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Solutions of Stochastic Ordinary Differential Equations Using Variable Step Size Runge-Kutta Methods

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بِّسْمِ اللهِ الرَّحْمَنِ الرَّحِيْمِ

﴿ إقْرأ بأسْمِ رَبِكَ الذي خَلَقَ * خَلَقَ

الانسانَ مِنْ عَلَقْ * إقْرأ وربُكَ

الأكْرَم * الذي عَلَمَ بالقَلَمْ * عَلَمْ

الانْسانَ مالَم يَعلَم * ﴾

صَدَقَ اللهُ العَظيم

سورة العلق

الإهداء

إلى من لا تراه العين ولا يصفه الواصفون .. الذي لولاهُ لما تمَ هذا العمل ...

أللّه "جلّ وعلا"

إلى سيد المرسلين وخاتم ألنبيين... إلى سيد ألكائنات...

محمد "صلى الله عليه واله وسلم"

إلى ذوي قربى ألرسول ألخاتم **محمد** " صلى الله عليه واله وسلم " ...

أئمتي ألأطهار "عليهم السلام"

إلى من ساند وأزر ألرسول ألخاتم " صلى الله عليه واله وسلم " ...

ألصحابة ألكرام " عليهم السلام "

إلى مثلي الأعلى واستاذي الأول .. إلى القلب الطيب والروح المتفانية المخلصة .. إلى من اهداني عمره وشبابه وعلمني اهداف ومعاني الحياة...

ابي

إلى منبع الحياة وخير الدنيا و صفائها .. إلى التي أخص الله سبحانه و تعالى الجنة تحت أقدامها .. إلى من كان بدعائها سر نجاحي وكان رضاها سر سعادتي والتي يغدو رد جميلها مستحيلاً ...

امي

إلى الأيادي التي ساندتني لكي أسير وأصل إلى ما أنا عليه ألان...

ألأُستاذين المُشرِفين

إلى الشموع التي اضائت لي طريق العلم...

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إلى سندي وفخري وعوني .. إلى توائم روحي ونور عيني .. إلى من لرؤيتهم يزول همي وتقر عيني...

إخوتي واخواتي

إلى من واكبوا سنين العمر بإخلاص ...

كل أصدقائي

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Supervisors Certifications

We certify that this thesis was prepared under our supervision at the department of Mathematics, College of Science, Al-Nahrain University, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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Examining Committee Certification

We certify that we have read this thesis entitled "*Solutions of Stochastic Ordinary Differential Equations Using Variable Step Size Runge-Kutta Methods*" and as examining committee examined the student *(Mustafa Mohammed Subhi)* in its contents and in what it connected with, and that, in our opinion, it meets the standards of a thesis for the degree of Master of Science in Mathematics.

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Abstract

The main objective of this thesis is divided in to three directions which are:

The first one is to study and overview the main and basic concepts of stochastic calculus, as well as, studying stochastic ordinary differential equations.

The second objective is to study explicit stochastic Runge-Kutta methods, then generalize this scheme for semi-explicit, implicit and mixed schemes and study theirs numerical stability.

The third objective is to introduce variable step size method for solving stochastic ordinary differential equations, which has the utility of improving the accuracy of the obtained results.

Basic Notations and Abbreviations

$-\alpha$	Delete the first components of a multi index α.
$\alpha-$	Delete the last components of a multi index α.
α^{+}	Delete all zero components of a multi index α.
$\alpha(t)$	Number of ways for labeling the vertices of t.
$\theta(t)$	The corresponding J-integral associated with tree t.
$\mathcal{B}$	The σ-algebra of Borel subsets of $\mathbb{R}^m$.
δ_{ij}	Kronecker delta function, where $\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$
$\Phi_{\cdot,t_0}, \Phi_{\cdot}$	The fundamental solution.
$G_{\cdot}$	Weak hierarchical set.
$\gamma(t)$	Density of tree.
$L_{\cdot}$	Strong hierarchical set.
$\rho(t)$	Number of vertices.
$\rho(r)$	The first characteristic Polynomial.
$\sigma(t)$	Symmetry of a tree.
ΔW_n	Random increment, such that $\Delta W_n = \Delta W_{t_n} = W_{t_{n+1}} - W_{t_n}$.
$\circ\, dW$	Stratonovich calculus integration symbol.
$\bullet$	Deterministic node.
$\circ$	Stochastic node.
$\mathcal{A}_t$	Filtration, which is an increasing family of σ-algebra fields.
$\mathcal{A}$	σ-Algebra.
$(\mathcal{A}_t)_{t\in I}$	Filtration satisfy the usual conditions, i.e., $(\mathcal{A}_t)_{t\in I}$ is a right-continuous filtration (satisfies $\mathcal{A}_t = \bigcap_{\varepsilon>0} \mathcal{A}_{t+\varepsilon}$ for all $t \geq 0$) and $\mathcal{A}_0$ contains all $\boldsymbol{P}$-negligible events in $\mathcal{A}$. Further, let $(\mathcal{A}_t)_{t\in I}$ be such that W is a martingale of $(\mathcal{A}_t)_{t\in I}$
e(t)	Local error coefficient.

$C(\mathbb{R}^m, \mathbb{R})$	The space of continuous functions $f : \mathbb{R}^m \longrightarrow \mathbb{R}$.
$C^k(\mathbb{R}^m, \mathbb{R})$	The space of k-times continuously differentiable functions $f : \mathbb{R}^m \longrightarrow \mathbb{R}$.
$C^{h,k}(I \times \mathbf{R}^d, \mathbf{R})$	The space of h-times and k-times (for time and stochastic process respectively) continuously differentiable functions $f : I \times \mathbb{R}^d \longrightarrow \mathbb{R}$.
$F(t)(y)$	Elementary differential.
H	Hierarchical set.
$I_{(.),t}$, $I_.$	Multiple Itô integrals.
$I_{\{.\}}$, I_A	The indicator function of a set A = {.}, where $I_A = \begin{cases} 1, & \text{if A is true} \\ 0, & \text{otherwise} \end{cases}$
$\hat{I}_1$, $\Delta\hat{W}_n$	Is a random variable and must be $A_{T_{n+1}}$-measurable, where $\Delta\hat{W}_n = \hat{I}_1 = \hat{W}_{n+1} - \hat{W}_n$ and $\hat{I}_1$ is another multiple Itô integrals.
$J_{(.),t}$, $J_.$	Multiple Stratonovich integrals.
$K_0(\alpha)$	The number of zero components in multi index α that precede the first non-zero components of α or until the end of α if all of its components are zeros.
$K_i(\alpha)$	The number of zeros between the i^{th} and $(i + 1)^{th}$ non-zero components of multi index α or the end of α, such that i = 1, 2, …, $L(\alpha^+)$.
L	The class of functions, such that $E\left(\int_0^t X_s^2\, ds\right) < \infty$, where E(.) is standing for expectation operators.
$L(\alpha)$	Length of multi index $\alpha = (j_1, j_2, \ldots, j_L)$.
L(v)	Length of multi index v and equal to zero.
L^0, L^1	The drift and the diffusion operators, respectively.
$L_2(\Omega, \mathbb{R}^n)$	The space of all square integrable functions defined from Ω to $\mathbb{R}^n$.
L_n	Local truncation error.
M	The set of all multi indices.

MSRKM's	Mixed Stochastic Runge-Kutta Methods.
MSRKM1	The method (3.15) will be referenced by this code.
MSRKM2	The method (3.19) will be referenced by this code.
MSRKM3	The method (3.23) will be referenced by this code.
MSRKM4	The method (3.27) will be referenced by this code.
MSRKM5	The method (3.31) will be referenced by this code.
MSRKM6	The method (3.35) will be referenced by this code.
$n(\alpha)$	Number of zero components of a multi index α.
P	All measurable functions, such that $\boldsymbol{P}\left(\int_0^t X_s^2\, ds < \infty\right) = 1$ where $P(.)$ is standing for probability set function.
$\boldsymbol{P}$	Probability measure.
PL	The method (2.55) will be referenced by this code.
R	The remainder set.
R_n	The remainder of deterministic part.
R2	The method (2.54) will be referenced by this code.
RKM	Runge-Kutta methods.
S	Stability region.
SODE	Stochastic ordinary differential equation.
SRKM's	Stochastic Runge-Kutta methods.
S_n	The remainder of stochastic part.
SLMM's	Stochastic Linear Multi-step Methods.
u, v	Variables, such that $u \le 0$, $v \ge 0$, and $\mathrm{Re}(u - v) + \mathrm{Re}(v) \le 0$.
W_t, W_t^1	The first Weiner process on time t.
W_t^n	The n^{th} Weiner process on time t.
w.p.1, ***P-w.p.1***	$\boldsymbol{P}$ converges with probability one.
X, X(ω)	Random variables, $\omega \in \Omega$.
$X_t(\omega)$, X_t	Stochastic process.
$X_{\cdot}(\omega)$	X as a function of the variables replaced by the dot for fixed ω.

$X_t(\bullet)$	X as a function of the variables replaced by the dot for fixed t.
$\|.\|_{L_2}$	The norm of $L_2(\Omega, \mathbb{R}^n)$ space and if $Z \in L_2(\Omega, \mathbb{R}^n)$ then $\|Z\|_{L_2} = (E\|Z\|^2)^{\frac{1}{2}}$.

CONTENTS

Introduction

In recent years, stochastic processes and stochastic calculus have been applied to a wide range of scientific disciplines such as physics, engineering, and finance. Stochastic calculus concerns with a specific class of stochastic processes that are stochastically integrable and are often expressed as solutions to the stochastic differential equations, [26]. They are typically describing the time dynamics of the evolution of a state vector, based on the (approximate) physics of the real system, together with a deriving noise process. It often represents processes not included in the model, but presented in real system, [2]. In the physical and engineering sciences, on the other hand, stochastic differential equations (SDE's for short) arise in a quite natural manner in the description of systems on which so-called "white noise" acts, [26], many physical systems are modeled by SDE's, where random effect are being modeled by a Wiener process (for more details, see for example [40]). A natural extension is given by systems of SDE's, where system noise is modeled by including a diffusion on term of some suitable form in the driving equations, [13]. Statistical inference for diffusion type processes satisfying the SDE's driven by Wiener process has been studied earlier and a comprehensive survey of various methods is given in, [34]. Recent years have witnessed that the most efficient and widely applicable approach in solving SDE's seems to be the simulation of sample paths of time discrete approximations on digital computers. This is based on a finite discretization of time interval [0, T] under consideration and generates an approximate values of sample paths step by step at the discretization times, [19].

Stochastic differential equations are differential equations in which one or more of its terms are stochastic processes, and therefore will give solutions which are itself stochastic process, [3]. SDE's are used in wide range of applications in environmental modeling, engineering and biological modeling, [20], and SDE's are a fundamental tool for mathematics and its applications, [18].

The types of SDE's incorporated into the systems are also very important; therefore, various authors have made extensive work on the analytic solution of SDE's, [14], [15], [28], [39] and the numerical solution of SDE's, [19], [27]. Since sometimes SDE's rarely have explicit solutions and hence in some cases accurate numerical methods are vital in order to make their implementation viable. Due to features of the stochastic calculus, the numerical analysis for solving SDE's differs in some key areas from the

already well-developed area of the numerical analysis of ordinary differential equations, [27].

There are two basic type of tasks connected with the simulation of solutions of SDE's. The first occurs in situation where a good pathwise approximation is required, for instance in direct simulations, filtering or testing statistical estimators. The second interest focuses on approximating expectations of functional of the Itô process, such as its probability distribution and its moments, [19]. As more realistic mathematical models become required to take into account random effects and influences in real world systems SDE's have become essential in the accurate description of such situations, [26].

Many SDE's have unknown analytical solution, so it is necessary to derive numerical methods to generate approximations to the exact solution, [7].

Nowdays, there are many researchers, who deals with numerical methods for solving SDE's. Yet the gap between the well-developed theory of SDE's and its application is still wide in range. A crucial task in bridging this gap is the development of an efficient numerical methods for solving SDE's [22], and in this regard one of such numerical methods is the Runge-Kutta methods, which was one of the most important of development numerical methods to give the optimal accuracy to the approximate solution, where Rümelin in (1982) [38], who is first investigated systematically stochastic Runge-Kutta type schemes of strong order. Further derivation of free schemes may be found by Platen in (1984) [32] and Kloeden and Platen in (1995) [22]. Some stochastic Runge-Kutta schemes of strong order 1.5 have been developed by Burrage ([5], [7]) using the rooted tree analysis for calculating the order conditions, [37]. By the same last style of construction with some modification, the derived SRK schemes of weak order 1 and 2 are developed by Rößler in (2003) [37] by using rooted tree analysis for calculating order conditions. At any rate, for simplicity, in this thesis we shall restrict our attention to use a one-dimensional Stratonovich or Itô process X_t with one Wiener process W_t .

In contrast to strong approximations, which require that the simulated paths are close to the exact solution X_t of the SDE, weak approximations do not necessarily need to approximate these paths.

This thesis is composed of three chapters. The first chapter gives a short introduction to some results of stochastic calculus, especially in view of the Wiener process. Rudiments for both Ito and Stratonovich stochastic integration bringing forth Itô and Stratonovich stochastic differential equations are considered. Itô's formula serves as a powerful tool applied for introduction of stochastic Taylor expansions due to Platen and Wagner [35], Representing a generalization of the deterministic Taylor formula. Truncated stochastic Taylor expansions can be used as a numerical schemes for strong convergence. That followed we gave the expectation of stochastic integrals at the end of this chapter.

In chapter two some concepts and definitions , which are needed to study the numerical solutions of stochastic differential equations and some formulae of Runge-Kutta methods are previewed, and used as approach to overcome the disadvantages of Taylor schemes. Therefore, the well known concepts of such methods are reflected for deterministic ordinary differential equations with the powerful theory of rooted trees. This concept builds a basis for the development of stochastic Runge-Kutta methods and the calculation of order conditions for strong convergence. The rooted tree theory was introduced for the stochastic setting by the application of two different kinds of coloured nodes. These representations are the key for the main theory of this chapter, giving general order conditions for stochastic Runge-Kutta methods which can be calculated very easily with the aid of rooted tree theory. It turns out that these conditions are a generalization of the well known deterministic order conditions. The stability of the numerical method was also studied.

In chapter three, the derivations of some semi-explicit, implicit and mixed stochastic Runge-Kutta methods are introduced individually and then studying the stability of such schemes which is given for completeness. Also in addition, variable step size method for stochastic version has been proposed and implemented in this chapter; some illustrative examples are given for comparison purpose between the given different schemes and that proposed in this study.

Chapter One

1

Basic Concepts of Stochastic Calculus

Chapter One

Basic Concepts of Stochastic Calculus

Introduction:

Stochastic calculus is concerned with the study of stochastic processes, which involve randomness or noise. Intuitively, this requires knowledge of random variables and probability measures. Therefore, this chapter provides the background definitions and concepts that will be required later.

Only those definitions which are of direct relevance to this exposition are given here. For more details see Arnold [3], Gard (1988) [17], Kloeden and Platen [22] for example.

This chapter consists of five sections. In section (1.1), some basic concepts related to the probability theory are given. In section (1.2), we discuss stochastic properties with some related properties are discussed. In section (1.3), theory of stochastic ODE's are given. In section (1.4) the theory of stochastic Taylor series expansion was discussed and giving some concepts about multiple (Itô and Stratonovich) integrals and multi-index basis and finally in section (1.5) the expectation of stochastic integrals was considered.

1.1 Probability Theory Background [3],[7]:

This section contains the background materials of probability theory that directly required for the work carried out in this thesis.

Definition (1.1):

The ***s-algebra*** A of subsets of a sample space Ω and satisfies the following:

1. $\Omega \in A$.
2. If $A \in A$, then $A^c = \{\omega \in \Omega \mid \omega \notin A\} \in A$.
3. For any sequence $\{A_n\} \subseteq A$, then $\mathbf{U}_{n=1}^{\infty} A_n \in A$ and $\mathbf{I}_{n=1}^{\infty} A_n \in A$.

The elements of A are called ***probability measurable sets*** and the pair (Ω, A) is called a probability ***measurable space***.

For example, a ***Borel Set (ß)*** in $\mathbb{R}^m$ is a σ-algebra generated by open sets in $\mathbb{R}^m$ (and so includes open, closed, half open and other m-dimensional intervals).

Definition(1.2):

A probability space triplet (Ω, $\mathcal{A}$, P[.]), where Ω is a sample space (set of all possible outcomes of random increment), $\mathcal{A}$ is class of all subset of Ω and p[.] is a probability set function whose domain is Ω and counter domain is the interval [0,1].

Definition (1.3):

A random vector $X=(X_1, X_2, \ldots, X_n)$ is said to have multivariate Normal (or Gaussian) distribution, denoted by $X \sim N(\mu, \Sigma)$ if X has p.d.f. is:

$$f(X) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(X-\mu)^T \Sigma (X-\mu)}$$

where $\mu=(\mu_1, \mu_2, \ldots, \mu_n)^T$ is the mean vector and $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \mathbf{K} & \sigma_{1n} \\ \mathbb{M} & & \mathbf{O} & \mathbb{M} \\ \sigma_{n1} & \sigma_{n2} & \mathbf{L} & \sigma_{nn} \end{pmatrix}$ is $n \times n$ symmetric positive definite variance-covariance matrix.

Conditional expectations are used to provide estimates of X based on possibly incomplete information about X. Thus if some event A has occurred, then the estimate of X can be improved upon by calculating the conditional expectation of X give the event A, where now only the values of X, as determined by the event A, need to be considered. For example, suppose that X is a continuous random variable with probability density function f(x), and let $A = \{\omega \in \Omega : a \le X(\omega) \le b\}$. Then the ***conditional density function of X given A*** is

$$f(x \mid A) = \begin{cases} f(x) / \int_a^b f(x)dx & ; a \le x \le b \\ 0 & ;\text{elsewhere} \end{cases}$$

The corresponding conditional expectation of X given event A is

$$E(X \mid A) = \int_a^b x\, f(x \mid A)\, dx$$
$$= \int_a^b x\, f(x)\, dx \Big/ \int_a^b f(x)\, dx$$

If there is an infinite sequence of random variables, then it is important to know how the sequence converges. There is a number of different modes of convergence, as it is give in the next definitions:

Definition (1.4):

A sequence of random variables $\{X_n(\omega)\}$ ***converges with probability one*** (denoted by ***P-w.p.1*** or ***w.p.1***) to $X(\omega)$ if

$$\boldsymbol{P}(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1.$$

This is also called ***almost sure convergence***.

Definition (1.5):

A sequence of random variables $\{X_n(\omega)\}$ such that $E(X_n^2) < \infty$ for all n is said to be ***converges in the mean square*** to $X(\omega)$ if:

$$\lim_{n\to\infty} E(|X_n - X|^2) = 0.$$

Definition (1.6):

A sequence of random variables $\{X_n(\omega)\}$ ***converges in probability*** (or ***stochastically***) to $X(\omega)$, if:

$$\lim_{n\to\infty} \boldsymbol{P}(\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\}) = 0, \ \forall\ \varepsilon > 0.$$

1.2 Stochastic Process:

In many physical applications, there are many processes in which the random variables depends on the space and/or time and this introductory material will be the main subject of the present section.

Definition (1.7), [7]:

A ***stochastic process*** $X(t, \omega)$ is a family of random variables which is denoted by $X_t(\omega)$ (or briefly X_t) of two variables t and ω, where $t \in [t_0, T] \subset [0, \infty)$, $T \in$ ¡ , $\omega \in \Omega$ on a probability space (Ω, A, ***P***), which

assumes real values and is ***P***-measurable as a function of ω for each fixed t. The parameter t is interpreted as a time and $X_t(.)$ represents a random variable on the above probability space Ω, while $X_.(\omega)$ is called a ***sample path*** or ***trajectory*** of the stochastic process.

Definition (1.8), [7]:

Let $X_t, t \in [a,b]$ be a stochastic process on probability space (Ω, A, ***P***) and let $\{A_t\}_{t\in[a,b]}$ be a non-decreasing family of σ-algebras of A, such that for each t ∈ [a, b], X_t is A_t-measurable. Then X_t is a ***martingale*** with respect to A_t, if:

$$E(X_{t+s} \mid A_t) = X_t, \quad \text{for all } s > 0.$$

Martingales are particularly relevant in the theory of stochastic integrals and it is another important class of stochastic process. Öttinger in (1996), [30] links martingale to a "courtroom process in which the truth is exposed in the course of time".

Now, an important class of stochastic processes is that with independent increments; that is, where the differences $X_{t_{i+1}} - X_{t_i}$ are independent, for any finite sequence $\{t_i\} \subseteq I$ with $t_i < t_{i+1}$ where I=[t_0, T].

Definition (1.9), [7]:

A stochastic process W_t, t ∈ [0, ∞), is said to be a ***Brownian motion*** or ***Wiener process***, if:

1. ***P***$(\{\omega \in \Omega \mid W_0(\omega) = 0\}) = 1$.
2. For $0 < t_0 < t_1 < \ldots < t_n$, the increments $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$ …, $W_{t_n} - W_{t_{n-1}}$ are independent.
3. For an arbitrary t and h > 0, $W_{t+h} - W_t$ has a Normal distribution with mean 0 and variance h.

Remark (1.1), [7]:

In general, a standard Weiner process has the properties that:

$$W_0 = 0 \ \textbf{\textit{w.p.1}}, \quad E(W_t) = 0, \quad Var(W_t - W_s) = t - s$$

for all $0 \le s \le t$; and so the increments are stationary.

The property $E(W_sW_t) = \min(s, t)$ can be used to demonstrate the independence of Wiener increments. Suppose that $0 \le t_0 < \ldots < t_{i-1} < t_i < \ldots < t_{j-1} < t_j < \ldots < t_n$; then:

$$E[(W_{t_i} - W_{t_{i-1}})\ (W_{t_j} - W_{t_{j-1}})] = E(W_{t_i}W_{t_j}) - E(W_{t_i}W_{t_{j-1}}) - E(W_{t_{i-1}}W_{t_j}) + E(W_{t_{i-1}}W_{t_{j-1}}) = t_i - t_i - t_{i-1} + t_{i-1} = 0$$

and hence the increments $W_{t_i} - W_{t_{i-1}}$ and $W_{t_j} - W_{t_{j-1}}$ are independent.

Definition (1.10), [3]:

A ($d \times m$ matrix)-valued function $G = G(t, \omega)$ defined on $I \times \Omega$ and measurable in (t, ω) is said to be ***nonanticipating*** (with respect to a family A_t of nonanticipating *s-**algebra*** A) if G(t, .) is A_t-measurable for all $t \in I$.

1.3 Theory of Stochastic Differential Equations.

This section contains some definitions and theorems about SDE's and their models that shows by studying the calculations of stochastic integrals and finally giving some analytical methods for solving several types of SDE's with different ways.

1.3.1 Stochastic Integrals and their Models [37]:

A sequence of node points is considered in the interval $I = [t_0, T]$ given by:

$$t_0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_{N_n}^{(n)} = T$$

with the property that they are refinements for increasing n and with:

$$\max_{0 \le i \le N_n - 1} \left\{ t_{i+1}^{(n)} - t_i^{(n)} \right\} \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty$$

If we define $\tau_i^{(n)} = \theta t_{i+1}^{(n)} + (1 - \theta) t_i^{(n)}$, for a fixed $\theta \in [0, 1]$, then the follwing series of random variables is called an ***approximation of stochastic integral***, viz ;

$$\int_{t_0}^{T} X_{\tau_i^{(n)}} dW_t = \sum_{i=0}^{N_n - 1} X_{\tau_i^{(n)}} (W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}}) \qquad \ldots (1.1)$$

converges as $n \longrightarrow \infty$ in probability if $W_{t^{(n)}}$, $t^{(n)} \geq 0$ be a Wiener process and X_t a real-valued stochastic process (sometimes, called a ***stochastic function*** or briefly a ***function***) with respect to Wiener process W_t. It is necessary that X and W are both defined on the same probability space (Ω, A, P).

Now, let A_t be an increasing family of σ-algebra fields, which is called also ***filtration***, for all $t \geq 0$, i.e., $A_{t_1} \subset A_{t_2}$ if $t_1 < t_2$, such that $A_t \subset A$, where $A(W_s,\ 0 \leq s \leq t)$ is in A_t, and $A(W_{\lambda+t} - W_t, \lambda > 0)$ is independent of A_t, for all $t \geq 0$. One can take, for instance, $A_t = A(W_s, 0 \leq s \leq t)$ [16]. The filtration $(A_t)_{t\in I}$ satisfy the usual conditions, i.e., $(A_t)_{t\in I}$ is a right-continuous filtration (satisfies $A_t = \bigcap_{\varepsilon>0} A_{t+\varepsilon}$ for all $t \geq 0$) and A_0 contains all ***P***-negligible events in A. Further, $(A_t)_{t\in I}$ be such that W is a martingale of $(A_t)_{t\in I}$, [37].

Definition (1.11), [7]:

Consider a probability space (Ω, A, P) with filtration $(A_t)_{t\in I}$ then a nonnegative random variable $\tau(\omega)$ on (Ω, A, P) is called a ***Markov time*** (or ***stopping time***) if the event $\{\omega \in \Omega : \tau(\omega) \leq t\} \in A_t$, for each $t \geq 0$.

Definition (1.12), [21]:

Let X_t, $t \in I$ be a stochastic process defined on a probability space (Ω, A, P) and let $(A_t)_{t\in I}$ be a filtration σ-algebra. The process X_t is ***adapted*** to the family $(A_t)_{t\in I}$ if X_t is A_t -measurable for every $t \in I$, or:

$$E(X_t \mid A_t) = X_t,\ t \in I$$

A_t -adapted random processes are also A_t -measurable.

It is also necessary that X_t be a non-anticipating (see [3]), by which it is meant that information about X_t at time t does not depend on events occurring after time t. In the next definition more details about X_t are given:

Definition (1.13), [37]:

The set L is the class of all $\mathcal{B}\times A$-measurable A_t-adapted processes $X_t : I \times \Omega \longrightarrow \mathbf{R}$,where $\mathcal{B}$ is the Borel set in $\mathbf{R}^m$ for which:

$$E\left(\int_0^t X_s^2 (\omega)\, ds \right) < \infty, \quad t > 0 \qquad \text{... (1.2)}$$

holds and the set P is the class of all $\mathcal{B}\times A$-measurable A_t-adapted processes $X_t : I \times \Omega \longrightarrow \mathbf{R}$, satisfying:

$$\boldsymbol{P}\left(\int_0^t X_s^2 (\omega)\, ds < \infty \right) = 1, \quad t > 0 \qquad \text{... (1.3)}$$

Now, back to equation (1.1), when equation (1.1) converges as $n \longrightarrow \infty$ in probability if $X_t \in P$ and in the mean-square sense given in equation (1.2) if $X_t \in L$.

However, the integral for dW_s are unlike the Riemann-Stieltjes integral, here is the selection of θ makes a difference. For $\theta = 0$, which is means that $\tau_i^{(n)}$ represent the left end point $t_i^{(n)}$, we have the Itô calculus. The limit of equation (1.1), denotes the first model given by:

$$\int_{t_0}^t X_s \, dW_s$$

and is called the ***Itô stochastic integral***. At Stratonovich calculus, we have to set $\theta = \dfrac{1}{2}$ and $\tau_i^{(n)}$ described the mid point of $[t_i^{(n)},\ t_{i+1}^{(n)}]$. Now, the limit of equation (1.1) denotes the second model which is given by:

$$\int_{t_0}^t X_s \circ dW_s$$

and is called the ***Stratonovich stochastic integral***.

To determine a value for the integral $\int_a^b W_t \, dW_t$, approximate W_t by the function $\varphi_n^{\lambda}(t)$, where:

$$\varphi_n^{\lambda}(t) = \lambda\ W_{t_{k+1}^{(n)}} + (1 - \lambda)\ W_{t_k^{(n)}}, \quad t_k^{(n)} \le t \le t_{k+1}^{(n)} \qquad \text{... (1.4)}$$

for $\lambda \in [0, 1]$, and then the integration of $\varphi_n^{\lambda}(t)$ in [a, b] equals to the approximate stochastic integral given in equation (1.1):

$$\int_a^b \varphi_n^{\lambda}(t)\, dW_t = \sum_{k=0}^{N_n-1} \varphi_n^{\lambda}(t_k)\,(W_{t_{k+1}^{(n)}} - W_{t_k^{(n)}}) \qquad \text{... (1.5)}$$

The right hand side of equation (1.5) may be written as:

$$\lambda \sum_{k=0}^{N_n-1} W_{t_{k+1}^{(n)}}\,(W_{t_{k+1}^{(n)}} - W_{t_k^{(n)}}) + (1-\lambda) \sum_{k=0}^{N_n-1} W_{t_k^{(n)}}\,(W_{t_{k+1}^{(n)}} - W_{t_k^{(n)}})$$

By arranging the terms algebraically, when $n \longrightarrow \infty$, then [1]:

$$\lim_{n\to\infty} \sum_{k=0}^{N_n-1} W_{t_k^{(n)}}[W_{t_{k+1}^{(n)}} - W_{t_k^{(n)}}] = \lim_{n\to\infty} \sum_{k=0}^{N_n-1} [W_{t_k^{(n)}} W_{t_{k+1}^{(n)}} - W^2_{t_k^{(n)}}]$$

$$= \frac{1}{2} \lim_{n\to\infty} \sum_{k=0}^{N_n-1} 2W_{t_k^{(n)}} W_{t_{k+1}^{(n)}} - 2W^2_{t_k^{(n)}} + W^2_{t_{k+1}^{(n)}} - W^2_{t_{k+1}^{(n)}}$$

$$= \frac{1}{2} \lim_{n\to\infty} \sum_{k=0}^{N_n-1} [W^2_{t_{k+1}^{(n)}} - W^2_{t_k^{(n)}} - [W_{t_{k+1}^{(n)}} - W_{t_k^{(n)}}]^2]$$

Such that:

$$\lim_{n\to\infty} \sum_{k=0}^{N_n-1} [W^2_{t_{k+1}^{(n)}} - W^2_{t_k^{(n)}}] = \lim_{n\to\infty} [W^2_{t_{N_n}^{(n)}} - W^2_{t_0^{(n)}}] = W_b^2 - W_a^2$$

where $\lim_{n\to\infty}$ is taken as the limit in probability, then [1]:

$$\lim_{n\to\infty} \sum_{k=0}^{N_n-1} W_{t_k^{(n)}}[W_{t_{k+1}^{(n)}} - W_{t_k^{(n)}}] = \frac{1}{2} W_b^2 - \frac{1}{2} W_a^2 - \frac{1}{2} \lim_{n\to\infty} \sum_{k=0}^{N_n-1} [W_{t_{k+1}^{(n)}} - W_{t_k^{(n)}}]^2$$

In a similar manner with $\sum_{k=0}^{N_n-1} W_{t_{k+1}^{(n)}}\,[W_{t_{k+1}^{(n)}} - W_{t_k^{(n)}}]$, such that:

$$\sum_{k=0}^{N_n-1} W_{t_{k+1}^{(n)}}\,[W_{t_{k+1}^{(n)}} - W_{t_k^{(n)}}] = \frac{1}{2} W_b^2 - \frac{1}{2} W_a^2 + \frac{1}{2} \lim_{n\to\infty} \sum_{k=0}^{N_n-1} [W_{t_{k+1}^{(n)}} - W_{t_k^{(n)}}]^2$$

According to [1]:

$$\sum_{k=0}^{N_n-1}\varphi_n^{\lambda}(t)[W_{t_{k+1}^{(n)}}-W_{t_k^{(n)}}]=\lambda[\frac{1}{2}W_b^2-\frac{1}{2}W_a^2$$
$$+\frac{1}{2}\lim_{n\to\infty}\sum_{k=0}^{N_n-1}[W_{t_{k+1}^{(n)}}-W_{t_k^{(n)}}]^2]+(1-\lambda)[\frac{1}{2}W_b^2$$
$$-\frac{1}{2}W_a^2-\frac{1}{2}\lim_{n\to\infty}\sum_{k=0}^{N_n-1}[W_{t_{k+1}^{(n)}}-W_{t_k^{(n)}}]^2]$$
$$=\frac{1}{2}W_b^2-\frac{1}{2}W_a^2+\frac{1}{2}(2\lambda-1)\lim_{n\to\infty}\sum_{k=0}^{N_n-1}[W_{t_{k+1}^{(n)}}-W_{t_k^{(n)}}]^2$$

... (1.6)

Now, by the next theorem, the last limit in probability equals to $b-a$.

Theorem (1.1), [1]:

Let X_t be a Wiener process, and Π_n a sequence of partitions $\{t_1^{(n)}, t_2^{(n)}\ldots, t_{N_n}^{(n)}\}$ of a finite closed interval [a, b] with $|\Pi_n| \longrightarrow 0$ if $n\to\infty$ Let:

$$S_n=\sum_{k=1}^{N_n}[X_{t_k^{(n)}}-X_{t_{k-1}^{(n)}}]^2$$

Then $S_n \to b-a$ in the mean.

