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Statistical Mechanics and Thermodynamic Properties of Bose-Einstein Condensation in Fractal Media

A Thesis

Submitted to the College of Science /Al-Nahrain University in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Physics

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سورة الجمعة

بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ

يُسَبِّحُ لِلَّهِ مَا فِى ٱلسَّمَٰوَٰتِ وَمَا فِى ٱلْأَرْضِ ٱلْمَلِكِ ٱلْقُدُّوسِ ٱلْعَزِيزِ ٱلْحَكِيمِ
﴿١﴾ هُوَ ٱلَّذِى بَعَثَ فِى ٱلْأُمِّيِّـۧنَ رَسُولًا مِّنْهُمْ يَتْلُوا۟ عَلَيْهِمْ ءَايَٰتِهِۦ
وَيُزَكِّيهِمْ وَيُعَلِّمُهُمُ ٱلْكِتَٰبَ وَٱلْحِكْمَةَ وَإِن كَانُوا۟ مِن قَبْلُ لَفِى ضَلَٰلٍ مُّبِينٍ
﴿٢﴾ وَءَاخَرِينَ مِنْهُمْ لَمَّا يَلْحَقُوا۟ بِهِمْ ۚ وَهُوَ ٱلْعَزِيزُ ٱلْحَكِيمُ ﴿٣﴾ ذَٰلِكَ
فَضْلُ ٱللَّهِ يُؤْتِيهِ مَن يَشَآءُ ۚ وَٱللَّهُ ذُو ٱلْفَضْلِ ٱلْعَظِيمِ ﴿٤﴾

صدق الله العلي العظيم

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Ibrahim Abdelmehdi Sadiq
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Summary

The phenomenon of Bose–Einstein condensation (BEC), which was predicted by Einstein in 1925 and experimentally realized in 1995, has been the subject of intensive research in the last decades. On the theoretical side, several approaches have been formulated. One of the important concerns of these approaches has been the conditions under which this phenomenon is realized. Among the known factors affecting this realization is the dimensionality of the bosons' confining medium. The main conclusions of previous studies are that BEC can occur in 3D and in 2D under a wide range of conditions. Among these conditions is the inhomogeneity of the Bose systems. But, to occur in 1D more stringent conditions are required, among them is the need for the treatment of a finite number of particles. Recently, there has been increased interest in BEC occurrence in media with fractional dimensions for two main reasons. The first is the experimental findings asserting the fractality of bosons' confining media. The second reason is the emergence of fractal geometry as a well–founded research discipline; whereas this emergence was indeed contemporary to the aforementioned experimental findings. However, the formulation of fractal models for the BEC phenomenon is still in its early stages.

The present work is mainly devoted to the formulation of such a model and also to the investigation of its thermodynamic behavior through symbolic computation by using the *MATHEMATICA®* software package as a computational environment. The model formulated in the present work assumes that a finite number of ideal bosons are harmonically trapped in a fractal medium. It also assumes that the applicable statistical mechanics ensemble is the grand canonical. The fractality of the confining medium has been introduced in the formulation by two distinct methods. The first is by adopting an idea due to Rovenchack; which assumes that the degeneracy factors of the energy levels can be extended to fractal dimensions by converting the factorial functions appearing in the expressions for the degeneracy factors into gamma functions. The second method is to use the well–established nonextensive Tsallis statistics; where the index of nonextensivity is related to the fractal dimension. It is important to mention here that both methods reduce to the standard case when dealing with the integer dimensions (1D, 2D and 3D).

To test the proposed *MATHEMATICA®* symbolic computational framework, computations of Bose–Einstein condensates for integer 1D, 2D and 3D were first carried out on the basis of the previously mentioned assumptions. The tests confirm the robustness of the computational scheme and the results obtained agree with previous ones.

Due to the success of the tests, computations on the basis of the two models were carried out for bosons harmonically trapped in fractal media which are embedded in 2D and in 3D dimensions. In general, it is found that the condensation temperature in the model based on Tsallis thermostatistics is lower than that obtained on the basis of Boltzmann–Gibbs thermostatistics. It is found that this result agrees with Salasnich result and observations by and other workers in the field.

In conclusion, the models presented in this thesis and the proposed symbolic computational scheme can be successfully used to treat the BEC phenomenon in fractal media and permit possible extension.

Symbols in Latin

$B_\nu(z)$: Bose function

C : heat capacity

D : general spatial dimension

D_E : Euclidean integer dimension

D_f : fractal dimension

g_n : degeneracy of the harmonic-oscillator

$g(\varepsilon)$: density of states

g : parameter that indicates the sort of g –ons particles.

$\hbar$: Planck's constant divided by 2π

i: running integer

j: running integer

k_B : Boltzmann's constant

L : linear size of a geometrical object

m : the mass of an individual particle

n : running integer

n_i : number of particles in the state i

$n(\varepsilon_i)$: number of particles in the state i

$n(p)$: distribution function of the particles in the momentum space

N : total number of particles

N_{ex} : number of particles in the excited states

N_0 : ground state population (number of particles in the 0-energy state)

$\mathcal{N}$: number of self–similar objects

$\mathcal{N}(e)$: The number of self–similar objects of size e

O : physical observable quantity

p : momentum

p_i : the probability of the state i

P_i : the escort probability of the state i

q : the index of the nonextensive entropy

r : spatial position

$\Re$: set of real numbers

S : entropy

S_{BG} : Boltzmann-Gibbs entropy

S_q : the Tsallis or the nonextensive entropy

§ : section

T : temperature

T_c : the condensation temperature or the critical temperature for condensation

T_0^{3D} : the condensation temperature in a 3D harmonic-oscillator for a gas in the thermodynamic limit

T_c^{3D} : the condensation temperature for a gas with finite number of particles in a 3D harmonic-oscillator

T_c^{2D} : the condensation temperature for a gas with finite number of particles in a 3D harmonic-oscillator

T_c^{1D} : the condensation temperature for a gas with finite number of particles in a 1D harmonic-oscillator

U : the internal energy

V : the volume of the gas

W : the weight of a configuration

$W_{\max}$: the maximum weight

$x, y \text{ or } z$: spatial positions

z : fugacity

Z_q : the partition function of the nonextensive statistics

Symbols in Greek

β : the inverse temperature ($1/k_B T$)

ε_i : energy of the state i

ε_0 : energy of the ground (zero) state

γ : coefficient which depends on the individual oscillator frequency

$\Gamma(\nu)$: the gamma function

List of Symbols

η : the power-law exponent

λ_T : de Broglie thermal wavelength

Λ : density of the number of particles

Λ_c : number of particles critical density for BEC

$\Lambda(r)$: number of particles density as a function of position

μ : the chemical potential

ρ : phase-space density

ρ_c : critical phase-space density for BEC

ω : angular frequency of the harmonic-oscillator

ω_m : arithmetic mean frequency of the harmonic-oscillator

$\bar{\omega}$: geometric mean frequency of the harmonic-oscillator

Ω : the total number of microstates

$\zeta(\nu)$: the Riemann zeta function, where $\nu > 1$.

Contents

Contents

Chapter 3

Condensation Temperature in Harmonic Traps of an Ideal Bose Gas with Finite Number of Particles

Chapter 4

Thermodynamic Properties of Finite Number of Bosons Harmonically Trapped Fractal Media

Contents

Chapter 5

Conclusions and Suggestions for Further Work

Appendices

References

Summary in Arabic: (المُلخَّــص)

Chapter One

What is Bose-Einstein Condensation?

1.1 The Invention of Bose–Einstein Statistics

In the second half of the 19th century, Maxwell and Boltzmann invented a statistical model to describe how ideal (non–interacting) classical particles (molecules) occupy energy levels in the state of thermodynamic equilibrium. This statistics was later on called Maxwell–Boltzmann statistics. In one of the most important papers in physics, entitled *"Plancks Gesetz und Lichtquantenhypothese"* whose English translation is *"Planck's Law and the Quantum Hypothesis"*, Satyendra Nath Bose derived Planck's Law by regarding the electromagnetic radiation in the Black–Body Radiation cavity as an ideal (non–interacting) gas of photons (quanta of light) [1–3]. Because *Philosophical Magazine*, in 1923, rejected Bose's derivation, he wrote a letter to Albert Einstein to consult him about his derivation. Einstein, not only recognized the importance of Bose’s work but also translated it into German and got it published in *Zeitschrift für Physik*, in 1924, upon Bose's request [2,3]. Bose's derivation has shown that photons are identical (indistinguishable) particles [1] and emphasized that photons (bosons) do not obey MBS. In two famous treatises entitled "*Quantentheorie des einatomigen idealen Gases"* whose English translation is "*Quantum Theory of the Ideal Monatomic Gas*", Einstein introduced a theory for an ideal quantum gas when he extended Bose's work to particles which possess mass [1–3]. Bose and Einstein's theoretical work, which describes how non–interacting quantum particles of integer spin value occupy energy levels in the state of thermodynamic equilibrium, is known as Bose–Einstein statistics. Particles which obey Bose–Einstein statistics are called bosons.

From the statistical point of view, the results obtained for the properties of an assembly of systems depend on whether the component systems are considered to obey classical or quantum mechanics. The differences in results will arise from the fundamental assumptions which are made regarding the behavior of the different types of systems [4]. The fundamental differences between Bose–Einstein statistics and Maxwell-Boltzmann statistics are exactly the reason beyond Bose's conclusions concerning photons in the black–body radiation cavity. Photons (bosons) are quantum particles, so, there will be only certain discrete energy levels which are available to bosons rather than the continuous spectrum which is available to classical systems. Systems that obey classical mechanics are considered completely distinguishable from other systems all belonging to the same species of particles, whilst two identical quantum systems must be taken as being completely indistinguishable unless they are considered to be localized in space, as in the case of atoms at particular sites in a crystal lattice [4]. Since bosons do not obey the Pauli exclusion principle, there is no restriction on the number of bosons that may occupy an energy state.

1.2 Prediction of Bose-Einstein Condensation

On the basis of the statistical description of photons enclosed in the black-body radiation cavity invented by Bose [1], Einstein predicted the phenomenon of condensation in ideal Bose gases [1]. He defined this phenomenon as a phase transition whereby a system composed of bosons will experience an abrupt avalanche into its ground state below a certain temperature [1], i.e., bosons condense. Today, this phase transition is called Bose–Einstein condensation (BEC) and the corresponding phase is known as Bose–Einstein condensate. According to the de Broglie hypothesis, quantum systems possess the property of wave–particle duality. The wave properties of a particle of mass m at a certain temperature T are determined by the thermal de Broglie wavelength $\lambda_T = \sqrt{2\pi\hbar^2 / mk_BT}$; where k_B is Boltzmann constant. When the temperature of the system is low enough, that λ_T is comparable to the average spacing between the particles, their thermal (de Broglie) waves overlap and the atoms behave coherently as a single giant atom [5]. Since the condensate particles are in a macroscopic coherent state, BEC is considered as the origin of superfluidity and superconductivity phenomena [5,6], where particles travel with no internal resistance (viscosity and electrical resistance). For these reasons, BEC is considered as the macroscopic manifestation of quantum mechanics [6].

1.3 Milestones of BEC from the First Clue to the First Experimental Realization

It is important to mention here that the prediction of the phenomenon of BEC by Einstein in 1924 was even earlier than the quantum theory being thoroughly formalized. So, it had been considered as an illusion for the extremely low phase transition temperature (T_c) an ideal Bose gas confined in a 3D container requires to undergo condensation. Fermi thought it was impossible for this phenomenon to be realized because at such low temperatures the interatomic interactions would suppress the phase transition. Even Einstein himself doubted about his prediction. In a private letter to Paul Ehrenfest he wondered if his prediction was true. He wrote "From a certain temperature on, the molecules condense without attractive forces, that they accumulate at the zero velocity. The theory is pretty, but is there also some truth to it?" [7].

In 1937, Jack Allen and Don Meisner, and independently Peter Kapitza, discovered that the viscosity of liquid helium vanishes suddenly below a certain temperature [8]. Independently, Fitzgerald London and Laszlo Tisza ascribed the disappearance of viscosity to a new phase transition of matter; an evidence for superfluidity [8]. In 1938, Fitzgerald London resurrected Einstein’s prediction when he interpreted this superfluid behavior as a manifestation of BEC [6]. Stimulated by Kapitza, Landau introduced a theory of helium II superfluidity in 1941 [8]. Motivated by London's phenomenological idea and Landau's theory of superfluidity, in 1947 Bogoliubov introduced the first microscopic theory of superfluidity in weakly interacting Bose particles [9]. In 1950, Ginzburg and Landau introduced a theory for sperconductors on the basis of quantum field theory and they found that superconductivity was associated with a specific type of order [10]. In 1956, Oliver Penrose and Lars Onsager also found that such type of order is associated with the superfluidity of liquid helium [10]. Penrose and Onsager proved their theory on the basis of a generalized mathematical description of BEC for interacting particles [9]. In 1960, the specific type of order, which has been associated with Ginzburg–Landau's superconductivity and Penrose–Onsager superfluidity, was defined by London calling it long–range order of the average momentum [10]. In 1962, Yang succeeded in connecting both superfluidity and superconductivity to BEC by introducing the concept of Off–Diagonal Long–Range Order (ODLRO) in which BEC is the simplest form of ODLRO [11]. In 1995, BEC was experimentally proved by using neutron scattering as a probe to investigate the momentum distribution of liquid helium [12]. Sokol justified the ground state with zero momentum which was occupied by a macroscopic fraction of the total number of particles by the existence of BEC [12,13].

The long–awaited achievement came up in 1995; BEC was experimentally realized with magnetically trapped dilute atomic gases of alkali atoms cooled by laser and evaporative cooling techniques [14–16]. For this eminent achievement and also for their fundamental studies of the properties of these condensates, Wolfgang Ketterle of the MIT, and Carl Wieman and Eric Cornell of JILA, an interdisciplinary research centre in Boulder, Colorado, USA, were awarded the 2001 Nobel Prize in physics [17].

To explain Bose–Einstein condensation comprehensively, one needs to understand the physical conditions, the required technology necessary to produce the physical phenomena related to its very special characteristics and features and its prospective aspects as a novel state of matter.

1.4 The Fifth State of Matter: Bose-Einstein Condensate

In our daily life we experience three phases of matter: solid, liquid and gas. Solids have fixed volumes and shapes, liquids have fixed volumes and varying shapes and gases have neither. This is ascribed to the differences in the strength of the intermolecular bonds. Solids have intermolecular bonds stronger than liquids which have intermolecular bonds stronger than gases. In terms of energy levels, solids possess the lowest energy levels, while liquids and gases possess higher energy levels. The state of matter that possesses the top of the energy levels is the fourth state of matter; plasma. It occurs when gases are exposed to extremely high temperatures, as in the case of fire flames or stars. The state of matter which is contradictory to the fourth state of matter is the fifth state of matter; the Bose–Einstein condensate. It occurs when gases are under extremely low temperatures. Phase transition of a gas into a liquid or into a solid is an ordinary condensation that takes place in the coordinate space while in BEC the condensation takes place in the momentum space [4]. So, a Bose–Einstein condensate is the state of matter with the lowest energy levels.

1.5 Physical Conditions to Produce BEC in Alkali Vapors

Bose–Einstein condensates in dilute atomic gases, first realized in 1995 for Rb [14], Li [15], and Na [16] vapors, differ from ordinary gases, liquids, and solids in a number of aspects. The densities of nucleons in atomic nuclei, liquids and solids, and molecules in air at room temperature and atmospheric pressure are, respectively, about 10^{38}cm^{-3}, 10^{22}cm^{-3} and 10^{19}cm^{-3}, while the particle (number) density at the center of a Bose–Einstein condensed atomic cloud is typically 10^{13}–10^{15}cm^{-3} [13]. This low number density, of this order, is indeed necessary to prevent phase transition into liquid or solid phases [13]. Therefore, contrast in the densities of the aforementioned systems is responsible for observing quantum phenomena in degeneracy temperatures of different orders. The effects of quantum degeneracy set in when the thermal de Broglie wavelength is comparable to the inter–particle separation.

To observe quantum phenomena in BEC atomic clouds, the temperature must be of order 10^{-5}K or less. For the helium liquids, the temperatures required for observing quantum phenomena are of order 1K. In solids, quantum effects can be observed for phonons below the Debye temperature, which is typically of order 10^2K, and these effects become strong for electrons in metals below the Fermi temperature, which is typically 10^4–10^5K. For the high particle density in atomic nuclei, the degeneracy temperature is about 10^{11}K [13].

1.6 How to Produce BEC Clouds

It was a long journey until the technology needed to realize BEC was developed. Methods for cooling alkali vapors by using lasers have been exploited since the mid–1970s [13]. Although the lasers used were powerful for producing and manipulating cold atomic vapors, however, laser cooling alone was not sufficient enough to realize the condensation temperatures. Therefore, it was followed by another technique of cooling [13]. This was the evaporative cooling in which the more energetic atoms are removed from the trap, thereby cooling the remaining atoms [13]. The following paragraph describes briefly the process involved in producing BEC clouds first realized in the mid–1990s [13]:

"A beam of sodium atoms emerges from an oven at a temperature of about 600 K, corresponding to a speed of about 800 m.s^{-1}, and is then passed through a so–called Zeeman slower, in which the velocity of the atoms is reduced to about 30 m.s^{-1} corresponding to a temperature of about 1K. In this slower, Zeeman interaction arises due to the magnetic moments of the electron and the nucleus with the external magnetic field. Also, in this slower, a laser beam propagates in the direction opposite to that of the atomic beam in order to retards the atoms by the radiation force resulting from absorption of photons. On emerging from the Zeeman slower, the atoms are slow enough to be captured by a magneto–optical trap, where they are further cooled by interactions with laser light to temperatures of order 100μK. The final step in achieving Bose–Einstein condensation is evaporative cooling; in which relatively energetic atoms leave the system, thereby lowering the average energy of the remaining atoms ".

1.7 Characteristics and Importance of BEC in Alkali Vapors

The BEC phenomenon can be viewed as the field where physics disciplines entangle. Concepts of statistical mechanics, atomic physics, nuclear physics and condensed matter physics are necessary to understand BEC qualitatively and quantitatively. Quantum statistics explains BEC as more than one atom sharing a phase space cell and how these atoms condense to the ground state. The thermodynamics point of view is that BEC occurs as a phase transition from gas to a new state of matter, while in quantum field theory BEC is commonly related to spontaneous symmetry breaking. Quantum mechanics views BEC as a matter–wave coherence arising from overlapping de Broglie waves of the atoms and draws an analogy between conventional and atom lasers. Without the use of novel low–temperature physics techniques, laser and optical instrumentation and the dexterous use of fluid dynamics and magnetism, BEC wouldn't be produced [5,18,19].

Despite the fact that BEC is essentially a microscopic (quantum) property, it is macroscopically manifested; superfliudity, superconductivity, nucleation of quantized vortices when the system is set in rotation and interference patterns of overlapping coherent matter waves are examples of BEC macroscopic phenomenological manifestations. These macroscopic phenomena emphasize the existence of a macroscopic wavefunction [5, 18, 19]**.** BEC of alkali vapors as a new state of matter has remarkably new physical features. The production of new physical systems requires number densities between 10^{14} cm^{-3} and 10^{15} cm^{-3} within a temperature range 2μK–0.5μK at pressure of order 10^{-11} torr [9, 13]. These condensates display coherent matter waves with well–defined amplitude and phase represented by a single wavefunction [18]. This fact makes the idea of matter waves with constructive/destructive interference possible [18]. These condensates have been shown to be an optically dense material where the measured speed of light is 17 $m.s^{-1}$ [20]. The speed of sound in these vapor condensates was shown to be a function of the number density, atomic mass and the trapping potential [21]. A recent estimation of the speed of sound in these vapor condensates pointed out that it is of order 10^{-7}m.s^{-1} [22]; which is very small in comparison with that in air under standard conditions (≈340 m.s^{-1}). From a phenomenological point of view, and due to the modification in the dispersion relation of microscopic particles, BEC with atomic vapors can be a promising laboratory means to test and explore many important theoretical issues such as quantum–gravity manifestation and space–time quantization [19,22].

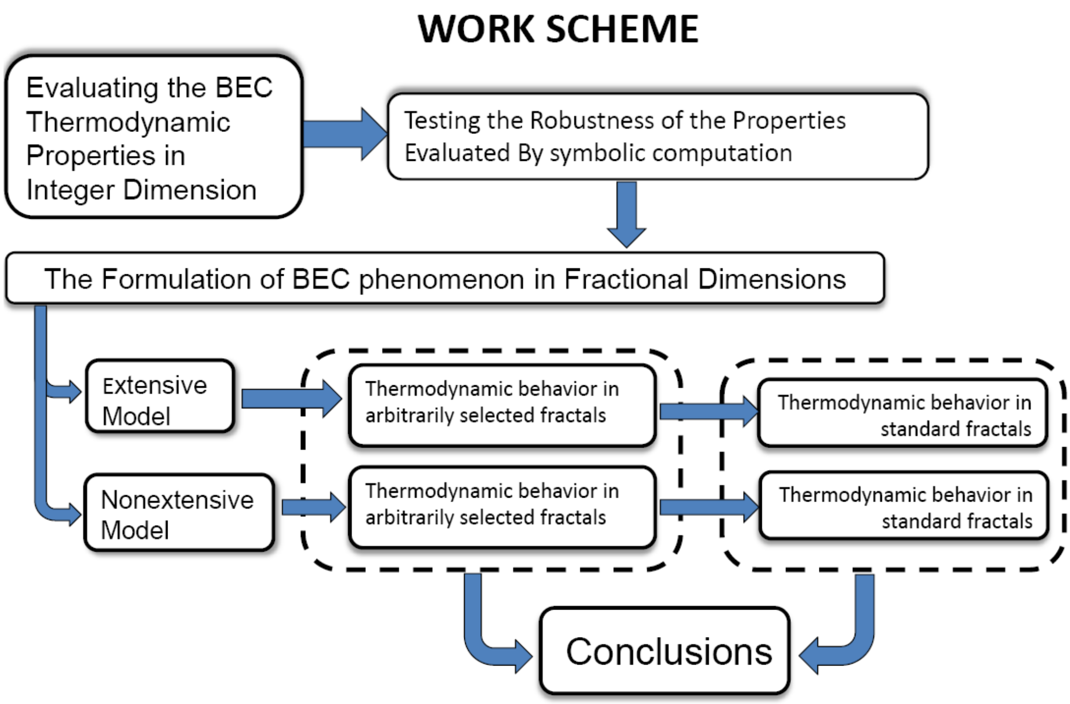
Besides the aforementioned characteristics, there are other interesting aspects of BEC: the possibilities of producing molecular Bose–Einstein condensates [19] and constructing atom lasers, amplification of matter waves [23] and the possibility of creating dibaryon lasers since in dense nuclear matter the Bose condensation of dibaryons can take place [19]. So, physicists have a lot of theoretical and experimental investigations to do; e.g. the proper type of cavities, mirrors, or lenses for these new types of lasers. Also, optical quantities such as refractive index and directionality, and optical relations and formulas such as Rayleigh scattering, Braggs' law, dispersion relations and others need to be investigated for BEC condensates. So, one can imagine how much BEC is a fertile field to explore and one can also imagine the significant role and importance of BEC in developing physical concepts for totally new state of matter.

1.8 The Aims of the Thesis

1) The aim of this thesis is to formulate a statistical mechanics model for the BEC phenomenon in fractal media and constructing a programming code for the main equations of the model using the software package *MATHEMATICA®*.

2) It also aims to investigate the thermodynamic behavior of the Bose–Einstein condensate properties in selected fractal geometries via evaluating the condensation temperature and the temperature dependence of the properties such as the fugacity, the condensate fraction, the internal energy and the heat capacity in these geometries.

3) The use of *MATHEMATICA®* as a computational environment, indeed, represents an aim by itself for the feature of symbolic computation; computing a sum over its terms is intended. The following chart exhibits a road map to realize the thesis aims.



1.9 Thesis Layout

This thesis is organized as follows. Chapter two is divided into two main parts. Part I reviews historically the development of the theory of BEC formation in Bose gases in such a way that generally reflects the influence of spatial dimension on BEC formation. Part II reviews how to analytically evaluate the thermodynamic properties of the condensate.

Chapter three is mainly devoted to the discussion of the symbolic computation in statistical mechanics problems. This utility is represented in a twofold way. The first is the correction of the condensation temperature (T_c) relative to the corresponding analytical expressions for a condensate in 1D, 2D and 3D harmonic traps. The second aspect is the evaluation of the condensate thermodynamic properties for the aforementioned systems.

Chapter four is mainly devoted to evaluate numerically (by symbolic computation) those condensate thermodynamic properties under investigation, for harmonically trapped bosons in fractal media. These properties are evaluated on the basis of two distinct types of thermostatistics; Boltzmann–Gibbs extensive thermostatistics and the Tsallis non–extensive thermostatistics.

Finally, chapter five gives the main conclusions and also gives suggestions for further work.

Chapter Two

BEC Theory and Thermodynamic Properties of Harmonically Trapped Ideal Bose-Einstein Condensate

This chapter is divided into two main parts. Part I reviews historically the development of the BEC theory in such a way that generally reflects the influence of spatial dimension. Part II reviews the analytic evaluation of the thermodynamic properties of the Bose–Einstein condensates in two separate sections. Properties when the thermodynamic limit is satisfied; i.e., when the Bose gas consists of a huge number of particles are discussed in § (2.5). The same properties when the Bose gas consists of a finite number of particles are discussed in § (2.6). The thermodynamic properties under investigation are the condensation temperature, the condensate fraction, the total energy and the heat capacity.

Part I:
Theory of BEC Formation in Integer Dimensions

2.1 Factors Affecting BEC Formation

In this section, the emphasis is on the roles of dimensionality, trapping potentials and the finiteness of the number of particles in BEC formation. Peierls was the first to notice that the properties of collective physical phenomena in an environment with a reduced number of dimensions could be dramatically changed with respect to our experience in three dimensions [24]. Since BEC is an eminent example of a collective physical behavior, dimensionality plays a very effective role in changing BEC properties. It was found that in low–dimensional systems, the number of spatial degrees of freedom affects the properties of the phase transitions and collective oscillations [25].

Einstein's prediction of BEC with non–interacting (ideal or perfect) free Bose gas (homogeneous or uniform system) in the thermodynamic limit, where both N and $V \rightarrow \infty$ but (N/V) is constant, was indeed for a system confined in a 3D container [1].

In 1966 and 1967, the idea of expecting the occurrence of the BEC phenomenon in 2D and 1D structures clashed with a proven statistical mechanics theorem (Mermin–Wagner–Hohenberg theorem [10,26]) for the absence of the long–range order which was previously observed in superconductivity (Ginzburg–Landau [10]) and superfluidity (Penrose–Onsagar [9]). It also clashed with the general result of BEC occurrence in an ideal Bose gas confined in dimensions $D \leq 2$ [27]. The same result was also observed on the basis of quantum field theory when Coleman found that there is no spontaneous symmetry breaking for an ideal homogeneous Bose system confined in 2D structure [28]. This means that BEC does not occur with homogeneous systems in 2D and 1D. In 1968, Widom [29] proved theoretically that the BEC phase transition is attainable in 2D and 1D provided that the Bose system is inhomogeneous (non–uniform system). Widom theoretical verification was based on investigating the BEC phenomenon in the presence of (i) external field (gravitational) and (ii) in rotational motion of an ideal Bose liquid [29].

Theoretical works that considered the ideal Bose gas as trapped in power–law potentials [30–32] and lower–dimensional systems [32–35] led to the conclusion that if the Bose gas is confined by a spatially varying potential (inhomogeneous system), BEC can occur for a sufficiently confining potential. It was shown that in a harmonically trapped ideal 2D Bose gas, occupation of the ground state becomes macroscopic below a critical temperature which depends on the number of particles and trap frequencies [31].

