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Optimality Conditions for Fuzzy Order Variational Problems

A Thesis

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سُورَةُ الفَاتِحَةِ

بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ ١
ٱلۡحَمۡدُ لِلَّهِ رَبِّ ٱلۡعَٰلَمِينَ ٢
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وَلَا ٱلضَّآلِّينَ ٧

Dedication

To my family with all love and respects

Basma

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Praise to Allah the lord of the worlds and peace and blessings be upon the master of human kind Muhammad and his pure progeny and his relatives and may God curse their enemies until the day of Judgment.

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Basma

October, 2014

Supervisors Certification

We certify that this thesis, entitled " ***Optimality Conditions for Fuzzy Order Variational Problems***" was prepared under our supervision at the Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University as a partial fulfillment of the requirements for the degree of master of science in Mathematics.

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SUMMARY

The main objectives of this thesis is oriented toward three directions: The first objective is to study fuzzy set theory with some basic properties related to the theory of variational problems.

The second objective is to study variational problems with fuzzy functions, fuzzy condition and fuzzy boundaries by using different approaches for defuzzification, such as centroid method, α-cut method, centroid point and expected interval in which fuzzy sets have been transformed into crisp sets.

The third objective is to find the necessary conditions for extremizing the fuzzy variational problems with fuzzy function and fuzzy boundaries.

LIST OF SYMBOLS

$A, B, \ldots$	Ordinary (or nonfuzzy) sets.
A_α	α-cut (α-Level) set of a fuzzy set $\tilde{A}$, $\alpha \in (0, 1]$.
$\tilde{A}, \tilde{B}, \ldots$	Fuzzy sets.
FVP	Fuzzy variational problem.
$hgt\ (\tilde{A})$	Height of a fuzzy set $\tilde{A}$.
O	Order.
R	Set of real numbers.
R_F	Set of all fuzzy numbers.
$S(\tilde{A}), Supp(\tilde{A})$	The support of a fuzzy set $\tilde{A}$.
$\tilde{x}, \tilde{y}, \ldots$	Fuzzy numbers.
$\mu_{\tilde{A}}, \mu_{\tilde{B}}, \ldots$	The membership functions of the fuzzy sets $\tilde{A}, \tilde{B}, \ldots$
$\tilde{\emptyset}$	The empty fuzzy set.
$*$	Binary operation.

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INTRODUCTION

In the basic sciences, such as engineering, chemistry or physics, we construct exact mathematical models of empirical phenomena, and then these models are used to make predictions. While some aspects of the real world problems always escape from such precise mathematical models and usually there is an elusive inexactness as a part of the original model. Also, the elements of the real world problems are perturbed by imperfection and thus, for example, there exists no elements that are perfectly round. Perfect notations or exact concepts correspond to the sort of things envisaged in pure mathematics, while inexact structures encounter us in real life problems, [2].

Moreover, everyday life, so many properties which can not be deals with satisfactory on a simple “belong” or “not belong” basis are used. Whether these properties perhaps best indicated by a shade of gray, rather than by the black or white. Assigning each individual in a population on a “belong” or “not belong” value, as is done in ordinary set theory is not an adequate way for dealing with properties of this type, [12].

Historically, the accepted birth date of the theory of fuzzy sets returns to 1965, when the first article entitled “fuzzy sets” submitted by Zadeh appeared in the journal of information and control. Also, the term “fuzzy” was introduced and coined by Zadeh for the first time. In which the original definition of fuzzy sets is to consider a class of objects with a continuum grades of membership, such a set is characterized by a membership (or characteristic) function which assigns to each object a grade of membership value ranging between zero and one. As the membership value approaches unity, the grade of membership of an event in the fuzzy set becomes higher. For example, the unit membership value indicates that the event x is strictly contained in the fuzzy set, and on the other hand, the zero membership value indicate strictly that x is strictly does not belong to the fuzzy set. Any intermediate value would reflects the degree on which x could be a member of the fuzzy set, [2].

In addition, the history of the calculus of variation is tightly interwoven with the history of mathematics. The field has drawn the attention of a remarkable range of mathematical luminaries, beginning with Newton, then initiated as a subject in its own right by the Bernoulli family. The first major developments appeared in the work of Euler, Lagrange and Laplace. In the nineteenth century, Hamilton, Dirichlet and Hilbert are among the outstanding contributors. In modern times, the subject of calculus of variations has continued to occupy center stage, witnessing major theoretical advances, along with wide-ranging applications in physics, engineering and all branches of mathematics, [22].

Calculus of variation is a branch of mathematics dealing with the optimization of physical quantities (such as time, area, or distance). It finds applications in many fields, such as aeronautics (maximizing the lift of an airplane wing), sporting equipment design (minimizing air resistance on a bicycle helmet, optimizing the shape of a ski), mechanical engineering (maximizing the strength of a column, a dam, or an arch), boat design (optimizing the shape of a boat hull), physics (calculating trajectories and geodesics in both classical mechanics and general relativity), [14].

A huge amount of problems in the calculus of variations have their origin in physics where one has to minimize the energy associated to the problem under consideration. Nowadays, many problems come from economics. Here is the main point that the resources are restricted. There is no economy without restricted resources, [22].

Minimization principles form one of the most wide-ranging means of formulating mathematical models governing the equilibrium configurations of physical systems. Moreover, many popular numerical integration schemes such as the powerful finite element method are also founded upon a minimization paradigm, [21].

The main objective of this thesis is to study fuzzy variational problem which are generalized from classical variational problem that are obtained by replacing real initial conditions and real boundaries by fuzzy ones.

This thesis consists of three chapters. In chapter one, some fundamental concepts of fuzzy set theory and variational problem are considered, in which this chapter consists of three sections. In the second section, some basic definitions, theorems and algebraic properties of fuzzy set theory are given with some illustrative examples. In the third section, basic definitions, theorems and algebraic properties of variational problems are also given for completeness purpose of this work.

In chapter two, variational problems with fuzzy function and variational problem with fuzzy boundary conditions are investigated. We will discuss the derivation of Euler-Lagrange equation of unconstrained variational problems with fuzzy function, and we will discuss the derivation of Euler-Lagrange equation of constrained variational problems with fuzzy function.

In chapter three, variational problems with fuzzy boundaries is investigated where in second section the Euler-Lagrange equation of such type of problems have been studied. In third section the centroid method for defuzzification have been discussed. In forth section the expected interval method is also discussed. In fifth section the centroid point method for defuzzification is introduced explained discussed with an illustrative example.

CHAPTER ONE

Basic Concepts of Fuzzy Set Theory and Variational Problems

1.1 Introduction:

In this chapter, the basic concepts, definitions and theorems related to fuzzy set theory and variational problems will be introduced, with some illustrative examples. These concepts includes for fuzzy set theory, the α-level sets, the extension principle, fuzzy relation and fuzzy functions which for variational problems: variational problems with simple form and Euler-Lagrange equation.

1.2 ***Fuzzy Sets Theory:***

Fuzzy set theory is a generalization of abstract set theory; it has a wider scope of applicability than abstract set theory for solving problems that involve to some degree subjective evaluation [2].

***Definition* (1.1), [6]:**

Let X be a classical set of objects, called the universal set, whose generic elements are denoted by x. The membership in a classical subset A of X is often viewed as a characteristic function μ_A from X into {0, 1}, such that:

$$\mu_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

{0, 1} is called a valuation set. If the valuation set is allowed to be the real interval [0, 1], then A is called a fuzzy set (which is denoted by $\tilde{A}$), and $\mu_{\tilde{A}}(x)$ is the grade of membership of x in $\tilde{A}$.

***Remarks* (1.1), [5]:**

1. Let X be a finite set, a fuzzy set on X is expressed as:

$$\tilde{A} = \mu_{\tilde{A}}(x_1)|x_1 + \mu_{\tilde{A}}(x_2)|x_2 + \ldots + \mu_{\tilde{A}}(x_n)|x_n$$

$$= \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)\Big|x_i$$

When X is infinite, we write:

$$\tilde{A} = \mu_{\tilde{A}}(x_1)|x_1 + \mu_{\tilde{A}}(x_2)|x_2 + \ldots$$

$$= \int_X \mu_{\tilde{A}}(x)\,|\,x$$

or:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X, \mu_{\tilde{A}}(x) \in [0, 1]\}$$

where the slash (|) is employed link the elements of the support with their grades of membership in $\tilde{A}$, and the plus sign (+) or the integral ($\int$) playing the role of "union" rather than arithmetic sum of integral [9].

2. The difference between crisp and fuzzy sets is that the former always have a unique membership, while every fuzzy sets have an infinite number of memberships that may represented them.

3. Functions that maps X into the unit interval may be fuzzy sets, but they become fuzzy set when, and only when, they match some intuitively plausible semantic description of imprecise properties of the objects in X.

   The following example illustrates this remark:

***Example* (1.1), [6]:**

Let:

$X = \{0, 1, 2, 3, 4, 5, 6\}$

be the set of number of children in a family may be choosed to have the fuzzy set:

$\tilde{B}$ = "desirable number of children in a family"

which may be described as follows:

$$\tilde{B} = \{(x, \mu_{\tilde{B}}(x)) \mid x \in X\}$$

and implies that:

$$\tilde{B} = \{(0, 0.1), (1, 0.3), (2, 0.7), (3, 1), (4, 0.7), (5, 0.3), (6, 0.1)\}$$

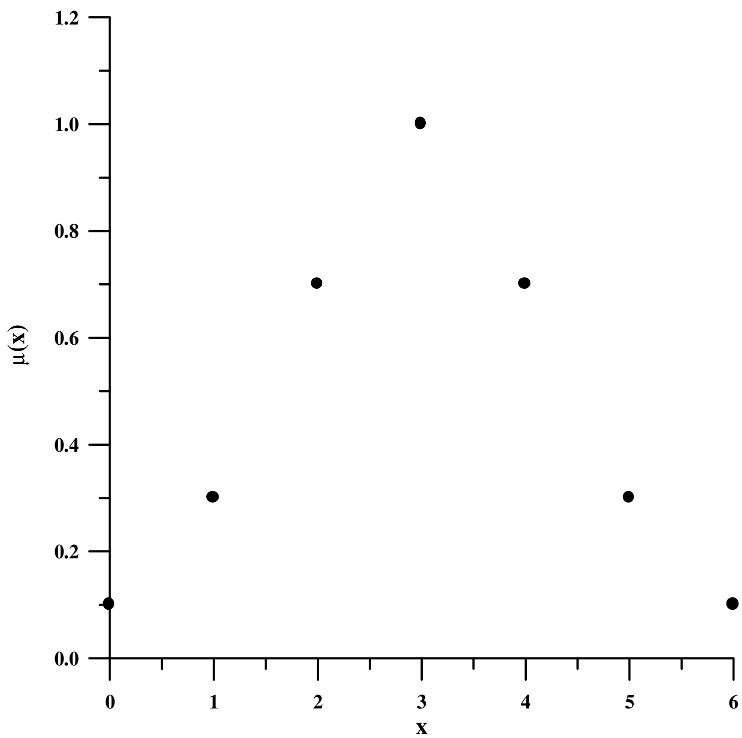

Figure (1.1) Membership function of example (1.1) with discrete universe, where $\mu(x)$ stands for the membership function on discrete universe.

***Example* (1.2), [8]:**

Let $X = R^+$ be the set of possible ages for human beings, then the fuzzy set:

$\tilde{C}$ = "About 50 years old"

may be expressed as:

$$\tilde{C} = \{(x,\ \mu_{\tilde{C}}(x)) \mid x \in X\}$$

where:

$$\mu_{\tilde{C}}(x) = \frac{1}{1+\left(\dfrac{x-50}{10}\right)^4}$$

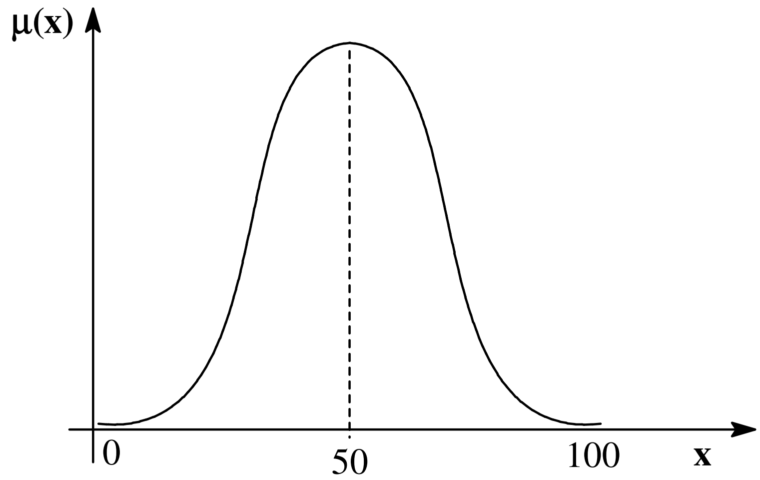

Fig.(1.2) Membership function of example (1.2) with continuous universe.

