1.087)	$(1.515\times10^{-3}$,0.966)
6	$(-1.11\times10^{-4}$,1.064)	(0.293,1.764)	$(-1.11\times10^{-4}$,1.098)	$(6.743\times10^{-3}$,0.951)
7	(–0.033,1.013)	(0.265,1.711)	(–0.033, 1.056)	(–0.013,0.9691)
8	(–0.038,1.011)	(0.227,1.689)	(–0.038,1.056)	(0.013,0.975)
9	(–0.038,1.006)	(0.201,1.624)	(–0.038,1.043)	(0.047,0.949)
10	$(5.535\times10^{-3}$,1.039)	(0.198,1.58)	$(5.535\times10^{-3}$,0.98)	(–0.014,0.984)
12	$(-7.051\times10^{-3}$,1.015)	(0.203,1.421)	$(-7.051\times10^{-3}$, 1.032)	(0.013,0.973)
14	(0.016,1.017)	(0.199,1.491)	(0.016,1.041)	$(7.801\times10^{-3}$,0.981)
16	(–0.02,0.993)	(0.107,1.424)	(–0.02,1.011)	$(1.936\times10^{-3}$,0.979)
18	$(-9.484\times10^{-3}$,1.01)	(0.101,1.373)	$(-9.484\times10^{-3}$,1.022)	(–0.035,0.992)
20	(–0.021,0.998)	(0.09,1.332)	(–0.021,1.016)	(0.032,1)
25	(0.021,1.004)	(0.081,1.269)	(0.021,1.005)	(–0.028,0.984
30	$(2.073\times10^{-3}$,1.007)	(0.07,1.223)	$(2.073\times10^{-3}$,1.002)	$(-5.046\times10^{-3}$,0.996)

Table(3.12) show that methods M.M, M.M.M ,and L.S.M give a good agreement between the true values of the parameters α=0 and β=1 with the estimators ($\hat{\alpha}$) and ($\hat{\beta}$) for all sample sizes, while the M.L.M give higher values for small and moderate sample sizes and become ad gate for large sample sizes.

3.9 The Bias of Estimators Using Procedure (L-2)

The biases of estimators $\hat{\alpha}$ and $\hat{\beta}$ which can be obtained by:

Bias $(\hat{\alpha}) = \hat{\alpha} - \alpha$

Bias $(\hat{\beta}) = \hat{\beta} - \beta$

Tables (3.13) and (3.14) show the biases of estimators $\hat{\alpha}$ and $\hat{\beta}$ obtained by the four methods of estimation:

Table (3.13)

Bias of Estimator ($\hat{a}$).

Sample size n	Bias of Estimation ($\hat{a}$)			
	M.M	**M.L.M**	**M.M.M**	**L.S.M**
5	-2.263×10^{-3}	0.31	-2.263×10^{-3}	1.515×10^{-3}
6	-1.11×10^{-4}	0.293	-1.11×10^{-4}	6.743×10^{-3}
7	–0.033	0.265	–0.033	–0.013
8	–0.038	0.227	–0.038	0.013
9	–0.038	0.201	–0.038	0.047
10	5.535×10^{-3}	0.198	5.535×10^{-3}	–0.014
12	-7.051×10^{-3}	0.203	-7.051×10^{-3}	0.013
14	0.016	0.199	0.016	7.801×10^{-3}
16	–0.02	0.107	–0.02	1.936×10^{-3}
18	-9.484×10^{-3}	0.101	-9.484×10^{-3}	–0.035
20	–0.021	0.09	–0.21	0.032
25	0.021	0.081	0.021	–0.028
30	2.072×10^{-3}	0.07	2.073×10^{-3}	-5.046×10^{-3}

Table (3.14)

Bias of Estimator ($\hat{b}$).

Sample size n	Bias of Estimation ($\hat{b}$)			
	M.M	M.L.M	M.M.M	L.S.M
5	0.048	0.802	0.087	–0.034
6	0.064	0.764	0.098	–0.049
7	0.013	0.711	0.056	–0.031
8	0.011	0.689	0.056	–0.025
9	0.006	0.624	0.043	–0.051
10	0.039	0.58	–0.02	–0.016
12	0.015	0.421	0.032	–0.027
14	0.017	0.491	0.041	–0.019
16	–0.007	0.424	0.011	–0.021
18	0.01	0.373	0.022	–0.008
20	–0.002	0.332	0.016	0
25	0.004	0.269	0.005	–0.016
30	0.007	0.223	0.002	–0.004

3.10 The Variance of Estimators Using Procedure (L-2)

The variances of estimator ($\hat{\alpha}$) are shown in table (3.15), where the true values of variances are given:

1- Equation (1.23) by moments method.

2- Equation (1.41) by modified moments method.

3- Equation (1.57) by least squares method.

Table (3.15) show the variance of estimator ($\hat{\alpha}$) where the true value of variance ($\hat{\alpha}$) are shown in parenthesis.

Table (3.15)

Variance of Estimator ($\hat{a}$).