The realization of BEC with dilute gases by means of evaporative cooling and laser trapping of alkali atoms [14–16] renewed the interest in studying different external potentials confining ideal quantum gases and the dimensions of the trapping potentials as well. Theoretically, BEC was examined in the presence of typical external potentials, for example, harmonic potential, toroidal potential and double–well potential [35]. These works confirmed that the trapping potential plays an essential role in condensation realization of for a given amount of a Bose gas. In this connection, a very important result concerning the types of external potentials confining ideal bosons and the dimensions of these trapping potentials, in the thermodynamic limit, was the one obtained by Salasnich [35]. Indeed, this result is a condition for BEC occurrence, whereas BEC can set up if and only if $[(D/2)+(D/\eta)]>1$, where D is a D–dimensional space and η is the power–law exponent of the generic external potential involved [35]. From this review it becomes clear that, in the thermodynamic limit, the occurrence of BEC with dilute atomic gases in 3D is attainable with homogeneous ideal Bose gases and it is also attainable in 2D but for inhomogeneous systems, i.e., Bose gases in sufficient confining potentials.

With the realization of BEC in experiments with dilute gases of alkali atoms [14—16], the two terms *finite size* and *finite number of particles* acquired an important interest. This is due to the fact that the number of particles and, hence, the size of the Bose systems in these experiments are indeed finite. Stimulated by this regard, a remarkable theoretical works that dealt with ideal Bose gas were confined to the case of finite number of particles [25,36–45]. The adoption of the treatment of finite number of particles has led to the prediction of BEC occurrence in 1D [25]. Prior to this adoption, the possibility of BEC occurrence in 1D was thought to be impossible even for harmonically trapped bosons [32]. The impossibility of expecting the BEC occurrence in 1D harmonic trap was ascribed [25] to the use of the semi–classical approach the work of Ref. [32] adopted. The occurrence of BEC with dilute gases in 1D was predicted to occur in a highly anisotropic harmonic potential [46]. Using highly anisotropic potentials enabled observing a crossover of BEC in 3D confinement into BEC in 2D or 1D confinement with varying condensation temperatures and other thermodynamic properties [47].

2.2 The Free Ideal Bose Gas: BEC Formation in 3D

The total number of particles in the system N is governed by the constraint [25]:

$$N = \sum_{i=0}{}^{'} n_i \tag{1.2}$$

where n_i is the number of particles in the i^{th} state (the occupation number). For Bose–Einstein statistics, the occupation number is defined as [48]:

$$n_i = n(\varepsilon_i) = \frac{1}{\exp[\beta(\varepsilon_i - \mu)] - 1} \tag{2.2}$$

where $\beta = 1/(k_B T)$, ε_i is the single particle energy in state i and μ is the chemical potential. The chemical potential should always be less than or equal to the ground state energy $(\mu \le \varepsilon_0)$ to ensure that the occupation numbers be all positive. As the system temperature drops, the inverse temperature β becomes larger and the chemical potential (μ) must correspondingly increase to prevent the occupation number, Eqn. (2.2), from being negative. Since the ground state energy is conveniently taken zero $(\varepsilon_0 = 0)$, therefore, μ is generally negative [25].

The classical gas theory is fully understood and described in the context of thermodynamics where the condition of the thermodynamic limit is satisfied [48]. For this condition, the molecules (classical particles) are distributed over energy states whose energy spacing is infinitesimal, i.e. $\Delta\varepsilon \to d\varepsilon$, and, hence, the energy spectrum for this system is continuous. Consequently, the evaluation of the thermodynamic quantities in classical gas theory was completely expressed in the notation of integrals instead of the notation of sums [48].

The quantum gas theory introduced by Einstein [1–3] was for an ideal free (homogeneous) Bose gas in the thermodynamic limit. This requires that the thermodynamic quantities of the quantum gas, which are evaluated on the basis of this theory, are in the notation of integrals. This is ascribed to the fact that when the discreteness of the energy levels is neglected, the sum of Eqn. (2.1) can be replaced by an integral. This is justified when the energy levels spacing is microscopic in comparison with the mean energy; and such a treatment is called the semi–classical approach [32]. Hence, when $N\to\infty$, the total number of particles, Eqn. (2.1), can be expressed as [48]:

$$N = \int_0^\infty n(p)dp \tag{2.3}$$

where $n(p)$ is the particles distribution function in the momentum–space. For an ideal free gas in a 3D box, the Hamiltonian is given by $\varepsilon = p^2/2m \equiv (p_x^2 + p_y^2 + p_z^2)/2m$, and equation (2.3) can be written as [49]:

$$N = \frac{V}{(2\pi\hbar)^3}\int_0^\infty \frac{4\pi p^2 dp}{\exp[\beta(p^2/2m) - \beta\mu] - 1} \tag{2.4}$$

Substituting $x = \beta p^2/2m$, gives [49]:

$$N = \frac{V2\pi}{(2\pi\hbar)^3}\left(\frac{2m}{\beta}\right)^{3/2}\int_0^\infty \frac{x^{1/2}dx}{\exp[x - \beta\mu] - 1} \tag{2.5}$$

By using de Broglie thermal wavelength, $\lambda_T = \sqrt{2\pi\hbar^2/mk_BT}$, and the Bose function $B_\nu(z)$ which is a function of the fugacity ($z = e^{\beta\mu}$), whose definition and properties are given in Appendix A, one can obtain the number density [49]:

$$\Lambda = \frac{N}{V} = \frac{B_{3/2}(e^{\beta\mu})}{\lambda_T^3} \tag{2.6}$$

Eqn. (2.6) can be used to introduce Einstein's condensation criterion; the phase–space density ρ that quantifies the creation of BEC [49] is:

$$\rho = \Lambda \lambda_T^3 = B_{3/2}(e^{\beta\mu}) \tag{2.7}$$

For the case of BEC, at some critical temperature T_c when $\mu = 0$ ($z = 1$), Eqn. (2.7) becomes [49]:

$$\rho_c = \Lambda_c \lambda_{T_c}^3 = B_{3/2}(1) \tag{2.8}$$

where the subscript c indicates the parameters at the critical temperature. From the properties of the Bose function, one has $B_\nu(1) = \zeta(\nu)$ for $\nu > 1$; where $\zeta(\nu)$ is the Riemann zeta function. Then, Einstein's criterion for condensation is expressed as [49]:

$$\left(\frac{2\pi\hbar^2}{mk_B}\right)^{3/2} \frac{\Lambda_c}{T_c^{3/2}} = \zeta(3/2) \approx 2.612 \tag{2.9}$$

From Eqns. (2.8) and (2.9), both ρ_c and T_c define the onset of the condensation of particles into the ground state. It is obvious that the right-hand side of equation (2.9) is definite. Since there is no theoretical reason that prevents the system temperature to drop below the critical temperature, T_c, therefore the critical number density, Λ_c, must necessarily be increasing by adding particles. An inconsistency then arises: there is nothing that prevents increasing the number of particles whilst total number of particles of the system N is governed by Eqn. (2.1); in other words where do the surplus particles go? Einstein resolved the problem by proposing that any surplus particles would occupy the ground state which had been ignored due to the use of the semi–classical approximation; Eqn. (2.4). Then, the critical temperature can be obtained from Eqn. (2.9) as [49]:

$$T_c = \frac{2\pi\hbar}{mk_B}\left(\frac{\Lambda_c}{\zeta(3/2)}\right)^{2/3} \tag{2.10}$$

From the properties of the Riemann zeta function, Eqn. (2.10) emphasizes that BEC for a perfect (ideal) free gas is attainable in a Bose gas in any dimension $D > 2$. It is clear that for an ideal free Bose gas confined in a 3D container the phase–space density and the critical temperature are both definite. It is also observed that in a 3D container, the single particle Hamiltonian of a free gas ($\varepsilon = p^2/2m \equiv (p_x^2 + p_y^2 + p_z^2)/2m$) is what led the analysis of Eqns. (2.9) and (2.10) to be expressed in terms of the Riemann zeta function, i.e., $\zeta(D/2)$. This observation about the single particle Hamiltonian will be used, from now on to analyze the BEC formation of a perfect Bose gas in integer dimensions and in harmonic potentials as well. To obtain similar expressions for the phase-space density and the critical temperature for an ideal free Bose gas confined in a 2D structure, the above analysis should be repeated with a single particle Hamiltonian which is defined by $\varepsilon = p^2/2m \equiv (p_x^2 + p_y^2)/2m$. The previous analysis would lead to obtain expressions in terms of $\zeta(2/2) \equiv \zeta(1)$. From such analysis, a more general rule can be deduced: the critical phase–space density for a free ideal Bose gas (homogeneous system) confined in a container of spatial dimension D is:

$$\rho_c(D) = \zeta(D/2), \qquad D > 2 \tag{2.11}$$

Mathematically, Eqn. (2.11) reads that when $D \leq 2$, the phase–space density and, consequently, the critical temperature are both indefinite; because the Riemann zeta function $\zeta(\nu)$ is only defined for $\nu > 1$. Physically, this means that for an ideal free Bose gas (homogeneous system) confined in structures with dimensions $D \leq 2$, BEC does not occur. This result completely agrees with Mermin–Wagner–Hohenberg theorem [10,26], which is mentioned in § (2.1) in connection with the absence of the long–range order observed previously in superconductivity. From Eqn. (2.11), now, it is clear that the critical phase–space density (ρ_c), responsible for BEC formation, is influenced by the spatial dimensions of the medium containing the Bose gas.

2.3 Ideal Bose Gas in Harmonic Traps: BEC Formation in 2D

Theoretically, the isotropic harmonic potential $(\omega_x = \omega_y = \omega_z)$ is the simplest model to demonstrate the effect of external potentials on physical systems. The single particle Hamiltonian is $\varepsilon = [(p^2/2m) + (m\omega^2 r^2/2)]$, where $p^2 \equiv (p_x^2 + p_y^2 + p_z^2)$ and $\omega^2 r^2 \equiv (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$. Substituting the Hamiltonian into Eqn. (2.4), one gets [49]:

$$N = \frac{1}{(2\pi\hbar)^3} \int d^3x \int_0^\infty \frac{4\pi p^2 dp}{\exp\left[\beta\left(\frac{p^2}{2m} + \frac{m\omega^2 r^2}{2}\right) - \beta\mu\right] - 1} = \left(\frac{k_B T}{\hbar\omega}\right)^3 B_3(e^{\beta\mu}) \tag{2.12}$$

Using the approach used to obtain Eqn. (2.7) from Eqn. (2.4), the phase–space density can be expressed as [49]:

$$\Lambda(r)\lambda_T^3 = B_3(e^{\beta(\mu - m\omega^2 r^2/2)}) \tag{2.13}$$

Eqn. (2.13) emphasizes that the phase–space density of the Bose gas in trapping potentials depends on the position "*r*". That is why Bose gases in external fields are denoted as inhomogeneous (non–uniform systems); for their densities are spatially dependent. Evaluating Eqn. (2.13) for $B_{3/2}(e^{\beta(\mu - m\omega^2 r^2/2)})$ at *r* = 0, recovers Einstein's criterion for free gas (Eqn. (2.9)). This means that the role of spatially varying potentials (harmonic potential in this case) is merely to concentrate the particles to the density at which BEC commences [32]. When the BEC condition ($N_0 = \mu = 0$) is applied, Eqn. (2.13) gives the critical temperature as [25,36–39,42]:

$$T_0^{3D} = \frac{\hbar\omega}{k_B}\left(\frac{N}{\zeta(3)}\right)^{1/3} \tag{2.14}$$

The subscript "0" indicates that the thermodynamic limit condition is satisfied while the superscript indicates the spatial dimension of the trapping potential. It becomes clear that for an ideal Bose gas (in the thermodynamic limit) trapped in a 3D harmonic potential, the analysis leads T_c to be expressed in terms of $\zeta(3) \equiv \zeta(D)$. An analogous analysis for a Bose gas with infinite number of particles trapped in an isotropic 2D harmonic trap, whose Hamiltonian is $\varepsilon = [(p^2/2m) + (m\omega^2 r^2/2)]$, where $p^2 = (p_x^2 + p_y^2)$ and $\omega^2 r^2 \equiv (\omega_x^2 x^2 + \omega_y^2 y^2)$, would lead the analysis to a critical temperature expressed in terms of $\zeta(2)$. Then, the critical temperature is given as [40,42]:

$$T_0^{2D} = \frac{\hbar\omega}{k_B}\left(\frac{N}{\zeta(2)}\right)^{1/2} \tag{2.15}$$

It is noticeable from Eqn. (2.15) that, in the thermodynamic limit, BEC is attainable in a 2D trap while BEC occurrence in 2D was impossible in the case of free ideal Bose gas (homogeneous system) confined in a 2D structure [32]; as mentioned in § (2.1). This means that BEC occurrence for a Bose gas with infinite number of particles (thermodynamic limit) is attainable in 2D provided that the system is inhomogeneous; which agrees with Widom [29] for BEC occurrence in 2D with inhomogeneous ideal Bose liquid. So, this illustrates exactly the role of spatially varying potential (harmonic potential in the case of Eqn. (2.15)).

2.4 Ideal Bose Gas with Finite Number of Particles: BEC Formation in 1D

From Eqns. (2.14) and (2.15), one might set a generalization for the expression of the condensation temperature for an ideal gas (in the thermodynamic limit) in a D–dimensional harmonic potential as [39]:

$$T_0^{\ D} = \frac{\hbar\omega}{k_B}\left(\frac{N}{\zeta(D)}\right)^{1/D} \tag{2.16}$$

Theoretically, the case of 1D system is a special one for which there is no condensation in the thermodynamic limit [39], i.e., in the semi–classical approach. In this case, the condensation temperature becomes indefinite because of the Riemann zeta function dependence, i.e., BEC is not attainable for ideal Bose gases even in the presence of the harmonic confinement when the thermodynamic limit condition is satisfied. The result of the aforementioned system was first predicted by Bagnato and Kleppner [32]. So, to investigate the possibility of BEC occurrence in a 1D harmonic trap, the semi–classical approximation is not proper anymore, i.e., the thermodynamic limit should be avoided. This requires that the total number of particles of Eqn. (2.3) has to be replaced by Eqn. (2.1).

In this section, the theoretical approach used in Ref. [25] to prove the possibility of BEC occurrence in 1D harmonic trap, will be reviewed. As a starting point, the occupation number of BEC given by Eqn. (2.2) reads [25]:

$$n(\varepsilon_i) = \frac{1}{e^{\beta(\varepsilon_i-\mu)}-1} = \frac{1}{z^{-1}e^{\beta\varepsilon_i}-1} = \frac{ze^{-\beta\varepsilon_i}}{1-ze^{-\beta\varepsilon_i}} \tag{2.17}$$

The energy of the ground state has been taken to be zero. The fugacity (z) can be determined by the total number of particles in the system, i.e. Eqn. (2.1). The degeneracy factors were avoided by accounting for degenerate states individually and the total number of particle is, then, given as [25]:

$$N = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} z^j \exp(-j\beta\varepsilon_i) \tag{2.18}$$

For the case of 1D harmonic trap, Eqn. (2.18) is expressed as [25]:

$$N = \frac{z}{1-z} + \sum_{j=1}^{\infty} z^j \frac{\exp(-j\beta\varepsilon_i)}{[1-\exp(-j\beta\varepsilon_i)]} \tag{2.19}$$

where $z/(1-z)$ = N_0 is the ground state population. For $\varepsilon = (\hbar\omega/k_BT) \ll 1$, Eqn. (2.19) is approximated by $[e^{-j\varepsilon}/(1-e^{-j\varepsilon})] \approx [e^{-j\varepsilon/2}/j\varepsilon]$ to give [25]:

$$N = \frac{z}{1-z} + \frac{1}{\beta\varepsilon}\sum_{j=1}^{\infty}\frac{\left[z\,\exp(-\beta\varepsilon)\right]^{j}}{j} \tag{2.20}$$

$$= \frac{z}{1-z} - \frac{k_B T}{\hbar\omega}\ln\left[1 - z\,\exp\left(-\frac{\hbar\omega}{2k_B T}\right)\right] \tag{2.21}$$

Applying the critical temperature condition, the total number of particles reads [25]:

$$N = \frac{k_B T}{\hbar\omega}\ln\left[\frac{2k_B T}{\hbar\omega}\right] \tag{2.22}$$

And, therefore, the critical temperature is expressed as [25, 39]:

$$T_c^{1D} = \frac{k_B T}{\hbar\omega}\frac{N}{\ln(2N)} \tag{2.23}$$

An alternative expression for T_c^{1D} is given by [40]:

$$T_c^{1D} = \frac{\hbar\omega}{k_B}\frac{N}{\ln(N)} \tag{2.24}$$

From Eqns. (2.14), (2.15) and (2.24), it is obvious that, for a harmonically trapped Bose gas having the same total number of particles , N, and the same trapping frequency, ω, the critical temperature increases with the reduction in the spatial dimension. Consequently, the condensate thermodynamic properties vary as $T_0^{3D} \propto N^{1/3}$, $T_0^{2D} \propto N^{1/2}$ and $T_c^{1D} \propto [N/\ln(2N)]$, i.e., the tighter the confinement, the higher the transition temperature [25].

Part II: Thermodynamic Properties of Harmonically Trapped Ideal Bose-Einstein Condensate

2.5 Thermodynamic Properties in the Thermodynamic Limit

In this section, the computation of the condensation temperature, T_c, and other thermodynamic properties will be reviewed for a perfect Bose gas trapped in a harmonic trap in the semi–classical approach, i.e., when the thermodynamic limit condition is satisfied. These properties represent the temperature dependence of the condensate fraction, the internal energy and the heat capacity. For non–interacting (ideal) bosons in thermodynamic equilibrium, the occupation number of is Eqn. (2.2)

$$n(\varepsilon_i) = \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1} \tag{2.25}$$

where ε_i denotes the single–particle energy of the state i for the particular trapping potential under consideration. When the gas is trapped in a 3D harmonic potential, the single particle energy levels are expressed as [13]:

$$\varepsilon(n_1,n_2,n_3)=\hbar\left\lfloor (n_1+\frac{1}{2})\omega_1+(n_2+\frac{1}{2})\omega_2+(n_3+\frac{1}{2})\omega_3 \right\rfloor \quad (2.26)$$

$$n_i = 0, 1, 2,\ldots$$

When the thermodynamic limit condition is satisfied, the total number of particles, Eqn. (2.1), is evaluated as [13]:

$$N=\int_0^\infty n(\varepsilon)g(\varepsilon)\,d\varepsilon \quad (2.27)$$

where $g(\varepsilon)$ is the density of states. For the case of a 3D harmonic trap, $g(\varepsilon)$ is given by [13]:

$$g(\varepsilon)=\frac{\varepsilon^2}{2\hbar^3\omega_1\omega_2\omega_3} \quad (2.28)$$

For a D–dimensional harmonic potential with frequencies ω_i, a generalization of Eqn. (2.28) can be expressed as [13]:

$$g(\varepsilon)=\frac{\varepsilon^{D-1}}{(D-1)!\prod_i \hbar\omega_i} \quad (2.29)$$

from which it is clear that the density of states varies as the power of the energy.

2.5.1 Condensation Temperature

The transition or condensation temperature, T_c, is defined as the highest temperature at which the macroscopic occupation of the lowest–energy state appears. When the number of particles, N, is sufficiently large, the zero–point energy in Eqn. (2.26) might be neglected and, thus, one can equate the lowest energy to zero [13]. Corrections to the transition temperature arising from the zero–point energy will be discussed in § (2.6). The number of particles in the excited states is given by [13]:

$$N_{ex}=\int_0^\infty n(\varepsilon)g(\varepsilon)d\varepsilon \quad (2.30)$$

The condensation temperature, T_c , is determined by the condition that the total number of particles can be accommodated in the excited states (when μ=0) that is [13]:

$$N=N_{ex}=\int_0^\infty \frac{g(\varepsilon)}{e^{\beta\varepsilon}-1}d\varepsilon \quad (2.31)$$

Eqn. (2.29) can be expressed in terms of the dimension of the harmonic potential (D) as [13]:

$$g(\varepsilon) = c_D \varepsilon^{D-1} \tag{2.32}$$

In the case of an isotropic 3D harmonic potential, $c_3 = 1/(2\hbar^3 \omega_1 \omega_2 \omega_3)$ and the geometric mean of the frequencies $\omega = (\omega_x \omega_y \omega_z)^{1/3}$ replaces the ω_i frequencies. Writing Eqn. (2.31) in terms of the dimensionless variable $x = \varepsilon / k_B T_c$, this equation becomes [13]:

$$N = c_D (k_B T_c)^{\alpha} \int_0^{\infty} \frac{x^{D-1}}{e^x - 1} dx = c_D (k_B T_c)^D \; \Gamma(D)\zeta(D) \tag{2.33}$$

where $\Gamma(\nu)$ is the gamma function of order ν and

$$\int_0^{\infty} \frac{x^{D-1}}{e^x - 1} dx = \Gamma(D)\zeta(D) \tag{2.34}$$

From Eqn. (2.33), the transition temperature is [13]:

$$k_B T_c = \left(\frac{N}{c_D \Gamma(D)\zeta(D)} \right)^{1/D} \tag{2.35}$$

Eqns. (2.14) and (2.15) are recovered when c_D of the isotropic case is used in Eqn. (2.35).

2.5.2 The Condensate Fraction

Below the transition temperature (when μ = 0), the number of particles in the excited states, N_{ex}, is given by [13]:

$$N_{ex}(T) = c_D \int_0^{\infty} \frac{\varepsilon^{D-1}}{e^{\varepsilon/k_B T} - 1} d\varepsilon \tag{2.36}$$

The evaluation of Eqn. (2.36) is similar to that for evaluating Eqn. (2.30); which gives [13]:

$$N_{ex} = c_D (k_B T)^D \; \Gamma(D)\zeta(D) \tag{2.37}$$

It is observed that the latter equation is independent of the total number of particles. Making use of Eqn. (2.35), Eqn. (2.37) becomes:

$$N_{ex} = N \left(\frac{T}{T_c} \right)^D \tag{2.38}$$

The number of particles in the ground state is then given by:

$$N_0(T) = N - N_{ex}(T) \tag{2.39}$$

and the condensate fraction is [13]:

$$\left(\frac{N_0}{N}\right) = 1 - \left(\frac{T}{T_c}\right)^D \tag{2.40}$$

For a 3D potential, $D = 3$ and the latter equation reads:

$$\left(\frac{N_0}{N}\right) = 1 - \left(\frac{T}{T_c}\right)^3 \tag{2.41}$$

2.5.3 The Total Internal Energy and the Heat Capacity

The energy of the macroscopically occupied state (the ground state) is taken to be zero and, therefore, only the excited states contribute to the total energy of the system. Below T_c, the chemical potential vanishes. By using the result of integral in Eqn. (2.33), the internal energy is given by [13]:

$$U = c_D \int_0^\infty \frac{\varepsilon}{e^{\varepsilon/k_B T} - 1} \varepsilon^{D-1} d\varepsilon = c_D \Gamma(D+1) \zeta(D+1) (k_B T_c)^{D+1} \tag{2.42}$$

The heat capacity $C = \partial U / \partial T$ is, therefore, given by [13]:

$$C = (D+1)\frac{U}{T} \tag{2.43}$$

Both Eqns. (2.42) and (2.43) do not depend on the total number of particles. Using the integral of Eqn. (2.34), Eqns. (2.42) and (2.43) are rewritten as [13]:

$$U = N k_B D \frac{\zeta(D+1)}{\zeta(D)} \frac{T^{D+1}}{T_c^D} \tag{2.44}$$

$$C = N k_B D(D+1) \frac{\zeta(D+1)}{\zeta(D)} \left(\frac{T}{T_c}\right)^D \tag{2.45}$$

For a 3D harmonic potential, the latter equations can be written as:

$$U = 3 N k_B \frac{\zeta(4)}{\zeta(3)} \frac{T^4}{T_c^3} \tag{2.46}$$

$$C = 12 N k_B \frac{\zeta(4)}{\zeta(3)} \left(\frac{T}{T_c}\right)^3 \tag{2.47}$$

2.6 Thermodynamic Properties for a Finite Number of Particles

In this section, the effect of a finite number of particles on T_c and on thermodynamic properties (the condensate fraction, the total energy and the heat capacity) will be reviewed for a Bose gas trapped in a 3D harmonic potential. Realization of BEC in dilute gases of alkali atoms [14–16] has disclosed that the semi–classical approach, which was commonly used prior to the BEC realization, e.g. [31,32], is an imprecise approach to treat the statistical problem of the BEC phenomena. This is because the semi–classical approach is only valid when the thermodynamic limit condition is satisfied. This is due to the fact that the number of particles and, hence, the size of the Bose gas are indeed finite in these experiments. In this respect, the pioneering works [25,36–39] pointed out that the condensation temperature for ideal Bose gases in a 3D harmonic trap witnesses a downward shift compared with that evaluated on the basis of the thermodynamic limit. Now and then, the condensation temperature when the thermodynamic limit condition is satisfied will be denoted by (T_c^0). This downward shift was shown to be proportional to $N^{-1/3}$ [25,36–39]. This is known as the first order shift of the critical temperature or the first order correction. Treatments of ideal Bose gases with finite number of particles in harmonic traps required making corrections to the condensation temperature and other thermodynamic properties (condensate fraction, internal energy, heat capacity, etc.) relative to those at the thermodynamic limit [25,36–39]. As a result, in an isotropic 3D harmonic trap when N=1000, the relative correction is found to be $(\Delta T_c / T_c^0) \approx -7.3\%$ [25,36–38].

For a gas trapped in a 3D harmonic potential, the single particle energy spectrum is given by Eqn. (2.26). Using the latter equation and Eqn. (2.25), the total number of particles reads [36]:

$$N = \sum_{n1,n2,n3} \frac{1}{\exp[\beta\hbar(n_1\omega_1 + n_2\omega_2 + n_3\omega_3) - \beta(\varepsilon_0 - \mu)] - 1} \qquad (2.48)$$

where $\varepsilon_0 = \hbar(\omega_1 + \omega_2 + \omega_3)/2$ is the zero–point energy. To evaluate Eqn. (2.48), the sum over the energy states has been replaced by integration over density of states and the density of states was parameterized as [36]:

$$g(\varepsilon) = \frac{1}{2}\frac{\varepsilon^2}{(\hbar\bar{\omega})^3} + \gamma\frac{\varepsilon}{(\hbar\bar{\omega})^2} \qquad (2.49)$$

where $\bar{\omega} = (\omega_1\omega_2\omega_3)^{1/3}$ is the geometric mean frequency and γ is a coefficient that depends on the individual oscillator frequency. For isotropic harmonic potential $\gamma = 3/2$ and for anisotropic potentials γ would be evaluated numerically [36]. After substituting the density of states, Eqn. (2.49), to convert the triple sum of Eqn. (2.48) into an integral, the total number of particles reads [36]:

$$N = \frac{z}{1-z} + B_3(\mathrm{z})\left(\frac{k_B T}{\hbar\bar{\omega}}\right)^3 + \gamma B_2(z)\left(\frac{k_B T}{\hbar\bar{\omega}}\right)^2 \qquad (2.50)$$

The works [25,37,39] also obtained the same result, Eqn. (2.50), by converting the sum of Eqn. (2.48) into an integral by using approximation techniques such as Euler–Maclaurian summation formula [39] or Mellin–Barnes integral representation [37].