Next, a summary of the most fundamental and necessary concepts in fuzzy set theory are given, [6], [5]:

1. The support of $\tilde{A}$ is the crisp set (or nonfuzzy set) of all $x \in X$, such that $\mu_{\tilde{A}}(x) > 0$ and is denoted by $S(\tilde{A})$ or $Supp(\tilde{A})$.

2. The height of a fuzzy set $\tilde{A}$ (denoted by hgt $(\tilde{A})$) is the supremum of $\mu_{\tilde{A}}(x)$ over all $x \in X$. If hgt $(\tilde{A}) = 1$, then $\tilde{A}$ is normal**,** otherwise it is subnormal, and a fuzzy set may be always normalized by defining the scaled membership function:

$$\mu^*_{\tilde{A}}(x) = \frac{\mu_{\tilde{A}}(x)}{\operatorname*{Sup}_{x \in X} \mu_{\tilde{A}}(x)}, \forall\, x \in X$$

3. A fuzzy singleton (or fuzzy point) x_α is a fuzzy set whose support is a single point $x \in X$, with membership function:

$$x_\alpha(y) = \begin{cases} \alpha, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$$

4. The crossover point of a fuzzy set $\tilde{A}$ is that point in X, whose grade of membership in $\tilde{A}$ is 0.5.

5. $\tilde{A} = \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$, $\forall\, x \in X$.

6. $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$, $\forall\, x \in X$.

7. $\tilde{A}^c$ is the complement of $\tilde{A}$ with membership function $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)$, for all $x \in X$.

8. The empty fuzzy set $\tilde{\varnothing}$ and the universal set X, have the membership functions $\mu_{\tilde{A}}(x) = 0$ and $\mu_{\tilde{A}}(x) = 1$, respectively, for all $x \in X$.

9. $\tilde{C} = \tilde{A} \cap \tilde{B}$ is a fuzzy set with membership function:

$$\mu_{\tilde{C}}(x) = \text{Min}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\},\ \forall\, x \in X$$

More generally, for any index set J, then $\bigcap_{j \in J} \tilde{A}_j$ is also a fuzzy set of X with membership function:

$$\mu_{\bigcap_{j \in J} \tilde{A}_j}(x) = \inf_{i \in J} \mu_{\tilde{A}_j}(x),\ \forall\, x \in X$$

10. $\tilde{D} = \tilde{A} \cup \tilde{B}$ is a fuzzy set with membership function:

$$\mu_{\tilde{D}}(x) = \text{Max}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\},\ \forall\, x \in X$$

More generally, for any index set J, then $\bigcup_{j\in J} \tilde{A}_j$ is also a fuzzy set of X with membership function:

$$\mu_{\bigcup_{j\in J} \tilde{A}_j}(x) = \sup_{i\in J} \mu_{\tilde{A}_j}(x), \forall x \in X$$

***Remark* (1.2), [21]:**

It is important to notice that the only law of contradiction is $\tilde{A} \cup \tilde{A}^c = X$ and the law of excluded middle $\tilde{A} \cap \tilde{A}^c = \varnothing$ are broken for the fuzzy sets, since $\tilde{A} \cup \tilde{A}^c \neq X$ and $\tilde{A} \cap \tilde{A}^c \neq \varnothing$. Indeed; for all $x \in \tilde{A}$, such that $\mu_{\tilde{A}}(x) = \alpha$, $0 < \alpha < 1$, then:

$$\mu_{\tilde{A}\cup\tilde{A}^c}(x) = \max\{\alpha, 1-\alpha\} \neq 1$$

$$\mu_{\tilde{A}\cap\tilde{A}^c}(x) = \min\{\alpha, 1-\alpha\} \neq 0.$$

1.2.1 *The Extension Principle*:

An important concept in fuzzy set theory that may be used to generalize crisp mathematical concepts to fuzzy sets, is the extension principle. In the elementary form, Zadeh already implied this principle in his first contribution in 1965, [6], [17].

***Definition* (1.2), [6]:**

Let X be the Cartesian product of universes $X_1, X_2, \dots, X_r$ and $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_r$ be r-fuzzy subsets of $X_1, X_2, \dots, X_r$, respectively, f is a mapping from X into a universe Y, $y = f(x_1, x_2, \dots, x_r)$. Then the extension principle allows us to define a fuzzy set $\tilde{B}$ in Y by:

$$\tilde{B} = \{(y, \mu_{\tilde{B}}) \mid y = f(x_1, x_2, \dots, x_r),\ x_1, x_2, \dots, x_r \in X\}$$

where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup \min\{\mu_{\tilde{A}_1}(x_1), \dots, \mu_{\tilde{A}_r}(x_r)\} & , \quad if\ f^{-1}(y) \neq \emptyset, (x_1, x_2, \dots, x_r) \in f^{-1}(y) \\ 0 & , \quad otherwise \end{cases}$$

where f^{-1} is the inverse image of f.

For $r = 1$, the fuzzy extension principle, of course reduces to:

$$\tilde{B} = f(\tilde{A}) = f\{(y, \mu_{\tilde{B}}(y)) \mid y = f(x), x \in X\}$$

where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x), & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

which is the definition of the fuzzy mapping.

The next examples illustrate the extension principle:

***Example* (1.3), [6]:**

Let f be a real valued function which is integrable over the interval $J = [a_0, b_0]$ where $a_0, b_0 \in R$ and $a_0 < b_0$, then according to the extension principle the membership function of the fuzzy integral $\int_{\tilde{a}}^{\tilde{b}} f$ is given by:

$$\mu_{\int_{\tilde{a}}^{\tilde{b}} f}(z) = \underset{\substack{x, y \in J \\ z = \int_x^y f}}{\mathrm{Sup}}\ \mathrm{Min}\left\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\right\}$$

In particular, let:

$$\tilde{a} = \{(4, 0.8), (5, 1), (6, 0.4)\}$$

$$\tilde{b} = \{(6, 0.7), (7, 1), (8, 0.2)\}$$

and

$$y = 2,\ x \in [a_0, b_0] = [4, 8]$$

where a_0 is the minimum of the supports of $\tilde{a}$ and b_0 is the maximum of the supports of $\tilde{b}$ and the problem is to find the fuzzy integration of f(x) over J = [4, 8], then:

$$\int_{\tilde{a}}^{\tilde{b}} f = \{(0, 0.4), (2, 0.7), (4, 1), (6, 0.8), (8, 0.2)\}$$

Example **(1.4), [6]:**

Let:

$$X = \{-1, 0, 1\}$$

and define a fuzzy subset of X, by:

$$\tilde{A} = \{(-1, 0.4), (0, 1), (1, 0.6)\}$$

If $f(x) = x^3$, then $\tilde{B} = f'(\tilde{A})$, is also a fuzzy set with membership function:

$$\mu_{f'(\tilde{A})}(y) = \sup_{f'(x)=y} \mu_{\tilde{A}}(x)$$

$$= \sup_{3x^2=y} \mu_{\tilde{A}}(x)$$

If y = 3, then $3x^2 = 3$ and hence $x = \pm 1$. Therefore:

$$\mu_{\tilde{A}}(1) = 0.6 \text{ and } \mu_{\tilde{A}}(-1) = 0.4$$

and hence sup {0.6, 0.4} = 0.6, which implies to $\mu_{f'(\tilde{A})}(3) = 0.6$.

Similarly:

$$\mu_{f'(\tilde{A})}(x) = \sup_{0=y} \mu_{\tilde{A}}(x) = 1$$

Therefore:

$$f'(\tilde{A}) = \{(3, 0.6), (0, 1)\}$$

1.2.2 *α-Cut Sets*:

The concept of this section is to cover some basic and most important properties of an ordinary set that can be derived from certain fuzzy set. These sets are called the α-level sets (or α-cuts), which corresponds to any fuzzy set. The α-level sets are those sets which collect between fuzzy sets and ordinary sets, that can be used to prove most of the result that are satisfied in ordinary sets are also satisfied here to fuzzy sets and vise versa, i.e., there is also another approach in which the classical sets and fuzzy sets are connected to each other [27].

***Definition* (1.3), [27]:**

The α-level (or α-cut) set of fuzzy set $\tilde{A}$, labeled by A_α, is the crisp set of all x in X such that $\mu_{\tilde{A}}(x) \geq \alpha$, i.e.,

$$A_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}, \alpha \in [0, 1]$$

One can notice that an α-level set discards those points of X whose membership values are less than α. Also, it is remarkable that in some literatures, if the equality is dropped in the definition of A_α then it is called a strong α-level set and is denoted byA_{α^+} or A_{α^-}.

The following properties are satisfied for all $\alpha \in [0, 1]$, [12]:

1. If $\tilde{A}$ is convex, $\alpha_1, \alpha_2 \in [0, 1]$, and $\alpha_1 \leq \alpha_2$, then $A_{\alpha_1} \supseteq A_{\alpha_2}$.
2. $(\tilde{A} \cup \tilde{B})_\alpha = A_\alpha \cup B_\alpha$.
3. $(\tilde{A} \cap \tilde{B})_\alpha = \tilde{A}_\alpha \cap \tilde{B}_\alpha$.
4. $\tilde{A} \subseteq \tilde{B}$, gives $A_\alpha \subseteq B_\alpha$.

5. $\tilde{A} = \tilde{B}$ if and only if $A_\alpha = B_\alpha$, $\forall\ \alpha \in [0, 1]$.

***Remarks* (1.3), [8]:**

1. The set of all levels $\alpha \in [0, 1]$, that represent distinct α-cuts of a given fuzzy set $\tilde{A}$ is called a level set of$\tilde{A}$, i.e.,

$$\Lambda(\tilde{A}) = \{\alpha \mid \mu_{\tilde{A}}(x) = \alpha, \text{ for some } x \in X\}$$

2. The support of $\tilde{A}$ is exactly the same as the strong α-cut of $\tilde{A}$ for $\alpha = 0$, $A_{0^+} = S(\tilde{A})$.

3. The core of $\tilde{A}$ is exactly the same as the α-cut of $\tilde{A}$ for $\alpha = 1$ (i.e., $A_1 = \text{core}\ (\tilde{A})$).

4. The height of $\tilde{A}$ may also be viewed as the supremum of α-cut for which $A_\alpha \neq \varnothing$.

1.2.3 *Convex Fuzzy Sets,* [8]:

An important property of fuzzy sets defined on R^n (for some $n \in N$) is their convexity; this property is viewed as a generalization of the classical concept of convexity of crisp sets. The definition of convexity for the fuzzy set does not necessarily mean that the membership function of a convex fuzzy set is also convex function.

***Definition* (1.4), [8]:**

A fuzzy set $\tilde{A}$ on R is convex if and only if:

$$\mu_{\tilde{A}}(\lambda\, x_1 + (1 - \lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$$

for all $x_1, x_2 \in R$, and all $\lambda \in [0, 1]$.

***Remark* (1.4), [8]:**

Assume that $\tilde{A}$ is a fuzzy set, we need to prove that for any $\alpha \in [0, 1]$, A_α is convex. Now, for any $x_1, x_2 \in A_\alpha$ and for any $\lambda \in [0, 1]$:

$$\mu_{\tilde{A}}(\lambda\, x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\} \geq \min\{\alpha, \alpha\} = \alpha$$

i.e., $\lambda x_1 + (1-\lambda)x_2 \in A_\alpha$, therefore A_α is convex for any $\alpha \in [0, 1]$, $\tilde{A}$ is convex.

1.2.4 *Fuzzy Number*:

Fuzzy number is expressed as a fuzzy set defining a fuzzy interval over the set of real numbers R. Since the boundary of this interval is ambiguous, the interval is also a fuzzy set. Generally, a triangular fuzzy interval is represented by two end points a_1 and a_3 and a peak point a_2 as $[a_1, a_2, a_3\ ]$.