Sample size n	Variance of Estimation ($\hat{a}$)			
	M.M	M.L.M	M.M.M	L.S.M
5	0.94(0.658)	0.879	0.94(0.658)	0.181(0.048)
6	0.88(0.548)	0.772	0.88(0.548)	0.164(0.033)
7	0.442(0.4703)	0.614	0.442(0.4703)	0.153(0.036)
8	0.49(0.4115)	0.651	0.49(0.4115)	0.089(0.034)
9	0.475(0.365)	0.551	0.475(0.365)	0.119(0.014)
10	0.345(0.3922)	0.475	0.345(0.3922)	0.171(0.011)
12	0.368(0.274)	0.451	0.368(0.274)	0.091(7.536×10^{-3})
14	0.273(0.235)	0.373	0.273(0.235)	0.098(5.719×10^{-3})
16	0.21(0.205)	0.201	0.21(0.205)	0.079(4.179×10^{-3})
18	0.19(0.1829)	0.173	0.19(0.1829)	0.093(3.308×10^{-3})
20	0.155(0.1646)	0.146	0.155(0.1646)	0.096(2.633×10^{-3})
25	0.132(0.1317)	0.114	0.132(0.1317)	0.075(1.676×10^{-3})
30	0.108(0.1097)	0.074	0.108(0.1097)	0.07(1.201×10^{-3})

Table (3.15) show that the variances values of the estimator $\hat{\alpha}$ are close to the true variance values given by eq.(1.23) and (1.41),while the variances values given by L.S.M are not satisfactory

The variance of estimator ($\hat{\beta}$) are shown in table (3.16), where the true values of variances are given:

1- Equation (1.25) by moments method.

2- Equation (1.43) by modified moments method.

3- Equation (1.55) by least squares method.

Table (3.16) show the variance of estimator ($\hat{\beta}$) where the true value of variance ($\hat{\beta}$) are shown in parenthesis.

Table (3.16)

Variance of Estimator ($\hat{b}$).

Sample size n	Variance of Estimation ($\hat{b}$)			
	M.M	M.L.M	M.M.M	L.S.M
5	0.358(0.688)	0.823	0.447(0.521)	0.024(0.036)
6	0.428(0.645)	0.688	0.362(0.42)	0.053(0.075)
7	0.334(0.609)	0.793	0.201(0.356)	0.047(0.015)
8	0.219(0.578)	0.63	0.215(0.31)	0.081(0.018)
9	0.284(0.55)	0.645	0.143(0.277)	0.057(0.011)
10	0.145(0.526)	0.417	0.118(0.252)	0.094(0.015)
12	0.118(0.487)	0.104	0.12(0.215)	0.041(3.214×10^{-3})
14	0.149(0.454)	0.101	0.107(0.19)	0.076(3.008×10^{-3})
16	0.091(0.428)	0.067	0.1(0.172)	0.056(1.535×10^{-3})
18	0.028(0.405)	0.062	0.078(0.158)	0.059(1.001×10^{-3})
20	0.029(0.386)	0.044	0.05(0.147)	0.04(1.499×10^{-3})
25	0.034(0.348)	0.037	0.041(0.127)	0.029(9.356×10^{-4}
30	0.028(0.319)	0.032	0.025(0.114)	0.032(2.779×10^{-4})

Table (3.16) show that the variances values of the estimator ($\hat{\beta}$) are less than true values given by eq.(1.25) and (1.43) while the L.S.M is higher than the values given by eq.(1.55).

3.11 The Skewness of Estimators Using Procedure (L-2)

The skewness of estimator ($\hat{\alpha}$) and ($\hat{\beta}$) which can be obtained by:

$$\text{Skewness }(\hat{\alpha}) = \frac{\frac{1}{n}\left[\sum_{i=1}^{n}(\alpha_i)^3 - 3\hat{\alpha}\sum_{i=1}^{n}(\alpha_i)^2 + 3\hat{\alpha}^2\sum_{i=1}^{n}(\alpha_i) - \hat{\alpha}^3\right]}{(\sigma^2)^{3/2}}$$

$$\text{Skewness }(\hat{\beta}) = \frac{\frac{1}{n}\left[\sum_{i=1}^{n}(\beta_i)^3 - 3\hat{\beta}\sum_{i=1}^{n}(\beta_i)^2 + 3\hat{\beta}^2\sum_{i=1}^{n}(\beta_i) - \hat{\beta}^3\right]}{(\sigma^2)^{3/2}}$$

Tables (3.17) and (318) show the skewness of estimators ($\hat{\alpha}$) and ($\hat{\beta}$) by the four method of estimation.

Table (3.17)

Skewness of Estimator ($\hat{a}$).

Sample size n	Skewness of Estimation ($\hat{a}$)			
	M.M	**M.L.M**	**M.M.M**	**L.S.M**
5	0.118	–0.649	0.118	–1.393
6	1.302	1.448	1.302	0.461
7	–1.216	–0.9	–1.216	–0.797
8	0.966	–0.048	0.966	–0.918
9	–1.437	–0.986	–1.437	–1.036
10	1.138	–1.099	1.138	–0.122
12	0.58	–1.377	0.58	0.663
14	1.362	–0.19	1.362	0.179
16	–0.498	–0.966	–0.498	–0.345
18	–0.578	–0.633	–0.578	–0.198
20	–0.296	–0.964	–0.296	–0.261
25	1.023	–1.337	1.023	1.364
30	–0.49	–0.387	–0.49	–0.423

Table (3.17) show that all methods give a very little skewness to left and to the right which indicate that the estimator ($\hat{\alpha}$) approach rapidly to normal distn.

Table (3.18)

Skewness of Estimator ($\hat{b}$).