Setting $z = 1$ and the ground state population $[z/(1-z)] \equiv N_0 = 0$, the Bose functions are bounded by the Riemann zeta function and the condensation temperature is obtained as [36]:

$$T_c \approx \frac{\hbar\bar{\omega}}{k_B}\left(\frac{N}{\zeta(3)}\right)^{1/3}\left(1-\frac{\gamma\zeta(2)}{3\zeta(3)^{2/3}N^{1/3}}\right) \tag{2.51}$$

For a large number of particles, $N^{-1/3}$ becomes much smaller than unity and the condensation temperature of Eqn. (2.14) is recovered. When the particles number is small, the downward shift of T_c of order $N^{-1/3}$, should be taken into account. The first order shift of the critical temperature in isotropic potential is [25, 36]:

$$\frac{T_c - T_c^0}{T_c^0} = \frac{\Delta T_c^{3D}}{T_c^0} = -\frac{B_2(1)B_3(1)^{-2/3}}{2}N^{-1/3} = -0.7275N^{-1/3} \tag{2.52}$$

Consequently, thermodynamic properties have to be corrected for finite number of particles. The condensate fraction and the total energy, given in Eqns. (2.41) and (2.46), were then corrected for finite number of particles as [36]:

$$\frac{N_0}{N} = 1-\left(\frac{T}{T_0}\right)^3 - \gamma\frac{\zeta(2)}{\zeta(3)^{2/3}N^{1/3}}\left(\frac{T}{T_0}\right)^2 \tag{2.53}$$

$$U - N\varepsilon_0 = 3k_BT\left(\frac{k_BT}{\hbar\bar{\omega}}\right)^3 B_4(z) + 2\gamma k_BT\left(\frac{k_BT}{\hbar\bar{\omega}}\right)^2 B_3(z) \tag{2.54}$$

The heat capacity expression in the semi–classical approximation, Eqn. (2.47), was also corrected by two distinct expressions as [36]:

$$\left(\frac{C}{Nk_B}\right)_{T<T_0} = 12\frac{\zeta(4)}{\zeta(3)}\left(\frac{T}{T_0}\right)^3 + 6\gamma\frac{\zeta(3)^{1/3}}{N^{1/3}}\left(\frac{T}{T_0}\right)^2 \tag{2.55}$$

and

$$\left(\frac{C}{Nk_B}\right)_{T>T_0} = 12\left(\frac{T}{T_0}\right)^3\frac{B_4(z)}{\zeta(3)} + \frac{6\gamma}{N^{1/3}}\left(\frac{T}{T_0}\right)^2\frac{B_3(z)}{\zeta(3)^{2/3}} + \left[3\left(\frac{T}{T_0}\right)^4\frac{B_3(z)}{\zeta(3)} + \frac{2\gamma}{N^{1/3}}\left(\frac{T}{T_0}\right)^3\frac{B_3(z)}{\zeta(3)^{2/3}}\right]\frac{T}{z}\frac{\partial z}{\partial T} \tag{2.56}$$

where

$$\frac{T}{z}\frac{\partial z}{\partial T} = -\frac{3T_0}{T}\frac{B_3(z) + (2\gamma T_0 / 3T)\left[\zeta(3)/N\right]^{1/3} B_2(z)}{B_2(z) + (\gamma T_0 / T)\left[\zeta(3)/N\right]^{1/3} B_1(z)} \tag{2.57}$$

Eqn. (2.55) expresses the heat capacity per particle below the condensation temperature ($T<T_0$) while Eqn. (2.56) expresses the heat capacity per particle above the condensation temperature ($T>T_0$). For a very large number of trapped particles, Eqn. (2.54) recovers the expression for the heat capacity in the thermodynamic limit given in Eqn. (2.46), while the expression of Eqn. (2.55) becomes [36]:

$$\left(\frac{C^{(\infty)}}{Nk_B}\right)_{T>T_0} = 3\left(4\frac{B_4(z)}{B_3(z)} - 3\frac{B_3(z)}{B_2(z)}\right) \tag{2.58}$$

The superscript " ∞ " denotes the case of very large N . Remarkably, for very large N, the heat capacity becomes discontinuous at T_0 [36]. The magnitude of the jump is quite significant [36]:

$$\left(\frac{\Delta C^{(\infty)}}{Nk_B}\right)_{T>T_0} = 9\left(\frac{\zeta(3)}{\zeta(2)}\right) \approx 6.577 \tag{2.59}$$

Chapter Three

Condensation Temperature in Harmonic Traps of an Ideal Bose Gas with Finite Number of Particles

This chapter brings out the utility of using *MATHEMATICA®* as a tool for statistical mechanics problems, especially those concerning the BEC phenomenon. This aim is exhibited by using the symbolic computation, *MATHEMATICA®* provides. This utility is represented by two main points. The first point is the correction of the condensation temperature (T_c) relative to the corresponding analytical expressions for a BEC in 1D, 2D and 3D harmonic trap. This aspect is dealt with in § (3.4.1), § (3.4.2) and § (3.4.3), while § (3.4) illustrates how to evaluate by symbolic computation the condensation temperature in these traps. The second point is the evaluation of the aforementioned BEC thermodynamic properties, which is given in § (3.5). The stimulus that led to bring out the utility of using symbolic computation in the two previous points was the investigation of the various correction orders contributions to T_c in a 3D harmonic potential. This investigation is discussed in § (3.3). The latter was motivated by the discrepancy concerning the accurate T_c determination in a 3D harmonic potential for the case of finite number of atoms (particles). This discrepancy is discussed in § (3.2).

3.1 Symbolic Computation

MATHEMATICA® as a general computing environment was first released in 1988 [50]. It was originally conceived by Stephen Wolfram and developed by a team of specialists that he assembled and led. Besides its capability to perform numeric computations, as many conventional programming languages do, one of *MATHEMATICA*'s advantages over the conventional computational approaches is its capability of symbolic computation (algebraic computation) [50]. The term symbolic computation relates to the use of computers to manipulate mathematical equations and expressions in a symbolic form, as opposed to manipulating the approximations of specific numerical quantities represented by those symbols. Thus, symbolic computation includes all of numerical calculation plus expressions; i.e., using variables and having variables in the outputs. Also, there are many symbolic equation solvers, for nearly all types of equations, linear, nonlinear, and differential equations included [50].

The first software program that facilitates symbolic mathematics was a special system known as computer algebra system [50]. The core functionality of this system is the manipulation of mathematical expressions in symbolic form. The expressions manipulated by the computer algebra system typically include polynomials in multiple variables, standard functions of expressions (trigonometric, exponential, etc.), various special functions Γ (gamma function), ζ (Riemann zeta function), error function, Bessel functions, etc., arbitrary functions of expressions, derivatives, integrals, simplifications, sums, products of expressions, and expressions of matrices [50]. In short, everything in *MATHEMATICA®* is represented symbolically.

The work achieved in this thesis is based on the feature of symbolic computation *MATHEMATICA®* provides. The evaluation of the condensation temperature and some thermodynamic properties for harmonically trapped bosons in this thesis are basically performed in symbolic form. The main symbolic manipulations involved in this work are the standard exponential function, the special functions Γ and ζ, sums, and the sums and products of these expressions. There has been a preliminary attempt in this direction [51]; and the adoption of symbolic computation in this chapter reveals an important observation that is clarified in more details.

3.2 Condensation Temperature in a 3D Harmonic Potential

The effect of finite particle number on the BEC phenomenon, discussed in § (2.6), showed that the first order relative correction to the condensation temperature in a 3D isotropic harmonic potential [25,36–38] is $(\Delta T_c / T_c^0) \approx -7.3\%$. In this regard, the condensate thermodynamic properties were corrected with respect to the shift in the condensation temperature, as mentioned in § (2.6). Subsequently, it has been claimed that the first order correction, whose relative value is approximately -7.3%, is inaccurate for atoms number $N < 10^5$ [52]. This result was justified by observing that, beyond the first order correction, the next order is not a second order correction, which would be proportional to $N^{-2/3}$, but rather a correction proportional to $N^{-1/2}$ [52]. Accordingly, the next order correction should be included for an accurate estimation of finite size corrections to T_c when $N \leq 10^5$ [52]. Due to this inclusion of the next order correction for $N = 1000$, the relative correction was found to be $(\Delta T_c / T_c^0) \approx -3.8\ \%$ [52].

Motivated by this discrepancy, an attempt is made, in this chapter, to study mainly three aspects. The first aspect is to examine the correction orders contributions, derived in Ref. [52], to T_c for an ideal Bose gas in a 3D harmonic potential. This aspect is discussed in § (3.2). The second aspect to be examined in § (3.3) is the validity of the first order correction; the correction on which the works of Refs. [25,36–38] relied on to determine the condensation temperature and other thermodynamic properties in the aforementioned Bose systems.

Despite the fact that results of the works [25,36–38] are indeed precise, the treatments used to determine the condensation temperature are still not the most precise statistical treatments for a finite number of particles. This is because the core idea in these works is, in general, to convert the sum into an integral as in the case of the thermodynamic limit treatment but by using a more accurate density of states and proper integration limits. Hence, the third aspect is the one which relied on evaluating the condensation temperature for harmonically trapped ideal bosons in 3D, 2D and 1D harmonic potentials by symbolic computation. This aspect is to be discussed in § (3.4). In the present work, the problem of the precise statistical evaluation is approached by using the key idea of computing the sum over all the energy states with no truncation or approximation. This goal is achieved by using the symbolic computation capability of *MATHEMATICA®* which enables computing a sum over its terms [50].

3.3 Correction Orders Contribution in a 3D Harmonic Potential

An expression for the relative correction for a finite number of ideal Bose particles in a power-law potential was derived in Ref. [52]. For the case of the harmonic potential, the relative correction was expanded in powers of x_0, where $x_0 = \varepsilon_0 / k_B T_c$, and the truncation was limited to second order in x_0 for $x_0 \le 0.1$ to give [52]:

$$\frac{\Delta T_c}{T_c^0} = -\frac{\zeta(2)}{3\zeta(3)} x_0 + \frac{\zeta(2/3)}{3\zeta(3)\Gamma(5/2)} x_0^{3/2} - \alpha(x_0) x_0^2 \tag{3.1}$$

$$\alpha(x_0) = 2\left(\frac{\zeta(2)}{3\zeta(3)}\right)^2 - \frac{1}{4\zeta(3)} + \frac{\ln(x_0)}{6\zeta(3)} \tag{3.2}$$

In terms of the atoms number, Eqn. (3.2) was expressed as [52]:

$$\frac{\Delta T_c}{T_c^0} \approx -\left(\frac{\zeta(2)\omega_m}{2\zeta(3)^{2/3}\bar{\omega}}\right)N^{-1/3} + \zeta(2/3)\left(\frac{2\omega_m{}^3}{3\pi\zeta(3)\bar{\omega}^3}\right)^{1/2} N^{-1/2}$$

$$\approx -0.7275\left(\frac{\omega_m}{\bar{\omega}}\right)N^{-1/3} + 1.098\left(\frac{\omega_m}{\bar{\omega}}\right)^{3/2} N^{-1/2} \tag{3.3}$$

$$\omega_m = \frac{1}{3}\sum_{i=1} n_i\omega_i, \qquad \bar{\omega} = \left(\prod_{i=1}\omega_i^{n_i}\right)^{1/3} \tag{3.4}$$

where ω_m and $\bar{\omega}$ are respectively the arithmetic and geometric mean frequencies of the harmonic oscillator, and $\omega_m = \bar{\omega}$ for an isotropic harmonic potential. It is obvious that the expression for x_0 in terms of atoms number is:

$$x_0 = \frac{3}{2}[\zeta(3)]^{1/3} N^{1/3} \tag{3.5}$$

It is also clear that the coefficient of third term in Eqn. (3.1), i.e., Eqn. (3.2), has been truncated in Eqn. (3.3). Eqn. (3.1) shows that the next correction term is not the second order correction, and from the present point of view, this result is beyond dispute. Based on this result, the first order correction has been considered as an inaccurate correction and it was suggested that the next term correction should be included [52]. Therefore, the correction up to $N^{-1/2}$, Eqn. (3.3), was considered as the accurate relative correction for non–interacting bosons harmonically confined in 3D [51]. It is argued here that the third term of Eqn. (3.2) should not be ignored because the expansion of the shift in powers of x_0, Eqn. (3.1), already involves the third term. In this view, the third term, which involves the second order, is also significant and it should be included for more accurate estimation of T_c. By substituting Eqn. (3.5) in Eqns. (3.1) and (3.2), a more accurate relative correction for T_c as a function of the atoms number N in a 3D harmonic potential can be expressed as:

$$\frac{\Delta T_c}{T_c^0} = -0.7275\left(\frac{\omega_m}{\bar{\omega}}\right)N^{-1/3} + 1.098\left(\frac{\omega_m}{\bar{\omega}}\right)^{3/2} N^{-1/2} - [0.529 + 0.35\ln(1.6N^{-1/3})]\left(\frac{\omega_m}{\bar{\omega}}\right)^2 N^{-2/3} \tag{3.6}$$

Fig. (3.1) shows that when $N = 1000$, the corrections up to the orders $N^{-1/2}$, $N^{-2/3}$ and $N^{-1/3}$ respectively give $(\Delta T_c / T_c^0) \approx -3.8\%$, $(\Delta T_c / T_c^0) \approx -5\%$ and $(\Delta T_c / T_c^0) \approx -7.3\%$. This illustrates that the contribution of the second order correction, the third term of Eqn. (3.5), which is proportional to $N^{-2/3}$, is also significant and it should not be neglected for an accurate determination of the condensation temperature when $N < 10^5$.

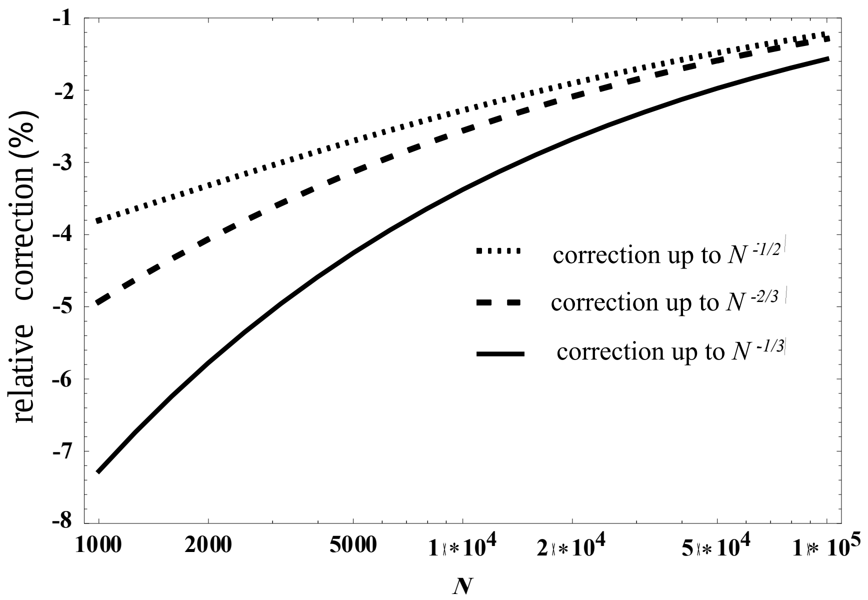


Fig. (3.1): The relative correction $(\Delta T_c / T_c^0)$ in an isotropic 3D harmonic potential as a function of atoms number $10^3 \leq N \leq 10^5$.

Fig. (3.2) shows that extrapolating the behavior of $(\Delta T_c / T_c^0)$ for atoms number $N < 1000$, results in an expected reduction when N decreases for the first order correction and for the correction of order up to $N^{-2/3}$. The insert, which exhibits the relative correction up to $N^{-1/2}$, shows an anomalous physical behavior because it becomes positive when $N \leq 10$; which means that the condensation temperature is greater than that for the thermodynamic limit; i.e., $T_c > T_c^0$. This behavior indicates that the correction up to $N^{-1/2}$ is not only an imprecise correction but it is also, physically, an improper correction for finite atoms number.

It is important to stress here that despite the fact that correction up to $N^{-2/3}$ is more accurate than that up to $N^{-1/2}$, it (the correction up to $N^{-2/3}$) is still not the precise correction for atom numbers $N < 10^5$. Thanks to the relative correction shift derivation of the work [52], it is clear that the first order and the second order corrections are respectively proportional to $-x_0$ and $-x_0^2$, and the next order correction is proportional to $+x_0^{3/2}$. Thus, it is concluded that the successive change in the sign of the correction orders, given by [52], would lead the relative correction $(\Delta T_c / T_c^0)$ to approach the first order relative correction as the order of the corrections increases. This is reflected in the reduction of relative correction from $(\Delta T_c / T_c^0) \approx -3.8\%$ given by Eqn. (3.7) to $(\Delta T_c / T_c^0) \approx -5\%$ given by Eqn. (3.9). In our opinion the first order correction $-[0.7275(\omega_m / \bar{\omega})]N^{-1/3}$ is still the most precise correction. The question that needs to be answered is: for which number of atoms the first order correction remains valid.

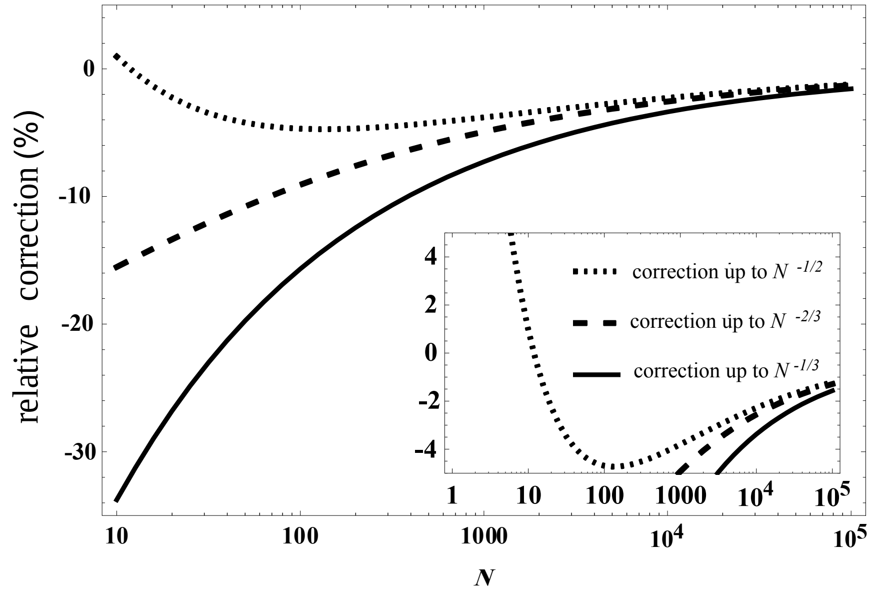


Fig. (3.2): The relative correction $(\Delta T_c / T_c^0)$ in an isotropic 3D harmonic potential as a function of atoms number $10 \le N \le 10^5$. The insert exhibits that the correction up to $N^{-1/2}$ gives $T_c > T_c^0$ when $N < 10$.

3.4 Evaluation of the Precise Condensation Temperature in Harmonic Potentials

On the basis of the grand canonical ensemble, the total atoms number for an ideal Bose gas confined in an isotropic harmonic potential is expressed as [40]:

$$N = \sum_{n=0}^{\infty} \frac{g_n}{\exp[\beta(\varepsilon_n - \mu)] - 1} = N_0 + \sum_{n=1}^{\infty} \frac{g_n}{z^{-1} e^{n/T} - 1}, \qquad N_0 = \frac{z}{1-z} \tag{3.7}$$

where the g_n's are the harmonic oscillator potential degeneracy factors defined as:

$$g_n = \begin{cases} 1, & \text{for a 1D trap} \\ (n+1), & \text{for a 2D trap} \\ (n+2)(n+1)/2, & \text{for a 3D trap} \end{cases}$$

To evaluate the critical temperature, T_c, in $(\hbar\omega / k_B)$ units for the aforementioned systems, Eqn. (3.7) is symbolically solved for T after setting $N_0 = 0$ and $z = 1$ for a given N. In computations, the upper limit of the sum in Eqn. (3.7) is indeed the integer at which the temperature T converges. The converged value of T is the condensation temperature T_c. In the following subsections, the relative corrections for the condensation temperature in harmonic traps will be evaluated.

3.4.1 The Relative Correction in a 3D Confinement

The T_0^{3D} obtained from Eqn. (2.14) and T_c evaluated by solving Eqn. (3.7) are used in in the form $[(T_c - T_0^{3D}) / T_0^{3D}]$ to evaluate the relative correction denoted by "present work". For this case, the used degeneracy is $g_n = (n+2)(n+1)/2$. The obtained relative correction is then compared with the relative correction up to $N^{-2/3}$ and with the first order relative correction namely $(\Delta T_c / T_0^{3D}) = -0.7275 N^{-1/3}$ [25,41].

Fig.(3.3) shows that the first order relative correction, $(\Delta T_c / T_0^{3D}) = -7.275 N^{-1/3}$, is in excellent agreement with the obtained relative correction (present work) for $10^3 \le N \le 10^5$. It also shows that when $N=10^3$, the numerical value of the obtained relative correction is approximately -7.347%. Hence, the first order correction makes a percentage difference of merely +0.98% in comparison with the obtained relative correction (present work).

Fig.(3.4) shows that extrapolating the behavior of $(\Delta T_c / T_0^{3D})$ for $N < 10^3$, the first order relative correction exhibits an excellent agreement with the obtained relative (present wok) correction even for atoms number $N > 100$. It is also obvious that the obtained relative correction becomes higher than that of the first order when $N < 100$. This means that the symbolically evaluated values of T_c are higher than those predicted by the first order correction. Also, this figure represents a correction to the first order correction for $N < 100$. Figures (3.3) and (3.4) also show that the correction up to $N^{-2/3}$ is far from the results for the obtained relative correction. Accordingly, this asserts the prediction in § (3.2) which indicates that the relative correction up to $N^{-2/3}$ is an imprecise relative correction for $N < 10^5$.

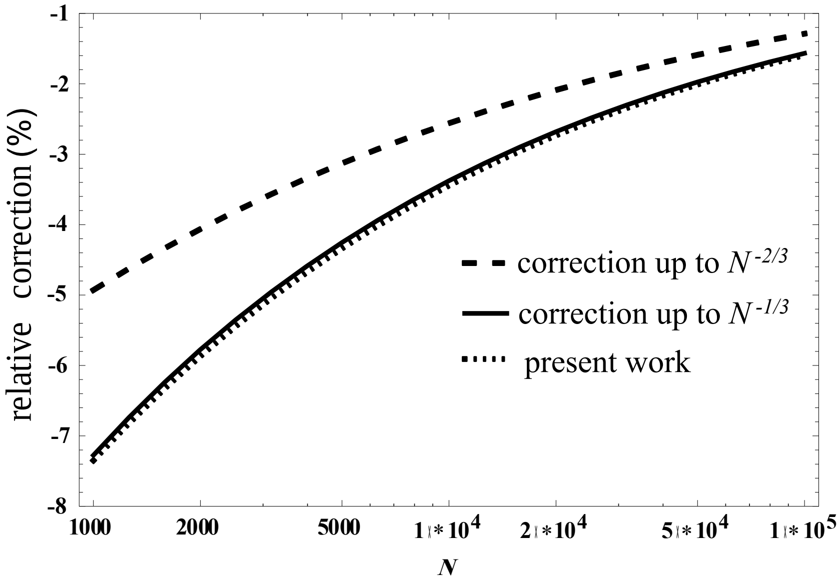


Fig.(3.3): The relative correction $(\Delta T_c / T_0^{3D})$ in an isotropic 3D harmonic potential as a function of atoms number $10^3 \leq N \leq 10^5$.

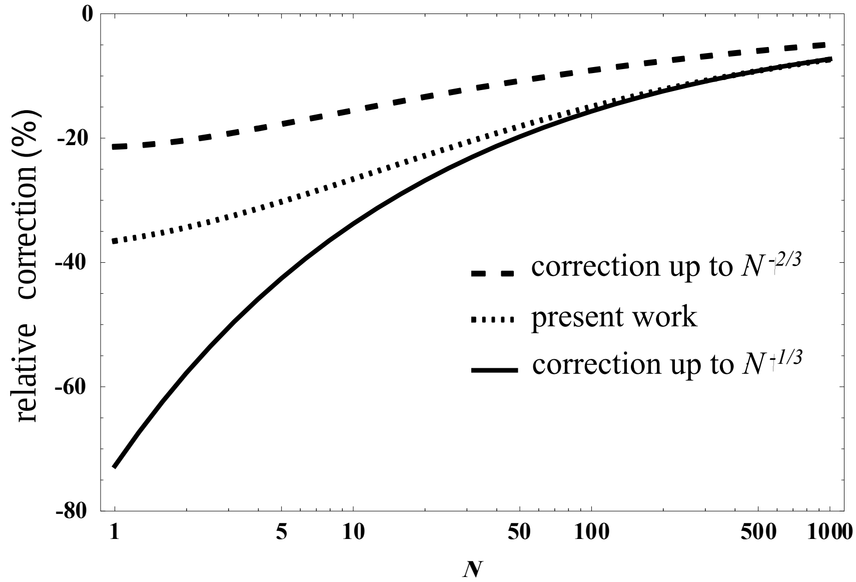


Fig. (3.4): The relative correction $(\Delta T_c / T_0^{3D})$ an isotropic 3D harmonic potential as a function of atoms number $1 \leq N \leq 1000$.

Figs. (3.3) and (3.4) emphasize that the first order relative correction $(\Delta T_c / T_0^{3D}) = -0.7275\,(\omega_m / \bar{\omega})\,N^{-1/3}$ is the most accurate correction for T_c; as it is clear from the excellent agreement with the obtained numerical relative correction (present work) for atoms number N >100. In order to precisely determine the condensation temperature, T_c, for ideal Bose gases harmonically confined in 2D and 1D, the use of symbolic evaluation for T_c in these bosonic systems is extended in the following subsections.

3.4.2 The Relative Correction in a 2D Confinement

The condensation temperature for a bosonic gas with infinite number of atoms in a 2D harmonic trap T_0^{2D}, given by Eqn. (2.15), and the T_c obtained symbolically from Eqn. (3.7), are plotted in Fig. (3.5). The used degeneracy for this case is $g_n = (n+1)$. Fig. (3.5) and its insert both show that evaluated condensation temperature denoted by "present work" is always less than that evaluated on the basis of the thermodynamic limit given by Eqn. (2.15).

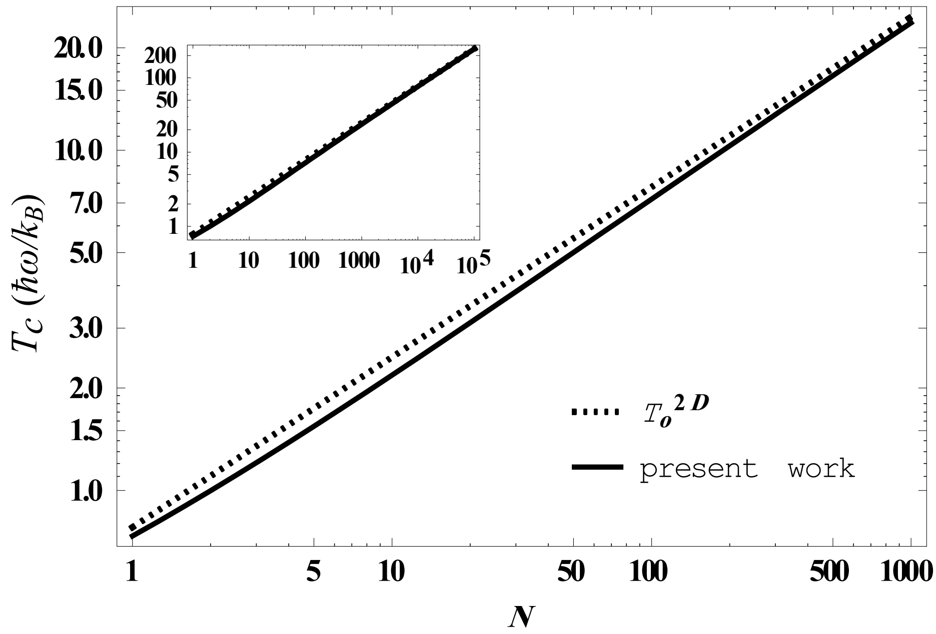


Fig. (3.5):The condensation temperature in an isotropic 2D harmonic potential as a function of atoms number $1 < N \leq 1000$. The insert extrapolates T_c for $1 < N < 10^5$.