The interval [a, b] has been partitioned into n-equal subintervals of length $\frac{b-a}{n}$, so for each k the expected value of $[W_{t_{k+1}^{(n)}} - W_{t_k^{(n)}}]^2$ is $t_{k+1}^{(n)} - t_k^{(n)}$, which is equal to $\frac{b-a}{n}$. Consequently, the mean-square limit of the sum of squares given in equation (1.6) is $n\left(\frac{b-a}{n}\right) = b - a$. Therefore, taking the mean-square limit of equation (1.6) as $t_{k+1}^{(n)}-t_k^{(n)}\to 0$, yields to:

$$\int_a^b W_t\,dW_t=\frac{1}{2}(W_b^2 - W_a^2) + (\lambda-\frac{1}{2})(b-a).$$

Thus, for any choice λ there are different results. In particular, if $\lambda = 0$, then the integral is known as Itô stochastic integral, and this leads to a calculus based on Itô chain rule, while taking $\lambda = \frac{1}{2}$, then the resulting integral is the Stratonovich stochastic integral (where the symbol $\circ$ is employed), and the

Stratonovich calculus follows the same results for the regular Riemann-Stieltjes calculus. The integral evaluation is:

$$\int_a^b W_t \, dW_t = \frac{1}{2}(W_b^2 - W_a^2) - \frac{1}{2}(b-a)$$

and

$$\int_a^b W_t \circ dW_t = \frac{1}{2}(W_b^2 - W_a^2)$$

for the Itô or Stratonovich calculus, respectively.

Remark (1.2), [37]:

The advantages of Stratonovich calculus is the availability of its rules similar to ordinary integration. However:

$$\int_0^t W_s \circ dW_s = \frac{1}{2}W_t^2, \qquad \forall\, W_0 = 0$$

whereas for Itô calculus:

$$\int_0^t W_s \, dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t, \qquad \forall\, W_0 = 0$$

A nice feature of the Itô stochastic integral is that it can be defined for a general class of non-anticipating random functions in such a way as to preserve various Wiener process properties, as well as, allowing easy calculation of moments of the solution of an SDE.

One of the main advantages of the Itô calculus in contrast to Stratonovich calculus is the fact that Itô integrals inherit some good properties of the Wiener process.

Definition (1.14) [16]:

A stochastic process X_t defined on [a, b] is called a ***step function*** if there exists a partition $a = t_0 < t_1 < \ldots < t_r = b$ of [a, b], such that:

$$X_t = X_{t_i}, \quad \text{if} \quad t_i \le t \le t_{i+1},\ i = 0,1,\ldots,r-1$$

Theorem (1.2), [16]:

If f is a step function in set L of interval [a, b], then:

$$E\left(\int_a^b f(t,\omega)\, dW_t\right) = 0,$$

$$E\left[\left|\int_a^b f(t,\omega)\, dW_t\right|^2\right] = E\int_a^b f^2(t,\omega)\, dt$$

Only for the non-anticipating Itô case ($\lambda = 0$) does the martingale property (and theorem (1.2)) hold. However, to offset this advantage, there is a disadvantage of the extra term in the chain rule, which is comes about as $(dW_t)^2$ behaves like dt in the mean square sense in theorem (1.2), [7].

Finally, there are many of theorems concerning stochastic integral but superior theorem with d-dimensional random variable is given next.

Theorem (1.3), [3]:

If f is independent of ω and belongs to P of interval $[t_0, T]$, it belongs to L of same interval for any sub σ-algebras $A_s \supset \beta [t_0, s]$, such that β is a Borel set of σ-algebra in the interval $[t_0, s]$ and f denotes $d \times m$-dimensional matrix valued function, then the Itô stochastic integral $\int_{t_0}^{T} f(s,\omega)\, dW_s$ is a normally distributed d-dimensional random vector with distribution:

$$N\left(0,\ \int_{t_0}^{t} f(s,\omega)\, f^{T}(s,\omega)\, ds\right)$$

1.3.2 Stochastic Differential Equations and their Models, [7], [22], [37].

Consider the SDE:

$$dy_t = f(t, y_t)\, dt + g(t, y_t)\, dW_t, \quad y_{t_0} = y_0 \qquad \ldots (1.7)$$

where $f : I \times \mathbf{R} \longrightarrow \mathbf{R}$, $g : I \times \mathbf{R} \longrightarrow \mathbf{R}$ be a Borel-measurable functions, we call f the ***drift function*** and g the ***diffusion function***.

The stochastic differential equation given in equation (1.7) may be written as an equivalent SDE of the form:

$$y_t = y_{t_0} + \int_{t_0}^{t} f(s, y_s)\, ds + \int_{t_0}^{t} g(s, y_s)\, dW_s \qquad \ldots (1.8)$$

However, the second integral given in equation (1.8) cannot be defined in a following meaning, where W_s is the Wiener. The variance of the Wiener process satisfies $Var(W_t) = t$, and so this increases as time increases even thought the mean stays at 0. Because of this, typical sample paths of a Wiener process attain larger values in magnitude as time progresses, and consequently the sample paths of the Wiener process are not bounded; hence the second integral in equation (1.8) cannot be considered as a Riemann-Stieltjes integral. Note that, more general process which has the martingale property can be used in place of W_s, but in this thesis only Wiener process will be used in the formulation of SDE. Also, note that the formulation in

equation (1.8) assumes initially that there is only a single scalar Wiener process, so the SDE is then represented by rewriting the integral equation given in equation (1.8), as:

$$y_t = y_{t_0} + \int_{t_0}^{t} f(s, y_s)\,ds + \int_{t_0}^{t} g(s, y_s) * dW_s \qquad \text{... (1.9)}$$

Or

$$d\,y_t = f(t,\ y_t)\,dt + g(t,\ y_t) * dW_t, \quad y_{t_0} = y_0 \qquad \text{... (1.10)}$$

where $\int_{t_0}^{t} g(s, y_s) * dW_s$ refers to either Itô stochastic integral $\int_{t_0}^{t} g(s, y_s)\,dW_s$ or Stratonovich stochastic integral $\int_{t_0}^{t} g(s, y_s) \circ dW_s$ such that the first integral in equation (1.9) is pathwise Lebsegue-integrable. Since the paths of Wiener process are almost sure of unbounded variation, we cannot interpret the second integral in equation (1.9) in the sense of a pathwise Riemann-Stieltijes integral.

Now, when considering the Itô and Stratonovich calculus, then we get a simple connection between the solution of an Itô SDE and that of a Stratonovich SDE. Let $(y_t)_{t\in I}$ be the solution of one-dimensional Itô SDE, such that:

$$d\,y_t = f(t,\ y_t)\,dt + g(t,\ y_t)\,dW_t, \quad y_{t_0} = y_0 \qquad \text{... (1.11)}$$

then:

$$y_t = y_0 + \int_{t_0}^{t} f(s, y_s)\,ds + \int_{t_0}^{t} g(s, y_s)\,dW_s$$

where W_t is a one-dimensional Wiener process. y_t is also a solution of the Stratonovich SDE, such that when:

$$d\,y_t = \underline{f}(t, y_t)\,dt + g(t,\ y_t) \circ dW_t, \quad y_{t_0} = y_0 \qquad \text{... (1.12)}$$

then:

$$y_t = y_0 + \int_{t_0}^{t} \underline{f}(s, y_s)\,ds + \int_{t_0}^{t} g(s, y_s) \circ dW_t$$

of Stratonovich calculus, where:

$$\underline{f}(t, y_t) = f(t,\ y_t) - \frac{1}{2}\frac{\partial g}{\partial y}(t,\ y_t)\,g(t,\ y_t) \qquad \text{... (1.13)}$$

Therefore, whichever interpretation of the SDE is appropriate in particular situation, we can switch to the corresponding SDE in the other calculus. For instance, we can apply the existence and uniqueness theorem for an Itô SDE given in equation (1.11) to obtain analogous results for the corresponding Stratonovich SDE given in equation (1.12).

Remark (1.3), [37]:

The class of Itô process may be introduced as follows:

A stochastic process Y of the form:

$$Y_t = Y_0 + \int_0^t f(s,\omega)\,ds + \int_0^t g(s,\omega)\,dW_s \qquad \text{... (1.14)}$$

with f and g are A_t-adapted, and:

$$P\left(\int_0^t |f(s,\omega)|\,ds < \infty, \text{ for all } t \geq 0\right) = 1 \qquad \text{... (1.15)}$$

and with g is a $\beta \times A$-measurable and

$$P\left(\int_0^t g^2(s,\omega)\,ds < \infty, \text{ for all } t \geq 0\right) = 1 \qquad \text{... (1.16)}$$

which is called ***Itô process***.

We are now in position to define the solution of an Itô SDE.

Definition (1.15), [37]:

A process $Y = (Y_t)_{t\in I}$ with values in $\mathbf{R}^d$ is called a ***strong solution of the stochastic differential equation*** given in equation (1.11) with respect to the fixed Wiener process $(W_t)_{t\in I}$ and the initial condition y_0, if the following properties hold:

1. Y is adapted to the filtration $(A_t)_{t\in I}$.
2. Y has continuous sample paths.
3. For multi-dimensions given in equation (1.11), such that i = 1, 2, …, d; j = 1, 2, …, m and $t \in I$ satisfy:

$$\int_0^t |f_i(s,Y_s)| + g_{ij}^2(s,Y_s)\,ds < \infty, \textbf{ P-w.p.1}.$$

4. Y_t satisfy with ***P-w.p.1***. the following stochastic integral equation given by:

$$Y_t = Y_0 + \int_0^t f(s,Y_s)\,ds + \int_0^t g(s,Y_s)\,dW_s, \ \forall\, t \in I$$

Theorem (1.4) (The Existence and Uniqueness Theorem), [3]:

Suppose that we have a stochastic differential equation given by:

$$dX_t = f(t, X_t)\,dt + G(t, X_t)\,dW_t\,, X_{t_0} = c, t_0 \le t \le T \le \infty,$$

where W_t is an R^m-valued Wiener process and c is the random variable independent of $W_t - W_{t_0}$ for $t \ge t_0$. Suppose that the R^d-valued function $f(t, X_t)$ and the ($d \times m$ matrix)-valued function $G(t, X_t)$ are defined and measurable on $I \times R^d$ where $I = [t_0, T]$ and have the following properties: There exists a constant K > 0 such that

a) (Lipschitz condition) for all $t \in I$, $x_t, y_t \in R^d$-valued random vector,

$$|f(t, x_t) - f(t, y_t)| + |G(t, x_t) - G(t, y_t)| \le K |x_t - y_t|,$$

$$(|G|^2 = \mathrm{tr}\, GG^T) \qquad \ldots (1.17)$$

b) (Restriction on growth) For all $t \in I$ and $x_t \in R^d$-valued random vector,

$$|f(t, x_t)|^2 + |G(t, x_t)|^2 \le K^2 (1 + |x_t|^2) \qquad \ldots (1.18)$$

Then the stochastic differential equation given in this theorem has on I a unique R^d-valued solution X_t, continuous ***w.p.1***, that satisfy the initial condition $X_{t_0} = c$; that is, if X_t and Y_t are continuous solution of SDE given above, with the same initial value c, then: $\boldsymbol{P}(\sup_{t_0 \le t \le T} |X_t - Y_t| > 0) = 0$.

In contrast to strong solution of stochastic differential equations, a notion of solvability for the equation (1.11) may be defined, which is a weaker condition.

Definition (1.16), [37]:

A ***weak solution of the stochastic differential equation*** given in equation (1.11) is a triple $((\Omega, \mathcal{A}, \boldsymbol{P}), (\mathcal{A}_t)_{t \in I}, (Y, W))$, such that:

1. $(\Omega, \mathcal{A}, \boldsymbol{P})$ is a probability space, $(\mathcal{A}_t)_{t \in I}$ is a right-continuous filtration in $\mathcal{A}$ and $\mathcal{A}_0$ contains all $\boldsymbol{P}$-negligible events in $\mathcal{A}$.
2. W is an m-dimensional Wiener process of $(\mathcal{A}_t)_{t \in I}$ and Y is a continuous, adapted $\mathbf{R}^d$-values process.

3. Conditions (3) and (4) of the definition (1.15) are satisfied.

Remarks (1.4):

1. If $f(t, y_t)$ and $g(t, y_t)$ satisfy the conditions of theorem (1.4), then the following holds; A solution (weak or strong) of the SDE given in equation (1.11) is weakly unique, where weak uniqueness means that solutions (weak or strong) have the same finite-dimensional distributions, [37].
2. In this work, we shall talk only about the strong solution of SDE's and the considered examples in chapters one and three are only of strong solution.

Consider the stochastic differential equation:

$$d y_t = f(t, y_t)\, dt + g(t, y_t)\, dW_t \qquad \text{... (1.19)}$$

Then, for a given function F and with certain smoothness, measurability and boundedness properties on f and g in equation (1.19) to guarantee the existence, pathwise-uniqueness and bounded second moments, the multi-dimensional stochastic chain rule gives:

$$dF(t, y_t) = \frac{\partial F}{\partial t}\, dt + \frac{\partial F}{\partial y_t} f\, dt + \frac{1}{2}\, \text{trace}\left(g\, g^T \frac{\partial^2 F}{\partial y_t^{\,2}}\right) dt + \frac{\partial F}{\partial y_t} g\, dW_t \qquad \text{... (1.20)}$$

which is known as the Itô formula in its condensed vector-matrix notation. The scalar case in equation (1.20) can be written as:

$$dF(t, y_t) = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial y_t} f + \frac{1}{2} \frac{\partial^2 F}{\partial y_t^{\,2}} g^2 \right) dt + \frac{\partial F}{\partial y_t} g\, dW_t \qquad \text{... (1.21)}$$

i.e., one of the most important tools for the stochastic calculus and especially for the Itô calculus is the ***Itô formula***.

Theorem (1.5) (The Itô Formula), [3], [22], [37]:

Let $dY_t = u\, dt + v\, dW_t$ be a d-dimensional Itô process of an m-dimensional Wiener process W_t. Suppose that each of the processes u_i and $v_{i,j}$ satisfy the conditions given in equations (1.15) and (1.16), for $1 \le i \le d$ and $1 \le j \le m$. Let $f(t, y) = (f_1(t, y), f_2(t, y), \ldots, f_p(t, y)) \in C^{1,2}(I \times \mathbf{R}^d, \mathbf{R}^p)$. Then:

$$X_t = f(t, Y_t)$$

is an Itô process, whose k-th component X_t^k is ***P-w.p.1***, given by:

$$d X_t^k = \frac{\partial f_k}{\partial t}(t,\ Y_t)\, dt + \sum_i \frac{\partial f_k}{\partial y_i(t)}(t,\ Y_t)\, d Y_t^{(i)} + \frac{1}{2}\sum_{i,j} \frac{\partial^2 f_k}{\partial y_t^{(i)} \partial y_t^{(j)}}$$

$$(t,\ Y_t)\, d Y_t^{(i)} d Y_t^{(j)} \qquad \text{... (1.22)}$$

where $d W_t^{(i)} d W_t^{(j)} = \delta_{ij}\, dt$, and δ_{ij} is the Kronecker delta function.

In fact equation (1.19) is sufficiently general to represent an m-dimensional, d-Wiener process system in which $g(t, y_t)$ is an m×d matrix and $W_t = (W_t^{(1)},\ W_t^{(2)}\ ,\ \ldots,\ W_t^{(d)})^T$ is a d-dimensional vector consisting of d independent Wiener processes. By letting the columns of $g(t,\ y_t)$ be labeled as $g_1(t,\ y_t)$, $g_2(t,\ y_t)$, …, $g_d(t,\ y_t)$; then the m-dimensional d-Wiener process system can also be written as:

$$d y_t = f(t,\ y_t)\, dt + \sum_{j=1}^{d} g_j(t,\ y_t)\, d W_t^{(j)} \qquad \text{... (1.23)}$$

In this case, the component-by-component version of Itô's formula is for $k = 1, 2, \ldots, m$:

$$dF_k(t,\ y) = \left(\frac{\partial F_k}{\partial t} + \sum_{i=1}^{m} f_i \frac{\partial F_k}{\partial y} + \frac{1}{2}\sum_{l=1}^{d}\sum_{i,j=1}^{m} g_{il}\, g_{jl} \frac{\partial^2 F_k}{\partial y_i \partial y_j} \right) dt +$$

$$\sum_{l=1}^{d}\sum_{i=1}^{m} g_{il} \frac{\partial F_k}{\partial y_i}\, dW_l \qquad \text{... (1.24)}$$

There has been much discussion about whether to use the Itô or Stratonovich interpretation of the integral and while both approaches are correct, the choice depends on modeling process that leads to the SDE formulation. Indeed, it is possible to convert from one interpretation to the other in order to take advantages of the particular features of one of the approaches as appropriate. In the scalar case, if the Itô SDE is as given in equation (1.19), then the related Stratonovich SDE is given by:

$$dy = \underline{f}\,(t, y)\, dt + g(t, y) \circ dW_t \qquad \text{... (1.25)}$$

where:

$$\underline{f}\,(t, y) = f(t, y) - \frac{1}{2}\frac{\partial g}{\partial y}(t, y)\, g(t, y) \qquad \text{... (1.26)}$$

In other words, the two equations (1.19) and (1.25) under different rules of calculus have the same solution. As an illustration, consider the following example:

Example (1.1), [7], [22]:

When the Itô SDE is:

$$d y_t = a\ y_t dt + b\ y_t\ dW_t$$

has the solution:

$$y_t = y_0 \exp((a - \frac{1}{2} b^2) t + b\ W_t)$$

as while Stratonovich SDE is:

$$d y_t = (a - \frac{1}{2} b^2)\ y_t dt + b\ y_t \circ dW_t$$

Which can be written as:

$$d y_t = \underline{a}\ y_t dt + b\ y_t \circ dW_t, \text{ where } \underline{a} = a - \frac{1}{2} b^2$$

has solution:

$$y_t = y_0 \exp(\underline{a}\ t + b\ W_t)$$

In the case of additive noise (where g is independent of y), the Itô and Stratonovich representations are equivalent.

In vector form, the relationship between the two integrals is:

$$\underline{f}_i(t, y) = f_i(t, y) - \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{d} g_{jk}(t, y) \frac{\partial g_{ik}}{\partial y_j}(t, y), i=1,2,\ldots,m \quad \ldots (1.27)$$

If Stratonovich calculus is used, then the chain rule becomes the familiar one from Riemann-Stieltjes calculus.

1.3.3 Analytical Methods for Solving SODE's [3], [22]:

Stochastic ordinary differential equations may be solved analytically dependent on the nature and the type of the stochastic differential equation, and these methods may be summarized as follows:

I. Linear stochastic differential equations:

1. The linear random differential equation given by:

$$X_t' = \frac{d X_t}{d t} = a(W_t)\ X_t + b(t, W_t) \quad \ldots (1.28)$$

has the solution for an initial value $X_0(W_t)$ at $t_0 = 0$ given by:

$$X(t, W_t) = e^{a(W_t)t}\left(X_0(W_s) + \int_0^t e^{-a(W_s)s}\, b(s, W_s)\, ds \right) \quad \text{... (1.29)}$$

2. The general linear stochastic differential equation given by:

$$dX_t = (A(t)\, X_t + a(t))\, dt + \sum_{i=1}^{m} (B_i(t)\, X_t + b_i(t))\, dW_t^i \quad \text{... (1.30)}$$

where A(t), $B_k(t)$ are d×d-matrix-valued and a(t), $b_k(t)$ are $\mathbf{R}^d$-valued since k=1,2,….m and $W_t = (W_t^1, ..., W_t^m)^T$ has the solution given by:

$$X_t = \Phi_t \left[c + \int_{t_0}^{t} \Phi_s^{-1} \left(a(s) - \sum_{i=1}^{m} B_i(s)\, b_i(s) \right) ds \right.$$

$$\left. + \sum_{i=1}^{m} \int_{t_0}^{t} \Phi_s^{-1}\, b_i(s)\, dW_s^i \right] \quad \text{... (1.31)}$$

Where; $c = X_{t_0}$ and

$$\Phi_t = \exp\left(\int_{t_0}^{t} \left(A(s) - \sum_{i=1}^{m} \frac{B_i(s)^2}{2} \right) ds + \sum_{i=1}^{m} \int_{t_0}^{t} B_i(s)\, dW_s^i \right) \quad \text{... (1.32)}$$

is the solution of the homogeneous equation related to equation (1.30).

$$d\Phi_t = A(t)\Phi_t\, dt + \sum_{i=1}^{m} B_i(t)\Phi_t\, dW_t^i \text{, with initial value } \Phi_{t_0} = 1.$$

(Note that, equation (1.30) is said to be homogeneous if $a(t)=b_k(t)=0$, k=1, 2, ... m, Also, it is said to be linear in the narrow sense if $B_k(t)=0$, k=1, 2, ... m. [3]).

3. The solution of the SDE in the narrow sense [3]:

$$dX_t = (A(t)\, X_t + a(t))\, dt + B(t)\, dW_t,\ X_{t_0} = c,\ t \in [t_0, T] \quad \text{... (1.33)}$$

where A(t) is d×d matrix, a(t) are vector with components in $\mathbf{R}^d$, B(t) is d×m-matrix and W_t is an m-dimensional Wiener process, is given by:

$$X_t = \Phi_t \left(c + \int_{t_0}^{t} \Phi_s^{-1}\, a(s)\, ds + \int_{t_0}^{t} \Phi_s^{-1}\, B(s)\, dW_s \right)$$

Here, c is constant or a Gaussian random variable, and Φ_t is the fundamental matrix solution of deterministic equation $X'_t = A(t)\, X_t$ is given by [3]:

$$\Phi_t = \exp\left[\int_{t_0}^{t} A(s)\, ds\right] \qquad \text{... (1.34)}$$

since $\Phi_{t_0} = I$.

Example (1.2) :

Consider the SDE:

$dX_t = b\, dW_t, \qquad X_{t_0} = X_0 \quad \text{at} \quad t_0 = 0$

where b is constant and $\Phi_t = 1$, then:

$$X_t = X_0 + \int_0^t 0 \times 1\, dt + \int_0^t b \times 1\, dW_t$$

$$= X_0 + b\,(W_t - W_0) = X_0 + b\, W_t, \text{ where } W_0 = 0.$$

Example (1.3):

Consider the SDE:

$$dX_t = b\, X_t\, dW_t$$

where b is constant, $B(t) \equiv b$ and $A(t) = a(t) = b(t) \equiv 0$ in equation (1.30) where m=1 and applying equation (1.32), then with $t_0 = 0$:

$$\Phi_t = \exp\left(\frac{1}{2} b^2\, t + b\, W_t\right), \text{ where } W_0 = 0$$

Now applying (1.31), yields to:

$$X_t = e^{\frac{1}{2} b^2 t + b\, W_t}\left[X_0 + \int_0^t (0 - b \times 0)\, e^{\frac{1}{2} b^2 t + b\, W_t} ds + \int_0^t 0 \times e^{\frac{1}{2} b^2 t + b\, W_t} dW_t\right]$$

$$= X_0\, e^{\frac{1}{2} b^2 t + b\, W_t}$$

Example (1.4), [3], [22], [41]:

The exponential process of the SDE

$dX_t = a\,X_t\,dt + b\,X_t\,dW_t$, $X_{t_0} = X_0$ at $t_0 = 0$

where a and b are constants , $t \geq 0$, $A(t) \equiv a$, $B(t) \equiv b$ and $a(t) = b(t) \equiv 0$ in equation (1.30) and applying equation (1.32), then:

$$\Phi_{t,t_0} = e^{a\,(t-t_0) - \frac{1}{2}b^2\,(t-t_0) + b\,(W_t - W_{t_0})}$$

Now, when $t_0 = 0$, $W_0 = 0$ then $\Phi_t = e^{a\,t - \frac{1}{2}b^2\,t + b\,W_t}$, and applying (1.31), yields to:

$$X_t = e^{a\,t - \frac{1}{2}b^2\,t + b\,W_t}\left[X_0 + \int_0^t (0 - b\times 0)\,e^{a\,t - \frac{1}{2}b^2\,t + b\,W_t}\,ds + \int_0^t 0\times e^{a\,t - \frac{1}{2}b^2\,t + b\,W_t}\,dW_s \right]$$

$$= X_0\,e^{a\,t - \frac{1}{2}b^2\,t + b\,W_t}$$

and when $X_0 = 1$, then $X_t = e^{a\,t - \frac{1}{2}b^2\,t + b\,W_t}$.

(Note that, in examples (1.3), (1.4), we take $d = m = 1$ in equations (1.30), (1.31) and (1.32).

II. Nonlinear stochastic differential equations [22]:

With an appropriate substitution $X_t = U(t, Y_t)$, certain nonlinear SDE's of the form:

$$dY_t = a(t, Y_t)\,dt + b(t, Y_t)\,dW_t \qquad \ldots (1.35)$$

can be reduced to a linear SDE in X_t of the form:

$$dX_t = (a_1(t)\,X_t + a_2(t))\,dt + (b_1(t)\,X_t + b_2(t))\,dW_t \qquad \ldots (1.36)$$

In general, if $\frac{\partial U}{\partial y}(t, y) \neq 0$, the inverse function theorem ensures the existence of a local inverse $y = V(t, x)$ of $x = U(t, y)$, that is with $x = U(t, V(t, x))$ and $y = V(t, U(t, y))$. A solution Y_t of equation (1.35)

then has the form $Y_t = V(t, X_t)$, where X_t is the solution of the SDE given in equation (1.36) for an appropriate coefficients a_1, a_2, b_1 and b_2.

Now, some formulae ready-made from reducible SDE's, in which the next three cases may be considered as a special case of the SDE given in equation (1.36), (when $a_k(t)$, $b_k(t)$ are scalars, k=1,2) which are given and discussed in [22] using different approaches.

Case I:

The Itô SDE given by:

$$dX_t = \frac{1}{2}g'(X_t)\, g(X_t)\, dt + g(X_t)\, dW_t,\ X_{t_0}(0) = X_0 \qquad \text{... (1.37)}$$

has the solution of the form:

$$\left.\begin{aligned} X_t &= U^{-1}(W_t + U(X_0)) \\ U(x) &= \int_0^x \frac{1}{g(s)}\, ds\ ;\ g(s) \neq 0 \end{aligned}\right\} \qquad \text{... (1.38)}$$

Case II:

The Itô SDE given by:

$$dX_t = \left(\alpha\, g(X_t) + \frac{1}{2} g(X_t)\, g'(X_t) \right) dt + g(X_t)\, dW_t \qquad \text{... (1.39)}$$

with $X_{t_0}(0) = X_0, \alpha \in$ ¡ , has the solution given by:

$$\left.\begin{aligned} X_t &= U^{-1}(\alpha t + W_t + U(X_0)) \\ U(x) &= \int_0^x \frac{1}{g(s)}\, ds\ ;\ g(s) \neq 0 \end{aligned}\right\} \qquad \text{... (1.40)}$$

Case III:

The Itô SDE given by:

$$dX_t = \left(\alpha\, g(X_t)\, U(X_t) + \frac{1}{2} g(X_t)\, g'(X_t) \right) dt + g(X_t)\, dW_t \qquad \text{... (1.41)}$$

where $X_{t_0}(0) = X_0, \alpha \in \mathbb{R}$, has the solution given by:

$$\left.\begin{aligned} X_t &= U^{-1}(U(X_0)e^{\alpha t} + e^{\alpha t}\int_0^t e^{-\alpha s}\, dW_s) \\ U(x) &= \int_0^x \frac{1}{g(s)} ds \; ; g(s) \neq 0 \end{aligned}\right\} \quad \text{... (1.42)}$$

Remarks (1.5):

1. The SDE in case I may be written as a Stratonovich SDE, such that the equivalent Stratonovich SDE is

$$dX_t = g(X_t) \circ dW_t$$

By ordinary calculus the separable ODE $dU(x) = \frac{dx}{g(x)}$ has the solution $U(x) = w(x) + U(x_0)$, where $w(x) = \int_0^x \frac{ds}{g(s)} ; g(s) \neq 0$. Hence, the Stratonovich SDE has also the solution $U(X_t) = W_t + U(X_0)$, that is $X_t = U^{-1}(W_t + U(X_0))$, where $U(x) = \int_0^x \frac{1}{g(s)} ds \; ; g(s) \neq 0$, [22].

2. Every example in case I may be modified to Cases II and III.

Example (1.5), [12] :

Using case I to solve the scalar nonlinear Itô SDE:

$dX_t = X_t^3\, dt + X_t^2\, dW_t$

when $f(X_t) = X_t^3, g(X_t) = X_t^2$ and since:

$$f(X_t) = \frac{1}{2} g(X_t)\, g'(X_t) = \frac{1}{2} 2X_t\, X_t^2 = X_t^3$$

Now when

$$y = U(x) = \int_{x_0}^{x} \frac{1}{g(s)} ds = \int_{x_0}^{x} \frac{1}{x^2} ds = -\frac{1}{x} + \frac{1}{x_0}$$

and since $x = U^{-1}(y) = -\frac{1}{y - \frac{1}{x_0}}$. and $X_t = U^{-1}(W_t + U(X_0))$, then:

$$X_t = \frac{X_0}{1 - X_0 W_t}, \text{ where } X_0 W_t \neq 0, \ \forall t \in I$$

Example (1.6), [21], [22]:

Using case II to solve the scalar nonlinear Itô SDE:

$$dX_t = \left(\frac{1}{2}X_t + \sqrt{X_t^2 + 1}\right)dt + \sqrt{X_t^2 + 1}\, dW_t, \ X_t(0) = 0, \text{ where:}$$

$$f(X_t) = \frac{1}{2}X_t + \sqrt{X_t^2 + 1}, g(X_t) = \sqrt{X_t^2 + 1}$$

and since:

$$f(X_t) = \alpha g(X_t) + \frac{1}{2} g(X_t)\, g'(X_t)$$

$$= \alpha\sqrt{X_t^2 + 1} + \frac{1}{2}\sqrt{X_t^2 + 1}\frac{X_t}{\sqrt{X_t^2 + 1}} = \frac{1}{2}X_t + \sqrt{X_t^2 + 1}, \text{ when}$$

$\alpha = 1$, then $f(X_t) = \frac{1}{2}X_t + \sqrt{X_t^2 + 1}$ .Now, when:

$$y = U(x) = \int_0^x \frac{1}{g(s)} ds = \int_0^x \frac{1}{\sqrt{s^2 + 1}} ds$$

then $y = U(x) = \ln\left(x + \sqrt{1 + x^2}\right)$ and since $x = U^{-1}(y)$, then

$$e^y = x + \sqrt{1 + x^2},$$

and thus

$$e^{2y} = \left(x + \sqrt{1 + x^2}\right)^2 = 2x^2 + 2x\sqrt{1 + x^2} + 1;$$

hence:

$$x(t) = \frac{e^y - e^{-y}}{2} = \sinh(y(t))$$

when $X_t = U^{-1}[\alpha\, t + W_t + U(X_0)]$, where $\alpha = 1$, then:

$$X_t = \sinh(t + W_t + \sinh^{-1}(X_0)).$$

The next example may be solved using all previous cases, which will give the same solution.

Example (1.7), [11] :

Consider the SDE:

$$dX_t = dt + 2\sqrt{X_t}\, dW_t \ , \ X_t(0) = 0 \ \text{at} \ t_0 = 0$$

Then using case I, with :

$$f(X_t) = \frac{1}{2}\, g(X_t)\, g'(X_t) = 1, \ g(x) = \sqrt{X_t} \ ,$$

the solution is found to be:

$$X_t = \left(\sqrt{X_0} + W_t\right)^2$$

and when using case II, with:

$$g(X_t) = 2\sqrt{X_t} \ , \ f(X_t) = \alpha g(X_t) + \frac{1}{2}\, g(X_t)\, g'(X_t)$$

$$= \alpha\sqrt{X_t} + 1 = 1$$

when $\alpha = 0$, and thus $f(X_t) = 1$, and since $y = U(x) = \sqrt{x}$ .Hence, the solution is found to be:

$$X_t = \left(\sqrt{X_0} + W_t\right)^2$$

Similarly, when using case III, with:

$$g(X_t) = 2\sqrt{X_t} \ , \ f(X_t) = \alpha g(X_t)\, U(X_t) + \frac{1}{2} g(X_t)\, g'(X_t)$$

such that $U(x) = y = \sqrt{x}$ and since

$$f\left(X_t\right) = \alpha\, 2\sqrt{X_t}\, \sqrt{X_t} + \frac{1}{2} 2\sqrt{X_t}\, \frac{1}{2} 2 \frac{1}{\sqrt{X_t}} = 1$$

when $\alpha = 0$. This will give the solution:

$$X_t = \left(\sqrt{X_0} + W_t\right)^2$$

1.4 Stochastic Taylor Series Expansion, [7], [22], [37].

Taylor series expansion is well-known for deterministic functions where they turn out to be useful tool especially in numerical analysis. This idea can be carried over the stochastic setting by applying Itô's formula. Thus, following Platen and Wagner [31], we get the stochastic Taylor formula, which represents a generalization of the deterministic Taylor formula. In the sequel, we distinguish the Itô Taylor expansion from the Stratonovich-Taylor expansion due to the different rules of calculus.