Sample size n	Skewness of Estimation ($\hat{b}$)			
	M.M	M.L.M	M.M.M	L.S.M
5	4.464	5.498	4.466	55.93
6	4.868	7.583	4.81	57.027
7	5.505	5.374	13.109	76.908
8	9.869	7.588	11.368	34.378
9	7.487	6.557	17.894	54.871
10	17.86	12.205	22.653	28.423
12	25.015	45.782	25.013	100.755
14	18.015	50.088	31.777	41.684
16	32.74	82.108	31.873	65.263
18	204.605	86.169	47.202	64.844
20	194.806	141.034	90.796	119.341
25	156.217	174.633	115.448	184.024
30	213.108	215.044	245.577	168.303

Table (3.18) show that when the sample size increase the skewness loose centrality.

3.12 The Kurtosis of Estimators Using Procedure (L-2)

The kurtosis of estimator ($\hat{\alpha}$) and ($\hat{\beta}$) which can be obtained by:

$$\text{Kurtosis }(\hat{\alpha})=\frac{\frac{1}{n}\left[\sum_{i=1}^{n}(\alpha_i)^4-4\hat{\alpha}\sum_{i=1}^{n}(\alpha_i)^3+6\hat{\alpha}^2\sum_{i=1}^{n}\alpha_i-4\hat{\alpha}^3\sum_{i=1}^{n}\alpha_i\right]}{(\sigma^2)^2}-3$$

$$\text{Kurtosis }(\hat{\beta})=\frac{\frac{1}{n}\left[\sum_{i=1}^{n}(\beta_i)^4-4\hat{\beta}\sum_{i=1}^{n}(\beta_i)^3+6\hat{\beta}^2\sum_{i=1}^{n}\beta_i-4\hat{\beta}^3\sum_{i=1}^{n}\beta_i\right]}{(\sigma^2)^2}-3$$

Tables (3.19) and (3.20) show the kurtosis of estimators ($\hat{\alpha}$) and ($\hat{\beta}$) by the four methods of estimation.

Table (3.19)

Kurtosis of Estimator ($\hat{a}$).

Sample size n	Kurtosis of Estimation ($\hat{a}$)			
	M.M	M.L.M	M.M.M	L.S.M
5	–2.041	–1.542	–2.041	–0.416
6	–0.43	0.063	–0.43	–1.822
7	–0.962	–1.951	–0.962	–1.082
8	–0.636	–1.886	–0.636	–1.403
9	–0.292	–1.217	–0.292	–0.822
10	–0.12	–1.307	–0.12	–1.491
12	–0.112	0.218	–0.112	–1.601
14	–0.543	–1.234	–0.543	–0.66
16	–0.302	–1.267	–0.302	–1.296
18	–0.181	–0.757	–0.181	–0.253
20	–0.39	–0.995	–0.39	–0.1
25	–0.092	–0.658	–0.092	–0.01
30	–0.179	–0.806	–0.179	–1.078

Table (3.19) show that all methods give a little kurtosis above and below the y-axis which indicate that the estimator ($\hat{\alpha}$) have limiting N(0,1) distn..

Table (3.20)

Kurtosis of Estimator ($\hat{b}$).

Sample size n	Kurtosis of Estimation ($\hat{b}$)			
	M.M	M.L.M	M.M.M	L.S.M
5	–9.666	–14.722	–6.761	–237.389
6	–5.981	–20.33	–11.27	–240.575
7	–9.426	–14.166	–24.931	–342.492
8	–19.614	–20.198	–24.343	–121.764
9	–11.104	–17.211	–52.764	–223.908
10	–51.418	–34.136	–57.584	–96.273
12	–69.197	–176.244	–74.28	–492.2
14	–44.836	–196.669	–93.867	–149.223
16	–110.737	–370.742	–96.69	–274.647
18	-1.234×10^3	–397.046	–141.146	–268.139
20	-1.15×10^3	–756.141	–412.213	–592.727
25	-8.25×10^3	–996.427	–573.247	-1.065×10^3
30	-1.288×10^3	-1.314×10^3	-1.547×10^3	–933.451

Table (3.20) show that all methods give higher kurtosis above and below the y-axis which indicate that maximum value of ($\hat{\beta}$) lies in the infinity.

3.13 Mean Square Error of Estimators Using Procedure (L-2)

The mean square error of estimators ($\hat{\alpha}$) and ($\hat{\beta}$) which can be obtained by:

$$\text{m.s.e}\ (\hat{\alpha}) = \text{Variance}(\hat{\alpha}) + [\text{bias}(\hat{\alpha})]^2$$

$$\text{m.s.e}\ (\hat{\beta}) = \text{Variance}(\hat{\beta}) + [\text{bias}(\hat{\beta})]^2$$

Tables (3.21) and (3.22) show the mean square error of estimators ($\hat{\alpha}$) and ($\hat{\beta}$) by the four methods of estimation.

Table (3.21)

Mean square error of Estimator ($\hat{a}$).

Sample size n	Mean square error of Estimation ($\hat{a}$)			
	M.M	M.L.M	M.M.M	L.S.M
5	0.94	0.975	0.94	0.181
6	0.88	0.858	0.88	0.164
7	0.443	0.684	0.443	0.153
8	0.491	0.703	0.491	0.089
9	0.476	0.592	0.476	0.122
10	0.345	0.514	0.345	0.171
12	0.368	0.492	0.368	0.091
14	0.273	0.412	0.273	0.098
16	0.21	0.212	0.21	0.079
18	0.19	0.183	0.19	0.094
20	0.156	0.154	0.156	0.097
25	0.132	0.121	0.132	0.075
30	0.108	0.079	0.108	0.07

Table (3.21)show that the m.s.e of ($\hat{\alpha}$) is very good by using L.S.M in comparison with other methods.