The evaluated relative correction for this case is obtained by a similar manner to that for the 3D harmonic potential, i.e. by using the two aforementioned condensation temperatures in the form $[(T_c - T_0^{2D})/T_0^{2D}]$. Fig. (3.6) exhibits the obtained relative correction as a function of atoms number and it also gives the precise T_c as a correction to Eqn. (2.15).

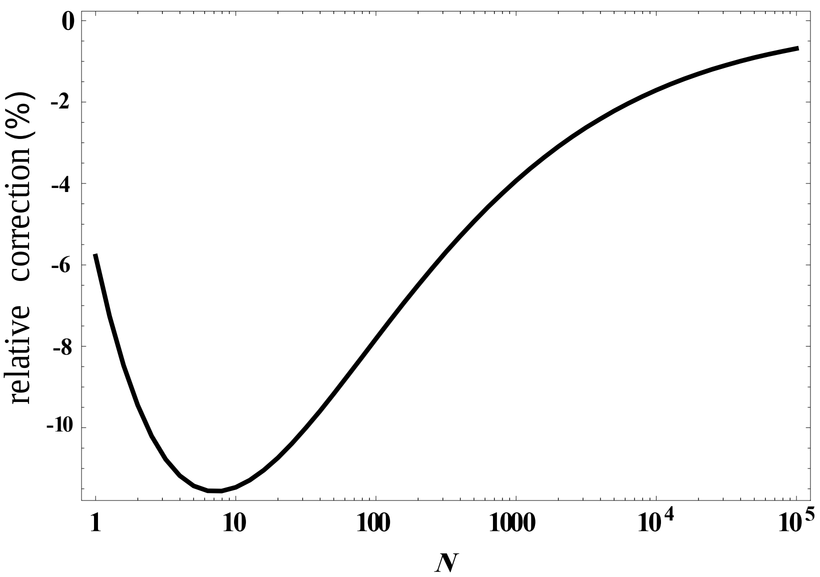


Fig. (3.6): The relative correction $(\Delta T_c / T_0^{2D})$ in an isotropic 2D harmonic potential as a function of atoms number $1 < N < 10^5$.

3.4.3 The Correction Factor in a 1D Confinement

For a Bose gas confined in 1D harmonic potential, the correction factor defined by (T_c / T_c^{1D}) is adopted to correct the T_c which Eqn. (2.23) gives. This is because in 1D harmonic potential, theoretically there is no BEC when the thermodynamic limit condition is satisfied [25]. The evaluation in the thermodynamic limit of the condensation temperature needs special manipulation because the Riemann zeta function is not defined when $D = 1$. For this case, the works [25,39] proposed two similar expressions given by Eqns. (2.23) and (2.24). The T_c^{1D} given by Eqns. (2.23) and (2.24) and the results obtained for T_c by solving Eqn. (3.7), for T where the degeneracy factors $g_n = 1$, are plotted as functions of atoms number in Fig. (3.7). The symbolically evaluated condensation temperature is denoted by "present work T_c". This figure clarifies that the symbolically evaluated T_c is remarkably higher than those predicted by Refs. [39] and [25]. The values of T_c in $(\hbar\omega/k_B)$ units which are evaluated by solving Eqn. (3.7) symbolically, and by Eqns. (2.23) and (2.24) are approximately 1300, 1100 and 1000 respectively for N=10000. For this case, the correction factor (T_c / T_c^{1D}) for T_c as a function of N is plotted in Fig.(3.8), where T_c is obtained from Eqn. (3.7), and T_c^{1D} is the condensation temperature given by Eqn. (2.23).

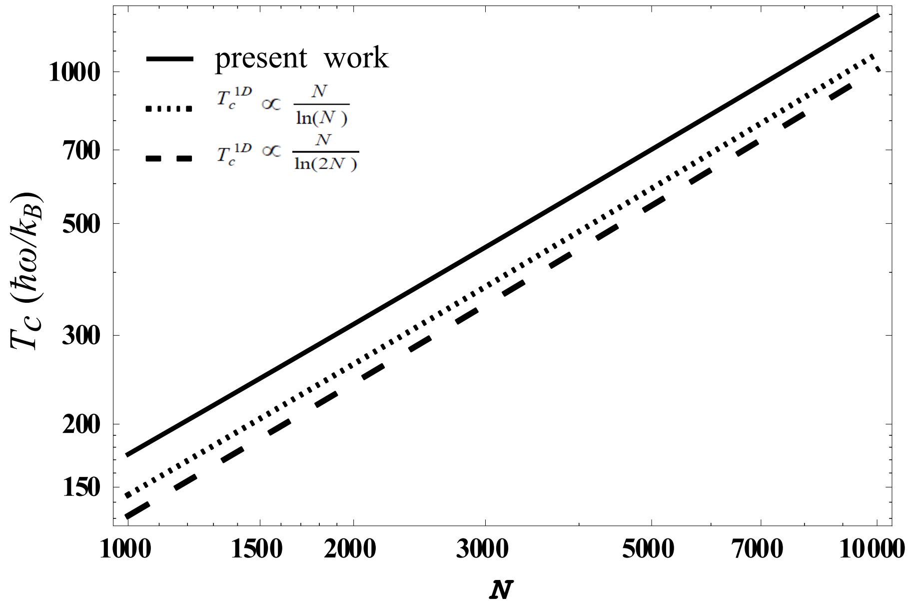


Fig. (3.7): The condensation temperature in a 1D harmonic trap as a function of atoms number $10^3 < N \leq 10^4$.

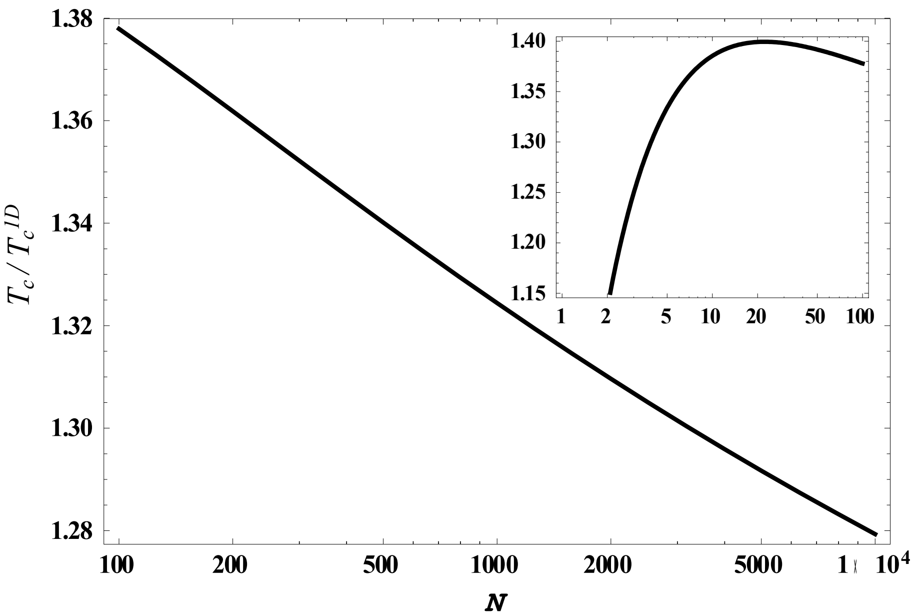


Fig. (3.8): The correction factor (T_c / T_c^{1D}) in a 1D harmonic potential as a function of atoms number $100 < N < 10^4$. The insert is (T_c / T_c^{1D}) for N <100.

3.5 Thermodynamic Properties Evaluation for Harmonically Trapped Bosons in Integer Dimensions

The evaluation of the temperature dependence of these properties is in the framework of the procedure given in Ref. [25]. The phenomenon of BEC for non–interacting particles is fully described by the occupation number and the total number of particles [25], namely; Eqns. (2.1) and (2.2). For the case of harmonically trapped Bose gas, the total number of particles is given by Eqn. (3.7). The nontrivial aspect in the evaluation of thermodynamic properties is the determination of the chemical potential μ [25]. Once μ is known, all the thermodynamic properties like condensate fraction, total energy and heat capacity follow directly from sums over the energy levels involving the occupation numbers [25]. In this framework, the condensation temperature and thermodynamic properties of a Bose–Einstein condensate in 3D, 2D and 1D harmonic potentials will be evaluated by means of symbolic computation. The condensation temperature and temperature dependence of these properties in the aforementioned harmonic traps for a given total atoms number N can be evaluated by using the harmonic oscillator degeneracy factors in Eqn. (3.7).

The evaluation of the thermodynamic properties requires, in the first step, the determination of the temperature dependence of the chemical potential, $\mu(T)$. This temperature dependence is evaluated from the temperature dependence of the fugacity $z(T) = \exp(\beta\mu)$. The latter is evaluated by solving Eqn. (3.7) for z taking into account the condensation temperature T_c, which is evaluated by symbolic computation. Hence, the temperature dependence of the condensate fraction (N_0/N) and the internal energy U can be evaluated from:

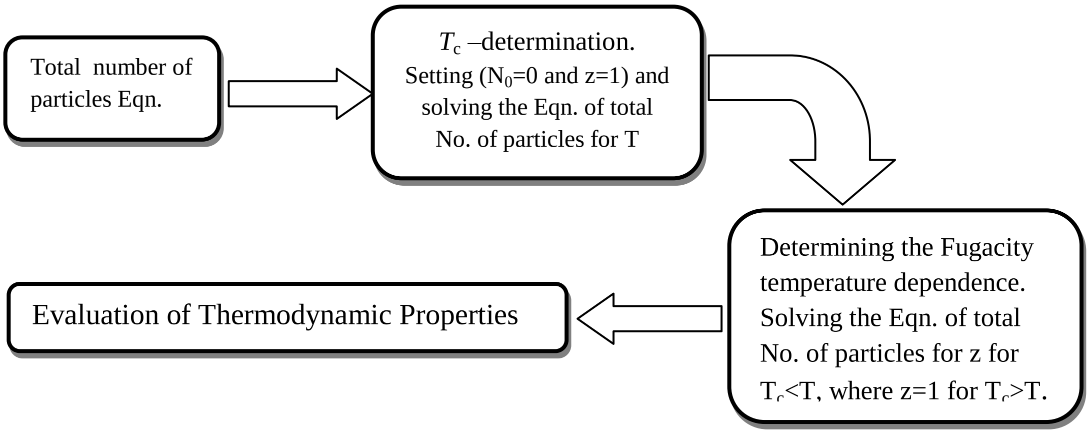
$$\frac{N_0}{N}(T,z) = 1 - \frac{1}{N}\sum_{n=1}^{\infty}\frac{g_n}{z^{-1}e^{n/T}-1} \tag{3.8}$$

$$U(T,z) = \hbar\omega\sum_{n=1}^{\infty} n\frac{g_n}{z^{-1}e^{n/T}-1} \tag{3.9}$$

The temperature dependence of the heat capacity, $C(T)$, is determined, by using the general definition of derivative, as:

$$C_V(T) = \frac{\partial U(T)}{\partial T} = \frac{U(T+\Delta T)-U(T)}{\Delta T} \tag{3.10}$$

The following chart may summarize the steps to evaluate the BEC thermodynamic properties.



The validity of symbolic computation adopted in this thesis for the evaluation of the thermodynamic properties for ideal bosons is verified in Figs. (3.9) and (3.10). It is necessary to indicate that the numerical results obtained in the latter figures were evaluated using the degeneracy factors $g_n = (n+2)(n+1)/2$. These figures illustrate that the symbolically evaluated results are in excellent agreement with the corresponding analytical expressions. Fig. (3.9) exhibits the condensate fraction (N_0/N) evaluated by symbolic computation of Eqn. (3.8) for N=1000, denoted by "present work", which agrees excellently with the analytical result, Eqn. (2.53), for the treatment of finite number of particles. This figure also exhibits the downward shift in T_c compared with previous results for the condensate fraction in the thermodynamic limit, Eqn. (2.41), when N= 1000. Fig. (3.10) exhibits the temperature dependence of the condensate fraction (N_0/N) for N=100. The symbolically evaluated condensate fraction denoted by "present work" is compared with the analytical results obtained from the treatment of finite number of particles, Eqn. (2.53), and with that result obtained from the thermodynamic limit given by Eqn. (2.41). This figure asserts the downward shift in T_c when compared with the result for the thermodynamic limit. It also shows that the symbolically obtained result agrees excellently with the analytical result obtained from the treatment of finite number of particles, Eqn. (2.53), apart from a small elevation in T_c. This elevation is because T_c is appreciably higher than that which Eqn. (2.53) yields for $N \leq 100$; as discussed in § (3.3.1).

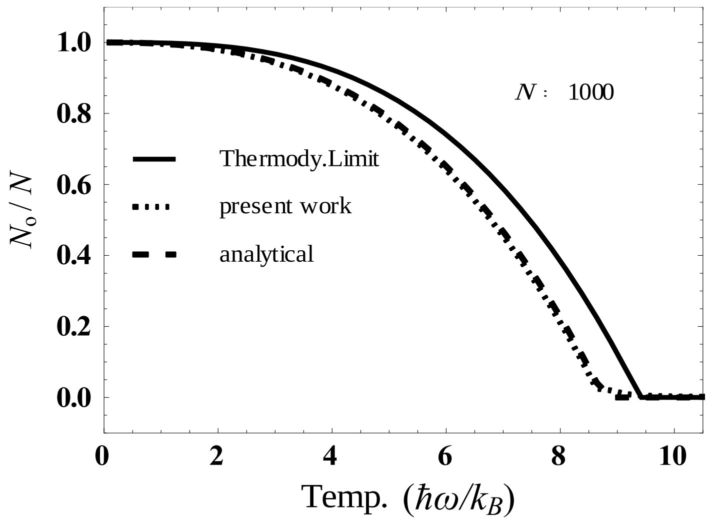


Fig. (3.9): The condensate fraction in a 3D isotropic harmonic potential for N =1000. The solid, dashed and dotted lines are respectively representing the thermodynamic limit, analytical treatment for finite number of particles and symbolic computations (present work).

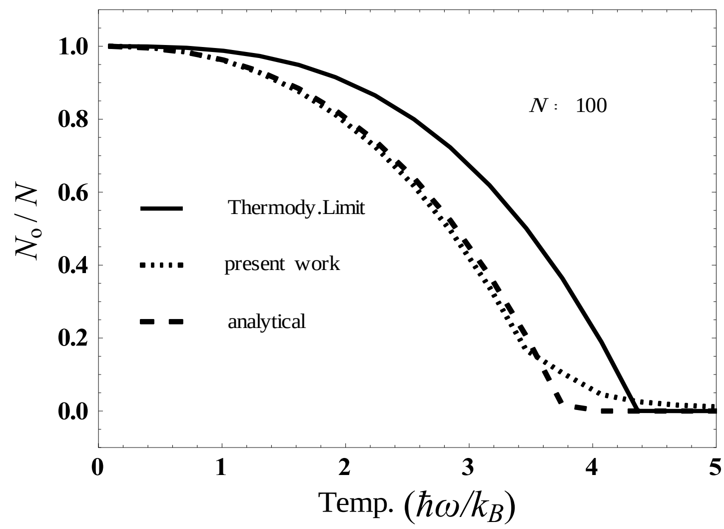


Fig. (3.10): The condensate fraction in a 3D isotropic harmonic potential for N =100. The solid, dashed and dotted lines are respectively representing the thermodynamic limit, analytical treatment for finite number of particles and symbolic computations (present work).

The following subsections exhibit comparisons for the condensation temperature and also for the thermodynamic properties (fugacity z, the total internal energy U, the c heat capacity C_V and the condensate fraction (N_o/N) for ideal bosons trapped harmonically in spatial dimensions in 1D, 2D and 3D. These results are evaluated symbolically and obtained by making use of the degeneracy factors in Eqns.(3.7), (3.8), (3.9) and (3.10) (for the aforementioned bosonic systems) for a fixed total number of particles N=100.

3.5.1 Condensation Temperature

The condensation temperatures for atoms number $N < 500$ in 1D, 2D and 3D harmonic traps are plotted in Fig. (3.11). This figure shows that the condensation temperature for a given atoms number increases remarkably with the reduction in spatial dimensions for the aforementioned traps. The increase in T_c with the decrease in in the dimension of trapping potential is ascribed to freezing (reducing) the degrees of freedom and this result agrees with that of Ref. [25]. This figure also exhibits results similar to the theoretical predictions, indicated in § (2.4), whereas the condensation temperature varies as $T_c^{1D} \propto (N/\ln 2N)$, $T_c^{2D} \propto N^{1/2}$ and $T_c^{3D} \propto N^{1/3}$.

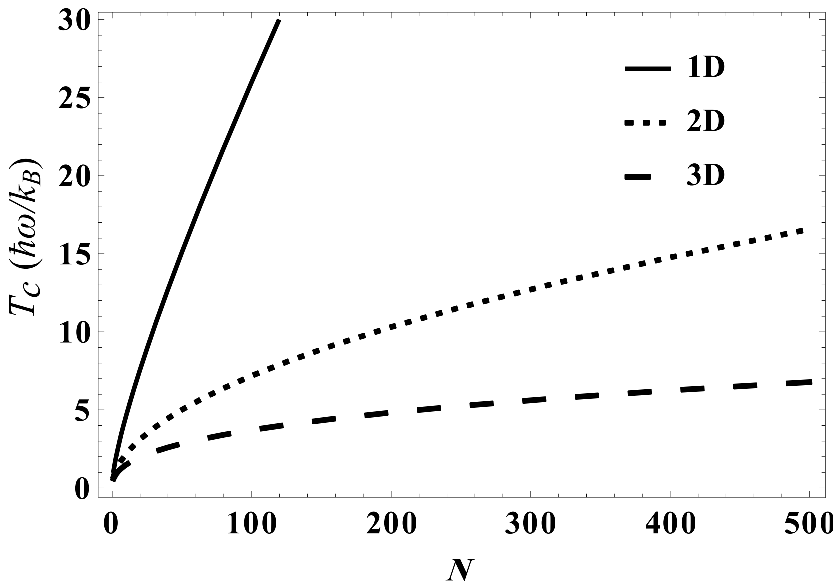


Fig. (3.11): The condensation temperature in 3D, 2D and 1D harmonic potentials as a function of atoms number N <500.

3.5.2 Temperature Dependence of the Fugacity

Since thermodynamic properties are temperature dependent and fugacity dependent, then determining the temperature dependence of fugacity, $z(T)$, comes in the first place after Tc is determined. If the temperature dependence of fugacity is determined, the temperature dependence of all thermodynamic quantities is attainable [25]. Figs. (3.12) and (3.13) exhibit temperature dependence of fugacity for ideal bosons trapped in 1D, 2D, and 3D harmonic traps for a given total number of particles N =100. These results are evaluated by making use of the corresponding degeneracy factors and solving Eqn. (3.7) for $z(T)$. Figs. (3.12) and (3.13) compare the effect of dimensionality on fugacity for harmonically trapped bosons. The advantage of these figures is not only because of their importance to exhibit the implicit dependence of thermodynamic properties on the fugacity but also in the determination of the chemical potential for these systems directly from the relationship $z = e^{\beta\mu}$, i.e. $\mu(T) = k_B T\,[\ln z\,(T)]$. Since the chemical potential value is constrained by $\mu \le \varepsilon_0$, where ε_0 is conventionally assumed to be equal to zero, hence the fugacity range is $0 \le z \le 1$. Thus, the temperature when the fugacity, z, becomes a unity indicates T_c.

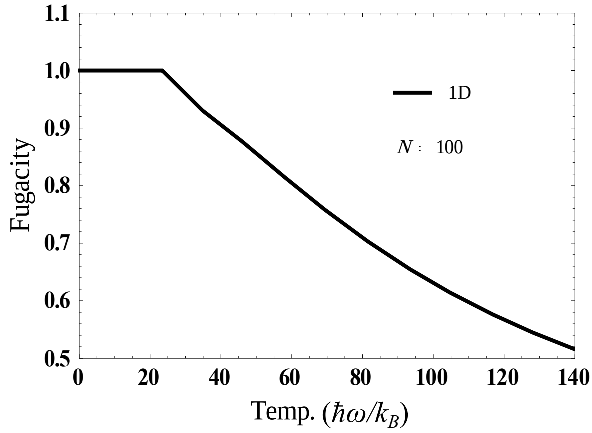


Fig. (3.12): Temperature dependence of fugacity for a Bose-Einstein condensate trapped in 1D harmonic potential.

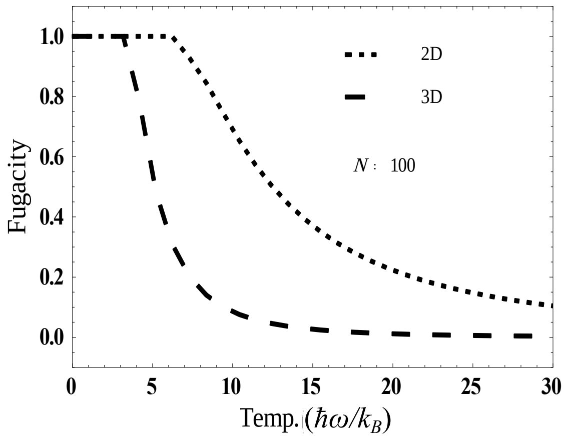


Fig. (3.13): Temperature dependence of fugacity for a Bose-Einstein condensate trapped in 3D and 2D harmonic potentials.

3.5.3 Temperature Dependence of the Condensate Fraction

Fig. (3.14) exhibits the influence of the degrees of freedom of the harmonic traps on the condensate fraction; whereas the harmonic trap with higher integer dimension possesses a higher condensate fraction at any temperature below the corresponding T_c.

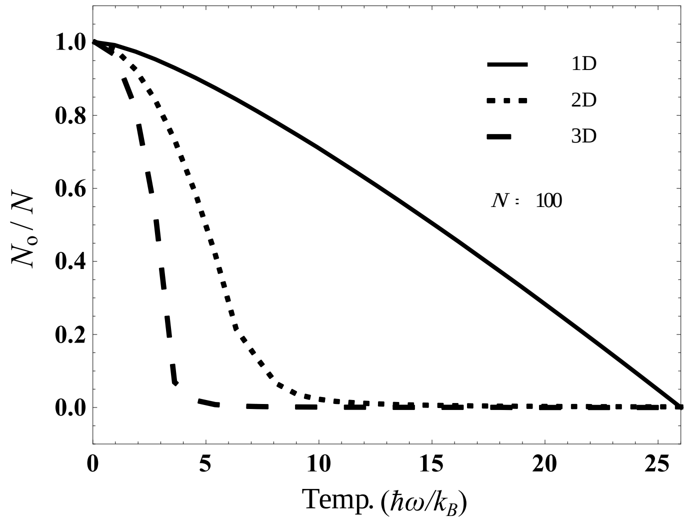


Fig. (3.14): Temperature dependence of the condensate fractions of Bose-Einstein condensates harmonically trapped in integer dimensions.

3.5.4 The Temperature Dependence of the Internal Energy and the Heat Capacity

The comparison between the figures (3.15), (3.16) and (3.17) exhibits the influence of dimensionality on total internal energy. This comparison shows that in a 3D harmonic trap, the drop in the internal energy for temperatures below the condensation temperature is deeper than the corresponding drop for a 2D harmonic trap and the drop in the latter is higher than that for a 1D harmonic trap. Consequently, the jumps in the values of the heat at the condensation temperature for the aforementioned traps, shown in Figs.(3.18) and (3.19), are remarkably influenced by dimensionality. The harmonic trap with higher dimension has the higher jump in the heat capacity. This is due to the fact that the higher dimensional trap has higher degrees of freedom.

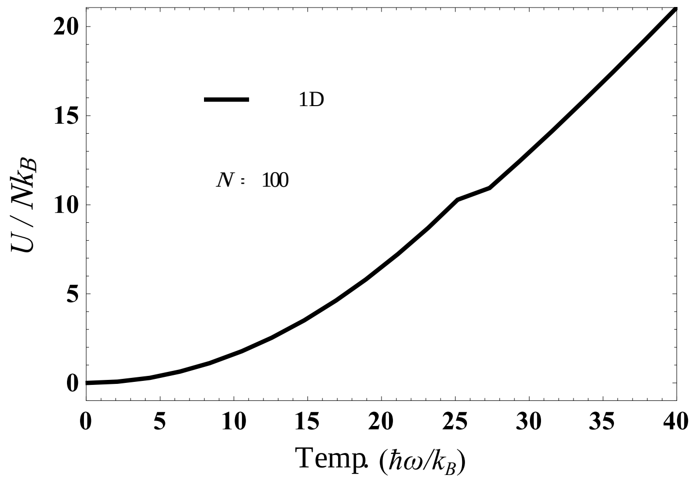


Fig. (3.15): Temperature dependence of the total internal energy of Bose-Einstein condensate trapped in a 1D harmonic potential.

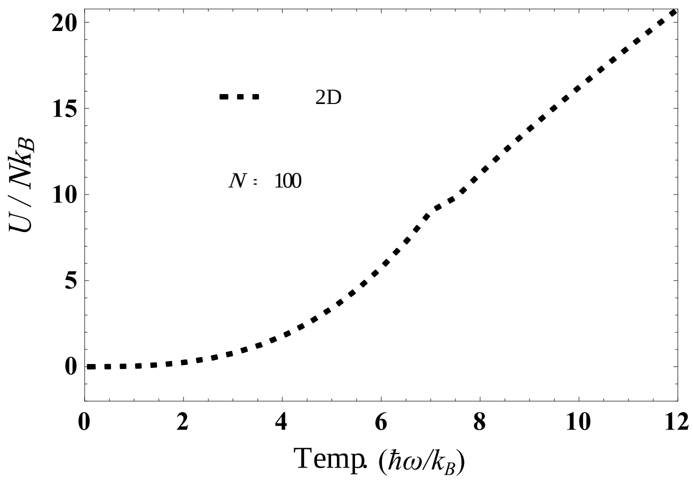


Fig. (3.16): Temperature dependence of the total internal energy of .Bose-Einstein condensate trapped in a 2D harmonic potential

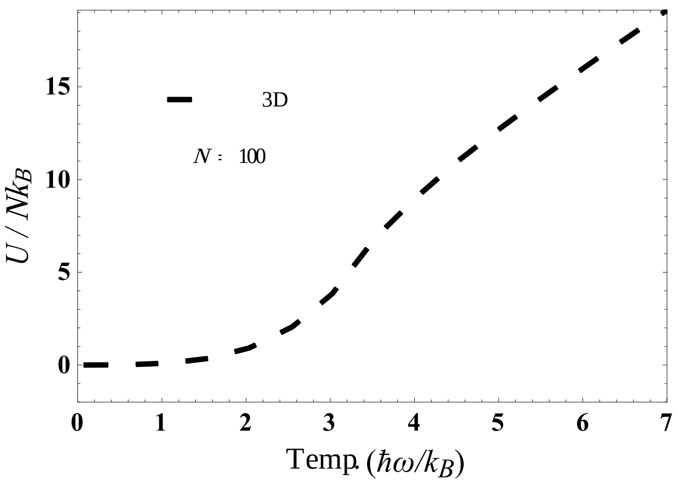


Fig. (3.17): Temperature dependence of the total internal energy of Bose-Einstein condensate trapped in a 3D harmonic potential.

