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# NUMERICAL SOLUTION VIA HAAR WAVELET APPROACH FOR BURGER'S FISHER EQUATION

**A Thesis**
**Submitted to the College of Science of Al-Nahrain University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics**

**By**

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿اقْرَأْ بِاسْمِ رَبِّكَ الَّذِي خَلَقَ ۝١ خَلَقَ الْإِنْسَانَ مِنْ عَلَقٍ ۝٢

اقْرَأْ وَرَبُّكَ الْأَكْرَمُ ۝٣ الَّذِي عَلَّمَ بِالْقَلَمِ ۝٤ عَلَّمَ الْإِنْسَانَ مَا لَمْ يَعْلَمْ ۝٥

كَلَّا إِنَّ الْإِنْسَانَ لَيَطْغَى ۝٦ أَنْ رَآهُ اسْتَغْنَى ۝٧ إِنَّ إِلَى رَبِّكَ الرُّجْعَى

۝٨ أَرَأَيْتَ الَّذِي يَنْهَى ۝٩ عَبْدًا إِذَا صَلَّى ۝١٠ ﴿أَرَأَيْتَ إِنْ كَانَ عَلَى الْهُدَى

۝١١ أَوْ أَمَرَ بِالتَّقْوَى ۝١٢ أَرَأَيْتَ إِنْ كَذَّبَ وَتَوَلَّى ۝١٣ أَلَمْ يَعْلَمْ بِأَنَّ اللَّهَ يَرَى

۝١٤ كَلَّا لَئِنْ لَمْ يَنْتَهِ لَنَسْفَعَنْ بِالنَّاصِيَةِ ۝١٥ نَاصِيَةٍ كَاذِبَةٍ خَاطِئَةٍ

۝١٦ فَلْيَدْعُ نَادِيَهُ ۝١٧ سَنَدْعُ الزَّبَانِيَةَ ۝١٨ كَلَّا لَا تُطِعْهُ وَاسْجُدْ وَاقْتَرِبْ﴾ ۝١٩

( صدق الله العظيم )

(سورة العلق)

# Dedication

To the man who has been beside me all the time and Gives me the power to complete my academic study when I need him. With all appreciation, respect and love to you.

My Father Dr. Hazim Mohammed

To the Women who sacrificed her life and future to reach me in this advanced stage of life. All love and appreciation for her.

My dear Mother

To my dear brothers, who were helping and supporting me in my studies. Thank you.

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*My deepest thanks to Allah, for his generous by providing me the strength to accomplish this work.*

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**Nawar Hazim Mohammed**

**March, 2017**

# Abstract

In this thesis, Haar wavelet method is implemented efficiently in finding the numerical solution of Burger's Fisher equation. This method shows rather rapid convergence than other existing methods. Illustrative examples are implemented to show the efficiency and the powerful of Haar wavelet approach. The comparison among the numerical results and the exact solution, and the solutions obtained by using some traditional methods such as variational iteration method (VIM) shows that the suggested scheme is fairly accurate and viable for solving Burger's Fisher problem.

# CONTENTS

# List of Figures and Tables

# List of Abbreviations

| Abbreviations | Meaning |
|---|---|
| DEs | Differential equations |
| ODEs | Ordinary differential equations |
| PDEs | Partial differential equations |
| HAM | Homotopy analysis method |
| HPM | Homotopy perturbation method |
| TFM | Tanh function method |
| VIM | Varitional iteration method |
| ADM | Adomian decomposition method |
| FDM | Finite difference method |
| FEM | Finite element method |
| WBM | Wavelet based method |
| WT | Wavelet transform |
| MRA | Multiresolution analysis |
| CWT | Continuous wavelet transform |
| DWT | Discrete wavelet transform |
| NPDEs | Nonlinear partial differential equations |

# INTRODUCTION

Differential equations (DEs) play a major role in describing a wide range of natural phenomena undergoing change. (DEs) relate an unknown function to its derivatives. These functions usually represent physical quantities whereas the derivatives represent their rates of change.

Mathematical models involving evolutionary partial differential equations (PDEs) as well as ordinary differential equations (ODEs) arise in many diverse applications, such as fluid flow, mechanical systems, image processing, physics, earth sciences, relativity, and mathematical finance [1].

The nature of these equations has been studied by mathematicians for hundreds of years and there are many well-developed solution techniques. The analytical solution for a given differential equation is always preferable. However, systems described by differential equations are often so complex or so large where a purely analytical solution to the equations is not tractable, or many of them cannot be solved exactly by using analytical methods.

In this case, the numerical methods are needed to be developed for solving differential equations. Methods for the numerical simulation of dynamic mathematical models have been the focus of intensive research. However, due to the increasing range of applications, there is a continuing demand today for better and more efficient methods [1].

Recently, the study of nonlinear partial differential equations modeling physical Phenomena has become an important tool. Nonlinear phenomena are of fundamental importance in various fields of science and engineering.

Most of nonlinear phenomena are models of our real life problems. The investigation of the travelling wave solutions plays an important role in nonlinear science. A variety of powerful methods have been presented, such as the inverse *scattering transform* (ISM) [2], *homotopy perturbation method* (HPM) [3], *homotopy analysis method* (HAM) [4, 5], *tanh function method* (TFM) [6], *variational iteration method* (VIM) [7-9], the *Adomian decomposition method* (ADM) [10-12], and ***the wavelet-based transform***(WBT) [13-16].

The wavelet-based transform has been introduced by Chen and Hsiao [13] for solving differential equations where the highest derivatives appearing in the differential equations are first expanded into Haar series. The lower order derivatives and the solutions can then be obtained quite easily using the Haar operational matrices of integration. The derivation for Haar operational matrix of integration and other operational matrix of an orthogonal function can be derived from block pulse operational matrix [18].

The ideas from Chen and Hsiao were later used by Maleknejad and Mirzaee[19], Razzaghi and Ordokhani[28], Lepik[14-15] and Shi *et al*. [21] to solve other differential and integral equations.

Their ideas were also applied by Dai and Cochran [22] to solve variational and optimal control problems. Although, the method has been applied successfully for numerical solution of linear ordinary differential equations by Chang and Piau[23], nonlinear differential equations by Hariharan*et al*. [16], Lepik[14-15] and fractional order differential equations by Li and Hu [24] and Li and Weiwei[25], but Haar wavelets or rather piecewise constant functions in general, are not vastly used for the higher order of partial differential equations, because of the difficulty in determining the accuracy and stability of the solution (see [26]). Due to the successful application of Haar operational matrix in numerical solution

of first order PDEs was proposed by Wu [27] and nonlinear evolution equations with only one-dimensional space by Lepik[15], It is clear that many authors have been attracted to use wavelets methods for solving differential equations.

This thesis is organized as follows:

In chapter one, a brief review of basic definitions and concepts relate to the work are introduced. It includes an overview of differential equations and their types as well as a review of some traditional numerical techniques for solving ordinary differential equations and partial differential equations, for comparison with the Haar wavelet method used in this thesis.

Chapter two gives insight of wavelet analysis theory and its application for solving differential equations. It includes a methodology for applying Haar wavelet method with numerical examples for solving Burger's Fisher equation.

Finally, chapter three contains the implementation of Haar wavelet based transform for solving Burger's Fisher equation numerically. The results are shown and compared to the given exact solution as well as to the results obtained from traditional existing methods such as VIM. Conclusions and future suggestions are given.

# CHAPTER ONE
# Basic Definitions and Concepts

## 1.1 Introduction

This chapter includes some basic definitions and concepts related to the work in this thesis. An overview of differential equations and their types is introduced. In addition, we review some traditional numerical techniques such as VIM for solving ordinary differential equations and partial differential equations, for comparison with the Haar wavelet method used in this thesis.

## 1.2 Overview Of Differential Equation

Differential equations appear in all fields of engineering and science. They relate some function of one or more variables with its derivatives. Many real physical, engineering, chemical and biological phenomena are modeled mathematically by differential equations. In general, most real engineering and science processes involve more than one independent variable, and the corresponding differential equations are called ***partial differential equations*** (PDEs). However, in many cases, simplifying assumptions are made which reduce the PDEs to ***ordinary differential equations*** (ODEs). An (ODE) is a differential equation for a function of a single variable.

Several different types of PDEs exist. These types depend on application where each application has its own special governing equation or equations and its own peculiarities which must be considered individually. The first order PDEs may be classified as: ***linear equation***, ***semi-linear equation***, ***quasi-linear equation***, and ***nonlinear equation***.

A first order PDE $f(x, y, u, u_x, u_y) = 0$ is said to be ***linear***, if it can be written as:

$$P(x,y)u_x + Q(x,y)u_y = Z(x,y)u + S(x,y) \quad (1.1)$$

where*P*, *Q* and *Z* are functions of *x* and *y* only and therefore, the PDE is ***linear*** in u.

A first order PDE $f(x,y,u,u_x,u_y) = 0$ is said to be ***semi-linear***, if it is linear in $u_x$ and $u_y$ .

A semi-linear first order PDE can be written as:

$$P(x,y)u_x + Q(x,y)u_y = Z(x,y,u) \quad (1.2)$$

where*P* and *Q* are functions of *x* and *y* only and *Z* is non-linear in *u*.

A first order PDE $f(x,y,u,u_x,u_y) = 0$ is said to be ***quasi-linear***, if it is linear in $u_x$ and $u_y$ and it can be written as:

$$P(x,y,u)u_x + Q(x,y,u)u_y = Z(x,y,u) \quad (1.3)$$

where at least one of *P* or *Q* is non-linear in *u*.

A first order PDE $f(x,y,u,u_x,u_y) = 0$ is said to be ***nonlinear*** PDE if it does not come under the Eqs. (1.1) to (1.3).

The general quasi-linear second-order nonhomogeneous PDE in two independent variables can be written as [28]:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \quad (1.4)$$

where the coefficients *A* to *C* may depend on x, y,$u_x$, and $u_y$; the coefficients *D* to *F* may depend on x, y, and *u*; and the nonhomogeneous term *G* may depend on *x* and *y*. Eq. (1.4) is said to be:

1) Elliptic if $B^2 - 4AC < 0$.
2) Parabolic if $B^2 - 4AC = 0$.
3) Hyperbolic if $B^2 - 4AC > 0$.

On the other hand, the PDE as the equation is supplemented by initial and / 0r boundary conditions. In order to find the solution, there are three types of boundary conditions:

1) Dirichlet boundary condition: numerical values of the function are specific of the boundary of the region.
2) Neumann boundary condition: specifies the values that the derivative of a solution to take on the boundary of the domain.
3) Mixed boundary conditions: defines a boundary value problem in which the solution of the given equation is required to satisfy different boundary conditions on disjoint parts of the boundary of the domain where the condition is stated. In effect, in a mixed boundary value problem, the solution is required to satisfy the Dirichlet or Neumann boundary conditions in a mutually exclusive way on disjoint parts of the boundary.

## 1.3 Analytical versus Numerical Methods

A solution to the set of equations is a mathematical model in order to understand its behavior by using simple mathematical techniques such a trigonometry or calculus is called the ***analytic solution***, because the analysis is used to figure it out. This gives an ***exact solution*** of how the model will behave under any circumstances. It is also referred to as a ***closed form*** solution.

For more complex models, the mathematical techniques become too much complicated. Then the ***numerical methods*** of solving the equations are the alternative.

For the differential equation that describes behavior over time, the numerical method starts with the initial values of the variables, and then use the equations to

figure out the changes in these variables over a short time period. It's only an approximation, but it can be a very good approximation under certain circumstances. The best is when the exact solution can be found out using trigonometry, calculus and other techniques. An analytic solution is preferred because it provides us a lot about the behavior of the systems.

On the other hand, there are many problems for which it is not possible to find an analytical solution. These are models that have non-linear equations. For these models there are methods such as Perturbation method which can be used to find an ***approximate analytical solution*** within a certain range.

For higher order or non-linear differential equations with complex coefficients, it becomes very difficult to find exact solution; therefore, the numerical methods for solving the equations are needed. There are several different numerical methods for solving PDEs, among these methods are: finite difference methods (FDM), finite element method (FEM), Adomain decomposition method (ADM), variational iterative method (VIM), and wavelet based method (WM).

## 1.4 The Variational Iteration Method

The variational iteration method (VIM) is a scheme that in many instances gives rapidly convergent successive approximations of the exact solution if such a solution exists. This method was developed by He J.H. [8] for solving linear, nonlinear and boundary value problems. The obtained approximations by this method are usually of high accuracy when the convergence is assured even if some iterations are used. The obtained approximate solution is given in an infinite series which converge very rapidly to accurate solution.

To illustrate the basic concept of the VIM , the following nonlinear partial differential equation is considered.

$$Lu + Nu = f(x) \tag{1.5}$$

where$L$ is a linear operator, $N$ is a nonlinear operator and $f$ is an inhomogeneous term which is an analytical function. According to VIM, we can construct a correction functional for Eq(1.5) as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\big(Lu_n(s) + N\tilde{u}_n(s) - f(s)\big)ds \tag{1.6}$$

where $\lambda$ is a Lagrange multiplier which can be identified optimally via variational iteration method. The subscript $n$ denote the $n$th approximation,$\tilde{u}_n$is considered as a restricted variation i.e, $\delta\tilde{u}_n = 0$.

It is obvious that the successive approximation $u_{n+1}, n \geq 0$of the solution $u$ will be obtained upon using the determined Lagrange multiplier and any selective function $u_0$ . Method of integration such as integration by parts is used to compute $\lambda(s)$ .

$$\int \lambda(s)\, u_n'(s)ds = \lambda(s)u_n(s) - \int \lambda'(s)u_n(s)ds \tag{1.7}$$

$$\int \lambda(s)\, u_n''(s)ds = \lambda(s)u_n'(s) - \lambda'(s)u_n(s) + \int \lambda''(s)u_n(s)ds \tag{1.8}$$

And so on. Consequently, the solution is given by:

$$u = \lim_{n\to\infty} u_n \tag{1.9}$$

Furthermore, the selective function $u_0(x)$ should be selected using the initial conditions, for fast convergence, as follows:

$$\left.\begin{array}{l} u_0(x) = u(0) \quad for\ the\ first\ order\ u_n' \\ u_0(x) = u(0) + xu'(0) \quad for\ the\ second\ order\ u_n'' \\ u_0(x) = u(0) + xu'(0) + \dfrac{1}{2!}x^2u''(0) \quad for\ the\ third\ order\ u_n''' \end{array}\right\} \tag{1.10}$$

In the following, the derivation of useful formulas of iteration for certain types of the first order and higher order DEs as well as determine of the Lagrange multiplier $\lambda(s)$ for each type will be given.

1) $$\begin{cases} u' + f(u,u') = 0 \\ u_{n+1}(x) = u_n(x) - \int_0^x [u_n'(s) + f(u_n,u_n')]ds \end{cases} \quad (1.11)$$

2) $$\begin{cases} u' + \alpha u + f(u,u') = 0 \\ u_{n+1}(x) = u_n(x) - \int_0^x e^{\alpha(s-x)}[u_n'(s) + \alpha u_n(s) + f(u_n,u_n')]ds \end{cases} \quad (1.12)$$

3) $$\begin{cases} u'' + f(u,u',u'') = 0 \\ u_{n+1}(x) = u_n(x) + \int_0^x (s-x)[u_n''(s) + f(u_n,u_n',u_n'')]ds \end{cases} \quad (1.13)$$

4) $$\begin{cases} u'' + \beta^2 u + f(u,u',u'') = 0 \\ u_{n+1}(x) = u_n(x) + \frac{1}{\beta}\int_0^x sin\beta(s-x)[u_n''(s) + \beta^2 u_n(s) + f(u_n,u_n',u_n'')]ds \end{cases} \quad (1.14)$$

5) $$\begin{cases} u'' + \alpha^2 u + f(u,u',u'') = 0 \\ u_{n+1}(x) = u_n(x) + \frac{1}{2\alpha}\int_0^x \left(e^{(s-x)} - e^{(x-s)}\right)[u_n''(s) + \alpha^2 u_n(s) + f(u_n,u_n',u_n'')]ds \end{cases} \quad (1.15)$$

6) $$\begin{cases} u''' + f(u,u',u'',u''') = 0 \\ u_{n+1}(x) = u_n(x) - \frac{1}{2}\int_0^x (s-x)^2[u_n'''(s) + f(u_n,u_n',u_n'',u_n''')]ds \end{cases} \quad (1.16)$$

7) $$\begin{cases} u^{(4)} + f\left(u,u',u'',u''',u^{(4)}\right) = 0 \\ u_{n+1}(x) = u_n(x) + \frac{1}{6}\int_0^x (s-x)^3\left[u_n^{(4)}(s) + f\left(u_n,u_n',u_n'',u_n''',,u^{(4)}\right)\right]ds \end{cases} \quad (1.17)$$

8) $$\begin{cases} u^{(n)} + f\left(u,u',u'',u''',\dots,u^{(n)}\right) = 0 \\ u_{n+1}(x) = u_n(x) + (-1)^n \int_0^x \frac{1}{(n-1)!}(s-x)^{n-1}\left[u_n^{(n)}(s) + f\left(u_n,u_n',u_n'',u_n''',,u^{(n)}\right)\right]ds \end{cases} \quad (1.18)$$

In summary, the VIM can be applied to solve linear and non-linear problems. The main step in this method is to determine the Lagrange multipliers $\lambda(s)$. The Lagrange multipliers $\lambda(s)$ for certain types of DEs as in the above discussion may be summarized as:

$$\begin{array}{lll} u' + f(u,u') = 0 & \lambda(s) = -1 & u_0(x) = u(0) \\ u'' + f(u,u',u'') = 0 & \lambda(s) = s - x & u_0(x) = u(0) + xu'(0) \\ u''' + f(u,u',u'',u''') = 0 & \lambda(s) = -\frac{1}{2!}(s-x) & u_0(x) = u(0) + xu'(0) + \frac{1}{2!}x^2u''(0) \\ \vdots & \vdots & \vdots \end{array} \quad (1.19)$$

And the solution is given by $u = \lim_{n\to\infty} u_n$ (1.20)

The selective function $u_0$ can be chosen by using the initial values $u(x,0), u_t(x,0)$ [17].

The following examples illustrate the implementation of the VIM.

**Example 2.1:** Consider the second order non-linear ordinary differential equation

$$y''(x) + y^2(x) = 0\,, \qquad y(0) = a\,, y'(0) = b \quad (1.21)$$

According to the above discussion, the Lagrange multiplier is $\lambda(s) = (s - x)$. Using Eq(2.94), the iteration formula is given by

$$y_{n+1}(x) = y_n(x) + \int_0^x (s-x)[(y_n)_{ss} + f_n(y_n, y_n', y_n'')]ds \quad (2.22)$$

By Taylor expansion and the initial conditions, we choose

$$y_0(x) = y(0) + xy'(0) = a + xb$$

$$y_1(x) = a + bx + \int_0^x (s-x)[(y_0(s))_{ss} + y_0^2(s)]ds \quad (1.23)$$

$$y_2(x) = y_1(x) + \int_0^x (s-x)[(y_1(s))_{ss} + y_1^2(s)]ds \tag{1.24}$$

Then, the solution is given by

$$y = \lim_{n\to\infty} y_n \tag{1.25}$$

**Example 2.2:** Consider the first order non-linear partial differential equation

$$\frac{\partial}{\partial x}y(x,t) + \frac{\partial}{\partial t}y(x,t) = y^2 y(x,t)\,, \qquad y(0,t) = a \tag{1.26}$$

Where a is constant.