Table (3.22)

Mean square error of Estimator ($\hat{b}$).

Sample size n	Mean square error of Estimation ($\hat{b}$)			
	M.M	M.L.M	M.M.M	L.S.M
5	0.36	1.466	0.455	0.056
6	0.432	1.271	0.371	0.056
7	0.334	1.298	0.204	0.048
8	0.219	1.104	0.218	0.082
9	0.284	1.034	0.145	0.06
10	0.147	0.753	0.119	0.095
12	0.119	0.281	0.121	0.042
14	0.15	0.342	0.108	0.076
16	0.091	0.246	0.101	0.057
18	0.028	0.201	0.079	0.059
20	0.029	0.154	0.05	0.04
25	0.034	0.109	0.041	0.029
30	0.028	0.032	0.025	0.032

Table (3.22) show that the m.s.e of ($\hat{\beta}$) is very good by using L.S.M in comparison with other methods.

Conclusions

1. The procedure (L-1) is superior (speed and accuracy) than the procedure (L-2) for generating samples of different sizes from Logistic distn.

2. In procedure (L-1), the values of maximum likelihood estimate are better than the values of procedure (L-2) and as the sample size increase the estimate values in both procedure, because closer the true parameters value.

3. In procedure (L-2), the M.M. and M.M.M. give estimate values to the estimator $\hat{\alpha}$ better than that of procedure (L-1).

4. In procedures (L-1) and (L-2), the M.M. and M.M.M. give variance to the estimator $\hat{\alpha}$ close to the true variance and at the same time these variances are better than the variances of the other methods. While the variances $\hat{\beta}$ in all methods give estimate value higher than the actual true value.

5. In both procedures (L-1) and (L-2), all methods of estimation the skewness of $\hat{\alpha}$ gives values close to zero and that indicate that $\hat{\alpha}$ is almost sure approach normality, while the skewness of $\hat{\beta}$ increase when the sample size increase and have positive values which lead to skewness to right and that indicate that $\hat{\beta}$ loose centrality.

6. In both procedures (L-1) and (L-2) the kurtosis values of the estimator $\hat{\alpha}$ lies in the interval $(-1,\ 0)$, while the kurtosis of $\hat{\beta}$

increase as the sample size increase and have negative value which lead to kurtosis to right as expected in comparison with its skewness mentioned above.

7. L.S.M. gives small m.s.e. of $\hat{\alpha}$ and $\hat{\beta}$ in comparison with other methods.
8. The disadvantage of Monte Carlo methods depends on generating pseudorandom variates and that might carry dirty data, and that might affect the results of M.L.M. of estimation $\hat{\alpha}$ and $\hat{\beta}$ when we use Newton-Raphson iteration.

Future Work and Recommendations

1. This work can be used for generalized Logistic distn. of three parameters and other life distn.
2. Another methods of estimation could be used to estimate the distn. parameters, such as minimum Chi-square, minimum distance, Bayesian method, etc.
3. It can generate r.v's. from Logistic distn. by other new procedures which can be compared with other used procedures.
4. True values of skewness and kurtosis could be found theoretically.
5. The skewness to right of $\hat{\beta}$ might be adjusted to behave normality.
6. We recommend procedure (L-1) and the priority method to be used respectively L.S.M., M.M., M.M.M., M.L.M.

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Appendices

Appendix A

Computer Programs of Estimation Methods

Program 1: Estimation by Moment Method

Enter your values of a , b ,n and m

$\alpha :=$ ▮ $\beta :=$ ▮ $m :=$ ▮ $n :=$ ▮

```
x := | for  j ∈ 0.. m − 1
     |   | for  i ∈ 0.. n − 1
     |   |   | u ← rnd(1)
     |   |   | r[i, j] ← α − β·ln((1 − u)/u)
     |   | r
     | r
```

$i := 0..\, n - 1$ $j := 0..\, m - 1$

$$xa_j := \frac{1}{n}\cdot\sum_{i=0}^{n-1} x_{i,j} \qquad \alpha 1 := \frac{1}{m}\cdot\sum_{i=0}^{m-1} xa_i \qquad \alpha 1 = \blacksquare$$

$$s_j := \sqrt{\frac{1}{n-1}\cdot\left[\sum_{i=0}^{n-1}\left(x_{i,j} - xa_i\right)^2\right]}$$

$$xb_j := \frac{s_j}{\pi}\cdot\sqrt{\frac{3\cdot(n-1)}{n}}$$

$$\beta 1 := \frac{1}{m}\cdot\sum_{j=0}^{m-1} xb_j$$

$\beta 1 = ▮$

$$\sigma := \frac{1}{n-1}\cdot\sum_{j=0}^{n-1}\left(xa_j - \alpha 1\right)^2$$

$\sigma = ▮$

$$sk := \frac{\frac{1}{n}\cdot\left[\sum_{j=0}^{n-1}\left(xa_j\right)^3 - 3\cdot\alpha 1\cdot\sum_{j=0}^{n-1}\left(xa_j\right)^2 + 3\cdot\alpha 1^2\cdot\sum_{j=0}^{n-1}\left(xa_j\right) - \alpha 1^3\right]}{\sigma^{\frac{3}{2}}}$$