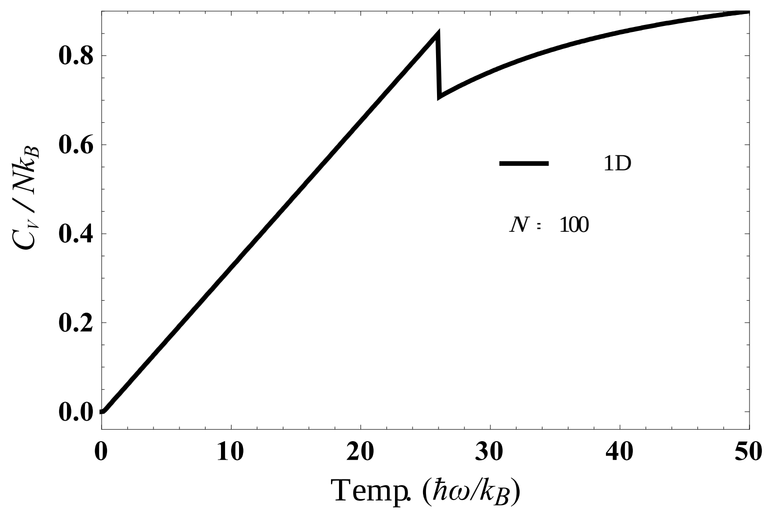


Fig. (3.18): Temperature dependence of the heat capacity of Bose-Einstein condensate trapped in a 1D harmonic potential.

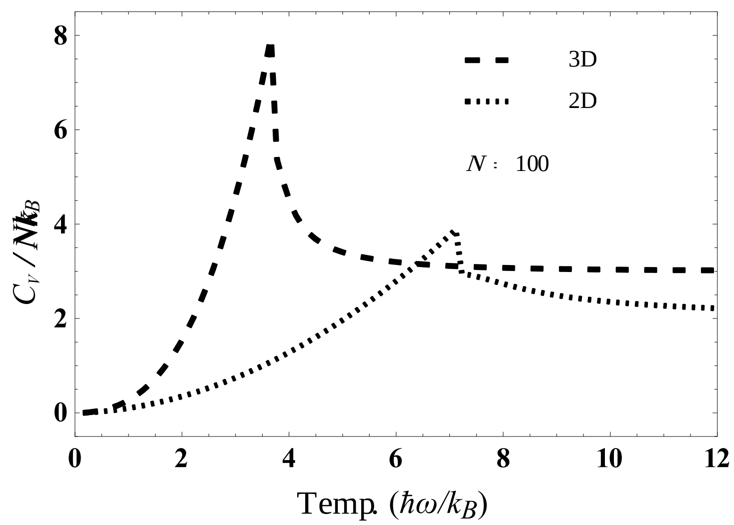


Fig. (3.19): Temperature dependence of the specific heat of Bose-Einstein condensate trapped in 2D and 3D isotropic harmonic potentials.

Chapter Four

Thermodynamic Properties for Finite Number of Bosons Harmonically Trapped in Fractal Media

This chapter is mainly devoted to introduce a theoretical model for the BEC phenomenon in fractal media and to investigate its thermodynamic behavior via evaluating the condensation temperature (T_c) and the BEC thermodynamic properties under investigation, fugacity, condensate fraction, internal energy and the heat capacity. This investigation has been carried out by symbolic computation for bosons harmonically trapped in fractal media for two different models. The first belongs to Boltzmann–Gibbs (BG) extensive statistical mechanics and the second belongs to the Tsallis (nonextensive) statistical mechanics. Therefore, the evaluated condensate properties for theses two models belong to two different thermostatistics. These two models are demonstrated in §(4.9) and §(4.12). The thermodynamic behaviors for these two thermostatistics are compared and exhibited in § (4.13) and § (4.14).

The term *fractal media* is briefly discussed in § **(4.1)**. Experimental evidences related to bosons confined in fractal media are reviewed in § (4.2). Theoretical approaches which have been used to treat these systems are discussed in § (4.3). The two types of thermostatistics used in this work are formalized upon two different types of entropies. The sort and the domain of applications for these two types of entropies are reviewed in § (4.4), § (4.5), and § (4.6). The literature related to the BEC phenomenon for the two types of thermostatistics is reviewed separately in § (4.7) and § (**4.**10). The importance of using the symbolic computation for treating bosons confined in fractal medium is discussed in § (4.8)**.**

4.1 Fractal Media

This term refers to structures (media) whose dimensions strictly exceed their topological dimensions. Topological dimension, for the present purpose, can be defined as the whole numbers 0, 1, 2 or 3 which, respectively, define the dimension of points, curves, surfaces and volumes in Euclidean geometry.

Mandelbrot's geometrical analysis of shapes, structures or patterns that are fragmented or irregular (rough), which cannot be described in terms of Euclidean geometry, led him to discover, in 1975, a novel sort of geometry [53]. This novel geometry which represents a generalization of Euclidean geometry is Fractal Geometry. It originated from Mandelbrot's work in 1967 on roughness (density of points on a certain set) in his endeavor to measure the length of a coastline [53]. Fractal geometry not only succeeded in describing these non–Euclidean structures, but it also succeeded in simulating natural phenomena in such a way that was previously impossible with Euclidean geometry [53].

In his fractal geometry, Mandelbrot introduced in 1975 the term *fractal* to refer to those objects which have the property of fractional dimensions and can be described by means of the rules of fractal geometry [53]. Besides the property of fractional dimension, the major mathematical feature of fractals is that they are non–differentiable although they are continuous [53]. Although the semblance of fractal objects is fragmented or irregular, it has common geometric characteristic properties. Mainly, the basic geometric characterization of fractals is the self–similarities on all scales of observation. This term means that the shape is made of smaller copies of itself. The copies are similar to the whole and the shape repeats but in a different size [53,54]; therefore, the *fractal dimension*, as a numerical measure, is preserved across the scales. In fractal geometry, there is no unique definition of fractal dimension [53].

The definition which is relevant to this work is based on the construction of fractal shape by the subsequent divisions of an original Euclidean shape. Thus, in general, if one starts with a regular shape of a linear size equal to L embedded in a space of Euclidean dimension D_E and then reduce its linear size by factor $(1/L)$ in each spatial direction, it is easy to notice that it takes $\mathcal{N} = L^D$ number of self–similar objects to fill the original object (or regular shape). Hence, the fractal dimension is given in terms of a logarithmic function of any base by [53–55]:

$$D = \frac{Log\,\mathcal{N}(L)}{Log\,L} \tag{4.1}$$

Applying Eqn. (4.1) to a fractal structure, one can obtain the dimension of the structure, or fractal (Hausdorff) dimension as [54]:

$$D_f = \lim_{\varepsilon \to 0} \frac{Log\,\mathcal{N}(e)}{Log\,(1/e)} \tag{4.2}$$

where $\mathcal{N}(e)$ is the number of self–similar structures of size e necessary to fill the fractal structure. It is found that the fractal dimension D_f is, in general, greater than the topological dimension D_T of the fractal object and less than its Euclidean dimension D_E [54]. The following figures are two well-known fractals; the Sierpinski carpet and the Menger sponge. The first is a fractal which covers an area and the second one is a fractal which fills a volume.

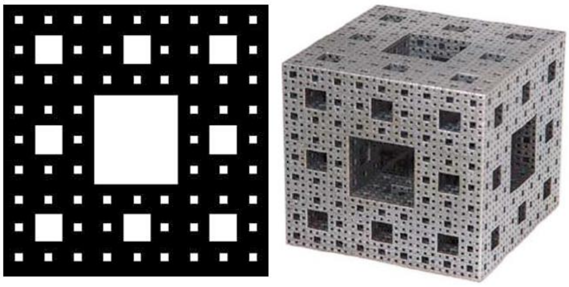


Fig.(4.1) The Sierpinski carpet (pores in white) and the Menger sponge (pores in black) [56].

For the Sierpinski carpet, there are 8 identical punched squares each of which has a linear size of one-third of the linear size of the entire shape of the object. According to Eqn. (4.2), the fractal dimension of the Sierpinski carpet which is embedded in $2D$ [53] is:

$$D_f^{Sierpinski} = \lim_{\varepsilon \to 0} \frac{Log\, \mathcal{N}(e)}{Log\,(1/e)}$$

$$= \lim_{k\to\infty} \frac{Log\, 8^k}{Log\, 3^k} = \frac{Log\, 8}{Log\, 3} = \frac{\ln 8}{\ln 3} = 1.8928... \qquad (4.3)$$

For the Menger sponge, there are 20 identical punched cubes each of which has a linear size of one-third of the linear size of entire shape of the object; the fractal dimension which is embedded in $3D$ is [53]:

$$D_f^{Menger} = \lim_{\varepsilon \to 0} \frac{Log\, \mathcal{N}(e)}{Log\,(1/e)}$$

$$= \lim_{k\to\infty} \frac{Log\, 20^k}{Log\, 3^k} = \frac{Log\, 20}{Log\, 3} = \frac{\ln 20}{\ln 3} = 2.7268... \qquad (4.4)$$

It is clear that the fractal dimensions of Sierpinski carpet which covers an area is less than the Euclidean dimension of area (D_E = 2). Similarly, the fractal dimension of Menger sponge which fills a volume is less than the Euclidean dimension of volume (D_E = 3).

Fractal geometry has found a large list of applications in a vast field of scientific disciplines and human knowledge [53, 54,57]. In physics, a large number of phenomena can be treated on the basis of the rules of fractal geometry [53, 57]. Historically, the first example of the fractional physical phenomena was the Brownian motion, whose paths are non-differentiable, self-similar curves that have a fractal dimension which is different from their topological dimension [53]. Furthermore, in view of considering fractal geometry as a generalization of the Euclidean geometry, the laws of physics known prior to the discovery of fractal geometry, which essentially rely on integer spatial dimensions, might need to be generalized by reformulations in terms of fractal geometry [57]. The importance of using the analysis of fractal geometry is not only due to these theoretical aspects, but also because this importance meets with findings in natural phenomena [53,54,57]. Some findings related to fractal geometry that are of relevance to the title of this thesis are highlighted in § (4.2). The rest of this chapter deals with bosons confined in fractal structures.

4.2 Experimental Findings Related to Bosons Confined in Fractal Media

Since the superfluidity of liquid ^{4}He was ascribed to the BEC phenomenon [8], liquid helium has been studied intensively in different situations. One of these situations was the confinement in a multiply connected geometry such as porous glasses, Vycor glass, aerogel glass and xerogel [58]. From 1975 to 1988, a series of experimental measurements of superfluid density of ^{4}He films in these porous glasses [58] revealed a sharp transition at T_c similar to that in bulk systems. In 1988, the first observation of a sharp heat capacity signature related to the superfluid transition in thin helium films adsorbed on porous Vycor and xerogel glasses was reported [58]. In the 1990s, ^{4}He in porous media was one of the preferred systems for studying BEC in an external potential. By confining ^{4}He in porous media, various experimental parameters such as dimension, topology, and disorder could be freely controlled [59]. Recently, the measurements of the heat capacity of ^{4}He confined in nanoporous (has nano diameter) Gelsil glass have been reported [59]. This was considered as an evidence for the formation of localized BE condensates on nanometer length scales [59]. The geometry of various glass surfaces and the geometry of liquid helium films adsorbed on those surfaces have been treated only in two dimensions [60]. The analysis of small–angle X–ray and neutron scattering data, in 1988 by Höhr *et al*, showed that the surface of Vycor porous glass has a fractal nature and that the Vycor glass surface has a fractal dimension larger than two [61]. According to this fact, it was suggested [60–63] that the presence of porous media should also be analyzed by finding an analogy with fractional dimensionality. The same suggestion was also recognized in the works of Ref. [32,35] for systems confined in external potentials.

Apart from the aforementioned fractal media in which bosons are confined, since 1983 there has been an increasing amount of clues that asserts space–time fractality [64]. It is found that space–time fractality extends from cosmology to the realm of quantum mechanics [64]. In 1985, an interesting paper has mentioned that the measured value of the dimension of space–time is $4-(5.3\pm 2.5)\times 10^{-7}$ [65]. It was also found by observation and experiments that the dimensionality is scale dependent; whereas different dimensionality is encountered from very large scales to very small scales [66]. The fractal properties of space–time have been found to be significantly manifested on the quantum scale [67]. References [64,66,67] contain well–known experiments and observations which record the fractality of space–time. Therefore, precise analyses of physical systems in actual experiments should not ignore the existence of space–time fractality especially within the quantum regime; the regime in which BE condensates form.

4.3 Theoretical Approaches for Bosons Confined in Fractal Media

The study of the BEC phenomenon in fractional dimensions demands looking at the Bose systems confined in these dimensions as ideal in order to investigate the effect of fractality. Motivated by the experimental evidence for bosons confined or trapped in fractal media [58–60], which was contemporary to the success of fractal geometry in a variety of applications [53,54,57], scientists were stimulated to look at these physical systems through " fractal eyes" [68]. This stimulus was also asserted by the dimension dependence of BEC occurrence discussed in chapter two. This, in turn, led to adopt theoretical approaches that consider the BEC phenomenon to evolve in spaces possessing fractional dimensions. It has been found that different approaches can be used to evaluate thermodynamic properties of quantum particles confined in fractional dimensions. Those approaches have different mathematical formalisms since they belong to different theoretical bases that introduce fractality. Thus, there are more than one thermostatistics for this endeavor. The common feature for these distinct approaches is the ability to introduce the effect of fractality of space or structures confining quantum particles. The main approaches which are adequate to treat the theory of BEC and to evaluate thermodynamic properties of ideal Bose gases (i.e., dilute atomic vapors) in fractal media are summarized in the paragraphs below.

The first approach that has been used is the one that adopts the concept of fractional dimension and with which the theory of the BEC phenomenon was extended into fractal dimensions [35,58,60,63,68,69] by functional analysis of the Bose gas function in fractal dimensions. These works investigated the Bose gas confined in fractal media for two cases; the homogeneous systems and inhomogeneous ones. The computations of the heat capacity within this analysis showed that there is a striking similarity with that corresponding to liquid ^{4}He in porous media [60]. In this approach, the parameter representing the integer dimension in the theory of BEC is denoted as *D*–dimension to indicate that *D* is not necessarily an integer. The first attempt in this framework was in 1988 whereas the specific heat, as phase–transition indicator, was studied for an ideal Bose gas in fractional dimension by Pfeifer [68]. In his study, Pfeifer pointed out that the heat capacity in these systems is similar to the model of the well–known bulk system but the low–temperature slopes and the high temperature limits are different [68].

The second approach is a special kind of invented quantum mechanics. It is a kind of deformed calculus (*D*–deformed calculus) that takes place in fractional–dimensional spaces which is invented in Ref. [70]. It was found that the *D*–deformed calculus is an appropriate tool for treating fractional dimensional systems and it is quite analogous to the corresponding one–dimensional partners [70].

In the context of the term "deformation", another (third) approach to evaluate thermodynamic properties of quantum particles known as "*q*–deformed thermostatistics" is also known. This approach is based on the *q*–deformed quantum algebra in which trapped quantum particles are considered as *q*–deformed oscillators. In many works, this approach is referred as *q*–deformed bosons; specifically when the considered quantum particles are bosons. The crucial idea of this theory is to deform the standard quantum algebra of the creation and annihilation operators of bosons [71]. In this formalism, the parameter *q*, where $q \in \Re$, represents the extent of deformation (deviation) from the standard quantum mechanics in such a way that when $q \rightarrow 1$, the standard quantum mechanics formalism is recovered [71]. The q–deformation whose extent is determined by the parameter q may represent different effects deforming the ideal or standard quantum system. Some of these effects are the fractality of spaces, interaction between quantum particles (fermions or bosons), impurities or any pure quantum effect. The works [71–85] are useful for obtaining more details concerning the types of effects the q parameter can represent and also for reviewing the theory and applications of the q–deformed analysis.

The fourth approach for investigating BEC phenomenon in fractal media is the one which is used in the works [86–88] and is called "fractional mathematical approach". In this "fractional mathematical approach", the order of the fractional derivative can represent different physical parameters; one of these parameters is the measure of the fractality of space [87]. A remarkable use for this approach is also observed in investigating the sort of interaction in the BE condensate of dilute atomic gases [88]. It is worth mentioning here that the fractional mathematical approach is based on the works [89–94].

In the works [90–94], the authors introduced a formalism that combines the standard statistics of Maxwell–Boltzmann (MB), Bose–Einstein (BE), and Fermi–Dirac (FD) on the basis of a generalized statistics known as "the Tsallis (nonextensive) statistics". This formalism introduces a unified statistics by using a fractional distribution function. This statistical unification is not only for the well–known distribution functions (of the MB, BE, FD statistics) but also for particles whose distribution functions interpolate in between [86]. This means that in the framework of this unified formalism, the MB, BE and the FD statistics are special cases. Another utility of using the analysis of this approach is the ability to introduce intermediate quantum statistics (wherever BE or FD statistics are not satisfied) [89]. Such statistics is necessary for what is called "quasi–particles" [86]. These particles are considered as virtual particles interpolate between bosons and fermions [86].

Upon the formalism of the works [86,90–94], a nonextensive model for the BEC phenomenon in fractal media will be formulated, in the present work, in § (4.12).

4.4 The Boltzmann-Gibbs (BG) Extensive Entropy

The concept of entropy introduced in 1865 by Rudolf Clausius was in the context of classical thermodynamics [95]. This concept had no connection with the microscopic world until Ludwig Boltzmann, one decade later, found that [96,4]:

$$S = k_B \ln(\Omega) \ , \qquad \Omega = W_{\max} \tag{4.5}$$

where Ω is the total number of microstates and $W_{\max}$ is the maximum weight (Planck's definition) [4]. Indeed, it was Max Planck who set Boltzmann's entropy in the form of Eq. (4.5) and gave the proportionality constant k_B the name of Boltzmann [4]. From Planck's definition of the maximum weight, Boltzmann's entropy only holds for systems in thermal equilibrium. The discrete form of equation (4.5) which is also known as Shannon's entropy is [96]:

$$S_{BG} = -k_B \sum_{i=1}^{W} p_i \ln p_i \tag{4.6}$$

where p_i is the propability associated with the i^{th} microscopic state of the system such that [96]:

$$\sum_{i=1}^{W} p_i = 1 \quad \text{and} \qquad p_i = 1/W \quad (\forall i) \tag{4.7}$$

The latter expressions of Eqn. (4.7) are the normalization condition and Boltzmann principle of equiprobablity [96]. The definition of entropy given by Eqns. (4.5) and (4.6) implies a magnificent connection of thermodynamics with the microscopic world; because entropy reflects microscopic information upon the physical systems. Both definitions (4.5) and (4.6) are also frequently called "Boltzmann–Gibbs (BG) entropy". The main characteristic feature of this entropy is the property of extensivity; i.e. the proportionality to the amount of matter involved which, in our present microscopic understanding, is interpreted as being proportional to the number of particles, *N*, involved in a system [95]. The term which is frequently used to stand for extensivity is "additivity"; i.e. extensive entropy means an additive one. From the statistical mechanics point of view, the property of extensivity (additivity) can be illustrated by the following example. When a physical system is composed of two statistically independent subsystems A and B with associated total number of microstates W_A and W_B, the total number of microstates in the composite system is

$(W_A\ W_B)$. Then, the entropy of the composite system is the sum of entropies of the individual subsystems; which follows directly from (4.5) and it is expressed as [96]:

$$S_{BG}(A \cup B) = k_B \ln(\ W_A\ W_B) = S(A) + S(B) \tag{4.8}$$

With this entropy, Josiah Willard Gibbs presented, in 1902, his last glorious achievement in statistical mechanics in the book entitled "Elementary Principles in Statistical Mechanics" [97]. In his last work, Gibbs introduced the foundations and the formalism of the well-known conventional or standard BG statistical mechanics which bears the names of Boltzmann and Gibbs. The core of Gibbs's achievement is a a firm bridge between the laws of mechanics and classical thermodynamics in such a way that thermodynamics can be viewed in a microscopic point of view [98]. From Eqns. (4.6) and (4.8), it is clear that BG statistical mechanics is basically constructed for extensive systems which are in thermal equilibrium.

The construction of BG statistical mechanics is basically oriented to thermodynamics that belongs to the entropy that bears the names of Boltzmann and Gibbs. The standard or conventional textbooks in statistical mechanics or statistical physics [4,48] are formalized upon this entropy. Applications of statistical mechanics methods to thermodynamics are denoted by the term thermostatistics. Thus, in this context, applications of the methods of the extensive (additive) BG statistical mechanics to thermodynamics are called BG thermostatistics.

4.5 Domain and Restrictions of BG Statistical Mechanics

Due to the overall success that BG statistical mechanics achieved (particularly the cases of MB, BE and FD statistics) and due to its elegant formalism as well, it was thought to be universal, eternal and infinitely precise [99,100]. In other words, BG statistical mechanics was thought to be applicable to all sorts of systems [101]. The fact that this belief is false was indeed recognized early even before 1902. Gibbs himself, in his book "Elementary Principles in Statistical Mechanics", explicitly pointed out to anomalies related to "*system or part of it* [which] *can be distributed in unlimited space* (*or in spaces which have limits*) *but still infinite in volume*" [102]; an example of such systems (anomalies) is gravitation [103]. This is because BG statistics can exactly describe the state of systems with short-range interaction (inter-particle forces) at thermal equilibrium; this means in systems whenever thermodynamic extensivity (additivity) holds [104,105]. Consequently, gravitating systems, which possess properties such as inhomogeneity and non–equilibrium due to the long–range nature of the gravitational force, are thought to be non–extensive [106]. However, many important phenomena in natural, artificial, and even social systems do not accommodate with the BG statistics. This is particularly frequent in physical sciences as well as in biology and economics, where non–equilibrium stationary states are the common rule [98]. In conclusion, the anomalies Gibbs addressed are those classes of physical systems whose ensembles lie outside the domain and the restrictions of BG statistics. The domain of BG statistics is the one whenever thermodynamic extensivity (additivity) holds and whose restrictions are [92,99,104,105]:

- short–range interactions (the range of the effective microscopic interactions must be small (or inexistent) compared to the linear size of the macroscopic systems);
- short–time memories (the time range of the microscopic memory must be small (or inexistent) compared to the time of observation, i.e., Marcovian processes);
- the system must evolve in an Euclidean–like space–time ("Euclidean-like" basically refers to a continuous and sufficiently differentiable variety, either curved or not).

When one (or more) of these restrictions is (are) violated, i.e. when the domain is not any more extensive (i.e., it is nonextensive), the formalism of BG thermostatistics fails. The latter statement of failure refers to the fact that standard sums (or integrals) that appear in the calculation of the BG thermostatistical quantities (e.g., partition function, internal energy, entropy and average square displacement) diverge [104]. Consequently, there will be ill–behaved mathematical prescriptions for calculating the quantities which are normally used for characterizing a system, and which enable meaningful comparisons with experimental data (always finite) [104].

4.6 The Tsallis (Nonextensive) Entropy

Systems which, either in their direct space–time description or in their phase space evolution, present a (multi–) fractal–like or unconventional structure exhibit serious mathematical untractability and unfamiliar scalings with size for large sizes or with time for long time intervals within the standard formalisms of BG thermostatistics [104]. Motivated by the behavior of these systems which is abnormal (anomalous) in the context of BG thermostatistics, Constantino Tsallis in 1988 proposed a theoretical formalism to generalize BG entropy [107]. This formalism has introduced a family of generalized entropy functionals with a single parameter (*q*) [108]:

$$S_q = k\ \frac{(1-\sum_i p_i^q)}{q-1} \quad , \qquad (q \in \Re,\ S_1 = S_{BG}) \qquad (4.12)$$

where k is a positive constant and $\{p_i\}$ are the probabilities of the *i* microscopic states. It is found that in the limit $q \to 1$, Shannon's entropy, Eqn. (4.7), is recovered [108]. From this, it is inferred that the Tsallis entropy includes BG entropy as a special single case; only when *q*=1. The generalized statistical mechanics which is based on this entropy basically relies upon two postulates; the first is the generalized entropy, Eqn. (4.12), and the second is the q –expectation value of an observable O, whose value in the state *i* is given by [92]:

$$\langle O \rangle_q = \sum_i^W p_i^q O_i \qquad (4.13)$$

where O_i is the value of the observable in the state *i*. In this formalism, it is noticeable that neither the entropy S_q of Eqns. (4.12) nor the observable O_q are extensive thermodynamic variables when $q \neq 1$. The interesting property of this entropy is that of pseudo–additivity (Nonextensivity) [92]. This term can be illustrated by the following example. When a system is composed of two statistically independent subsystems *A* and *B*, the entropy of the composite system is [92]:

$$S_q(A \cup B) = S_q(A) + S_q(B) + (1-q)\, S_q(A) S_q(B) \qquad (4.14)$$

From the latter expression, it is clear that S_q is generally nonadditive (nonextensive) [105], and that is why the Tsallis entropy is called nonextensive entropy. Therefore, $(1-q)$ represents the measure of nonextensivity (or the measure of the departure from BG entropy) [105].

In this view, the nonextensivity index (q) may represent a physical effect that stands beyond the deviation from the BG thermostatistics. Eqn. (4.14) also implies that S_q is superadditive (entropy of the system is greater than the sum of the constituting subsystems) for q <1 and subadditive (entropy of the system is smaller than the sum of the constituting subsystems) for $q > 1$ [92]. Other properties of the Tsallis entropy are [92]:

- Positivity ($S_q > 0$ for any arbitrary set of $\{p_i\}$ and for any value of q).
- Concavity (S_q is concave for $q > 0$ and convex for $q < 0$).
- Equiprobability of the microcanonical ensemble (i.e., $p_i = 1/W, \forall i$) and S_q attains its extremal value:

$$S_q = k(W^{1-q} - 1)/(1-q) \tag{4.15}$$

- The optimization of S_q is under the two constraints [109]

$$\sum_i^W p_i = 1 \tag{4.16}$$

$$\sum_i^W P_i \; \varepsilon_i = U_q \tag{4.17}$$

where $\{\varepsilon_i\}$ is the set of the eigenvalues of the Hamiltonian (energy spectrum) and P_i is the escort probability (to normalize the energy U_q) which is defined as [109]:

$$P_i = p_i^q / \sum_{j=1}^W p_j^q \;, \qquad \left(\sum_i^W P_i = 1 \right) \tag{4.18}$$

From the above, the grand canonical probability distribution and the associated generalized partition function are respectively [110]:

$$p_i = \frac{1}{Z_q} \left[1 - (1-q) \beta (\varepsilon_i - \mu N) \right]^{\frac{1}{1-q}} \tag{4.19}$$

$$Z_q = \sum_i^W \left[1 - (1-q) \beta (\varepsilon_i - \mu N) \right]^{\frac{1}{1-q}} \tag{4.20}$$

It is important, here, to focus on the probability distribution, Eqn. (4.19). For the case when $q < 1$, Tsallis complemented the probability distribution by an auxiliary condition (cut–off condition) that $p_i = 0$ whenever the argument of the function becomes negative [100], i.e., this cut–off condition is expressed as:

$$p_i = \begin{cases} \frac{1}{Z_q} [1-(1-q)\beta(\varepsilon_i - \mu N)]^{\frac{1}{1-q}} \;, & [1-(1-q)\beta(\varepsilon_i - \mu N)] > 0 \\ 0 \;, & \text{elsewhere} \end{cases} \tag{4.21}$$

Within the above postulates and the properties of nonextensivity or pseudo–additivity, irreversibility, positivity, concavity, equiprobability of the microcanonical ensemble, and Onsager reciprocity, it was found that the Tsallis entropy preserves and generalizes the relevant features of the BG entropy [91,111]. It has also been shown to be compatible with the information theory foundations of statistical mechanics given by Jaynes and with the dynamical thermostatistics approach to statistical ensembles [107]. More important, it has been found that the Legendre–transform structure of thermodynamics is invariant for all values of q, indicating that the entire formalism of thermodynamics can be extended to be nonextensive [91]. In addition to this, it has been proved that the thermodynamical stability, the H–theorem of Boltzmann as well as the Ehrenfest theorem hold for all q values [91,92]. Since applications of the methods of statistical mechanics to thermodynamics are denoted by the term thermostatistics, applications of the methods of the Tsallis (or nonextensive) statistical mechanics to thermodynamics are denoted by Tsallis thermostatistics.