With deterministic differential equation methods, a numerical method may be derived by comparing the expansion of the method and the solution of the ordinary differential equation in a Taylor series; and exactly the same procedure can take place in the stochastic setting, using a stochastic version of Taylor series. The Itô -Taylor expansion was first established by Wanger and Platen in 1978 and 1982 [31], [42], and full details are given by Kloeden and Platen in 1995 [22]. It allows y_t (or any function of y_t to be expanded about the point y_{t_0} up to the required degree of accuracy) in terms of multiple stochastic integrals along with function evaluations at y_{t_0}. In order to derive the expansion, the Itô formula is applied successively to the SDE given in equation (1.19) as it is represented in the autonomous integral form:

$$y_t = y_{t_0} + \int_{t_0}^{t} f(y_s)\, ds + \int_{t_0}^{t} g(y_s)\, dW_s \qquad \text{... (1.43)}$$

From the stochastic chain rule (1.21) in autonomous form:

$$F(y_t) - F(y_{t_0}) = \int_{t_0}^{t} \left(\frac{dF}{dy_s} f + \frac{1}{2} \frac{d^2F}{dy_s^2} g^2 \right) ds + \int_{t_0}^{t} \frac{dF}{dy_s} g\, dW_s$$

$$= \int_{t_0}^{t} L^0 F(y_s)\, ds + \int_{t_0}^{t} L^1 F(y_s)\, dW_s \qquad \text{... (1.44)}$$

where the operators for scalar problems are:

$$L^0 F(y_t) = \frac{dF}{dy} f + \frac{1}{2} \frac{d^2F}{dy_t^2} g^2,$$

$$L^1 F(y_t) = \frac{dF}{dy_t} g$$

Applying the Itô formula given by equation (1.44) for f and g in equation (1.43), then one application gives:

$$y_t = y_{t_0} + \int_{t_0}^{t} \left(f(y_{t_0}) + \int_{t_0}^{s} L^0(f(y_u)\,du + \int_{t_0}^{s} L^1(f(y_u)\,dW_u \right) ds$$
$$+ \int_{t_0}^{t} \left(g(y_{t_0}) + \int_{t_0}^{s} L^0(g(y_u)\,du + \int_{t_0}^{s} L^1(g(y_u)\,dW_u \right) dW_s \quad \text{... (1.45)}$$

Consequently, by applying the Itô formula and using $L^0 f$, $L^1 f$, $L^0 g$ and $L^1 g$, the Itô-Taylor expansion is given by:

$$y_t = y_{t_0} + f(y_{t_0}) \int_{t_0}^{t} ds + g(y_{t_0}) \int_{t_0}^{t} dW_s + \int_{t_0}^{t} \int_{t_0}^{s} L^0 f(y_u)\,du\,ds$$
$$+ \int_{t_0}^{t} \int_{t_0}^{s} L^1 f(y_u)\,dW_u\,ds + \int_{t_0}^{t} \int_{t_0}^{s} L^0 g(y_u)\,du\,dW_s$$
$$+ \int_{t_0}^{t} \int_{t_0}^{s} L^1 g(y_u)\,dW_u\,dW_s$$

Similarly, the Stratonovich-Taylor expansion is developed, using this time the following operators:

$$\underline{L}^0 a(y_t) = \frac{da}{dy_t} f,$$

$$\underline{L}^1 a(y_t) = \frac{da}{dy_t} g$$

Then the first few terms in the Stratonovich-Taylor series are thus given by:

$$y_t = y_{t_0} + f(y_{t_0}) \int_{t_0}^{t} ds + g(y_{t_0}) \int_{t_0}^{t} \circ\, dW_s + \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^0 f(y_u)\,du\,ds +$$
$$\int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^1 f(y_u) \circ dW_u\,ds + \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^0 g(y_u)\,du \circ dW_s + \int_{t_0}^{t} \int_{t_0}^{s}$$
$$\underline{L}^1 g(y_u) \circ dW_u \circ dW_s$$

and applying the standard chain rule in Stratonovich calculus to $\underline{L}^0 f$, $\underline{L}^1 f$, $\underline{L}^0 g$ and $\underline{L}^1 g$ will yields to the next terms in the expansion.

Remarks (1.6), [37]:

1. The above discussion is given for one-dimensional autonomous SDE's, and we shall consider next the nonautonomous SDE's, and deriving its related stochastic Taylor series expansion. Let X_t be the solution of the Itô SDE in general form:

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)\, ds + \int_{t_0}^{t} b(s, X_s)\, dW_s \qquad \text{... (1.46)}$$

and let $f : I \times \mathbf{R} \longrightarrow \mathbf{R}$ with $f \in C^{1,2}(I \times \mathbf{R}, \mathbf{R})$. By applying the Itô formula, we get for $Y_t = f(t, X_t)$, the following equation:

$$Y_t = Y_{t_0} + \int_{t_0}^{t} \left(\frac{\partial f}{\partial t}(s, X_s) + a(s, X_s)\frac{\partial f}{\partial x}(s, X_s) + \frac{1}{2} b^2(s, X_s) \frac{\partial^2 f}{\partial x^2}(s, X_s) \right) ds + \int_{t_0}^{t} b(s, X_s)\frac{\partial f}{\partial x}(s, X_s)\, dW_s.$$

For simplicity of notations, the operators $L^0 . = \frac{\partial .}{\partial t} + a\frac{\partial .}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 .}{\partial x^2}$ and $L^1 . = b\frac{\partial .}{\partial x}$ are introduced and rewriting the above mentioned equations as:

$$Y_t = Y_{t_0} + \int_{t_0}^{t} L^0 f(s, X_s)\, ds + \int_{t_0}^{t} L^1 f(s, X_s)\, dW_s$$

and by applying the Itô formula given in equation (1.21) to the functions $f = a$ and $f = b$ in equation (1.46), getting:

$$X_t = X_{t_0} + \int_{t_0}^{t} \left(a(t_0, X_{t_0}) + \int_{t_0}^{s} L^0 a(u, X_u)\, du + \int_{t_0}^{s} L^1 a(u, X_u)\, dW_u \right) ds + \int_{t_0}^{t} \left(b(t_0, X_{t_0}) + \int_{t_0}^{s} L^0 b(u, X_u)\, du + \int_{t_0}^{s} L^1 b(u, X_u)\, dW_u \right) dW_s \qquad \text{... (1.47)}$$

which may be also written as:

$$X_t = X_{t_0} + a(t_0, X_{t_0}) \int_{t_0}^{t} ds + b(t_0, X_{t_0}) \int_{t_0}^{t} dW_s + R$$

where R denotes the remainder. Continuing in this way by applying the Itô formula to the functions $f = L^i a$ and $f = L^i b$, for $i = 0, 1$ in equation (1.47) to get the Itô-Taylor series expansion.

2. The ***Stratonovich-Taylor expansion*** works now analogously to the Itô-Taylor expansion. Let X_t be the solution of the Stratonovich SDE in integral form:

$$X_t = X_{t_0} + \int_{t_0}^{t} \underline{a}(s, X_s)\, ds + \int_{t_0}^{t} b(s, X_s) \circ dW_s \qquad \text{... (1.48)}$$

Since a Stratonovich SDE transforms according to the deterministic chain rule, we get for $f: I \times \mathbf{R} \longrightarrow \mathbf{R}$ with $f \in C^{1,2}(I \times \mathbf{R}, \mathbf{R})$ the equation:

$$f(t, X_t) = f(t_0, X_{t_0}) + \int_{t_0}^{t} \left(\frac{\partial f}{\partial t}(s, X_s) + \underline{a}(s, X_s) \frac{\partial f}{\partial x}(s, X_s) \right) ds +$$

$$\int_{t_0}^{t} b(s, X_s) \frac{\partial f}{\partial x}(s, X_s) \circ dW_s \qquad \text{... (1.49)}$$

Introducing the operators $\underline{L}^0 . = \frac{\partial .}{\partial t} + \underline{a} \frac{\partial .}{\partial x}$ and $\underline{L}^1 . = b \frac{\partial .}{\partial x}$, then one can also write:

$$f(t, X_t) = f(t_0, X_{t_0}) + \int_{t_0}^{t} \underline{L}^0 f(s, X_s)\, ds + \int_{t_0}^{t} \underline{L}^1 f(s, X_s) \circ dW_s$$

Continuing in the same way as for the Itô calculus and applying equation (1.49) to the functions $f = \underline{a}$ and $f = b$ in equation (1.48), getting:

$$X_t = X_{t_0} + \int_{t_0}^{t} [\, \underline{a}(t_0, X_{t_0}) + \int_{t_0}^{s} \underline{L}^0 \underline{a}(u, X_u)\, du + \int_{t_0}^{s} L^1 \underline{a}(u, X_u) \circ$$

$$dW_u \,]\, ds + \int_{t_0}^{t} [\, b(t_0, X_{t_0}) + \int_{t_0}^{s} \underline{L}^0 b(u, X_u)\, du +$$

$$\int_{t_0}^{s} \underline{L}^1 b(u, X_u) \circ dW_u] \circ dW_s$$

$$= X_{t_0} + \underline{a}(t_0, X_{t_0}) \int_{t_0}^{t} ds + b(t_0, X_{t_0}) \int_{t_0}^{t} \circ dW_s$$

$$+ R \qquad \text{... (1.50)}$$

with remainder R. Again, continue in the expansion applying for instance equation (1.49) to integrands $\underline{L}^i \underline{a}$ and $\underline{L}^i b$, for i = 0, 1 in equation (1.50) to get the Stratonovich Taylor expansion recursively.

Leaving the one-dimensional case treating the general setting with a d-dimensional solution of SDE's of m-dimensional Wiener process, we make use of a convenient notation applying the concept of hierarchical sets. In the following, let $a: I \times \mathbf{R}^d \longrightarrow \mathbf{R}^d$ be a drift vector and $b: I \times \mathbf{R}^d \longrightarrow \mathbf{R}^{d \times m}$ be the diffusion matrix and let X_t be the solution of the SDE:

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)\, ds + \int_{t_0}^{t} b(s, X_s)\, dW_s \qquad \text{... (1.51)}$$

In order to describe the stochastic Taylor series expansion, a multi-dimensional and for multi Wiener process setting, the following terminology will be used:

1. A ***multiple Itô integral*** is given by:

$$I_{(j_1,j_2,\ldots,j_L),t} = \int_0^t \int_0^{s_L} \ldots \int_0^{s_2} dW_{s_1}^{j_1} \ldots dW_{s_L}^{j_L} \qquad \ldots (1.52)$$

whereas above $j_i \in \{0, 1, \ldots, m\}$ for m -Weiner processes, and where $dW_{s_i}^0 = ds_i$.

2. A ***multiple Stratonovich integral*** is given by:

$$J_{(j_1,j_2,\ldots,j_L),t} = \int_0^t \int_0^{s_L} \ldots \int_0^{s_2} \circ dW_{s_1}^{j_1} \circ \ldots \circ dW_{s_L}^{j_L} \qquad \ldots (1.53)$$

where $j_i \in \{0, 1, .., m\}$ form m -Weiner process; the brackets and the dependence on t will be dropped when the meaning is clear. Here, also $o\, dW_{s_i}^0 = ds_i$.

For more explanation to this context, we start with the definition of multi-indices and hierarchical sets which provide an efficient notation in the following. Let:

$$M = \{\alpha = (j_1, j_2, \ldots, j_L) \in \{0, 1, \ldots, m\}^L : L \in \mathbb{N}\} \cup \{v\} \qquad \ldots (1.54)$$

be set of all ***multi-indices***. The length $L(\alpha)$ of a multi-index $\alpha = (j_1, j_2, \ldots, j_L)$, where $j_i \in \{0, 1, .., m\}$, $i \in \{0, 1, .., L\}$ and $m = 1, 2, \ldots$ may be defined as:

$$L(\alpha) = L, \; L \in \mathbb{N} \qquad \ldots (1.55)$$

Where v is the multi-index of length 0, such that:

$$L(v) = 0 \qquad \ldots (1.56)$$

Thus, for example $L((1, 0)) = 2$ and $L((1, 0, 1)) = 3$.

In addition let $n(\alpha)$ denote the number of components of a multi-index α, which are equal to 0, such that:

$$n(\alpha) = n \qquad \ldots (1.57)$$

where n is the number of zero components of α, for example $n((1, 0, 1)) = 1$, $n((0, 1, 0)) = 2$, $n((0, 0)) = 2$.

Now, for $\alpha = (j_1, j_2, \ldots, j_L) \in M$ with $L = L(\alpha) \geq 1$, define:

$$-\alpha = (j_2, j_3, \ldots, j_L) \text{ and } \alpha- = (j_1, j_2, \ldots, j_{L-1}) \qquad \ldots (1.58)$$

by deleting the first and the last components of α, respectively. For example:

$-(1, 0) = (0), (1, 0)- = (1), -(0, 1, 1) = (1, 1), (0, 1, 1)- = (0, 1)$

A subset $H \subset M$ is called a ***hierarchical set*** if $H \neq \varnothing$ and if:

$$\sup_{\alpha \in H} L(\alpha) < \infty \quad \text{and} \quad -\alpha \in H, \text{ for each } \alpha \in H \setminus \{v\} \qquad \text{... (1.59)}$$

For example, the sets {v}, {v, (0), (1)}, {v, (0), (1), (1, 1)} are hierarchical sets.

The corresponding ***remainder set*** $R(H)$ for the hierarchical set H is defined as:

$$R(H) = \{\alpha \in M \setminus H : -\alpha \in H\} \qquad \text{... (1.60)}$$

For example:

$R(\{v\}) = \{(0), (1)\}$, $R(\{v, (0), (1)\}) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, and $R(\{v, (0), (1), (1, 1)\}) = \{(0, 0), (0, 1), (1, 0), (0, 1, 1), (1, 1, 1)\}$

and consists of all next following multi-indices with respect to the given hierarchical set H.

We are now able to define multiple stochastic integrals. Let us introduce three classes of adapted right continuous stochastic processes $(f_t)_{t \in I}$ with left hand limits. We say:

$$f \in H_v \quad \text{if } |f(t, \omega)| < \infty, \quad \textbf{\textit{P-w.p.1}} \quad \text{for each } t \geq 0 \qquad \text{... (1.61)}$$

and we say for each $t \geq 0$, $f \in H_{(0)}$ if f satisfies condition given by:

$$\boldsymbol{P}\left(\int_0^t |f(s, \omega)|\, ds < \infty\right) = 1, \ \textbf{\textit{P-w.p.1}} \qquad \text{... (1.62)}$$

In addition, define $H_{(1)}$ is the set of all f satisfying the condition:

$$\int_0^t |f(s, \omega)|^2 \, ds < \infty, \ \textbf{\textit{w.p.1}} \text{ for each } t \geq 0 \qquad \text{... (1.63)}$$

Furthermore, define $f \in H_{(j)}$ for each $j \in \{2, \ldots, m\}$ if $f \in P$ holds.

Now, let ρ and τ be two stopping times with:

$$0 \leq \rho(\omega) \leq \tau(\omega) \leq T, \ \textbf{\textit{P-w.p.1}} \qquad \text{... (1.64)}$$

For a multi-index $\alpha = (j_1, j_2, \ldots, j_L) \in M$ and a process $f \in H_\alpha$, we define the ***multiple Itô integral*** $I_\alpha[f(.)]_{\rho,\tau}$ with respect to the m-dimensional Wiener process $W = (W^1, W^2, \ldots, W^m)$ recursively by:

$$I_\alpha[f(.)]_{\rho,\tau} = \begin{cases} f(\tau), & \text{if } L = 0 \\ \int_\rho^\tau I_{\alpha-}[f(.)]_{\rho,s}\, ds, & \text{if } L \geq 1 \text{ and } j_L = 0 \\ \int_\rho^\tau I_{\alpha-}[f(.)]_{\rho,s}\, dW_s^{j_L}, & \text{if } L \geq 1 \text{ and } j_L \geq 1 \end{cases} \quad \ldots (1.65)$$

Here, we note the H_α with $\alpha = (j_1, j_2, \ldots, j_L)$ and $L \geq 2$ describes the totality of adapted right continuous process f with left hand limits, such that the integral process $(I_{\alpha-}[f(.)]_{\rho,t})_{t\in I}$ considered as a function of t satisfies $I_{\alpha-}[f(.)]_{\rho,.} \in H_{(j_L)}$. If the integrand is constant, i.e., $f(t, \omega) \equiv 1$, we abbreviate $I_\alpha[f(.)]_{\rho,\tau}$ as I_α if the limits ρ and τ are obvious from the context. In the following, we denote $W_t^0 = t$, $dW_t^0 = dt$ and $I_{\alpha,t} = I_\alpha[1]_{0,t}$ when $\rho = 0$ and $\tau = t$.

As an illustration of this terminology, consider the following examples:

$$I_v[f(.)]_{0,t} = f(t), \quad I_{(0)}[f(.)]_{\tau_i,\tau_{i+1}} = \int_{\tau_i}^{\tau_{i+1}} f(s)\, ds,$$

$$I_{(1)}[f(.)]_{0,t} = \int_0^t f(s)\, dW_s^1, \; I_{(0,1)}[f(.)]_{0,t} = \int_0^t \int_0^{s_1} f(s_2)\, ds_2\, dW_{S_1}^1$$

<u>Remarks (1.7), [7]:</u>

By the same way, the ***multiple Stratonovich integral*** may be defined which is denoted by $J_\alpha[f(.)]_{\rho,\tau}$ with respect to m-dimensional Weiner process and use the same modifications of multiple Itô integral about the function and the limits ρ and τ, i.e., $f(t, \omega) = 1$, $J_\alpha[f(t)]_{\rho,\tau} = J_\alpha = J_\alpha[1]_{0,t}$, when $\rho = 0$, $\tau = t$, which will be denoted by $W_t^0 = t$ and $\circ\, dW_t^0 = dt$, such that:

$$J_{\alpha}[f(.)]_{\rho,\tau} = \begin{cases} f(\tau), & \text{if } L = 0 \\ \int_{\rho}^{\tau} J_{\alpha-}[f(.)]_{\rho,s}\, ds, & \text{if } L \geq 1 \text{ and } j_L = 0 \\ \int_{\rho}^{\tau} J_{\alpha-}[f(.)]_{\rho,s}\, \mathbf{0}\, dW_s^{j_L}, & \text{if } L \geq 1 \text{ and } j_L \geq 1 \end{cases} \quad \text{... (1.66)}$$

For example:

$$J_{(0,1,1)}[1]_{0,t} \equiv J_{(0,1,1),t} = \int_0^t J_{(0,1)} \circ dW_{s_1}^1$$

$$= \int_0^t \int_0^{s_1} J_{(0)} \circ dW_{s_2}^1 \circ dW_{s_1}^1$$

$$= \int_0^t \int_0^{s_1} \int_0^{s_2} ds_3 \circ dW_{s_2}^1 \circ dW_{s_1}^1$$

$$J_{(0,1,0)}[1]_{0,t} \equiv J_{(0,1,0),t} = \int_0^t J_{(0,1)} ds_1 = \int_0^t \int_0^{s_1} J_{(0)} \circ dW_{s_2}^1\, ds_1$$

$$= \int_0^t \int_0^{s_1} \int_0^{s_2} ds_3 \circ dW_{s_2}^1\, ds_1$$

Lemma (1.1), [7], [22]:

Let $\alpha \in (j_1, j_2, \ldots, j_L)$, $\beta \in (j_1', j_2', \ldots, j_p')$, with L, p = 1, 2, …; and where $j_i,\ j_k' \in \{0, 1, \ldots, m\}$, for m-Wiener processes, then:

$$I_{\alpha,t}\, I_{\beta,t} = \int_0^t I_{\alpha,s}\, I_{\beta-,s}\, dW_s^{j_p'} + \int_0^t I_{\alpha-,s}\, I_{\beta,s}\, dW_s^{j_L} + \int_0^t I_{\alpha-,s}\, I_{\beta-,s}\, I_{\{j_L = j_p' \neq 0\}}\, ds$$

for $t \geq 0$ and where the indicator function:

$$I_A = I_{\{j_L = j_p' \neq 0\}} = \begin{cases} 1, & \text{if A is true} \\ 0, & \text{otherwise} \end{cases}$$

Lemma (1.2), [7], [22]:

$$W_t^j\, I_{\alpha,t} = \sum_{i=0}^{L} I_{(j_1, j_2, \ldots, j_i, j, j_{i+1}, \ldots, j_L),t} + \sum_{i=1}^{L} I_{\{j_i = j \neq 0\}} I_{(j_1, j_2, \ldots, j_{i-1}, 0, j_{i+1}, \ldots, j_L),t}$$

for all $t \geq 0$, where $\alpha = \{j_1, j_2, \ldots, j_L\}$, $j_i \in \{0, 1, \ldots, m\}$ for m-Wiener processes.

Example (1.8), [7]:

Use lemma (1.2) to evaluate $I_0\, I_{01}$, when $\alpha = (j_1, j_2)$, since $j_1, j_2 \in \{0, 1\}$, $L(\alpha) = 2$

$$W^{j}_{(t)}\, I_{\alpha,t} = \sum_{i=0}^{2} I_{(j_1, j_2, \ldots, j_i, j, j_{i+1}, \ldots, j_L),t}$$

$$+ \sum_{i=1}^{2} I_{\{j_i = j \neq 0\}} I_{(j_1, j_2, \ldots, j_{i-1}, 0, j_{i+1}, \ldots, j_L),t}$$

$$= I_{(j, j_1, j_2),t} + I_{(j_1, j, j_2),t} + I_{(j_1, j_2, j),t}$$

$$+ I_{\{j_1 = j \neq 0\}} I_{(0, j_2),t} + I_{\{j_2 = j \neq 0\}}\, I_{(j_1, 0),t}$$

Now, when $j_1 = 0$, $j_2 = 1$, $j = 0$, and for $i = 1, 2$, then:

$I_{\{j_1 = j \neq 0\}} = I_{\{0=0\neq 0\}} = 0$, $I_{\{j_2 = j \neq 0\}} = I_{\{1=0\neq 0\}} = 0$, and thus:

$$I_0\, I_{01} = t\, I_{01} = W^{0}_{(t)}\, I_{(0,1),t} = I_{(0,0,1),t} + I_{(0,0,1),t} + I_{(0,1,0),t}$$

Then $I_0\, I_{01} = 2\, I_{001} + I_{010}$.

Also, using lemma (1.1), such that $\alpha = (0)$, $\beta = (0, 1)$, then:

$$I_{(0),t}\, I_{(0,1),t} = \int_0^t I_{(0),s}\, I_{(0),s}\, d\, W^{1}_{s} + \int_0^t (1)\, I_{(0,1),s}\, d\, W^{0}_{s} + \int_0^t (1) I_{(0),s}\, I_{\{0=1\neq 0\}}$$

$$ds$$

Where $I_0 = \int_0^t ds$, $d\, W^{0}_{s} = dt$, $I_{010} = \int_0^t I_{(0,1),s}\, ds$, $I_{\{0=1\neq 0\}} = 0$; and by using the indicator function $I_A = I_{\{j_i = j'_p \neq 0\}} = \begin{cases} 1, & \text{if A is true} \\ 0, & \text{otherwise} \end{cases}$.

Now, when $\alpha = (0)$, $\beta = (0)$, then:

$$I_0\, I_0 = \int_0^t (1)\, I_0\, d\, W^{0}_{s} + \int_0^t (1)\, I_0\, d\, W^{0}_{s} + \int_0^t (1)\, (1)\, I_{\{0=0\neq 0\}}\, ds$$

$$= 2 \int_0^t I_0\, ds = 2\, I_{00}$$

Then $I_{(0),t}\, I_{(0,1),t} = \int_0^t 2\, I_{(0,0),s}\, d\, W^{1}_{s} + I_{010} = 2\, I_{001} + I_{010}$.

We also have the following special cases of lemmas (1.1) and (1.2).

$$\left.\begin{aligned} I_0\, I_1 &= I_{01} + I_{10} \\ I_0\, I_{11} &= I_{110} + I_{101} + I_{011} \\ I_1\, I_{01} &= 2\, I_{011} + I_{101} + I_{00} \\ I_1\, I_{10} &= I_{101} + 2\, I_{110} + I_{00} \end{aligned}\right\} \quad \text{... (1.67)}$$

Lemma (1.3), [22]:

Suppose that $\alpha = (j_1, j_2, \ldots, j_L)$ with $j_1 = j_2 = \ldots = j_L = j \in \{0, 1, \ldots, m\}$ for m-Wiener processes; where $L \geq 2$. Then for $t \geq 0$

$$I_{\alpha,t} = \begin{cases} \dfrac{1}{L!}\, t^L, & \text{if } j = 0 \\ \dfrac{1}{L}\left(W^j_{(t)}\, I_{\alpha-,t} - t\, I_{(\alpha-)-,t} \right), & \text{if } j \geq 1 \end{cases}$$

Example (1.9), [22]:

Using lemma (1.3) to evaluate I_{11}, I_{111} and I_{000}, yields to:

1. $I_{000} = \dfrac{1}{3!} t^3$, where $\alpha = (0, 0, 0)$, $L(\alpha) = 3$.
2. When $\alpha = (1, 1)$, $L(\alpha) = 2$, $\alpha- = (1)$, then:

$$I_{11} = \frac{1}{2}\left(W^1_{(t)}\, I_{(1),t} - t\,(1) \right) = \frac{1}{2}\left(I^2_{(1),t} - t \right), \text{ where } I_{(j),t} = W^j_{(t)},$$

$j = 1, 2, \ldots$.

3. When $\alpha = (1, 1, 1)$, $L(\alpha) = 3$, $\alpha- = (1, 1)$, $(\alpha-)- = (1)$, then:

$$I_{111} = \frac{1}{3}\left(W^1_{(t)}\, I_{(1,1),t} - t\, I_{(1),t} \right) = \frac{1}{3}\left(I_{(1),t}\left(\frac{1}{2} I^2_{(1),t} - \frac{t}{2} \right) - t\, I_{(1),t} \right)$$

$$= \frac{1}{3}\left(\frac{1}{2} I^3_{(1),t} - \frac{1}{2} I_{(1),t}\, t - I_{(1),t} \right) = \frac{1}{3!}\left(I^3_{(1),t} - 3\, I_{(1),t}\, t \right)$$

Now, lemmas (1.2) and (1.3) may be rewritten in another form to obtain relationships between multiple Stratonovich integrals, as follows:

Lemma (1.4), [7], [22]:

$$W^j_t\, J_{\alpha,t} = \sum_{i=0}^{L} J_{(j_1, j_2, \ldots, j_i, j, j_{i+1}, \ldots, j_L),t}\,, \text{ for all } t \geq 0,\ \alpha = (j_1, j_2, \ldots, j_L).$$

Lemma (1.5), [22]:

$J_{\alpha,t} = \frac{1}{L!}(J_{(j),t})^L$, for all $t \geq 0$, $\alpha = (j_1, j_2, \ldots, j_L)$, $j_1 = j_2 = \ldots = j_L = j$ and $L \geq 0$

For the later use, we state now some special cases of the last two lemmas:

$$\left.\begin{aligned} & J_0 J_1 = J_{01} + J_{10} \\ & J_0 J_{11} = J_{110} + J_{101} + J_{011} \\ & J_1 J_{01} = 2J_{011} + J_{101} \\ & J_1 J_{10} = J_{101} + 2J_{110} \\ & J_{11} = \frac{1}{2} J_1^2 \\ & J_{111} = \frac{1}{6} J_1^3 \end{aligned}\right\} \quad \ldots (1.68)$$

Also, the following results obtained in [22], expresses the relationship between multiple Stratonovich and Itô integrals:

$$\left.\begin{aligned} & J_\alpha = I_\alpha, && \text{where } L(\alpha) = 0 \text{ or } 1 \\ & J_\alpha = I_\alpha + \frac{1}{2} I_{\{j_1 = j_2 \neq 0\}} I_0, && \text{where } L(\alpha) = 2 \\ & J_\alpha = I_\alpha + \frac{1}{2}\left(I_{\{j_1 = j_2 \neq 0\}} I_{(0,j_3),t} + I_{\{j_2 = j_3 \neq 0\}} I_{(j_1,0),t} \right), && \text{where } L(\alpha) = 3 \\ & J_\alpha = I_\alpha + \frac{1}{4} I_{\{j_1 = j_2 \neq 0\}} I_{\{j_3 = j_4 \neq 0\}} I_{(0,0),t} + \frac{1}{2}\left(I_{\{j_1 = j_2 \neq 0\}} I_{(0,j_3,j_4),t} + \right. \\ & \left. I_{\{j_2 = j_3 \neq 0\}} I_{(j_1,0,j_4),t} + I_{\{j_3 = j_4 \neq 0\}} I_{(j_1,j_2,0),t} \right), && \text{where } L(\alpha) = 4 \end{aligned}\right\}$$

... (1.69)

Kloeden and Platen in (1995), [22] also express this relationship as recursive formula for J_α. Thus, if $L(\alpha) \geq 2$ and $\alpha = (j_1, j_2, \ldots, j_L)$, then:

$$J_\alpha = I_{j_L}[J_{\alpha-}] + \frac{1}{2} I_{\{j_L = j_{L-1} \neq 0\}} I_0 [J_{(\alpha-)-}] \quad \ldots (1.70)$$

where:

$$I_{j_L}[f] = \begin{cases} \int f(s)\, ds, & \text{if } j_L = 0 \\ \int f(s)\, dW_s^{j_L}, & \text{if } j_L > 0 \end{cases}$$

Example (1.10), [7], [22]:

1. When $\alpha = (1)$, then $J_1 = I_1$.

2. When $\alpha = (1, 1)$, $L(\alpha) = 2$, $\alpha- = (1)$ and $I_{\{j_L = j_{L-1} \neq 0\}} = I_{\{1=1\neq 0\}} = 1$

when applying equation (1.70), then $J_{(1,1),t} = I_1[J_{(1),t}] + \frac{1}{2}(1)\, I_0[(1)]$,

since $J_{(1),t} = I_{(1),t}$, $I_1[I_{(1),t}] = I_{(1,1),t}$, i.e.,

$$I_1[I_{(1),t}] = \int_0^t I_{(1)} d W_S^1 = \int_0^t \int_0^s d W_u^1 d W_S^1 = I_{(1,1),t}$$

Hence, $J_{11} = I_{11} + \frac{1}{2} I_0$.

3. Similarly, $J_{111} = I_{111} + \frac{1}{2} I_{01} + \frac{1}{2} I_{10}$.

Theorem (1.6) (The Itô -Taylor Expansion), [22], [37]:

Let $H \subseteq M$ be a hierarchical set, let ρ and τ be two stopping times with $t_0 \leq \rho(\omega) \leq \tau(\omega) \leq T < \infty$ ***P-w.p.1*** and let $f : I \times \mathbb{R}^d \to \mathbb{R}$, then for the solution $(X_t)_{t\in I}$ of the Itô SDE given in equation (1.51). The Itô -Taylor expansion:

$$f(\tau, X_\tau) = \sum_{\alpha \in H} I_\alpha[f_\alpha(\rho, X_\rho)]_{\rho,\tau} + \sum_{\alpha \in R(H)} I_\alpha[f_\alpha(., X_.)]_{\rho,\tau} \quad \text{... (1.71)}$$

holds, provided that all of the derivatives of f, a and b and all of the multiple Itô integrals appearing in (1.71) exist.

Theorem (1.7) (The Stratonovich-Taylor Expansion), [22]:

Let $H \subseteq M$ be a hierarchical set, let ρ and τ be two stopping times with $t_0 \leq \rho(\omega) \leq \tau(\omega) \leq T < \infty$ ***P-w.p.1*** and let $\underline{f} : I \times \mathbb{R}^d \to \mathbb{R}$, then for the solution $(X(t))_{t\in I}$ of the Stratonovich SDE given in equation:

$$X_t = X_{t_0} + \int_0^t \underline{a}(s, X_s)\, ds + \int_0^t b(s, X_s) \circ dW_s$$

Then, the Stratonovich-Taylor expansion:

$$f(\tau, X_\tau) = \sum_{\alpha \in H} J_\alpha[\underline{f}_\alpha(\rho, X_\rho)]_{\rho,\tau} + \sum_{\alpha \in R(H)} J_\alpha[\underline{f}_\alpha(., X_.)]_{\rho,\tau} \quad \text{... (1.72)}$$

holds, provided that all of the derivatives of $\underline{f}$, $\underline{a}$ and b and all of the multiple Stratonovich integrals appearing in equation (1.72) exist.

1.5 Expectation of Stochastic Integrals, [7], [22]:

It is also necessary when studying the order of convergence of a numerical method, to be able to determine the expected value of multiple stochastic integrals. The following lemmas, adopted from [7] and [22] provide a means for calculating these expected values. Firstly, however, some additional terminologies is required.

If $\alpha = (j_1, j_2, \ldots, j_L)$ is the index of the multiple stochastic integral, then α^+ is the index obtained from α by deleting all zero components, for example, if $\alpha = (1, 0, 2, 1)$, then we have:

$$\alpha^+ = (1, 0, 2, 1)^+ = (1, 2, 1)$$

and $K_0(\alpha)$ will denote the number of zero components in α that precede the first non-zero component of α or until the end of α if all of its components are zeros, and $K_i(\alpha)$, for $i = 1, 2, \ldots, L(\alpha^+)$, will count the number of zeros between the i^{th} and $(i + 1)^{th}$ non-zero components of α For example, if $\alpha = (0, 1, 1, 0, 2)$, then $\alpha^+ = (1, 1, 2)$, $L(\alpha) = 5$, $L(\alpha^+) = 3$, $K_0(\alpha) = 1$, $K_1(\alpha) = 0$, $K_2(\alpha) = 1$, $K_3(\alpha) = 0$; and for $\alpha = (0, 1, 2, 0)$, we have $\alpha^+ = (1, 2)$, $L(\alpha) = 4$, $L(\alpha^+) = 2$ and $K_0(\alpha) = 1$, $K_1(\alpha) = 0$, $K_2(\alpha) = 1$.