$sk = ▮$

$$ku := \frac{\frac{1}{n}\cdot\left[\sum_{j=0}^{n-1}\left(xa_j\right)^4 - 4\cdot\alpha 1\cdot\sum_{j=0}^{n-1}\left(xa_j\right)^3 + 6\cdot\alpha 1^2\cdot\sum_{j=0}^{n-1}\left(xa_j\right)^2 - 4\cdot\alpha 1^3\cdot\sum_{j=0}^{n-1}\left(xa_j\right) + \alpha 1^4\right]}{\sigma^2} - 3$$

$ku = ▮$

$$mse := \alpha 1^2 + \sigma$$

$mse = ▮$

$$\sigma 2 := \frac{1}{n-1}\cdot\sum_{j=0}^{n-1}\left(xb_j - \beta 1\right)^2$$

$\sigma 2 = ▮$

$$sk2 := \frac{\frac{1}{n}\cdot\left[\sum_{j=0}^{n-1}\left(xb_j\right)^3 - 3\cdot\beta 1\cdot\sum_{j=0}^{n-1}\left(xb_j\right)^2 + 3\cdot\beta 1^2\cdot\sum_{j=0}^{n-1}\left(xb_j\right) - \beta 1^3\right]}{\sigma 2^{\frac{3}{2}}}$$

$sk2 = ▮$

$$\chi := \frac{\frac{1}{n}\cdot\left[\sum_{j=0}^{n-1}\left(xb_j\right)^4 - 4\cdot\beta 1\cdot\sum_{j=0}^{n-1}\left(xb_j\right)^3 + 6\cdot\beta 1^2\cdot\sum_{j=0}^{n-1}\left(xb_j\right)^2 - 4\cdot\beta 1^3\cdot\sum_{j=0}^{n-1}\left(xb_j\right) + \beta 1^4\right]}{\sigma 2^2} - 3$$

$\chi = ▮$

$\theta := \beta 1 - 1$ $\theta = ▮$

$mse2 := \theta^2 + \sigma 2$ $mse2 = ▮$

Program 2: Estimation by Maximum Likelihood Method

Enter your values of a , b and n

$\alpha := ▮$ $\beta := ▮$ $n := ▮$

```
x := | for  j ∈ 0.. n − 1
     |   | for  i ∈ 0.. n − 1
     |   |   | u ← rnd(1)
     |   |   | r[i, j] ← α − β·ln((1 − u)/u)
     |   | r
     | r
```

$$p := \left|\begin{array}{l} n \leftarrow ▮ \\ \alpha \leftarrow ▮ \\ \beta \leftarrow ▮ \\ \text{for } j \in 0..n-1 \end{array}\right.$$

$$\quad \text{for } i \in 0..n-1$$

$$f1 \leftarrow \frac{n}{\beta} - \frac{2}{\beta}\cdot\sum_{i=0}^{n-1}\left(1 - \frac{1}{1+e^{-\frac{x_{i,j}-\alpha}{\beta}}}\right)$$

$$f2 \leftarrow \frac{1}{\beta^2}\cdot\sum_{i=0}^{n-1}\left(x_{i,j}-\alpha\right) - \frac{n}{\beta} - \frac{2}{\beta^2}\cdot\sum_{i=0}^{n-1}\left[\left(x_{i,j}-\alpha\right)\cdot\left(1 - \frac{1}{1+e^{-\frac{x_{i,j}-\alpha}{\beta}}}\right)\right]$$

$$a \leftarrow \frac{-2}{\beta^2}\cdot\sum_{i=0}^{n-1}\frac{e^{-\frac{x_{i,j}-\alpha}{\beta}}}{\left(1+e^{-\frac{x_{i,j}-\alpha}{\beta}}\right)^2}$$

$$b \leftarrow \frac{-n}{\beta^2} - \frac{2}{\beta^3}\cdot\sum_{i=0}^{n-1}\left[\frac{\left(x_{i,j}-\alpha\right)\cdot e^{-\frac{x_{i,j}-\alpha}{\beta}}}{\left(1+e^{-\frac{x_{i,j}-\alpha}{\beta}}\right)^2}\right] + \frac{2}{\beta^2}\cdot\sum_{i=0}^{n-1}\left(1 - \frac{1}{1+e^{-\frac{x_{i,j}-\alpha}{\beta}}}\right)$$

$$c \leftarrow \frac{-2}{\beta^3}\cdot\sum_{i=0}^{n-1}\left(x_{i,j}-\alpha\right) + \frac{n}{\beta^2} - \frac{2}{\beta^4}\cdot\sum_{i=0}^{n-1}\frac{\left(x_{i,j}-\alpha\right)^2\cdot e^{-\frac{x_{i,j}-\alpha}{\beta}}}{\left(1+e^{-\frac{x_{i,j}-\alpha}{\beta}}\right)^2} + \frac{4}{\beta^3}\cdot\sum_{i=0}^{n-1}\left(x_{i,j}-\alpha\right)\cdot\left(1 - \frac{1}{1+e^{-\frac{x_{i,j}-\alpha}{\beta}}}\right)$$

$$h_j \leftarrow \beta - \frac{-b\cdot f1 + a\cdot f2}{a\cdot c - \left(b^2\right)}$$

$$z_j \leftarrow \alpha - \frac{c\cdot f1 - b\cdot f2}{a\cdot c - \left(b^2\right)}$$

$$\beta \leftarrow h_j$$

$$\alpha \leftarrow z_j$$

$$\alpha$$

$$p = ▮$$

$k :=$
$n \leftarrow \blacksquare$
$\alpha \leftarrow \blacksquare$
$\beta \leftarrow \blacksquare$
$\text{for } j \in 0..n-1$
$\quad \text{for } i \in 0..n-1$