Fuzzy number should be normalized and convex. Here the condition of normalization implies that maximum membership value is 1 (i.e., there exist $x_0 \in R$, such that $\mu_{\tilde{A}}(x_0) = 1$).

Operations over fuzzy numbers can be generalized from that of crisp interval. Based on the extension principle, arithmetic operations on fuzzy numbers are defined as follows, [28]:

If $\widetilde{M}$ and $\widetilde{N}$ are fuzzy numbers, the membership function of $\widetilde{M}(*)\widetilde{N}$ is defined as follow:

$$\mu_{\widetilde{M}(*)\widetilde{N}}(z) = \sup_{z = x * y} \min\{\mu_{\widetilde{M}}(x), \mu_{\widetilde{N}}(y)\}$$

where * stands for any of the four arithmetic operations, namely addition, subtraction, multiplication and division.

The procedure of addition or subtraction is simple, but the procedure of multiplication or division is more difficult.

***Definition* (1.5), [24]:**

A fuzzy number is a fuzzy set like $\widetilde{M}: R \rightarrow I = [0, 1]$ which satisfies:

1. $\widetilde{M}$ is upper semi-continous,
2. $\widetilde{M}(x) = 0$ outside some interval $[a, d]$,
3. There are real numbers a, d such that $a \leq b \leq c \leq d$ and
   a. $\widetilde{M}(x)$ is monotonic increasing on $[a, b]$,
   b. $\widetilde{M}(x)$ is monotonic decreasing on $[c, d]$,
   c. $\widetilde{M}(x) = 1, b \leq x \leq c$.

The membership function of $\widetilde{M}$ may be expressed as:

$$\mu_{\widetilde{M}}(x) = \begin{cases} \widetilde{M}_L(x), & a \leq x \leq b, \\ 1, & b \leq x \leq c, \\ \widetilde{M}_R(x), & c \leq x \leq d, \\ 0, & otherwise \end{cases}$$

where $\widetilde{M}_L : [a, b] \rightarrow [0, 1]$ and $\widetilde{M}_R: [c, d] \rightarrow [0, 1]$ are left and right membership functions of fuzzy number $\widetilde{M}$.

A Trapezoidal fuzzy number $\widetilde{M}$ is defined as $[a, b, c, d]$ where the membership function:

$$\mu_{\tilde{M}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x \le b, \\ 1, & b \le x \le c, \\ \dfrac{d-x}{d-c}, & c \le x \le d, \\ 0, & otherwise \end{cases}$$

when $b = c$, $\tilde{M}$ is called triangular fuzzy number and defined as $[a, b, d]$.

***Remarks* (1.5), [17]:**

1. A fuzzy number $\tilde{M}$ may be represented in terms of its α-level sets, as the following closed intervals of the real line:

   $$M_\alpha = [x - \sqrt{1-\alpha}\,, x + \sqrt{1-\alpha}\,]$$

   or

   $$M_\alpha = [\alpha\, x, \frac{1}{\alpha} x]$$

   where $x \in R$, which is called the mean value of $\tilde{M}$ and $\alpha \in [0, 1]$. This fuzzy number may be written as $M_\alpha = [\underline{M}, \overline{M}]$, where $\underline{M}$ refers to the greatest lower bound of M_α and $\overline{M}$ to the least upper bound of M_α.

2. If the sides of the fuzzy number $\tilde{M}$ are strictly monotone then one can see easily that $\underline{M}$ and $\overline{M}$ are inverse functions of $\tilde{M}_L(x)$ and $\tilde{M}_R(x)$, respectively. We denote by $F(R)$ the set of all fuzzy numbers.

1.2.5 *Fuzzy Functions on Fuzzy Sets,* [5]:

A fuzzy function is a generalization of the classical function in which a classical function *f* is a mapping (correspondence) from the domain *D* of definition of the function into a space S; $f(D) \subseteq S$ is called the range of f. Different features of the classical concept of a function can be considered to be fuzzy rather than crisp. Therefore, different "degrees" of fuzzification of the classical notion of a function are conceivable:

1. There can be a crisp mapping from a fuzzy set, which carries along the fuzziness of the domain and therefore generates a fuzzy set. The image of a crisp argument would again be crisp.
2. The mapping itself can be fuzzy, thus blurring the image of a crisp argument. Thus, we shall call a fuzzy function. These are called "fuzzifying functions".
3. Ordinary functions can have fuzzy properties or be constrained by fuzzy constraints.

***Definition* (1.6), [5]:**

A classical function $f : X \to Y$ maps from a fuzzy domain $\tilde{A}$ in X into a fuzzy range $\tilde{B}$ in Y if and only if:

$$\mu_{\tilde{B}}(f(x)) \geq \mu_{\tilde{A}}(x), \ \forall x \in X.$$

Given a classical function $f: X \to Y$ and a fuzzy domain $\tilde{A}$ in X, the extension principle yields the fuzzy range $\tilde{B}$ with the membership function:

$$\mu_{\tilde{B}}(y) = \sup_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x)$$

hence f is a function according to definition (1.5).

***Definition* (1.7), [5]:**

Let X and Y be two universes and $\tilde{P}(Y)$ be the set of all fuzzy subsets of Y (power set), $\tilde{f} : X \to \tilde{P}(Y)$ is a mapping, then $\tilde{f}$ is a fuzzy function if and only if:

$$\mu_{\tilde{f}(x)}(y) = \mu_{\tilde{R}}(x, y), \forall (x, y) \in X \times Y$$

where $\mu_{\tilde{R}}(x, y)$ is the membership function of a fuzzy relation.

***Examples* (1.5), [5]:**

1. Let X be the set of all workers of a plant, $\tilde{f}$ the daily output, and y be the number of processed work pieces. A fuzzy function may then be defined as $\tilde{f}(x) = y$.
2. If $\tilde{a}, \tilde{b}$ are two fuzzy subsets of R, $X = R$, $\tilde{f} : X \longrightarrow \tilde{a}X \oplus \tilde{b}$, is a fuzzy function.
3. $X =$ set of all 1-mile runners, $\tilde{f}=$ possible recorded times, $\tilde{f}(x) = \{y \mid y: achieved\ record\ times\}$.

***Definition* (1.8), [20]:**

The gH-difference of two fuzzy numbers u, $v \in R_F$, is the fuzzy number w, if it exists, such that:

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} (i) u = v + w, \\ or\ (ii) v = u + (-1)w \end{cases}.$$

If $w = u \ominus_{gH} v$ exists as a fuzzy number, its level sets $[\underline{w}(\alpha), \overline{w}(\alpha)]$ are obtained by:

$$\underline{w}(\alpha) = \min\{\underline{u}(\alpha) - \underline{v}(\alpha), \overline{u}(\alpha) - \overline{v}(\alpha)\}$$

and

$$\overline{w}(\alpha) = \max\{\underline{u}(\alpha) - v(\alpha), \overline{u}(\alpha) - \overline{v}(\alpha)\} \text{ for all } \alpha \in [0, 1].$$

***Definition* (1.9), [20]:**

Let $x_0 \in [a,b]$ and h be such that $x_0 + h \in (a,b)$, then the gH-derivative of a function $f:(a,b) \rightarrow \mathrm{R}_F$ at x_0 is defined as:

$$f'_{gH}(x_0) = \lim_{h \to 0} \frac{1}{h}[f(x_0 + h) \ominus_{gH} f(x_0)] \qquad \ldots(1.1).$$

If $f'_{gH}(x_0) \in R_F$ satisfying (1.1) exists, we say that f is gH-differentiable at x_0.

***Theorem* (1.1), [20]:**

Let $f:(\mathrm{a, b}) \rightarrow R_F$ be such that:

$$f(x)_\alpha = [\underline{f}(x,\alpha), \overline{f}(x,\alpha)].$$

Suppose that functions $\underline{f}(x,\alpha)$ and $\overline{f}(x,\alpha)$ are real-valued functions, differentiability with respect to x, uniformly with respect to $\alpha \in [0, 1]$. Then the function $f(x)$ is gH-differentiable at a fixed $x \in (a,b)$ if and only if one of the following two cases holds:

(a) $(\underline{f})'(x,\alpha)$ is increasing, $(\overline{f})'(x,\alpha)$ is decreasing as functions of α, and $(\underline{f})'(x,1) \leq (\overline{f})'(x,1)$,

or

(b) $(\underline{f})'(x,\alpha)$ is decreasing, $(\overline{f})'(x,\alpha)$ is increasing as functions of α, and $(\overline{f})'(x,1) \leq (\underline{f})'(x,1)$.

Also, $\forall \alpha \in [0, 1]$ we have:

$$f'_{gH}(x)[\alpha] = \left[\min\left\{\left(\underline{f}\right)'(x,\alpha), \left(\overline{f}\right)'(x,\alpha)\right\}, max\left\{\left(\underline{f}\right)'(x,\alpha)\left(\overline{f}\right)'(x,\alpha)\right\}\right].$$

***Definition* (1.10), [20]:**

Let $f : [a,b] \rightarrow R_F$ and $x_0 \in (a,b)$ with $\underline{f}(x,\alpha)$ and $\overline{f}(x,\alpha)$ both differentiable at x_0.

$-\ f$ is (i)-gH-differentiable at x_0 if:

$$(i) f'_{gH}(x_0)[\alpha] = \left[\left(\underline{f}\right)'(x_0,\alpha), \left(\overline{f}\right)'(x_0,\alpha)\right], \forall\alpha \in [0,1]$$

$-\ f$ is (ii)-gH-differentiable at x_0 if:

$$(ii) f'_{gH}(x_0)[\alpha] = \left[\left(\overline{f}\right)'(x_0,\alpha), \left(\underline{f}\right)'(x_0,\alpha)\right], \forall\alpha \in [0,1]$$

It is possible that $f : [a,b] \rightarrow R_F$ is gH-differentiable at x_0 and not (i)-gHdifferentiable nor (ii)-gH-differentiable.

1.3 *Basic Concepts of Calculus of Variation:*

The subjects of calculus of variation is concerned with solving extremal problems for a functionals. That is to say the maximum and minimum problems for functions whose domain contains functions, $Y(x)$ (or $Y(x_0,\cdots x_1)$, or n-tuples of functions). The range of the functional will be the real numbers R, [13].

***Definition* (1.10), [1]:**

Let R be the set of real numbers and Ω a set of functions. Then the function $J : \Omega \rightarrow R$ is called a functional.

***Example* (1.6), [13]:**

Given two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the plane, which are joined by a curve $y = f(x)$. The length functional is given by:

$$L_{1,2}(y) = \int_{x_1}^{x_2} \underbrace{\sqrt{1+(y')^2}\,dx}_{ds}.$$

The domain is the set of all curves, $y(x) \in C^1$, such that:

$$y(x_i) = y_i, i = 1,2.$$

The minimum problem for $L_{1,2}[y]$ is solved by the straight line segment $\overline{P_1P_2}$.

***Example* (1.7), [1]:**

Let A and B be two fixed points in a space. Then we want to find the shortest distance between these two points. From basic geometry (i.e., Pythagoras Theorem), it is know that:

$$ds^2 = dx^2 + dY^2$$

$$= \{1 + (Y')^2\}\, dx^2 \qquad \ldots(1.2)$$

The second line of this is achieved by noting $Y\,' = \frac{dY}{dx}$. Now to find the path joining the two points A and B, we integrate ds between A and B, i.e., $\int_A^B ds$. We however replace ds using equation (1.2) and hence get the expression of the length of our curve:

$$J(y) = \int_a^b \sqrt{1 + (y')^2}\, dx$$

To find the shortest path, i.e., to minimise J, the extremal function is needed.

1.3.1 *Variational Problem with Simple Form*, [14]:

Setting:

$$v(y) = \int_{x_0}^{x_1} F(x, y(x), y'(x))dx$$

and for given $y_0, y_1 \in R$

$$Y = \{y \in C^1[x_0, x_1]: y(x_0) = y_0, y(x_1) = y_1\}$$

where $-\infty < x_0 < x_1 < \infty$ and F is sufficiently regular. One of the basic problems in the calculus of variation is

$$min_{y \in Y}\, v(y) \qquad \ldots(1.3)$$

that is, we seek for a function y, such that:

$$y \in Y : v(y_1) \leq v(y_2) \text{ for all } y_2 \in Y$$

1.3.2 *Euler-Lagrange Equation*, [14]:

All solutions of the variational Problem (1.3) above satisfy the following so-called ***Euler-Lagrange equations***:

$$F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') = 0, \forall\, x \in (x_0, x_1)$$

with the boundary conditions $y(x_0) = y_0,\ \ y(x_1) = y_1$.