The Lagrange multiplier is $(s) = -1$ , then

$$y_{n+1}(x,t) = y_n(x,t) - \int_0^x \left[\frac{\partial}{\partial s}y_n(s,t) + \frac{\partial}{\partial s}y_n(s,t) - y_n^2(s,t)\right] ds \tag{1.27}$$

By Taylor expansion and the initial condition, we choose

$$y_0(x,t) = y(0,t) = a$$

$$y_1(x,t) = a - \int_0^x \left[\frac{\partial}{\partial s}y_0(s,t) + \frac{\partial}{\partial s}y_0(s,t) - y_0^2(s,t)\right] ds \tag{1.28}$$

$$y_2(x,t) = y_1(x,t) - \int_0^x \left[\frac{\partial}{\partial s}y_1(s,t) + \frac{\partial}{\partial s}y_1(s,t) - y_1^2(s,t)\right] ds \tag{1.29}$$

Then, the solution is given by

$$y = \lim_{n\to\infty} y_n \tag{1.30}$$

In what follows, some concepts of wavelet theory are introduced. These concepts are useful in illustrating the usage of wavelet based transform as an efficient numerical method for solving differential equations.

# CHAPTER TWO
# Wavelet Analysis

## 2.1 Introduction

In this chapter we introduce the basic idea of the wavelet transform (WT) and its properties, as well as, its applications. Wavelet application for solving differential equations will be discussed so as to observe how wavelet is implemented efficiently as a numerical method to be applied to linear and non-linear problems in the process of approximation, and also how mathematical aspects of wavelet affect the approximate solution and the results of it.

The insights of how wavelets in mathematics are implemented in a way to fit the engineering and science models will be studied. Then the concept of Multiresolution Analysis (MRA) is explained.

## 2.2 Wavelet Transform

The wavelets are mathematical functions defined over a finite interval and having an average value of zero that transform data into different frequency components, representing each component with a resolution matched to its scale [29]. It was first introduced by A. Grossmann and J. Morlet in 1984 [30]. The basic idea of the (WT) is to represent any arbitrary function $f(t)$ as a superposition of the set of such wavelets or basis functions. These basis functions are obtained from a single wavelet called ***mother wavelet*** $\psi(t)$because all other wavelet functions within the family are obtained by dilating and translating of $\psi(t)$ by amounts $s$ and $\tau$ respectively as given bellow [31]:

$$\psi_{s,\tau}(t) = \left\{ \psi\left(\frac{t-\tau}{s}\right) ;\ (s,\tau) \in \mathbb{R}^{+} \times \mathbb{R} \right\} \tag{2.1}$$

The process of changing the two parameters $s$ and $\tau$ that result in the basis functions are shown in Figure 2.1.

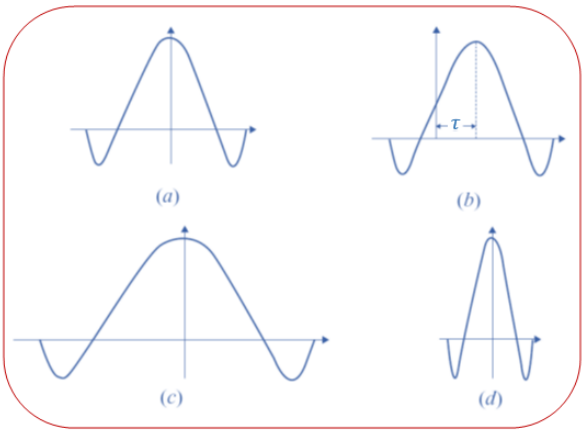



**Figure. 2.1: Effect of Time Dilation and Translation on the Mother Wavelet [31]**

**(a)** Mother wavelet $\psi(t) = \psi_{1,0}\,; s = 1, \tau = 0$. **(b)** Wavelet $\psi_{1,\tau}(t)\,; s = 1, \tau \neq 0$.

**(c)** Wavelet $\psi_{2,0}(t); at\ scale\ s = 2, \tau = 0$. **(d)** Wavelet $\psi_{0.5,0}(t); at\ scale\ s = 0.5, \tau = 0$.

These wavelets are distinguishing by compactly supported functions defined over a finite interval and having an average value of zero, and that leads to efficient implementation. The mother wavelet $\psi(t)$ is the function with zero translation and a dilation of one.

In wavelet transform the basis functions are wavelets. Wavelets tend to be irregular and symmetric. All wavelet functions, $\psi(2st - \tau)$, are derived from a single mother wavelet, $\psi(t)$. This wavelet is a small wave or pulse, like the one shown in Figure 2.2.

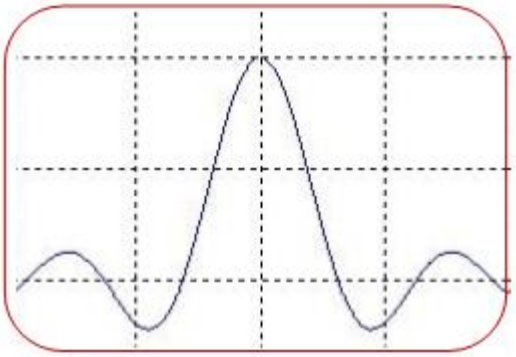


**Fig. 2.2: Mother Wavelet $\psi(t)$ [32]**

Normally it starts at time $t$ = 0 and ends at $t = T$. The ***shifted*** wavelet $\psi(t-\tau)$ starts at $t = \tau$and ends at $t = \tau + T$. The ***scaled*** wavelets $\psi(2st)$ start at $t$ = 0 and end at $t = T/2s$. Their graphs are $\psi(t)$ compressed by the factor of $2s$ as shown in Figure 2.3. For example, when $s$ = 1, the wavelet is shown in Figure 2.3(a). If $s$ = 2 and 3, they are shown in (b) and (c), respectively [29].

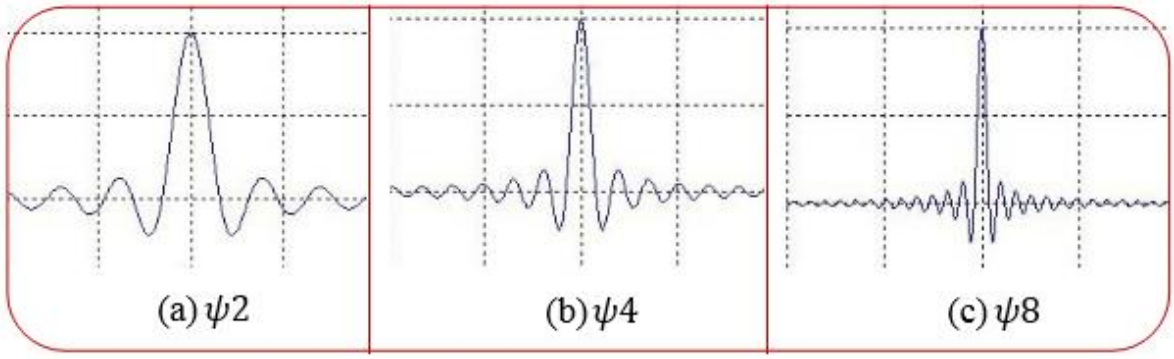



**Fig. 2.3: Scaling the Wavelets [32]**

By a dilation we mean a scaling of the argument; so, given any function $\psi(t)$ and a parameter $s$> 0, $\psi(\frac{t-\tau}{s})$ is a dilation of $\psi(t)$. Consequently, a dilation of a function corresponds to either a spreading out or contraction of the function. The factor $\frac{\mathbf{1}}{\sqrt{|s|}}$ is introduced with $\psi_{s,\tau}(t)$ equation to keep the energy of the mother wavelets constant that it yields normalization necessary to have an orthonormal wavelet basis.

The translation simply means a shift of the argument along the real axis, that is, given, the translation of $\psi(t)$ by $\tau$is $\psi(t-\tau)$. For any analyzing wavelet $\psi(t)$ we thus define a family of functions $\psi_{s,\tau}(t)$ by the dilations and translations of $\psi(t)$ as given by the Eq. (3.2):

$$\psi_{s,\tau}(t) = \left\{\frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right) ;\ \ s,\tau \in \mathbb{R}^{+} \times \mathbb{R}\right\} \tag{2.2}$$

Each $\psi_{s,\tau}$ is called a wavelet.

Where, $\psi(t)$ is a mother wavelet,

$s$ is coefficient of expansions or scaling (dilation),

$\tau$ is a coefficient of translation.

The translation $\tau$and dilation $s$ allow the wavelet transform to be localized in time and frequency. Also, wavelet basis functions can represent functions with discontinuities and spikes in a more compact way than sine and cosine [33].

The wavelets are called ***orthogonal*** when their inner products are zero. The smaller the scaling factor is, the wider the wavelet is. Wide wavelets are comparable to the low-frequency sinusoids and narrow wavelets are comparable to high frequency sinusoids [29].

The following are some important definitions which are related to the wavelet transform.

**Definitions 2.1:**

**Translation 2.1.1 [34] :**

Let $f: \mathbb{R} \to \mathbb{R}$ and y $\in \mathbb{R}$ , then the translation $T_y: \mathbb{R} \to \mathbb{R}$ , is an operator defined by

$$T_y f(x) = f(x-y) \tag{2.3}$$

**t-Dilation 2.1.2 [34] :**

Let $f: \mathbb{R} \to \mathbb{R}$ and $t > 0$ , then the t-dilation $f_t: \mathbb{R} \to \mathbb{R}$ is defined by

$$f_t(x) = \frac{1}{t} f\left(\frac{x}{t}\right) \tag{2.4}$$

**Orthogonally 2.1.3 [35] :**

The expansion functions $\{\psi_{s,\tau}(x)\}$ form an orthogonal of basis functions:

$$\langle \psi_{s,\tau}(x), \psi_{s_1,\tau_1}(x) \rangle = \int \psi_{s,\tau}(x)\psi_{s_1,\tau_1}(x)dx$$

$$= \begin{cases} 1 & for \quad s = s_1, \ \tau = \tau_1 \\ 0 & for \quad s \neq s_1, \ \tau \neq \tau_1 \end{cases} \tag{2.5}$$

Where, $\psi_{s,\tau}(x)$ : is a decomposition wavelet,$s$ and $s_1$: are dilation (scaling) factors,$\tau$ and $\tau_1$: are translating (shifting) factors.

**Compact Support 2.1.4 [36]:**

We say that $\psi(x)$ has compact support on interval *I*, if it has zero values (vanish) outside this interval, so it is limited in time domain.

**Admissibility Condition 2.1.5 [36] :**

The inverse wavelet transform is held if the wavelet function is satisfied for any $\psi(x)$, then:

$$c = \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty \tag{2.6}$$

Where $\Psi(x)$ is the Fourier transform of the fundamental mother wavelet $\psi(x)$, $\Psi(\omega) = \int_{-\infty}^{\infty} \psi(x)\, e^{-j\omega x} dx$, and $c$ represents how closely correlated of the wavelet with this section of the signal, so the higher value of *c* is the more of

similarity. Then $c$ is positive and finite, in most cases, this simply means that $\Psi(0) = 0$ and $\Psi(\omega) \to 0\ as\ \omega \to \infty$ fast enough to make $c < \infty$.

The requirement that $c$ be positive and finite imposes another restriction of the choice of wavelet.

**Normalized 2.1.6 [35] :**

A wavelet function is defined as a function with a zero average

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 0 \ \ and \ \ \|\psi(x)\| = 1 \tag{2.7}$$

It is normalized, and centered in the neighborhood for $x = 0$

**Series Expansion 2.1.7 [37] :**

A function or signal $f(x)$ can often be better analyzed as a linear combination of expansion function.

$$f(x) = \sum_{k} \alpha_k \emptyset_k(x) \tag{2.8}$$

Where, $k$ is an integer index of the finite or infinite sum, $\alpha_k$are real valued expansion coefficients, $\emptyset_k(x)$are real valued expansion functions and called (basis functions).

The expressible functions form a function space that is referred to as the closed span of the expansion set, denoted by:

$$V = \overline{span\{\emptyset_k(x)\}} \tag{2.9}$$

**Scaling Functions 2.1.8 [37] :**

The set of expansion functions composed of the integer translations and binary scaling of the real, square integrable function $\emptyset(x)$; is the set $\{\emptyset_{s,\tau}(x)\}$

where

$$\emptyset_{s,\tau}(x) = 2^{s/2}\emptyset(2^s x - \tau), \forall s, \tau \in \mathbb{Z}\ and\ \emptyset(x) \in L^2(\mathbb{R}) \qquad (2.10)$$

Where, $s$ : is $\emptyset_{s,\tau}(x)$'s width, how broad or narrow it is along *x*-axis,

$\tau$ : is the position of $\emptyset_{s,\tau}(x)$ along the *x*-axis,

$2^{\frac{s}{2}}$: is controls $\emptyset_{s,\tau}(x)$'s height or amplitude.

Because the shape of $\emptyset_{s,\tau}(x)$ changes with *s*, $\emptyset(x)$ is called a ***scaling function***. By Eq. (2.9), we will denote the subspaces spanned over $\tau$for any *s* as,

$$V_s = \overline{span_\tau\{\emptyset_{s,\tau}(x)\}} \qquad (2.11)$$

A $V_s$ sequence of nested closed subspaces functions spanned by $\emptyset_{s,\tau}$ over τ for any *s*; by Eq. (2.8), if $f(x) \in V_s$ then, it can be written as

$$f(x) = \sum_\tau \alpha_\tau \emptyset_{s,\tau}(x) \qquad (2.12)$$

It is noted that increasing *s* increases the size of $V_s$ , which implies that functions with smaller variations will be included in the subspace. This is because when s is increasing, the set $\{\emptyset_{s,\tau}(x)\}$ becomes narrower and separated by smaller changes in *x*, as will be seen in the following example:

**Example 2.1 [37]:** Consider the unit-height, unit-width scaling function $\emptyset(x)$, which is called Haar scaling function, where;

$$\emptyset(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & otherwise \end{cases} \qquad (2.13)$$

Figures 2.4 (a)-(d) depicts some of the many expansion functions by substituting the given scaling function into Eq. (2.10).

Note that the expansion functions for $s$ = 1 in Figures 2.4(c) and (d) are half as wide as those for $s$ = 0 in Figures 2.4(a) and (b). In addition, it can be defined twice as many $V_1$ scaling functions as $V_0$ scaling functions (for example:$\emptyset_{1,0}$ and $\emptyset_{1,1}$ of $V_1$ versus $\emptyset_{0,0}$ of $V_0$for $x \in [0,1)$.

Figure 2.4(e) shows a member of subspace $V_1$ . This function does not belong to $V_0$ because the $V_0$ expansion functions in Figures 2.4(a) and (b) are too coarse to represent it. Higher resolution functions like those in Figures 2.4(c) and (d) are required. They can be used, as shown in (e), to represent the function by the expansion

$$f(x) = \frac{1}{2}\emptyset_{1,0}(x) + \emptyset_{1,1}(x) - \frac{1}{4}\emptyset_{1,4}(x) \tag{2.14}$$

Also, the decomposition of $\emptyset_{0,0}(x)$ as a sum of $V_1$ expansion functions is included in Figure 2.4(f). Similarly, any $V_0$ expansion function can be decomposed using the following relation:

$$\emptyset_{0,\tau}(x) = \frac{1}{\sqrt{2}}\emptyset_{1,2\tau}(x) + \frac{1}{\sqrt{2}}\emptyset_{1,2\tau+1}(x) \tag{2.15}$$

Thus, if $f(x) \in V_0$then $f(x) \in V_1$. This means $V_0 \subset V_1$ .

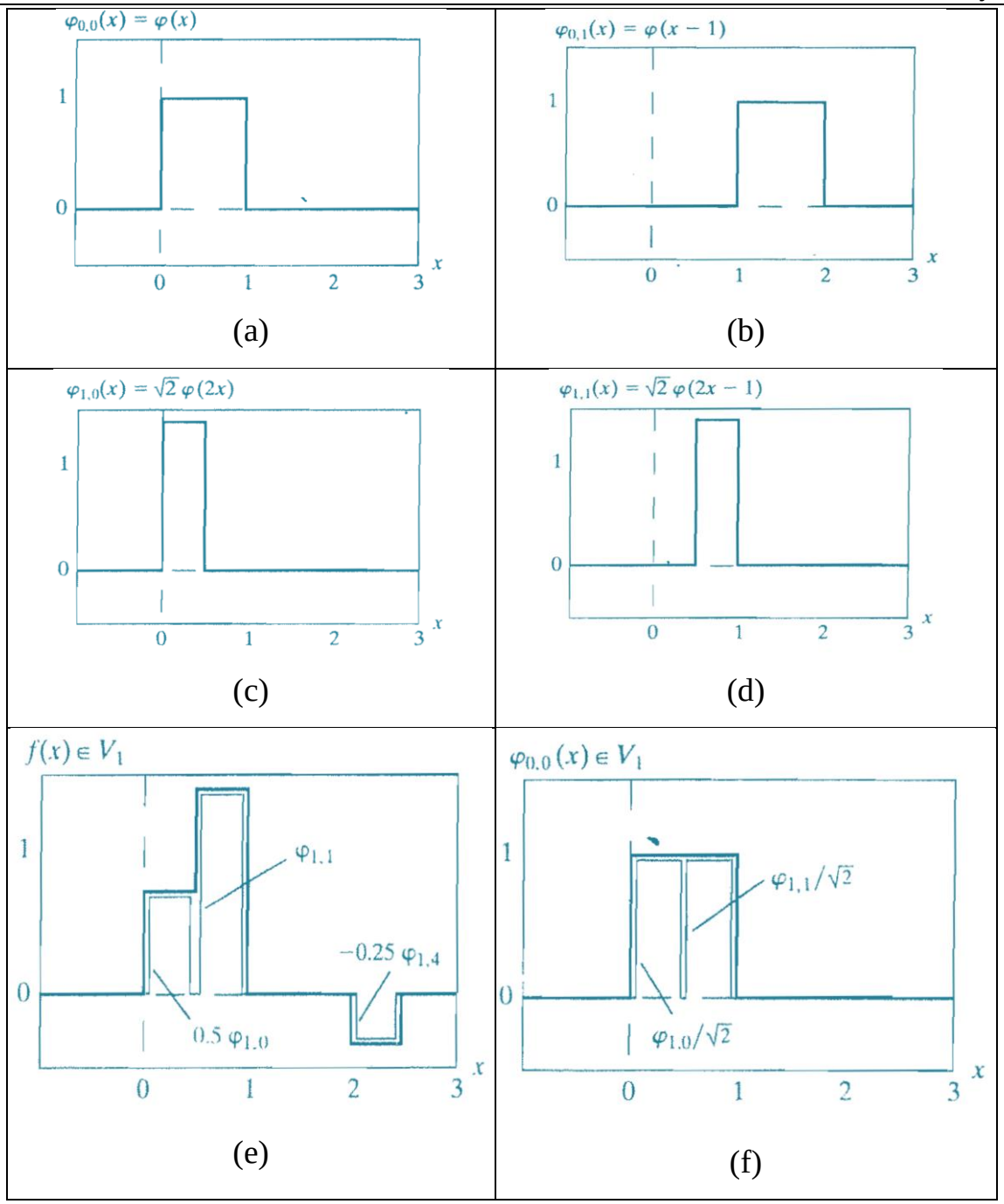



**Fig. 2.4 : Some Haar Scaling Functions [37]**

**Wavelet Function 2.1.9 [37] :**

A wavelet function $\psi_{s,\tau}(x)$ can be defined as spans for the difference between any two adjacent scaling subspaces, $V_s$and $V_{s+1}$.