$$f1 \leftarrow \frac{n}{\beta} - \frac{2}{\beta}\cdot\sum_{i=0}^{n-1}\left(1 - \frac{1}{1+e^{-\frac{x_{i,j}-\alpha}{\beta}}}\right)$$

$$f2 \leftarrow \frac{1}{\beta^2}\cdot\sum_{i=0}^{n-1}\left(x_{i,j}-\alpha\right) - \frac{n}{\beta} - \frac{2}{\beta^2}\cdot\sum_{i=0}^{n-1}\left[\left(x_{i,j}-\alpha\right)\cdot\left(1 - \frac{1}{1+e^{-\frac{x_{i,j}-\alpha}{\beta}}}\right)\right]$$

$$a \leftarrow \frac{-2}{\beta^2}\cdot\sum_{i=0}^{n-1}\frac{e^{-\frac{x_{i,j}-\alpha}{\beta}}}{\left(1+e^{-\frac{x_{i,j}-\alpha}{\beta}}\right)^2}$$

$$b \leftarrow \frac{-n}{\beta^2} - \frac{2}{\beta^3}\cdot\sum_{i=0}^{n-1}\left[\frac{\left(x_{i,j}-\alpha\right)\cdot e^{-\frac{x_{i,j}-\alpha}{\beta}}}{\left(1+e^{-\frac{x_{i,j}-\alpha}{\beta}}\right)^2}\right] + \frac{2}{\beta^2}\cdot\sum_{i=0}^{n-1}\left(1 - \frac{1}{1+e^{-\frac{x_{i,j}-\alpha}{\beta}}}\right)$$

$$c \leftarrow \frac{-2}{\beta^3}\cdot\sum_{i=0}^{n-1}\left(x_{i,j}-\alpha\right) + \frac{n}{\beta^2} - \frac{2}{\beta^4}\cdot\sum_{i=0}^{n-1}\frac{\left(x_{i,j}-\alpha\right)^2\cdot e^{-\frac{x_{i,j}-\alpha}{\beta}}}{\left(1+e^{-\frac{x_{i,j}-\alpha}{\beta}}\right)^2} + \frac{4}{\beta^3}\cdot\sum_{i=0}^{n-1}\left(x_{i,j}-\alpha\right)\cdot\left(1 - \frac{1}{1+e^{-\frac{x_{i,j}-\alpha}{\beta}}}\right)$$

$$h_j \leftarrow \beta - \frac{-b\cdot f1 + a\cdot f2}{a\cdot c - \left(b^2\right)}$$

$$z_j \leftarrow \alpha - \frac{c\cdot f1 - b\cdot f2}{a\cdot c - \left(b^2\right)}$$

$\alpha \leftarrow z_j$
$\beta \leftarrow h_j$
β

$k = \blacksquare$

$$\sigma := \frac{1}{n-1}\cdot\sum_{j=0}^{n-1}\left(xa_j - p\right)^2 \qquad \sigma = ▮$$

$$sk := \frac{\frac{1}{n}\cdot\left[\sum_{j=0}^{n-1}\left(xa_j\right)^3 - 3\cdot\alpha1\cdot\sum_{j=0}^{n-1}\left(xa_j\right)^2 + 3\cdot\alpha1^2\cdot\sum_{j=0}^{n-1}\left(xa_j\right) - \alpha1^3\right]}{\sigma^{\frac{3}{2}}}$$

$$sk = ▮$$

$$ku := \frac{\frac{1}{n}\cdot\left[\sum_{j=0}^{n-1}\left(xa_j\right)^4 - 4\cdot\alpha1\cdot\sum_{j=0}^{n-1}\left(xa_j\right)^3 + 6\cdot\alpha1^2\cdot\sum_{j=0}^{n-1}\left(xa_j\right)^2 - 4\cdot\alpha1^3\cdot\sum_{j=0}^{n-1}\left(xa_j\right) + \alpha1^4\right]}{\sigma^2} - 3$$

$$ku = ▮$$

$$mse := \alpha1^2 + \sigma \qquad mse = ▮$$

$$\sigma2 := \frac{1}{n-1}\cdot\sum_{j=0}^{n-1}\left(xb_j - \beta1\right)^2 \qquad \sigma2 = ▮$$

$$sk2 := \frac{\frac{1}{n}\cdot\left[\sum_{j=0}^{n-1}\left(xb_j\right)^3 - 3\cdot\beta1\cdot\sum_{j=0}^{n-1}\left(xb_j\right)^2 + 3\cdot\beta1^2\cdot\sum_{j=0}^{n-1}\left(xb_j\right) - \beta1^3\right]}{\sigma2^{\frac{3}{2}}} \qquad sk2 = ▮$$

$$\chi := \frac{\frac{1}{n}\cdot\left[\sum_{j=0}^{n-1}\left(xb_j\right)^4 - 4\cdot\beta1\cdot\sum_{j=0}^{n-1}\left(xb_j\right)^3 + 6\cdot\beta1^2\cdot\sum_{j=0}^{n-1}\left(xb_j\right)^2 - 4\cdot\beta1^3\cdot\sum_{j=0}^{n-1}\left(xb_j\right) + \beta1^4\right]}{\sigma2^2} - 3$$

$$\theta := \beta1 - 1 \qquad \chi = ▮$$

$$\theta = ▮$$

$$mse2 := \theta^2 + \sigma2 \qquad mse2 = ▮$$

Program 3: Estimation by Modified Moment Method

Enter your values of a , b ,n and m

$\alpha :=$ ▮ $\beta :=$ ▮ $m :=$ ▮ $n :=$ ▮

```
x := | for j ∈ 0.. m − 1
     |   | for i ∈ 0.. n − 1
     |   |   | u ← rnd(1)
     |   |   | r[i,j] ← α + β·ln( u / (1 − u) )
     |   | r
     | r
```