***Lemma* (1.1), [13]:**

Let $M(x) \in C^n[x_0, x_1], 0 \leq n \leq \infty$. If $\int_{x_0}^{x_1} M(x)\, \eta(x) dx = 0$ for all $\eta(x)$ such that $\eta(x_0) = \eta(x_1) = 0, \eta(x) \in C^n,$ on $[x_0, x_1]$ then $M(x) = 0$ at all points of continuity.

***Theorem* (1.2), [1]**:

A necessary condition for $J(Y)$ to have an extremum (maximum or minimum) at y is:

$$\delta J = \langle \varepsilon\xi, J'(y) \rangle = 0 \qquad \ldots(1.4)$$

for all admissible functions ξ.

***Theorem* (1.3), [1]:**

A necessary condition for

$$J(y) = \int_{x_0}^{x_1} F(x, y, y')dx$$

with $y(x_0) = y_0$and $y(x_1) = y_1$, to have an extremum at y is that y is a solution of

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

with $x_0 < x < x_1$ and $F = F(x, y, y')$. This is known as the **Euler-Lagrange equation**.

CHAPTER TWO

Variational Problems with Fuzzy Integrands

2.1 *Introduction:*

In this chapter, variational problems with fuzzy function and variational problem with fuzzy boundary conditions are investigated. In the second section, we will discuss the derivation of Euler-Lagrange equation of unconstrained variational problems with fuzzy function, while in the third section, we will discuss the derivation of Euler-Lagrange equation of constrained variational problems with fuzzy function.

2.2 *Unconstrained Fuzzy Variational Problems with Fuzzy Function*, [20]:

Consider the fuzzy variational problem (FVP) with fuzzy function $\tilde{y}: A \to B$, which is to minimize the functional:

$$J(\tilde{y}) = \int_{x_0}^{x_1} F(x, \tilde{y}, \tilde{y}')dx \qquad \ldots (2.1)$$

where A and B are subsets of R, x_0 and x_1 are crisp fixed points and the boundary conditions $\tilde{y}(x_0) = \tilde{y}_0$ and $\tilde{y}(x_1) = \tilde{y}_1$ are fixed fuzzy numbers.
Where the fuzzy curve $\tilde{y} = \tilde{y}(x)$ is a fuzzy function of $x \in [x_0, x_1] \subseteq R$ and belongs to the class of fuzzy functions with continuous first gH-derivatives

with respect to $x \in [x_0,\ x_1]$ and F assigns a fuzzy number to the fuzzy point $(x, \tilde{y}(x), \tilde{y}'(x)) \in\ R_F \times\ R_F \times R$. We assume that the integrand F has continuous first and second partial gH-derivatives with respect to all of its arguments.

The goal of FVP is to find an admissible fuzzy curve $\tilde{y}^*$ in a fuzzy weak neighborhood, if any exists, such that minimize *J*. The fuzzy curve $\tilde{y}^*\ =\ \tilde{y}^*(x)$ is a minimizing curve for the FVP if for all admissible fuzzy curves $\tilde{y}^*$ in the fuzzy weak neighborhood, i.e.,

$$J(\tilde{y}) \geq\ J(\tilde{y}^*) \qquad \ldots (2.2)$$

Now, by using the α −cuts of fuzzy curve $\tilde{y}^*(x)$, in terms of $\underline{y}^*(x,\alpha)$ and $\overline{y}^*(x,\alpha)$, by using α −cuts of an arbitrary twice continuously gH-differentiable fuzzy function $\tilde{\eta}(x)$ as follows:

$$\underline{y}(x,\alpha) = \underline{y}^*(x,\alpha) + \epsilon\ \underline{\eta}(x,\alpha) \qquad \ldots (2.3)$$

$$\overline{y}(x,\alpha) = \overline{y}^*(x,\alpha) + \epsilon\ \overline{\eta}(x,\alpha) \qquad \ldots (2.4)$$

such that $y(x)_\alpha =\ [\underline{y}(x,\alpha), \overline{y}(x,\alpha)]$ is admissible for any real number ϵ and further $\tilde{\eta}(x_0) \approx\ \tilde{\eta}(x_1) \approx 0$. Given ε > 0, to be sufficiently small, we are able to make $\tilde{y}^*(x)$ lie in the fuzzy weak neighborhood.

The inequality (2.2) holds if and only if:

$$\underline{J}(\tilde{y},\alpha) \geq\ \underline{J}(\tilde{y}^*,\alpha) \text{ and } \overline{J}(\tilde{y},\alpha) \geq\ \overline{J}(\tilde{y}^*,\alpha) \qquad \ldots (2.5)$$

for all $\alpha\ \in [0,1]$.

From (2.5), inequality (2.2) holds if and only if the lower and upper-increments are non-negative (in the sense of α-cuts), that is,

$$\Delta\, \underline{J}\left(\underline{y}^*(t,\alpha), \overline{y}^*(t,\alpha), \alpha\right) = (\underline{\Delta}\, \underline{J}, \overline{\Delta}\, \underline{J}) \geq (0,0) \qquad \ldots (2.6)$$

$$\Delta\, \overline{J}\left(\underline{y}^*(t,\alpha), \overline{y}^*(t,\alpha), \alpha\right) = (\underline{\Delta}\, \overline{J}, \overline{\Delta}\, \overline{J}) \geq (0,0) \qquad \ldots (2.7)$$

for all $\alpha\ \in [0,1]$ and all admissible curve close to y^*. We see that (2.6) and (2.7) hold if and only if

$$\underline{\Delta}\,\underline{J} = \underline{J}\left(\underline{y}(x,\alpha), \overline{y}^*(x,\alpha), \alpha\right) - \underline{J}\left(\underline{y}^*(x,\alpha), \overline{y}^*(x,\alpha), \alpha\right) \geq 0 \quad \ldots (2.8)$$

$$\overline{\Delta}\underline{J} = \underline{J}\left(\underline{y}^*(x,\alpha), \overline{y}(x,\alpha), \alpha\right) - \underline{J}\left(\underline{y}^*(x,\alpha), \overline{y}^*(x,\alpha), \alpha\right) \geq 0 \quad \ldots (2.9)$$

$$\underline{\Delta}\,\overline{J} = \overline{J}\left(\underline{y}(x,\alpha), \overline{y}^*(x,\alpha), \alpha\right) - \overline{J}\left(\underline{y}^*(x,\alpha), \overline{y}^*(x,\alpha), \alpha\right) \geq 0 \ldots (2.10)$$

$$\overline{\Delta}\,\overline{J} = \overline{J}\left(\underline{y}^*(x,\alpha), \overline{y}(x,\alpha), \alpha\right) - \overline{J}\left(\underline{y}^*(x,\alpha), \overline{y}^*(x,\alpha), \alpha\right) \geq 0 \ldots (2.11)$$

and consider only equation (2.8), then by using (2.8) and (2.3), one can easily verify that:

$$\underline{\Delta}\,\underline{J}\left(\underline{y}^*, \overline{y}^*, \alpha\right) = \int_{x_0}^{x_1} \left(\underline{F}\left(\underline{y}^* + \epsilon\,\underline{\eta}, (\underline{y}^*)' + \epsilon\,\underline{\eta}', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \right.$$

$$\left.\underline{F}\left(\underline{y}^*, (\underline{y}^*)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right)\, dx \;\geq 0 \quad \ldots (2.12)$$

Corresponding to definition (1.10), the following two cases just can occur:

***<u>Case (i)</u>*:** F is (*i*)-gH differentiable ((*ii*)-gH differentiable) with respect to y and y'.

Expanding the integrant $\underline{F}\left(\underline{y}^* + \epsilon\,\underline{\eta}, (\underline{y}^*)' + \epsilon\,\underline{\eta}', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)$ of (2.12) in a Taylor series about the point $\left(\underline{y}^*, (\underline{y}^*)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)$ gives:

$$\underline{\Delta}\,\underline{J}\left(\underline{y}^*, \overline{y}^*, \alpha\right) = \epsilon\,\underline{J_1}\left(\underline{y}^*, \overline{y}^*, \alpha\right) + \epsilon^2\,\underline{J_2}\left(\underline{y}^*, \overline{y}^*, \alpha\right) + O(\epsilon^3) \quad \ldots (2.13)$$

where;

$$\underline{J_1}\left(\underline{y}, \overline{y}, \alpha\right) = \int_{x_0}^{x_1} \left(\underline{\eta}\,\frac{\partial \underline{F}}{\partial \underline{y}} + \underline{\eta}'\,\frac{\partial \underline{F}}{\partial \underline{y}'}\right) dx$$

$$\underline{J_2}\left(\underline{y}, \overline{y}, \alpha\right) = \frac{1}{2}\int_{x_0}^{x_1} \left(\underline{\eta}^2\,\frac{\partial^2 \underline{F}}{\partial \underline{y}^2} + 2\,\underline{\eta}\,\underline{\eta}'\,\frac{\partial^2 \underline{F}}{\partial \underline{y}\,\underline{y}'} + (\underline{\eta}')^2\,\frac{\partial^2 \underline{F}}{\partial (\underline{y}')^2}\right) dx$$

The integral $\underline{J_1}\left(\underline{y}, \overline{y}, \alpha\right)$ is called the first variation of $\underline{J}$, since it is expressed in terms containing the first-order change in $\underline{J}$ with respect to the deformations $\underline{y}^* + \epsilon\,\underline{\eta}$ and $\overline{y}^* + \epsilon\,\overline{\eta}$. Similarly, the integral $\underline{J_2}\left(\underline{y}, \overline{y}, \alpha\right)$ is called the second variation of $\underline{J}$. The notation $O(\epsilon^3)$ denotes terms in the expansion of order 3 and greater in ϵ.

By (2.12), the right-hand side of equation (2.13) is non-negative. On the other hand, ϵ is an arbitrary and may be positive or negative. Hence, dividing the right-hand side of (2.13) by ϵ, the two following inequalities can be taken into consideration:

$$\underline{J}_1\left(\underline{y}^*, \overline{y}^*, \alpha\right) + \epsilon\, \underline{J}_2\left(\underline{y}^*, \overline{y}^*, \alpha\right) + O(\epsilon^2) \geq 0, if\ \epsilon > 0 \quad \ldots (2.14)$$

$$\underline{J}_1\left(\underline{y}^*, \overline{y}^*, \alpha\right) + \epsilon\, \underline{J}_2\left(\underline{y}^*, \overline{y}^*, \alpha\right) + O(\epsilon^2) \leq 0, if\ \epsilon < 0 \quad \ldots (2.15)$$

Now, the two inequalities (2.14) and (2.15) can be reduced to $\underline{J}_1\left(\underline{y}^*, \overline{y}^*, \alpha\right) \geq 0$ and $\underline{J}_1\left(\underline{y}^*, \overline{y}^*, \alpha\right) \leq 0$, respectively, as ϵ approaches zero. This means that:

$$\underline{J}_1\left(\underline{y}^*, \overline{y}^*, \alpha\right) = \int_{x_0}^{x_1} \left(\underline{\eta} \frac{\partial F}{\partial \underline{y}}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) + \underline{\eta}' \frac{\partial F}{\partial \underline{y}'}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) \right) dx$$
$$= 0 \quad \ldots (2.16)$$

for all admissible $\underline{\eta}(x, \alpha)$. Since $\underline{\eta}(x_0, \alpha) = \underline{\eta}(x_1, \alpha) = 0$, solving integral involves integration by part, the equation (2.16) becomes:

$$\int_{x_0}^{x_1} \left(\underline{\eta} \left(\frac{\partial F}{\partial \underline{y}}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial \underline{y}'}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) \right) \right) \right) dx$$
$$= 0 \quad \ldots (2.17)$$

for all admissible $\underline{\eta}(x, \alpha)$.