Lemma (1.6), [22]:

Let $\alpha \in M \setminus \{v\}$ with $L(\alpha) \neq n(\alpha)$, let $f \in H_\alpha$ and let ρ and τ be two stopping times with $t_0 \le \rho \le \tau \le T < \infty$, ***w.p.1***. Then:

$$E(I_\alpha[f(.)]_{\rho,\tau} | A_\rho) = 0, \ \textbf{\textit{w.p.1}} \qquad \ldots (1.73)$$

Lemma (1.7), [7], [22]:

Let $\alpha, \beta \in M$, let $f \in H_\alpha$, $g \in H_\beta$ and let ρ and τ be two stopping times with $t_0 \le \rho \le \tau \le T < \infty$, ***w.p.1.***,where τ is A_ρ-measurable Then:

$$E(I_\alpha[f(.)]_{\rho,\tau}, I_\beta[g(.)]_{\rho,\tau} \mid A_\rho) \begin{cases} = 0, & \text{if } \alpha^+ \neq \beta^+ \\ \le K_{f,g} \dfrac{(\tau-\rho)^{\omega(\alpha,\beta)}}{\omega(\alpha,\beta)!} \displaystyle\prod_{i=0}^{L(\alpha^+)} \frac{(k_i(\alpha)+k_i(\beta))!}{k_i(\alpha)!\ k_i(\beta)!}, & \text{if } \alpha^+ = \beta^+ \end{cases}$$

$$\ldots (1.74)$$

where $\omega(\alpha, \beta) = L(\alpha^+) + \sum_{i=0}^{L(\alpha^+)} (k_i(\alpha) + k_i(\beta))$ and

$K_{f,g} = \sup_{S_1, S_2 \in [\rho, \tau]} E(|f1(S_1) g(S_2)|| A_\rho)$. Moreover, (1.74) holds with $f \equiv g \equiv 1$.

Remark (1.8), [22]:

By using lemma (1.7) and the moment estimates given in equations (1.73) and (1.74) for multiple Itô integrals, the values of the conditional expectations may be evaluated as:

$$E(I_{\alpha_1} I_{\alpha_2} \dots I_{\alpha_k}) = E\left(\prod_{i=1}^{k} I_{\alpha_i, t_0, t_0+h} \middle| A_0 \right)$$

for $h \in [0, T - t_0]$ and $\alpha_1, \alpha_2, \dots, \alpha_k \in \{\alpha \in M : L(\alpha) \le \beta\} \setminus \{v\}$; where $\beta \in \{1, 2, \dots\}$, we have:

$$E(I_{\alpha_1} I_{\alpha_2} \dots I_{\alpha_k}) = 0 \qquad \dots (1.75)$$

When the number of nonzero components of the multi-indices involved $\lambda := \sum_{i=1}^{k} (L(\alpha_i) - n(\alpha_i))$ is odd. Furthermore, when λ is even we find that:

$$|E(I_{\alpha_1} I_{\alpha_2} \dots I_{\alpha_k})| \le K h^\rho, \text{ where } \rho = \frac{1}{2}\lambda + \sum_{i=1}^{k} n(\alpha_i) \qquad \dots (1.76)$$

We shall say that the expectation given in equation (1.76) has an order ρ in the time increment h. Excluding those already given in equation (1.75) and those of constants $I_{(0)} = h$, $I_{(0,0)} = \frac{1}{2}h^2$, $I_{(0,0,0)} = \frac{1}{3!}h^3$,

Example (1.11), [7]:

Let $\alpha = (1)$ and $\beta = (1, 0)$. Then $\alpha^+ = (1) = \beta^+$, and so by lemma (1.7) the expectation $E(I_1 I_{10})$ is non-zero. Counting the zero components, $K_0(\alpha) = K_1(\alpha) = 0$ and $K_0(\beta) = 0$, $K_1(\beta) = 1$, such that:

$$\omega(\alpha, \beta) = L(\alpha^+) + \sum_{i=0}^{L(\alpha^+)} (K_i(\alpha) + K_i(\beta))$$

$$= 1 + K_0(\alpha) + K_0(\beta) + K_1(\alpha) + K_1(\beta) = 2$$

Therefore, from lemma (1.7), with $\tau = h$, $\rho = 0$ and where $\alpha^+ = \beta^+$ and $f \equiv g \equiv 1$

$$E(I_1 I_{10}) \le K_{f,g} \frac{(\tau-\rho)^{\omega(\alpha,\beta)}}{\omega(\alpha,\beta)!} \prod_{i=0}^{L(\alpha^+)} \frac{(K_i(\alpha)+K_i(\beta))!}{K_i(\alpha)!\ K_i(\beta)!}$$

$$\le K_{1,1} \frac{h^2}{2!} \frac{(K_0(\alpha)+K_0(\beta))!}{K_0(\alpha)!\ K_0(\beta)!} \frac{(K_1(\alpha)+K_1(\beta))!}{K_1(\alpha)!\ K_1(\beta)!} = \frac{1}{2}h^2$$

where, $K_{1,1} = \delta_{i,j}$ and upon taking the supremum value of $E(I_1\ I_{10})$, we get $E(I_1\ I_{10}) = \frac{1}{2}h^2$. Then:

$$E(I_{(j_1),t}\ I_{(j_2,0),t}) = E(I_{(j_1),t}\ I_{(0,j_2),t}) = \frac{1}{2}\ h^2\ \delta_{j_1,j_2} \qquad \ldots\ (1.77)$$

Example (1.12), [7]:

In order to calculate the expectation of $I_1 I_{10} I_{101}$, first apply lemma (1.7) and from equation (1.67), hence:

$$I_1 I_{10} = 2\ I_{110} + I_{101} + I_{00}$$

Then:

$$E(I_1 I_{10} I_{101}) = E((2\ I_{110} + I_{101} + I_{00})\ I_{101})$$

$$= 2\ E(I_{110} I_{101}) + E(I_{101}^2) + E(I_{00} I_{101})$$

Letting $\alpha = (1, 1, 0)$ and $\beta = (1, 0, 1)$, then $L(\alpha^+) = 2$ and $\alpha^+ = \beta^+$. Noting that $K_0(\alpha) = K_1(\alpha) = 0$, $K_2(\alpha) = 1$, $K_0(\beta) = 0$, $K_1(\beta) = 1$, $K_2(\beta) = 0$, then:

$$\omega(\alpha, \beta) = L(\alpha^+) + \sum_{i=0}^{L(\alpha^+)} (K_i(\alpha) + K_i(\beta)) = 2 + 2 = 4$$

and so with $\tau = h$, $\rho = 0$ and $f \equiv g \equiv 1$, and thus $K_{f,g} = 1$, we have:

$$E(I_{110} I_{101}) \le K_{f,g} \frac{(\tau-\rho)^{\omega(\alpha,\beta)}}{\omega(\alpha,\beta)!} \prod_{i=0}^{L(\alpha^+)} \frac{(K_i(\alpha)+K_i(\beta))!}{K_i(\alpha)!K_i(\beta)!}$$

$$\le K_{1,1} \frac{h^4}{4!} \prod_{i=0}^{2} \frac{(K_i(\alpha)+K_i(\beta))!}{K_i(\alpha)!K_j(\beta)!} = (1)\ \frac{h^4}{4!}(1)(1)(1) = \frac{h^4}{4!}$$

Now, if $\alpha = \beta = (1, 0, 1)$ and $L(\alpha^+) = L(\beta^+) = 2$, then $K_0(\alpha) = K_0(\beta) = 0$, $K_1(\alpha) = K_1(\beta) = 1$, $K_2(\alpha) = K_2(\beta) = 0$, $\omega(\alpha, \beta) = 2 + 2 = 4$, and:

$$E(I_{101}^2) = E(I_{101}I_{101}) \le K_{1,1}\frac{h^4}{4!}\prod_{i=0}^{2}\frac{(K_i(\alpha)+K_i(\beta))!}{K_i(\alpha)!\ K_i(\beta)!}$$

$$= (1)\ \frac{h^4}{4!}(1)\ (2)\ (1) = \frac{h^4}{12}$$

Finally, $E(I_{00}\, I_{101}) = 0$, since $\alpha(0, 0)$ and $\beta = (1, 0, 1)$ gives $\alpha^+ \neq \beta^+$. Consequently:

$$E(I_1\, I_{10}\, I_{101}) \le\ 2\ \frac{h^4}{4!} + \frac{h^4}{12} = \frac{h^4}{6}$$

Example (1.13), [7]:

Let $\alpha = \beta = (1, 1)$ and thus $\alpha^+ = \beta^+ = (1, 1)$. Then by using lemma (2.2) to find $E(J_{11}^2)$, such that $J_{11} = I_{11} + \frac{1}{2}\, I_0$, then:

$$E(J_{11}^2) = E((I_{11} + \frac{1}{2}\, I_0)^2) = E(I_{11}^2) + 2\, E(I_0\, I_{11}) + \frac{1}{4}\, E(I_0^2)$$

To find $E(I_{11}^2)$ use lemma (1.7), such that $K_0(\alpha) = K_1(\alpha) = K_2(\alpha) = K_0(\beta) = K_1(\beta) = K_2(\beta) = 0$, and then $\omega(\alpha, \beta) = L(\alpha^+) + \sum_{i=0}^{L(\alpha^+)} (K_i(\alpha) + K_i(\beta)) = 2$, and so with $\tau = h$, $\rho = 0$ and $f \equiv g \equiv 1$,

$$E(I_{11}\, I_{11}) \le K_{f,g}\frac{(\tau-\rho)^{\omega(\alpha,\beta)}}{\omega(\alpha,\beta)!}\prod_{i=0}^{L(\alpha^+)}\frac{(K_i(\alpha)+K_i(\beta))!}{K_i(\alpha)!\ K_i(\beta)!}$$

$$\le K_{1,1}\frac{h^2}{2!}\prod_{i=0}^{2}\frac{(K_i(\alpha)+K_i(\beta))!}{K_i(\alpha)!K_i(\beta)!} = \frac{1}{2}h^2$$

and by [22] then:

$$E(I_{(j_1,j_2),t}\, I_{(j_3,j_4),t}) = \frac{1}{2}\, h^2\, \delta_{j_1,j_3}\, \delta_{j_2,j_4} \qquad \dots (1.78)$$

and $E(I_0\, I_{11}) = 0$, where $\alpha = (0) = \beta = (1, 1)$, then $\alpha^+ \neq \beta^+$. Finally, $E(I_0^2) = h^2$, such that $I_0 = \int_0^h dt = h$ and $E(h^2) = h^2$. Consequently:

$$E(J_{11}^2) \le \frac{1}{2}\, h^2 + 0 + \frac{1}{4}\, h^2 = \frac{3}{4}\, h^2.$$

Remark (1.9):

For all examples considered in this section, we can apply remark (1.8), for example, when back to example (1.11), such that $E(I_1 I_{10})$ and since $\alpha_1 = (1)$, $\alpha_2 = (1, 0)$ and $L(\alpha_1) = 1$, $L(\alpha_2) = 2$, $n(\alpha_1) = 0$, $n(\alpha_2) = 1$, then:

$$\lambda = \sum_{i=1}^{2} (L(\alpha_i) + n(\alpha_i)) = 3 - 1 = 2$$

and since 2 is even, then:

$$\rho = \frac{1}{2}\lambda + \sum_{i=1}^{2} n(\alpha_i) = \frac{1}{2}(2) + 1 = 2$$

and thus $|E(I_1 I_{10})| \leq K h^2$, where $K = \frac{1}{2}$ according to the solution of this example in lemma (1.7).

Since the result of lemma (1.7) is applied only to the Itô integral and because in this work, we will study the order of convergence of the numerical methods will be analyzed when solving SDE's in Stratonovich formulation, therefore in this section we will includes three additional important theorems which give the results concerning the expectation of products of multiple Stratonovich integrals. These theorems are applied in the derivation of stochastic Runge-Kutta methods in chapter two. The known result from [22], which is $E(I_{\alpha_1} I_{\alpha_2} \dots I_{\alpha_k}) = 0$ if the total number of non-zero indices in α_1, α_2, …, α_k is odd is applied in the following theorem which extends the result to Stratonovich integrals:

Theorem (1.8), [7]:

If the total number of non-zero indices in α_1, α_2, …, α_k is odd, then $E\left(\prod_{L=1}^{k} J_{\alpha_L}\right) = 0.$

Theorem (1.9), [7]:

Let J_α be a Stratonovich integral with $L(\alpha) = k$, $n(\alpha) = k - t$, i.e., $k - t$ zeros. Then $E(J_\alpha^2) = O(h^{2k-t})$.

The result in theorem (1.9) may be extended to the product of an arbitrary number of Stratonovich integrals, as in the next theorem:

Theorem (1.10), [7]:

$E(J_{\alpha_1} J_{\alpha_2} \dots J_{\alpha_k}) = O(h^{\rho})$, for λ is even, where

$$\lambda = \sum_{i=1}^{k} (L(\alpha_i) - n(\alpha_i)) \text{ and } \rho = \frac{1}{2}\lambda + \sum_{i=1}^{k} n(\alpha_i).$$

Now, when back to the example (1.13), then:

$$E(J_{11}^2) = E\left(\left(\frac{1}{2}J_1^2\right)^2\right) \text{ (by lemma (1.5))}$$

$$= \frac{1}{4}E(J_1^4) = \frac{1}{4}E(J_1 J_1 J_1 J_1)$$

By theorem (1.10), when $\alpha_i = (1)$, $L(\alpha_i) = 1$, $n(\alpha_i) = 0$, for all i = 1, 2, 3, 4, then $\lambda = \sum_{i=1}^{4} (L(\alpha_i) - n(\alpha_i)) = 4$ and since 4 is an even number, then $\rho = \frac{1}{2}\lambda + \sum_{i=1}^{2} n(\alpha_i) = 2$ and thus $E(J_1^4) = O(h^2)$.

In equations (1.69), when $J_1 = I_1$ then by remark (1.8):

$$|E(I_1 I_1 I_1 I_1)| \le K h^2$$

and K = 3 according to [22], such that:

$$E(I_{(j_1)}I_{(j_2)}I_{(j_3)}I_{(j_4)}) = \begin{cases} 3h^2, & \text{if } j_1 = j_2 = j_3 = j_4 \\ h^2, & \text{if } \{j_1, j_2, j_3, j_4\} \text{ consists of 2 distinct pairs of identical numbers} \\ 0, & \text{otherwise} \end{cases} \quad \dots (1.79)$$

Then $E(J_{11}^2) = \frac{1}{4} E(J_1^4) = \frac{1}{4} E(I_1^4) = \frac{3}{4} h^2$.

Example (1.14), [7]:

Using the above theorems, it can be shown that $E(J_1 J_2 J_{21}) \le \frac{1}{2} h^2$.

Consequently, as $J_1 J_2 = J_{12} + J_{21}$, then by lemma (1.4):

$$E(J_1 J_2 J_{21}) = E((J_{12} + J_{21})J_{21}) = E(J_{12} J_{21} + J_{21}^2)$$

Now, by theorem (1.10), $E(J_{12} J_{21}) = O(h^2)$,

but when $J_{12} = I_{12}$, $J_{21} = I_{21}$ (by equation (1.69)), such that $I_{\{j_1=j_2\neq 0\}} = 0$), then $E(I_{12}\, I_{21}) = 0$ (by equation (1.78)) such that $\delta_{j_1,j_3} = \delta_{j_2,j_4} = 0$,

or, according to equation (1.75) and $E(J_{21}^2) = O(h^2)$ (by theorem (1.10)),

or when $J_{21} = I_{21}$ (by equation (1.69), since $I_{\{j_1=j_2\neq 0\}} = 0$)

Therefore, $E(J_{21}^2) = E(I_{21}^2) = \frac{1}{2} h^2$ (by equation (1.78) such that $\delta_{j_1,j_3} = \delta_{j_2,j_4} = 1$, or according to equation (1.76)).

Hence:

$$E(J_1\, J_2\, J_{21}) = E(J_{12}\, J_{21} + J_{21}^2) = E(J_{21}^2) = \frac{1}{2}h^2.$$

These examples demonstrate the systematic way that the lemmas can be applied to calculate the expectation of products of stochastic integrals.

Finally, it is noted that if two random variables X_1 and X_2 are independent, then $E(X_1\, X_2) = E(X_1)\, E(X_2)$. This fact can be used to calculate, for example $E(J_1^2\, J_2^2)$ as J_1 and J_2 are independent.

Example (1.15), [7], [1]:

First approach:

$$E(J_1^2\, J_2^2) = E(J_1^2)\, E(J_2^2)$$

When $\alpha = (1) = \alpha^+$, $\beta = (2) = \beta^+$ and $L(\alpha^+) = L(\beta^+) = 1$, then $K_0(\alpha) = K_0(\beta) = K_1(\alpha) = K_1(\beta) = 0$, $\omega(\alpha, \beta) = L(\alpha^+) + \sum_{i=0}^{L(\alpha^+)} (K_i(\alpha) + K_i(\beta)) = 1$ with $\tau = h$, $\rho = 0$ and $f = g = 1$, $I_1 = J_1$ (by equation (1.69), since:

$$E(I_1\, I_1) \leq K_{f,g} = h$$

In this example when by [22], $K_{1,1} = \delta_{i,j}$ in equation (1.77), then:

$$E(I_{(j_1),t}\, I_{(j_2),t}) = h\, \delta_{j_1,j_2} \qquad \ldots (1.80)$$

and by the same way followed previously $E(J_2^2) = h$ and by theorem (1.10), $E(J_1^2) = E(J_2^2) = O(h)$.

Second approach:

Also, using theorem (1.2) with $J_1 = I_1$ (by equation (1.69)) and with $f(t) = 1$, $a = 0$, $b = h$, hence:

$$E(J_i^2[f(.)]_{,a,b}) = E(I_i^2[f(.)]_{,a,b}) = E\left[\left|\int_a^b f(t)\, dW_t\right|^2\right] = E\left[\int_a^b f^2(t)\, dt\right],$$

hence

$$E(J_i^2) = E\left[\left|\int_0^h dW_t\right|^2\right] = E\left[\int_0^h dt\right] = E(h) = h, \qquad i = 1, 2.$$

Therefore, $E(J_1^2 J_2^2) = E(J_1^2)\, E(J_2^2) = h^2$.

Third approach:

This result may also be obtained as follows:

By equation (1.69), $J_1^2 = I_1^2$ and $I_1^2 = 2 I_{11} + I_0$. Also, $J_2^2 = I_2^2 = 2 I_{22} + I_0$ Hence:

$$E(J_1^2 J_2^2) = E((2 I_{11} + I_0)(2 I_{11} + I_0))$$

$$= 4 E(I_{11} I_{22}) + 2 E(I_0 I_{11}) + 2 E(I_0 I_{22}) + E(I_0^2)$$

such that from equation (1.78) $E(I_{11} I_{22}) = 0$ and since $\delta_{j_1,j_3} = \delta_{j_2,j_4} = 0$. According to equation (1.75) or using lemma (1.7) when $\alpha = (0)$, $\beta = (1, 1)$, then $\alpha^+ \neq \beta^+$. Also, $E(I_0 I_{22}) = E(I_0 I_{11}) = 0$.

Finally, $E(I_0^2) = E\left(\left[\int_0^h dt\right]^2\right) = E(h^2) = h^2$. Then:

$$E(J_1^2 J_2^2) = 4 E(I_{11} I_{22}) + 2 E(I_0 I_{11}) + 2 E(I_0 I_{22}) + E(I_0^2) = h^2$$

Fourth approach:

Also, we can use equation (1.79) to find $E(J_1^2 J_2^2)$, so

$E(J_1^2 J_2^2) = E(J_1 J_1 J_2 J_2) = h^2$, such that $J_1 = I_1$, $J_2 = I_2$ (by equation (1.69) and $\{\alpha_i\}$, $i = 1, 2, 3, 4$ of J_{α_i} consists of two distinct pairs of identical numbers).

Extending this to N random variables yields to the result:

$$E(J_1^2 J_2^2 \dots J_N^2) = h^N \qquad \dots (1.81)$$

The independence of random variables also results in the following examples:

1. $E(J_1^4 J_2^2) = E(J_1^4) E(J_2^2)$ (by equations (1.79) and (1.81))

$= (3h^2) (h) = 3h^3$

2. $E(J_1^4 J_2^4) = E(J_1^4) E(J_2^4)$ (by equation (1.79))

$= (3h^2) (3h^2) = 9h^4$

3. $E(J_1^6 J_2^2) = E(J_1^6) E(J_2^2)$

$= (15h^3) (h) = 15h^4$

Such that $E(J_2^2) = h$, hence using theorem (1.10) $E(J_1^6) = O(h^3)$ and when $J_1 = I_1$ in equation (1.69), then by remark (1.8)

$$|E(I_1 I_1 I_1 I_1 I_1 I_1)| \leq K h^3$$

and K = 15 according to [22], such that:

$$E(I_{j1} I_{j2} I_{j3} I_{j4} I_{j5} I_{j6}) = \begin{cases} 15h^3, & \text{if } j_1 = j_2 = \ldots = j_6 \\ 3h^2, & \text{if 1 pair and 1 quadruple of identical } j_i \\ h^3, & \text{if 3 different pairs of identical } j_i \\ 0, & \text{otherwise} \end{cases}$$

… (1.82)

Chapter Two

2

Runge-Kutta Methods

For Solving

Ordinary Stochastic Differential Equation

Chapter Two

Runge-Kutta Methods for Solving Ordinary Stochastic Differential Equations

Introduction:

In this chapter, we give some conceptions about strong and weak convergence and give some models of stochastic Runge-Kutta methods.

Pamela in 1999 [7] derived some models with multiples stages stochastic Runge-Kutta methods (SRKM's for short) with strong order of convergence and proved that those models are stable using the mean square stability and asymptotically stability sachems.

This chapter consists of four sections. In section 2.1 some conceptions about strong and weak convergence are given. In section 2.2 the previous work of stochastic Runge-Kutta schemes are illustrated in details. In section 2.3 the derivation and analysis of stochastic Runge-kutta methods are given. Finally in section 2.4 the stability of the explicit SRKM's using the mean square concept of stability.

It is remarkable that in this chapter, we shall discuss the derivation of explicit Stratonovich stochastic Runge-Kutta methods for solving autonomous SDE's with strong order 1 according to the style of Burrage in (1999) [7], and its stability also will be considered.

2.1 Fundamental Concepts, [7], [22],[39].

In order to determine the quality of the scheme, i.e., the accuracy of the discrete time approximation, we have to specify a criterion for the type of the convergence, since there are mainly two different objectives connected with the approximation of solutions of SDE's of the form:

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t \; ; \;\; X_{t_0} = X_0 \qquad \dots (2.1)$$

Definition (2.1) (Strong Convergence), [22], [37]:

A discrete time approximation Y^h is said to be converges strongly with order $p > 0$ at time T if there exists a positive constant C, which does not depend on the step size h, and $\delta_0 > 0$, such that:

$$E(\| X_T - Y_T^h \|) \le Ch^p \qquad \dots (2.2)$$

where $h = \frac{T - t_0}{N} \in (0, \delta_0)$, N is the number of subintervals of the interval $I = [t_0, T]$, X_T is the actual solution at T and Y_T^h is the numerical solution at T.

Remark (2.1), [7], [22], [37]:

If we want to construct the Itô-Taylor scheme of strong order $p = 0.5, 1.0, 1.5, \dots$; i.e., the stochastic Taylor expansion is constructed using definition (2.1).

We need to take the association with the hierarchical set defined by:

$$\Lambda_p = \{\alpha \in M : L(\alpha) + n(\alpha) \le 2p \text{ or } L(\alpha) = n(\alpha) = p + \frac{1}{2}\} \qquad \dots (2.3)$$

where M, $L(\alpha)$, $n(\alpha)$ are defined in section (1.4).

In the general multidimensional case, with a d-dimensional SDE and m-dimensional Wiener process, the Itô-Taylor scheme of strong order p defined by the vector equation:

$$Y_{n+1} = Y_n + \sum_{\alpha \in \Lambda_p \setminus \{v\}} f_\alpha(t_n, Y_n)\, I_{\alpha, t_n, t_{n+1}}; Y_0 = x_0 \qquad \dots (2.4)$$

recursively with $F(t, x) = x$, for all $(t, x) \in I \times \mathbf{R}^d$ and for $n = 0, 1, \dots, N-1$; $\{v\}$ is defined in chapter one (see equation (1.56)), provided that all derivatives of f and g appearing in equation (2.1) exist, [22].

It is remarkable that, the above discussion is given for the Itô-Taylor formula, and by the same way, the Stratonovich-Taylor scheme of strong order $p = 0.5, 1.0, 1.5, \dots$; may be discussed only by replacing $f(t, X_t)$ in Itô SDE's with $\underline{f}(t, X_t)$ in Stratonovich SDE's and $I_{\alpha, t_n, t_{n+1}}$ in in equation (2.4) with $J_{\alpha, t_n, t_{n+1}}$.

However, in some cases it is not necessary to find an accurate path wise approximation of an Itô process. Instead, only some of the moments which may be of interest, or more generally E(f(X)) for some function f. This is, a much weaker condition, which give the reason for the next definition:

Definition (2.2) (Weak Convergence), [22], [37]:

A discrete time approximation Y^h with step size h is said to be converges weakly with order p > 0 to X_T at time T as $h \longrightarrow 0$, if for each $f \in C^{2(p+1)}(\mathbf{R}^d, \mathbf{R})$ there exists positive constant C, which does not depend on h and a finite number $\delta_0 > 0$, such that:

$$|E(f(X_T)) - E(f(Y_T^h))| \leq Ch^p, \forall h \in (0, \delta_0).$$

Remarks (2.2), [22], [37]:

1. In order to construct a weak Taylor scheme of order p = 1, 2, 3,…; for the general d-dimensional SDE, one have to consider the hierarchical set:

   $$\Gamma_p = \{\alpha \in M : L(\alpha) \leq p\} \qquad \ldots (2.5)$$

   then an m-dimensional Wiener process, the Itô-Taylor scheme of weak order p is recursively defined by the vector equation:

   $$Y_{n+1} = Y_n + \sum_{\alpha \in \Gamma_p \setminus \{v\}} f_\alpha(t_n, Y_n)\, I_{\alpha, t_n, t_{n+1}},\ Y_{t_0} = x_0 \qquad \ldots (2.6)$$

   with F(t, x) = x, for all (t, x) ∈ I × $\mathbf{R}^d$ and for n = 0, 1, …, N − 1; provided that all derivatives of f and g appearing in equation (2.1) exists.

   Similarly, as in the case of Itô-Taylor expansion, we may use the Stratonovich-Taylor expansion of weak order p = 1, 2, 3, …; by replacing f(t, X_t) in the Itô SDE (2.1) with $\underline{f}(t, X_t)$ in Stratonovich SDE's and $I_{\alpha, t_n, t_{n+1}}$ in equation (2.6) with $J_{\alpha, t_n, t_{n+1}}$.

2. On the one hand, the partial derivatives appearing in the Taylor scheme must be calculated and on the other hand to simulate the correlated random variables.

   $$I_{\alpha, t_n, t_{n+1}} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{S_k} \ldots \int_{t_n}^{S_2} dW_{S_1}^{\alpha_1} dW_{S_2}^{\alpha_2} \ldots dW_{S_k}^{\alpha_k}$$

where $dW_S^{\alpha} = dt$ when $\alpha = 0$, for all $\alpha \in \Lambda_p$ for strong scheme and $\alpha \in \Gamma_p$ for weak scheme.

For the weak approximation of order p, the multiple Itô integrals I_α may be replaced by other random variables $\hat{I}_\alpha$, if there exists a constant C > 0, such that the moment condition:

$$\left| E\left(\prod_{k=1}^{L} I_{\alpha_k, t_n, t_{n+1}} - \prod_{k=1}^{L} \hat{I}_{\alpha_k, t_n, t_{n+1}} \middle| A_{t_n} \right) \right| < C\,(t_{n+1} - t_n)^{p+1} \qquad \ldots (2.7)$$

holds for all choices of multi-indices $\alpha_k \in \Gamma_p \backslash \{v\}$, for k = 1, 2, …, L and with L = 1, 2, …, 2 (p + 1).

In the next section, some models which are formulated for autonomous SRK methods according to strong or weak convergence are given and thereafter, models of autonomous and non-autonomous SRK methods have been discussed.

2.2 Previous Work on Stochastic Runge-Kutta Methods.

In this section, the SRKM are classified into two types:

1. Autonomous SRKM's which may be also classified into two types:
   (a) Autonomous SRK methods of strong convergence and following some well-known models for this type:

- When the Itô SDE is given by:

  $$dY_t = f(Y_t)\,dt + g(Y_t)\,dW_t,\ Y_{t_0} = Y_0 \qquad \ldots (2.8)$$

  then the strong order 1.0 SRKM is:

  $$Y_{n+1} = Y_n + f(Y_n)\,h + g(Y_n)\,\Delta W_n + \frac{1}{2}[g(Y_n + g(Y_n)\sqrt{h}) - g(Y_n)]$$
  $$(\Delta W_n^2 - h) \qquad \ldots (2.9)$$

  where $h = t_{n+1} - t_n$, $\Delta W_n = W_{t_{n+1}} - W_{t_n}$. This model is proposed by Rümelin in (1982), [38].

- When the Stratonovich SDE is given by:

  $$dY_t = \underline{f}\,(Y_t)\,dt + g(Y_t) \circ dW_t \qquad \ldots (2.10)$$

  where:

$$\underline{f}\,(Y_t) = f(Y_t) - \frac{1}{2}\, g(Y_t)\, g'(Y_t),\ y_{t_0} = y_0$$

The strong order 1.0 SRKM are:

$$\left.\begin{aligned} Y_i &= y_n + h\sum_{j=1}^{s} a_{ij}\,\underline{f}\,(Y_j) + +J_1\sum_{j=1}^{s} b_{ij}\, g(Y_j)\ ;i = 1,2,...,s \\ y_{n+1} &= y_n + h\sum_{j=1}^{s} \alpha_j\,\underline{f}\,(Y_j) + J_1\sum_{j=1}^{s} \gamma_j\, g(Y_j) \end{aligned}\right\} \quad \ldots (2.11)$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are s×s-matrices of real elements with $\alpha^T = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ and $\gamma^T = (\gamma_1, \gamma_2, \ldots, \gamma_s)$ are row the vectors in $\mathbf{R}^s$, $h = t_{n+1} - t_n$, $J_1 = \Delta W_n = W_{t_{n+1}} - W_{t_n}$ and s is the stages of the method.

If both A and B are strictly triangular matrices, then equation (2.11) is said to be explicit, otherwise it is implicit. This model is proposed by Burrage and Burrage in (1996-1999), [4], [5], [7].

(b) Autonomous SRK methods of weak convergence and next some of the most well-known models of this type:

- When the SDE is as given in equation (2.8), then the weak order 2 SRKM is:

$$Y_{t_0} = Y_0$$

$$Y_{n+1} = Y_n + \frac{1}{2}\,[f(u) + f(Y_n)]\,h + \frac{1}{4}\,[g(u_+) + g(u_-) + 2\,g(Y_n)]\,\Delta\hat{W}_n +$$

$$\frac{1}{4}[g(u_+) - g(u_-)]\frac{\Delta\hat{W}_n^2 - h}{\sqrt{h}} \quad \ldots (2.12)$$

where:

$h = t_{n+1} - t_n$.

$\Delta\hat{W}_n = \hat{I}_1 = \hat{W}_{t_{n+1}} - \hat{W}_{t_n}$.

$u = Y_n + f(Y_n)\,h + g(Y_n)\,\Delta\hat{W}_n$.

$u_+ = Y_n + f(Y_n)\,h + g(Y_n)\,\sqrt{h}$.

$u_- = Y_n + g(Y_n)\,h - g(Y_n)\,\sqrt{h}$.

Such that $\Delta\hat{W}_n$ is a random variable and must be $A_{T_{n+1}}$-measurable and satisfy the moment condition given in equation (2.7).

This model is proposed by Platen and Kloeden in (1992), [22] and Platen in (1987), [33].

- When the Stratonovich SDE is given by equation (2.10), the weak order 2.0 SRKM is given by:

$$Y_{n+1} = Y_n + \sum_{j=1}^{s} \alpha_j \underline{f}(H_i^{(0)})\, h_n + \sum_{j=1}^{s} \gamma_i^{(1)}\, g(H_i^{(1)})\, J_1;\ Y_{t_0} = Y_0 \quad \ldots (2.13)$$

for n = 0, 1, …, N − 1; with supporting values:

$$H_i^{(0)} = Y_n + \sum_{j=1}^{i-1} A_{ij}^{(0)}\, \underline{f}(H_j^{(0)})\, h_n + \sum_{j=1}^{i-1} B_{ij}^{(1)(0)}\, g(H_j^{(1)})\, J_1$$

$$H_i^{(1)} = Y_n + \sum_{j=1}^{i-1} A_{ij}^{(1)}\, \underline{f}(H_j^{(0)})\, h_n + \sum_{j=1}^{i-1} B_{ij}^{(1)(1)}\, g(H_j^{(1)})\, J_1$$

where $h_n = t_{n+1} - t_n$, $J_1 = \Delta W_n = W_{t_{n+1}} - W_{t_n}$ and $A_{ij}^{(0)}$, $A_{ij}^{(1)}$, $B_{ij}^{(1)(0)}$, $B_{ij}^{(1)(1)}$ are s×s matrices of real elements with $\alpha^T = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ and $\gamma^T = (\gamma_1, \gamma_2, \ldots, \gamma_s)$ are row vectors in $\mathbf{R}^s$. This model was proposed by Rößler in (2003), [37].