$i := 0..n-1$ $j := 0..m-1$

$y := \min(x)$ $xa_j := \frac{1}{n}\cdot\sum_{i=0}^{n-1} x_{i,j}$ $\alpha 1 := \frac{1}{m}\cdot\sum_{i=0}^{m-1} xa_i$

$\alpha 1 =$ ▮

$$\psi(n) := \ln(n) - \frac{1}{2\cdot n} - \frac{1}{12\cdot\left(n^2\right)} + \frac{1}{120\cdot\left(n^4\right)} - \frac{1}{252\cdot\left(n^6\right)} + \frac{1}{240\cdot\left(n^8\right)}$$

$\beta 1 := \frac{y - \alpha 1}{-0.577 - \psi(n)}$ $\beta 1 =$ ▮

$\sigma := \frac{1}{n-1}\cdot\left[\sum_{j=0}^{n-1}\left(xa_j\right)^2 - n\cdot\alpha 1^2\right]$ $\sigma =$ ▮

$$sk := \frac{\frac{1}{n}\cdot\left[\sum_{j=0}^{n-1}\left(xa_j\right)^3 - 3\cdot\alpha1\cdot\sum_{j=0}^{n-1}\left(xa_j\right)^2 + 3\cdot\alpha1^2\cdot\sum_{j=0}^{n-1} xa_j - \alpha1^3\right]}{\sigma^{\frac{3}{2}}} - 3$$

$sk = ▮$

$$ku := \frac{\frac{1}{n}\cdot\sum_{j=0}^{n-1}\left[xa_j + 4 - 4\cdot\left(xa_j\right)^3 + 6\cdot\alpha1^2\cdot\left(xa_j\right)^2 - 4\cdot\left(xa_j\right)\cdot\alpha1^3 + \alpha1^4\right]}{\sigma^2}$$

$ku = ▮$

$mse := \alpha1^2 + \sigma$

$mse = ▮$

$$\sigma22 := \frac{1}{n-1}\cdot\sum_{j=0}^{n-1}\left(xb_j - \beta1\right)^2$$

$\sigma22 = ▮$

$$sk2 := \frac{\frac{1}{n}\cdot\left[\sum_{j=0}^{n-1}\left(xb_j\right)^3 - 3\cdot\beta1\cdot\sum_{j=0}^{n-1}\left(xb_j\right)^2 + 3\cdot\beta1^2\cdot\sum_{j=0}^{n-1} xb_j - \beta1^3\right]}{\sigma2^{\frac{3}{2}}} - 3$$

$sk2 = ▮$

$$ku2 := \frac{\frac{1}{n}\cdot\sum_{j=0}^{n-1}\left[\left(xb_j\right)^4 - 4\cdot\left(xb_j\right)^3 + 6\cdot\beta1^2\cdot\left(xb_j\right)^2 - 4\cdot\left(xb_j\right)\cdot\beta1^3 + \beta1^4\right]}{\sigma2^2}$$

$ku2 = ▮$

$\theta := \beta1 - 1$

$\theta = ▮$

$mse2 := \theta^2 + \sigma2$

$mse2 = ▮$

Program 4: Estimation by Least Square Method

Enter your values of a , b ,n and m

$\alpha :=$ ▮ $\beta :=$ ▮ $m :=$ ▮ $n :=$ ▮

```
x := | for j ∈ 0.. m − 1
     |   | for i ∈ 0.. n − 1
     |   |   | u ← rnd(1)
     |   |   | r[i, j] ← α + β·ln( u / (1 − u) )
     |   | r
     | r
```

$j := 0..\, m - 1$

$u := \mathrm{runif}(n, 0, 1)$

$$t := \ln\left(\frac{1-u}{u}\right) \qquad y_j := \frac{1}{n}\cdot\left(\sum_{i=0}^{n-1} x_{i,j}\right) \qquad \tau_j := \frac{1}{n}\cdot\left(\sum_{i=0}^{n-1} t_i\right)$$

$$\alpha11_j := \frac{\tau_j\cdot\sum_{i=0}^{n-1}\left(t_i\cdot x_{i,j}\right) - y_j\cdot\sum_{i=0}^{n-1}\left(t_i\right)^2}{\left(\tau_j\right)\cdot\sum_{i=0}^{n-1}\left(t_i\right) - \sum_{i=0}^{n-1}\left(t_i\right)^2}$$

$$\alpha1 := \frac{1}{m}\cdot\sum_{j=0}^{m-1}\alpha11_j$$

$\alpha1 =$ ▮

$$\beta11_j := -\frac{y_j \cdot \sum_{i=0}^{n-1} t_i - \sum_{i=0}^{n-1} \left(t_i \cdot x_{i,j}\right)}{\left(\tau_j\right) \cdot \sum_{i=0}^{n-1} \left(t_i\right) - \sum_{i=0}^{n-1} \left(t_i\right)^2}$$

$$\beta1 := \frac{1}{m} \cdot \sum_{j=0}^{m-1} \beta11_j$$

$\beta1 = ▮$

$$\sigma := \frac{1}{n-1} \cdot \sum_{j=0}^{n-1} \left(xa_j - \alpha1\right)^2$$

$\sigma = ▮$

$$sk := \frac{\frac{1}{n} \cdot \left[\sum_{j=0}^{n-1} \left(xa_j\right)^3 - 3 \cdot \alpha1 \cdot \sum_{j=0}^{n-1} \left(xa_j\right)^2 + 3 \cdot \alpha1^2 \cdot \sum_{j=0}^{n-1} \left(xa_j\right) - \alpha1^3\right]}{\sigma^{\frac{3}{2}}}$$