In the framework of the Tsallis nonextensive entropy (with $q \neq 1$), a generalized statistical mechanics, whereas BG statistics (with $q = 1$) is a special case, has been developed rapidly by many researchers [112,113].
The development of the nonextensive statistical mechanics was accompanied by an increasing amount of confirmations indicating that remarkable processes and systems, not only in physics but also in other disciplines as well, are better described in terms of the Tsallis distribution [113]. This, in turn, has initiated a new stream in the foundations of statistical mechanics which produced a huge amount of research works and a universal success in a variety of applications (direct experimental and observations data) on this subject have been witnessed [86–94,96,98,100,103–107,109–120].

An automatically updated bibliography which exhibits research works concerning this rapidly growing field is given in Ref. [121]. Before closing this section, it is necessary to mention that there is an acute and non–ignorable connection between the Tsallis nonextensivity and quantum group theory [122–126]; the theory to which the q –deformrd oscillators belong.

4.7 Literature Related to BEC in Fractal Media within BG Thermostatistics

It is necessary to mention, here, that FD and BE quantum statistics, as well as MB classical statistics are based on the framework of BG statistical mechanics [95]. Thus, the theory of BEC occurrence previously reviewed in chapter two and the thermodynamic properties evaluated in chapter three for bosons harmonically trapped in integer dimensions are based on BE statistics; hence thermodynamic properties of these Bose systems were evaluated in the framework of BG thermostatistics.

It is also worth mentioning here that the analysis of the works [35,59,60,63,68,69], reviewed in § (4.3), which adopted the functional analysis approach for treating Bose systems confined in fractal media, is based on the BE statistics. Hence, these works belong to the standard BG statistical mechanics; i.e., these works treat extensive systems. In the works [35,59,60,63,68,69], the analysis used is based on integrals, and this in turn led to evaluate the BEC properties in terms of the Bose function, gamma function and the Riemann zeta function. This indeed asserts that the properties computed for these Bose systems are for gases within the thermodynamic limit; hence, one is dealing with an analysis similar to that discussed in chapter two for treating the BEC phenomena.

4.8 The Problem of Finite Number of Particles

Despite the fact that the analyses used to compute of the heat capacity within the works [35,58,60,63,68,69] showed that there is a striking similarity with that corresponding to liquid ^{4}He in porous media [60], the analyses of these works are inadequate for treating Bose systems with finite number of particles. The justification for this inadequacy concerning the case of finite number of particles is discussed in § (2.6) and § (3.2), and it is also asserted by the results obtained in § (3.4.1) and thereafter. Furthermore, in view of the rapid development in nanotechnology and the possibility of constructing complex structures or networks [127–131], confining a few dozens of particles (bosons) becomes plausible. The most recent works [127–131] assert the possibility of the occurrence of BEC for ideal Bose gases confined in these networks. The most interesting feature of these networks is that BEC can occur within dimensions $D<2$ even in the absence of external potentials (see § (2.1), § (2.2) and § (2.3)). This occurrence of BEC in these networks is called "topology–induced BEC"; whereas the complex structure of these media (networks) plays the role of external potentials in producing inhomogeneous Bose systems.

In conclusion, the above discussion emphasizes the necessity to adopt the treatment of finite number of particles for the BEC phenomenon in fractal media. The rest of this chapter is mainly devoted for the BEC phenomenon with finite number of particles in fractal media.

4.9 BEC in Harmonic Fractal Traps within BG Thermostatistics: An Extensive Model

A remarkable approach for BEC with finite number of particles is that of A. Rovenchack [132]. Motivated by the works such as [25,36,40,41], which rely on the case of finite number, Rovenchack presented an approach for treating the BEC phenomenon for bosons harmonically trapped on the well–known fractal Sierpinski carpet. He introduced the effect of fractality into the main equation for computing the properties, total number of particles, by replacing the discrete representation of harmonic degeneracy factors by a continuum representation [132]; i.e. generalizing the degeneracy factors of the harmonic oscillator from the discrete form into a continuous one. Mathematically, this will be illustrated below. In his endeavor, Rovenchack [132], determined the condensation temperature and thermodynamic properties by using Euler–MacLaurin formula as a mathematical technique for computing the sum by converting the latter into an integral (i.e., an approximation).

Encouraged by Rovenchack's approach [132], this section discusses investigating the thermodynamic behavior of an ideal Bose gas with finite number of particles in the framework of grand canonical ensemble. The system is assumed to be harmonically trapped in a fractal medium. This investigation is carried out by using the symbolic computation as a mathematical technique for evaluating the sums entering in the expression of the properties of the BE condensate. The manner used for evaluating these properties is similar to that illustrated in § (3.5) for bosons harmonically trapped in integer dimensions. The starting point is the equation of the total number of particles which is related to the temperature, T, and the fugacity, z ,[132]:

$$N = N_o + \sum_{n=1}^{\infty} \frac{g_n}{z^{-1} e^{\varepsilon_n / k_B T} - 1} \;, \qquad N_o = \frac{z}{1-z} \tag{4.22}$$

where N_0 is the occupation of the lowest energy level, ε_n is the single-particle energy spectrum, and g_n is the harmonic degeneracy factor of the n^{th} level. For a system consisting of N harmonic oscillators confined in an isotropic harmonic trap, the single-particle energy spectrum is given by $\varepsilon_n = n\hbar\omega$ (neglecting the zero point energy). In one dimension (1D), the degeneracy g_n = 1, in 2D the degeneracy $g_n = n+1$, and in 3D it is given by $g_n = (n+2)(n+1)/2$. So, the degeneracy in the D-integer dimensions is given by the binomial coefficient [132]:

$$g_n = C_{n+D-1}^{D-1} = \frac{(n+D-1)!}{n!\,(D-1)!} \tag{4.23}$$

Extending Eqn. (4.23) to a continuous D_f, Eqn.(4.23) becomes [136]:

$$g_n = \frac{\Gamma(n+D_f)}{\Gamma(n+1)\Gamma(D_f)} \tag{4.24}$$

Inserting Eqn. (4.24) into Eqn. (4.22), yields the total number of bosons harmonically trapped in fractal media as [132]:

$$N = N_0 + \frac{z}{\Gamma(D_f)} \sum_{n=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+D_f)} \frac{1}{e^{n\hbar\omega/k_B T} - 1} \tag{4.25}$$

Now, Eqn. (4.25) is the main equation that would be used for evaluating the condensation temperature, T_c, fugacity, z, the condensate fraction, (N_0/N), the total internal energy, U, and the heat capacity at constant volume, C_V, for a finite number of bosons harmonically trapped in a fractal medium. Determining the condensation temperature, T_c, for a given total number N requires solving Eqn. (4.25) for T numerically by applying the condition for Bose-Einstein condensation $(N_0 = 0, z = 1)$.

Evaluating the fugacity temperature dependence $z(T)$ requires solving Eqn. (4.25) numerically for z by taking into account the evaluated T_c, where the fugacity is customarily taken as $z = 1$ when $T < T_c$.

The temperature dependence of the condensate fraction (N_0/N) for these Bose systems is obtained by using Eqns. (3.8) and (4.25) to be in the form:

$$(N_0/N) = 1 - \frac{1}{N}\left(\frac{z}{\Gamma(D_f)} \sum_{n=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+D_f)} \frac{1}{e^{n\hbar\omega/k_B T} - 1} \right) \tag{4.26}$$

The temperature dependence of the internal energy is obtained by using Eqns. (3.7) and (4.25) to get:

$$U(T,z) = \hbar\omega\left(\frac{z}{\Gamma(D_f)}\sum_{n=1}^{\infty} n\ \frac{\Gamma(n+1)}{\Gamma(n+D_f)}\frac{1}{e^{n\hbar\omega/k_BT}-1}\right) \tag{4.27}$$

The temperature dependence of the heat capacity (C_V) is obtained by using the result of Eqn. (4.25) in Eqn. (3.9) as:

$$C_V = \frac{\partial}{\partial T}U(T,z) = \hbar\omega\,\frac{\partial}{\partial T}\left(\frac{z}{\Gamma(D_f)}\sum_{n=1}^{\infty} n\ \frac{\Gamma(n+1)}{\Gamma(n+D_f)}\frac{1}{e^{n\hbar\omega/k_BT}-1}\right) \tag{4.28}$$

4.10 Literature Related to Nonextensive Bosons

It has been discussed in § (4.5) and § (4.6) where and why it is better to treat certain systems with the Tsallis statistics. In this respect, the BEC phenomenon has been also reviewed in the framework of the Tsallis thermostatistics, where the thermodynamic behavior and several statistical quantities were generally investigated as q –dependent quantities [90–94,110,112,120,124,133–136]. A significant use of the nonextensivity index, q , has been witnessed in the work of A. Lawani *et al.* [137]. In the latter work, the index q was used as a parameter that stands for interparticle interaction of the BEC vapors.

It seems feasible and also interesting to investigate the phenomenon of BEC in fractal media within the Tsallis statistics, especially, when there is a proposed definition for the fractional dimension within the Tsallis nonextinsivity [115]. For this goal, it is intended to formulate a suitable theoretical quantum statistical model for the BEC phenomenon in fractal media. The starting point for the formalism of the intended nonexensive model is the distribution function to be discussed below.

4.11 The Distribution Function of Nonextensive Bosons

The works of Fevzi Büyükkiliç *et al* [90–94] introduced a statistics that unifies the statistics for classical and quantum gases. In the framework of this statistics, bosons and fermions are regarded as g –ons which obey fractional exclusion statistics. With this point of departure, the thermostatistical relations concerning the Bose and Fermi systems are unified under the g –on formulation [90]. This unified statistics (g –ons) is essentially constructed in the framework of the Tsallis thermostatistics [94–98]. In the grand canonical ensemble, the g-ons distribution function is given by [90,91]:

$$n(\varepsilon_i\,,\,g\,,q) = \frac{1}{[1+(q-1)\beta(\varepsilon_i-\mu)]^{\frac{1}{q-1}}+2g-1} \tag{4.29}$$

where g, whose value $0 \le g \le 1$, is the parameter which specifies the sort of statistics the g-ons belong to. To simplify matter, the role of g can be illustrated in the extensive BG thermostatistics and by using the generalized q–exponential functions given by [138]:

$$\exp_q(x) \equiv \mathrm{e}_q^x \equiv [1+(1-q)x]^{1/(1-q)}\;,\; (q \to 1,\;\; e_q^x = e_1^x) \tag{4.30}$$

Replacing x by $\beta(\varepsilon_i-\mu)$ yields:

$$[1+(1-q)\beta(\varepsilon_i-\mu)]^{1/(1-q)} \equiv \mathrm{e}_q^{\beta(\varepsilon_i-\mu)} \tag{4.31}$$

Substituting Eqn. (4.31) into Eqn. (4.29) gives:

$$n(q,g,\varepsilon_i) = \frac{1}{[\mathrm{e}_q^{\beta(\varepsilon_i-\mu)}]+2g-1} \tag{4.32}$$

For the case of extensive statistics, $(q \to 1,\;\; e_q^x = e_1^x)$, Eqn. (4.32) reads:

$$n_{FD}(\varepsilon_i\,,\,g=1) = \frac{1}{e^{\beta(\varepsilon_i-\mu)}+1} \tag{4.33}$$

$$n_{MB}(\varepsilon_i\,,\,g=\frac{1}{2}) = \frac{1}{e^{\beta(\varepsilon_i-\mu)}} \tag{4.34}$$

$$n_{BE}(\varepsilon_i\,,\,g=0) = \frac{1}{e^{\beta(\varepsilon_i-\mu)}-1} \tag{4.35}$$

From Eqns. (4.33), (4.34) and (4.35), it is clear that within the BG statistics, (q=1), the standard distribution functions for ideal particles in FD, MB and BE statistics are recovered from the nonextensive distribution function for the values g = {1, 1/2, 0}.

From Eqns. (4.29) and (4.35), and when $(q \neq 1,\;\; g = 0)$, the nonextensive Bose–Einstein distribution function of the grand canonical ensemble is obtained as [94–98]:

$$n(\varepsilon_i) = \frac{1}{[1+(q-1)\beta(\varepsilon_i-\mu)]^{\frac{1}{(q-1)}}-1} \tag{4.36}$$

4.12 BEC in Harmonic Fractal Traps within Tsallis Thermostatistics: A Nonextensive Model

The key idea of introducing fractality into the medium confining the Bose gas is to use the nonextensivity index q. Tsallis, on the basis of the nonextensive analysis, proposed a direct connection between this index and the fractal dimension as [115]:

$$q = D_f \,/\, D_E \tag{4.37}$$

where D_f and D_E are respectively the fractal dimension and the Euclidean dimension in which the fractal dimension is embedded. Tsallis also indicated [115] that porous structures (fractal media) have $q < 1$ because ($D_f < D_E$). So, the auxiliary cut–off condition, Eqn. (4.21), is extremely necessary in this formalism to preserve the probability of the Tsallis statistics.

In view of the conclusion of § (4.8), the intended nonextensive model for BEC in fractal media is the one that addresses a Bose gas with finite number of particles. Therefore, the symbolic computation is the mathematical technique to investigate the thermodynamic behavior. The total number of particles, Eqn. (2.2), is defined as [25]:

$$N = \sum_{i=0} n_i \;\equiv\; \sum_{i=0} g_i \; n(\varepsilon_i) \tag{4.38}$$

where g_i is the degeneracy of the i^{th} state and $n(\varepsilon_i)$ is the distribution function of the ideal nonextensive bosons. By substituting Eqn. (4.36) in Eqn. (4.38), one gets:

$$N = \sum_{n=0} \frac{g_n}{[1+(q-1)\beta(\varepsilon_n-\mu)]^{\frac{1}{q-1}} - 1} \tag{4.39}$$

Then, the ground state population, N_0, is expressed as:

$$N_o = \frac{g_0}{[1+(q-1)\beta(\varepsilon_0-\mu)]^{\frac{1}{q-1}} - 1} = \frac{1}{[1+(1-q)\beta\mu]^{\frac{1}{q-1}} - 1} \quad ,\varepsilon_0 = 0,\; g_0 = 1 \tag{4.40}$$

Isolating the ground state population, Eqn. (4.40) reads:

$$N = \frac{1}{[1+(1-q)\beta\mu]^{\frac{1}{q-1}} - 1} + \sum_{n=1} \frac{g_n}{[1+(q-1)\beta(\varepsilon_n-\mu)]^{\frac{1}{q-1}} - 1} \tag{4.41}$$

To determine the thermodynamic properties for this nonextensive Bose system, Eqn. (4.41) is expressed in terms of fugacity z ($z = e^{\beta\mu} \Leftrightarrow \beta\mu = \ln z$) as:

$$N = \frac{1}{[1+(1-q)\ln z]^{\frac{1}{q-1}} - 1} + \sum_{n=1} \frac{g_n}{[1+(q-1)(\frac{n\hbar\omega}{k_B T} - \ln z)]^{\frac{1}{q-1}} - 1} \tag{4.42}$$

$$\text{where } \beta\varepsilon_n = \frac{n\hbar\omega}{k_B T}$$

The validity for this formulation is verified only if it is proved that the ground state population, Eqn. (4.40), recovers its standard form, $N_o = \frac{z}{1-z}$, when $q \to 1$; in other words:

$$\lim_{q\to 1} \frac{1}{\left[1+(1-q)\ln z\right]^{\frac{1}{(1-q)}} - 1} = \frac{z}{1-z} \tag{4.43}$$

The proof of Eqn. (4.43) is given in Appendix B. Now, Eqn. (4.42) is the one to be used for evaluating the condensation temperature and the temperature dependence for the properties of the nonextensive condensate, where the total number of particle is written as a function of T and z

To determine T_c, it is required to solve Eqn.(4.42) for T, after setting $N_0 = 0$ and $z = 1$. Evaluating the fugacity temperature dependence, $z(T)$, requires solving Eqn. (4.42) for z by taking into account the determined T_c, where the fugacity is customarily taken as $z = 1$ when $T < T_c$. The temperature dependence of the condensate fraction, (N_0 / N), is obtained by using Eqns. (3.8) and (4.42) to be in the form:

$$(N_o/N) = 1 - \frac{1}{N}\sum_{n=1}^{\infty} \frac{g_n}{[1+(q-1)(\frac{n\hbar\omega}{k_B T} - \ln z]^{\frac{1}{(q-1)}} - 1} \tag{4.44}$$

The temperature dependence of the total internal energy, $U(T,z)$, is obtained by using Eqns. (3.7) and (4.42) to get:

$$U(T,z) = \hbar\omega \sum_{n=1}^{\infty} n \frac{g_n}{[1+(q-1)(\frac{n\hbar\omega}{k_B T}\ln z]^{\frac{1}{(q-1)}} - 1} \tag{4.45}$$

The temperature dependence of the heat capacity is obtained by using the result of Eqn. (4.45) in Eqn. (3.9) as:

$$C_V = \frac{\partial}{\partial T} U(T,z) = \hbar\omega \frac{\partial}{\partial T} \left[\sum_{n=1}^{\infty} n \frac{g_n}{[1+(q-1)(\frac{n\hbar\omega}{k_B T} - \ln z]^{\frac{1}{(q-1)}} - 1}\right] \tag{4.46}$$

4.13 Thermodynamic Behaviors for the Extensive and Nonextensive BE Condensates in Harmonic Fractal Traps

This section exhibits the thermodynamic behaviors of the two models (the extensive and the nonextensive) formulated in § (4.9) and § (4.13) for the BEC phenomenon in a fractal medium. The temperature dependence for thermodynamic properties of the Bose Einstein condensate will be evaluated for a given total number of particles N=10000. The bosons' confining fractal media are selected with arbitrary fractional dimensions; D_f =2.7, 2.8 and 2.9. It is assumed that these fractal media are embedded in a Euclidean space of 3D. Hence, the corresponding q values, which represent the fractal dimensions according to Eqn. (4.37), for the nonextensive model are respectively (2.7/3), (2.8/3) and (2.9/3).

Figs. (4.2) and (4.3) show the effect of fractional dimension $(1 \leq D_f \leq 3)$ on the condensation temperature of the extensive bosonic system for two different values of total number of particles (5000 and 10000). It is clear from these figures that for N harmonically trapped bosons, the condensation temperature T_c increases with decreasing the fractional dimension (D_f).

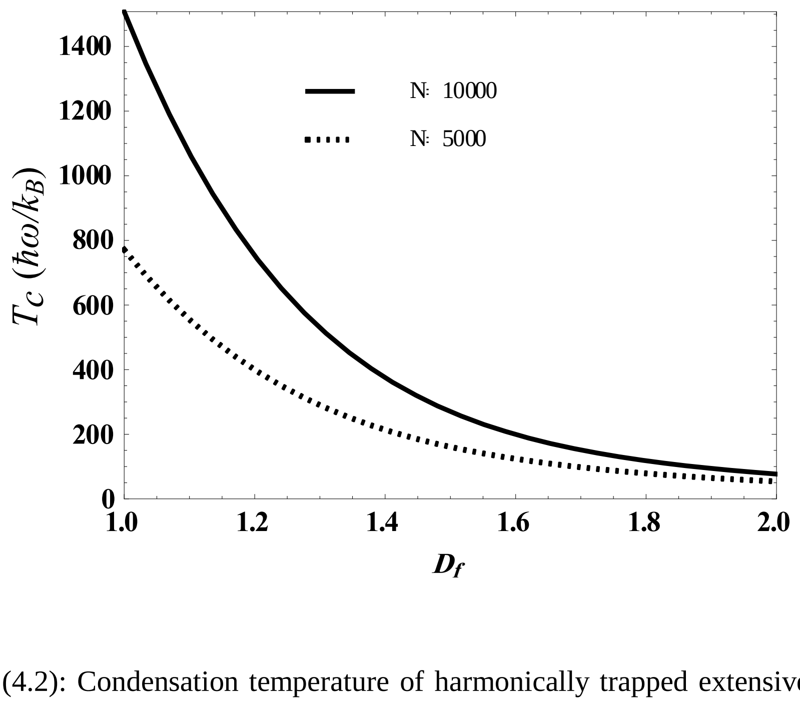


Fig. (4.2): Condensation temperature of harmonically trapped extensive bosons in fractal media whose fractional dimensions are in the range $(1 \leq D_f \leq 2)$.

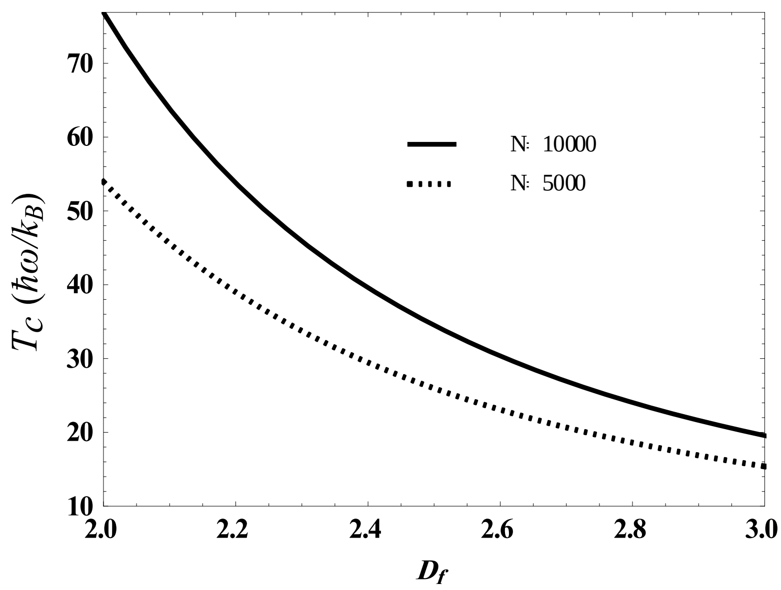


Fig. (4.3): Condensation temperature of harmonically trapped extensive bosons in fractal media whose fractional dimensions are in the range $(2 \leq D_f \leq 3)$.

Figs. (4.4) and (4.5), below, show the effect of the fractional dimension D_f on the fugacity temperature dependence. It is clear from these figures that at a certain temperature when $T > T_c$, the lower the fractional dimension is the higher the fugacity.

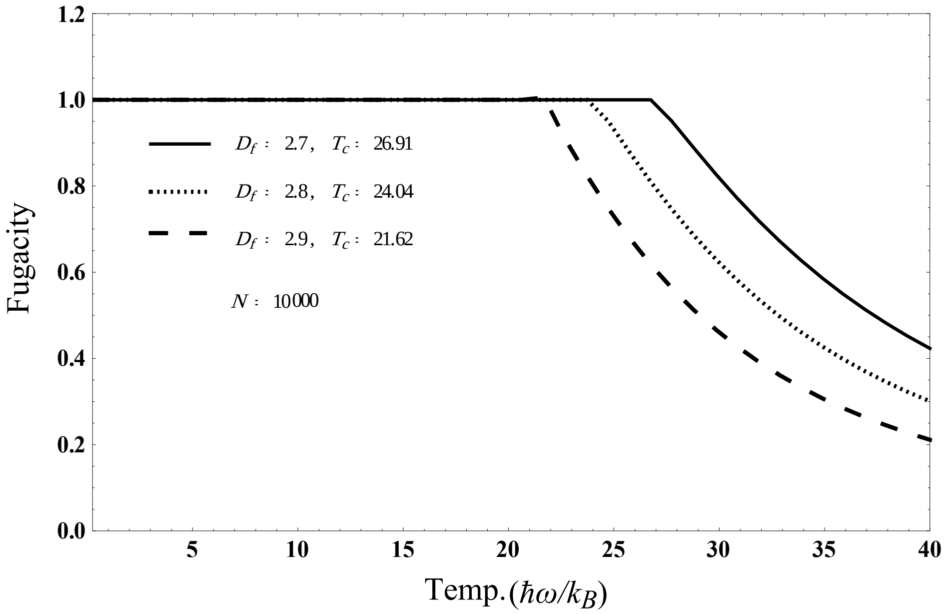


Fig. 4.4: Temperature dependence of the fugacity for harmonically trapped extensive bosons in fractal media.

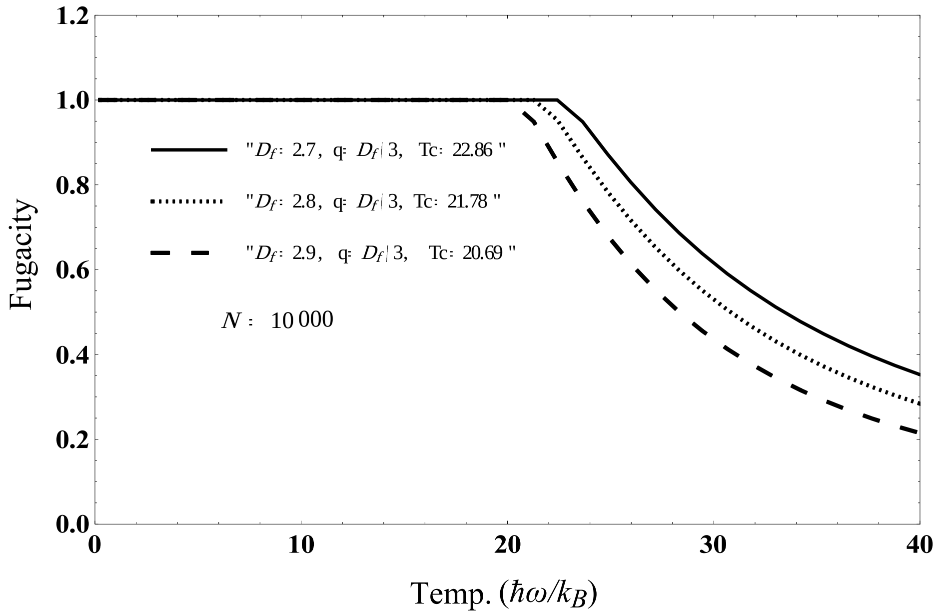


Fig. 4.5: Temperature dependence of the fugacity for harmonically trapped nonextensive bosons in fractal media.

Figs. (4.6) and (4.7) show the effect of the fractional dimension D_f on the condensate fractions. These figures show that, for a given total number of particles N, when $T < T_c$, the lower the fractional dimension is the higher the condensate fraction.