2. Non-autonomous SRKM's, which may be also classified into two types:

(a) Non-autonomous SRK methods which is worked according to the strong or weak convergence and following some well-known models:

- When the Itô SDE is given by:

$$dY_t = f(t, Y_t)\, dt + g(t, Y_t)\, dW_t \quad \ldots (2.14)$$

then the strong order 1.0 SRKM is:

$$\left.\begin{aligned} Y_{t_0} &= Y_0 \\ Y_{n+1} &= Y_n + f(t_{n+1}, Y_{n+1})\, h + g(t_n, Y_n)\, \Delta W_n + \frac{1}{2\sqrt{h}} [g(t_n, Y_n) \\ &\quad + f(t_n, Y_n)\, h + g(t_n, Y_n)\sqrt{h}) - g(t_n, Y_n)][(\Delta W_n)^2 - h] \end{aligned}\right\} \quad \ldots (2.15)$$

Where $h = t_{n+1} - t_n$, $\Delta W_n = W_{t_{n+1}} - W_{t_n}$. This model was proposed by

Platen and Kloeden in (1992), [22].

- When the SDE is as in equation.(2.14), then the weak order 2.0 SRKM is:

$$Y_{n+1} = Y_n + \frac{1}{2} g(t_n, Y_n)\, \Delta\hat{W}_n + \frac{1}{2}[f(t_n, Y_n) - g(t_n, Y_n) g'(t_n, Y_n)]\, h +$$

$$\frac{1}{2}\, g(t_n, Y_n)\, g'(t_n, Y_n)\, (\Delta\hat{W}_n)^2 + \frac{1}{2} f(t_{n+1}, Y_n + f(t_n, Y_n)\, h +$$

$$g(t_n, Y_n)\, \Delta\hat{W}_n)\, h + \frac{1}{4} g(t_{n+1}, Y_n + f(t_n, Y_n)\, h + \frac{1}{\sqrt{3}} g(t_n, Y_n)$$

$$\Delta\hat{W}_n)\, \Delta\hat{W}_n + \frac{1}{4} g(t_{n+1}, Y_n + f(t_n, Y_n)\, h - \frac{1}{\sqrt{3}} g(t_n, Y_n)$$

$$\Delta\hat{W}_n)\, \Delta\hat{W}_n, \; Y_{t_0} = Y_0 \qquad \ldots (2.16)$$

Where $Y_{t_0} = Y_0$, $h = t_{n+1} - t_n$ and $\Delta\hat{W}_n = \hat{I}_1 = \hat{W}_{t_{n+1}} - \hat{W}_{t_n}$.

This model was proposed by Milstein in (1992), [22].

(b) Non-autonomous SRKM which is derived without introducing the strong or weak convergence (i.e., it is derived in a similar manner as in the ordinary derivation of deterministic RKM) in order to throw off expectations complications, i.e., we shall deal with deterministic terms separately from the stochastic terms and this model is called stochastic Runge-Kutta Maruyama methods and it is proposed by Buckwar , Rößler and Winkler, [35] [36], which is given by:

$$Y_{n+1} = Y_n + h \sum_{i=1}^{s} \alpha_i f(t_n + c_i h, K_i) + g(t_n, Y_n) \Delta W_n, \; Y_{t_0} = Y_0 \qquad \ldots (2.17)$$

with stage values for $i = 1, 2, \ldots, s$ and $h = t_{n+1} - t_n$,

$$\Delta W_n = W_{n+1} - W_n, \; K_i = Y_n + h \sum_{j=1}^{s} a_{ij}\, f(t_n + c_i\, h, K_j)$$

The previous classifications are worked due to what we find by a large class of models to SRKM's in orders specific above, and this does not mean that there are no other models of SRKM's. There are also other models for

SRK of higher order. In addition, it is possible to make some modification on the models of autonomous SRKM to be transformed into non-autonomous SRKM, and vice versa.

2.3 Analysis and Derivation of Stochastic Runge-Kutta Methods.

In the case of RKM for deterministic problems, the order of accuracy is found by comparing the computed solution with the exact solution over one step assuming exact initial values. This is accomplished by expanding both the actual and the numerical solutions in Taylor series expansion, and this approach may be carried over SDE's when a stochastic Taylor series expansion (using either the Itô or Stratonovich calculus) is applied [5], [7].

As mentioned earlier in chapter one, because of the simplified nature of the Stratonovich calculus, only the Stratonovich form of the stochastic Taylor series will be used here in this section.

2.3.1 Taylor Series Expansion for the Actual Solution [7].

Consider the autonomous, one Weiner process and Stratonovich SDE given in equation (2.10), which may be rewritten as:

$$d y_t = f(y_t)\, dt + g(y_t) \circ dW_t,\ y_{t_0} = y_0 \qquad \dots (2.18)$$

where f refers to $\underline{f}$ for simplicity, and as an integral equation form:

$$y_t = y_{t_0} + \int_{t_0}^{t} f(y_s)\, ds + \int_{t_0}^{t} g(y_s) \circ dW_s \qquad \dots (2.19)$$

Itô's formula states that a given function a of the solution y may be written as:

$$a(y_t) = a(y_{t_0}) + \int_{t_0}^{t} L^0 a(y_s)\, ds + \int_{t_0}^{t} L^1 a(y_s) \circ dW_s \qquad \dots (2.20)$$

where in the Itô form, the operators L^0 and L^1 are given by:

$$L^0 a(y) = \frac{da}{dy} f + \frac{1}{2} \frac{d^2 a}{dy^2} g^2, \quad L^1 a(y) = \frac{da}{dy} g$$

While in Stratonovich form:

$$L^0a(y) = \frac{da}{dy}f, \qquad L^1a(y) = \frac{da}{dy}g \qquad \dots (2.21)$$

Applying theorem (1.7), and using equations (2.19) and (2.21) with $a(y) \equiv y$ and one Weiner process, and letting $y_{t_0} = y_0$, then the actual solution given by, [7]:

$$\begin{aligned} y_t = {} & y_0 + f(y_0)J_0 + g(y_0)J_1 + f'(y_0)(f(y_0))J_{00} + f'(y_0)(g(y_0))J_{10} + \\ & g'(y_0)f(y_0))J_{01} + g'(y_0)(g(y_0))J_{11} + f''(y_0)(f(y_0)f(y_0))J_{000} + \\ & f'(y_0)(f'(y_0)(f(y_0)))J_{000} + f''(y_0)(f(y_0)g(y_0))J_{100} + f'(y_0) \\ & (f'(y_0)\ (g(y_0)))J_{100} + f''(y_0)\ (g(y_0)f(y_0))J_{010} + f'(y_0)(g'(y_0) \\ & (f(y_0)))J_{010} + f''(y_0)(g(y_0)\ g(y_0))J_{110} + f'(y_0)\ (g'(y_0)(g(y_0)))\ J_{110} + \\ & g''(y_0)(f(y_0)f(y_0))J_{001} + g'(y_0)\ (f'(y_0)(f(y_0)))J_{001} + g''(y_0)(f(y_0)g(y_0)) \\ & J_{101} + g'(y_0)(f'(y_0)\ (g(y_0)))J_{101} + g''(y_0)\ (g(y_0)f(y_0))J_{011} + g'(y_0) \\ & (g'(y_0)(f(y_0)))J_{011} + g''(y_0)(g(y_0)\ g(y_0)J_{111} + g'(y_0)(g'(y_0)\ (g(y_0)))J_{111} \\ & + R \qquad \dots (2.22) \end{aligned}$$

where R refers to the remainder term and $J_{j_1 j_2 \cdots j_k}$ represents the Stratonovich multiple integral, which are with respect to ds if $j_i = 0$ or $\circ dW(s)$ if $j_i = 1$.

It should be noticed that there is a multiplicity factor associated with some of the higher derivative terms. For example:

$$L^0f = f'\,f$$

$$L^0L^0f = L^0(f'\,f) = (f'\,f)'\,f = f''\,f^{\,2} + (f')^2\,f$$

$$\begin{aligned} L^0L^0L^0f &= L^0(f''\,f^{\,2} + (f')^2\,f) = (f''\,f^{\,2})'\,f + ((f')^2\,f)'\,f \\ &= f'''\,f^{\,3} + 3\,f''\,f'\,f^{\,2} + f'\,f''\,f^{\,2} + (f')^3\,f \qquad \dots (2.23) \end{aligned}$$

Thus, in the expression of $L^0L^0L^0f$, there is a factor 3 associated with the term $f''f'f^{\,2}$.

Equation (2.22) is the generalization of Taylor series expansion for deterministic equations with $g \equiv 0$. By comparing this expansion with the application of the numerical method being considered, it is possible to choose the coefficients of the numerical method to obtain a particular order.

2.3.2 Rooted Tree Theory [7]:

For the analysis of order conditions for RKM for ordinary differential equations, Butcher in (1987) [9] demonstrated the correspondence between elementary differentials and rooted tree theory in the Taylor series expansion of the actual solution. This technique (which is first introduced by Butcher in (1963) [8]) simplifies markedly the derivation of order conditions for higher order RKM.

The main results on rooted tree theory (for deterministic ordinary differential equations) are presented in [9]. Then, this theory was extended to the stochastic setting, so that although the order conditions for lower-order RKM may be obtained by comparing directly the RK scheme with stochastic Stratonovich Taylor expansion, this theory may be used to develop the order conditions for more general RKM's.

Order conditions for deriving RKM may be obtained by expanding both the actual and numerical solutions of the differential equation in a Taylor series. By formally differentiating $y'(x) = f(y(x))$, the higher derivatives of y may be represented as follows:

$$y''(x) = f'(y(x))\, y'(x) = f'f,\ y'''(x) = f''\, f^{2} + (f')^{2} f,\ \ldots$$

The ordinary derivatives f' and f'' are linear and bilinear, respectively. In vector notation (and for m-dimensional system), the i^{th} component $f'f$ and $f''f^{2}$ can be written as:

$$(f'(f))^{i} = f^{i}_{j}\, f^{j} = \sum_{j=1}^{m} \frac{\partial f^{i}}{\partial x_{j}} f^{j},$$

$$(f''(f\,f))^{i} = f^{i}_{jk} f^{i} f^{k} = \sum_{j,k=1}^{m} \frac{\partial^{2} f^{i}}{\partial x_{j} \partial x_{k}} f^{j} f^{k}$$

The complexity of these expressions builds up very quickly, so it simplifies matters to use the pattern of rooted trees for constructing these expressions. In the deterministic setting, τ is used to denote the tree node ●, and other trees can be build up recursively by defining a new tree t (which is formed by joining trees $t_1, t_2, \ldots, t_k$ to a new root τ) as:

$t = [t_1, t_2, \ldots, t_k]$

each tree corresponds to an elementary differential F(t)(y(x)), defined by:

$$F(t)(y(x)) = f^{(k)}(F(t_1)(y(x))\, F(t_2)(y(x)) \ldots F(t_k)(y(t)))$$

and consequently forms part of Taylor series. Full details of this are in Butcher (1987), [9] and (1994), [10], but here just a brief summary will be provided so as to form a basis for the extension of the concept to the stochastic area.

Remarks (2.3):

1. Consider the s-stages RKM (for solving the deterministic problem) given by:

$$\left.\begin{aligned} Y_i &= y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j);\ i = 1,2,...,s \\ y_{n+1} &= y_n + h\sum_{j=1}^{s} b_j f(Y_j) \end{aligned}\right\} \quad \ldots (2.24)$$

since $b_j = \sum_{i=1}^{s} a_{ij};\ j = 1,2,...,s$, which may be represented in tabulated form, as:

C	A
	b^T

where $C = A\,e$, $e = (1, 1, \ldots, 1)^T$.

2. Each rooted tree of the above formulation has a corresponding elementary weight $\Phi_D(t) = b^T\psi(t)$, where $\psi(t)$ is defined recursively by:

$$\psi(\tau) = e,\ \psi([t_1, t_2, \ldots, t_k]) = (A\psi(t_1)) * (A\psi(t_2)) * \ldots * (A\psi(t_k))$$

where $*$ denotes the component-wise multiplication (see [10]).

3. A number of functions on trees may be defined recursively. The order of a tree $\rho(t)$ or number of vertices of t satisfies the recursion, [9]:

$$\rho(\tau) = 1,\ \rho([t_1, t_2, \ldots, t_k]) = 1 + \sum_{j=1}^{k} \rho(t_j)$$

4. The function $\gamma(t)$ is used to represent the density of a tree (which is a measure of its non-bushiness) and is defined recursively by:

$$\gamma(\tau) = 1,\ \gamma([t_1, t_2, \ldots, t_k]) = \rho([t_1, t_2, \ldots, t_k]) \prod_{j=1}^{k} \gamma(t_j)$$

5. The symmetry of a tree, $\beta(\tau)$, can also be defined recursively, as:

$$\beta(\tau) = 1,\ \beta([\, t_1^{n_1}, t_2^{n_2}, \ldots, t_k^{n_k}\,]) = n_1!\ n_2! \ldots n_k! \prod_{j=1}^{k} \beta(t_j)^{n_j}$$

6. The function $\alpha(t)$ is the number of ways to label the vertices of t (with labels 1, 2, …, $\rho(t)$) so that the labels increase outwardly along the arcs, symmetry of the tree must be taken into account, and hence $\alpha(t)$ takes the form, [9]:

$$\alpha(t) = \frac{\rho(t)!}{\gamma(t)\ \beta(t)};\ \gamma(t)\ \beta(t) \neq 0, \forall t$$

7. By using some of the above functions on trees, the formal deterministic Taylor series for the actual solution is [10]:

$$y(x_0 + h) = y_0 + \sum_{t \in T} \alpha(t)\ F(t)(y_0)\ \frac{h^{\rho(t)}}{\rho(t)!}$$

where T is the set of all rooted trees.

8. For the numerical solution given by equation (2.24), Butcher in (1987) has shown that, [9]:

$$\hat{y}\,(x_0 + h) = y_0 + \sum_{t \in T} \alpha(t)\ \gamma(t)\ \Phi_D(t)\ F(t)(y_0)\ \frac{h^{\rho(t)}}{\rho(t)!}$$

9. By comparing the expansions of the actual and the numerical solutions term by term up to and including trees with p vertices, an RKM will have an order p if and only if $\Phi_D(t) = \dfrac{1}{\gamma(t)}$.

10. Noting also that the local truncation error over one step give an exact initial value which may be written as:

$$L(x_0 + h) = y(x_0 + h) - \hat{y}\,(x_0 + h)$$

$$= \sum_{t \in T} \alpha(t) \left(\frac{1}{\rho(t)!} - \frac{\gamma(t)}{\rho(t)!} \Phi_D(t) \right) h^{\rho(t)}\ F(t)(y_0)$$

and the term:

$$e(t) = \frac{1}{\rho(t)!} - \frac{\gamma(t)}{\rho(t)!}\Phi_D(t)$$

will be called the ***local truncation error*** coefficient for a tree t.

The extension of this rooted tree theory to the stochastic setting with one Wiener process is based on the consideration of the set of ***bi-colored*** rooted trees T, where the roots of each tree is either ● (τ for a deterministic node) or ○ (σ for stochastic node), see for more details [24].

If $t_1, t_2, \ldots, t_m$ are bi-colored trees, then $[t_1, t_2, \ldots, t_m]$ and $\{t_1, t_2, \ldots, t_m\}$ are trees in which $t_1, t_2, \ldots, t_m$ are each joined by a single branch ● or ○, respectively. For example, if $t_1 = [\sigma]$ and $t_2 = \sigma$, then figures (2.1) and (2.2) shows the two trees $[t_1, t_2]$ and $\{t_1, t_2\}$, respectively.

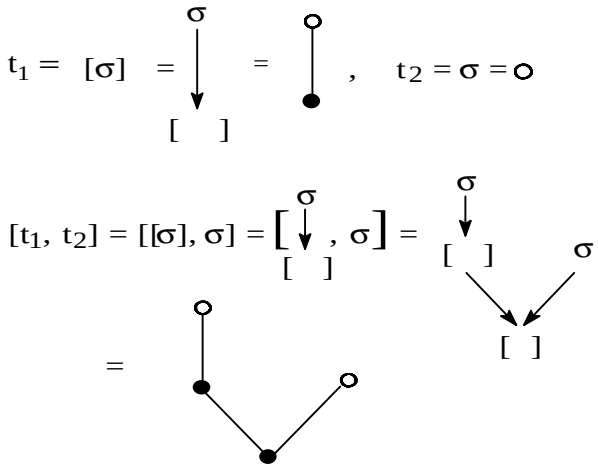

Figure (2.1) The tree $[t_1, t_2]$.

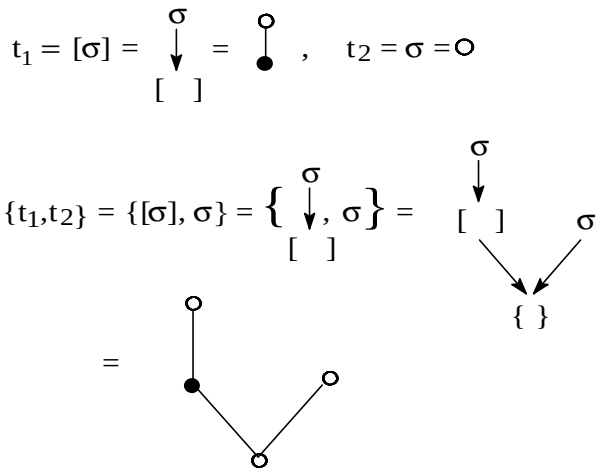

Figure (2.2) The tree $\{t_1, t_2\}$.

In a similar manner to the deterministic case, an elementary differential may be associated with any $t \in T$, such that:

$$\left.\begin{aligned} &F(\tau)(y) = f(y),\ F(\sigma)(y) = g(y) \\ &F(t)(y) = f^{(m)}(y)(F(t_1)(y) \dots F(t_m)(y)), t = [t_1, \dots, t_m] \\ &F(t)(y) = g^{(m)}(y)(F(t_1)(y) \dots F(t_m)(y)), t = \{t_1, \dots, t_m\} \end{aligned}\right\} \quad \dots (2.25)$$

For example, $t = [\tau, \sigma]$ is associated with $f''(y)(f(y)g(y))$, i.e., t reads as $(f''(y)(F(\tau)(y)F(\sigma)(y)))=(f''(y)(f(y)g(y)))$, and $t=[[\sigma]]$ is associated with $f'(y)$ $(f'(y)$ $((g(y)))$, i.e., when $t = [\sigma]$, then $f'(y)$ $(F(\sigma)(y))$ $=f'(y)(g(y))$ and when $t = [[\sigma]]$, then $f'(y)(f'(y)(F(\sigma)(y)))=f'(y)(f'(y)$ $(g(y)))$.

In addition, an elementary weight may be associated with each elementary differential by associating the integer 0 with deterministic node ● and the integer 1 with a stochastic node ○. These elementary weights are in fact Stratonovich integrals.

An easy way to determine the J-integral associated with each tree is to read the tree from top to bottom and simultaneously from right to left, replacing τ-nodes with 0's and σ-nodes with 1's. This is equivalent to read the bracket representation from the inside to outside, and within a bracket from right to left (just the left bracket of the pairs makes a contribution) and then writing the index of the J-integral from left to right. For this last example, $t = \{\tau, \sigma\}$ is associated with J_{101}, i.e., $t = \{\tau, \sigma\}$ reads as $(\sigma \equiv 1 \Rightarrow \tau \equiv 0 \Rightarrow \{\ \} \equiv 1)$, whereas 101 is to consider the subscript of J_{101} and $t = \{\{\tau\}\}$ is associated with J_{011}, i.e., $t = \{\{\tau\}\}$ reading as $(\tau \equiv 0 \Rightarrow \{\ \}_{inside} \equiv 1 \Rightarrow$

$\{\ \}_{outside} \equiv 1)$ whereas 011 is to consider the subscript of J_{011}. Hence, the following remark is easily shown to be true.

Remark (2.4), [7]:

The Stratonovich Taylor series for the actual solution of the SDE given by equation (2.18) is:

$$y(t) = \sum_{t \in T} \alpha(t)\, F(t)(y(t_0))\, \theta(t) \quad \dots (2.26)$$

where $\theta(t)$ represents the corresponding J-integral associated with tree t as described above. A direct comparison of entries in table (2.1) with the terms in the Stratonovich Taylor series given by equation (2.22) demonstrates this compact way of representing y(t)

Table (2.1)

Trees and elementary differentials.

t	**F(t)**	**q(t)**	**t**	**F(t)**	**q(t)**
ϕ	y_0	1	$\{\tau, \sigma\}$	$g''(y_0)\,(f(y_0)\,g(y_0))$	J_{101}
τ	$f(y_0)$	J_0	$\{\sigma, \tau\}$	$g''(y_0)\,(g(y_0)\,f(y_0))$	J_{011}
σ	$g(y_0)$	J_1	$\{\sigma, \sigma\}$	$g''(y_0)\,(g(y_0)\,g(y_0))$	J_{111}
$[\tau]$	$f'(y_0)\,(f(y_0))$	J_{00}	$[[\tau]]$	$f'(y_0)\,(f'(y_0)\,(f(y_0)))$	J_{000}
$[\sigma]$	$f'(y_0)\,(g(y_0))$	J_{10}	$[[\sigma]]$	$f'(y_0)\,(f'(y_0)\,(g(y_0)))$	J_{100}
$\{\tau\}$	$g'(y_0)\,(f(y_0))$	J_{01}	$[\{\tau\}]$	$f'(y_0)\,(g'(y_0)\,(f(y_0)))$	J_{010}
$\{\sigma\}$	$g'(y_0)\,(g(y_0))$	J_{11}	$[\{\sigma\}]$	$f'(y_0)\,(g'(y_0)\,(g(y_0)))$	J_{110}
$[\tau, \tau]$	$f''(y_0)\,(f(y_0)\,f(y_0))$	J_{000}	$\{[\tau]\}$	$g'(y_0)\,(f'(y_0)\,(f((y_0)))$	J_{001}
$[\tau, \sigma]$	$f''(y_0)\,(f\,(y_0)\,g(y_0))$	J_{100}	$\{[\sigma]\}$	$g'(y_0)\,(f'(y_0)\,(g(y_0)))$	J_{101}
$[\sigma, \tau]$	$f''(y_0)\,(g(y_0)\,f(y_0))$	J_{010}	$\{\{\tau\}\}$	$g'(y_0)\,(g'(y_0)\,(f(y_0)))$	J_{011}
$[\sigma, \sigma]$	$f''(y_0)\,(g(y_0)\,g(y_0))$	J_{110}	$\{\{\sigma\}\}$	$g'(y_0)\,(g'(y_0)\,(g(y_0)))$	J_{111}
$\{\tau, \tau\}$	$g''(y_0)\,(f(y_0)\,f(y_0))$	J_{001}			

The one-Wiener process case hides a lot of the potential complexities that can arise with multiple Wiener processes.

2.3.3 Order Conditions for Runge-Kutta Methods.

In the preceding subsection, the Stratonovich Taylor series for the actual solution was established. In this subsection, the corresponding Stratonovich Taylor series for the numerical solution needs to be derived. The numerical methods under considerations will belong to the general family of s-stages SRKM's, where there can be an arbitrary number of random variables included in the formulation of the method. Previously, however, there have been other classes of methods using just one random variable J_1.

In [17], certain classes of s-stages methods have been introduced for solving the autonomous SDE given by equation (2.18) along with the initial value $y_{t_0} = y_0$. Perhaps the most general class of methods considered so far takes the form:

$$\left.\begin{aligned} Y_i &= y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j) + J_1 \sum_{j=1}^{s} b_{ij} g(Y_j), i = 1,2,\ldots,s \\ y_{n+1} &= y_n + h\sum_{j=1}^{s} \alpha_j f(Y_j) + J_1 \sum_{j=1}^{s} \gamma_j g(Y_j) \end{aligned}\right\} \quad \ldots (2.27)$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are s×s-matrices of real elements while $\alpha^T = (\alpha_1, \ldots, \alpha_s)$ and $\gamma^T = (\gamma_1, \ldots, \gamma_s)$ are row vectors in $\mathbf{R}^s$. If both A and B are strictly lower triangular, then equation (2.27) is said to be explicit, otherwise it is implicit. The stochastic component comes from the J_1 integral $J_1 = \int_{t_n}^{t_{n+1}} o\,dW_s$ associated with B and γ. Most researchers, such as Rümelin in [38] and Gard in [17] consider only for simplicity explicit methods.

Rümelin in (1982) [38], has shown that if f and g and the necessary partial derivatives of f and g are bounded then equation (2.27) converges uniformly on $[t_0, T]$ in the quadratic mean sense to the Itô solution of:

$$dy = (f(y) + \lambda \frac{\partial g}{\partial y} g(y))\, dt + g(y)\, dW$$

where:

$$\lambda = \gamma^T B\, e.$$

Furthermore, if $\lambda = 1/2$, then equation (2.27) converges to the solution of the corresponding Stratonovich equation.

Remarks (2.5), [7]:

1. It will be seen that $\gamma^T B\, e = \frac{1}{2}$ is a necessary condition for equation (2.27) to have strong order 1, so that any method of strong order 1 or higher will converge to the solution of the Stratonovich equation.
2. In particular Rümelin in (1982), [38] has proven:

If f and g are arbitrary functions and have continuous and bounded partial derivatives up to the sixth order, then the strong order of equation (2.27) cannot exceed 1.

An example of SRKM of the form given in equation (2.27) with strong order 1 includes the method of Platen which can be written with $s = 2$, as:

$$A = B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \alpha^T = (1, 0),\ \gamma^T = (\frac{1}{2}, \frac{1}{2}) \qquad \text{... (2.28)}$$

Thus if higher stronger order methods are required, equation (2.27) needs to be modified in some way so as to include other stochastic elements as a part from just J_1. This will be done by the introduction of an arbitrary matrix $Z^{(1)}$ and vector $z^{(1)T}$ whose elements are themselves random variables. Since the stepsize h is just $J_0 = \int_{t_0}^{t_0+h} ds$, for consistency the stepsize will be included in the parameter matrix associated with the deterministic components (so $Z^{(0)} = h\ A$ and $z^{(0)T} = h\ \alpha^T$). Hence, the general family of s-stages SRKM for one Wiener process case will be given by:

$$\left.\begin{aligned} Y_i &= y_n + \sum_{j=1}^{s} Z_{ij}^{(0)} f(Y_j) + \sum_{j=1}^{s} Z_{ij}^{(1)} g(Y_j), i = 1, 2, .., s \\ y_{n+1} &= y_n + \sum_{j=1}^{s} z_j^{(0)} f(Y_j) + \sum_{j=1}^{s} z_j^{(1)} g(Y_j) \end{aligned}\right\} \qquad \text{… (2.29)}$$

By studying the general strong order properties of equation (2.29) for an arbitrary random variable elements within $Z^{(1)}$ and $z^{(1)}$, the order condition will be used to construct two-stages method of the form given in equation (2.27), which is optimal in terms of minimizing the local truncation error coefficients.

Of course, equation (2.29) is a very general representation, and so a simplifying assumption which often be placed on the $Z_{ij}^{(1)}$ and $z_j^{(1)}$ in that it will be assumed that each of these random variables can be written as a linear combination of p similar random variables $\theta_1, \theta_2, \ldots, \theta_p$, such as J_1 (i.e., $\theta_1 = \theta_2 = \ldots = \theta_p = J_1$) in this case $Z_{ij}^{(0)}$, $z_j^{(0)}$, $Z_{ij}^{(1)}$ and $z_j^{(1)}$ will be written as:

$$\left.\begin{aligned} Z_{ij}^{(0)} &= h\, a_{ij};\ i,\ j=1,2,\ldots,s \\ z_j^{(0)} &= h\alpha_j,;\ j=1,\ 2,\ \ldots,s \\ Z_{ij}^{(1)} &= \sum_{L=1}^{p} b_{ij}^{(L)}\theta_L = \sum_{L=1}^{p} b_{ij}^{(L)} J_1, i,\ j=1,\ 2,\ldots,s \\ z_j^{(1)} &= \sum_{L=1}^{p} b_j^{(L)}\theta_L = \sum_{L=1}^{p} b_j^{(L)} J_1, i,\ j=1,\ 2,\ldots,s \end{aligned}\right\} \quad \ldots (2.30)$$

and equation (2.29) can be written as generalization of equation (2.27) (SRKM with a multi-Weiner process).

$$\left.\begin{aligned} Y_i &= y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j) + \sum_{L=1}^{p}\left(\sum_{j=1}^{s} b_{ij}^{(L)} g(Y_j)\right) J_1, i=1,2,\ldots,s \\ y_{n+1} &= y_n + h\sum_{j=1}^{s} \alpha_j f(Y_j) + \sum_{L=1}^{p}\left(\sum_{j=1}^{s} \gamma_j^{(L)} g(Y_j)\right) J_1 \end{aligned}\right\} \quad \ldots (2.31)$$

This family of methods may be characterized by the following table:

	A	$B^{(1)}$	**L**	$B^{(p)}$
	α	$\gamma^{(1)}$	**L**	$\gamma^{(p)}$

In order to study the order conditions in equation (2.27), which is associated with equation (2.29), then equation (2.29) will be written as a function of t:

$$\left.\begin{aligned} Y_t^{(i)} &= y_{t_0} + \sum_{j=1}^{s} Z_{ij}^{(0)} f(Y_t^{(j)}) + \sum_{j=1}^{s} Z_{ij}^{(1)} g(Y_t^{(j)}), i=1,2,..,s \\ y_{n+1} &= y_{t_0} + \sum_{j=1}^{s} z_i^{(0)} f(Y_t^{(i)}) + \sum_{j=1}^{s} z_i^{(1)} g(Y_t^{(i)}) \end{aligned}\right\} \quad \ldots (2.32)$$

By substituting $Y_t^{(i)} = \left[Y_t^1, Y_t^2, ..., Y_t^s\right]$ in the expression for Y(t), the $f(Y_t^{(i)})$ and $g(Y_t^{(i)})$ analogously, but with operator L^0 and L^1, can be expanded in a Taylor series representation, as:

$$\left.\begin{aligned} f(Y_t^{(i)}) &= f(Y_{t_0}^{(i)}) + \sum_{k=1}^{\infty} \frac{(L^0)^k f(Y_{t_0}^{(i)})}{k!} \\ g(Y_t^{(i)}) &= g(Y_{t_0}^{(i)}) + \sum_{k=1}^{\infty} \frac{(L^1)^k g(Y_{t_0}^{(i)})}{k!} \end{aligned}\right\} \quad \text{... (2.33)}$$

where the operators L^0 and L^1 are as in equation (2.21) and where it is assumed that f and g are sufficiently differentiable.

By recursively substituting for the $Y_t^{(i)}$ back into equation (2.32) and using equation (2.33) (and a similar formula for $g(Y_t^{(i)})$) it is seen that:

$$L^0 f(Y_{t_0}^{(i)}) = \sum_{j=1}^{s} Z_{ij}^{(0)} \frac{df(Y_{t_0}^{(i)})}{dY_{t_0}^{(i)}} f(Y_{t_0}^{(i)}) + \sum_{j=1}^{s} Z_{ij}^{(1)} \frac{df(Y_{t_0}^{(i)})}{dY_{t_0}^{(i)}} g(Y_{t_0}^{(i)})$$

$$L^1 g(Y_{t_0}^{(i)}) = \sum_{j=1}^{s} Z_{ij}^{(0)} \frac{dg(Y_{t_0}^{(i)})}{dY_{t_0}^{(i)}} f(Y_{t_0}^{(i)}) + \sum_{j=1}^{s} Z_{ij}^{(1)} \frac{dg(Y_{t_0}^{(i)})}{dY_{t_0}^{(i)}} g(Y_{t_0}^{(i)}).$$

Now, for the second derivative terms, a similar analysis gives:

$$\begin{aligned} (L^0)^2 f(Y_{t_0}^{(i)}) &= L^0 L^0 f(y_0) \\ &= L^0 \left[\sum_{j=1}^{s} Z_{ij}^{(0)} f'(y_0) f(y_0) + \sum_{j=1}^{s} Z_{ij}^{(1)} f'(y_0) g(y_0) \right] \\ &= \left[\sum_{j=1}^{s} Z_{ij}^{(0)} f'(y_0) f(y_0) + \sum_{j=1}^{s} Z_{ij}^{(1)} f'(y_0) g(y_0) \right]' \\ &\quad \left[\sum_{k=1}^{s} Z_{jk}^{(0)} f'(y_0) f(y_0) + \sum_{k=1}^{s} Z_{jk}^{(1)} f'(y_0) g(y_0) \right] \end{aligned}$$

when back to equation (2.23), recall that $L^0 L^0 f = (f'f)'f = f''f\,f + f'f'f$; then:

$$(L^0)^2 f(Y_{t_0}^{(i)}) = \left(\sum_{\substack{\text{when} \\ j=k}} Z_{ij}^{(0)} \right)^2 f''(y_0)\,(f(y_0)\,f(y_0)) + 2 \sum_{\substack{\text{when} \\ j\neq k}} Z_{ij}^{(0)}\, Z_{jk}^{(0)} \quad f'(y_0)$$

$$(f'(y_0)\,f(y_0)) + \sum_j Z_{ij}^{(0)} \sum_j Z_{ij}^{(1)}\, f''(y_0)\,(f(y_0)\,g(y_0)) +$$

$$2 \sum_{\substack{\text{when} \\ j\neq k}} Z_{ij}^{(0)}\, Z_{jk}^{(1)}\, f'(y_0)\,(f'(y_0)\,g(y_0)) + 2 \sum_{\substack{\text{when} \\ j\neq k}} Z_{ij}^{(1)}\, Z_{jk}^{(0)}\, f'(y_0)$$

$$(g'(y_0)\,f(y_0)) + \sum_j Z_{ij}^{(1)} \sum_j Z_{ij}^{(0)} \quad f''(y_0)\,(g(y_0)\,f(y_0)) +$$

$$\left(\sum_{\substack{\text{when} \\ j=k}} Z_{ij}^{(1)} \right)^2 f''(y_0)\,(g(y_0)\,g(y_0)) + 2 \sum_{\substack{\text{when} \\ j\neq k}} Z_{ij}^{(1)}\, Z_{jk}^{(1)} \quad f'(y_0)$$

$$(g'(y_0)\,g(y_0)).$$

Analogous results hold for $(L^1)^2 g(Y_{t_0}^{(i)})$.