The set $\psi_{s,\tau}(x)$ of wavelets can be defined as a basis wavelet from the mother function $\psi(x)$ such as:

$$\psi_{s,\tau}(x) = 2^{\frac{s}{2}}\psi(2^s x - \tau) \tag{2.16}$$

Where $s, \tau \in \mathbb{Z}$ and $\psi(x) \in L^2(\mathbb{R})$ that spans the $W_s$ spaces as

$$W_s = \overline{span_\tau\{\psi_{s,\tau}(x)\}} \tag{2.17}$$

Where $W_s$ is a sequence of closed subspaces of functions spanned by $\psi_{s,\tau}(x)$ over $\tau$ for any $s$, by Eq. (3.8), if $f(x) \in W_s$ then, it can be written as

$$f(x) = \sum_{\tau} \alpha_\tau \psi_{s,\tau}(x) \tag{2.18}$$

►**Remark 2.1 [37]:**

The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales. So, the subspaces containing high-resolution function must also contain all lower resolution functions, that is,

$$V_{-\infty} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_\infty = L^2(R) \tag{2.19}$$

Figure 3.5 explains this remark.

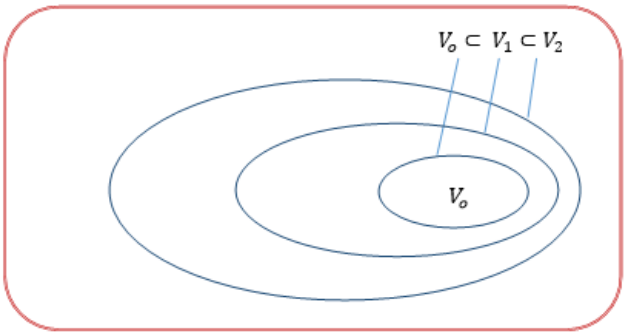



**Figure. 2.5: The Nested Function Spaces Spanned By a Scaling Function [37].**

**► Remark 2.2 [37]:**

The scaling and wavelet functions subspace are related by

$$V_{s+1} = V_s \oplus W_s \;\; and \;\; L^2(R) = V_{s_0} \oplus W_{s_0} \oplus W_{s_0+1} \oplus \cdots \tag{2.20}$$

Where $\oplus$ denotes the union of spaces (like the union of sets) and $V_s \perp W_s$.

For constructing orthogonal wavelets, the theory of ***multiresolution analysis*** (*MRA*) explained a systematic method to achieve this task. The first basic concept of (*MRA*) was introduced by Mallat [38]. This concept is the material of the following section.

## 2.3 Multiresolution Analysis

The (*MRA*) is related to the representation and analysis of signals at more than one resolution. It is to approximate a function *f(x)* at various levels of resolution by analyzing a function at different scales [37]. In (*MRA*), two functions are taking into consideration: the scaling function $\phi(x)$ which structures, using Eq.(2.10), a number of scaling functions $\phi_{s,\tau}(x)$ by the dilated (scaled) and translated (shifted); and the mother wavelet $\psi(x)$, which structures, using Eq.(2.16), a number of orthogonal wavelet basis functions $\psi_{s,\tau}(x)$.

$\|\psi_{s,\tau}\|_{L^2}^2 = \|\psi\|_{L^2}^2$ , because if we let $= (2^s x - \tau)$ , we obtain

$$\|\psi_{s,\tau}\|_{L^2}^2 = \int_{\mathbb{R}} \left|2^{s/2}\psi(2^s x - \tau)\right|^2 dx = \int_{\mathbb{R}} 2^s |\psi(2^s x - \tau)|^2 \, dx$$

$\int_{\mathbb{R}} |\psi(u)|^2 \, du = \|\psi\|_{L^2}^2$ .

Similarly, $\|\phi_{s,\tau}\|_{L^2}^2 = \|\phi\|_{L^2}^2$ . In fact, $\|\phi_{s,\tau}\|_{L^2}^2 = \|\psi_{s,\tau}\|_{L^2}^2$ , $\forall s, \tau \in \mathbb{Z}$ .

To design a multiresolution analysis, we need a set of nested spaces, by selecting the functions $\phi_{s,\tau}, \forall s, \tau \in \mathbb{Z}$ ; we determine the nested spaces $V_s$ , $\forall s \in \mathbb{Z}$ . For fixed *s* , the set of scaling functions $\phi_{s,\tau}$ are orthonormal.

Now consider $V_s$ to be the vector spaces corresponding to spanning set $\{\phi_{s,\tau}(x)\}$, assuming that the resolution increases with decreasing *s*, and these vector spaces characterize successive approximation vector spaces, ( i.e, each space $V_s$ is contained in the next resolution space $V_{s+1}$), this is depicted in Figure 2.5 and each with resolution $2^s$ such as in Eq.(2.19) [38].

To say that $f(x) \in V$ means that *f*(x) belongs to the closed span of $\phi_{s,\tau}(x)$ and can be written in the form of Eq.(2.12).

Thus, a (*MRA*) with scaling function $\phi$ consists of a sequence of closed subspaces $\{V_s\,,\ s \in \mathbb{Z}\}$ of $L^2(R)$ which have the following properties [37]:

1. $V_s \subset V_{s+1}\,, \forall s \in \mathbb{Z}$. (Nested)
2. $\overline{\bigcup_{s\in\mathbb{Z}} V_s} = L^2(\mathbb{R})$ . (Density)
3. $\bigcap_{s\in\mathbb{Z}} V_s = \{0\}$ . (Trivial intersection)
4. The following scale relations exist:

$f(x) \in V_s \leftrightarrow f(2x) \in V_{s+1}$ . (Scaling invariance property)
$f(x) \in V_0 \leftrightarrow f(x-n) \in V_0\,, \forall n \in \mathbb{Z}$ . (Scaling invariance property)

5. There exists a scaling function $\phi \in V_0$ such that its integer translates, $\{\phi(x-n)\}$ is an orthonormal basis of $V_0$, where $\phi_{s,\tau}(x)$ as in Eq.(2.10).

This means that the basic rule of multiresolution analysis is that whenever the above properties are satisfied, there exists an orthonormal wavelet basis and scaling basis such that any $f(x) \in L^2(\mathbb{R})$ can be expanded as a linear combination of both the scaling basis function$\phi_{s,\tau}(x)$ and the wavelet basis functions$\psi_{s,\tau}(x)$ .

In a (MRA), since $\phi(x) \in V_0 \subset V_1$ and $\{\phi_{s,\tau}(x), \tau \in \mathbb{Z}\}$ is an orthonormal basis of $V_1$, there exists some set of coefficients $\{a_n\,, n \in \mathbb{Z}\}$ such that the function $\phi(x) \in V_0$ can be represented as a linear combination of the functions from $V_1$ and so [37],

$$\phi(x) = \sum_n a_n \, \phi(2x - n) \tag{2.21}$$

Alternatively, if we consider property (4: Scaling invariance property) of (MRA) written as a linear combination of $\phi(x)$ in the scaled form defined in Eq.(2.3.1), the recursion for $\phi(x)$ can be written in terms of a new set of coefficients $\{h_n\}$ as:

$$\phi(x) = \sqrt{2} \sum_n h_n \, \phi(2x - n) \tag{2.22}$$

where,

$$\sum_n a_n = \sqrt{2} \sum_n h_n \tag{2.23}$$

For some coefficients $\{h_n \, , n \in \mathbb{Z}\}$, using the fact that $\{\phi_{s,\tau}(x)\}$ are orthonormal. The coefficients $\{h_n \, , n \in \mathbb{Z}\}$ can be obtained by computing the inner product:

$$h_n = \sqrt{2} \int_{-\infty}^{\infty} \phi(x) \, \phi(2x - n) \, dx \tag{2.24}$$

Where the function $\phi(x)$ is the scaling function.

It is interesting to notice that the scaling function $\phi(x)$ can be recursively generated by scaled (shrunk to half) and shifted versions of itself, as it is described by Eq. (2.21). This means that the scaling function $\phi(x)$ has the self similarity property.

For each (*MRA*), it is also possible to define a mother wavelet, $(x)$ , which will explain the detail at each level *s* . Assume $W_s$ to be the orthogonal

complement of $V_s$ in $V_{s+1}$ ; that is $W_s$ is the difference between the function space $V_{s+1}$ spanned by scaling functions $\phi_{s+1,\tau}(x)$ ; and the function space $V_s$ spanned by $\phi_{s,\tau}(x)$ , so that

$$V_{s+1} = V_s \oplus W_s \tag{2.25}$$

Where $\oplus$ represents the union of the two spaces. The space $W_s$ is composed of all functions representable in $V_{s+1}$ but not representable in $V_s$ . This can be carried out recursively to get:

$$L^2(\mathbb{R}) = V_\infty = V_s \oplus W_s \oplus W_{s+1} \oplus W_{s+2} \oplus \ldots \tag{2.26}$$

Similar to a function space$V_s$ spanned by the scaling functions$\phi_{s,\tau}(x)$, the functionspace $W_s$ is also spanned by a set of basis function, called the wavelet functions. Then the fundamental result is that$\{\psi(x-\tau),\ \tau \in \mathbb{Z}\}$ forms an orthonormal basis for $W_s$ , and$W_0 \subset V_1$ , orthogonal to all functions in $V_s$ , where $\psi_{s,\tau}(x)$ is defined as in Eq. (2.16).

The wavelet functions$\psi_{s,\tau}(x)$ can be expanded in the space $V_{s+1}$ as:

$$\psi_{s,\tau}(x) = \sum_n g_n\, \phi_{s+1,n}(x) = \sum_n g_n\, 2^{(s+1)/2} \phi(2^{s+1}x - n) \tag{2.27}$$

Where $g_n$ are the expansion coefficients. Usually we let $s = 0$ and drop the subscripts $s$ and $\tau$ to indicate that any wavelet function $\psi_s(x) \in V_1$ can be expressed as a linear combination of the basis scaling functions $\phi_{s+1}$ of the functions from $V_1$ ;

$$\psi(x) = \sqrt{2} \sum_n g_n\, \phi(2x - n) \tag{2.28}$$

This is in the same form for the scaling functions Eq.( 2.22).

Eq. (2.28) can be expressed as:

$$\psi(x) = \sum_{n} (-1)^n \, a_{1-k} \phi(2x - n) \tag{2.29}$$

Where,

$$g_n = (-1)^n 2^{-1/2} a_{1-k} \tag{2.30}$$

The coefficients $\{g_n\}$ can be obtained by computing the inner product:

$$g_n = \sqrt{2} \int_{-\infty}^{\infty} \psi(x) \, \phi(2x - \tau) dx \tag{2.31}$$

These coefficients are called coefficients of highpass filter. Coefficients of highpass filter can be calculated from coefficients of lowpass filter using this Equation [37]:

$$g_n = (-1)^n h_{1-n} \tag{2.32}$$

**Remark 2.3:** The scaling function in example (2.1) satisfies the requirements of (*MRA*).

**Example 2.2:** The Haar scaling coefficients was defined with the boundary conditions:

$h_\phi(0) = h_\phi(1) = \frac{1}{\sqrt{2}}$. Using Eq.( 2.32), the coefficients for the wavelet function are:

$g_0 = (-1)^0 h_{1-0} = h_1 = \frac{1}{\sqrt{2}}$ and $g_1 = (-1)^1 h_{1-1} = -h_0 = -\frac{1}{\sqrt{2}}$.

Substituting these values into Eq.(2.28), we obtained the wavelet functions as:

$$\psi(x) = \sqrt{2}\sum_n g_n\,\phi(2x-n) = \sqrt{2}\frac{1}{\sqrt{2}}\phi(2x) - \sqrt{2}\frac{1}{\sqrt{2}}\phi(2x-1)$$

$$= \phi(2x) - \phi(2x-1)\ .$$

Which is plotted in Figure 2.6(a), thus, the Haar wavelet function is

$$\psi(x) = \begin{cases} 1 & 0 \le x < 0.5 \\ -1 & 0.5 \le x < 1 \\ 0 & otherwise \end{cases}$$

Using Eq. (2.16), we can generate the universe of scaled and translated Haar.Figure 2.6(a) explains $\psi_{0,0}(x) = 2^0\psi(2^0x - 0) = \psi(x)$, and two such wavelets $\psi_{0,2}(x)$ and $\psi_{1,0}(x)$ are plotted in Figure 2.6(b) and Figure 2.6(c) respectively, where;

$$\psi_{0,2}(x) = 2^0\psi(2^0x - 2) = \psi(x-2)$$

$$\psi_{1,0}(x) = 2^{\frac{1}{2}}\psi(2^1x) = \sqrt{2}\psi(2x)$$

Finally, Figure 2.6(d) shows a function of space $V_1$that is not in the subspace $V_0$ . However, Eq.( 2.25) indicates that it can be expanded using $V_0$ and $W_0$ expansion functions as:

$$f(x) = f_a(x) + f_d(x) \qquad (2.33)$$

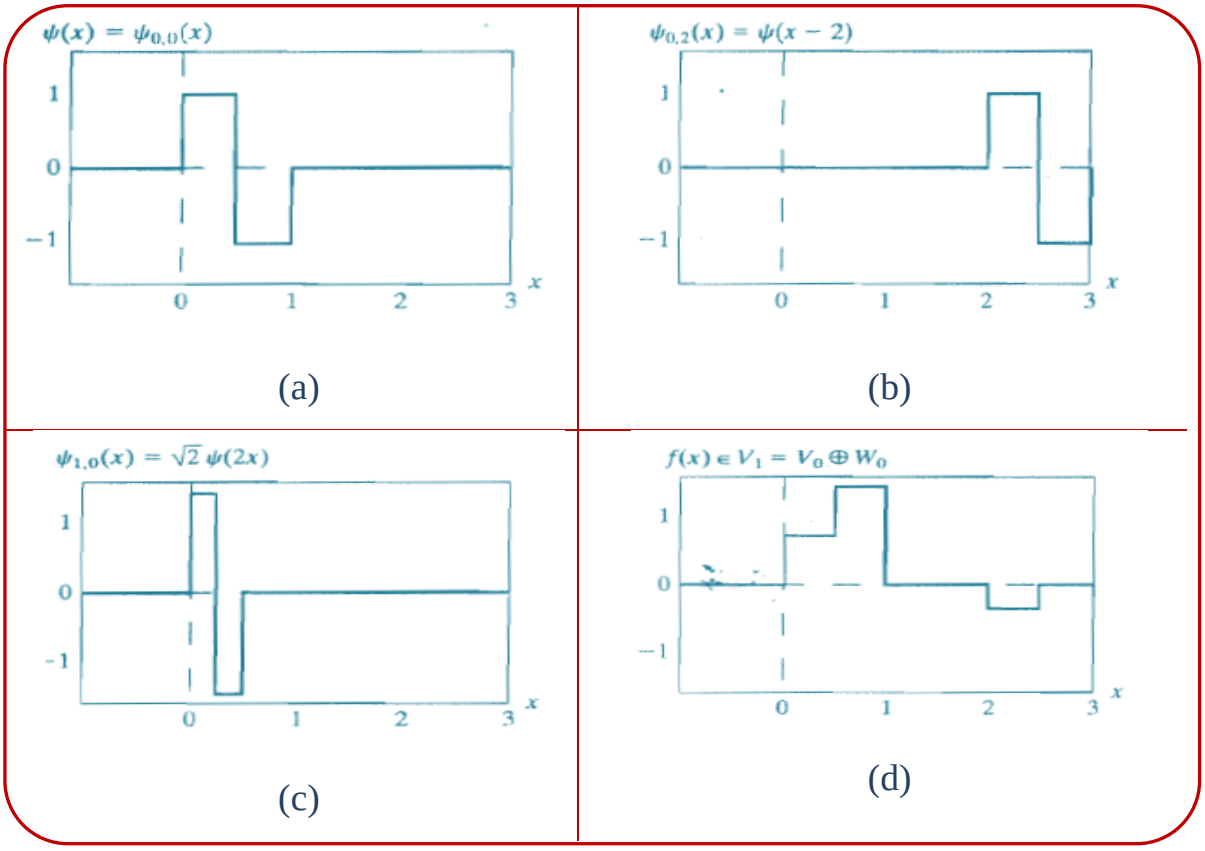
Where,

$$f_a(x) = \frac{3\sqrt{2}}{4}\phi_{0,0}(x) - \frac{\sqrt{2}}{8}\phi_{0,2}(x)$$

and

$$f_d(x) = \frac{-\sqrt{2}}{4}\psi_{0,0}(x) - \frac{\sqrt{2}}{8}\psi_{0,2}(x).$$

Here, $f_a(x)$is an approximation of $f(x)$using $V_0$ scaling functions, while $f_d(x)$ is the difference $f(x) - f_a(x)$ as a sum of $W_0$ wavelets. The two expansions are shown in Figures 2.6(e) and (f) [37].



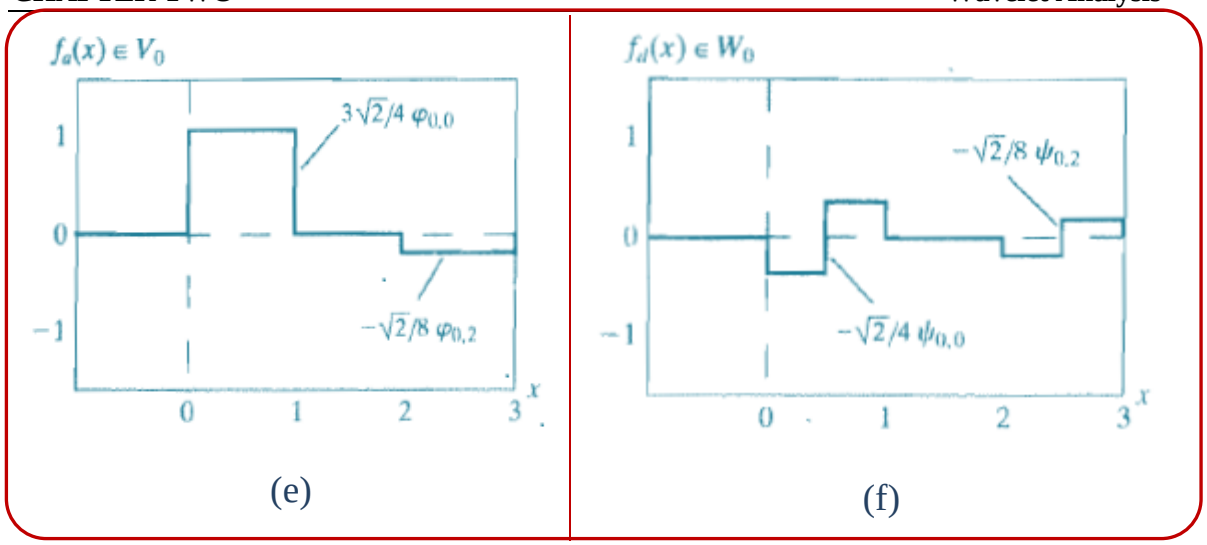



**Figure. 2.6: Haar Wavelet Functions in $W_0$ and $W_1$ [37]**

The wavelet transform has two types of transforms that are explained in the following section.