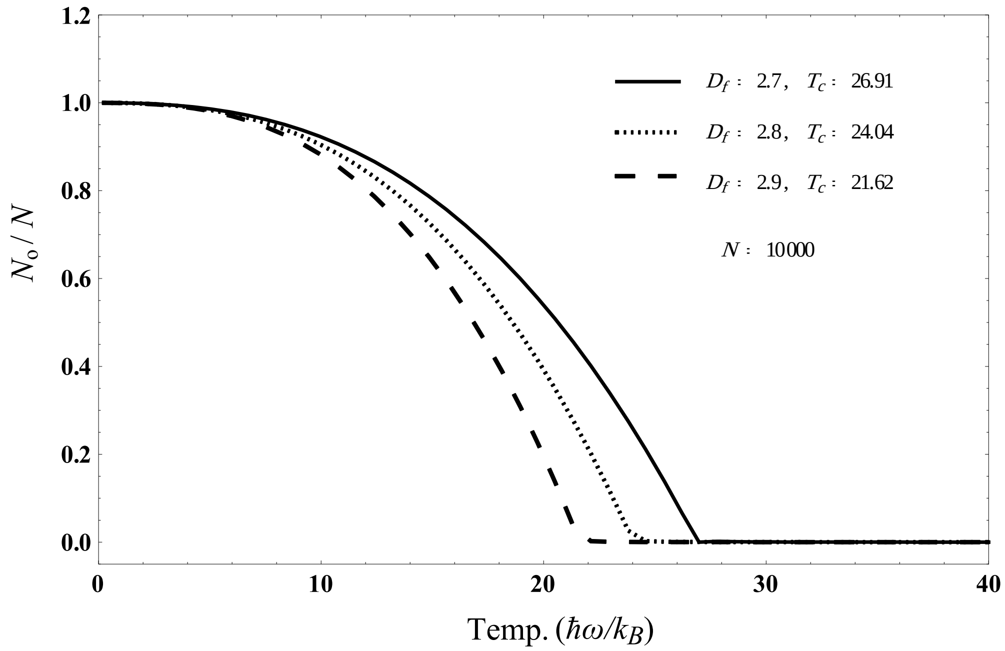


Fig. (4.6): Temperature dependence of the condensate fraction for harmonically trapped extensive bosons in fractal media.

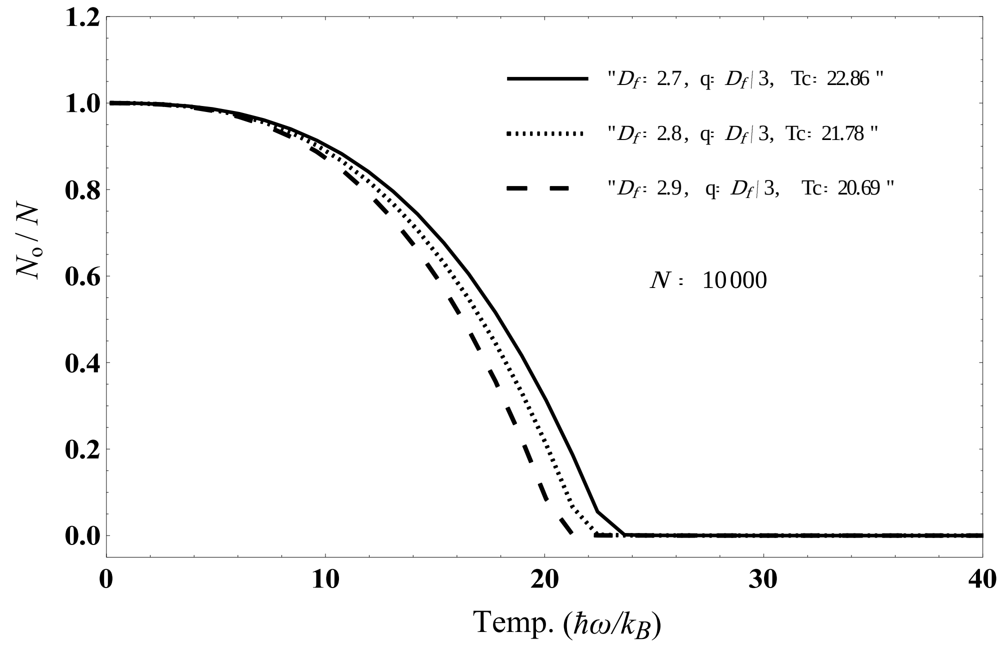


Fig. (4.7): Temperature dependence of the condensate fraction for harmonically trapped nonextensive bosons in fractal media.

Figs. (4.8) and (4.9) show the effect of the fractional dimension D_f on the temperature dependence of the internal energy. These figures assert that the lower the fractional dimension is the lower the internal energy. This result emphasizes that these particles possess higher degrees of freedom when trapped in higher fractional dimensions.

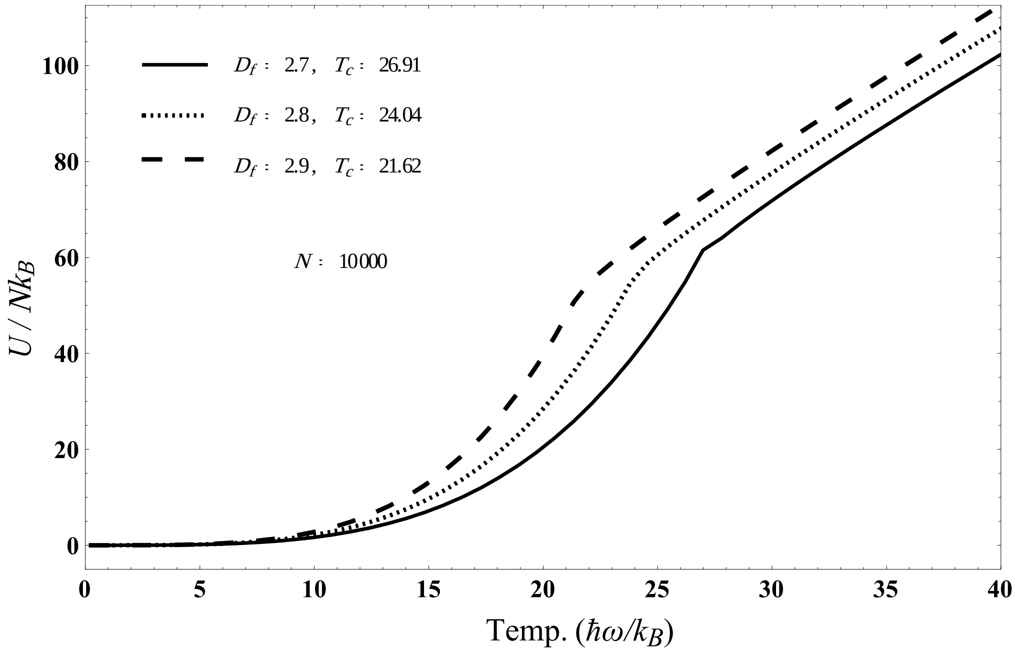


Fig. (4.8): Temperature dependence of the internal energy for harmonically trapped extensive bosons in fractal media.

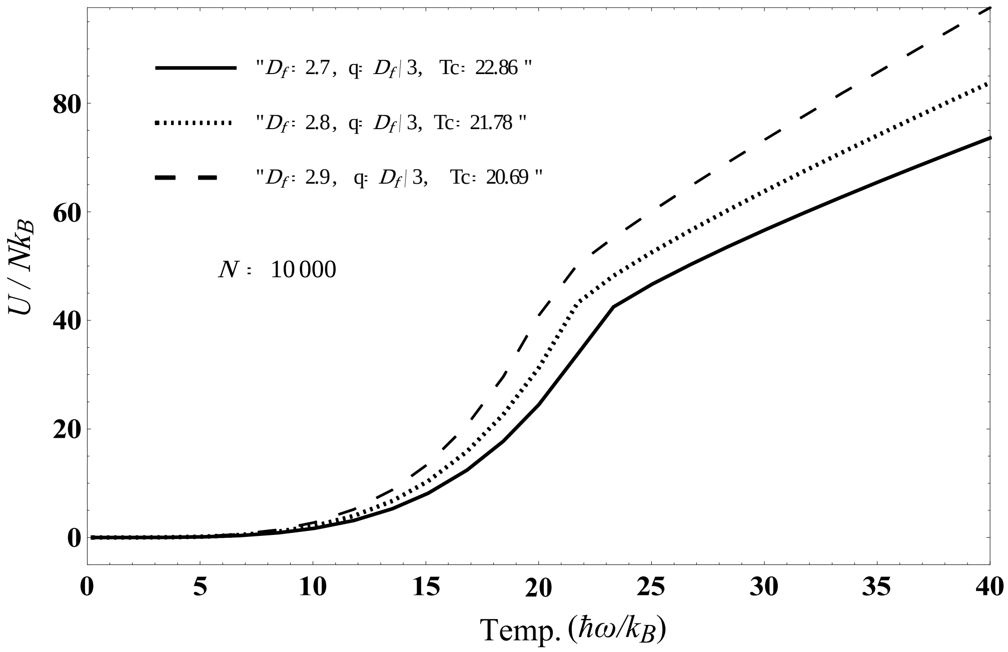


Fig. (4.9): Temperature dependence of the internal energy for harmonically trapped nonextensive bosons in fractal media.

Figs. (4.10) and (4.11) show the effect of the fractional dimension D_f on the temperature dependence of the heat capacity for these Bose systems. These figures show that the lower the fractional dimension is the lower the heat capacity.

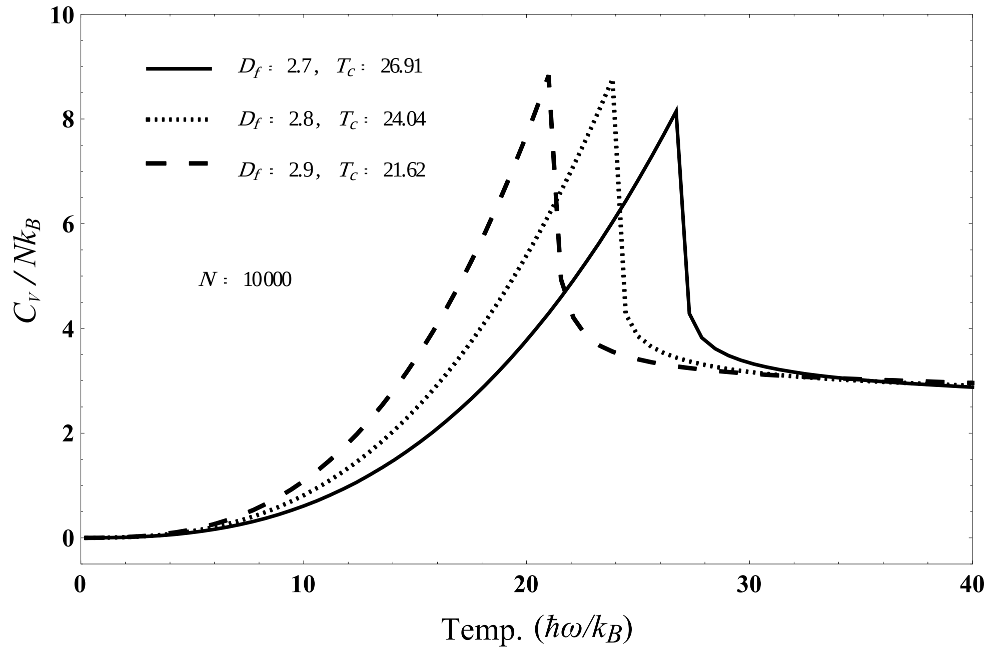


Fig.(4.10): Temperature dependence of the heat capacity for harmonically trapped extensive bosons in fractal media.

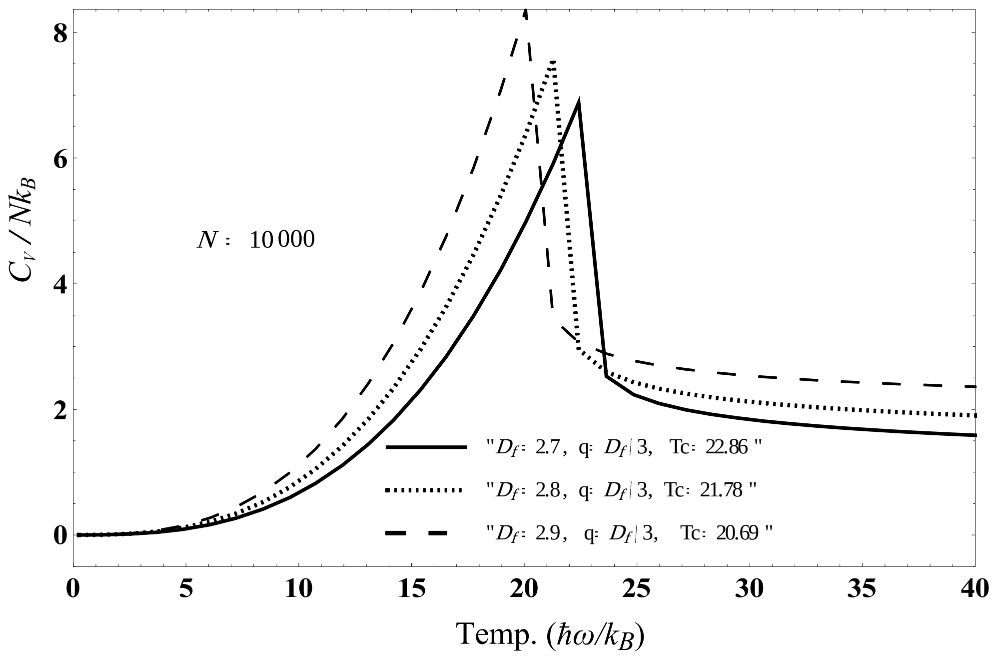


Fig.(4.11): Temperature dependence of the heat capacity for harmonically trapped nonextensive bosons in fractal media.

In Figs. (4.4)–(4.11), the effect of fractional dimensions on the condensates thermodynamic properties for the two models (the extensive and the nonextensive) exhibits a similarity with the corresponding properties evaluated in chapter three for bosons harmonically trapped in integer dimensions. The comparison of the effect of fractal dimension on the condensate properties for the extensive bosons and the nonextensive bosons leads to the following two observations. The first observation is that the condensation temperature within the Tsallis's thermostatistics (T_c^T) is always less than the corresponding one in BG thermostatistics (T_c^{BG}); i.e., $T_c^T < T_c^{BG}$. The reason that stands behind $T_c^T < T_c^{BG}$ is the Tsallis cut–off condition, which is given by Eqn. (4.21). This is because the structure of porous media has values for q<1. It is found that this result agrees with Salasnich's result [133]. Consequently, the condensate thermodynamic properties in the Tsallis thermostatistics are always shifted to lower temperatures. The second observation is that despite the fact that the thermodynamic behaviors of the two thermostatistics are, in general, similar, the condensate thermodynamic properties seem to possess different temperature responses (slopes).

4.14 BEC in the Sierpinski Carpet and the Menger Sponge: A Comparison between Extensive and Non-extensive Thermostatistics

In § (4.13), the obtained results compare between the thermodynamic behaviors of the extensive and the nonextensive bosons, where these bosons are trapped harmonically in fractal media. But the fractal dimensions of the bosons' confining media are arbitrarily selected. In this section, it is also intended to compare the thermodynamic behaviors of the extensive and the nonextensive harmonically trapped bosons but within standard fractals, the Sierpinski carpet and the Menger sponge. These fractals have fractal dimensions defined in Eqns. (4.3) and (4.4). The Siepinski carpet whose $D_f = \log 8 / \log 3$, is embedded in a Euclidean space of dimension 2D, while the Menger sponge, whose $D_f = \log 20 / \log 3$, is embedded in a Euclidean space of dimension 3D. The application of the two formulated models for bosons trapped in harmonic fractal traps is due to their characteristic features. These fractals have connected interior, are highly symmetric, and most of all, are infinitely ramified [139]. Applying the two models to these fractals yields the results from Fig. (4.13) to Fig.(4.20).

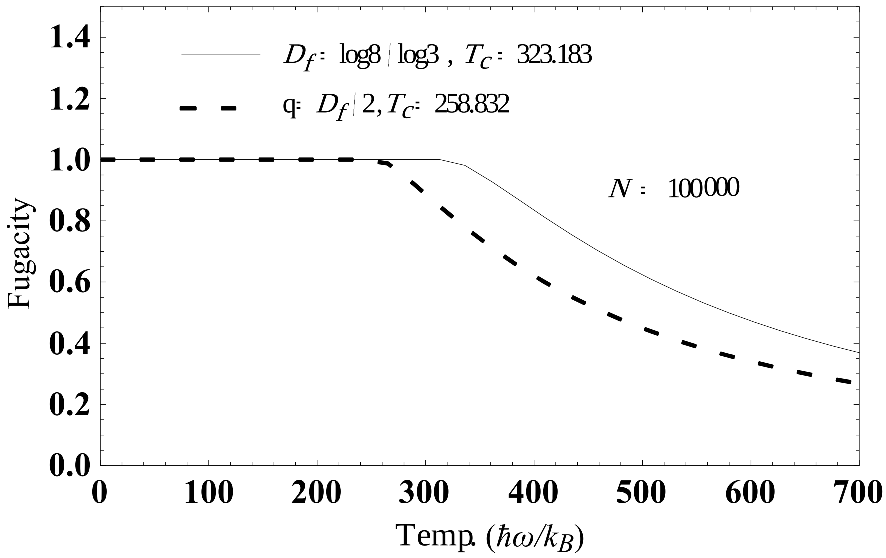


Fig. (4.12): Temperature dependence of the fugacity for harmonically trapped extensive and nonextensive bosons in the Sierpinski carpet.

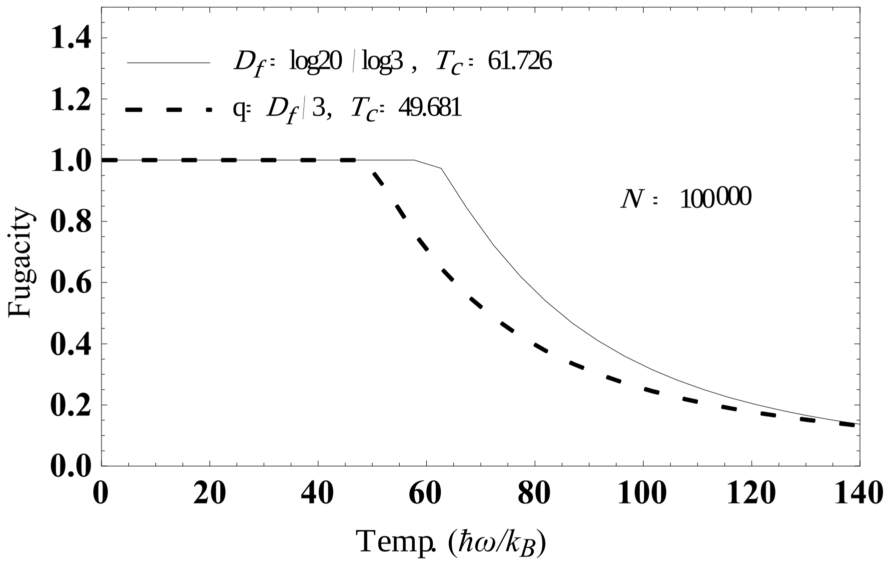


Fig. (4.13): Temperature dependence of the fugacity for harmonically trapped extensive and nonextensive bosons in the Menger sponge.

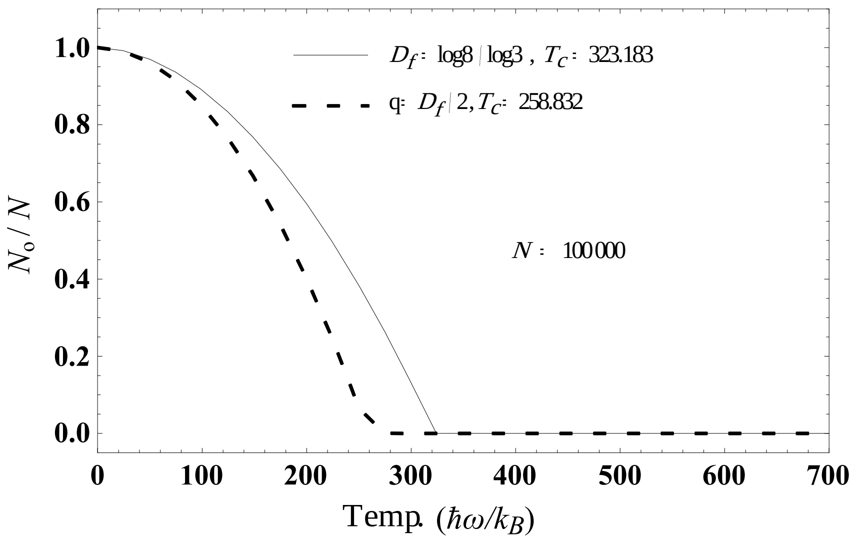


Fig. (4.14): Temperature dependence of the condensate fractions for harmonically trapped extensive and nonextensive bosons in the Sierpinski carpet.

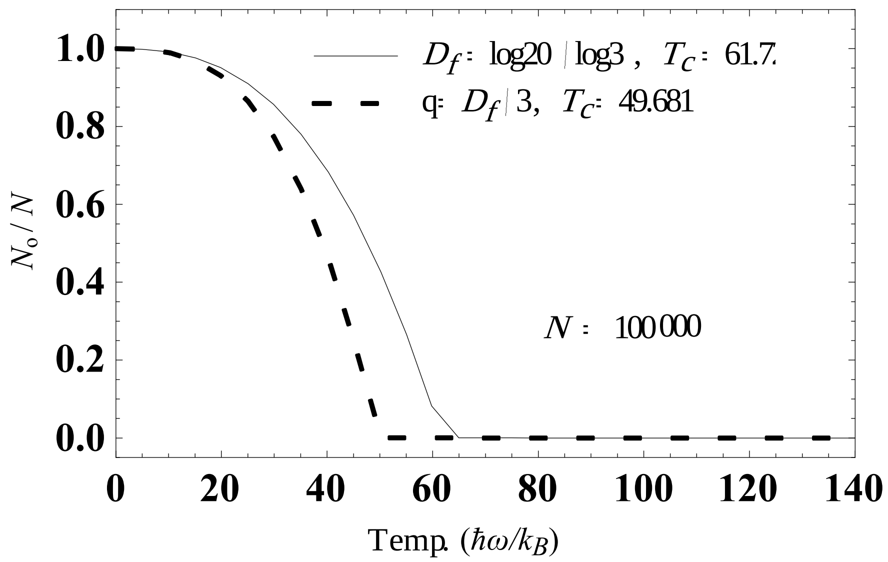


Fig. (4.15): Temperature dependence of the condensate fractions for harmonically trapped extensive and nonextensive bosons in the Menger sponget.

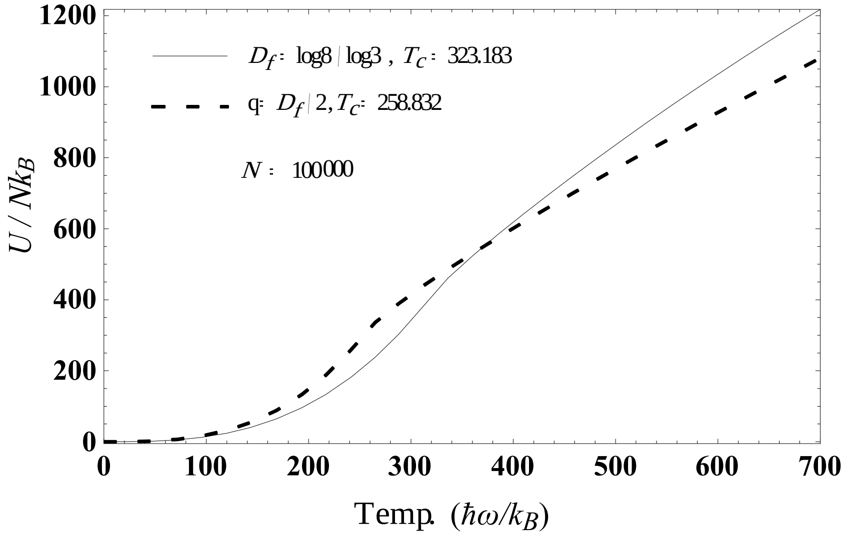


Fig. (4.16): Temperature dependence of the internal energy for harmonically trapped extensive and nonextensive bosons in the Sierpinski carpet.

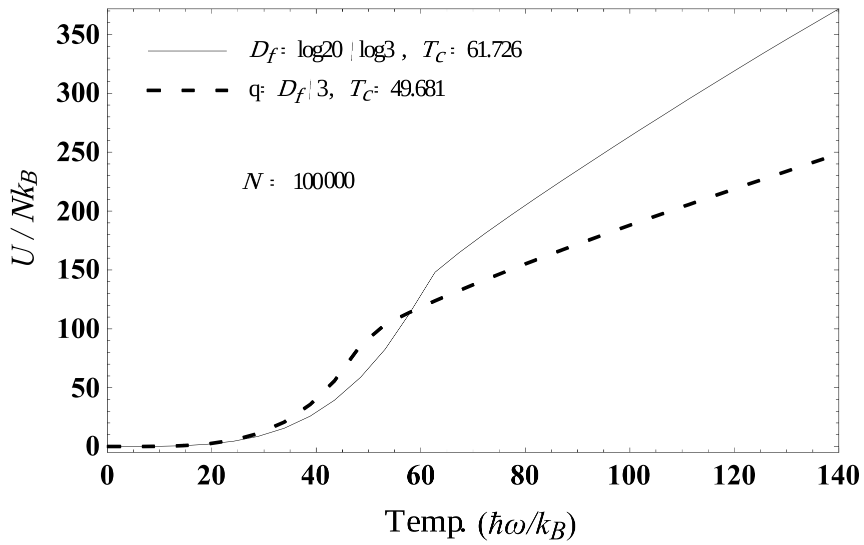


Fig. (4.17): Temperature dependence of the internal energy for harmonically trapped extensive and nonextensive bosons in the Menger sponge.

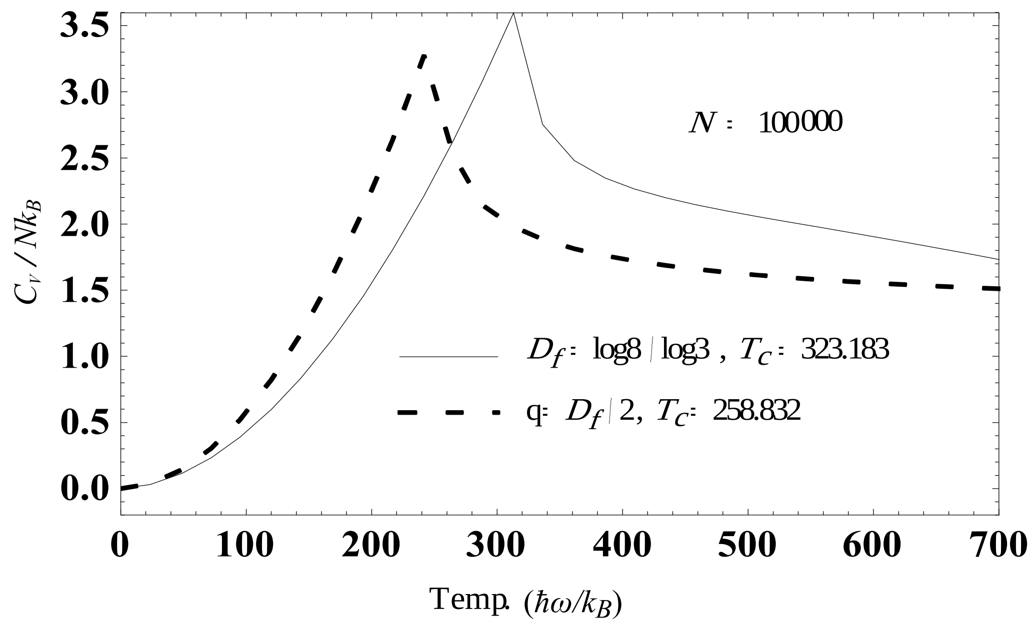


Fig. (4.18): Temperature dependence of the heat capacity for harmonically trapped extensive and nonextensive bosons in the Sierpinski carpet.