Now, the numerical method may be expanded in a Taylor series, as:

$$Y_t = y_{t_0} + \sum_{i=1}^{s} z_i^{(0)} f(Y_t^{(i)}) + \sum_{i=1}^{s} z_i^{(1)} g(Y_t^{(i)})$$

$$= y_{t_0} + \sum_{i=1}^{s} z_i^{(0)} \left[f(y_{t_0}) + L^0 f(y_{t_0}) + \frac{1}{2!}(L^0)^2 f(y_{t_0}) + \ldots \right] +$$

$$\sum_{i=1}^{s} z_i^{(1)} \left[g(y_{t_0}) + L^1 g(y_{t_0}) + \frac{1}{2!}(L^1)^2 g(y_{t_0}) + \ldots \right]$$

Now, when $e^T = (1, 1, \ldots, 1)^T$ contains s of elements 1. Then:

$$\sum_{i=1}^{s} z_i = \begin{pmatrix} z_1 \\ z_2 \\ \mathbb{M} \\ z_s \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \\ \mathbb{M} \\ 1 \end{pmatrix} = z^T e$$

and in a similar manner:

$$\sum_{j=1}^{s} Z_i = (Z_1, Z_2, \ldots, Z_s)\begin{pmatrix}1\\1\\\mathbb{M}\\1\end{pmatrix} = Ze.$$

Then:

$$Y_t = y_{t_0} + z^{(0)T} e\, f(y_0) + z^{(0)T} Z^{(0)} e\, f'(y_0) f(y_0) + z^{(0)T} Z^{(1)} e\, f'(y_0) g(y_0) + \frac{1}{2}$$

$$z^{(0)T} (Z^{(0)}e)^2 f''(y_0) (f(y_0) f(y_0)) + z^{(0)T} (Z^{(0)})^2 e\, f'(y_0)$$

$$(f'(y_0) f(y_0)) + \frac{1}{2} z^{(0)T} Z^{(0)} e\, Z^{(1)} e\, f''(y_0) (f(y_0) g(y_0)) + z^{(0)T}$$

$$Z^{(0)} Z^{(1)} e\, f'(y_0) (f'(y_0) g(y_0)) + z^{(0)T} Z^{(1)} Z^{(0)} e\, f'(y_0) (g'(y_0) f(y_0))$$

$$+ \frac{1}{2} z^{(0)T} Z^{(1)} e\, Z^{(0)} e\, f''(y_0) (g(y_0) f(y_0)) + \frac{1}{2} z^{(0)T} (Z^{(1)} e)^2 f''(y_0) (g(y_0)$$

$$g(y_0)) + z^{(0)T} (Z^{(1)})^2 e\, f'(y_0) (g'(y_0) g(y_0)) + \ldots + z^{(1)T} e\, g(y_0)$$

$$+ z^{(1)T} Z^{(0)} e\, g'(y_0) f(y_0) + z^{(1)T} Z^{(1)} e\, g'(y_0) g(y_0) + \frac{1}{2} z^{(1)T} (Z^{(0)}e)^2$$

$$g''(y_0) (f(y_0) f(y_0)) + z^{(1)T} (Z^{(0)})^2 e\, g'(y_0) (f'(y_0) f(y_0)) + \frac{1}{2} z^{(1)T} Z^{(0)} e$$

$$Z^{(1)} e\, g''(y_0) (f(y_0) g(y_0)) + z^{(1)T} Z^{(0)} Z^{(1)} e\, g'(y_0) (f'(y_0) g(y_0))$$

$$+ z^{(1)T} Z^{(1)} Z^{(0)} e\, g'(y_0) (g'(y_0) f(y_0)) + \frac{1}{2} z^{(1)T} Z^{(1)} e\, Z^{(0)} e\, g''(y_0) (g(y_0)$$

$$f(y_0)) + \frac{1}{2} z^{(1)T} (Z^{(1)} e)^2 g''(y_0) (g(y_0) g(y_0)) + z^{(1)T} (Z^{(1)})^2 e\, g'(y_0)$$

$$(g'(y_0) g(y_0)) + \ldots .$$

Let $\phi(t)$ be defined recursively, as:

$$K(\phi) = e$$

$$\left.\begin{aligned}\phi(t) &= \rho(t)\, z^{(0)T} \prod_{L=1}^{m} K(t_L),\ t = [t_1, t_2, \ldots, t_m] \\ \phi(t) &= \rho(t)\, z^{(1)T} \prod_{L=1}^{m} K(t_L),\ t = \{t_1, t_2, \ldots, t_m\}\end{aligned}\right\} \quad \ldots (2.34)$$

where:

$$\left.\begin{aligned}K(t) &= \rho(t)\, Z^{(0)} \prod_{L=1}^{m} K(t_L),\ t = [t_1, t_2, \ldots, t_m] \\ K(t) &= \rho(t)\, Z^{(1)} \prod_{L=1}^{m} K(t_L),\ t = \{t_1, t_2, \ldots, t_m\}\end{aligned}\right\} \quad \ldots (2.35)$$

where the multiplication of vectors is considered component wise.

As an illustration, consider the following examples:

Examples (2.6) [1], [7]:

1. By taking $z^{(0)T}\, e\, f(y_0)$, then $\rho(\tau) = 1$, because τ is one node, $\phi(t) = \rho(t)\, z^{(0)T} \prod_{L=1}^{m} K(t_L)$ implies to $\phi(\tau) = 1\, z^{(0)T} K(\phi)$, such that $K(\phi) = e$, then $\phi(\tau) = z^{(0)T} e$ and $f(y) = F(\tau)(y(t_0))$ (back to table (2.1)), and hence:

   $z^{(0)T} e f(y_0) = \phi(\tau)\, F(\tau)(y(t_0))$

2. By taking $z^{(0)T}\, Z^{(1)}\, e\, f'(y_0)\, g(y_0)$, then $f'(y_0)\, g(y_0) = F([\sigma])(y(t_0))$ and $\rho([\sigma]) = 2$ (because $[\sigma]$ is two nodes $(\sigma + [\ ])$), and hence:

   $$\phi([\sigma]) = 2\, z^{(0)T} \prod_{L=1}^{1} K(t_L) = 2\, z^{(0)T} K(t_1) = 2\, z^{(0)T} K(\sigma)$$

   such that $K(\sigma) = \rho(\sigma)\, Z^{(1)} K(\phi) = 1\, Z^{(1)} e$, implies that $\phi([\sigma]) = 2\, z^{(0)T}\, Z^{(1)} e$ and thus $z^{(0)T}\, Z^{(1)}\, e\, f'(y_0)\, g(y_0) = \frac{1}{2}\, \phi([\sigma])\, F([\sigma])(y(t_0))$.

3. By taking $\frac{1}{2}$ $z^{(1)T} Z^{(0)} e Z^{(1)} e g''(y_0)$ $(f(y_0)\, g(y_0))$, then $g''(y_0)$ $(f(y_0)\, g(y_0)) = F(\{\tau, \sigma\})y(t_0)$ and $\rho(\{\tau, \sigma\}) = 3$(because$\{\tau, \sigma\}$ is three nodes $(\tau+\sigma+\{\})$). Now, when $\phi(\{\tau, \sigma\})=3\, z^{(1)T} \prod_{L=1}^{2} K(t_L) = 3\, z^{(1)T} (K(\tau)\, K(\sigma))$, such that $K(\tau) = \rho(\tau)\, Z^{(0)} K(\phi)$, $\rho(\tau) =1$, then $K(\tau) = Z^{(0)} e$ and $K(\sigma) = \rho(\sigma)\, Z^{(1)} K(\phi)$, $\rho(\sigma) = 1$, then $K(\sigma) = Z^{(1)} e$ and thus $\phi(\{\tau,\sigma\})=3\, z^{(1)T}\, Z^{(0)} e\, Z^{(1)} e$, implies that $\frac{1}{2} z^{(1)T} Z^{(0)} e\, Z^{(1)} e\, g''(y_0)\, (f(y_0)\, g(y_0)) = \frac{1}{3!}\, \phi(\{\tau, \sigma\})\, F(\{\tau, \sigma\})(y(t_0))$.

4. By taking $z^{(1)T}\, Z^{(0)}\, Z^{(1)} e\, g'(y_0)$ $(f'(y_0)\, g(y_0))$, then $g'(y_0)$ $(f(y_0)\, g(y_0)) = F(\{[\sigma]\})(y(t_0))$ and $\rho(\{[\sigma]\}) = 3$, (because $\{[\sigma]\}$ is three nodes $(\sigma+[\,]+\{\,\})$. Now, when $\phi(\{[\sigma]\}) = 3\, z^{(1)T} \prod_{L=1}^{2} K(t_L) = 3\, z^{(1)T}\, K([\sigma])$, such that when backing to example 2 above, $\phi([\sigma]) = 2z^{(0)T} Z^{(1)} e$ and thus $K([\sigma]) = 2Z^{(0)}Z^{(1)}e$ (by substituting $z^{(0)T}$ in $\phi([\sigma])$ with $Z^{(0)}$ in $K([\sigma])$, because applying in (3,36)), then $\phi(\{[\sigma]\}) = 3!\, z^{(1)T}\, Z^{(0)}\, Z^{(1)} e$, implies that $z^{(1)T}\, Z^{(0)}\, Z^{(1)} e\, g'(y_0)$ $(f'(y_0)\, g(y_0)) = \frac{1}{3!}\phi(\{[\sigma]\})\, F(\{[\sigma]\})(y(t_0))$.

Thus, for trees with up to 3 nodes and using F(t) from table (2.1), the Taylor series expansion of the numerical method may be written as:

$$Y_t = y_{t_0} + \phi(\tau)\, F(\tau)(y_{t_0}) + \phi(\sigma)\, F(\sigma)(y_{t_0}) + \frac{1}{2!}\, \phi([\tau])\, F([\tau])(y_{t_0}) + \frac{1}{2!}\, \phi(\{\tau\})\, F(\{\tau\})(y_{t_0}) + \frac{1}{2!}\phi([\sigma])\, F([\sigma])(y_{t_0}) + \frac{1}{2!}\, \phi(\{\sigma\})\, F(\{\sigma\})(y_{t_0}) + \frac{1}{3!}\, \phi([\tau,\tau])\, F([\tau,\tau])(y_{t_0}) + \frac{1}{3!}\, \phi([\tau,\sigma])\, F([\tau,\sigma])(y_{t_0}) + \frac{1}{3!}\, \phi([\sigma,\tau])\, F([\sigma,\tau])(y_{t_0}) + \frac{1}{3!}\, \phi([\sigma,\sigma])$$

$$F([\sigma,\sigma])(y_{t_0}) + \frac{1}{3!}\phi(\{\tau,\tau\})\,F(\{\tau,\tau\})(y_{t_0}) + \frac{1}{3!}\phi(\{\tau,\sigma\})$$

$$F(\{\tau,\sigma\})(y_{t_0}) + \frac{1}{3!}\phi(\{\sigma,\tau\})\,F(\{\sigma,\tau\})(y_{t_0}) + \frac{1}{3!}\phi(\{\sigma,\sigma\})$$

$$F(\{\sigma,\sigma\})(y_{t_0}) + \frac{1}{3!}\phi([[\tau]])\,F([[\tau]])(y_{t_0}) + \frac{1}{3!}\phi([[\sigma]])$$

$$F([[\sigma]])(y_{t_0}) + \frac{1}{3!}\phi([\{\tau\}])\,F([\{\tau\}])(y_{t_0}) + \frac{1}{3!}\phi([\{\sigma\}])$$

$$F([\{\sigma\}])(y_{t_0}) + \frac{1}{3!}\phi(\{[\tau]\})\,F(\{[\tau]\})(y_{t_0}) + \frac{1}{3!}\phi(\{[\sigma]\})$$

$$F(\{[\sigma]\})(y_{t_0}) + \frac{1}{3!}\phi(\{\{\tau\}\})\,F(\{\{\tau\}\})(y_{t_0}) + \frac{1}{3!}\phi(\{\{\sigma\}\})$$

$$F(\{\{\sigma\}\})(y_{t_0}) + \dots$$

and in general:

$$Y(t) = \sum_{t\in T} \frac{\alpha(t)\phi(t)F(t)(y_{t_0})}{\rho(t)!} \qquad \dots (2.36)$$

where $\alpha(t)$ is the multiplicity factor associated with some of the higher derivative terms, exactly as was required in the expansion of the actual solution of the SDE as given in equation (2.26).

Hence, the local truncation error at $t = t_n$ of an SRKM can be written as:

$$L_n = \sum_{t\in T} \alpha(t)\left(\theta(t) - \frac{\phi(t)}{\rho(t)!}\right)F(t)(y_{t_n}) \qquad \dots (2.37)$$

Thus, if $\sqrt{E(|L_n|)^2} \le C h^{p+\frac{1}{2}}$, then a method will have strong global order p. Writing L_n as:

$$L_n = \sum_{t\in T} e(t)\,F(t)(y_{t_n}) \qquad \dots (2.38)$$

and letting:

$$c = Z^{(0)} e,\ \lambda = Z^{(1)} e \qquad \dots (2.39)$$

then table (2.2) gives e(t) for all trees with $\rho(t) \le 3$.

Table (2.2)

Local error coefficients.

No.	t	e(t)	No.	t	e(t)
1	τ	$J_0 - z^{(0)T} e$	12	$[\{\tau\}]$	$J_{010} - z^{(0)T} Z^{(1)} c$
2	σ	$J_1 - z^{(1)T} e$	13	$[\{\sigma\}]$	$J_{110} - z^{(0)T} Z^{(1)} \lambda$
3	$[\tau]$	$J_{00} - z^{(0)T} e$	14	$[\sigma,\sigma]$	$J_{110} - \frac{1}{2} z^{(0)T} \lambda^2$
4	$[\sigma]$	$J_{10} - z^{(0)T} \lambda$	15	$\{[\tau]\}$	$J_{001} - z^{(1)T} Z^{(0)} c$
5	$\{\tau\}$	$J_{01} - z^{(1)T} c$	16	$\{[\sigma]\}$	$J_{101} - z^{(1)T} Z^{(0)} \lambda$
6	$\{\sigma\}$	$J_{11} - z^{(1)T} \lambda$	17	$\{\tau,\tau\}$	$J_{001} - \frac{1}{2} z^{(1)T} c^2$
7	$[\tau,\tau]$	$J_{000} - \frac{1}{2} z^{(0)T} c^2$	18	$\{\tau,\sigma\}$	$J_{101} - \frac{1}{2} z^{(1)T} c \lambda$
8	$[[\tau]]$	$J_{000} - z^{(0)T} Z^{(0)} c$	19	$\{\sigma,\tau\}$	$J_{011} - \frac{1}{2} z^{(1)T} c \lambda$
9	$[[\sigma]]$	$J_{100} - z^{(0)T} Z^{(0)} \lambda$	20	$\{\sigma,\sigma\}$	$J_{111} - \frac{1}{2} z^{(1)T} \lambda^2$
10	$[\tau,\sigma]$	$J_{100} - \frac{1}{2} z^{(0)T} c \lambda$	21	$\{\{\tau\}\}$	$J_{011} - z^{(1)T} Z^{(1)} c$
11	$[\sigma,\tau]$	$J_{010} - \frac{1}{2} z^{(0)T} c \lambda$	22	$\{\{\sigma\}\}$	$J_{111} - z^{(1)T} Z^{(1)} \lambda$

2.3.4 Derivation of the Methods.

Using the order condition discussed earlier, it is now possible to analyse the parameters for different classes of methods. In comparison with the deterministic case. There are many more trees to be analysed for comparable order in the stochastic setting. For every tree t in the deterministic case, there are in the one Weiner process case $2^{\rho(t)}$ trees (for $\rho(t) = 1$ or 2), $2^{\rho(t)+1}$ trees (for $\rho(t) = 3$) and $2^{\rho(t)+2}$ trees (for $\rho(t) = 4$) that must be considered in the stochastic case. Table (2.3) illustrates how quickly the number of trees grows [7].

Table (2.3)

Number of trees.

ρ(t)	1	2	3	4
Deterministic	1	1	2	4
Stochastic	2	4	16	64

When back to table (2.2) one can see the first two components of t is the $\rho(t) = 1$ and the four of t after the first two is the $\rho(t) = 2$ and so on.

First, consider the class of methods given by equation (2.27) in which:

$Z^{(0)} = h\,A$, $z^{(0)T} = h\,\alpha^T$, $Z^{(1)} = J_1\,B$, $z^{(1)T} = J_1\,\gamma^T$, $b = B\,e$, $a = A\,e$.

In an attempt to get strong global order 1.5, it is necessary that the order conditions corresponding to trees 1 - 6, 20 and 22 in table (2.2) vanish as these are the trees whose corresponding J-integals have expectation behaving as $O(h^p)$, $p < 2$, i.e., when back to equation (2.3) and the definition of the strong convergence, then the hierarchical set:

$$\Lambda_p = \{\alpha \in M : L(\alpha) + n(\alpha) \le 2\,p \text{ or } L(\alpha) = n(\alpha) = p + \frac{1}{2}\}$$

Now, (1, 1) is the unique element which give strong order 1.0 in addition elements set $\Lambda_{0.5}$ (when back to tree 6 in table (2.2), then (1, 1) will be find in the subscript J_{11}). Hence when $\alpha = (1, 1)$, then $L(\alpha) = 2$, $n(\alpha) = 0$ implies to fulfill the condition given in (2.3) such that

$L(\alpha) + n(\alpha) \le 2\,p$, i.e., $2 + 0 \le 2\,p$, then $p = 1$, and thus:

$$\Lambda_{1.0} = \Lambda_{0.5} \cup \{(1, 1)\}$$

and also the elements in the set β, given by $\beta = \{(0, 1), (1, 0), (0, 0), (1, 1, 1)\}$, which are uniquely determined which gives strong order 1.5 in addition elements set $\Lambda_{1.0}$ (look for the subscript J in trees (3, 4, 5, 20, 22) existing in table (2.2)). Now, we shall apply the elements of the set β in condition (2.3) to prove their elements have a strong order 1.5.

When $\alpha = (0, 1)$, $L(\alpha) = 2$, $n(\alpha) = 1$, then $L(\alpha) + n(\alpha) = 2 + 1 \le 2\,p$ and thus $p = 1.5$.

When $\alpha = (1, 0)$, $L(\alpha) = 2$, $n(\alpha) = 1$, then $L(\alpha) + n(\alpha) = 2 + 1 \leq 2\ p$ and thus $p = 1.5$.

When $\alpha = (0, 0)$, $L(\alpha) = 2$, $n(\alpha) = 2$, then $L(\alpha) = n(\alpha) = 2 = p + \frac{1}{2}$ and thus $p = 1.5$.

When $\alpha = (1, 1, 1)$, $L(\alpha) = 3$, $n(\alpha) = 0$, then $L(\alpha) + n(\alpha) = 3 + 0 \leq 2\ p$ and thus $p = 1.5$.

Then $\Lambda_{1.5} = \Lambda_{1.0} \cup \{(0, 1), (1, 0), (0, 0), (1, 1, 1)\}$

As an illustration, the order condition associated with tree 4 in table (2.2) should be after taking the square value of the trace and hence its expectation, is:

$$E((J_{10} - h\, J_1\, \psi)^2) = O(h^4) \text{ or } 0 \qquad \ldots (2.40)$$

where $\psi = \alpha^T b$.

To analyze this condition, first expand the left hand side, given by [1]:

$$E(J_{10}^2) - 2\, \psi\, h\, E(J_{10}\, J_1) + \psi^2\, h^2\, E(J_1^2) \qquad \ldots (2.41)$$

But $J_1 \sim N(0, h)$ and using the results of chapter one, section (1.5), we get:

$$E(J_{10}^2) = \frac{1}{3}\, h^3,\ E(J_{10}\, J_1) = \frac{1}{2}\, h^2,\ E(J_1^2) = h \qquad \ldots (2.42)$$

and so equation (2.41) becomes:

$$\frac{1}{3}\, h^3 - 2\, \psi\, h\, (\frac{1}{2}\, h^2) + \psi^2\, h^2\, h = h^3\, (\frac{1}{3} - \psi + \psi^2) \neq 0$$

because when $f(\psi) = \frac{1}{3} - \psi + \psi^2 = 0$, then ψ is complex number

Hence, for resultant on minimum real solution of ψ, we must derive $f(\psi)$ and equating the result to zero, yields to $\psi = \frac{1}{2}$.

Thus in fact the minimum of the quadratic occurs when $\psi = \frac{1}{2}$ in which case the minimum value is $\frac{1}{12}$.

Applying this analysis to the order conditions, leads to a complete characterization of the class of explicit SRKM of the form given in equation (2.27) with strong order 1 and minimum principle local truncation error.

In particular, trees 1, 2 and 6 have associated J-integrals which are of order h, $\sqrt{h}$ and h, respectively, so for a method to have strong order 1, it is necessary that to consider for [7]:

1. For tree 1 in table (2.2)

   $E(J_0 - z^{(0)T} e)^2 = E((J_0)^2 - 2 J_0 z^{(0)T} e + (z^{(0)T} e)^2$, when $J_0 = h$, $z^{(0)T} = h \alpha^T$, and thus $E(J_0 - z^{(0)T} e)^2 = h^2 ((\alpha^T e)^2 - 2 (\alpha^T e) + 1) = 0$, when $h^2 \neq 0$. Then:

   $$\alpha^T e = 1 \quad \ldots (2.43)$$

2. For tree 2 in table (2.2)

   $E(J_1 - z^{(1)T} e)^2 = E((J_1)^2 - 2 J_1 z^{(1)T} e + (z^{(1)T} e)^2$, when $z^{(1)T} = J_1 \gamma^T$, $E(J_1^2) = h$. Hence:

   $E(J_1 - z^{(1)T} e)^2 = h ((\gamma^T e)^2 - 2 (\gamma^T e) + 1) = 0$, when $h \neq 0$

   Then:

   $$\gamma^T e = 1 \quad \ldots (2.44)$$

3. For tree 6 in table (2.2)

   $E(J_{11} - z^{(1)T} \lambda)^2 = E(J_{11} - z^{(1)T} Z^{(1)} e)^2$, when $\lambda = Z^{(1)} e$

   $= E(J_{11} - J_1^2 \gamma^T B e)^2 = E(J_{11} - J_1^2 \gamma^T b)^2$,

   when $z^{(1)T} = J_1 \gamma^T$, $Z^{(1)} = J_1 B$, $b = B e$

   Now:

   $$E(J_{11} - J_1^2 \gamma^T b)^2 = E(J_{11}^2) - 2 b \gamma^T E(J_{11} J_1^2) + b^2 (\gamma^T)^2 E(J_1^4) \quad \ldots (2.45)$$

   Since $J_{1\ldots1} = \frac{J_1^p}{p!}$, then [22]:

   $$E(J_{11}^2) = E\left(\left(\frac{J_1^2}{2}\right)^2\right) = \frac{1}{4} E(J_1^4) = \frac{3}{4} h^2,\ E(J_1^4) = 3h^2,\ E(J_{11} J_1^2) = \frac{3}{2} h^2$$

   and equation (2.45) implies to $h^2 (\frac{3}{4} - 3\gamma^T b + 3(\gamma^T b)^2) = 0$, when $h^2 \neq 0$, then:

$$\gamma^T b = \frac{1}{2} \quad \dots (2.46)$$

Now, when back to definition remainder set in (1.60) then:

$$R(\Lambda_{1.0}) = \{(0, 0), (0, 1), (1, 0), (0, 1, 1), (1, 1, 1)\}$$

and thus, the terms corresponding to the $h^{1.5}$ terms arise from trees 4,5,20 and 22:

$$E(J_{10} - h J_1 \alpha^T b)^2 = \left(\frac{1}{3} - \alpha^T b + (\alpha^T b)^2\right) h^3 \quad \dots (2.47)$$

$$E(J_{01} - h J_1 \gamma^T a)^2 = \left(\frac{1}{3} - \gamma^T a + (\gamma^T a)^2\right) h^3 \quad \dots (2.48)$$

$$E(J_{111} - \frac{1}{2} J_1^3 \gamma^T b^2)^2 = \left(\frac{1}{9} - \frac{2}{3}\gamma^T b^2 + (\gamma^T b^2)^2\right) \frac{15}{4} h^3 \quad \dots (2.49)$$

$$E(J_{111} - J_1^3 \gamma^T B b)^2 = \left(\frac{1}{36} - \frac{1}{3}\gamma^T B b + (\gamma^T B b)^2\right) 15 h^3 \quad \dots (2.50)$$

These four equations are minimized if:

$$\alpha^T b = \frac{1}{2}, \gamma^T a = \frac{1}{2}, \gamma^T b^2 = \frac{1}{3}, \gamma^T B\, b = \frac{1}{6} \quad \dots (2.51)$$

in this case, the respective minimum in (2.47), (2.48), (2.49), (2.50) are:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, 0 \quad \dots (2.52)$$

For a 2-stages explicit method, the last equation for equations (2.51):

$$\gamma^T B\, b = (\gamma_1, \gamma_2) \begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ b_2 \end{pmatrix} = 0$$

then the equation (2.50) is:

$$\left(\frac{1}{36} - \frac{1}{3}\gamma^T B b + (\gamma^T B b)^2\right) 15 h^3 = \frac{5h^3}{12}$$

and so in this case, the principal error constants are:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, \frac{5h^3}{12} \quad \dots (2.53)$$

From equations (2.46) and (2.51), for this 2-stages explicit method:

$$(\gamma_1, \gamma_2)\begin{pmatrix} 0 \\ b_{21} \end{pmatrix} = \gamma_2 b_{21} = \frac{1}{2}, \qquad (\alpha_1, \alpha_2)\begin{pmatrix} 0 \\ b_{21} \end{pmatrix} = \alpha_2 b_{21} = \frac{1}{2},$$

$$(\gamma_1, \gamma_2)\begin{pmatrix} 0 \\ a_{21} \end{pmatrix} = \gamma_2 a_{21} = \frac{1}{2}$$

while the following equation:

$$\gamma^T b^2 = \frac{1}{3} \text{ or } \gamma^T b^2 = (\frac{1}{3}, \frac{1}{3})$$

is obtained in general as follows:

since $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and $e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then:

$$b = Be = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} b_{11} + b_{12} \\ b_{21} + b_{22} \end{pmatrix}$$

and hence

$$b^2 = b\, b^T = \begin{pmatrix} b_{11} + b_{12} \\ b_{21} + b_{22} \end{pmatrix}(b_{11} + b_{12}, b_{21} + b_{22})$$

$$= \begin{pmatrix} (b_{11} + b_{12})^2 & (b_{11} + b_{12})(b_{21} + b_{22}) \\ (b_{11} + b_{12})(b_{21} + b_{22}) & (b_{21} + b_{22})^2 \end{pmatrix}$$

therefore

$$\gamma^T b^2 = (\gamma_1, \gamma_2)\begin{pmatrix} (b_{11} + b_{12})^2 & (b_{11} + b_{12})(b_{21} + b_{22}) \\ (b_{11} + b_{12})(b_{21} + b_{22}) & (b_{21} + b_{22})^2 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_1(b_{11} + b_{12})^2 + \gamma_2(b_{11} + b_{12})(b_{21} + b_{22}) \\ \gamma_2(b_{21} + b_{22})^2 + \gamma_1(b_{11} + b_{12})(b_{21} + b_{22}) \end{pmatrix}^T$$

and since

$$\gamma^T b^2 = (\frac{1}{3}, \frac{1}{3})$$

so

$$\begin{pmatrix} \gamma_1(b_{11} + b_{12})^2 + \gamma_2(b_{11} + b_{12})(b_{21} + b_{22}) \\ \gamma_2(b_{21} + b_{22})^2 + \gamma_1(b_{11} + b_{12})(b_{21} + b_{22}) \end{pmatrix}^T = (\frac{1}{3}, \frac{1}{3})$$

hence we get:

$$\gamma_1(b_{11}+b_{12})^2+\gamma_2(b_{11}+b_{12})(b_{21}+b_{22})=\frac{1}{3}$$

$$\gamma_2(b_{21}+b_{22})^2+\gamma_1(b_{11}+b_{12})(b_{21}+b_{22})=\frac{1}{3}$$

Now, for explicit case, we get $\gamma_2\, b_{21}^2=\frac{1}{3}$

i.e., to obtain such as γ_2, then [1]: $b_{21}=\frac{1}{2\gamma_2}$, $\gamma_2\, b_{21}^2=\gamma_2\frac{1}{4\gamma_2^2}=\frac{1}{3}$, implies to $\gamma_2=\frac{3}{4}$, and this produces the solution $b_{21}=\frac{2}{3}$, $\gamma_2=\frac{3}{4}$, $a_{21}=\frac{2}{3}$, $\alpha_2=\frac{3}{4}$. Then $\alpha_1=1-\alpha_2=\frac{1}{4}$, $\gamma_1=1-\gamma_2=\frac{1}{4}$, and so the 2-stages method (with maximum possible strong order equals 1) with minimum principle error constants is represented by the tableau [7]:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array} \qquad \begin{array}{c|cc} 0 & 0 & 0 \\ \frac{2}{3}J_1 & 0 & 0 \\ \hline & \frac{1}{4}J_1 & \frac{3}{4}J_1 \end{array} \qquad \dots (2.54)$$

This method will be referenced by the code "R2".

Note that, the Platen method (referenced subsequently by the code "PL"), is given by [7]:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline & 1 & 0 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & 0 \\ J_1 & 0 & 0 \\ \hline & \frac{1}{2}J_1 & \frac{1}{2}J_1 \end{array} \qquad \dots (2.55)$$

and has principal error constants

$$\frac{h^3}{3},\ \frac{h^3}{3},\ \frac{h^3}{36},\ \frac{5h^3}{12} \qquad \dots (2.56)$$

2.4 Stability of Stochastic Runge-Kutta Methods, [7].

Having established the procedures for determining the stability of an SDE, it is now appropriate to investigate the stability of the stochastic numerical method when applied to such a problem.

An important concept is that the absolute stability (or A-stable), where Kloeden and Platen in (1992) [22], consider the complex-valued linear test equation with just additive noise:

$$dy_t = \lambda\, y_t + dW_t,\ \ Re(\lambda) < 0.$$

The numerical update by a one-step stochastic method can then be written as:

$$y_{n+1} = R(h\,\lambda)y_n + Z_n$$

where Z_n is the random variable sampled to model the Wiener process W(t), and the ***region of absolute stability*** of the numerical scheme is defined to be:

$$S = \{h\,\lambda \in \pounds : Re(\lambda) < 0, |R(h\,\lambda)| < 1\}$$

They, then declare the numerical scheme to be A-stable if its region of absolute stability contains the entire negative half complex plane. Thus, the stability of the stochastic method is inherited from the stability of the deterministic components.

In the following analysis, the deterministic equation $y' = \lambda y$ is extended in a natural way to the following Stratonovich scalar linear test equation:

$$dy_t = a\ y_t\ dt + b\ y_t \circ dW_t \qquad \ldots (2.57)$$

this equation has multiplicative noise, and the solution $(t_0 = 0, y_0 = 1)$ is:

$$y_t = \exp(a\ t + b\ W_t)$$

At first, when back to section (2.3) and precisely to Pamela's model which is given in equation (2.27), one can recall the general form of an s-stages SRKM applied to the general scalar Stratonovich SDE:

$$dy_t = f(y_t)\,dt + g(y_t) \circ dW_t$$

yields to:

$$\left.\begin{aligned} Y &= y_n e + h A f(Y) + B J_1 g(Y) \\ y_{n+1} &= y_n + h \alpha^T f(Y) + \gamma^T J_1 g(Y) \end{aligned}\right\} \quad \dots (2.58)$$

where A, B are s×s matrices, Y is an s×1 vector of "intermediate calculations" and e is the s×1 unit vector. The s×1 vectors α, γ contain the weights of the method.