## 2.4 Wavelet Transform Types

There are two types of wavelet transform: The continuous wavelets transform (CWT), and the discrete wavelets transform (DWT).

### 2.4.1 The continuous wavelets transform (CWT):

The CWT of a function $f \in L^2(R)$ , includes a mother wavelet $\psi(x)$. The mother wavelet can be any continuous function, real or complex, that satisfies the following properties [36]:

1. The total area under the curve of the function is zero, such that :

$$\int_{-\infty}^{\infty} \psi(x)dx = 0 \qquad (2.34)$$

2. The total area of $|\psi(x)|^2$ is finite , such that

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx \; < \infty \tag{2.35}$$

The CWT in 1-D of a square integrable function *f*(x) with respect to the wavelet $\psi(x)$ is a function $W_\psi(s,\tau)$ of two variables $s,\tau$ , and defined as:

$$W_\psi(s,\tau) = \int_{-\infty}^{\infty} f(x)\psi_{s,\tau}(x)dx \tag{2.36}$$

Where,

$$\psi_{s,\tau}(x) = \frac{1}{\sqrt{s}}\psi\left(\frac{x-\tau}{s}\right), s,\tau \in \mathbb{R}; s \neq 0 \tag{2.37}$$

$W_\psi(s,\tau)$ : is the wavelet coefficient of the function *f(x)*.

$s$ : is the scale parameter.

$\tau$ : is the position parameter.

The quantity $\frac{1}{\sqrt{s}}$ is a normalizing factor that guarantees the energy of $\psi(x)$ remains independent of *s* and $\tau$, such that [36]:

$$\int_{-\infty}^{\infty} \left|\psi_{s,\tau}(x)\right|^2 dx = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx \tag{2.38}$$

For any $s$, $\psi_{s,\tau}(x)$ is a copy of $\psi_{s,0}(x)$ shifted $\tau$ along the time axis. Setting $\tau = 0$, $\tau$ is said to be ***translation*** parameter, such that :

$$\psi_{s,0}(x) = \frac{1}{\sqrt{s}}\psi\left(\frac{x}{s}\right) \quad for\ any\ s \tag{2.39}$$

The parameter $s$ is said to be a ***scaling*** (dilation) parameter.

If $s > 1$ stretch the wavelet, while $0 < s < 1$ shrink the wavelet, as shown in Figure 2.7 [36].

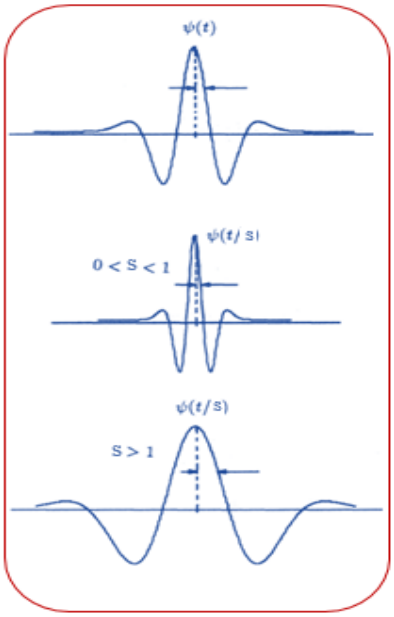



**Figure. 2.7: Typical Wavelet Family in Time and Frequency Domains**

The inverse continuous wavelet transformation of the function *f*(*x*) is given by the formula:

$$f(x) = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^\infty \frac{1}{s^2} W_\psi(s,\tau)\psi_{s,\tau}(x) d\tau\, ds \tag{2.40}$$

Where,

$W_\psi(s,\tau)$: is a given wavelet coefficient

$\psi_{s,\tau}(x)$ : is Wavelet function.

$C_\psi$ : is defined as

$$C_\psi = \int_{-\infty}^\infty \frac{|\Psi(\omega)|^2}{|\omega|} d\omega \tag{2.41}$$

And $\Psi(\omega)$ is the Fourier transform of $\psi(x)$, such that:

$$\Psi(\omega) = \int_{-\infty}^\infty \psi(x) e^{-i\omega x} dx \tag{2.42}$$

The inverse CWT exists if $C_\psi$ is positive and finite. Since $C_\psi$ is defined by means of $\Psi$, which itself is defined by means of the wavelet $\psi(x)$, the requirement that $C_\psi$ be positive and finite imposes another restriction, called the ***admissibility condition***, on the choice of wavelet. So the wavelet is called admissible if $C_\psi < \infty$ [36].

In many applications in engineering and science, the sampled data are discrete in time. Thus, a discrete representation of time and frequency is needed, which is called the **D*iscrete Wavelet Transform*** (DWT). Before giving the definition of DWT, we need to explain the concept of ***wavelet series expansion***.

### 2.4.2 Wavelet Series Expansion[37]:

For a specific value $s = s_0$ , by means of these subspaces, the Eq. (2.25) discussed in section (2.3) can be decomposed as

$$L^2(\mathbb{R}) = V_\infty = V_{s_0} \oplus W_{s_0} \oplus W_{s_0+1} \oplus W_{s_0+2} \oplus \dots \tag{2.43}$$

This is shown in Figure 2.8.

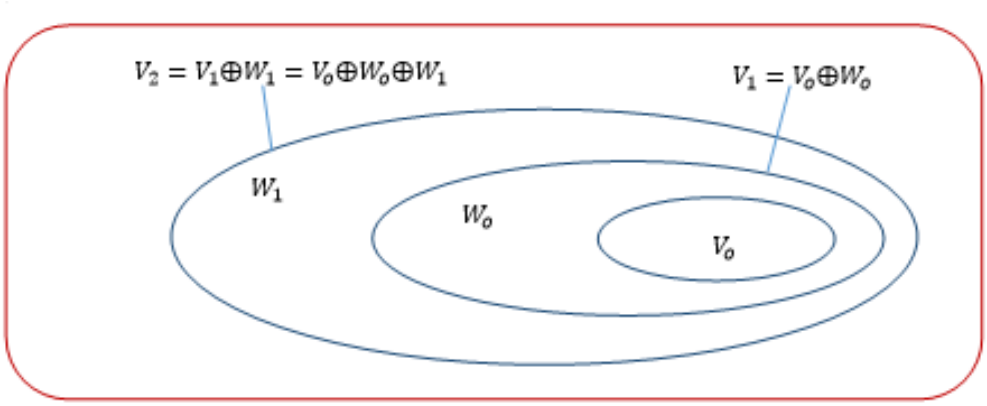



**Fig. 2.8: The Relationship Between Scaling and Wavelet Function Space**

It indicates that any square-integrable function $f(x) \in L^2(\mathbb{R})$ can be expanded as a linear combination of both the scaling basis functions $\emptyset_{s_0,\tau}(x)$ and the wavelet basis functions $\psi_{s,\tau}(x)$ , $s = s_0, s_0 + 1, \dots$ . That is, each function $f(x) \in L^2(\mathbb{R})$ can be represented in its wavelet series expansion:

$$f(x) = \sum_{\tau} c_{s_0,\tau} \emptyset_{s_0,\tau}(x) + \sum_{s=s0}^{\infty} \sum_{\tau} d_{s,\tau}\, \psi_{s,\tau}(x) \tag{2.44}$$

where $c_{s_0,\tau}$ is called ***approximation coefficient*** or ***scaling coefficients*** defined as:

$$c_{s_0,\tau}(\tau) = \langle f(x), \emptyset_{s_0,\tau}(x) \rangle = \int f(x) \emptyset_{s_0,\tau}(x)\, dx \qquad (2.45)$$

and $d_{s,\tau}$ is called ***detail coefficient*** or ***wavelet coefficients*** defined as:

$$d_{s,\tau}(\tau) = \langle f(x), \psi_{s,\tau}(x) \rangle = \int f(x) \psi_{s,\tau}(x)\, dx \qquad (2.46)$$

The first term contained in the wavelet expansion of the function $f(x)$ Eq. (2.44) represents the ***approximation*** of the function at scale level $s_0$ by the linear combination of the scaling functions $\emptyset_{s_0,\tau}(x)$ , and the summation with index *s* in the second term in the expansion is for the ***details*** of different levels contained in the function $f(x)$ approximated by the linear combination of the wavelet functions of progressively higher scales $s_0 + 1, s_0 + 2, \ldots$ .

**Example 2.3:** Consider a continuous function $f(x)$, which is defined over the period $0 \leq x \leq 1$:

$$f(x) = \begin{cases} x^2 & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

As shown in Figure 3.9. Using Harr wavelet and a starting scale $s_0 = 0$, each individual space $V_0, W_0, W_1, \ldots$ is spanned by different number of basis functions. Eqs. (2.45) and (2.46)can be used to compute the expansion coefficients:

Since $c_{s_0}(\tau) = \int f(x), \emptyset_{s_0,\tau}(x)\, dx$ , then;

$$c_0(0) = \int_0^1 x^2\, \emptyset_{0,0}(x)dx = \int_0^1 x^2\, dx = \frac{1}{3}$$

$$d_0(0) = \int_0^1 x^2\, \psi_{0,0}(x)dx = \int_0^{\frac{1}{2}} x^2\, dx \; - \int_{\frac{1}{2}}^1 x^2\, dx \; = -\frac{1}{4}$$

$$d_1(0) = \int_0^1 x^2\, \psi_{1,0}(x)dx = \int_0^{\frac{1}{4}} x^2\, \sqrt{2}\, dx \; - \int_{\frac{1}{4}}^{\frac{1}{2}} x^2\, \sqrt{2}\, dx \; = -\frac{\sqrt{2}}{32}$$

$$d_1(1) = \int_0^1 x^2\, \psi_{1,1}(x)dx = \int_0^{\frac{3}{4}} x^2\, \sqrt{2}\, dx \; - \int_{\frac{3}{4}}^1 x^2\, \sqrt{2}\, dx \; = -\frac{3\sqrt{2}}{32}$$

Substituting these values into Eq. (3.44), we get the wavelet series expansion

$$f(x) = y = \underbrace{\underbrace{\frac{1}{3}\emptyset_{0,0}(x)}_{V_0} + \underbrace{\left[-\frac{1}{4}\psi_{0,0}(x)\right]}_{W_0}}_{V_1 = V_0 \oplus W_0} + \underbrace{\left[-\frac{\sqrt{2}}{32}\psi_{1,0}(x) - -\frac{3\sqrt{2}}{32}\psi_{1,1}(x)\right]}_{W_1}}_{V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1}$$

$$+ \cdots$$

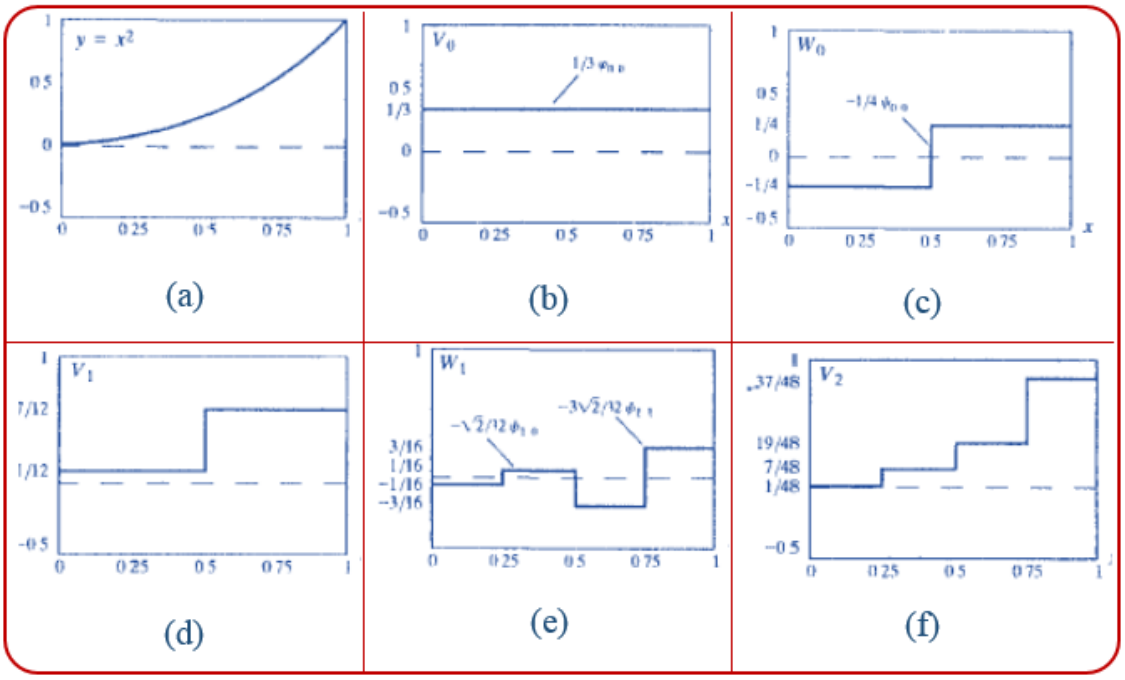



**Figure. 2.9:Wavelet Series Expansion of $f(x)=x^2$ Using Haar Wavelets [37]**

The first term in this expansion uses $c_0(0)$ to generate a subspace $V_0$ approximation of the function being expanded. This approximation is shown in Figure 2.9(b) and is the average value of the original function. The second term uses $d_0(0)$ to refine the approximation by adding a level of detail from subspace $W_0$. The added detail and resulting $V_1$ approximation are shown in Figures 2.9(c) and (d), respectively. Another level of detail is added by the subspace $W_1$ cofficients $d_1(0)$ and $d_1(1)$ . This additional detail is shown in Figure 2.9(e) and the resulting $V_2$ approximation is depicted in Figure 2.9(f).

In summary, the series expansion of a function in terms of a given set of simple function $\{f_n\}_{n=0}^{\infty}$ , under certain conditions, is given by

$$f(x) = \sum_{n=0}^{\infty} a_n f_n(x)$$

These conditions are differ and depending on the function $f(x)$ and the set $\{f_n\}_{n=0}^{\infty}$, and they enable us finding the coefficients $a_n$ . For example, Fourier series expands a square integrable function using the orthogonal trigonometric system $\left\{\sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right\}$ where $L$ is the function's period. The orthogonality of such system on $[-L, L]$ enable us to determine the coefficients $a_n$ . On the other hand, the Taylor series expands a periodic function of period $L$ that is analytic at point $x_0$ using the orthogonal system $\{(x - x_0)^n\}_{n=0}^{\infty}$ .

### 2.4.3 The Discrete Wavelet Transform:

The DWT was proposed by Mallat (1989) [38], is an efficient algorithm for calculating the coefficients of the wavelet transform of a discrete series. It is like the Fourier series expansion, the wavelet series expansion of the previous section maps the function of a continuous variable into a sequence of coefficients. If the function being expanded is discrete, the resulting coefficients are called the DWT [37].

The DWT in one – dimension (1-D) is given by:

$$W_{\emptyset}(s_0, \tau) = \frac{1}{\sqrt{M}} \sum_{x} f(x) \emptyset_{s_0,\tau}(x) \qquad (2.47)$$

called the ***approximation*** or scaling coefficients, and

$$W_{\psi}(s,\tau) = \frac{1}{\sqrt{M}} \sum_{x} f(x)\psi_{s,\tau}(x) \ , For\ s \geq s_0 \qquad (2.48)$$

called the ***detail*** or wavelet coefficients.

The inverse of DWT in (1-D) is given by:

$$f(x) = \frac{1}{\sqrt{M}} \sum_{\tau} W_{\emptyset}(s_0,\tau)\emptyset_{s_0,\tau}(x) + \frac{1}{\sqrt{M}} \sum_{s=s_0}^{\infty} \sum_{\tau} W_{\psi}(s,\tau)\psi_{s,\tau}(x) \qquad (2.49)$$

Here $f(x)$, $\emptyset_{s_0,\tau}(x)$, and $\psi_{s,\tau}(x)$ are functions of discrete variable *x*, for example

$f(x) = f(x_0 + k\Delta x)$and $k = 0,1,2,\dots,M-1$. The factor $\frac{1}{\sqrt{M}}$ is normalizing factor.

Normally, we let $s_0 = 0$ and select *M* to be a power of 2 (i.e. $M = 2^s$) so that the summations are performed over $k = 0,1,2,\dots,M-1; s = 0,1,2,\dots,s-1$ and $\tau = 0,1,2,\dots,2^s - 1$ [37].

The transform itself is a composed of *M* coefficients, the minimum scale is *0* and the maximum scale is ($s$ – 1).

The $W_{\emptyset}(s_0,\tau)$ and $W_{\psi}(s,\tau)$of Eqs.(2.47) and (2.48) correspond to the $c_{s_0}(\tau)$ 's and $d_s(\tau)$ 's of the ***wavelet series expansion*** that explained in the previous section, that is Eqs. (2.45) and (2.46).

**Example 2.4:** To compute the 1-D DWT coefficients, consider the discrete function of four points: *f*(0) = 1, *f*(1) = 4, *f*(2) = -3 and *f*(3) = 0. *M* = 4, *s* = 2

and $s_0 = 0$. The summations are performed over $x$ = 0,1,2,3 ;$s$ = 0,1 and $\tau$ = 0 for $s$ = 0 or $\tau$ =0,1 for $s$ = 1. We will use the Haar scaling and wavelet functions and assume that the four sampling of $f(x)$are distributed over the support of the basis functions, which is 1. Substituting the four samples into Eqs. (2.47) and (2.48), we find :

$$W_{\emptyset}(0,0) = \frac{1}{2}\sum_{x=0}^{3} f(x)\emptyset_{0,0}(x)$$

$$= \tfrac{1}{2}[1.1 + 4.1 - 3.1 + 0.1] = 1 \text{ (since } \emptyset_{0,0}(x) = 1 \text{ for } x = 0,1,2,3)$$

$$W_{\psi}(s,\tau) = \frac{1}{\sqrt{M}}\sum_{x} f(x)\psi_{s,\tau}(x)$$

$$W_{\psi}(0,0) = \frac{1}{2}[1.1 + 4.1 - 3.(-1) + 0.(-1)] = 4$$

$$W_{\psi}(1,0) = \frac{1}{2}\left[1.\sqrt{2} + 4.(-\sqrt{2}) - 3.0 + 0.0\right] = -1.5\sqrt{2}$$

$$W_{\psi}(1,1) = \frac{1}{2}\left[1.0 + 4.0 - 3.\sqrt{2} + 0.(-\sqrt{2})\right] = -1.5\sqrt{2}$$

Thus, the DWT of the given four sample function relative to the Haar wavelet and scaling function is $[1,4,-1.5\sqrt{2},-1.5\sqrt{2}]$, where the transform coefficients have been arranged in the order in which they were calculated.