Applying lemma (1.1) to equation (2.17), we have:

$$\frac{\partial F}{\partial \underline{y}}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial \underline{y}'}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) \right) = 0 \quad \ldots (2.18)$$

Following the scheme of obtaining (2.18) and adapting it to the cases (2.9), (2.10) and (2.11), one can easily show that:

$$\frac{\partial F}{\partial \overline{y}}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial \overline{y}'}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) \right) = 0$$

$$\frac{\partial \overline{F}}{\partial \underline{y}}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial \overline{F}}{\partial \underline{y}'}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) = 0$$

$$\frac{\partial \overline{F}}{\partial \overline{y}}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial \overline{F}}{\partial \overline{y}'}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) = 0$$

***Case (ii)*:** F is (i)-gH differentiable ((ii)-gH differentiable) with respect to y and (ii)-gH differentiable ((i)-gH differentiable) with respect to y'.

Similar the procedure of obtaining the fuzzy Euler-Lagrange conditions for case (i), one can show that the conditions for this case are:

$$\frac{\partial \underline{F}}{\partial \underline{y}}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial \overline{F}}{\partial \underline{y}'}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) = 0 \quad \ldots (2.19)$$

$$\frac{\partial \underline{F}}{\partial \overline{y}}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial \overline{F}}{\partial \overline{y}'}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) = 0 \ldots (2.20)$$

$$\frac{\partial \overline{F}}{\partial \underline{y}}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial \overline{F}}{\partial \underline{y}'}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) = 0 \quad \ldots (2.21)$$

$$\frac{\partial \overline{F}}{\partial \overline{y}}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial \underline{F}}{\partial \overline{y}'}\left(\underline{y}^*, \left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) = 0 \ldots (2.22)$$

Now, we consider the following example as an illustration:

***Example* (2.1):**

Find the minimum of:

$$J(\tilde{y}) = \int_0^1 -(\tilde{y}')^2 dx$$

with:

$$\tilde{y}(0) \approx \tilde{2} = \langle 0, 2, 4 \rangle, \qquad \tilde{y}(1) \approx \tilde{4} = \langle 2, 4, 6 \rangle$$

First from the α −level set of J, we get:

$$J(\tilde{y})_\alpha = \left[\underline{\mathrm{J}}(y, \alpha), \overline{\mathrm{J}}(y, \alpha)\right] = \int_{x_0}^{x_1} \left[-(\underline{y}')^2, -(\overline{y}')^2\right] dx$$

And therefore the integrand is:

$$F(x,\tilde{y}\,,\tilde{y}')_\alpha = \left[\underline{F}(x,\tilde{y}\,,\tilde{y}'),\overline{F}(x,\tilde{y}\,,\tilde{y}')\right] = \left[-\,(\underline{y}')^2,-\,(\overline{y}')^2\right]$$

and hence:

$$\underline{F}_{\underline{y}'}(x,y\,,y') = -2\underline{y}'(x,\alpha),$$

$$\underline{F}_{\underline{y}}\ \left(x,\underline{y}\,,\underline{y}'\right) = 0$$

$$\overline{F}_{\overline{y}'}\left(x,\overline{y}\,,\overline{y}'\right) = -2\overline{y}'(x,\alpha),$$

$$\overline{F}_{\overline{y}}\ \left(x,\overline{y}\,,\overline{y}'\right) = 0$$

Suppose that case (*ii*) is satisfied in $J(y)$, i.e., *F* is (*i*)-gH differentiable ((*ii*)-gH differentiable) with respect to *y* and (*ii*)-gH differentiable ((*i*)-gH differentiable) with respect to y'.

Using the fuzzy Euler-Lagrange equations given by (2.19) and (2.20):

$$-\frac{d}{dx}\left(-2\underline{y}^{'}(x,\alpha)\right) = 0$$

$$-\frac{d}{dx}\left(-2\overline{y}^{'}(x,\alpha)\right) = 0$$

by solving this differential equations, one may get:

$$\underline{y}(x,\alpha) = \underline{c}_1\,x + \underline{c}_2$$

$$\overline{y}(x,\alpha) = \overline{c}_1\,x + \overline{c}_2$$

By representing the endpoint conditions in its the α −cut sets:

$$2_\alpha = [2\alpha\,,2\frac{1}{\alpha}]$$

$$4_\alpha = [4\alpha\,,4\frac{1}{\alpha}]$$

using the endpoint conditions, we have:

$$\underline{y}(x,\alpha) = 2\alpha\,x + 2\alpha$$

$$\overline{y}(x,\alpha)=\frac{2}{\alpha}\,x+\frac{2}{\alpha}$$

and at $\alpha=1$, $\underline{y}(x,1)=\overline{y}(x,1)=2x+2$.

Then:

$$y(x)_\alpha=\left[\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right]=\left[2\alpha\,x+2\alpha,\frac{2}{\alpha}\,x+\frac{2}{\alpha}\right]$$

that defines the α-level sets of a fuzzy number which minimizes J in the fuzzy sense.

2.3 *Constrained Fuzzy Variational Problem with Fuzzy Function*, [20]:

The problem involving minimization of a fuzzy functional with fuzzy integral constraints is called the ***constrained fuzzy variational problem*** and it is stated as follows:

Minimize $J(\tilde{y})=\int_{x_0}^{x_1}F(x,\tilde{y},\tilde{y}')dx$

Subject to $I(\tilde{y})=\int_{x_0}^{x_1}H(x,\tilde{y},\tilde{y}')dx\approx c$

$$\tilde{y}(x_0)\approx\tilde{y}_0,\tilde{y}(x_1)\approx\tilde{y}_1$$

where c is a given fuzzy number.

For constrained fuzzy variational problems consider the following deformations of the $\alpha-$cuts of fuzzy curve y^* by taking into consideration $\alpha-$cuts of an arbitrary twice continuously gH-differentiable fuzzy function $\delta(t)$ as:

$$\underline{y}(x,\alpha)=\underline{y}^*(x,\alpha)+\epsilon\,\underline{\delta}(x,\alpha) \qquad \ldots (2.23)$$

$$\overline{y}(x,\alpha)=\overline{y}^*(x,\alpha)+\epsilon\,\overline{\delta}(x,\alpha) \qquad \ldots (2.24)$$

where ϵ is a small real number and $\delta(t) = \sigma\eta(t) + \beta\zeta(t)$. In the last equation σ, β are real constants and the arbitrary independent fuzzy functions $\eta(t)$ and $\zeta(t)$ vanish in the fuzzy sense at the endpoints.

Since I is equal to the fuzzy number c, therefore its increment is identically zero, particularly, the first variation must be zero. According to definition (1.10), and therefore, eight cases can be occur.

***Case (i)*:** F and H are both (i)-gH differentiable ((ii)-gH differentiable) with respect to y and y'. In this case, for all $\alpha \in [0,1]$, we have:

$$\int_{x_0}^{x_1}\left(\underline{\delta}\frac{\partial\underline{H}}{\partial\underline{y}}\left(\underline{y}^*,\left(\underline{y}^*\right)',\overline{y}^*,(\overline{y}^*)',x,\alpha\right)+\underline{\delta}'\frac{\partial\underline{H}}{\partial\underline{y}'}\left(\underline{y}^*,\left(\underline{y}^*\right)',\overline{y}^*,(\overline{y}^*)',x,\alpha\right)\right)dx=0 \qquad \dots (2.25)$$

$$\int_{x_0}^{x_1}\left(\overline{\delta}\frac{\partial\underline{H}}{\partial\overline{y}}\left(\underline{y}^*,\left(\underline{y}^*\right)',\overline{y}^*,(\overline{y}^*)',x,\alpha\right)+\overline{\delta}'\frac{\partial\underline{H}}{\partial\overline{y}'}\left(\underline{y}^*,\left(\underline{y}^*\right)',\overline{y}^*,(\overline{y}^*)',x,\alpha\right)\right)dx=0 \qquad \dots (2.26)$$

$$\int_{x_0}^{x_1}\left(\underline{\delta}\frac{\partial\overline{H}}{\partial\underline{y}}\left(\underline{y}^*,\left(\underline{y}^*\right)',\overline{y}^*,(\overline{y}^*)',x,\alpha\right)+\underline{\delta}'\frac{\partial\overline{H}}{\partial\underline{y}'}\left(\underline{y}^*,\left(\underline{y}^*\right)',\overline{y}^*,(\overline{y}^*)',x,\alpha\right)\right)dx=0 \qquad \dots (2.27)$$

$$\int_{x_0}^{x_1}\left(\overline{\delta}\frac{\partial\overline{H}}{\partial\overline{y}}\left(\underline{y}^*,\left(\underline{y}^*\right)',\overline{y}^*,(\overline{y}^*)',x,\alpha\right)+\overline{\delta}'\frac{\partial\overline{H}}{\partial\overline{y}'}\left(\underline{y}^*,\left(\underline{y}^*\right)',\overline{y}^*,(\overline{y}^*)',x,\alpha\right)\right)dx=0 \qquad \dots (2.28)$$

Consider only equation (2.25) and by integrating by parts, the terms involving $\underline{\delta}'$ and letting $\underline{\delta}(t) = \sigma\underline{\eta}(t) + \beta\underline{\zeta}(t)$, we may find that:

$$\int_{x_0}^{x_1}\left(\left(\sigma\underline{\eta}(t)+\beta\underline{\zeta}(t)\right)\left\{\frac{\partial\underline{H}}{\partial\underline{y}}\left(\underline{y}^*,\left(\underline{y}^*\right)',\overline{y}^*,(\overline{y}^*)',x,\alpha\right)+\frac{d}{dx}\left(\frac{\partial\underline{H}}{\partial\underline{y}'}\left(\underline{y}^*,\left(\underline{y}^*\right)',\overline{y}^*,(\overline{y}^*)',x,\alpha\right)\right)\right\}\right)dx$$
$$=0 \qquad \dots (2.29)$$

Let us define the fuzzy operator $\underline{L}(*)$ as:

$$\underline{L}(*)=\frac{\partial *}{\partial\underline{y}}-\frac{d}{dx}\left(\frac{\partial *}{\partial\underline{y}'}\right) \qquad \dots (2.30)$$

Hence, we see that (2.25) can be written as:

$$\int_{x_0}^{x_1}\left(\sigma\underline{\eta}(t)+\beta\underline{\zeta}(t)\right)\underline{L}\left(\underline{H}\right)dx=0 \qquad \dots (2.31)$$

Observe that y^* is not the minimizer of I therefore, $\underline{L}(\underline{H}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right))$. Furthermore, for any $\underline{\eta}, \underline{\zeta}$, the constants σ, β are related together by (2.27).

The assumption that y^* is the minimizer of J grantees the increment of J must be non-negative in the fuzzy sense with respect to the deformation given in (2.19).

Consequently, the first variation is zero and after integrating by parts, it holds that:

$$\int_{x_0}^{x_1} \left(\sigma\underline{\eta}(t) + \beta\underline{\zeta}(t)\right) \underline{L}\left(\underline{F}\right) dx = 0 \qquad \dots (2.32)$$

where σ, β are those satisfy (2.31). Solving by elimination σ and β between (2.31) and (2.32), for every independent and twice continuously differentiable functions $\underline{\eta}$ and $\underline{\zeta}$, one can show that:

$$\frac{\int_{x_0}^{x_1} \underline{\eta}\, \underline{L}\left(\underline{F}\right) dx}{\int_{x_0}^{x_1} \underline{\eta}\, \underline{L}\left(\underline{H}\right) dx} = \frac{\int_{x_0}^{x_1} \underline{\zeta}\, \underline{L}\left(\underline{F}\right) dx}{\int_{x_0}^{x_1} \underline{\zeta}\, \underline{L}\left(\underline{H}\right) dx} \qquad \dots (2.33)$$

Introducing the constant $-\underline{\lambda}_1 = -\underline{\lambda}_1(\alpha)$which is equal to both sides of the equality in (2.33) gives that:

$$\int_{x_0}^{x_1} \underline{\eta}\, \underline{L}\left(\underline{F}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) + \underline{\lambda}_1(\alpha)\underline{H}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) dx = 0 \quad \dots (2.34)$$

for any admissible $\underline{\eta} = \underline{\eta}(x, \alpha)$.

Applying Lemma 1.1, we derive from (2.34) that:

$$\underline{L}\left(\underline{F}\left(\underline{y}^*, (\underline{y}^*)',, \overline{y}^*, (\overline{y}^*)', x, \alpha\right) + \underline{\lambda}(\alpha)\underline{H}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) = 0 \qquad \dots (2.35)$$

Taking into account the structure of $\underline{L}$ in (2.30), we get from (2.31) that:

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z)+\underline{\lambda}_1\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z)+\underline{\lambda}_1\underline{H}(z)\right)\right)=0 \qquad \dots (2.36)$$

where $\underline{z}=(\underline{y}^*,(\underline{y}^*)^{'},\overline{y}^*,(\overline{y}^*)^{'},x,\alpha)$.