If the SRKM given in equation (2.58) are applied to the test equation given in equation (2.57), then:

$$Y = y_n e + (\boldsymbol{a} A h + \boldsymbol{b} B J_1) Y$$

$$Y - (\boldsymbol{a} A h + \boldsymbol{b} B J_1) Y = y_n e$$

which yields to:

$$Y = (I - \boldsymbol{a} A h - \boldsymbol{b} B J_1)^{-1} y_n e$$

and consequently:

$$\begin{aligned} y_{n+1} &= y_n + h \alpha^T \boldsymbol{a} Y + J_1 \gamma^T \boldsymbol{b} Y \\ &= y_n + (h \alpha^T \boldsymbol{a} + J_1 \gamma^T \boldsymbol{b}) (I - \boldsymbol{a} A h - \boldsymbol{b} B J_1)^{-1} y_n e \\ &= [1 + (h \alpha^T \boldsymbol{a} + \gamma^T J_1 \boldsymbol{b}) (I - \boldsymbol{a} A h - \boldsymbol{b} B J_1)^{-1} e] y_n \\ &= R(h, \boldsymbol{a}, \boldsymbol{b}) y_n \end{aligned} \quad \dots (2.59)$$

where the stability function R is defined to be:

$$R(h, \boldsymbol{a}, \boldsymbol{b}) = 1 + (h \alpha^T \boldsymbol{a} + \gamma^T J_1 \boldsymbol{b}) (I - \boldsymbol{a} A h - \boldsymbol{b} B J_1)^{-1} e.$$

To analyze the mean-square stability of the SRKM for the above models, it is necessary to evaluate $E(R^2(h, \boldsymbol{a}, \boldsymbol{b}))$, where a method is mean square stable if, when $Re(\boldsymbol{a}) + (Re(\boldsymbol{b}))^2 \leq 0$, then:

$$E(R^2(h, \boldsymbol{a}, \boldsymbol{b})) < 1 \quad \dots (2.60)$$

Note that this analysis is based on the stability function calculated for just one step of the SRKM. In the scalar case, and due to the independence of the Wiener increments over successive time steps, this is equivalent to the analysis over h steps. Letting R_i to denote the evaluation of the stability functional at the i^{th}-time step, then application of n steps of SRKM, gives:

$$y_{n+1} = R_{n+1} R_n \dots R_1 y_0 = R y_0$$

and so after n-time steps:

$$\mathrm{E}(R^{\mathrm{T}}R) = \mathrm{E}(R_1^T\ R_2^T\ \ldots R_{\mathrm{n}}^{\mathrm{T}}\ R_{\mathrm{n+1}}^{\mathrm{T}} R_{\mathrm{n+1}} R_{\mathrm{n}} \ldots R_1)$$

In the scalar case, this can be written as $\mathrm{E}\left(\prod_{\mathrm{i=1}}^{\mathrm{n}} (R_{\mathrm{i}}^{\mathrm{T}} R_{\mathrm{i}})\right)$, which by independence is $\prod_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{E}(R_{\mathrm{i}}^{\mathrm{T}} R_{\mathrm{i}})$, so that equation (2.60) must be hold.

Also, note that, because its mean-square convergence that is used in the stochastic case, the polynomial in h, a and b to be analyzed is of a higher degree, then its counterpart in the deterministic setting.

Now, the stability functions for various methods will be derived. In order to aid visualization of the regions of stability, the stability plots are restricted to the case when a and b are real numbers. To plot the stability regions, a change of variables will be used, namely:

$$z = \mathrm{h}\, a,\ v = \mathrm{h}\, b^{\,2},\ u = z + v \qquad \ldots (2.61)$$

and the stability region, will be:

$$S = \{(u, v) : v \geq 0,\ u \leq 0,\ \mathrm{E}(R^2) < 1\}$$

which will be plotted in the (u, v) -plane. Note that we have $u \leq 0$, since the condition $\mathrm{Re}(\mathrm{h}\, a) + \mathrm{Re}(\mathrm{h}\, b^{\,2}) \leq 0$ is necessary for the mean-square stability.

2.4.1 Stability of 2-Stages Explicit SRKM [7].

Now, backing to Pamela's model given in section (2.4), it will be assumed that in general deterministic terms A, α^{T} and general stochastic terms B and γ^{T}, as follows:

$$\mathrm{A} = \begin{pmatrix} 0 & 0 \\ \mathrm{a}_{21} & 0 \end{pmatrix},\ \alpha^{\mathrm{T}} = (\alpha_1 \quad \alpha_2)$$

$$\mathrm{B} = \begin{pmatrix} 0 & 0 \\ \mathrm{b}_{21} & 0 \end{pmatrix},\ \gamma^{\mathrm{T}} = (\gamma_1 \quad \gamma_2)$$

Moreover:

$$e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and thus satisfying those for $R(h, \boldsymbol{a}, \boldsymbol{b})$ in equation (2.59), then:

$$R(h, \boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{a}\, a_{21}\, h + \boldsymbol{b}\, b_{21}\, J_1)(\boldsymbol{a}\, \alpha_2\, h + \boldsymbol{b}\, J_1\, \gamma_2) + \boldsymbol{a}\, \alpha_1\, h + \boldsymbol{a}\, \alpha_2\, h + \boldsymbol{b}\, J_1\, \gamma_1 + \boldsymbol{b}\, J_1\, \gamma_2 + 1$$

Then [1]:

$$R^2(h, \boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{a}^2\, a_{21}^2\, \boldsymbol{b}^2 h^2\, J_1^2\, \gamma_2^2 + \boldsymbol{a}^2 \boldsymbol{b}^2\, b_{21}^2\, \alpha_2^2\, h^2\, J_1^2 + a_{21}^2\, \alpha_2^2\, \boldsymbol{a}^4\, h^4 + b_{21}^2\, \gamma_2^2\, \boldsymbol{b}^4\, J_1^4 + 4\, a_{21}\, b_{21}\, \alpha_2\, \gamma_2\, \boldsymbol{a}^2 \boldsymbol{b}^2 h^2 J_1^2 + 2\, \boldsymbol{a}\, a_{21}\, b_{21}\, h\, \gamma_2^2\, \boldsymbol{b}^3 J_1^3 + 2\, \boldsymbol{a}\, \alpha_2\, h\, \gamma_2\, b_{21}^2\, \boldsymbol{b}^3 J_1^3 + 2\, a_{21}\, \boldsymbol{b}\, b_{21}\, J_1\, \alpha_2^2\, \boldsymbol{a}^3 h^3 + 2\, \boldsymbol{b}\, \alpha_2\, J_1\, \gamma_2\, a_{21}^2\, \boldsymbol{a}^3 h^3 + 2\, \boldsymbol{a}\, a_{21}\, h\, \gamma_1\, \gamma_2\, \boldsymbol{b}^2 J_1^2 + 2\, \boldsymbol{a}\, b_{21}\, \alpha_1\, h\, \gamma_2\, \boldsymbol{b}^2 J_1^2 + 2\, \boldsymbol{a}\, b_{21}\, \alpha_2\, h\, \gamma_1\, \boldsymbol{b}^2 J_1^2 + 2\, a_{21}\, \boldsymbol{b}\, \alpha_1\, J_1\, \gamma_2\, \boldsymbol{a}^2 h^2 + 2\, a_{21}\, \boldsymbol{b}\, \alpha_2\, J_1\, \gamma_1\, \boldsymbol{a}^2 h^2 + 2\, \boldsymbol{b}\, b_{21}\, \alpha_1\, \alpha_2\, J_1\, \boldsymbol{a}^2 h^2 + 4\, \boldsymbol{a}\, b_{21}\, \alpha_2\, h\, \gamma_2\, \boldsymbol{b}^2 J_1^2 + 4\, a_{21}\, \boldsymbol{b}\, \alpha_2\, J_1\, \gamma_2\, \boldsymbol{a}^2 h^2 + 2\, \boldsymbol{a}\, a_{21}\, h\, \boldsymbol{b}^2\, J_1^2\, \gamma_2^2 + 2\, \boldsymbol{b}\, b_{21}\, J_1\, \boldsymbol{a}^2\, \alpha_2^2\, h^2 + 2\, a_{21}\, \alpha_1\, \alpha_2\, \boldsymbol{a}^3 h^3 + 2\, b_{21}\, \gamma_1\, \gamma_2\, \boldsymbol{b}^3 J_1^3 + 2\, a_{21}\, \alpha_2^2\, \boldsymbol{a}^3 h^3 + 2\, b_{21}\, \gamma_2^2\, \boldsymbol{b}^3\, J_1^3 + \boldsymbol{a}^2\, \alpha_1^2\, h^2 + \boldsymbol{a}^2\, \alpha_2^2\, h^2 + \boldsymbol{b}^2 J_1^2\, \gamma_1^2 + \boldsymbol{b}^2\, J_1^2\, \gamma_2^2 + 2\, \boldsymbol{a}\, a_{21}\, \boldsymbol{b}\, h\, J_1\, \gamma_2 + 2\, \boldsymbol{a}\, \boldsymbol{b}\, b_{21}\, \alpha_2\, h\, J_1 + 2\, \boldsymbol{a}\, \boldsymbol{b}\, \alpha_1\, h\, J_1\, \gamma_1 + 2\, \boldsymbol{a}\, \boldsymbol{b}\, \alpha_1\, h\, J_1\, \gamma_2 + 2\, \boldsymbol{a}\, \boldsymbol{b}\, \alpha_2\, h\, J_1\, \gamma_1 + 2\, \boldsymbol{a}\, \boldsymbol{b}\, \alpha_2\, h\, J_1\, \gamma_2 + 2\, a_{21}\, \alpha_2\, \boldsymbol{a}^2 h^2 + 2\, b_{21}\, \gamma_2\, \boldsymbol{b}^2\, J_1^2 + 2\, \alpha_1\, \alpha_2\, \boldsymbol{a}^2 h^2 + 2\, \gamma_1\, \gamma_2\, \boldsymbol{b}^2\, J_1^2 + 2\, \boldsymbol{a}\, \alpha_1\, h + 2\, \boldsymbol{a}\, \alpha_2\, h + 2\, \boldsymbol{b}\, J_1\, \gamma_1 + 2\, \boldsymbol{b}\, J_1\, \gamma_2 + 1$$

Now, recall that $E(J_1) = 0$, $E(J_1^2) = h$, $E(J_1^3) = 0$ and $E(J_1^4) = 3\, h^2$, then the expectation of $R^2(h, \boldsymbol{a}, \boldsymbol{b})$ is:

$$E(R^2(h, \boldsymbol{a}, \boldsymbol{b})) = a_{21}^2\, \alpha_2^2\, \boldsymbol{a}^4 h^4 + \boldsymbol{a}^2\, a_{21}^2\, \boldsymbol{b}^2\, \gamma_2^2\, h^3 + \boldsymbol{a}^2 \boldsymbol{b}^2\, b_{21}^2\, \alpha_2^2\, h^3 + 4\, a_{21}\, b_{21}\, \alpha_2\, \gamma_2\, \boldsymbol{a}^2 \boldsymbol{b}^2 h^3 + 3\, b_{21}^2\, h^2\, \gamma_2^2\, \boldsymbol{b}^4 + 2\, a_{21}\, \alpha_1\, \alpha_2\, \boldsymbol{a}^3 h^3 + 2\, a_{21}\, \alpha_2^2\, \boldsymbol{a}^3 h^3 + 2\, \boldsymbol{a}\, a_{21}\, \gamma_1\, \gamma_2\, \boldsymbol{b}^2 h^2 + 2\, \boldsymbol{a}\, b_{21}\, \alpha_1\, \gamma_2\, \boldsymbol{b}^2 h^2 + 2\, \boldsymbol{a}\, b_{21}\, \alpha_2\, \gamma_1\, \boldsymbol{b}^2 h^2 + 4\, \boldsymbol{a}\, b_{21}\, \alpha_2\, \gamma_2\, \boldsymbol{b}^2 h^2 + 2\, \boldsymbol{a}\, a_{21}\, \boldsymbol{b}^2 h^2\, \gamma_2^2 + \boldsymbol{a}^2\, \alpha_1^2\, h^2 + \boldsymbol{a}^2\, \alpha_2^2\, h^2 + 2\, a_{21}\, \alpha_2$$

$$a^2 h^2 + 2\,\alpha_1 \alpha_2\, a^2 h^2 + h\, b^2\, \gamma_1^2 + h\, b^2\, \gamma_2^2 + 2\, b_{21}\, h\, \gamma_2\, b^2 + 2\, h\, \gamma_1 \gamma_2\, b^2 + 2\, a\, \alpha_1 h + 2\, a\, \alpha_2 h + 1 \quad \ldots (2.62)$$

and when using equation (2.61) in equation (2.62), i.e., substituting $a\,h = z$, $b^2 h = v$, $z = u - v$, then we shall obtain:

$$\begin{aligned} E(R^2(u, v)) &= a_{21}^2 \alpha_2^2 z^4 + a_{21}^2 \gamma_2^2 z^2 v + b_{21}^2 \alpha_2^2 z^2 v + 4 a_{21} b_{21} \alpha_2 \gamma_2 z^2 v + 3 b_{21}^2 \gamma_2^2 v^2 + 2 a_{21} \alpha_1 \alpha_2 z^3 + 2 a_{21} \alpha_2^2 z^3 + 2 a_{21} \gamma_1 \gamma_2 z v + 2 b_{21} \alpha_1 \gamma_2 z v + 2 b_{21} \alpha_2 \gamma_1 z v + 4 b_{21} \alpha_2 \gamma_2 z v + 2 a_{21} \gamma_2^2 z v + \alpha_1^2 z^2 + \alpha_2^2 z^2 + 2 a_{21} \alpha_2 z^2 + 2 \alpha_1 \alpha_2 z^2 + \gamma_1^2 v + \gamma_2^2 v + 2 b_{21} \gamma_2 v + 2 \gamma_1 \gamma_2 v + 2 \alpha_1 z + 1. \\ &= a_{21}^2 \alpha_2^2 (u - v)^4 + a_{21}^2 \gamma_2^2 (u - v)^2 v + b_{21}^2 \alpha_2^2 (u - v) v + 4 a_{21} b_{21} \alpha_2 \gamma_2 (u - v)^2 v + 2 a_{21} \alpha_1 \alpha_2 (u - v)^3 + 2 a_{21} \alpha_2^2 (u - v)^3 + \alpha_1^2 (u - v)^2 + \alpha_2^2 (u - v)^2 + 3 b_{21}^2 \gamma_2^2 v^2 + 2 a_{21} \gamma_1 \gamma_2 (u - v) v + 2 b_{21} \alpha_1 \gamma_2 (u - v) v + 2 b_{21} \alpha_2 \gamma_1 (u - v) v + 4 b_{21} \alpha_2 \gamma_2 (u - v) v + 2 a_{21} \gamma_2^2 (u - v) v + 2 a_{21} \alpha_2 (u - v)^2 + 2 \alpha_1 \alpha_2 (u - v)^2 + \gamma_1^2 v + \gamma_2^2 v + 2 b_{21} \gamma_2 v + 2 \gamma_1 \gamma_2 v + 2 \alpha_1 (u - v) + 2 \alpha_2 (u - v) + 1 \quad \ldots (2.63) \end{aligned}$$

Now, to solve equation (2.63), we shall take $v = 0$ to obtain a function of u only (i.e., to obtain the u-intercept), where the stability regions S, will be:

$$R_1 = E(R^2(u, 0)) = a_{21}^2 \alpha_2^2 u^4 + 2 a_{21} \alpha_1 \alpha_2 u^3 + 2 a_{21} \alpha_2^2 u^3 + \alpha_1^2 u^2 + \alpha_2^2 u^2 + 2 a_{21} \alpha_2 u^2 + 2 \alpha_1 \alpha_2 u^2 + 2 \alpha_1 u + 2 \alpha_2 u + 1 \quad \ldots (2.64)$$

Also, in order to find the v-intercept, take $u = 0$ in equation (2.63), to obtain a function of v only, which is:

$$R_2 = E(R^2(0, v)) = a_{21}^2 \alpha_2^2 v^4 + a_{21}^2 \gamma_2^2 v^3 + b_{21}^2 \alpha_2^2 v^3 + 4 a_{21} b_{21} \alpha_2 \gamma_2 v^3 + 3 b_{21}^2 \gamma_2^2 v^2 + 2 a_{21} \alpha_1 \alpha_2 v^3 - 2 a_{21} \gamma_1 \gamma_2 v^2 - 2 b_{21} \alpha_1 \gamma_2 v^2 - 2 b_{21} \alpha_2 \gamma_1 v^2 -$$

$$4b_{21}\alpha_2\gamma_2 v^2 - 2a_{21}\gamma_2^2 v^2 + 2a_{21}\alpha_2 v^2 + 2\alpha_1\alpha_2 v^2 + \gamma_1^2 v + \gamma_2^2 v + 2b_{21} v + 2\gamma_1\gamma_2 v - 2\alpha_1 v - 2\alpha_2 v + 1 \quad \dots (2.65)$$

Applying PL-model given by equation (2.55) in equation (2.64) and equation (2.65), i.e., when $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\alpha^T = (1 \quad 0)$, $\gamma^T = (\frac{1}{2} \quad \frac{1}{2})$, yields to the stability subregions:

$R_1 = u^2 + 2u + 1 < 1$, and thus $v = 0$, $u \geq -2$

$R_2 = \frac{v^3}{4} - \frac{v^2}{4} + 1 < 1$, and thus $u = 0$, $v \leq 1$

Thus, the stability region S obtained from $R_1 \cap R_2$ related to this model is given in Fig (2.3).

Now, applying R2-model given by equation (2.54) in equation (2.64) and equation (2.65), i.e., with $A = \begin{pmatrix} 0 & 0 \\ 2/3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 2/3 & 0 \end{pmatrix}$, $\alpha^T = (\frac{1}{4} \quad \frac{3}{4})$, $\gamma^T = (\frac{1}{4} \quad \frac{3}{4})$, then the stability subregions are:

$R_1 = \frac{u^4}{4} + u^3 + 2u^2 + 2u + 1 < 1$, and thus $v = 0$, $u \geq -2$

$R_2 = \frac{v^4}{4} + \frac{v^3}{2} + \frac{v^2}{2} + 1 < 1$, and thus $u = 0$, $v \leq \sqrt{2} - 1$

Then, the stability region S obtained from $R_1 \cap R_2$ related to R2-model are given in Fig.(2.4)

Now, the next two figures represent the stability regions of R2 and PL-models respectively, such that, in this figures the function f refers to R_1 and g to R_2, while x refers either for u or v.

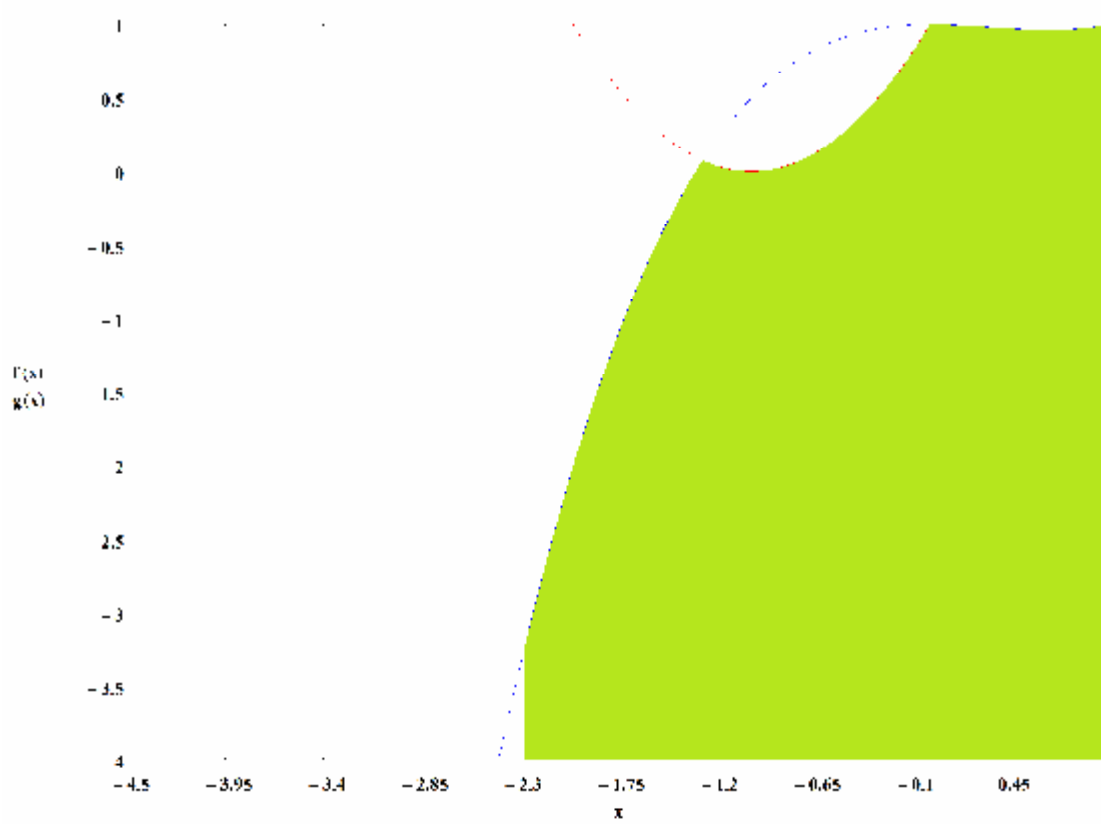

Figure (2.3) Stability region of PL-Model

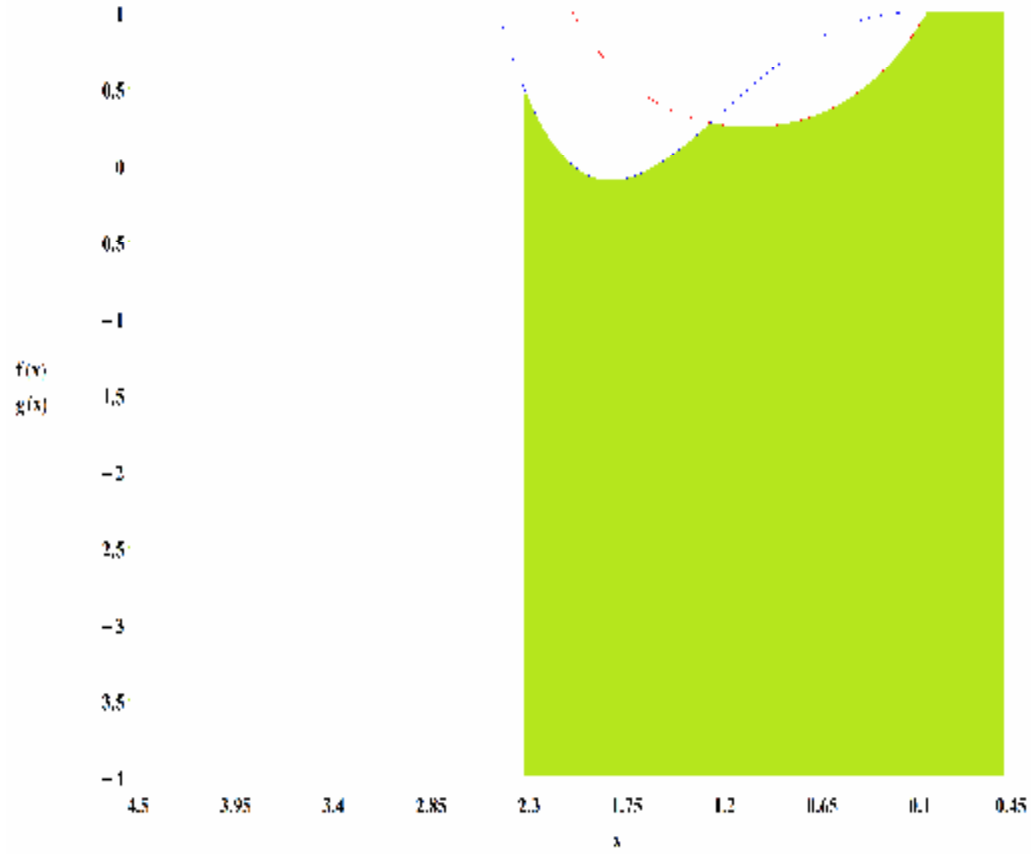

Figure (2.4) Stability region of R2-Model.

Chapter Three

3

Semi-Explicit, Implicit and Mixed Stochastic Runge-Kutta Methods

Chapter three

Semi-Explicit, Implicit and Mixed stochastic Runge-Kutta Methods

Introduction:

In this chapter, we will discuss in details two-stages semi-explicit and implicit stochastic Runge-Kutta Methods of strong order 1, then mixing these methods to obtain what is called the mixed stochastic Runge-Kutta methods. All of the presented methods are presented with minimum principal error coefficients. Also, numerical results are presented in order to compare between the convergence properties of the considered stochastic Runge-Kutta methods.

It is remarkable that the stochastic differential equation which will be considered here is given by:

$$dy_t = f(y_t)dt + g(y_t) \circ dW_t,\ y_{t_0} = y_0 \qquad \dots (3.1)$$

where $t \in [t_0,T]$, $y_t \in R^m$ and W_t is the Wiener process whose increment $\Delta W(t) = W_{t+\Delta t} - W_t$ is a Gaussian random variable with mean 0 and variance Δt. It is assumed that equation (3.1) is autonomous in order to simplify notations.

Runge–Kutta methods are one of the most efficient classes of methods used for solving ordinary differential equations (ODEs). Runge-Kutta methods resemble their structure in the discretization methods for ODEs of that name. Much work has been made on designing stochastic Runge–Kutta methods in recent years; see, for example, [4, 5, 6, 7, 23, 25, 29].

This chapter consist five sections. In section 3.1 and section 3.2 the attentions was paid toward finding the formulations of semi-explicit, implicit and mixing the explicit, semi explicit and implicit stochastic Runge-Kutta methods to obtain the so called mixed stochastic Rung-Kutta methods that is abbreviated by MSRKM. In section 3.3 the stability of the derived SRKM's will be studied, in section 3.4 some illustrative examples which are solved

numerically are considered. Finally, in section 3.5 the Variable Step Size method have been introduced for the first time for solving SRKM's in order to improve the accuracy of the obtained methods.

3.1 Derivation of Semi Explicit and Implicit Methods:

In this section the derivation and the formulations of 2-stages semi explicit and implicit Runge-Kutta methods will be considered, as well as, studying the principal errors of derived methods.

For $s=2, i=1,2$ ، $n=0.1,2,...$, equation (2.27) may be written as :

$$\left.\begin{aligned} Y_i &= y_n + h\sum_{j=1}^{2} a_{ij} f(Y_j) + J_1 \sum_{j=1}^{2} b_{ij} g(Y_j), i=1,2 \\ y_{n+1} &= y_n + h\sum_{j=1}^{2} \alpha_j f(Y_j) + J_1 \sum_{j=1}^{2} \gamma_j g(Y_j) \end{aligned}\right\} \quad \dots (3.2)$$

suppose that $A=(a_{ij})=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B=(b_{ij})=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, $\gamma^T=[\gamma_1 \quad \gamma_2]$, and $\alpha^T=[\alpha_1 \quad \alpha_2]$, when a_{ij}, b_{ij}, α_j and γ_j which are constants to be evaluated with $i,j=1,2$, $h=t_{n+1}-t_n$, $J_1=\Delta W_n=W_{t_{n+1}}-W_{t_n}$, then equation (3.2) for all $n=0.1,2,...$ becomes:

$$\left.\begin{aligned} Y_1 &= y_n + h\,[a_{11}f(Y_1)+a_{12}f(Y_2)] + J_1[b_{11}g(Y_1)+b_{12}g(Y_2)] \\ Y_2 &= y_n + h[a_{21}f(Y_1)+a_{22}f(Y_2)] + J_1[b_{21}g(Y_1)+b_{22}g(Y_2)] \\ y_{n+1} &= y_n + h[\alpha_1\, f(Y_1)+\alpha_2\, f(Y_2)] + J_1[\gamma_1\, g(Y_1)+\gamma_2\, g(Y_2)] \end{aligned}\right\} \quad \dots (3.3)$$

By the same process in deriving the explicit method that is given in chapter two, we can construct the next cases:

Remark (3.1):

It is easy to classify the SRKM's from it's matrices A and B that are given above as follows:

1. If $a_{ij}=b_{ij}=0; \forall\, i<j$ when $i,j=1,2$, then the method is called semi explicit SRKM.
2. If $a_{ij}=b_{ij}=0; \forall\, i\le j$ when $i,j=1,2$, then the method is called explicit SRKM, otherwise it is called implicit SRKM.

In this chapter, a mixture between these matrices will be made in order to get a new SRKM's which are called Mixed Stochastic Runge-Kutta Methods (or breviate by MSRKM) that considered in section (3.2).

Now, consider the following two cases:

Case 1: For 2- stages semi-explicit methods, then the matrices A, B, α and γ will be given as:

$$A=(a_{ij})=\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix},\ B=(b_{ij})=\begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix} \text{ and } \gamma^T=[\gamma_1 \quad \gamma_2],$$

$$\alpha^T=[\alpha_1 \quad \alpha_2].$$

Then equations (3.3) will take the form:

$$\left.\begin{aligned} Y_1 &= y_n + h\,a_{11}f(Y_1) + J_1\,b_{11}\,g(Y_1) \\ Y_2 &= y_n + h[a_{21}f(Y_1) + a_{22}f(Y_2)] + J_1[b_{21}g(Y_1)+b_{22}g(Y_2)] \\ y_{n+1} &= y_n + h[\alpha_1\ f(Y_1) + \alpha_2\ f(Y_2)] + J_1[\gamma_1\ g(Y_1) + \gamma_2\ g(Y_2)] \end{aligned}\right\} \quad \dots (3.4)$$

Now; from equations (2.43), (2.44), (2.46) and (2.51) the following system of nonlinear algebraic equations related to equation (3.4) may be derived:

$$\left.\begin{aligned} &\alpha_1+\alpha_2=1, \gamma_1+\gamma_2=1 \\ &\gamma_1 b_{11}+\gamma_2(b_{21}+b_{22})=\frac{1}{2} \\ &\alpha_1 b_{11}+\alpha_2(b_{21}+b_{22})=\frac{1}{2} \\ &\gamma_1 a_{11}+\gamma_2(a_{21}+a_{22})=\frac{1}{2} \\ &\gamma_1 (b_{11})^2+\gamma_2 b_{11}(b_{21}+b_{22})=\frac{1}{3} \\ &\gamma_2 (b_{21}+b_{22})^2+\gamma_1 b_{11}(b_{21}+b_{22})=\frac{1}{3} \\ &\gamma_1 b^2{}_{11}+\gamma_2(b_{11}b_{21}+b_{21}b_{22}+b_{22}{}^2)=\frac{1}{6} \end{aligned}\right\} \quad \dots (3.5)$$

Thus, by solving the above system, the following non unique results will be obtained:

$$\alpha_1=0.1233, \alpha_2=0.8639, \gamma_1=0.125, \gamma_2=0.8605, a_{11}=-0.5007, a_{21}=0.2454,$$

$$a_{22}=0.435,\ b_{11}=0.4936, b_{21}=0.3513, b_{22}=0.1802,$$

or may be written in tabulated from as:

$$\begin{array}{c|cc} -0.5007 & 0 \\ 0.2454 & 0.435 \\ \hline 0.1233 & 0.8639 \end{array} \qquad \begin{array}{c|cc} 0.4936J_1 & 0 \\ 0.3513J_1 & 0.1802J_1 \\ \hline 0.125J_1 & 0.8605J_1 \end{array} \qquad \ldots (3.6)$$

Now; equation (3.4) becomes:

$$\left.\begin{aligned} Y_1 &= y_n - 0.5007\,h\,f(Y_1) + 0.4936\,J_1 g(Y_1) \\ Y_2 &= y_n + h[0.2454\,f(Y_1) + 0.435\,f(Y_2)] + J_1[0.3513\,g(Y_1) \\ &\quad +0.1802\,g(Y_2)] \\ y_{n+1} &= y_n + h[0.1233\,f(Y_1) + 0.8639\,f(Y_2)] + J_1[0.125\,g(Y_1) \\ &\quad +0.8605\,g(Y_2)] \end{aligned}\right\} \qquad \ldots (3.7)$$

which represent the semi-explicit SRKM.