To reconstruct the original function from its transform, using the Eq.(2.49):

$$f(x) = \frac{1}{\sqrt{M}} \sum_{\tau} W_{\emptyset}(s_0, \tau) \emptyset_{s_0,\tau}(x) + \frac{1}{\sqrt{M}} \sum_{s=s_0}^{\infty} \sum_{\tau} W_{\psi}(s, \tau) \psi_{s,\tau}(x)$$

Then;

$$f(x) = \frac{1}{2}\big[W_{\emptyset}(0,0)\emptyset_{0,0}(x) + W_{\psi}(0,0)\psi_{0,0}(x) + W_{\psi}(1,0)\psi_{1,0}(x) + W_{\psi}(1,1)\psi_{1,1}(x)\big]$$

For $x$ = 0,1,2,3. If $x$ = 0,

$$f(0) = \tfrac{1}{2}\big[1.1 + 4.1 - \big(1.5\sqrt{2}\big).\big(\sqrt{2}\big) - \big(1.5\sqrt{2}\big).0\big] = 1\ .$$

The one dimensional transforms (1-D) are extended easily to two dimensional transform (2-D) by using a 2-D scaling function $\phi(x, y)$, and three 2-D wavelets, $\psi^H(x, y), \psi^V(x, y), and\ \psi^D(x, y)$, so to reconstruct the signal $f(x, y)$ of size $M \times N$ in form the 2-D discrete wavelet coefficient$\big(W_f(s, m, n)\big)$, we found 2-D DWT pair becomes as [37]:

$$W_{\emptyset}(s_0, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \emptyset_{s_0,m,n}(x, y) \qquad (2.50)$$

$$W_{\psi}^{i}(s, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \psi_{s,m,n}^{i}(x, y) \qquad (2.51)$$

where,

$$\emptyset_{s_0,m,n}(x, y) = 2^{\frac{s_0}{2}} \phi(2^{s_0}x - m, 2^{s_0}y - n) \qquad (2.52)$$

$$\psi^{i}_{s,m,n}(x,y) = 2^{\frac{s}{2}}\psi^{i}(2^{s}x - m, 2^{s}y - n) \qquad (2.53)$$

For $s > s_0$ and $i = \{H, V, D\}$ .

As in 1-D case, $s_0$ is an arbitrary starting scale and the$W_{\emptyset}(s_0, m, n)$ coefficients define an approximation of $f(x,y)$ at scale $s_0$. The $W^{i}_{\psi}(s,m,n)$ coefficients add horizontal, vertical, and diagonal details for scales; $s > s_0$.

Usually, let $s_0 = 0$ and $N = M = 2^{s}$ so that $s$= 0,1,2, … , $s - 1$and $M = N =$ 0,1,2, … , $2^{s} - 1$. The inverse discrete wavelet ransform IDWT for 2-D is given by [37] :

$$f(x,y) = \frac{1}{\sqrt{MN}}\sum_{m}\sum_{n} W_{\phi}(s_0,m,n)\phi_{s_0,m,n}(x,y) + \frac{1}{\sqrt{MN}}\sum_{i-H,V,D}\sum_{s=s_0}^{\infty}\sum_{m}\sum_{n} W^{i}_{\psi}(s,m,n)\psi^{i}_{s,m,n}(x,y) \qquad (2.54)$$

### 2.4.4 How the Wavelets Transform Works:

In order to explain how the wavelets transform works, we take the Haar wavelet transform, which is one of the simplest and basic transformations from the space domain to a local frequency domain.

The Haar wavelet becomes an efficient method for solving many problems in engineering and science due to its simplicity. Therefore, we will use this technique for solving differential equations in the next chapter.

A Haar transform decomposes each signal into two components, one is called ***average*** (approximation) and the other is known as ***difference*** (detail) [36]. The following steps illustrate how the Haar transform can be used to calculate of a matrix of $m \times n$ size.

***Step 1***: Find the average of each pair of elements.

***Step 2***: Find the difference between each average and the elements it was calculated from.

***Step 3***: Fill the first half of the matrix with averages.

***Step 4***: Fill the second half of the matrix with differences.

***Step 5***: Repeat the process on the first half of the matrix.

**Step 6**: If the dimension is odd number, we can add row (coulum) of zero elements.

In order to give an idea of its implementation, the procedure of its application may be explained with the help of a simple example as shown below.

**Example 2.5:** Consider the 8x8 matrix ,

$$M = \begin{bmatrix} 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \\ 9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\ 17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\ 40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\ 32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\ 41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\ 49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\ 8 & 58 & 59 & 5 & 4 & 62 & 63 & 1 \end{bmatrix}$$

We start with an arbitrary vector representing one row of an $8\times 8$ matrix.

***Step 1: Average***

$$y = \begin{bmatrix} 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \end{bmatrix}$$

$$\frac{64+2}{2} \quad \frac{3+61}{2} \quad \frac{60+6}{2} \quad \frac{7+57}{2} \Rightarrow \quad \mathrm{B} = \begin{bmatrix} 33 & 32 & 33 & 32 \end{bmatrix}$$

Where *B* is called ***approximation coefficients***, and the results become the first four entries of our modified string $y_1$ .

***Differencing***

$$64 - 33 \quad 3 - 32 \quad 60 - 33 \quad 7 - 32 \Rightarrow \quad \mathrm{C} = \begin{bmatrix} 31 & -29 & 27 & -25 \end{bmatrix}$$

Where $C$ is called ***detail coefficients***, and the results become the last four entries of $y_1$ .

$y_1 = [B \quad C]$

$y_1 = [33 \quad 32 \quad 33 \quad 32 \quad 31 \quad -29 \quad 27 \quad -25]$

***Step 2: Average***

$y_1 = [\mathbf{33 \quad 32 \quad 33 \quad 32} \quad 31 \quad -29 \quad 27 \quad -25]$

$$\frac{33+32}{2} \quad \frac{33+32}{2} \quad \Rightarrow \quad \mathrm{B} = [32.5 \quad 32.5]$$

***Differencing***

$$33 - 32.5 \quad 33 - 32.5 \quad \Rightarrow \quad \mathrm{C} = [0.5 \quad 0.5]$$

$y_2 = [\mathbf{32.5 \quad 32.5} \quad 0.5 \quad 0.5 \quad 31 \quad -29 \quad 27 \quad -25]$

***Step 3: Average***

$y_2 = [\mathbf{32.5 \quad 32.5} \quad 0.5 \quad 0.5 \quad 31 \quad -29 \quad 27 \quad -25]$

$$\frac{32.5+32.5}{2} \quad \Rightarrow \quad \mathrm{B} = [32.5]$$

***Differencing***

$$32.5 - 32.5 \quad \Rightarrow \quad \mathrm{C} = [0]$$

$y_3 = [\mathbf{32.5 \quad 0} \quad 0.5 \quad 0.5 \quad 31 \quad -29 \quad 27 \quad -25]$

The Haar wavelet does this transformation to each row of the matrix, and then again to every column in the matrix. The resulting matrix is known as the ***Haar wavelet transform*** of the original matrix. It is important to note at this point that this process is completely reversible. It is this fact that makes it possible to retrieve the original matrix from the Haar wavelet transform of the matrix.

Apply average and differencing to the entire matrix*M*:

***First***: the rows of matrix*M* is:

$$M=\begin{bmatrix}64&2&3&61&60&6&7&57\\9&55&54&12&13&51&50&16\\17&47&46&20&21&43&42&24\\40&26&27&37&36&30&31&33\\32&34&35&29&28&38&39&25\\41&23&22&44&45&19&18&48\\49&15&14&52&53&11&10&56\\8&58&59&5&4&6&63&1\end{bmatrix}\Rightarrow\begin{bmatrix}32.50&.5&.5&31&-29&27&-25\\32.50&-.5&-.5&-23&21&-19&17\\32.50&-.5&-.5&-15&13&-11&9\\32.50&.5&.5&7&-5&3&-1\\32.50&.5&.5&-1&3&-5&7\\32.50&-.5&-.5&9&-11&13&-15\\32.50&-.5&-.5&17&-19&21&-23\\32.57&.5&-13.5&-25&27&-1&31\end{bmatrix}$$

***Second:*** The columns of matrix *M* is:

$$\begin{bmatrix}32.50&.5&.5&31&-29&27&-25\\32.50&-.5&-.5&-23&21&-19&17\\32.50&-.5&-.5&-15&13&-11&9\\32.50&.5&.5&7&-5&3&-1\\32.50&.5&.5&-1&3&-5&7\\32.50&-.5&-.5&9&-11&13&-15\\32.50&-.5&-.5&17&-19&21&-23\\32.57&.5&-13.5&-25&27&-1&31\end{bmatrix}\Rightarrow\begin{bmatrix}32.50&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&4&-4&4&-4\\0&0&0&0&4&-4&4&-4\\0&0&.5&.5&27&-25&23&-21\\0&0&-.5&-.5&-11&9&-7&5\\0&0&.5&.5&-5&7&-9&11\\0&0&-.5&-.5&21&-23&25&-27\end{bmatrix}=N$$

A matrix $N$ can be represented as a more concise manner, with one overall average in the upper left-hand corner of the matrix is called ***approximation coefficients***. The remaining components are all ***detail coefficients*** that now represent the amount of detail in that area of the matrix. Because we know this, we can eliminate some information from the given matrix and still be capable of attaining a fairly good approximation of the original matrix. Doing this it can choose some number $\delta$ and set equal to zero all elements with a magnitude less than $\delta$.

Choosing $\delta = 5$, then eighteen of the detail coefficients (bold) are eliminated.

$$\begin{bmatrix} 32.50 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & \mathbf{0} & \mathbf{0} & 27 & -25 & 23 & -21 \\ 0 & 0 & \mathbf{0} & \mathbf{0} & -11 & 9 & -7 & \mathbf{0} \\ 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & 7 & -9 & 11 \\ 0 & 0 & \mathbf{0} & \mathbf{0} & 21 & -23 & 25 & -27 \end{bmatrix}$$

**Remark 2.4:**

The averaging and differencing method that we just discussed is very effective. Using linear algebra, we can use three matrices $(A_1, A_2, A_3)$ that perform each of the three steps of the averaging anddifferencing process. In our previous Example (2.5), the transformation of $y$ to $y_1$ can describe as

$$y_1 = yA_1$$

$$= [64 \;\; 2 \;\; 3 \;\; 61 \;\; 60 \;\; 6 \;\; 7 \;\; 57] \begin{bmatrix} 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & -1/2 \end{bmatrix}$$

$$= [\,33 \;\; 32 \;\; 33 \;\; 32 \;\; 31 \;\; -29 \;\; 27 \;\; -25]$$

It can next be shown that the transformation from $y_1$ to $y_2$ can be written as $y_2 = y_1 A_2$

$$= [\,33 \;\; 32 \;\; 33 \;\; 32 \;\; 31 \;\; -29 \;\; 27 \;\; -25] \begin{bmatrix} 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \end{bmatrix}$$

$$= [\,32.5 \;\; 32.5 \;\; 0.5 \;\; 0.5 \;\; 31 \;\; -29 \;\; 27 \;\; -25]$$

and lastly we can show that $y_3 = y_2 A_3$

$$= [\,32.5 \;\; 32.5 \;\; 0.5 \;\; 0.5 \;\; 31-29 \;\; 27-25] \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [\,32.5 \;\; 0 \;\; 0.5 \;\; 0.5 \;\; 31 \;\; -29 \;\; 27 \;\; -25]$$

This whole transformation can be done in one step by multiplying these three matrices together to obtain a single transform matrix $W = A_1 A_2 A_3$ . We can now multiply the original string by just one transform matrix to go directly from the original string to the final results of step 3.where $W$ is,

$$W = \begin{bmatrix} 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \end{bmatrix} \begin{bmatrix} 1/8 & 1/8 & 1/4 & 0 & 1/2 & 0 & 0 & 0 \\ 1/8 & 1/8 & 1/4 & 0 & -1/2 & 0 & 0 & 0 \\ 1/8 & 1/8 & -1/4 & 0 & 0 & 1/2 & 0 & 0 \\ 1/8 & 1/8 & -1/4 & 0 & 0 & -1/2 & 0 & 0 \\ 1/8 & -1/8 & 0 & 1/4 & 0 & 0 & 1/2 & 0 \\ 1/8 & -1/8 & 0 & 1/4 & 0 & 0 & -1/2 & 0 \\ 1/8 & -1/8 & 0 & -1/4 & 0 & 0 & 0 & 1/2 \\ 1/8 & -1/8 & 0 & -1/4 & 0 & 0 & 0 & -1/2 \end{bmatrix}$$

$$= [\,32.5 \;\; 0 \;\; 0.5 \;\; 0.5 \;\; 31 \;\; -29 \;\; 27 \;\; -25]$$

The following equations simplify this process of matrix multiplication of the averaging and differencing.

$T = ((MW)^T W)^T$

$T = (W^T M^T W)^T$

$T = W^T (M^T)^T (W^T)^T$

$T = W^T MW$

It is also important to note that the since every column of the individual matrices that comprise $W$ is orthogonal to every other column, the matrices are invertible. Thus,

$W^{-1} = A_3^{-1} A_2^{-1} A_1^{-1}$, then

$N = W^T MW$

$M = (W^T)^{-1} NW^{-1}$

Where, $M$ is original matrix

$W$ is transforming matrix

$N$ is compressed matrix (approximate matrix)

## 2.5 Wavelet Collocation Method:

Collocation method [39] is used in numerical solution of DEs. It involves numerical operators doing in point values (collocation points) in the physical space or dividing the domain as a number of points and determines the

solution about it. By chosen a wavelet and some types of grid structure that will be computational adapted, wavelet collocation methods may be created.

## 2.6 Haar Wavelets and Its Integration:

The Haar Wavelet family for $x \in [0,1)$is defined by

$$h_i(x) = 2^{\frac{j}{2}}\psi\left(2^j x - k\right) = \begin{cases} 1 & for\ x \in [\beta_1, \beta_2) \\ -1 & for\ x\ \in [\beta_2, \beta_3) \\ 0 & otherwise \end{cases} \tag{2.55}$$

Where

$$\beta_1 = \frac{k}{m}, \qquad \beta_2 = \frac{k+0.5}{m}, \qquad \beta_3 = \frac{k+1}{m}$$

And

$$m = 2^j \ , \qquad j = 0,1,2,\dots,J \ , \quad k = 0,1,2,\dots,m-1$$

$j$ indicates the level of wavelet, or the dilation parameter, $k$ denotes translation parameter, $J$ denotes the maximum level of resolution.

The index $i$ in $h_i(x)$ is computed by $i = m + k + 1$.

In the case of minimum values $m = 1$, $k = 0$, we have $i = 2$.

The maximum value of $I$ is $i = 2M = 2^{J+1}$.

For $i = 1$, the function $h_1(x)$ is the scaling function for the family of the Haar wavelets which is defined as

$$h_1(x) = \begin{cases} 1 & for\ x \in [0,1) \\ 0 & otherwise \end{cases} \tag{2.56}$$

For $i = 2$, the function $h_2(x)$ is the mother wavelet for the family of the Haar wavelet which is defined as

$$h_2(x) = \begin{cases} 1 & for\ x \in [0, \frac{1}{2}) \\ -1 & for\ x \in [\frac{1}{2}, 1) \\ 0 & otherwise \end{cases} \tag{2.57}$$

For $i = 3$, the function $h_3(x)$ is defined as

$$h_3(x) = \begin{cases} 1 & for\ x \in [0, \frac{1}{4}) \\ -1 & for\ x \in [\frac{1}{4}, \frac{1}{2}) \\ 0 & otherwise \end{cases} \tag{2.58}$$

For $i = 4$, the function $h_4(x)$ is defined as

$$h_4(x) = \begin{cases} 1 & for\ x \in [\frac{1}{2}, \frac{3}{4}) \\ -1 & for\ x \in [\frac{3}{4}, 1) \\ 0 & otherwise \end{cases} \tag{2.59}$$

The collocation points $x_l = \frac{l-0.5}{2m}$, $l = 0,1,2,\ldots,2m$ are obtained by discretizing Haar function $h_i(x)$ by dividing the interval [0,1] into 2m parts of equal length $\Delta t = \frac{1}{2m}$ to get coefficient matrix $H$ of order $2m \times 2m$.

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The Haar wavelets are orthogonal.

That is,

$$\int_0^1 h_i(x)h_l(x)dx = \begin{cases} \frac{1}{2m} & i = l \\ 0 & i \neq l \end{cases}$$

Let

$$p_i(x) = \int_0^x h_i(t)\,dt \qquad and \qquad q_i(x) = \int_0^x p_i\,(t)dt$$

When $i = 1$

$$h_1(x) = \begin{cases} 1 & for\ x \in [0,1) \\ 0 & otherwise \end{cases}$$

$$p_1(x) = \begin{cases} x & for\ x \in [0,1) \\ 0 & otherwise \end{cases}$$

$$q_1(x) = \begin{cases} \frac{2}{2} & for\ x \in [0,1) \\ 0 & otherwise \end{cases}$$

else

$$h_i(x) = \begin{cases} 1 & for\ x \in [\beta_1, \beta_2) \\ -1 & for\ x\ \in [\beta_2, \beta_3) \\ 0 & otherwise \end{cases}$$

$$p_i(x) = \begin{cases} (x-\beta_1) & for\ x \in [\beta_1, \beta_2) \\ \beta_3 - x & for\ x\ \in [\beta_2, \beta_3) \\ 0 & otherwise \end{cases}$$

$$q_i(x) = \begin{cases} \dfrac{(x-\beta_1)^2}{2} & for\ x \in [\beta_1, \beta_2) \\ \dfrac{(\beta_1-\beta_2)^2 + (\beta_2-\beta_3)^2 - (\beta_3-x)^2}{2} & for\ x\ \in [\beta_2, \beta_3) \\ \dfrac{(\beta_1-\beta_2)^2 + (\beta_2-\beta_3)^2}{2} & for\ x\ \in [\beta_3, 1) \\ 0 & otherwise \end{cases}$$

It is usually needed to perform integrations in studying differential equation models of dynamic systems in order to get the solution of dynamic problem. Chen and Hsiao introduced an efficient method to integrate differential equations [13].

The following integrals have to evaluate:

The operational matrix *P* which is a 2*m* square matrix is defined by

$$P_{i,1} = \int_0^x h_i(t)dt \tag{2.60}$$

Often, we need the integrals

$$P_{i,\gamma}(x) = \underbrace{\int_A^x \int_A^x \dots \int_A^x h_i(t)dt^{\gamma}}_{\gamma-time} = \frac{1}{(\gamma-1)!}\int_A^x (x-t)^{\gamma-1}\, h_i(t)dt \tag{2.61}$$

Where $\gamma = 2,3,\dots,n$ and $i = 1,2,3,\dots,2m$.