$sk = ▮$

$$ku := \frac{\frac{1}{n} \cdot \left[\sum_{j=0}^{n-1} \left(xa_j\right)^4 - 4 \cdot \alpha1 \cdot \sum_{j=0}^{n-1} \left(xa_j\right)^3 + 6 \cdot \alpha1^2 \cdot \sum_{j=0}^{n-1} \left(xa_j\right)^2 - 4 \cdot \alpha1^3 \cdot \sum_{j=0}^{n-1} \left(xa_j\right) + \alpha1^4\right]}{\sigma^2} - 3$$

$ku = ▮$

$$mse := \alpha1^2 + \sigma$$

$mse = ▮$

$$\sigma2 := \frac{1}{n-1} \cdot \sum_{j=0}^{n-1} \left(xb_j - \beta1\right)^2$$

$\sigma2 = ▮$

$$sk2 := \frac{\frac{1}{n}\cdot\left[\sum_{j=0}^{n-1}\left(xb_j\right)^3 - 3\cdot\beta1\cdot\sum_{j=0}^{n-1}\left(xb_j\right)^2 + 3\cdot\beta1^2\cdot\sum_{j=0}^{n-1}\left(xb_j\right) - \beta1^3\right]}{\sigma2^{\frac{3}{2}}}$$

$sk2 = ▮$

$$\chi := \frac{\frac{1}{n}\cdot\left[\sum_{j=0}^{n-1}\left(xb_j\right)^4 - 4\cdot\beta1\cdot\sum_{j=0}^{n-1}\left(xb_j\right)^3 + 6\cdot\beta1^2\cdot\sum_{j=0}^{n-1}\left(xb_j\right)^2 - 4\cdot\beta1^3\cdot\sum_{j=0}^{n-1}\left(xb_j\right) + \beta1^4\right]}{\sigma2^2} - 3$$

$\chi = ▮$

$\theta := \beta1 - 1$

$\theta = ▮$

$mse2 := \theta^2 + \sigma2$

$mse2 = ▮$

Appendix B
Computer Programs for
Generating Random Variates of
Logistic Distribution

*Program 5: procedure (**L**-1)*

Enter your values of a , b ,n and m

$\alpha :=$ ▮ $\beta :=$ ▮ $m :=$ ▮ $n :=$ ▮

```
x := | for  j ∈ 0.. m − 1
     |   | for  i ∈ 0.. n − 1
     |   |   | u ← rnd(1)
     |   |   | r_{i,j} ← α − β·ln((1 − u)/u)
     |   | r
     | r
```

$$x := \text{for } j \in 0..m-1 \;\; \Big| \; \text{for } i \in 0..n-1 \;\; \Big| \; u \leftarrow \text{rnd}(1), \; r_{i,j} \leftarrow \alpha - \beta \cdot \ln\left(\frac{1-u}{u}\right); \; r; \; r$$

*Program 6: procedure (**L**-2)*

Enter your values of a , b ,n and m

$\alpha :=$ ▮ $\beta :=$ ▮ $m :=$ ▮ $n :=$ ▮

```
x :=  for  j ∈ 0..m − 1
         for  i ∈ 0..n − 1
             u ← rnd(1)
             b[i,j] ← −ln(1 − u)
             w[i,j] ← b[i,j] + ln(1 − e^(−b[i,j]))
             r[i,j] ← α + β·w[i,j]
         r
      r
```

المستخلص

تطرقنا في هذه الرسالة الى توزيع لوجستك ذو المعلمتين لأهميتـــه فـــي مجـــالات الاحصاء. ثم أستعراض وتوحيد خواص التوزيع الرياضية والاحصائية والعـــزوم والعـــزوم العليا. تطرقنا الى التخمين حيث تم مناقشة أربعة طرق تخمينية لمعالم التوزيع وهي طريقـــة العزوم، طريقة الترجيح الاعظم، طريقة العزوم المعدلة وطريقة المربعات الصغرى حيـــث نوقشت هذه الطرق نظرياً وطبقت عملياً بأستخدام أسلوبين من محاكاة مونت كـــارلو لتوليـــد المتغيرات العشوائية من التوزيع اللوجستي، ثم جدولة خواص المخمنات لمعلمات التوزيـــع والتي هي التحيز، التباين، معامل الالتواء، معامل التقلص وقياس مربعات الخطأ.

جمهـورية العـراق
وزارة التعليم العالي والبحث العلمي
جــــامعة النهــــــريـــن
كليــــة العلوم
قسم الريــاضيات وتطبيقات الحاسوب

توليد المتغيرات العشوائية لتخمين معلمات توزيع لوجستك باستخدام محاكاة مونت كارلو

رسالة
مقدمة إلى كلية العلوم - جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل
زهراء أموري علي الحجار
(بكالوريوس علوم، جامعة النهرين، ٢٠٠٦)

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