Now following the scheme of obtaining (2.36) and adapting it to the case under consideration involving (2.26), (2.27) and (2.28), it may be shown that:

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z)+\overline{\lambda}_2\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z)+\overline{\lambda}_2\underline{H}(z)\right)\right)=0 \qquad \dots (2.37)$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z)+\underline{\lambda}_2\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z)+\underline{\lambda}_2\overline{H}(z)\right)\right)=0 \qquad \dots (2.38)$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z)+\overline{\lambda}_1\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z)+\overline{\lambda}_1\overline{H}(z)\right)\right)=0 \qquad \dots (2.39)$$

Following the scheme of obtaining the Euler-Lagrange conditions for case (*i*), one can show that these conditions for other cases are as follows:

<u>Case (ii)</u>**:** *H*, *F* with respect to y and *F* with respect to y' are both (*i*)-gH differentiable ((*ii*)-gH differentiable) and *H* is (*ii*)-gH differentiable ((*i*)-gH differentiable) with respect to y'.

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z)+\underline{\lambda}_1\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z)+\underline{\lambda}_1\overline{H}(z)\right)\right)=0 \qquad \dots (2.40)$$

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z)+\overline{\lambda}_2\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z)+\overline{\lambda}_2\overline{H}(z)\right)\right)=0 \qquad \dots (2.41)$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z)+\underline{\lambda}_2\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z)+\underline{\lambda}_2\underline{H}(z)\right)\right)=0 \qquad \dots (2.42)$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z)+\overline{\lambda}_1\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z)+\overline{\lambda}_1\underline{H}(z)\right)\right)=0 \qquad \dots (2.43)$$

<u>Case (iii)</u>**:** *H* with respect to y and y' is (*ii*)-gH differentiable ((*i*)-gH differentiable) and *F* is (*i*)-gH differentiable ((*ii*)-gH differentiable) with respect to y and y' .

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z)+\underline{\lambda}_1\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z)+\underline{\lambda}_1\overline{H}(z)\right)\right)=0 \qquad \dots (2.44)$$

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z)+\overline{\lambda}_2\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z)+\overline{\lambda}_2\overline{H}(z)\right)\right)=0 \qquad \dots (2.45)$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z)+\underline{\lambda}_2\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z)+\underline{\lambda}_2\underline{H}(z)\right)\right)=0 \qquad \dots (2.46)$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z)+\overline{\lambda}_1\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z)+\overline{\lambda}_1\underline{H}(z)\right)\right)=0 \qquad \dots (2.47)$$

***Case (iv)*:** *H, F* with respect to y' and *F* with respect to y are both (*i*)-gH differentiable ((*ii*)-gH differentiable) and *H* is (*ii*)-gH differentiable ((*i*)-gH differentiable) with respect to y.

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z)+\underline{\lambda}_1\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z)+\underline{\lambda}_1\underline{H}(z)\right)\right)=0 \qquad \dots (2.48)$$

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z)+\overline{\lambda}_2\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z)+\overline{\lambda}_2\underline{H}(z)\right)\right)=0 \qquad \dots (2.49)$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z)+\underline{\lambda}_2\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z)+\underline{\lambda}_2\overline{H}(z)\right)\right)=0 \qquad \dots (2.50)$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z)+\overline{\lambda}_1\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z)+\overline{\lambda}_1\overline{H}(z)\right)\right)=0 \qquad \dots (2.51)$$

***Case (v)*:** *H, F* with respect to y and *H* with respect to y' are both (*i*)-gH differentiable ((*ii*)-gH differentiable) and *F* is (*ii*)-gH differentiable ((*i*)-gH differentiable) with respect to y'.

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z)+\underline{\lambda}_1\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z)+\underline{\lambda}_1\underline{H}(z)\right)\right)=0 \qquad \dots (2.52)$$

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z)+\overline{\lambda}_2\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z)+\overline{\lambda}_2\underline{H}(z)\right)\right)=0 \qquad \dots (2.53)$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z)+\underline{\lambda}_2\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z)+\underline{\lambda}_2\overline{H}(z)\right)\right)=0 \qquad \dots (2.54)$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z)+\overline{\lambda}_1\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z)+\overline{\lambda}_1\overline{H}(z)\right)\right)=0 \qquad \dots (2.55)$$

***Case (vi)*:** *H, F* with respect to y are both (*i*)-gH differentiable ((*ii*)-gH differentiable) and *H, F* are both (*ii*)-gH differentiable ((*i*)-gH differentiable) with respect to y'.

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z)+\underline{\lambda}_1\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z)+\underline{\lambda}_1\overline{H}(z)\right)\right)=0 \qquad \dots (2.56)$$

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z)+\overline{\lambda}_2\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z)+\overline{\lambda}_2\overline{H}(z)\right)\right)=0 \qquad \dots (2.57)$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z)+\underline{\lambda}_2\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z)+\underline{\lambda}_2\underline{H}(z)\right)\right)=0 \qquad \dots (2.58)$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z)+\overline{\lambda}_1\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z)+\overline{\lambda}_1\underline{H}(z)\right)\right)=0 \qquad \dots (2.59)$$

***<u>Case (vii)</u>*:** *H* with respect to y' and *F* with respect to *y* are both (*i*)-gH differentiable ((*ii*)-gH differentiable) and H with respect to *y* and *F* with respect to y' are both (*ii*)-gH differentiable ((*i*)-gH differentiable).

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z)+\underline{\lambda}_1\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z)+\underline{\lambda}_1\underline{H}(z)\right)\right)=0 \qquad \dots (2.60)$$

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z)+\overline{\lambda}_2\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z)+\overline{\lambda}_2\underline{H}(z)\right)\right)=0 \qquad \dots (2.61)$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z)+\underline{\lambda}_2\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z)+\underline{\lambda}_2\overline{H}(z)\right)\right)=0 \qquad \dots (2.62)$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z)+\overline{\lambda}_1\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z)+\overline{\lambda}_1\overline{H}(z)\right)\right)=0 \qquad \dots (2.63)$$

***<u>Case (viii)</u>*:** *F* with respect to y is (*i*)-gH differentiable ((*ii*)-gH differentiable) and *H*, *F* with respect to y' and *H* with respect to y are both (*ii*)-gH differentiable ((*i*)-gH differentiable).

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z)+\underline{\lambda}_1\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z)+\underline{\lambda}_1\underline{H}(z)\right)\right)=0 \qquad \dots (2.64)$$

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z)+\overline{\lambda}_2\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z)+\overline{\lambda}_2\underline{H}(z)\right)\right)=0 \qquad \dots (2.65)$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z)+\underline{\lambda}_2\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z)+\underline{\lambda}_2\overline{H}(z)\right)\right)=0 \qquad \dots (2.66)$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z)+\overline{\lambda}_1\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z)+\overline{\lambda}_1\overline{H}(z)\right)\right)=0 \qquad \dots (2.67)$$

As an illustration, we consider the following example, [20]:

***Example* (2.2):**

Find the minimum of:

$$J(\tilde{y}) = \int_0^1 -(\tilde{y}')^2 dx$$

Subject to

$$I(\tilde{y}) = \int_0^1 \tilde{y}(x) dx \approx c = \langle 0, 1, 3 \rangle$$

$$\tilde{y}(0) \approx \tilde{2} = \langle 0, 2, 4 \rangle, \qquad \tilde{y}(1) \approx \tilde{4} = \langle 2, 4, 6 \rangle$$

In order to find the optimal solution of the above problem, it suffices to find the optimal solution of:

$$\mathcal{J}(\tilde{y}) = \int_0^1 (-(\tilde{y}')^2 + \lambda\, \tilde{y}(x)) dx$$

given that:

$$\tilde{y}(0) \approx \tilde{2} = \langle 0, 2, 4 \rangle, \qquad \tilde{y}(1) \approx \tilde{4} = \langle 2, 4, 6 \rangle$$

We first derive the α-level set of $\mathcal{J}$ as follows:

$$J(y)_\alpha = \int_0^1 \left[-\left(\overline{y}'\right)^2 (x, \alpha) + \underline{\lambda}(\alpha)\, \underline{y}(x, \alpha), -\left(\underline{y}'\right)^2 (x, \alpha) + \overline{\lambda}(\alpha)\, \overline{y}(x, \alpha) \right] dx \qquad \dots (2.68)$$

We have $F = -(\tilde{y}')^2$ and $H = \tilde{y}(x)$. Suppose that case (i) is fulfilled in this constrained fuzzy variational problem, i.e., F and H are (i)-gH differentiable ((ii)-gH differentiable) with respect to y and y'. Using equation (2.34) for equation (2.68), we have $\left(\underline{y}''\right)^2 (x, \alpha) = 0$. Hence, by virtue of the classical differential equation theory, we may solve it analytically for fixed $\alpha \in [0,1]$ to get:

$$\underline{y}(x, \alpha) = k_1 x + k_2$$

Here, the constants of integration, i.e., k_1 and k_2, might be given by the end-point conditions, so:

$$\underline{y}(x, \alpha) = 2x + 2\alpha$$

On the other hand, in view of $< 0,1,3 >_{[\alpha]}= [\alpha, 3-2\alpha]$, the above left-hand endpoint of the α-level set of extremal must satisfy the fuzzy constraint $I(x)$. That is:

$$\int_0^1 (2x+2\alpha)\,dx = \alpha$$

which is contradiction with $\alpha \in [0,1]$. Then this problem has no solution with Euler-Lagrange conditions obtained in.

In case (*ii*), by using (2.37), we have $\left(\underline{y}''\right)^2(x,\alpha) = 0$. Similar previous case, we cannot find the solution of problem.

Now, suppose that H with respect to y and y' be (*ii*)-gH differentiable ((*i*)-gH differentiable) and F be (*i*)-gH differentiable ((*ii*)-gH differentiable) with respect to y and y' (according to case (*iii*)). In this case, the fuzzy Euler-Lagrange conditions (2.44), (2.45), (2.46) and (2.47) say:

$$\overline{\lambda}_2(\alpha) - \frac{d}{dx}\left(-2\overline{y}'(x,\alpha)\right) = 0$$

$$\underline{\lambda}_2(\alpha) - \frac{d}{dx}\left(-2\underline{y}'(x,\alpha)\right) = 0$$

From the classical differential equation theory and the endpoint conditions, we have:

$$\overline{y}(x,\alpha) = -\frac{\overline{\lambda}_2(\alpha)}{4}x^2 + \left(2+\frac{\overline{\lambda}_2(\alpha)}{4}\right)x + 4 - 2\alpha \qquad \dots (2.69)$$

$$\underline{y}(x,\alpha) = -\frac{\underline{\lambda}_2(\alpha)}{4}x^2 + \left(2+\frac{\underline{\lambda}_2(\alpha)}{4}\right)x + 4 - 2\alpha \qquad \dots (2.70)$$

Now by virtue of $< 0,1,3 > [\alpha] = [\alpha, 3-2\alpha]$ and the fact that the above left-hand and right-hand endpoints of α-level set of extremal must satisfy the fuzzy constraint $I(x), \overline{\lambda}_2(\alpha), \underline{\lambda}_2(\alpha)$ are determined by considering:

$$3-2\alpha = \int_0^1 \left(-\frac{\overline{\lambda}_2(\alpha)}{4}x^2 + \left(2+\frac{\overline{\lambda}_2(\alpha)}{4}\right)x + 4 - 2\alpha\right) dx$$

$$\alpha = \int_0^1 \left(-\frac{\underline{\lambda}_2(\alpha)}{4} x^2 + \left(2 + \frac{\underline{\lambda}_2(\alpha)}{4}\right) x + 2\alpha \right) dx$$

which result in $\overline{\lambda}_2(\alpha) - 48$ and $\underline{\lambda}_2(\alpha) = -24(\alpha + 1)$. According to this results (2.68) and (2.69) turn to:

$$\overline{y}(x, \alpha) = 12x^2 - 10x + 4 - 2\alpha$$

$$\underline{y}(x, \alpha) = 6(\alpha + 1)x^2 - (4 + 6\alpha)x + 2\alpha$$

One can easily show that:

$$\frac{\partial \overline{y}(x, \alpha)}{\partial \alpha} = -2 \leq 0, \forall x \in [0, 1]$$

$$\frac{\partial \underline{y}(x, \alpha)}{\partial \alpha} = 6x^2 - 6x + 2 \geq 0, \forall x \in [0, 1]$$

that is, $\overline{y}(x, \alpha)$ and $\underline{y}(x, \alpha)$ are continuous nonincreasing and nondecreasing functions of α respectively.