The case $\gamma = 0$corresponds to function $h_i(t)$ . Taking into account Eq (2.55), these integrals can be calculated analytically and we obtain

$$P_{i,\gamma}(x) = \begin{cases} 0 & x < \beta_1 \\ \frac{1}{\gamma!}[x-\beta_1]^\gamma & x \in [\beta_1, \beta_2] \\ \frac{1}{\gamma!}[x-\beta_1]^\gamma - 2[x-\beta_2]^\gamma & x \in [\beta_2, \beta_3] \\ \frac{1}{\gamma!}[x-\beta_1]^\gamma - 2[x-\beta_2]^\gamma + [x-\beta_3]^\gamma & x > \beta_3 \end{cases} \quad (2.62)$$

These formulas hold for *i*> 1.

In case $i = 1$ we have $\beta_1 = A,\ \beta_2 = \beta_3 = B,\ x \in [A, B]$ and $P_{1,\gamma}(x) = \frac{1}{\gamma!}[x-A]^\gamma$.

Carrying out these integrals, the following integrals are used in the Haar wavelet method:

$$P_{i,1} = \int_0^x h_i(t)dt \quad (2.63)$$

$$p_{i,1}(x) = \begin{cases} (x-\beta_1) & for\ x \in [\beta_1, \beta_2) \\ \beta_3 - x & for\ x \in [\beta_2, \beta_3) \\ 0 & otherwise \end{cases} \quad (2.64)$$

$$P_{i,\gamma+1} = \int_0^x P_{i,\gamma}(t)dt \ , \qquad \gamma = 1,2,3,\dots \quad (2.65)$$

Therefore,

$$p_{i,2}(x) = \begin{cases} 0 & x \in [0, \beta_1) \\ \dfrac{(x-\beta_1)^2}{2} & x \in [\beta_1, \beta_2) \\ \dfrac{1}{4m^2} - \dfrac{(\beta_3 - x)^2}{2} & x \in [\beta_2, \beta_3) \\ \dfrac{1}{4m^2} & x \in [\beta_3, 1) \end{cases} \tag{2.66}$$

$$p_{i,3}(x) = \begin{cases} 0 & x \in [0, \beta_1) \\ \dfrac{(x-\beta_1)^3}{6} & x \in [\beta_1, \beta_2) \\ \dfrac{(x-\beta_2)}{4m^2} - \dfrac{(\beta_3 - x)^3}{6} & x \in [\beta_2, \beta_3) \\ \dfrac{x-\beta_2}{4m^2} & x \in [\beta_3, 1) \end{cases} \tag{2.67}$$

$$p_{i,4}(x) = \begin{cases} 0 & x \in [0, \beta_1) \\ \dfrac{(x-\beta_1)^4}{24} & x \in [\beta_1, \beta_2) \\ \dfrac{(x-\beta_2)}{8m^2} - \dfrac{(\beta_3 - x)^4}{24} + \dfrac{1}{192m^4} & x \in [\beta_2, \beta_3) \\ \dfrac{x-\beta_2}{8m^2} + \dfrac{1}{192m^4} & x \in [\beta_3, 1) \end{cases} \tag{2.68}$$

Similarly we can evaluate $p_{i,\gamma}(x)$ for $\gamma = 5,6,7,\ldots$

The Haar Matrix $H_{2m\times 2m}$ which is the coefficient matrix at the collocation points may be defined as

$$H_{(m)}(i,l) = h_i(x_l) \tag{2.69}$$

To evaluate H, we define a vector of Haar functions

$$H_{(m)}(x) = [h_0(x), h_1(x), h_2(x), \dots., h_{m-1}(x)]^T \tag{2.70}$$

where m is the dimension of the vector. The vectors are used to compute each column of the Haar matrix as

$$H_{(m)} = \left[h_{(m)}\left(\frac{1}{4m}\right), h_{(m)}\left(\frac{3}{4m}\right), \dots, h_{(m)}\left(\frac{(4m-1)}{4m}\right)\right]^T \tag{2.71}$$

The integral matrices have the same elements $P_\gamma(i,l) = P_{i,\gamma}(l)$.

Chen and Hsiao[13] calculated $P_{(m)}$ from the equation

$$\int_0^x h_{(m)}(t)dt = P_{(m)}h_{(m)}(x), \quad x \in [0,1) \tag{2.72}$$

This results in the square operational matrix $P_{(m)} = P_{(m\times m)}$ of integration. The following recursive formula satisfies:

$$P_{(m)} = \frac{1}{2m}\begin{bmatrix} 2mP_{(\frac{m}{2}\times\frac{m}{2})} & -H_{(\frac{m}{2}\times\frac{m}{2})} \\ H^{-1}{}_{(\frac{m}{2}\times\frac{m}{2})} & O_{(\frac{m}{2}\times\frac{m}{2})} \end{bmatrix} \tag{2.73}$$

where $O_{(\frac{m}{2}\times\frac{m}{2})}$ is a null matrix.

$$H_{(m)} = \left[h_{(m)}(x_0), h_{(m)}(x_1), \dots, h_{(m)}(x_{m-1}),\right]^T \tag{3.74}$$

Where

$$\frac{1}{m} \le x_l \le \frac{l+1}{m}\,, \quad H^{-1}{}_{\left(\frac{m}{2}\times\frac{m}{2}\right)} = \frac{1}{m} H^T{}_{(m\times m)} diag(r)$$

The calculation for $P_{(m)}$ and $H_{(m)}$ have to be carried out only once. Since $H$ and $H^{-1}$ contain many zeros, this makes the Haar wavelet series converges rapidly.

## 2.7 The Method of Function Approximation:

Any square integrable function $y(x)$ over [0,1) can be expanded into Haar wavelets series since Haar wavelets are orthogonal as [40]:

$$y(x) = \sum_{i=0}^{\infty} a_i\, h_i(x) \tag{2.75}$$

where $a_i's$ are Haar wavelet coefficients which are determined as

$$a_0 = \int_0^1 y(x)\, h_0(x)dx \quad , \quad a_i = 2^j \int_0^1 y(x)\, h_i(x)dx\,, \quad i = 2^j + k, j \ge 0, 0 \le k \le 2^j, x \in [0,1)$$

Such that the following error norm is minimized.

$$E = \int_0^1 \left[ y(x) - \sum_{i=0}^{m-1} a_i h_i(x) \right]^2 , \quad m = 2^j \ , j \geq 0 \tag{2.76}$$

The series expansion of $y(x)$ is usually containing infinite number of terms. If $y(x)$ be piecewise constant, then the sum can be terminated to finite number of terms, that is

$$y(x_l) = \sum_{i=1}^{2M} a_i h_i(x_l) = a^T H \tag{2.77}$$
$$a^T = [a_1, a_2, \dots, a_{2M}] \ , \quad H = \{h_1(x), h_2(x), \dots, h_{2M}(x)\}^T$$

## 2.8 Convergence Analysis of Haar Wavelets:

Suppose that $u(x)$ is a differentiable function such that

$$|u(x)| \leq L, \qquad x \in (a, b), \qquad |u'(x)| \leq L,$$
$$and\ L\ is\ a\ positive\ constant$$

Then Haar wavelet approximation for the function u(x) is given by

$$u_L(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{2.78}$$

The square of the error norm of the wavelet approximation is given by [41]:

$$\|u(x) - u_M(x)\|^2 = \frac{L^3}{12M^2} \tag{3.79}$$

That is $\|u(x) - u_M(x)\|^2 = O(\frac{1}{M})$.

The maximum absolute error and the maximum relative error are given respectively as

$$E_\infty = max.\left|u_l^{exa} - u_l^{app}\right|$$
$$E_{mre} = \frac{E_\infty}{\left|u_l^{exa}\right|} \qquad (3.80)$$

where$u_l^{exa}$ and $u_l^{app}$ are the exact and Haar solution respectively at the $l^{th}$ collocation points $x_l$, and $l = 1,2,3,\dots,2m$.

**Example 3.6:**If $\boldsymbol{x = \frac{1}{4}}$ and $l = 1,3$ ; the first two Haar function vectors can be written as

$$h_{(2)}\left(\frac{1}{4}\right) = \left[h_0\left(\frac{1}{4}\right), h_1\left(\frac{1}{4}\right)\right]^T = [1 \quad 1]^T$$

$$h_{(2)}\left(\frac{3}{4}\right) = \left[h_0\left(\frac{3}{4}\right), h_1\left(\frac{3}{4}\right)\right]^T = [1 \quad -1]^T$$

In matrix form $H_{(2)}$, the Haar matrix of order two can be expressed as

$$H_{(2)} = \left[h_{(2)}\left(\frac{1}{4}\right), h_{(2)}\left(\frac{3}{4}\right)\right]^T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The Haar matrix of order four can be generated from the four Haar function vectors with $\boldsymbol{x = \frac{1}{8}}$and $l = 1,3,5,7$ as

$$h_{(4)}\left(\frac{1}{8}\right)=\left[h_0\left(\frac{1}{8}\right),h_1\left(\frac{1}{8}\right),h_2\left(\frac{1}{8}\right),h_3\left(\frac{1}{8}\right)\right]^T=[1\quad 1\quad 1\quad 0]^T$$

$$h_{(4)}\left(\frac{3}{8}\right)=\left[h_0\left(\frac{3}{8}\right),h_1\left(\frac{3}{8}\right),h_2\left(\frac{3}{8}\right),h_3\left(\frac{3}{8}\right)\right]^T=[1\quad 1\quad -1\quad 0]^T$$

$$h_{(4)}\left(\frac{5}{8}\right)=\left[h_0\left(\frac{5}{8}\right),h_1\left(\frac{5}{8}\right),h_2\left(\frac{5}{8}\right),h_3\left(\frac{5}{8}\right)\right]^T=[1\quad -1\quad 0\quad 1]^T$$

$$h_{(4)}\left(\frac{7}{8}\right)=\left[h_0\left(\frac{7}{8}\right),h_1\left(\frac{7}{8}\right),h_2\left(\frac{7}{8}\right),h_3\left(\frac{7}{8}\right)\right]^T=[1\quad -1\quad 0\quad -1]^T$$

Then

$$H_{(4)}=\left[h_4\left(\frac{1}{8}\right),h_4\left(\frac{3}{8}\right),h_4\left(\frac{5}{8}\right),h_4\left(\frac{7}{8}\right)\right]^T=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1\end{bmatrix}$$

Similarly, The Haar matrix of order eight can be written as

$$H_{(8)}=\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1\\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1\end{bmatrix}$$

**Example 3.7:** The operational matrices of integration with different order and ranks are:

1) The operational matrix of order one is $P_{(1)} = \left[\frac{1}{2}\right]$.
2) By recursive formula, The operational matrix of order two is obtained, and

   The operational matrix of rank one is defined as

$$P_{(2)} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

3) By recursive formula, The operational matrix of order four is obtained, and the operational matrix of rank two is defined as

$$P_{(4)} = \frac{1}{16}\begin{bmatrix} 2 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

4) By recursive formula, The operational matrix of order eight is obtained,

   and the operational matrix of rank four is defined as

$$P_{(8)} = \frac{1}{64}\begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

## 2.9 Application of Wavelet Method for Differential Equations:

Unlike the power series method which starts with the expansion of the solution then the expansion of the derivatives obtain through the differentiation; the Haar wavelet method begins with the wavelet expansion of the highest derivative to obtain the wavelet expansion of the lower order derivatives and the solution itself through integration.

Consider the $m^{th}$ order linear differential equation:

$$Lu(x) = f(t)\,, A \le t \le B, \qquad L\ is\ differentiable\ operator \tag{2.81}$$

**Step 1:**

$$u^m(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{2.82}$$

**Step 2:** For $n < m$

$$u^m(x) = \sum_{i=1}^{2M} a_i P_{i,m-n}(x) + \sum_{j=1}^{m-n-1} \frac{1}{j!}(x-A)^j u_0^{n+j} \tag{3.83}$$

**Step 3:** Put the obtained derivatives from steps (1) and (2) into eq.(2.75). This step provides us with the expansion of the solution $u(x)$ and all of its derivatives that appear in the equation.

**Step 4:** Put the expansion of the solution and all its derivatives obtained in the above steps into the given equation and compute at the collocation points $x_l = \frac{2l-1}{4m}$ or $x_l = \frac{l-0.5}{2m}$ , $l = 1,2,3,\dots,2m$ for a given resolution *M*. We get system of algebraic equations.

**Step 5:** The wavelets coefficients $a_i's$ are obtained by solving the system of algebraic equations obtained in step (4).

**Step 6:** Substitute the coefficients $a_i's$ in the expansion of the solution to get the numerical solution with resolution *M*.

As it is mentioned above, according to the property of Haar wavelet transform, the function $u''(x)$ which is a function of *x* can be approximated by the Haar wavelet function as

$$u''(t) = \sum_{i=1}^{2M} a_i h_i(t) \tag{2.85}$$

Integrating Eq(3.85) we obtain

$$u'(t) = \int u''(t)\,dt = \int \left[\sum_{i=1}^{2M} a_i h_i(t)\right] dt = \sum_{i=1}^{2M} \int a_i h_i(t)\,dt \tag{2.86}$$

Then

$$u(t) = \int u'(t)\,dt = \int\int \left[\sum_{i=1}^{2M} a_i h_i(t)\right] dtdt$$

$$= \sum_{i=1}^{2M} \int\int a_i h_i(t)\,dtdt \tag{2.87}$$

## 2.10 Solution Methodology for Initial Value Problem:

Consider the second order initial value problem as

$$u''(t) = \varphi(t, u, u') \tag{2.88}$$

with initial conditions $u(a) = \beta_1$ , $u'(a) = \beta_2$ .

The following cases are discussed:

**Case 1:** if $a \in [0,1)$ , then

$$u''(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{2.89}$$

$$u'(x) - u'(a) = \int_a^t \sum_{i=1}^{2M} a_i h_i(x)dx$$

$$= \sum_{i=1}^{2M} a_i P_i(x) - \int_0^a \sum_{i=1}^{2M} a_i h_i(x)dx \tag{2.90}$$

$$u(x) - u(a) = \int_a^t \sum_{i=1}^{2M} a_i P_i(x)dx = (x-a)u'(a) + \sum_{i=1}^{2M} a_i q_i(x)$$

$$-\int_0^a \sum_{i=1}^{2M} a_i P_i(x)dx - (x-a)A\,, \quad where\ A = \int_a^a \sum_{i=1}^{2M} a_i h_i(x)dx \tag{2.91}$$

When $a = 0$ ,

$$u'(x) = u'(0) + \sum_{i=1}^{2M} a_i P_i\,(x) \tag{2.92}$$

$$u(x) = u(0) + xu'(0) + \sum_{i=1}^{2M} a_i q_i\,(x) \tag{2.93}$$

In order to obtain $u(x)$ , the unknown $a_i'$s have to be first determined by solving the system of $2M$ equations. That is,

$u''(t) = \varphi(t, u, u')$ at $t_j's$ which are the collocation points; $j = 1,2, \dots, 2M$.

Putting Eq(3.89), Eq(2.92) and Eq(2.93) in Eq(2.88), we obtain
$\sum_{i=1}^{2M} a_i h_i\,(x) = \varphi\big(x_j, u(a) + (x-a)u'(a) + \sum_{i=1}^{2M} a_i q_i\,(x_j) -$
$\int_0^a \sum_{i=1}^{2M} a_i P_i\,(x_j)dx - (x_j - a)A, u'(a) + \sum_{i=1}^{2M} a_i P_i\,(x_j) -$
$\int_a^a \sum_{i=1}^{2M} a_i h_i\,(x_j)dx\ \big)$ (2.94)

**Case 2:** if $a \in (-\infty, 0) \cup [1, \infty)$ , then $u(x)$ and $u'(x)$ can be found by integrating $u''(x)$ from $a$ to $t$ using interval summation rule of integration.

**Example 2.8:** Consider the second order initial value linear ordinary differential equation

$u''(x) + u(x) = f(x), \qquad u(0) = u'(0) = 0, x \in [0,1]$ (2.95)

The exact solution of the given equation is $u(x) = 1 - \cos(t)$.

Now we want to find the numerical solution of Eq(2.95) using wavelet method. We shall take only four collocation points and hence $J = 4$ and

$M = 2$ . Therefore, we are going to define $h_i(t)$, $P_i(t)$, $q_i(t)$ for $i = 1,2,3,4$ .

$i = m + k + 1$ , when $i = 1$ then $m = k = 0$ .

$$h_1(x) = \begin{cases} 1 & x \in [0,1) \\ 0 & otherwise \end{cases}$$

$$P_1(x) = \begin{cases} x & x \in [0,1) \\ 0 & otherwise \end{cases}$$

$$q_1(x) = \begin{cases} \frac{x^2}{2} & x \in [0,1) \\ 0 & otherwise \end{cases}$$

when $i = 2$ then $m = 1,\ k = 0$; then $\beta_1 = 0$ , $\beta_2 = \frac{1}{2}$ , $\beta_3 = 1$ and hence

$$h_2(x) = \begin{cases} 1 & x \in [0,\frac{1}{2}) \\ -1 & x \in [\frac{1}{2},1) \\ 0 & otherwise \end{cases}$$

$$P_2(x) = \begin{cases} x - 0 & x \in [0,\frac{1}{2}) \\ 1 - x & x \in [\frac{1}{2},1) \\ 0 & otherwise \end{cases}$$

$$q_2(x) = \begin{cases} \frac{x^2}{2} & x \in [0,\frac{1}{2}) \\ \frac{-2x^2 + 4x - 1}{4} & x \in [\frac{1}{2},1) \\ \frac{1}{4} & x = 1 \\ 0 & otherwise \end{cases}$$

when $i = 3$ then $m = 2,\ k = 0$; then $\beta_1 = 0\ , \beta_2 = \frac{1}{4}\ , \beta_3 = \frac{1}{2}$ and hence

$$h_3(x) = \begin{cases} 1 & x \in [0, \frac{1}{4}) \\ -1 & x \in [\frac{1}{4}, \frac{1}{2}) \\ 0 & otherwise \end{cases}$$

$$P_3(x) = \begin{cases} t - 0 & x \in [0, \frac{1}{4}) \\ \frac{1}{2} - x & x \in [\frac{1}{4}, \frac{1}{2}) \\ 0 & otherwise \end{cases}$$

$$q_3(x) = \begin{cases} \frac{x^2}{2} & x \in [0, \frac{1}{4}) \\ \frac{-8x^2 + 8x - 1}{16} & x \in [\frac{1}{4}, \frac{1}{2}) \\ \frac{1}{16} & x \in [\frac{1}{2}, 1) \\ 0 & otherwise \end{cases}$$

when $i = 4$ then $m = 2,\ k = 1$; then $\beta_1 = \frac{1}{2}\ , \beta_2 = \frac{3}{4}\ , \beta_3 = 1$ and hence

$$h_4(x) = \begin{cases} 1 & x \in [\frac{1}{2}, \frac{3}{4}) \\ -1 & x \in [\frac{3}{4}, 1) \\ 0 & otherwise \end{cases}$$

$$P_4(x) = \begin{cases} x - \frac{1}{2} & x \in [\frac{1}{2}, \frac{3}{4}) \\ 1 - x & x \in [\frac{3}{4}, 1) \\ 0 & otherwise \end{cases}$$

$$q_4(x) = \begin{cases} \dfrac{4x^2 - 4x + 1}{8} & x \in [\frac{1}{2}, \frac{3}{4}) \\ \dfrac{-7 - 8x^2 + 16x}{16} & x \in [\frac{3}{4}, 1) \\ \dfrac{1}{16} & x = 1 \\ 0 & otherwise \end{cases}$$