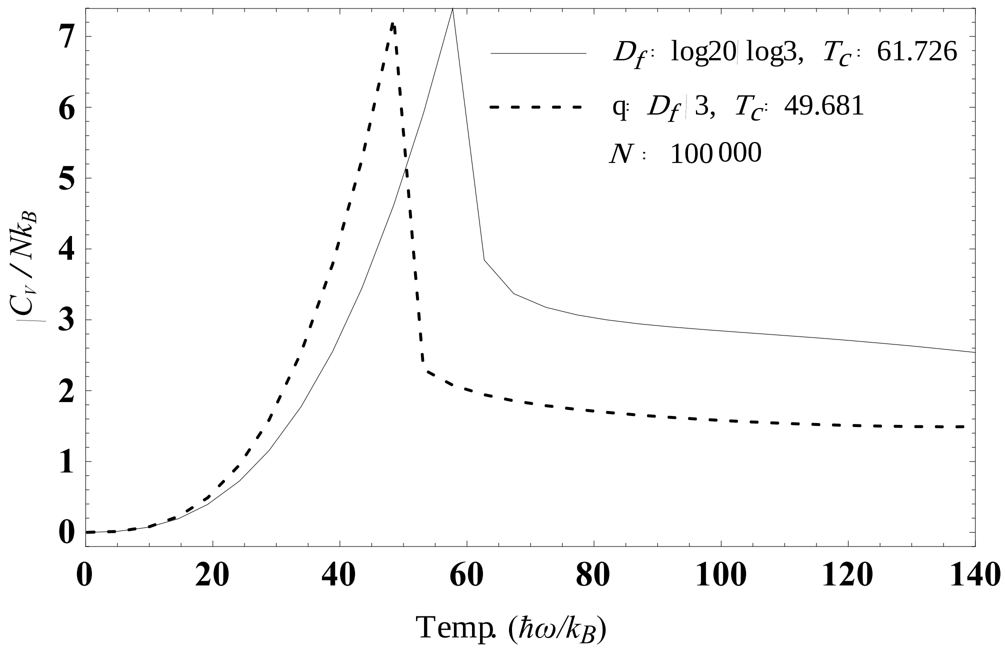


Fig. (4.19): Temperature dependence of the heat capacity for harmonically trapped extensive and nonextensive bosonsin the Menger sponge.

These comparisons, Figures from (4.12) to (4.19), were carried out for total number of particles $N = 100000$. These figures assert the two main results concerning the condensation temperature and the thermodynamic properties observed in § (4.13) with arbitrarily selected fractal dimensions (2.7, 2.8 and 2.9). Fig. (4.20) compares the condensation temperatures obtained by the extensive and the nonextensive thermostatistics in the Sierpinski carpet and the Menger sponge for varying number of particles (bosons). The two upper curves (with "SC" super script) belong to the Sierpinski carpet, while the two lower curves (with "MS" super script) belong to the Menger sponge. This figure also indicates that the condensation temperature in extensive thermostatistics model is always higher than that in the nonextensive one. Finally, from the thermodynamic behavior of the BEC phenomenon in the present section, and the thermodynamic behavior of the arbitrarily selected fractals, the results of § (4.13), it can be concluded that these two models (the extensive and the nonextensive) are adequate to represent the BEC phenomenon in fractal media.

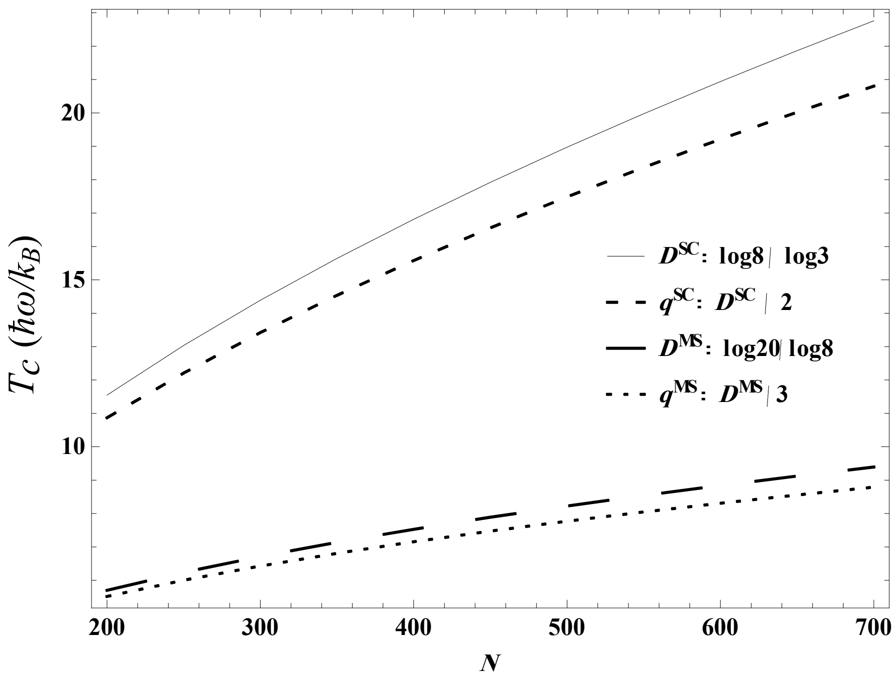


Fig. (4.20): The condensation temperature for harmonically trapped extensive and nonexrensive bosons as a function of varying total number of particles.

Chapter Five

Conclusions and Suggestions for Further Work

5.1 Conclusions

1. A theoretical quantum statistical mechanics model for the BEC phenomenon in fractal media is formulated. This model is based upon the generalized Tsallis (nonextensive) thermostatistics. The thermodynamic behavior computed on the basis of this model (the temperature dependence of the condensate thermodynamic properties) is also investigated.
2. In addition to this nonextensive model, another model for bosons harmonically trapped in a fractal medium based on the extensive thermostatistics is also adopted and its thermodynamic behavior is investigated.
3. The thermodynamic behaviors of these two models (the extensive and the nonextensive) are compared using the symbolic *MATHEMATICA®* computational scheme. The comparisons were carried out for arbitrarily selected fractal dimensions and also for two standard fractals; the Sierpinski carpet and the Menger sponge. One of the standard fractals the Sierpinski carpet is embedded in Euclidean space of 2D and the other (the Menger sponge) is embedded in Euclidean space of 3D. The comparisons lead to the following observations:

- The condensation temperature within the Tsallis thermostatistics is always less than the corresponding one in the Boltzmann–Gibbs thermostatistics. Consequently, the condensate thermodynamic properties in the Tsallis thermostatistics are always shifted to lower temperatures. The disagreement in the condensation temperature between the two thermostatistics is also justified. The justification is ascribed to the Tsallis cut–off condition for $q < 1$ when representing a fractal structures.

- Despite the fact that the thermodynamic behaviors of the two thermostatistics are, in general, similar, the condensate thermodynamic properties seem to possess different temperature responses (slopes).

4. The use of *MATHEMATICA®* software package as a computing environment, in this thesis, has shown an important utility. This utility is represented mainly in its feature of symbolic computation for computing the sums over their terms; i.e. evaluating the condensate properties by computing the sums over the energy states. In addition to the utility of using *MATHEMATICA®* for carrying out the aforementioned investigations, the use of *MATHEMATICA®* has disclosed an important observation from which significant results are obtained. This observation is related to the essential difference between computing quantities by using integrals or by using sums; i.e., the problem of finite number of particles. This observation explicitly appears in chapter three when investigating the accurate condensation temperature correction order for the case of ideal Bose gases trapped in a 3D isotropic harmonic potentials. The latter investigation reveals significant results:

- The first is that the first order correction is the upper bound correction for the case indicated above. This result is justified for the excellent agreement between the numerically (by symbolic computation) obtained relative correction and the first order relative correction $(\Delta T_c / T_0^{3D}) = -0.7275\,(\omega_m / \overline{\omega})\, N^{-1/3}$. Accordingly, the evaluation of the thermodynamic properties of an ideal Bose gas confined in a 3D harmonic trap on the basis of the first order correction is more accurate than the properties evaluated by any other correction order. Hence, the excellent agreement of the first order correction with the obtained results (by symbolic computation) for atoms number $100 < N < \infty$ exhibits the extent of precision in the density of states used in Refs. [25,36]. The departure of the first order relative correction from the obtained (by symbolic computation) one when $N < 100$ uncovers the fundamental difference between the use of discrete sums and continuous spectrum together with a correctly approximated density of states.
- The other two results are numerical corrections to analytical expressions (equations) which correspond to the condensation temperature in 2D and in 1D harmonic potentials. The most interesting observation in the result of these two cases is that in 1D harmonic oscillator, the

condensation temperature is significantly higher than in previous predictions [25,39].

5. In view of the rapid development in nanotechnology and the possibility of constructing complex structures or networks [127–131], confining few dozens of bosons becomes plausible. This, in turn, emphasizes the necessity and the utility for using the symbolic computation in statistical treatment of finite number of particles.
6. Taking into account effects such as the size and the shape of the container and boundary conditions, which are relevant to the finiteness of the Bose system size, on the properties of non–interacting bosons, the interaction influence may be precisely understood.
7. A final comment on the use of symbolic computation is in order here. This work may reveal its role and its importance in different statistical mechanics issues not only those related to the BEC phenomenon but also in other cases with finite number of particles. Hence, the symbolic computation method, employed in this thesis, warrants extension to the domain of other statistical mechanics issues not only related to the BEC phenomenon but others that are affected by the finiteness of the number of particles.

5.2 Suggestions for Further work

A number of suggestions for further work related to the theoretical and computational aspects dealt with in this thesis can be enlisted here:

1. The case of rotating bosons confined in fractal geometries, rotation is highly recommended as a further area of application for its relevant connection with experimental findings in superfluids. It is important to comment here that formulating such a model for this aspect seems possible based on the Tsallis statistics but not upon Rovenchack's idea. This is because the latter model is restricted to the case of an isotropic potential while rotation leads to anisotropy. Furthermore, the model based on the Tsallis statistics is not only applicable for bosons in fractal media under external potentials but also to the case of free bosons in such fractal media. On the other side, the model based on Rovenchack's idea is not applicable to the case of free bosons.

2. The case of q–deformed bosons, referred to in section § (4.3), is also a promising approach for treating the BEC phenomenon in fractal media. Furthermore, it is applicable to the cases of rotating bosons, free bosons, and bosons trapped in a potential as well. It is suggested that further work can be done in this direction. However, a drawback in this approach which emerged in a preliminary investigation is absence of a direct and well–defined relation between the fractal dimension and the index of q–deformation, which is not the case for the Tsallis statistics.
3. It is important to indicate here that the models formulated in the present work, and also the approaches reviewed in § (4.3), are for treating ideal bosons in fractal media. But to treat real systems the interaction should be introduced. To introduce the effect of interaction simultaneously with the effect of fractality, it is suggested to use a two–parameter q–deformation model; which is an extension to the q–deformed quantum algebra since it involves two parameters rather than one parameter. In this case, the interaction can be treated as an additional deformation to the case of ideal bosons. The treatment of BEC as a general case with two–parameter deformation, but not taking into account either the fractality or the interaction, is found in work of A. Algin and B Deviren [140].
4. The two models formulated in the present work, and also the approaches reviewed in § (4.3), for treating bosons in fractal media, are indeed phenomenological approaches. Most recently, a remarkable rigorous mathematical treatment of the BEC phenomenon in fractal media has been suggested by Chen [139]. In the latter work, spectral geometry analysis based on a rigorous mathematical formulation was used to formulate a quantum statistical mechanics of the BEC phenomenon. In this approach, the fractality of the medium confining the bosons is the starting point. The treatments of interacting bosons and massless bosons are also involved. It is also worth mentioning that the latter work is devoted to extensive bosons only. It is suggested that some of the ideas adopted in the present thesis, such as the symbolic computation method and the nonextensivity extension, may find some applications here as well.

Appendices

Appendix A

The Bose Function

The general form of the Bose function is [141]:

$$B_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \nu \frac{x^{\nu-1} dx}{z^{-1} e^x} \tag{A.1}$$

where $0 \le z \le 1$, $\nu \in R$, and Γ(ν) is the gamma function. For small values of *z*, the integrand can be expanded as:

$$\frac{1}{z^{-1}e^x - 1} = ze^{-x} \frac{1}{1 - z^{-1}e^x} = ze^{-x} \sum_{k=0}^{\infty} (ze^{-x})^k$$

$$= \sum_{k=1}^{\infty} z^k e^{-kx} \tag{A.2}$$

Using equation (A.2) in equation (A.1) yields:

$$B_\nu(z) = \frac{1}{\Gamma(\nu)} \sum_{k=1}^{\infty} z^k \int_0^\infty x^{\nu-1} e^{-kx} dx \tag{A.3}$$

Changing the variable, *kx*=*y*; equation (A.3) becomes:

$$B_\nu(z) = \frac{1}{\Gamma(\nu)} \sum_{k=1}^{\infty} \frac{z^k}{k^\nu} \int_0^\infty y^{\nu-1} e^{-y} dy \tag{A.4}$$

The last integral is the definition of the gamma function Γ(ν), and equation (A.4) becomes

$$B_\nu(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\nu} \quad , \quad 0 \le z \le 1 \tag{A.5}$$

When z=1 (μ=0) , equation (A.5) leads to the Riemann zeta function which is defined as:

$$\zeta(\nu) = B_\nu(1) = \sum_{k=1}^{\infty} \frac{1}{k^\nu} \quad , \quad \nu > 1 \tag{A.6}$$

The integral form of (A.6) is given by [145]:

$$\zeta(\nu) = B_\nu(1) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} dx}{e^x - 1} \quad , \quad \nu > 1 \tag{A.7}$$

Both (A.6) and (A.7) are finite (converging) for ν>1 for all $0 \le z \le 1$.
Some special values of the Riemann zeta functions are given below:

$\zeta(1) \to \infty$	$\zeta(3/2) = 2.612$	$\zeta(2) = 1.645$
$\zeta(5/2) = 1.341$	$\zeta(3) = 1.202$	$\zeta(7/2) = 1.127$

Appendix B

Evaluation of the Limit Related to the Occupation Number of the Ground State in the Nonextensive Model

It is required to prove that $\underset{q\to 1}{Lim}\ \dfrac{1}{\left[1+(1-q)\ln z\right]^{1/(q-1)}-1}=\dfrac{z}{1-z}$

Let $q-1=b$, then $q\to 1 \Leftrightarrow\ b\to 0$,

$$\therefore\ \underset{q\to 1}{Lim}\ \frac{1}{\left[1+(1-q)\ln z\right]^{1/q-1}-1}=\underset{b\to 0}{Lim}\frac{1}{(1-b\ln z)^{1/b}-1} \qquad \text{(B1)}$$

Define $b\ \ln z=\dfrac{1}{x}$, then *as* $b\to 0 \Leftrightarrow\ x\to\infty$, and

$$\underset{b\to 0}{Lim}\frac{1}{(1-b\ln z)^{1/b}-1}=\underset{x\to\infty}{Lim}\frac{1}{\left(1-\frac{1}{x}\right)^{x\ln z}-1}=\underset{x\to\infty}{Lim}\frac{1}{\left(\left(1-\frac{1}{x}\right)^{x}\right)^{\ln z}-1} \qquad \text{(B2)}$$

Now, it is required to determine

$$\underset{x\to\infty}{Lim}\left(1-\frac{1}{x}\right)^{x} \qquad \text{(B3)}\ .$$

To find this limit, define $t=\dfrac{1}{x}$, then, $x\to\infty \Leftrightarrow \mathrm{t}\to 0$.

One also notices that

$$\underset{x\to\infty}{Lim}\ln\left(1-\frac{1}{x}\right)^{x}=\underset{t\to 0}{Lim}\ln(1-t)^{1/t}=\underset{t\to 0}{Lim}\frac{\ln(1-t)}{t} \qquad \text{(B4)}$$

which can be found by L' Hospital's rule as

$$\underset{t\to 0}{Lim}\frac{\ln(1-t)}{t}=-\underset{t\to 1}{Lim}\frac{1}{1-t}=-1 \qquad \text{(B5)}$$

It is also noted that

$$f(x)\to 0 \Leftrightarrow \ln(f(x))\to\infty;$$
$$f(x)\to 1 \Leftrightarrow \ln(f(x))\to 0;$$
$$f(x)\to e \Leftrightarrow \ln(f(x))\to 1;$$

Hence, by mathematical induction, $\ln(f(x))\to -1 \Leftrightarrow f(x)\to e^{-1}$

Using this result in Eqn. (B4), yields

$$\underset{x\to\infty}{Lim}\left(1-\frac{1}{x}\right)^{x}=e^{-1}$$

Hence, $\underset{q\to 1}{Lim}\ \dfrac{1}{\left[1+(1-q)\ln z\right]^{1/(q-1)}-1}=\dfrac{1}{\left(e^{-1}\right)^{\ln z}-1}=\dfrac{z}{1-z}$.

Appendix C

Abstracts of Papers Submitted for Publication

Condensation Temperature in Harmonic Traps for Ideal Bose Gas with Finite Number of Particles*

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Department of Physics, College of Science, Al-Nahrain University, Baghdad, IRAQ.

Abstract

For the case of ideal Bose gas harmonically trapped in a 3D potential, three correction orders for an accurate condensation temperature determination are numerically investigated. This investigation reveals that the corrections up to $N^{-1/2}$ and up to $N^{-2/3}$ do not give a better agreement with the (numerically obtained) accurate result. Instead, the first order correction (the lowest order) which is proportional to $N^{-1/3}$, yields results closest to the numerically accurate result. This observation is obtained by using the accurate statistical treatment for computing the sum over the energy states. This treatment is also extended to the cases of an ideal Bose gas harmonically trapped in 2D potential and 1D potential in order to evaluate the accurate condensation temperature for these Bose systems. In the case of 1D harmonic potential, the numerically accurate condensation temperature evaluated in this work uncovers that the condensation temperature is higher than previous predictions.

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*Submitted for publication to the 1st. International Conference on Physics for Sustainable Development, Al-Nahrain University, Baghdad, Oct. 2014.

Bose-Einstein Condensation in Fractal Traps: A Comparison between Extensive and Non-extensive Thermostatistics*

Ibrahim A. Sadiq, M.A.Z Habeeb and Ayad A. Al-Ani,
Department of Physics, College of Science, Al-Nahrain University, Baghdad, IRAQ.

Abstract

A quantum statistical mechanics model for the BEC phenomenon in fractal media is formulated. This model is based upon the generalized nonextensive Tsallis thermostatistics. The thermodynamic behavior this formulation (the temperature dependence of the condensate thermodynamic properties) is also investigated. In addition to the formulated nonextensive model, a model for bosons harmonically trapped in fractal media based on the extensive thermostatistics is also adopted and its thermodynamic behavior is investigated.

The thermodynamic behaviors of the two models (the extensive and the nonextensive) are compared. The comparisons are carried out for two standard fractals (the Sierpinski carpet and the Menger sponge). One of them is topologically embedded in 2D and the other is embedded in 3D. The comparisons reveal that the condensation temperature within the Tsallis (nonextensive) thermostatistics is always less than the corresponding one in the Boltzmann–Gibbs (extensive) thermostatistics. Consequently, the condensate thermodynamic properties in Tsallis thermostatistics are always shifted to lower temperatures. These comparisons also show that despite the thermodynamic behaviors for the two thermostatistics are, in general, similar; the condensate thermodynamic properties seem to possess different temperature responses (slopes). The disagreement in the condensation temperature between the two thermostatistics is also justified.

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*Submitted for publication to the 1st. International Conference on Physics for Sustainable Development, Al-Nahrain University, Baghdad, Oct. 2014.

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المُـلخَص

لَـقدْ أضحَتْ ظاهِرةُ تكاثـُف بـوز- آيـنشتايـن(BEC)، التي تنبـأ بها آينشتاين عام 1925 والتي تحقّقـتْ عَملياً في عام 1995، موضُوعاً للكثـيرِ مِن البحوثِ فـي العقودِ الأخيرة. مِن الجانب النـّـظري، صِـيغَتْ العديـدُ مِـن الأسـاليب لِدراسـة هـذهِ الظاهــرة. أحـدُ أهـمِّ المواضـيعِ التـي تـتعرّضُ لهـا تلـكَ الدراسـات هِـيَ الظـروفُ التـي وِفـقها تَـتحقّـقُ هـذهِ الظـّاهِـرة. مِـن بيـن العوامِــلِ المعروفَــة والتـــي تُـــؤثِرُ فـــي تَحقيقِهـــا هـــو مِقـــدارُ بُعـــدِ الفـــضاء المكـــاني (dimensionality) لِـــلوسطِ الحاضِــن لجُـسيماتِ بــوز (البوزونــات). لَـــقدْ أشــارَتْ نتــائجُ الدراساتِ السّابقة، في هذا الخصوص، إلى إمكانيةِ حُـدوثِ هـذهِ الظاهِـرة في فضاءٍ مكانـيٍ ذيِ ثــلاثةِ أبعـادٍ أو آخـرَ ذي بـعُـدين ولكـن تحـت مـدىً أوسـع مِـن الشّــروطِ الواجِـب تَحقـقُها. فـي الدراساتِ النَّـظرية، لِكـي تتحقق هـذهِ الظاهِـرة في فضـاءٍ مكاني ذي بُعدٍ واحدٍ فإنَّـهُ يتوجَّبُ تَحقّقُ المزيدِ من الشروط الصارِمة ومِنها وجوبُ إستخدامُ المُعالجةِ الخاصة بالتـعامُلِ مـعَ عددٍ محدودٍ مِن الجسيمات. مُــؤخراً، شَهِـدَتْ حالـةُ حُـدوثِ ظاهِـرة تكاثـُـف بـوز- آيـنشتاين فـي الأوسـاطِ ذات البُعـدِ الفضـائي الكِـسوري المزيـدَ مِـن الإهتِـمام لِـسببــين رئيسَيــن. الأولُ هـوَ الأكتشــافاتُ المُختبريِّــة التــي أكـدّتْ الطـّـبيعة الكِسـورية للفضــاءِ المكـاني لأوسـاطٍ حاضـنةٍ للبوزونات. السببُ الثـاني هو ظـُهورُ الهندسـةِ الكِسورية كمجـالِ بحثٍ لأحد فـروعِ المعرفـة، حيث كان إنبثـاقُ الهندسةِ الكسورية متـزامناً مـع الإكتشـافات المُختبريِّـة المذكـورةِ أعلاه. معَ ذلِـك، فأنَّ الصِياغَة النـّظـرية لنماذجٍ كسورية لِظاهـرِة تكاثـُف بوز-آيـنشتاين لَـمّــا تــزلْ فـي مراحِلِها المُبكِّرة.

إنَّ العمـلَ الحـالي مكرسٌ ، بشـكلٍ أساسـي، لِصِـياغَةِ نَمُوذَجٍ نـَظري يُحـاكِي مِثـلَ هـذهِ الحــالات وكـذلِكَ لِأستقصــاءِ السـلوك الثـرموديـناميـكي لِهـذا النـّموذج مِـن خِـلال الاحتسـاب الرمزي (symbolic computation) الّذي تُوفرُهُ حُزمَةُ برامجيات *MATHEMATICA*®. إنَّ النَّموذجَ المُراد صياغَتُهُ لِلعمل الحـالي يَفترِضُ وجودَ عددٍ محدودٍ من البوزونـات المثاليـة (ideal) عالِـقةٌ في مِصيَدةِ مُتذبذبٍ توافقي(harmonic trap) داخِلَ وسَطٍ ذي بُعدٍ كسوري، و كَذلِكَ يَفترِضُ النـّموذج الخضـوعَ لميكانيكيـةِ (grand canonical ensemble) الإحصـائية. لقد تمَّ إدخالُ الطبعيةِ الكسورية لِـلأوساطِ الحاضِنةِ للـبوزونات بأستخدامِ طريقتـين متمايزتيـن . الأولـى هي التي تـتبنّى فِكرةَ Rovenchack، والتـي تَـفـترِضُ إمكانيـة إستخدام عوامِل تحلل مستويات الطاقة (energy levels degeneracy factors) لِتشمِل الأبعادِ الكسورية مِن خِلالِ تَحويل الـدَّوال ذات العوامل المضـروبة (factorial functions)، والتـي تِظهرُ فـي عوامِلِ تحلل مُستويات الطـّاقة، الـى دوالِّ گامـا (gamma functions). الطريقـةُ الثـانـيةُ هـي التـي تستخدِمُ إحصاءَ Tsallis، حيث يكون مُـؤشرَ هذا الأحصاءِ (index of nonextensivity) ذو صلةٍ بالبعدِ الكسوري. مِن المُهمِ الأشـارةُ اليـهِ، فـي هذا المقام، إنَّ كِـلتي الطريقتـين تختـزلُ، أي تعـودُ، الــى الحالــةِ القياسـيـة عِنـدَ التعامُـلِ مـعَ أبعــادِ الفضــاءِ المكــاني الصـحيحةِ العـدد (1D و2D و3D).

لِغــرض إختبــار الإطــار البرمجـــي للحســـابِ الرمـــزي لِحُزَمـــةِ برامجيـــات MATHEMATICA®، تمَّ إحتسابُ الخصـائصِ الثـرموديـناميـكة لمُتكاثـِف بـوز- آيـنشتاين في أبعاد الفضاء المكاني الصحيحةِ العدد (1D و2D و3D) وعلى أساس ذات الأفتراضات المذكورة سابقا.

لقد أثـبتـتْ هذهِ الأختبارات جَودةِ النتـائج الحسابية وإتفاقِها معَ سابِقاتِها مِما شجَّعَ علـى إجـراءِ هـذهِ الحسـابات باستخدام النمـوذجين لبـــوزونات عالِــقةٍ فـي مَصـائدَ مُتذبـذبٍ تـوافقي (harmonic traps) داخِل أوساطٍ لها أبعادٌ كِسورية ، حيث تكون هذهِ الأوساط قابِعةُ داخِل فضـاءاتٍ مكانيـةٍ ذات أبعـادٍ صـحيحةِ العـدد (2Dو3D). و مـن خـلال المقارنـة بـين النتـائج، بشكلٍ عـام، فقد تبيَّنَ أنَّ درجـةَ الحـرارةِ الحرجـة للتكاثـُف فـي النمـوذج المبنـي علـى إحصـاءِ Tsallis تكونُ أوطَـأ مِن تِلكَ التي تنتُجُ عن النمــوذجِ المبنـي علـى فِكرةَ Rovenchack والـذي ينتمي لِإحصاء Boltzmann-Gibbs. لقد لـُوحِظَ أنَّ هذهِ النتيجة تتـَّفِقُ معَ Salasnich ونتـائج باحثـين آخرين في هذا المجال.

يُستَـنـتــجُ مِـن ذلـك إنَّ النمـوذجـين االــذين تعرضـهما هـذهِ الاطروحـة وكـذلك المـنهج المقترحَ لإستخدام الأحتساب الرمزي، يمكن أسـتخدامها بنجـاحٍ للتعامُلِ معَ ظاهِـرةِ تـكاثـُف بوز- آيـنشتاين في الأوساطِ الكسورية وبشكلٍ قابلٍ للتـطـويــر.

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الميكانيك الإحصائي والخصائص الثرموديناميكية لتكاثف بوز- آينشتاين فيالوساط الكسورية

أطروحة

مقدمة إلى كلية العلوم / جامعة النهرين

وهي جزء من متطلبات نيل درجة الدكتوراه في الفيزياء

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