Moreover, for all $0 \leq \alpha \leq 1$,

$$\underline{y}(x, 1) = 12x^2 - 10x + 2,$$

And

$$\overline{y}(x, 1) = 12x^2 - 10x + 2$$

Hence, it holds

$$\underline{y}(x, 1) = \overline{y}(x, 1).$$

Consequently, y parameterized by:

$$y(x)_\alpha = \left[\underline{y}(x, \alpha), \overline{y}(x, \alpha)\right]$$
$$= [6(\alpha + 1)x^2 - (4 + 6\alpha)x + 2\alpha, 12x^2 - 10x + 4 - 2\alpha]$$

that defines α-level set of a fuzzy function which minimizes J in the fuzzy sense.

CHAPTER THREE

Variational Problems with Fuzzy Boundary Conditions

3.1 *Introduction*:

In this chapter, the variational problems with fuzzy boundary conditions is investigated. In second section the Euler-Lagrange equation of such type of problems have been studied. In third section the centroid method for defuzzification have been discussed. In forth section the expected interval method is also discussed. In fifth section the centroid point method for defuzzification is introduced explained discussed with an illustrative example.

3.2 *Fuzzy Variational Problem*:

Consider the problem of minimizing the fuzzy functional:

$$J(y) = \int_{\tilde{x}_0}^{\tilde{x}_1} F(x, y, y')dx \qquad \ldots (3.1)$$

and $\tilde{x}_0$ or $\tilde{x}_1$ are fuzzy numbers with fuzzy boundary conditions:

$$y(\tilde{x}_0) = \tilde{y}_0, y(\tilde{x}_1) = \tilde{y}_1$$

where y is real valued function.

For simplicity, it is assumed that one of the boundaries $\tilde{x}_0$ or $\tilde{x}_1$, for instance, $\tilde{x}_0$ is fixed non fuzzy number while the other end point $\tilde{x}_1$is a fuzzy number.

Consider the following well-known description of a fuzzy number $\tilde{x}_1$:

$$\tilde{x}_1(x) = \begin{cases} l_1(x), & a_1 \leq x \leq a_2 \\ 1 \quad , & a_2 \leq x \leq a_3 \\ l_2(x), & a_3 \leq x \leq a_4 \\ 0 \quad , & otherwise \end{cases}$$

where $a_1, a_2, a_3, a_4 \in R, l_1: [a_1, a_2] \to [0, 1]$ is nondecreasing upper semi-continuous function, $l_1(a_1) = 0, l_1(a_2) = 1$, called the left side of the fuzzy number and $l_2: [a_3, a_4] \to [0, 1]$ is nonincreasing upper semi-continuous function, $l_2(a_3) = 1, l_2(a_4) = 0$, called the right side of the fuzzy number.

We transform the fuzzy number $\tilde{x}_1$ into a crisp number by using many approaches that will be explained next.

Fuzzy numbers may be also represented using the α −cut or α −level sets and for this purpose, we represent $\tilde{x}_1$ in a parametric form where $\tilde{x}_1 = [\underline{x}_1(\alpha), \overline{x}_1(\alpha)]$, where $\underline{x}_1(\alpha)$ and $\overline{x}_1(\alpha)$ are the endpoints of the α-level set and $\alpha \in [0, 1]$. In case of the trapezoidal fuzzy number $\tilde{x}_1$=[x, y,δ, β] with $\delta > 0$ and $\beta > 0$. The parametric form of $\tilde{x}_1$ is:

$$\underline{x}_1(\alpha) = x - \delta + \delta\alpha,$$

$$\overline{x}_1(\alpha) = y + \beta - \beta\alpha$$

where $\alpha \in [0, 1]$ provided that if $x = y$ then $\tilde{x}_1$ is a triangular fuzzy number, and we write $\tilde{x}_1 = [x, \delta, \beta]$. Also, by setting the value of α, we obtain the crisp value of lower and upper bounds $\underline{x}_1(\alpha)$ and $\overline{x}_1(\alpha)$ which are corresponding to $\tilde{x}_1$.

3.3 *The Centroid Method for Defuzzification*, [9]:

The centroid method is the most popular method for defuzzification, i.e., transforming fuzzy problems into nonfuzzy problems.

This method determines the center of the area of the combined membership functions.

We use this approach when the fuzzy number $\tilde{x}_1$ is the union of two or more fuzzy numbers. So, the fuzzy number $\tilde{x}_1 = [a_1, a_2, a_3, a_4]$ will be transformed into a crisp number x^*, by using the centroid method as follows:

$$x^* = \frac{\int_{\tilde{x}_1} \mu_{\tilde{x}_1}(x)\, x\, dx}{\int_{\tilde{x}_1} \mu_{\tilde{x}_1}(x)\, dx},\ a_1 \le x \le a_4 \qquad \ldots (3.2)$$

where $\int_{\tilde{x}_1}$ means that the integration of the membership function is carried over each line segment of the produced union fuzzy number.

Now, equation (3.1) can be solved by the same as in the problem presented in chapter two.

***Example* (3.1):**

Find the minimum of the functional:

$$J(y) = \int_0^{\tilde{x}_1} -(y')^2\, dx \qquad \ldots (3.3)$$

with

$$y(0) \approx \tilde{2} = \langle 0, 2, 4 \rangle, \quad y(\tilde{x}_1) \approx \tilde{4} = \langle 2, 4, 6 \rangle$$

with $\tilde{x}_1 = \tilde{A}_1 \cup \tilde{A}_2 \cup \tilde{A}_3$, where $\tilde{A}_1 =< 0, 1, 4, 5 >$, $\tilde{A}_2 =< 3, 4, 6, 7 >$,

$\tilde{A}_3 =< 5, 6, 7, 8 >$, then:

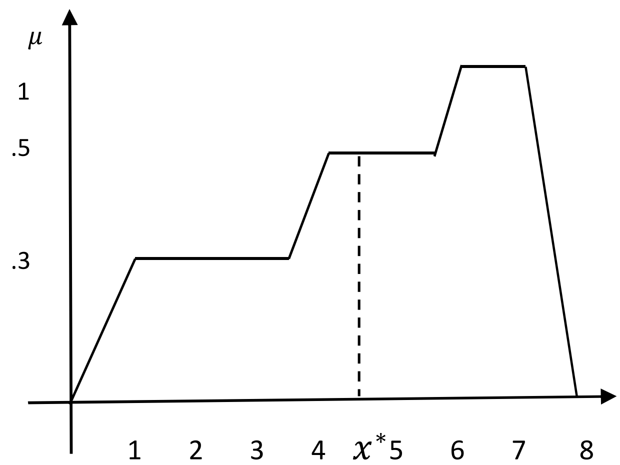

***Figure* (3.1) *Membership function of example* (3.1)**

$$x^* = \frac{\int_{\tilde{x}_1} \mu_{\tilde{x}_1}(x)\, t\, dt}{\int_{\tilde{x}_1} \mu_{\tilde{x}_1}(x)\, dt}, \qquad 0 \le x \le 8$$

$$x^* = \left[\int_0^1 (0.3\,x)xdx + \int_1^{3.6} (0.3\,x)\,dx + \int_{3.6}^4 \left(\frac{x-3}{2}\right) xdx + \int_4^{5.5} (0.5)xdx \right.$$
$$\left. + \int_{5.5}^6 (x-5)xdx + \int_6^7 xdx + \int_7^8 (8-x)xdx \right]$$
$$\div \left[\int_0^1 (0.3\,x)dx + \int_1^{3.6} (0.3)dx + \int_{3.6}^4 \left(\frac{x-3}{2}\right) dx + \int_4^{5.5} (0.5)dx + \int_{5.5}^6 (x-5)\,dx \right.$$
$$\left. + \int_6^7 dx + \int_7^8 (8-x)dx \right] = 4.9$$

By representing the endpoint conditions in its α −cut sets:

$$2_\alpha = [2\alpha, 2\frac{1}{\alpha}]$$

$$4_\alpha = [4\alpha, 4\frac{1}{\alpha}]$$

using x^* which is a crisp number, then equation (3.3) becomes:

$$J(y) = \int_0^{4.9} -(y')^2 \, dx$$

with

$$y(0) \approx \tilde{2}, \quad y(4.9) \approx \tilde{4}$$

then the Euler-Lagrange equation is:

$$2y''(x) = 0$$

then:

$$y(x) = c_1 \, x + c_2$$

Using the boundary conditions:

$$\underline{y}(0, \alpha) = \underline{c_2} = 2\alpha$$

$$\overline{y}(0, \alpha) = \overline{c}_2 = 2\frac{1}{\alpha}$$

and;

$$\underline{y}(4.9, \alpha) = \underline{c_1}(4.9) + 2\alpha = 4\alpha$$

$$\underline{c_1} = 0.41\alpha$$

$$\overline{y}(4.9, \alpha) = \overline{c}_1(4.9) + 2\frac{1}{\alpha} = 4\,\frac{1}{\alpha}$$

$$\overline{c}_1 = 0.41\frac{1}{\alpha}$$

hence:

$$\underline{y}(x,\alpha) = 0.41\alpha\, x + 2\alpha$$

$$\overline{y}(x,\alpha) = 0.41\frac{1}{\alpha}\, x + 2\frac{1}{\alpha}$$

Then:

$$y(x)_\alpha = \left[\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right] = \left[0.41\alpha\, x + 2\alpha\,, 0.41\frac{1}{\alpha}\, x + 2\frac{1}{\alpha}\right]$$

defines the α-level sets of a fuzzy function which minimizes J.

3.4 *The Expected Interval for Defuzzification*, [26]:

The interval of defuzification can be used as a crisp approximation set with respect to a fuzzy number or any fuzzy quantity.

The α-cut of a fuzzy number $\tilde{x}_1$ (for simplicity set $A = \tilde{x}_1$) is:

$$A_\alpha = \left[\underline{A}(\alpha), \overline{A}(\alpha)\right],$$

where $\alpha \in [0,1]$ and :

$$\underline{A} = inf\{x \in R: \mu_A \geq \alpha\},$$

$$\overline{A} = sup\{x \in R: \mu_A \geq \alpha\},$$

The expected interval $EI(A)$ of a fuzzy number A is defined by:

$$EI(A) = [E_*(A), E^*(A)] = [\int_0^1 \underline{A}(\alpha)\, d\alpha,\ \int_0^1 \overline{A}(\alpha)\, d\alpha]$$

Fuzzy numbers with simple membership functions are preferred in practice. The most used such fuzzy numbers are the trapezoidal fuzzy numbers. A trapezoidal fuzzy number T, $T_\alpha = [\underline{T}(\alpha), \overline{T}(\alpha)]$, $\alpha \in [0,1]$, is given by:

$$\underline{T}(\alpha) = x_1 + (x_2 - x_1)\alpha,$$

and

$$\overline{T}(\alpha) = x_4 + (x_4 - x_3)\alpha,$$

where $x_1, x_2, x_3, x_4 \in R, x_1 \le x_2 \le \ x_3 \le x_4$. When $x_2 = x_3$, we obtain a triangular fuzzy number. We denote:

$$T = [x_1, x_2, x_3, x_4],$$

to be a trapezoidal fuzzy number and by $F^T(R)$ the set of all trapezoidal fuzzy numbers. The expected interval for a trapezoidal fuzzy number T is:

$$EI(T) = \left[\frac{x_1+x_2}{2}, \frac{x_3+x_4}{2}\ \right] \qquad \ldots (3.4)$$

and by using (3.4), problem (3.3) can be solved.