If $f(x) = 1$ in the given problem, eq.(2.95) becomes

$$u''(x) + u(x) = 1 \tag{2.96}$$

$$u''(x) = \varphi(x, u, u') = 1 - u(x) \tag{2.97}$$

By Eq.(3.94) and Eq.(3.97) , we obtain

$$\varphi(x, u, u') = 1 - \left[ u(a) + (x_j - a)u'(a) + \sum_{i=1}^{4} a_i q_i(x_j) - \int_0^a \sum_{i=1}^{4} a_i P_i(x_j) dx - (x_j - a)A \right]$$

Now $a = 0$ is given, then

$$\varphi(x, u, u') = 1 - \left[ u(0) + (x_j - 0)u'(0) + \sum_{i=1}^{4} a_i q_i(x_j) \right]$$

$$\sum_{i=1}^{4} a_i h_i(x_j) = 1 - \sum_{i=1}^{4} a_i q_i(x_j)$$

$$\sum_{i=1}^{4} a_i h_i(x_j) + \sum_{i=1}^{4} a_i q_i(x_j) = 1$$

$$\sum_{i=1}^{4} a_i (h_i(x_j) + q_i(x_j)) = 1 \tag{2.98}$$

Substitute the values of $h_i's$ and $q_i's$ in Eq.(2.98) for $i = 1,2,3,4$ and if $x = 0.125$, then

$a_1 + a_2 + a_3 = 0.992248$

when $x_j = 0.375$ , then

$a_1 + a_2 - 0.88321a_3 = 0.9343066$

when $x_j = 0.625$ , then

$1.1953125a_1 - 0.8203125a_2 + 0.0625a_3 + 1.0078125a_4 = 1$

when $x_j = 0.875$ , then

$1.3828125a_1 - 0.7578125a_2 + 0.0625a_3 + 0.9453125a_4 = 1$

Solving above equations we obtain the values of $a_1, a_2, a_3, and\ a_4$ as

$a_1 = 0.844076138,$ $a_2 = 0.117404501,$

$a_3 = 0.0307673599,$ $a_4 = 0.0847882533$

Substitute the obtained values of $a_i's$ in the relation

$$u(x) = u(0) + xu'(0) + \sum_{i=1}^{4} a_i q_i\,(x)$$

We obtain

$$u(x) = \sum_{i=1}^{2M} a_i q_i\,(x)$$

$$u(x) = 0.844076138\left(\frac{x^2}{2}\right) + 0.117404501\left(\frac{4x - 2x^2 - 1}{4}\right)$$

$$+0.0307673599\left(\frac{1}{16}\right) + 0.0847882533\left(\frac{16x - 8x^2 - 7}{16}\right)$$

Table 2.1 shows the exact and the approximate solutions at different points

| $x_j$ | Exact sol. | Haar sol. | Error |
|---|---|---|---|
| 0.1 | 0.00499583 | 0.00496124 | 0.00003459 |
| 0.2 | 0.01984495998 | 0.01984495998 | 0.000088462 |
| 0.3 | 0.0446635 | 0.04457424 | 0.000089269 |
| 0.4 | 0.07868757 | 0.07868757 | 0.00000000 |

## 2.11 Solution Methodology for Boundary Value Problem:

We shall discuss the solution methodology of Haar wavelet method for solving boundary value problems (BVBs) appearing in the mathematical modeling of various engineering applications. There are different types of boundary conditions.

**Case1:**

$u''(x) = \varphi(x, u, u')$ ,

$u'(0) = a$ , $u'(1) = b$ ; $a, b$ *are real constants* (2.99)

$$u''(x) = \sum_{i=1}^{2M} a_i h_i(x) \quad (2.100)$$

By Integrating Eq.(3.100) from 0 to *t*, one can obtain

$$u'(x) - u'(0) = \sum_{i=1}^{2M} a_i P_{i,1}(x)$$

$$u'(x) - a = \sum_{i=1}^{2M} a_i P_{i,1}(x) \tag{2.101}$$

Now, integrate Eq.(3.100) from *x* to 1, we obtain

$$\int_x^1 u''(s)\,ds = \int_x^1 \sum_{i=1}^{2M} a_i h_i(s)\,ds = \sum_{i=1}^{2M} \left[ \int_0^1 a_i h_i(s) ds - \int_0^x a_i h_i(s) ds \right]$$

$$u'(1) - u'(x) = a_1 - \sum_{i=1}^{2M} a_i P_{i,1}(x)$$

$$b - u'(x) = a_1 - \sum_{i=1}^{2M} a_i P_{i,1}(x) \tag{2.102}$$

From Eq.(2.101) and Eq.(2.102) we get

$$a_1 = b - a$$

$$u''(x) = a_1 h_1(x) + \sum_{i=1}^{2M} a_i h_i(x)$$

Thus, the corresponding approximations are

$$u''(x) = (b-a)h_1(x) + \sum_{i=1}^{2M} a_i h_i(x) \tag{2.103}$$

By integrating Eq.(2.103) from 0 to *x*, we obtain

$$u'(x) = a + \sum_{i=1}^{2M} a_i P_{i,1}\,(x)$$

$$u'(x) = a + a_1 P_{1,1} + \sum_{i=2}^{2M} a_i P_{i,1}\,(x)$$

$$u'(x) = a + (b-a)P_{1,1} + \sum_{i=2}^{2M} a_i P_{i,1}\,(x) \qquad (2.104)$$

By integrating Eq(2.104) from 0 to *x*, we get

$$\int_0^x u'(s)\,ds = \int_0^x a\,ds + \int_0^x (b-a)P_{1,1}(s)\,ds + \int_0^x \sum_{i=2}^{2M} a_i P_{i,1}\,(s)\,ds$$

$$u(x) = u(0) + ax + (b-a)P_{1,2}(x) + \sum_{i=2}^{2M} a_i P_{i,2}\,(x) \qquad (2.105)$$

$$u(t) = u(0) + at + (b-a)q_1(t) + \sum_{i=2}^{2M} a_i q_i\,(t) \qquad (2.106)$$

Now, substituting the values of $u(x)$, $u'(x)$, and $u''(x)$ in the given DE we get system of equations, simplifying and computing at the collocation points $x_j = \frac{j-0.5}{2m}$ , $j = 1, \ldots, 2$

$$(b-a)h_1(x_j)+\sum_{i=1}^{2M} a_i h_i\,(x_j)$$

$$= \varphi\begin{pmatrix} x_j, u(0)+ax_j+(b-a)P_{1,2}(x_j)+\sum_{i=2}^{2M} a_i P_{i,2}\,(x), \\ a+(b-a)P_{1,1}(x_j)+\sum_{i=2}^{2M} a_i P_{i,1}\,(x_j) \end{pmatrix}$$

$$(b-a)h_1(x_j)+\sum_{i=}^{2M} a_i h_i\,(x_j)$$

$$= \varphi\begin{pmatrix} x_j, u(0)+ax_j+(b-a)q_1(x_j)+\sum_{i=2}^{2M} a_i q_i\,(x_j), \\ a+(b-a)P_1(x_j)+\sum_{i=2}^{2M} a_i P_i\,(x_j) \end{pmatrix}$$

Solve the above system m terms of unknowns $u(0)$ and $a_i's$ for $i \neq 1$ to get the numerical solution for Eq.(2.99).

**Case2:**

$u''(x)=\varphi(x,u,u')$ ,

$u(0)=a$ , $u(1)=b$ ; $a, b$ *are real constants* (2.107)

$$u''(x)=\sum_{i=1}^{2M} a_i h_i\,(x) \qquad (2.108)$$

By Integrating Eq.(2.108) from 0 to *x*, we obtain

$$u'(x) - u'(0) = \sum_{i=1}^{2M} a_i P_{i,1}(x) \tag{2.109}$$

Now, integrate Eq.(2.109) from 0 to *x*, we obtain

$$u(x) - u(0) - xu'(0) = \sum_{i=1}^{2M} a_i P_{i,2}(x) \tag{2.110}$$

Putting $x = 1$ in Eq.(3.110), we get

$$u(1) - u(0) - u'(0) = \sum_{i=1}^{2M} a_i P_{i,2}(1) \tag{2.111}$$

$$b - a - u'(0) = \sum_{i=1}^{2M} a_i P_{i,2}(1) \tag{2.112}$$

$$u'(0) = b - a - \sum_{i=1}^{2M} a_i P_{i,2}(1) \tag{2.113}$$

$$u(x) = a + x\left(b - a - \sum_{i=1}^{2M} a_i P_{i,2}(1)\right) + \sum_{i=1}^{2M} a_i P_{i,2}(x) \tag{2.114}$$

$$u(x) = a + xb - xa - x\sum_{i=1}^{2M} a_i P_{i,2}(1) + \sum_{i=1}^{2M} a_i P_{i,2}(x) \tag{2.115}$$

$$u'(x) = b - a - \sum_{i=1}^{2M} a_i P_{i,2}(1) + \sum_{i=1}^{2M} a_i P_{i,2}(x) \qquad (2.116)$$

Now, substituting Eqs(2.115), (2.116) and (2.108) in the given DE we get system of equations, simplifying and computing at the collocation points $x_j = \frac{j-0.5}{2m}$ , $j = 1, \dots, 2M$

$$\sum_{i=1}^{2M} a_i h_i(x_j)$$

$$= \varphi \begin{pmatrix} x_j, a + x_j b - x_j a - x_j \sum_{i=1}^{2M} a_i P_{i,2}(1) + \sum_{i=1}^{2M} a_i P_{i,2}(x_j), \\ b - a - \sum_{i=1}^{2M} a_i P_{i,2}(1) + \sum_{i=1}^{2M} a_i P_{i,2}(x_j) \end{pmatrix} \qquad (2.117)$$

Solving the above system for unknowns $a_i's$ to get the numerical solution for Eq.(2.107).

**Example 2.9:** Consider the second order boundary value ordinary differential equation

$$u''(x) - 5u'(x) = 0, \qquad u(0) = 1, u(1) = 0, x \in (0,1) \qquad (2.118)$$

The exact solution is

$$u(x) = \frac{e^5 - e^{5x}}{e^5 - 1}$$

If the levels of Haar wavelet are taken to be 3 (J = 3), then

$$u''(x) = \sum_{i=1}^{2M} a_i h_i(x) \qquad (2.119)$$

By integrating Eq.(2.119) from 0 to *x*, we obtain

$$\int_0^x u''(s)\,ds = \int_0^x \sum_{i=1}^{2M} a_i h_i(s)\,ds \tag{2.120}$$

$$u'(x) - u'(0) = \sum_{i=1}^{2M} a_i \int_0^x h_i(s)ds \tag{2.121}$$

$$u'(x) = u'(0) + \sum_{i=1}^{2M} a_i p_{i,1}(x) \tag{2.122}$$

Now, integrate Eq.(2.122) from 0 to *x*, we get

$$u(x) = u(0) + xu'(0) + \sum_{i=1}^{2M} a_i P_{i,2}(x) \tag{2.123}$$

Using the value of $u(1)$, we can compute $u'(0)$, that is, if *x* = 1,

$$u(1) = u(0) + u'(0) + \sum_{i=1}^{2M} a_i P_{i,2}(1) \tag{2.124}$$

$$u'(0) = -1 - \sum_{i=1}^{2M} a_i P_{i,2}(1) \tag{2.125}$$

Therefore,

$$u(x) = u(0) - x - x\sum_{i=1}^{2M} a_i P_{i,2}(1) + \sum_{i=1}^{2M} a_i P_{i,2}(x) \tag{2.126}$$

$$u(x) = 1 - x - x\sum_{i=1}^{2M} a_i P_{i,2}(1) + \sum_{i=1}^{2M} a_i P_{i,2}(x) \tag{2.127}$$

By substituting $u'(x)$ and $u''(x)$ in the given equation Eq(2.118), we obtain

$$\sum_{i=1}^{2M} a_i\left[h_i(x_j) + 5P_{i,2}(1) - 5P_{i,2}(x_j)\right] = -5 \ ,$$

$$j = 1,2,\dots,16 \qquad (2.128)$$

After evaluating at the collocation points $x_j$ and solving the system of linear equations for the wavelets coefficients $a_i$, we substitute the obtained coefficients in Eq(2.127) to get the solution.

Table 2.2 shows the exact and the approximate solutions at different points

| $x/_{32}$ | Exact Solution | Approximate Solution | Error |
|---|---|---|---|
| 1 | 0.99885276 | 0.99885507 | 0.00000231 |
| 3 | 0.99594341 | 0.99595121 | 0.00000780 |
| 5 | 0.99196679 | 0.99198147 | 0.00001468 |
| 7 | 0.98653140 | 0.98655462 | 0.00002322 |
| 9 | 0.97910211 | 0.97913582 | 0.00003371 |
| 11 | 0.96894747 | 0.96899391 | 0.00004644 |
| 13 | 0.95506772 | 0.95512935 | 0.00006163 |
| 15 | 0.93609635 | 0.93617573 | 0.00007938 |
| 17 | 0.91016556 | 0.91026508 | 0.00009952 |
| 19 | 0.87422238 | 0.87484378 | 0.00012141 |
| 21 | 0.82627729 | 0.82642091 | 0.00014362 |
| 23 | 0.76006071 | 0.76022417 | 0.00016346 |
| 25 | 0.66955337 | 0.66972955 | 0.00017617 |
| 27 | 0.54584451 | 0.54601839 | 0.00017388 |
| 29 | 0.37675454 | 0.37689837 | 0.00014383 |
| 31 | 0.14563596 | 0.14570195 | 0.00006598 |

**Remark 2.1:** The same procedure may be followed for other cases of boundary conditions such as $(u'(0) = a\,, u(1) = b)$ and $(u(0) = a\,, u'(1) = b)$.

# CHAPTER THREE

# Haar Wavelet Method for Solving Burger's Fisher Equation

## 3.1 Introduction

In this chapter, the method of Haar wavelet [15] for solving partial differential equations is introduced. We investigate the Haar wavelets method for the approximation solution of nonlinear evolution (hyperbolic- parabolic) problem.

The method is applied in the case of Burger's Fisher equation. The numerical solutions of nonlinear partial differential equations (NPDEs), such as the Burger's-Fisher equation, have attracted much attention due to their potential applications in various fields of fluid dynamics, heat conduction, gas dynamic, traffic flow, applied mathematics and some other fields of science [42-44].

The Burger-Fisher equation is a mixed hyperbolic-parabolic type of NPDEs which describes the interaction between the reaction mechanism, convection effect, and diffusion transport [45] is considered in this thesis. Many numerical schemes have been proposed for obtaining approximate solutions of the Burger's-Fisher equation [46-49].

The wavelets method shows rather rapid convergence than other existing methods. Illustrative examples suggest that using wavelet based method providing a powerful approach to find numerical solutions of Burger's Fisher equation.

The comparison of numerical results with the exact solutions, and the solutions obtained using some traditional methods such as variational iteration method (VIM) [49] show that the suggested scheme is fairly accurate and viable for solving such problems.

All results are obtained by using MATLAB software.

## 3.2 Problem formulation and solution

Let us consider the Burgers's Fisher equation [48] is as follows:-

$$u_t - u_{xx} + \alpha u^{\gamma} u_x + \beta u(u^{\gamma} - 1) = 0, \qquad x \in [0,1], \qquad t \geq 0 \tag{3.1}$$

Where α, β ≥ 0 and $\gamma > 0$ are given constants. If $\gamma$ = 1, Eq(3.1) is called the ***Burger's Fisher*** equation. When α =0, $\gamma$ = 1, Eq(3.1) is reduced to the ***Huxley*** equation which describes nerve pulse propagation in nerve fibre and wall motion in liquid crystals [50]. Generalized Burger's equation will be obtained when β =0. This equation when β = 0, has been used to investigate sound waves in a viscous medium by Lighthill [51]. However, it was originally introduced by Burger [52] to model one-dimensional turbulence.

### Exact Solution 3.2.1

In [53 ], The authors found the exact solution to Burger's Fisher equation via using new expansion method, as discussed briefly below:

consider the Burger's Fisher equation [48] as in eq.(3.1):

$$u_t - u_{xx} + \alpha u^{\gamma} u_x + \beta u(u^{\gamma} - 1) = 0, \tag{3.2}$$

The traveling wave variable

$$u(x, t) = u(\xi),\ \xi = x - ct, \tag{3.3}$$

permits us converting Eq. (3.2) into the following ODE:

$$-cu' - u'' + \alpha u^{\gamma} u' + \beta u^{\gamma+1} - \beta u = 0, \tag{3.4}$$

Consider the homogeneous balance between $u''$ and $u^{n+1}$ in (3.2) we get $m = \frac{2}{n}$. It should be noted that *m* is not a positive integer. However, with reference to [54] we may still choose the solution of Eq. (3.4) in the form.

$$u(\xi) = A\left(\frac{G'}{G}\right)^{\frac{2}{n}}, n > 0, \tag{3.5}$$

where *A* is a constant to be determined and *G* satisfies the following equation,

$G''+\lambda G'+ \mu G = 0$

Substituting (3.5) into (3.4) we obtain the polynomial

$$[2Anc\mu + 2Ab\lambda\mu(4-n)]\left(\frac{G'}{G}\right)^{\frac{n+2}{n}} + [Abn^2 + 2Anc\lambda + 4A\lambda^2 + 8Ab\mu]\left(\frac{G'}{G}\right)^{\frac{n+2}{n}}$$

$$+ [2Anc + 2Ab\lambda(4+n) - 2aA^{1+n}n\lambda]\left(\frac{G'}{G}\right)^{\frac{3n+2}{n}} + [2Ab\mu^2(2+n) - A^{1+n}dn^2 -$$

$$2aA^{1+n}n\lambda]\left(\frac{G'}{G}\right)^{\frac{4n+2}{n}} + [2aA^{1+n}n]\left(\frac{G'}{G}\right)^{\frac{5n+2}{n}} + [2Ab\mu^2(2-n)]\left(\frac{G'}{G}\right)^{\frac{2}{n}} = 0 \tag{3.6}$$

On equating the coefficients of the polynomial (3.6) to be zero, we get a system of algebraic equations, which can be solved by Mathematica to obtain the following results:

$$A = \lambda^{-\frac{2}{n}}, c = -\frac{dn(4+n)}{2\lambda(2+n)}, \mu = 0, a = 0, b = -\frac{dn^2}{\lambda^2(4+2n)}, \lambda \neq 0 \tag{3.7}$$

From eq.(3.5) and (3.7) the exact traveling wave solution can obtained

$$u(x,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha\gamma}{2(\gamma+1)}\left(x - \left(\frac{\alpha}{\gamma+1} + \frac{\beta(\gamma+1)}{\alpha}\right)t\right)\right)\right)^{\frac{1}{\gamma}} \tag{3.8}$$

For computational and simplicity point of view, we have taken the initial as well as the boundary conditions of Burger's Fisher equation from the exact solution to study the comparison between the suggestion approach and the exact solution [47-48].

with the initial condition given by:

$$u(x,0) = (\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\alpha\gamma}{2(\gamma+1)}x\right))^{\frac{1}{\gamma}} \tag{3.9}$$

and the boundary conditions:

$$\left.\begin{aligned} u(0,t) &= \left(\frac{1}{2} + \frac{1}{2}tanh\left(\frac{\alpha\gamma}{2(\gamma+1)}\left(\frac{\alpha}{\gamma+1} + \frac{\beta(\gamma+1)}{\alpha}\right)t\right)\right)^{\frac{1}{\gamma}}, \quad t \geq 0 \\ u(1,t) &= \left(\frac{1}{2} + \frac{1}{2}tanh\left[\frac{-\alpha\gamma}{2(\gamma+1)}\left(1 - \left(\frac{\alpha}{\gamma+1} + \frac{\beta(\gamma+1)}{\alpha}\right)t\right)\right]\right)^{\frac{1}{\gamma}}, \quad t \geq 0 \end{aligned}\right\} \tag{3.10}$$

The Haar wavelet family for $x \in [0,1)$ and its integration is defined in section (2.6). Now, consider an initial boundary value problem (IBVP) for the nonhomogeneous Burger's-Fisher equation (3.1) with the initial and boundary conditions:

$$u(x,0) = f(x), \qquad x \in [0,1] \tag{3.11}$$

$$u(0,t) = g_1(t), \qquad t \geq 0 \tag{3.12}$$

$$u(1,t) = g_2(t), \qquad t \geq 0 \tag{3.13}$$

It showed be noted that, the computability of first order may be suggested as:

$$f(x) = g_1(0) \; and \; f(1) = g_2(0)$$

Let us divide the interval $[0,T]$ into N equal parts of length $\Delta t = \frac{T}{N}$ and denote $t_s = (s-1)\Delta t, \; s = 1,2,\dots,N$. One can assume that a Haar wavelet solution for eq.(3.1) in the form:

$$\ddot{u}(x,t) = \sum_{i=1}^{2M} a_s(i) h_i(x) \tag{3.14}$$

Where the dot and prime denote the differentiation with respect to $t$ and $x$ respectively, the row vector $a_s$ is constant in the sub-interval $t \in [t_s, t_{s+1}]$. Integrating equation (3.14) with respect to $\boldsymbol{t}$ in the limits $[t_s$,t] then by integrating the resultant equation with respect to $x$ in the limits [0, $x$] , and the resultant equation is again integrating with respect to $\boldsymbol{x}$ in the limits [0,$x$] and differentiating the resultant equation with respect to $\boldsymbol{t}$ , the following equations are obtained respectively.

$$u''(x,t) = (t - t_s)\sum_{i=1}^{2M} a_s\,(i)h_i(x) + u''(x,t_s) \tag{3.15}$$

$$u'(x,t) = (t - t_s)\sum_{i=1}^{2M} a_s\,(i)P_i(x) + u'(x,t_s) - u'(0,t_s) + u'(0,t) \tag{3.16}$$

$$u(x,t) = (t - t_s)\sum_{i=1}^{2M} a_s\,(i)q_i(x) + u(x,t_s) - u(0,t_s) + u(0,t)$$

$$+x[u'(0,t) - u'(0,t_s)] \tag{3.17}$$

$$\dot{u}(\,x,t) = \sum_{i=1}^{2M} a_s(i)q_i(x) + \dot{u}\,(0,t) + x\dot{u}'(0,t) \tag{3.18}$$

One can use the boundary conditions

$$u(0,t_s) = g_1(t_s)\,, \qquad u(1,t_s) = g_2(t_s) \tag{3.19}$$

$$\dot{u}(0,t) = \dot{g_1}(t)\,, \qquad \dot{u}(1,t) = \dot{g_2}(t) \tag{3.20}$$

Putting $x = 1$ in equations (3.15) and (3.16) and using the conditions in (3.17) and (3.18) one can obtain,

$$u'(0,t) - u'(0,t_s) = -(t-t_s)\sum_{i=1}^{2M} a_s\,(i)q_i(1) + g_2(t) - g_1(t) - g_2(t_s)$$

$$+g_1(t_s) \tag{3.21}$$

$$\dot{u}'(0,t) = -\sum_{i=1}^{2M} a_s(i)q_i(1) + \dot{g}_2(t) - \dot{g}_1(t) \tag{3.22}$$

The wavelet collocation points are defined as:

$$x_I = \frac{I-0.5}{2M}, \qquad I = 1,2,\dots,2M \tag{3.23}$$

Substituting equations (3.17) - (3.20) in equations (3.13 - 3.16) and taking $x \to x_I$ , $t \to t_{s+1}$ , one can get.

$$u"(x_I,t_{s+1}) = \Delta t \sum_{i=1}^{2M} a_s(i)h_i(x) + u"(x_I,t_s) \tag{3.24}$$

$$u'(x_I,t_{s+1}) = \Delta t \sum_{i=1}^{2M} a_s(i)\,[P_i(x_I) - q_i(1)] + u'(x_I,t_s) + g_2(t_{s+1})$$

$$-g_1(t_{s+1}) - g_2(t_s) + g_1(t_s) \tag{3.25}$$

$$u(x_I,t_{s+1}) = \Delta t \sum_{i=1}^{2M} a_s(i)\,[P_i(x_I) - x_I q_i(1)] + u(x_I,t_s) + x_I[g_2(t_{s+1}) - g_2(t_s)]$$

$$+(1-x_I)[g_1(t_{s+1}) - g_1(t_s)] \tag{3.26}$$

$$\dot{u}(x_I,t_{s+1}) = \sum_{i=1}^{2M} a_s(i)[q_i(x_I) - x_I q_i(1)] + \dot{g}_1(t_{s+1}) + x_I[\dot{g}_2(t_{s+1})$$

$$-\dot{g}_1(t_{s+1})] \tag{3.27}$$

Now will have the following scheme,

$$\dot{u}(x_I, t_{s+1}) = u"(x_I, t_{s+1}) - u(x_I, t_{s+1})u'(x_I, t_{s+1}) + u(x_I, t_{s+1})[1 - u(x_I, t_{s+1})] \quad (3.28)$$

This leads us from the time layer $t_s$ to $t_{s+1}$ .

Taking the collocation points $x \to x_I$ into Eq.(3.28) and using Eqs (3.23) to (3.27), one gets.

$$\sum_{i=1}^{2M} a_s(i)[q_i(x_I) - x_I q_i(1)] + \dot{g}_1(t_{s+1}) + x_I[\dot{g}_2(t_{s+1}) - \dot{g}_1(t_{s+1})$$

$$= u"(x_I, t_{s+1}) - u(x_I, t_{s+1})u'(x_I, t_{s+1}) + u(x_I, t_{s+1})[1 - u(x_I, t_{s+1})] \quad (3.29)$$

The wavelet coefficients $a_s(i)$ , $i = 1,2, \ldots, 2M$ can be successively calculated from Eq(3.29). This process is started with the initial condition (3.11). These coefficients are then substituted into Eqs (3.24 - 3.26) to obtain the approximate solutions at different time levels.

## 3.3 Illustration

In this section, two examples are considered to check the efficiency and accuracy of the Haar wavelet method. The entire computational work has been done with the help of MATLAB software.

**Example 3.1:**

Let's consider the Burger's Fisher equation (3.1), for $\alpha = 0.001, \beta = 0.001$ and $\gamma = 1$ with the initial and boundary conditions in Eqs (3.2) and (3.3) respectively; and the exact solution given in Eq.(3.4).

Table (3.1) shows a comparison between the absolute error of the solution by the proposed method and the absolute error by using the (VIM) [50]. The figure (3.1) shows the Comparison of the absolute errors of the proposed method with the VIM, whereas for different values of *x* and *t*.

| x | T | Exact Sol. | Haar Sol. | Haar Err. | VIM Err. |
|---|---|---|---|---|---|
| 0.01 | 0.02 | 0.500015019999996 | 0.499132084264304 | 0.000882935735692 | 0.0025031102 |
| | 0.04 | 0.500035039999943 | 0.498697734403682 | 0.001337305596261 | 0.0025081138 |
| | 0.06 | 0.50005509999777 | 0.498312881454306 | 0.001742218543464 | 0.0025131170 |
| | 0.08 | 0.500075079999436 | 0.497977989064286 | 0.00209709093515 | 0.0025181206 |
| 0.04 | 0.02 | 0.500000020000000 | 0.509811685266022 | 0.009811665266022 | 0.0099961959 |
| | 0.04 | 0.500020039999989 | 0.509360739108795 | 0.009340699108806 | 0.0100011899 |
| | 0.06 | 0.500040059999914 | 0.508960366604204 | 0.00892030660429 | 0.0100061907 |
| | 0.08 | 0.500060079999711 | 0.508611049610424 | 0.008550969610713 | 0.0100111915 |

**Table (3.1): Numerical Results of the nonlinear Burger's Fisher equation.**

**Example 3.2:**

Consider the Burger's Fisher equation (3.1), for $\alpha = 0.01, \beta = 0.01$ and $\gamma = 1$ with the initial and boundary conditions in Eqs (3.2) and (3.3) respectively; and the exact solution given in Eq (3.4).

Table (3.2) shows the approximate solution of the given equation using the proposed method against the exact solution, and figure (3.2) shows Comparison between the proposed method and the Exact solution.. whereas for different values of *x* and *t*.

| X | T | Exact solution | Haar solution | Absolute Error |
|---|---|---|---|---|
| 0.01 | 0.02 | 0.500037624999929 | 0.499132084264304 | 0.000905540735625320 |
| 0.02 | | 0.500025124999979 | 0.502892009223431 | 0.002866884223452 |
| 0.03 | | 0.500012624999997 | 0.506445479270577 | 0.006432854270580 |
| 0.04 | | 0.500000125000000 | 0.509811685266022 | 0.009811560266022 |
| 0.05 | | 0.499987625000003 | 0.513007881958884 | 0.013020256958881 |
| 0.01 | 0.03 | 0.500062687499672 | 0.498439354576423 | 0.001623332923248 |
| 0.02 | | 0.500050187499831 | 0.502378619998906 | 0.002328432499075 |
| 0.03 | | 0.500037687499929 | 0.506101165599605 | 0.006063478099676 |
| 0.04 | | 0.500025187499979 | 0.509627059589430 | 0.009601872089451 |
| 0.05 | | 0.500012687499997 | 0.512974341469135 | 0.012961653969137 |
| 0.01 | 0.04 | 0.500087749999099 | 0.497075433069911 | 0.003012316929188 |
| 0.02 | | 0.500075249999432 | 0.501141883179817 | 0.001066633180385 |
| 0.03 | | 0.500062749999671 | 0.50498498853876 | 0.004921748854205 |
| 0.04 | | 0.500050249999831 | 0.508623956709160 | 0.008573706709329 |
| 0.05 | | 0.500037749999928 | 0.512078840448603 | 0.012041090448675 |
| 0.01 | 0.05 | 0.500112812498086 | 0.495074644318046 | 0.005038168180040 |
| 0.02 | | 0.500100312498654 | 0.499212224708285 | 0.000888087790369 |
| 0.03 | | 0.500087812499097 | 0.503122230345461 | 0.003034417846364 |
| 0.04 | | 0.500075312499430 | 0.506825658317006 | 0.006750345817576 |
| 0.05 | | 0.500062812499670 | 0.510341379000510 | 0.010278566500841 |
| 0.01 | 0.06 | 0.500137874996505 | 0.492492007972452 | 0.007645867024053 |
| 0.02 | | 0.500125374997372 | 0.496642393806129 | 0.003482981191243 |
| 0.03 | | 0.500112874998083 | 0.500564964433619 | 0.000452089435537 |
| 0.04 | | 0.500100374998652 | 0.504280738884366 | 0.004180363885715 |
| 0.05 | | 0.500087874999095 | 0.507808607216788 | 0.007720732217693 |
| 0.01 | 0.07 | 0.500162937494232 | 0.489399727599114 | 0.010763209895119 |
| 0.02 | | 0.500150437495461 | 0.493504097865546 | 0.006646339629915 |
| 0.03 | | 0.500137937496501 | 0.497383931385718 | 0.002754006110783 |
| 0.04 | | 0.500125437497386 | 0.501059969090885 | 0.000934531593517 |
| 0.05 | | 0.500112937498079 | 0.504550852381356 | 0.004437914883276 |
| 0.01 | 0.08 | 0.500187999991140 | 0.485882659102875 | 0.014305340888266 |
| 0.02 | | 0.500175499992793 | 0.489883504232251 | 0.010291995760542 |
| 0.03 | | 0.500162999994226 | 0.493666521190520 | 0.006496478803706 |
| 0.04 | | 0.500150499995455 | 0.497251880445489 | 0.002898619549966 |
| 0.05 | | 0.500137999996496 | 0.500657713270939 | 0.000519713274444 |

**Table (3.2): Numerical Results of the nonlinear Burger's Fisher equation.**

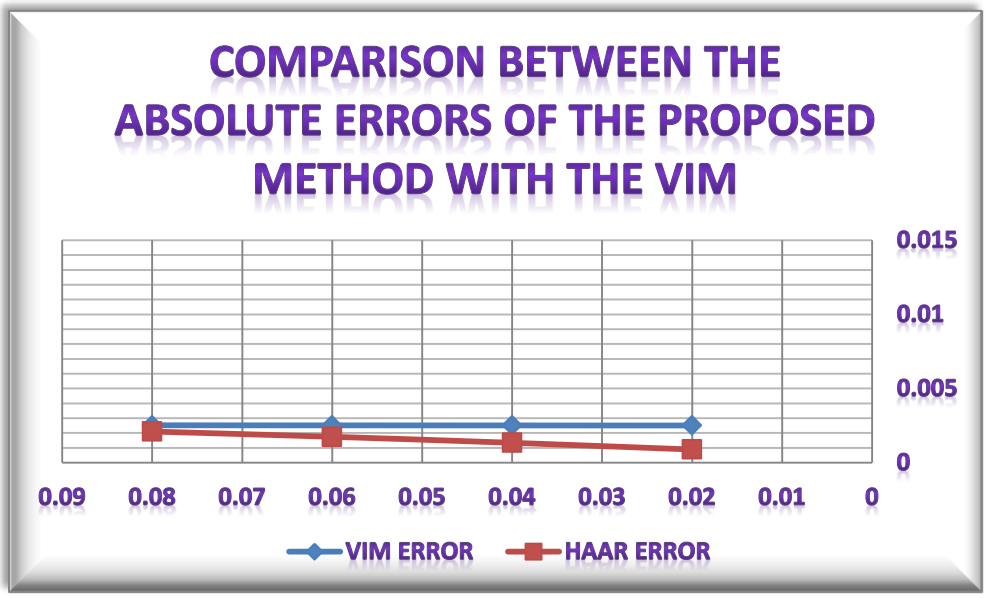



**Figure. 3.1: Comparison between the absolute errors of the proposed method with the VIM.**

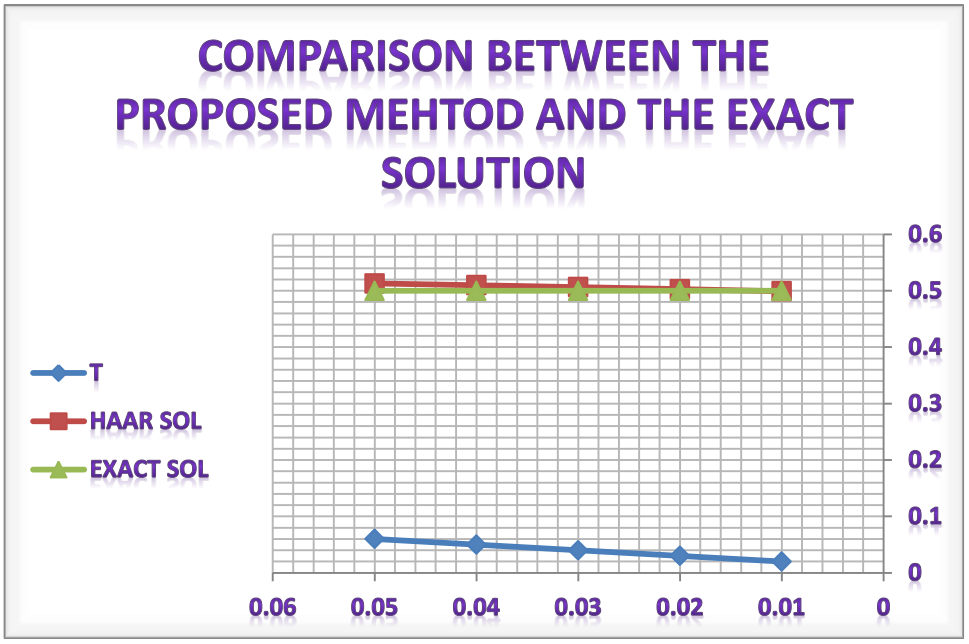



**Figure. 3.2: Comparison between the proposed method and the Exact solution.**

# Conclusions and Future Work

1. The Burger's Fisher equation has been analyzed using the Haar wavelet method, and the results of this method have been compared with the variational iteration method. The proposed method shows that it is in good agreement with the exact solution and it is better than variational iteration method. The experimental results show that the Haar wavelet method is computationally efficient for solving evolution problems and can easily be implemented on computer.
2. The fast convergence and simple applicability of this method provides excellent foundation for using these functions in numerical approximation of variety problems.
3. On the other hand, the Haar wavelet method has distinctive property in terms of its ability of dealing with the IVPs and BVPs without transformation the BVPs into IVPs as needed in some numerical methods.

Based on the results of the proposed method and its illustrative, the following future work may be suggested.

1. One can use multiwavelets methods for solving highly nonlinear partial differential equations.
2. One can use a hybrid method to improve the scheme of numerical approximation methods for solving differential equations.

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# الملخص

**في هذه الرسالة، تم تطبيق طريقة مويجات هار بكفاءة في إيجاد الحل العددي لمعادلة برغر فيشر. اظهرت هذه الطريقة تقارباً سريعا بالنسبة الى الطرائق الأخرى. وتشير الأمثلة التوضيحية إلى أن استخدام طريقة المويجات تزود بطريقة قوية لايجاد الحلول العددية لمعادلة برغر فيشر. أظهرت المقارنة بين النتائج العددية والحل التام والحلول التي تم الحصول عليها باستخدام بعض الطرق التقليدية مثل طريقة التكرار التباييني (VIM)، أن الطريقة المقترحة تعطي نتائج دقيقة إلى حد ما لحل مسألة برجر فيشر.**

جمهـورية العـراق
وزارة التعليم العالي والبحث العلمي
جـامعـة النــهرين
كليـــة العلوم
قسم الرياضيات وتطبيقات الحاسوب

# حل عددي لمعادلة بُرجر فِيشر باستخدام إسلوب مويجات هار

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وهي جزء من متطلبات نيل درجة ماجستير في علوم الرياضيات

من قبل

**نوار حازم محمد**

(بكلوريوس كلية التربية/ الجامعة المستنصرية، ٢٠١٠ )

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