***Example* (3.2):**

Find the minimum of:

$$J(y) = \int_0^{\tilde{x}_1} -(y')^2 \, dx$$

with

$$y(0) \approx \tilde{2}, \quad y(\tilde{x}_1) \approx \tilde{4}$$

where $\tilde{x}_1 = [0,1,3,4]$ is a trapezoidal fuzzy number.

$$EI(\tilde{x}_1) = \left[\frac{0+1}{2}, \frac{3+4}{2}\right] = \left[\frac{1}{2}, \frac{7}{2}\right]$$

using $EI(\tilde{x}_1)$ which is a crisp number, then equation (3.3) becomes:

$$J(y) = \int_0^{0.5} -(y')^2 \, dx$$

with

$$y_\alpha(0) \approx 2_\alpha \approx [\alpha\, 2, \frac{1}{\alpha}\; 2], \quad y_\alpha(\tilde{x}_1) \approx \left[\underline{y}\left(\frac{1}{2}\right), \overline{y}\left(\frac{7}{2}\right)\right] \approx 4_\alpha \approx [\alpha\, 4, \frac{1}{\alpha}\; 4]$$

then the Euler-Lagrange equation is:

$$2y''(x) = 0$$

then:

$$y(x) = c_1\, x + c_2$$

Using the boundary conditions:

$$\underline{y}(0,\alpha) = \underline{c_2} = 2\alpha$$

$$\overline{y}(0,\alpha) = \overline{c}_2 = 2\frac{1}{\alpha}$$

and;

$$\underline{y}\left(\frac{1}{2},\alpha\right) = \underline{c_1}\left(\frac{1}{2}\right) + 2\alpha = 4\alpha$$

$$\underline{c_1} = 4\alpha$$

$$\overline{y}\left(\frac{7}{2},\alpha\right) = \overline{c}_1\left(\frac{7}{2}\right) + 2\frac{1}{\alpha} \;\; = 4\,\frac{1}{\alpha}$$

$$\overline{c}_1 = \frac{4}{7}\,\alpha$$

hence:

$$\underline{y}(x,\alpha) = 4\alpha\, x + 2\alpha$$

$$\overline{y}(x,\alpha) = 0.6\frac{1}{\alpha}\,x + 2\frac{1}{\alpha}$$

Then:

$$y(x)_\alpha = \left[\underline{y}(x,\alpha), \overline{y}(x,\alpha)\,\right] = \left[4\alpha\, x + 2\alpha\,, 0.6\frac{1}{\alpha}\,x + 2\frac{1}{\alpha}\;\right]$$

defines the α-level sets of a fuzzy function which minimizes J.

3.5 Centroid Point Method for Defuzzification, [25]:

This method is used for extended fuzzy number. An extended fuzzy number $\tilde{A}$ is described as any fuzzy subset of the universe set U with membership function $\mu_{\tilde{A}}$ defined as follow:

- $\mu_{\tilde{A}}$ is a continuous mapping from U to the closed interval $[0,w], 0 < w \leq 1$.
- $\mu_{\tilde{A}}(x) = 0$, for all $x \in (-\infty, a_1]$.
- $\mu_{\tilde{A}}$ is strictly increasing between $[a_1, a_2]$.
- $\mu_{\tilde{A}}(x) = w$, for all $x \in [a_2, a_3]$, w is a constant and $0 < w \leq 1$.
- $\mu_{\tilde{A}}$ is strictly decreasing between $[a_3, a_4]$.
- $\mu_{\tilde{A}}(x) = 0$, for all $x \in [a_1, +\infty]$.

In the above situations a_1, a_2, a_3 and a_4 are real numbers. If $a_1 = a_2 = a_3 = a_4$, $\tilde{A}$ becomes a crisp real number.

The membership function $\mu_{\tilde{A}}$ of the extended fuzzy number $\tilde{A}$ may be expressed as:

$$\mu_{\tilde{A}} = \begin{cases} f_{\tilde{A}}^{L}(x), & when\ a_1 \leq x \leq a_2 \\ w, & when\ a_2 \leq x \leq a_3 \\ f_{\tilde{A}}^{R}(x), & when\ a_3 \leq x \leq a_4 \\ 0, & otherwise \end{cases}$$

where $f_{\tilde{A}}^{L}(x): [a_1, a_2] \longrightarrow [0, w]$ and $f_{\tilde{A}}^{R}(x): [a_3, a_4] \longrightarrow [0, w]$. Based on the basic theories of fuzzy numbers, is a normal fuzzy number if $w = 1$, whereas $\tilde{A}$ is a non-normal fuzzy number if $0 < w \leq 1$. Therefore, the extended fuzzy number $\tilde{A}$ can be denoted as $[a_1, a_2, a_3, a_4; w]$. The image $-\tilde{A}$ of $\tilde{A}$ can be expressed by $[-a_1, -a_2, -a_3, -a_4; w]$.

If $\tilde{x}_1$ is extended fuzzy number, let $g_{\tilde{x}_1}^{L}(y): [0, w] \rightarrow [a_1, a_2]$ and $g_{\tilde{x}_1}^{R}(y): [0, w] \rightarrow [a_3, a_4]$ be the inverse functions of $f_{\tilde{x}_1}^{L}$ and $f_{\tilde{x}_1}^{R}$, respectively. Then $g_{\tilde{x}_1}^{L}(y)$ and $g_{\tilde{x}_1}^{R}(y)$ should be integrable on the closed interval $[0, w]$. In the other words, both $\int_0^w g_{\tilde{x}_1}^{L}(y)\, dy$ and $\int_0^w g_{\tilde{x}_1}^{R}(y)\, dy$ should exist.

In the case of trapezoidal fuzzy number, the inverse functions $g_{\tilde{x}_1}^{L}(y)$ and $g_{\tilde{x}_1}^{R}(y)$ can be analytically expressed as:

$$g_{\tilde{x}_1}^{L}(y) = a_1 + \frac{(a_2 - a_1)y}{w}, 0 \leq y \leq w$$

$$g_{\tilde{x}_1}^{R}(y) = a_4 + \frac{(a_4 - a_3)\, y}{w}, 0 \leq y \leq w.$$

In order to determine the centroid point $(\overline{x}_0(\tilde{x}_1), \overline{y}_0(\tilde{x}_1))$ of a fuzzy number $\tilde{x}_1$. provided the following centroid formulae, [23]:

$$\overline{x}_0(\tilde{x}_1) = \frac{\int_{a_1}^{a_2} x\, f_{\tilde{x}_1}^L(x)\, dx + \int_{a_2}^{a_3} (x\, w)\, dx + \int_{a_3}^{a_4} x\, f_{\tilde{x}_1}^R(x)\, dx}{\int_{a_1}^{a_2} f_{\tilde{x}_1}^L(x)\, dx + \int_{a_2}^{a_3} (w)\, dx + \int_{a_3}^{a_4} f_{\tilde{x}_1}^R(x)\, dx} \qquad \dots (3.5)$$

$$\overline{y}_0(\tilde{x}_1) = \frac{\int_0^w y\left(g_{\tilde{x}_1}^R(y) - g_{\tilde{x}_1}^L(y)\right) dy}{\int_0^w \left(g_{\tilde{x}_1}^R(y) - g_{\tilde{x}_1}^L(y)\right) dy} \qquad \dots (3.6)$$

For this trapezoidal fuzzy number, the following results are derived from (3.5) and (3.6),

$$\overline{x}_0 = \frac{1}{3}\left[a_1 + a_2 + a_3 + a_4 - \frac{a_4 a_3 - a_1 a_2}{(a_4 + a_3) - (a_{1} - a_2)}\right] \qquad \dots (3.7)$$

$$\overline{y}_0 = w \frac{1}{3}\left[1 + \frac{a_3 - a_2}{(a_4 + a_3) - (a_{1} - a_2)}\right] \qquad \dots (3.8)$$

So, by using the results of (3.7) and (3.8), we can solve equation (3.3) by the same way as in chapter two.

***Example* (3.3):**

Find the minimum of the functional:

$$J(y) = \int_0^{\tilde{x}_1} -(y')^2\, dx$$

with

$$y(0) = y_0, \quad y(\tilde{x}_1) = y_1$$

where $\tilde{x}_1 = [0, 1, 3, 4; 1]$ is a trapezoidal fuzzy number, then the centroid point $(\overline{x}_0(\tilde{x}_1), \overline{y}_0(\tilde{x}_1))$ of a fuzzy number $\tilde{x}_1$ is:

$$\overline{x}_0 = \frac{1}{3}\left[a_1 + a_2 + a_3 + a_4 - \frac{a_4 a_3 - a_1 a_2}{(a_4 + a_3) - (a_{1} - a_2)}\right]$$

$$= \frac{1}{3}\left[0 + 1 + 3 + 4 - \frac{(4)(3) - (0)(1)}{(4 + 3) - (0 + 1)}\right] = 3.33$$

$$\overline{y}_0 = w\frac{1}{3}\left[1 + \frac{a_3 - a_2}{(a_4 + a_3) - (a_{1-}a_2)}\right]$$

$$= (1)\frac{1}{3}\left[1 + \frac{3 - 1}{(4 + 3) - (0 - 1)}\right] = 0.42$$

Now, using the centroid point $(3.33, 0.42)$ of a fuzzy number $\tilde{x}_1$, equation (3.3) can be solve as in chapter two.

using x^* which is a crisp number, then equation (3.3) becomes:

$$J(y) = \int_0^{3.3} -(y')^2\, dx$$

with

$$y(0) \approx \tilde{2}, \quad y(3.33) \approx \tilde{4}$$

then the Euler-Lagrange equation is:

$$2y''(x) = 0$$

then:

$$y(x) = c_1\, x + c_2$$

Using the boundary conditions:

$$\underline{y}(0, \alpha) = \underline{c}_2 = 2\alpha$$

$$\overline{y}(0, \alpha) = \overline{c}_2 = 2\frac{1}{\alpha}$$

and;

$$\underline{y}(3.33,\alpha) = \underline{c}_1(3.33) + 2\alpha = 4\alpha$$

$$\underline{c}_1 = 1.2\alpha$$

$$\overline{y}(3.33,\alpha) = \overline{c}_1(3.33) + 2\frac{1}{\alpha} = 4\,\frac{1}{\alpha}$$

$$\overline{c}_1 = 0.6\frac{1}{\alpha}$$

hence:

$$\underline{y}(x,\alpha) = 1.2\alpha\, x + 2\alpha$$

$$\overline{y}(x,\alpha) = 0.6\frac{1}{\alpha}\, x + 2\frac{1}{\alpha}$$

Then:

$$y(x)_\alpha = \left[\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right] = \left[1.2\alpha\, x + 2\alpha\,, 0.6\frac{1}{\alpha}\, x + 2\frac{1}{\alpha}\right]$$

defines the α-level sets of a fuzzy function which minimizes J.

CONCLUSIONS AND FUTURE WORKS

From the present study, we can conclude the following:

1. As it is expected, there is a very strong relationship between fuzzy variational problems and it level sets in ordinary form (parametric form).
2. The validity of the results obtained by solving the Euler-Lagrange equation may be checked by setting $\alpha = 1$ in the α-level of solution of related to the fuzzy solution, in which the upper and lower solutions must be equal when setting $\alpha = 1$. The crisp solution may be obtained from the fuzzy solution, by setting $\alpha = 1$ in the α-level solution related to the fuzzy solution.
3. Fuzzy variational problems may be considered as a generalization to the nonfuzzy variational problems.

Also, we can recommend the following for future work:

1. Studying real life problems, in which the governing mathematical modeling is fuzzy variational problems.
2. Studying fuzzy variational problems using other definition for differentiation.
3. Solving non-linear fuzzy variational problems using variational approach.
4. Solving fuzzy variational problems with moving boundaries.

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الملخص

الاهداف الرئيسيه لهذه الاطروحه تصب في ثلاث اتجاهات: الهدف الاول هو دراسة نظرية المجاميع الضبابيه مع بعض المفاهيم الاساسيه لنظرية مسائل التغاير.

الهدف الثاني هو دراسة مسائل التغاير ذات الدوال الضبابيه والشروط الضبابيه والحدود الضبابيه باستخدام طرق مختلفه لتحويل المجاميع الضبابيه الى مجاميع حقيقيه على سبيل المثال طريق: المركزيه، ال(α – قطع)، النقطة المركزيه، الفتره المتوقعه.

الهدف الثالث هو ايجاد الحلول لمسائل التغاير الضبابيه ذات الدوال الضبابيه و الحدود الضبابيه.

جمهـورية العـراق
وزارة التعليم العالي والبحث العلمي
جـــامعة النهـــريـن
كليــة العلوم
قسم الريـاضيات وتطبيقات الحاسوب

الشروط المثاليه للمسائل التغايريه ذات الرتب الضبابيه

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وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل
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