In the presented case, the respective minimum given in equations (2.47), (2.48), (2.49) and (2.50) are:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, 0 \qquad \ldots (3.8)$$

and for a 2-stages semi-explicit method the last equation of system (2.51) becomes:

$$\gamma^T B\, b = (\gamma_1, \gamma_2) \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} + b_{22} \end{pmatrix} = 0.2621$$

hence equation (2.50) will take the form:

$$\left(\frac{1}{36} - \frac{1}{3}\gamma^T B b + (\gamma^T B b)^2\right) 15h^3 = 0.1366\, h^3$$

and so in this case, the principal error constants are:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, 0.1366\, h^3 \qquad \ldots (3.9)$$

Case 2: For 2- stages implicit method, the matrices A and B will be given as:

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = (b_{ij}) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \text{ and}$$

$\gamma^T = [\gamma_1 \quad \gamma_2]$, $\alpha^T = [\alpha_1 \quad \alpha_2]$, yields to a set of equations which is identical with equation (3.3) and from the equations (2.43), (2.44), (2.46) and (2.51) in this case we get:

$$\left.\begin{aligned}
&\alpha_1+\alpha_2=1,\ \gamma_1+\gamma_2=1\\
&\gamma_1(b_{11}+b_{12})+\gamma_2(b_{21}+b_{22})=\frac{1}{2}\\
&\alpha_1(b_{11}+b_{12})+\alpha_2(b_{21}+b_{22})=\frac{1}{2}\\
&\gamma_1(a_{11}+a_{12})+\gamma_2(a_{21}+a_{22})=\frac{1}{2}\\
&\gamma_1(b_{11}+b_{12})^2+\gamma_2(b_{11}+b_{12})(b_{21}+b_{22})=\frac{1}{3}\\
&\gamma_2(b_{21}+b_{22})^2+\gamma_1(b_{11}+b_{12})(b_{21}+b_{22})=\frac{1}{3}\\
&\gamma_1(b^2{}_{11}+b_{12}b_{21}+b_{11}b_{12}+b_{12}b_{22})+\gamma_2(b_{11}b_{21}\\
&\qquad +b_{21}b_{22}+b_{12}b_{21}+b_{22}{}^2)=\frac{1}{6}
\end{aligned}\right\} \quad \text{.. (3.10)}$$

Thus, by solving the above system of nonlinear equations we get the following results:

$\alpha_1=0.2222,\alpha_2=0.7591,\gamma_1=0.0569,\gamma_2=0.9434,a_{11}=-0.2495,a_{12}=0.05,$
$a_{21}=0.2127,,a_{22}=0.3293,b_{11}=0.4115,b_{12}=0.1861,\ b_{21}=-1.8567,$
$b_{22}=2.387,$

which may be written in tabulated from as:

$$\begin{array}{cc|cc}
-0.2495 & 0.05 & 0.4115J_1 & 0.1861J_1\\
0.2127 & 0.3293 & -1.8567J_1 & 2.387J_1\\
\hline
0.2222 & 0.7591 & 0.0569J_1 & 0.9434J_1
\end{array} \quad \text{… (3.11)}$$

and hence equation (3.3) becomes :

$$\left.\begin{aligned}
Y_1&=y_n+h[-0.2495\ f(Y_1)+0.05\ f(Y_2)]+J_1[0.4115\ g(Y_1)\\
&\quad +0.1861\ g(Y_2)]\\
Y_1&=y_n+h[0.2127\ f(Y_1)+0.3293\ f(Y_2)]+J_1[-1.8567\ g(Y_1)\\
&\quad +2.387\ g(Y_2)]\\
y_{n+1}&=y_n+h[0.2222\ f(Y_1)+0.7591\ f(Y_2)]+J_1[0.0569\ g(Y_1)\\
&\quad +0.9434\ g(Y_2)]
\end{aligned}\right\} \text{… (3.12)}$$

Equations (3.12) represent the implicit SRKM.

In this case, the respective minimum given in equations (2.47), (2.48), (2.49) and (2.50) are:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, 0 \qquad \dots (3.13)$$

For a 2-stage implicit method the last equation of equations (2.51) becomes:

$$\gamma^T B\, b = (\gamma_1, \gamma_2) \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} b_{11}+b_{12} \\ b_{21}+b_{22} \end{pmatrix} = 0.16703$$

and hence equation (2.50) will take the form:

$$\left(\frac{1}{36} - \frac{1}{3}\gamma^T B b + (\gamma^T B b)^2\right) 15h^3 = 1.93197 \times 10^{-7} h^3$$

and so in this case, the principal error constants are:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, 1.93197 \times 10^{-7} h^3 \qquad \dots (3.14)$$

3.2 Mixed Stochastic Runge-Kutta Methods:

In these methods the deterministic and stochastic results are mixed in the cases of explicit, semi explicit and implicit methods in order to obtain new schemes for solving SODE's using SRKM's, and as follows :

Case1: MSRKM1-Semi-Explicit - Explicit:

In this case, consider the semi-explicit form for the deterministic part and explicit form for the stochastic part, i.e. /

$$A = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ b_{21} & 0 \end{bmatrix}, \; \gamma^T = \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix}, \; \alpha^T = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix}$$

and similarly by the same process followed and described previously, one may get:

$\alpha_1 = 0.2389, \alpha_2 = 0.7602, \gamma_1 = 0.2385, \gamma_2 = 0.761, b_{21} = 0.6594, a_{11} = -0.4435,$

$a_{21} = 0.2727, a_{22} = 0.5236,$

which may be written in tabulated from as:

-0.4435	0		0	0
0.2727	0.5236		$0.6594\,J_1$	0
0.2389	0.7602		$0.2385J_1$	$0.761J_1$

… (3.15)

Then equations (3.3) will take the form:

$$\left.\begin{aligned} Y_1 &= y_n - 0.4435\ h \\ Y_2 &= y_n + h[0.2727\ f(Y_1) + 0.5236\ f(Y_2)] + 0.6594\ J_1\, g(Y_1) \\ y_{n+1} &= y_n + h[0.2389\ f(Y_1) + 0.7602\ f(Y_2)] + J_1[0.2385\ g(Y_1) \\ &\quad + 0.761\ g(Y_2)] \end{aligned}\right\} \quad \ldots (3.16)$$

Equations (3.16) represent the MSRKM1- semi-explicit - explicit.

Also in this case, the respective minimum given in equations (2.47), (2.48), (2.49) and (2.50) are:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, 0 \quad \ldots (3.17)$$

and for a 2-stage MSRKM1-semi-explicit – explicit, the last equation for equations (2.51) becomes:

$$\gamma^T B\, b = (\gamma_1, \gamma_2)\begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ b_{21} \end{pmatrix} = 0$$

then equation (2.50) will be:

$$\left(\frac{1}{36} - \frac{1}{3}\gamma^T Bb + (\gamma^T Bb)^2\right)15h^3 = \frac{5h^3}{12}$$

and so in this case, the principal error constants are:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, \frac{5h^3}{12} \quad \ldots (3.18)$$

<u>Case 2:</u> MSRKM2-Implicit - Explicit:

In this case, consider the implicit form for the deterministic part and explicit form for the stochastic part, i.e. /

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ b_{21} & 0 \end{bmatrix},\ \gamma^T = [\gamma_1 \quad \gamma_2],\ \alpha^T = [\alpha_1 \quad \alpha_2],$$

and similarly by the same process followed and described previously, one may get:

$\alpha_1 = 0.2513, \alpha_2 = 0.749, \gamma_1 = 0.2504, \gamma_2 = 0.7496, a_{11} = -0.3876, a_{12} = 0.0253,$

$a_{21} = 0.5527\ a_{22} = 0.2357, b_{21} = 0.6671,$

which may be written in tabulated from as:

$$\begin{array}{cc|cc} -0.3876 & 0.0253 & 0 & 0 \\ 0.5527 & 0.2357 & 0.6671J_1 & 0 \\ \hline 0.2513 & 0.749 & 0.2504J_1 & 0.7496J_1 \end{array} \quad \dots (3.19)$$

Then equations (3.3) will take form:

$$\left.\begin{aligned} Y_1 &= y_n + h\,[-0.3876\, f(Y_1) + 0.0253\, f(Y_2)] \\ Y_2 &= y_n + h\,[0.5527\, f(Y_1) + 0.2357\, f(Y_2)] + 0.6671\, J_1\, g(Y_1) \\ y_{n+1} &= y_n + h\,[0.2513\, f(Y_1) + 0.749\, f(Y_2)] + J_1\,[0.2504\, g(Y_1) \\ &\quad + 0.7496\, g(Y_2)] \end{aligned}\right\} \dots (3.20)$$

Equations (3.20) represent the MSRKM2- implicit - explicit.

Also in this case, the respective minimum given in equations (2.47), (2.48), (2.49) and (2.50) becomes:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, 0 \quad \dots (3.21)$$

and for a 2-stage MSRKM2- implicit - explicit, the last equation for equations (2.51) will be:

$$\gamma^T B\, b = (\gamma_1, \gamma_2)\begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ b_{21} \end{pmatrix} = 0$$

then equation (2.50) become:

$$\left(\frac{1}{36} - \frac{1}{3}\gamma^T Bb + (\gamma^T Bb)^2\right) 15h^3 = \frac{5h^3}{12}$$

and so in this case, the principal error constants are:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, \frac{5h^3}{12} \quad \dots (3.22)$$

<u>Case 3:</u> MSRKM3-Explicit – Semi-Explicit.

In this case, consider the explicit form for the deterministic part and semi-explicit form for the stochastic part, i.e. /

$$A = \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix},\ B = \begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix},\ \gamma^T = [\gamma_1 \quad \gamma_2],\ \alpha^T = [\alpha_1 \quad \alpha_2]$$

and similarly by the same process followed and described previously, one may get:

$\alpha_1 = 0.4583, \alpha_2 = 0.5345, \gamma_1 = 0.4604, \gamma_2 = 0.5363, a_{21} = 0.9105, b_{11} = 0.5638,$
$b_{21} = -0.2, b_{22} = 0.6763,$

which may be written in tabulated from as:

$$\begin{array}{c|cc} 0 & 0 & \\ 0.9105 & 0 & \\ \hline & 0.4583 & 0.5345 \end{array} \quad \begin{array}{c|cc} 0.5638J_1 & 0 & \\ -0.2\,J_1 & 0.6763J_1 & \\ \hline & 0.4604J_1 & 0.5363J_1 \end{array} \qquad \ldots (3.23)$$

Then equation (3.3) will take the form:

$$\left.\begin{aligned} Y_1 &= y_n + 0.5638\, J_1\, g(Y_1) \\ Y_2 &= y_n + 0.9105\, h\, f(Y_1) + J_1[-0.2\, g(Y_1) + 0.6763\, g(Y_2)] \\ y_{n+1} &= y_n + h\,[0.4583\, f(Y_1) + 0.5345\, f(Y_2)] + J_1[0.4604\, g(Y_1) \\ &\quad + 0.5363\, g(Y_2)] \end{aligned}\right\} \ldots (3.24)$$

Equations (3.24) represent the MSRKM3- explicit – semi-explicit.

Also in this case, the respective minimum given in equations (2.47), (2.48), (2.49) and (2.50) becomes:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, 0 \qquad \ldots (3.25)$$

and for a 2-stage MSRKM3- explicit - semi-explicit, the last equation for equations (2.51) will be:

$$\gamma^T B\, b = (\gamma_1, \gamma_2)\begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} b_{11} \\ b_{21} + b_{22} \end{pmatrix} = 0.2586$$

then equation (2.50) becomes:

$$\left(\frac{1}{36} - \frac{1}{3}\gamma^T B b + (\gamma^T B b)^2\right) 15h^3 = 0.1269\, h^3$$

and so in this case, the principal error constants are:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, 0.1269\, h^3 \qquad \ldots (3.26)$$

<u>Case 4:</u> MSRKM4-Implicit - Semi-Explicit.

In this case, consider the implicit form for the deterministic part and semi-explicit form for the stochastic part, i.e. /

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ B = \begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix},\ \gamma^T = [\gamma_1 \quad \gamma_2],\ \alpha^T = [\alpha_1 \quad \alpha_2]$$

and similarly by the same process followed and described previously, one may get:

$\alpha_1 = 0.0968, \alpha_2 = 0.8907, \gamma_1 = 0.1676, \gamma_2 = 0.8277, a_{11} = 0.6529, a_{12} = 0.0965,$

$a_{21} = 0.201,\ a_{22} = 0.2528, b_{11} = 0.628, b_{21} = -1.2303, b_{22} = 1.7462,$

which may be written in tabulated from as:

$$\begin{array}{c|cc|cc} & 0.6529 & 0.0965 & 0.628J_1 & 0 \\ & 0.201 & 0.2528 & -1.2303J_1 & 1.7462J_1 \\ \hline & 0.0968 & 0.8907 & 0.1676J_1 & 0.8277J_1 \end{array} \quad \ldots (3.27)$$

Then equations (3.3) will take the form:

$$\left.\begin{aligned} Y_1 &= y_n + h[0.6529\ f(Y_1) + 0.0965\ f(Y_2)] + 0.628\ J_1\ g(Y_1) \\ Y_2 &= y_n + h[0.201\ f(Y_1) + 0.2528\ f(Y_2)] + J_1[-1.2303\ g(Y_1) \\ &\quad + 1.7462\ g(Y_2)] \\ y_{n+1} &= y_n + h[0.0968\ f(Y_1) + 0.8907\ f(Y_2)] + J_1[0.1676\ g(Y_1) \\ &\quad + 0.8277\ g(Y_2)] \end{aligned}\right\} \quad \ldots (3.28)$$

Equations (3.28) represent the MSRKM4- implicit - semi-explicit**.**

Also in this case, the respective minimum given in equations (2.47), (2.48), (2.49) and (2.50) becomes:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, 0 \quad \ldots (3.29)$$

and for a 2-stage MSRKM4- implicit – semi-explicit, the last equation for equations (2.51) will be:

$$\gamma^T B\, b = (\gamma_1, \gamma_2)\begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} b_{11} \\ b_{21} + b_{22} \end{pmatrix} = 0.1722$$

then equation (2.50) becomes:

$$\left(\frac{1}{36} - \frac{1}{3}\gamma^T Bb + (\gamma^T Bb)^2\right)15h^3 = 4.659\times10^{-4}h^3$$

and so in this case, the principal error constants are:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, 4.659\times10^{-4}h^3 \quad \ldots (3.30)$$

Case 5: MSRKM5-Explicit- Implicit:

In this case, consider the explicit form for the deterministic part and implicit form for the stochastic part, i.e. /

$$A=\begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix},\ B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},\ \gamma^T=[\gamma_1 \quad \gamma_2],\ \alpha^T=[\alpha_1 \quad \alpha_2],$$

and similarly by the same process followed and described previously, one may get:

$$\alpha_1=0.1223, \alpha_2=0.8628, \gamma_1=0.1223, \gamma_2=0.8758, a_{21}=0.5709, b_{11}=0.4864,$$

$$b_{12}=0.1228, b_{21}=-1.6469, b_{22}=2.1729,$$

which may be written in tabulated from as:

$$\begin{array}{c|cc} 0 & 0 & \\ 0.5709 & 0 & \\ \hline & 0.1223 & 0.8628 \end{array} \qquad \begin{array}{c|cc} & 0.4864J_1 & 0.1228J_1 \\ & -1.6469J_1 & 2.1729J_1 \\ \hline & 0.1223J_1 & 0.8758J_1 \end{array} \qquad \dots (3.31)$$

Then equations (3.3) will take the form:

$$\left.\begin{aligned} Y_1 &= y_n + J_1[0.4864\ g(Y_1)+0.1228\ g(Y_2)] \\ Y_2 &= y_n + 0.5709\ h\ f(Y_1) + J_1[-1.6469\ g(Y_1) \\ &\quad +2.1729\ g(Y_2)] \\ y_{n+1} &= y_n + h[0.1223\ f(Y_1) + 0.8628\ f(Y_2)] + J_1[0.1223\ g(Y_1) \\ &\quad + 0.8758\ g(Y_2)] \end{aligned}\right\} \dots (3.32)$$

Equations (3.32) represent the MSRKM5- explicit - implicit.

Also in this case, the respective minimum given in equations (2.47), (2.48), (2.49) and (2.50) are:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, 0 \qquad \dots (3.33)$$

and for a 2-stage MSRKM5-explicit - implicit, the last equation for equations (2.51) becomes:

$$\gamma^T B\, b = (\gamma_1, \gamma_2)\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} b_{11}+b_{12} \\ b_{21}+b_{22} \end{pmatrix} = 0.1664$$

then equation (2.50) will be:

$$\left(\frac{1}{36}-\frac{1}{3}\gamma^T Bb+(\gamma^T Bb)^2\right)15h^3 = 7.178\times10^{-7}h^3$$

and so in this case, the principal error constants are:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, 7.178\times10^{-7} h^3 \quad \ldots (3.34)$$

***Case 6:* MSRKM6-Semi-Explicit - Implicit:**

In this case, consider the semi-explicit form for the deterministic part and implicit form for the stochastic part, i.e. /

$$A=\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix},\ B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},\ \gamma^T=[\gamma_1 \quad \gamma_2],\ \alpha^T=[\alpha_1 \quad \alpha_2],$$

and similarly by the same process followed and described previously, one may get:

$\alpha_1=0.1253, \alpha_2=0.8571, \gamma_1=0.1134, \gamma_2=0.8553, a_{11}=-0.2521, a_{21}=0.3601,$

$a_{22}=0.1967,\ b_{11}=0.52, b_{12}=0.1821, b_{21}=-0.313, b_{22}=0.8113,$

which may be written in tabulated from as:

$$\begin{array}{c|cc} -0.2521 & 0 \\ 0.3601 & 0.1967 \\ \hline 0.1253 & 0.8571 \end{array} \qquad \begin{array}{c|cc} 0.52J_1 & 0.1821J_1 \\ -0.313J_1 & 0.8113J_1 \\ \hline 0.1134J_1 & 0.8553J_1 \end{array} \quad \ldots (3.35)$$

Then equations (3.3) will take the form:

$$\left.\begin{aligned} Y_1 &= y_n - 0.2521\ h\ f(Y_1) + J_1[0.52\ g(Y_1)+0.1821\ g(Y_2)] \\ Y_2 &= y_n + h[0.3601\ f(Y_1)+0.1967\ f(Y_2)] + J_1[-0.313\ g(Y_1) \\ &\quad +0.8113\ g(Y_2)] \\ y_{n+1} &= y_n + h[0.1253\ f(Y_1) + 0.8571\ f(Y_2)]+J_1[0.1134\ g(Y_1) \\ &\quad + 0.8553\ g(Y_2)] \end{aligned}\right\} \ldots (3.36)$$

Equations (3.36) represent the MSRKM6- semi-explicit - implicit.

Also in this case, the respective minimum given in equations (2.47), (2.48), (2.49) and (2.50) are:

$$\frac{h^3}{12}, \frac{h^3}{12}, 0, 0 \quad \ldots (3.37)$$

and for a 2-stage MSRKM6-semi-explicit - implicit, the last equation for equations (2.51) becomes:

$$\gamma^T B\ b = (\gamma_1, \gamma_2)\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} b_{11}+b_{12} \\ b_{21}+b_{22} \end{pmatrix} = 0.2095$$

then equation (2.50) will be:

$$\left(\frac{1}{36}-\frac{1}{3}\gamma^{T}Bb+(\gamma^{T}Bb)^{2}\right)15h^{3}=0.0275h^{3}$$

and so in this case, the principal error constants are:

$$\frac{h^{3}}{12},\ \frac{h^{3}}{12},\ 0,\ 0.0275h^{3} \qquad \ldots (3.38)$$

Remarks (3,2):

1. In all SRKM's were described in chapter two and three of this work and given in the form of equation (2.27) have maximum strong order 1.0 for any number of stages s, and all these methods have optimal principal error coefficients for s = 2 which are given by equations (2.54), (2.55), (3.6), (3.11), (3.15), (3.19), (3.23), (3.27), (3.31) and (3.35), respectively.
2. The 1-norm is used to estimate the contribution of all the error terms to the principal error, note that as an illustration, the calculation of $\|P.E.\|_1$ for PL model have been located by finding the sum of principal error constants terms which are given in equation (2.56) as a coefficients of h^3. i.e., The $\|P.E.\|_1$ for PL model is:

$$\|P.L.\|_1=\left|\frac{1}{3}\right|+\left|\frac{1}{3}\right|+\left|\frac{1}{36}\right|+\left|\frac{5}{12}\right|=\frac{10}{9}\ ;\ 1.111111.$$

The following table (3.1) gives these values for PL, R2 and all new methods considered in sections two and three above.

Table (3.1)
Error Coefficients

Models	$\|\mathbf{P.E.}\|_1$
PL	1.11111
R2	0.58333
Semi explicit	0.30323
Implicit	0.16667
MSRKM1	0.58333
MSRKM2	0.58333
MSRKM3	0.29352
MSRKM4	0.16713
MSRKM5	0.16667
MSRKM6	0.19419

3.3 Stability of Stochastic Rung-Kutta Methods:

Consider the SODE in Stratonovih form with one Wiener process:

$$dy_t = f(t, y_t)\,dt + g(t, y_t)\,odW_t \qquad \dots (3.39)$$

In this section, the stability of the semi-explicit scheme, implicit scheme and all MSRK schemes that are described above will be discussed, and this will be made by comparing the stochastic linear one step method in the autonomous Stratonovich case with our schemes (semi explicit, implicit and mixed stochastic Runge-Kutta methods) and finally the consistency conditions are also derived.

3.3.1 Modified Stochastic Linear Multi-Step Methods:

Consider the modified stochastic linear k-step method for approximating the solution of the SODE given in equation (3.39), for $n = k, k+1, ..., N$ which takes the form:

$$\sum_{j=0}^{k} \alpha_j Y_{n-j} = h \sum_{j=0}^{k} \beta_j f(t_{n-j}, Y_{n-j}) + \sum_{j=1}^{k} \gamma_j g(t_{n-j}, Y_{n-j}) J_1^{t_{n-j}, t_{n-j+1}} \qquad \dots (3.40)$$

where α_j, β_j and γ_j are constants to be evaluated and set without loss of generality $\alpha_0 = 1$ and we require given initial and starting values $Y_0, Y_1, \dots, Y_{k-1} \in L_2(\Omega, \mathfrak{F}^n)$.

As in the deterministic case, usually only $Y_0 = Y(t_0)$ is given by the stochastic initial value problem and the values $Y_1, Y_2, \dots, Y_{k-1}$ need to be computed numerically as the starting values. This can be made by any suitable one-step method, where one has to be careful to achieve the desired accuracy.

Definition (3.1):

The local error of the stochastic linear multisteps methods given in equation (3.40) for the approximation of the solution of the SODE given in equation (3.39) for n = k, k+1, …, N, may be written as:

$$L_n = \begin{cases} \sum_{j=0}^{k} \alpha_j Y_{n-j} - h\sum_{j=0}^{k} \beta_j f(t_{n-j}, Y_{n-j}) - \sum_{j=1}^{k} \gamma_j\, g(t_{n-j}, Y_{n-j}) \\ \qquad\qquad J_1^{t_{n-j}, t_{n-j+1}}; \quad \text{for } n = k,..,N \\ Y(t_n) - Y_n \qquad\qquad ; \text{for } n = 0,...,k-1 \end{cases} \quad \dots (3.41)$$

and represent the local error in the following form:

$$L_n = R_n + S_n ; L_n = R_n + \sum_{j=1}^{k} S_{j,n-j+1} \quad \dots (3.42)$$

where $n = k, k+1,..., N$ and $S_{j,n-j+1}$ is $A_{t_{n-j+1}}$-measurable with $E(S_{j,n-j+1} \mid A_{t_{n-j}}) = 0$, $\forall n = k, k+1,..., N$, $j = 1, 2,..., k$.

3.3.2 Stochastic Linear One Step Method for Stratonovich Case:

In this subsection the consistent conditions for stochastic linear one step method with one Wiener process will be studied.

The consistent conditions for one step Stratonovich stochastic linear multistep method with one wiener process will be derived next.
First, rewrite equation (3.40) with k=1 as follows:

$$\alpha_0 Y_n + \alpha_1 Y_{n-1} = h\,[\beta_0 f(t_n, Y_n) + \beta_1 f(t_{n-1}, Y_{n-1})] + \gamma_1 g(t_{n-1}, Y_{n-1})$$
$$J_1^{t_{n-1}, t_n} \quad ; n = 1, \ldots, N \quad \dots (3.43)$$

Now, consider the operators:

$$L^0 a = \frac{\partial a}{\partial y} f \text{ and } L^1 a = \frac{\partial a}{\partial y} g \quad \dots (3.44)$$

For the present case; the local error for equation (3.40) may be rewritten as:

$$L_n = \begin{cases} [\alpha_0 Y_n + \alpha_1 Y_{n-1}] - h[\beta_0 f(t_n, Y_n) + \beta_1 f(t_{n-1}, Y_{n-1})] - \gamma_1 g(t_{n-1}, Y_{n-1}) \\ \qquad\qquad J_1^{t_{n-1}, t_n} \quad ; \text{for } n = 1,...,N \\ Y(t_n) - Y_n \qquad\qquad ; \text{for } n = 0 \end{cases} \quad \dots (3.45)$$

To derive the consistent conditions for Stratonovich case, we need to the next formulas:

$$f(s,Y(s)) = f(t_{n-j}, Y(t_{n-j})) + J_0^{t_{n-j},s}(L^0 f) + J_1^{t_{n-j},s}(L^1 f) \quad \text{... (3.46)}$$

$$g(s,Y(s)) = g(t_{n-j}, Y(t_{n-j})) + J_0^{t_{n-j},s}(L^0 g) + J_1^{t_{n-j},s}(L^1 g) \quad \text{... (3.47)}$$

Equations (3.46) and (3.47) represent the application of the Itô-formula on the corresponding interval to the drift coefficient f, as well as, to the diffusion coefficient g, yields for $s \in [t_{n-1}, t_n]$:

$$f(t_{n-1}, Y(t_{n-1})) = f(t_{n-2}, Y(t_{n-2})) + J_0^{t_{n-2},t_{n-1}}(L^0 f) + J_1^{t_{n-2},t_{n-1}}(L^1 f) \quad \text{... (3.48)}$$

$$f(t_n, Y(t_n)) = f(t_{n-1}, Y(t_{n-1})) + J_0^{t_{n-1},t_n}(L^0 f) + J_1^{t_{n-1},t_n}(L^1 f) \quad \text{... (3.49)}$$

by the analysis of the local error L_n given by equation (3.45) of the scheme (3.43) for solving the SODE given in equation (3.39), the consistency condition may be derived. The following lemma relates this result which is modified and improved here for Stratonovich case.

Lemma (3.1):

Assume that the coefficients f and g of the SODE given in equation (3.39) belong to the class $C^{1,2}$ with $L^0 f, L^0 g, L^1 f$ and $L^1 g \in C^k$ .Then the local error given in equation (3.45) of the stochastic linear one step method given in equation (3.43) allows the representation:

$$L_n = R_n^0 + S_n^0 ; \text{ for } n = 1, \ldots, N \quad \text{... (3.50)}$$

Where R_n^0 and S_n^0 are A_{t_n} -measurable with $E\left(S_n^0 \mid A_{t_n}\right) = 0$ and

$$R_n^0 = [\alpha_0 + \alpha_1] Y(t_{n-1}) + [\alpha_0 - \beta_0 - \beta_1] h f(t_{n-1}, Y(t_{n-1})) + \text{ℛ}_n^0$$
$$S_n^0 = [\alpha_0 - \gamma_1]\, g(t_{n-1}, Y(t_{n-1}))\Delta W_{n-1} + \text{𝒮}_n^0$$

with

$$\| \text{ℛ}_n^0 \|_{L_2} = O(h^2) ; \| \text{𝒮}_n^0 \|_{L_2} = O(h) \quad \text{... (3.51)}$$

Proof:

To derive a representation of the local error in the form (3.50), the deterministic parts is evaluated and resumed at the point $(t_{n-1}, Y(t_{n-1}))$ and separate the stochastic terms carefully over the subinterval $[t_{n-1}, t_n]$. This ensures the independence of the random variables. It does make the calculations more complicated, since:

$$\alpha_0 Y(t_n) + \alpha_1 Y(t_{n-1}) = \alpha_0 Y(t_n) + \alpha_1 Y(t_{n-1}) - \alpha_0 Y(t_{n-1}) + \alpha_0 Y(t_{n-1})$$

$$\alpha_0 Y(t_n) + \alpha_1 Y(t_{n-1}) = [\alpha_0 + \alpha_1] Y(t_{n-1}) + \alpha_0 [Y(t_n) - Y(t_{n-1})] \quad \ldots (3.52)$$

then the local error for linear one-step method given in equation (3.41) for k=1 may be expresses for $n = 1, 2, \ldots, N$ as:

$$L_n = [\alpha_0 + \alpha_1] Y(t_{n-1}) + \alpha_0 [Y(t_n) - Y(t_{n-1})] - h[\beta_0 f(t_n, Y_n) + \beta_1 f(t_{n-1}, Y_{n-1})] - \gamma_1 g(t_{n-1}, Y_{n-1}) J_1^{t_{n-1}, t_n} \quad \ldots (3.53)$$

Hence, the SODE given in equation (3.39) implies the identities:

$$Y(t_n) - Y(t_{n-1}) = \int_{t_{n-1}}^{t_n} f(s, Y(s))\, ds + \int_{t_{n-1}}^{t_n} g(s, Y(s)) \circ dw(s)$$

or equivalently

$$Y(t_n) - Y(t_{n-1}) = J_0^{t_{n-1}, t_n}(f) + J_1^{t_{n-1}, t_n}(g)$$

Substituting equations (3.46) and (3.47) for $J_0^{t_{n-1}, t_n}(f)$ and $J_1^{t_{n-1}, t_n}(g)$, respectively, to obtain:

$$Y(t_n) - Y(t_{n-1}) = hf(t_{n-1}, Y(t_{n-1})) + J_{00}^{t_{n-1}, t_n}(L^0 f) + J_{01}^{t_{n-1}, t_n}(L^1 f) + g(t_{n-1}, Y(t_{n-1})) J_1^{t_{n-1}, t_n} + J_{10}^{t_{n-1}, t_n}(L^0 g) + J_{11}^{t_{n-1}, t_n}(L^1 g) \quad \ldots (3.54)$$

Inserting equation (3.54) and the expansion given in equation (3.49); into the local error formula given in equation (3.53) and reordering the terms yields to:

$$
\begin{aligned}
L_n &= [\alpha_0+\alpha_1]Y(t_{n-1}) + \alpha_0[hf(t_{n-1},Y(t_{n-1})) + J_{00}^{t_{n-1},t_n}(L^0f) \\
&\quad + J_{01}^{t_{n-1},t_n}(L^1f) + g(t_{n-1},Y(t_{n-1}))J_1^{t_{n-1},t_n} + J_{10}^{t_{n-1},t_n}(L^0g) \\
&\quad + J_{11}^{t_{n-1},t_n}(L^1g)] - h[\beta_0\{f(t_{n-1},Y(t_{n-1})) + J_0^{t_{n-1},t_n}(L^0f) \\
&\quad + J_1^{t_{n-1},t_n}(L^1f)\} + \beta_1 f(t_{n-1},Y(t_{n-1})] - \gamma_1 g(t_{n-1},Y_{n-1})J_1^{t_{n-1},t_n} \\
&= [\alpha_0+\alpha_1]Y(t_{n-1}) + h\alpha_0 f(t_{n-1},Y(t_{n-1})) + \alpha_0 J_{00}^{t_{n-1},t_n}(L^0f) \\
&\quad + \alpha_0 J_{01}^{t_{n-1},t_n}(L^1f) + \alpha_0 g(t_{n-1},Y(t_{n-1}))J_1^{t_{n-1},t_n} + \alpha_0 J_{10}^{t_{n-1},t_n}(L^0g) \\
&\quad + \alpha_0 J_{11}^{t_{n-1},t_n}(L^1g) - h\beta_0 f(t_{n-1},Y(t_{n-1})) - h\beta_0 J_0^{t_{n-1},t_n}(L^0f) \\
&\quad - h\beta_0 J_1^{t_{n-1},t_n}(L^1f) - h\beta_1 f(t_{n-1},Y(t_{n-1})] - \gamma_1 g(t_{n-1},Y_{n-1})J_1^{t_{n-1},t_n} \\
&= [\alpha_0+\alpha_1]Y(t_{n-1}) + [\alpha_0-\beta_0-\beta_1]hf(t_{n-1},Y(t_{n-1})) + \{\alpha_0 J_{00}^{t_{n-1},t_n} \\
&\quad (L^0f) - h\beta_0 J_0^{t_{n-1},t_n}(L^0f)\} + [\alpha_0-\gamma_1]g(t_{n-1},Y(t_{n-1}))J_1^{t_{n-1},t_n} \\
&\quad + \{\alpha_0 J_{11}^{t_{n-1},t_n}(L^1g) - h\beta_0 J_1^{t_{n-1},t_n}(L^1f) + \alpha_0 J_{10}^{t_{n-1},t_n}(L^0g) \\
&\quad + \alpha_0 J_{01}^{t_{n-1},t_n}(L^1f)\}
\end{aligned}
$$

hence

$$
\begin{aligned}
L_n &= [\alpha_0+\alpha_1]Y(t_{n-1}) + [\alpha_0-\beta_0-\beta_1]hf(t_{n-1},Y(t_{n-1})) + \text{ℝ}_n^0 \\
&\quad + [\alpha_0-\gamma_1]g(t_{n-1},Y(t_{n-1}))\Delta W_{n-1} + \text{𝕊}_n^0
\end{aligned}
\qquad \dots (3.55)
$$

where

$$
\text{ℝ}_n^0 = \alpha_0 J_{00}^{t_{n-1},t_n}(L^0f) - h\beta_0 J_0^{t_{n-1},t_n}(L^0f) \qquad \dots (3.56)
$$

and

$$
\begin{aligned}
\text{𝕊}_n^0 &= \alpha_0 J_{11}^{t_{n-1},t_n}(L^1g) - h\beta_0 J_1^{t_{n-1},t_n}(L^1f) + \alpha_0 J_{10}^{t_{n-1},t_n}(L^0g) \\
&\quad + \alpha_0 J_{01}^{t_{n-1},t_n}(L^1f)
\end{aligned}
\qquad \dots (3.57)
$$

■

Remark (3.3):

The consistency conditions for the stochastic one step method in the Stratonovich case are then from lemma (3.1) read as follows:

$$
[\alpha_0+\alpha_1]=0,\ [\alpha_0-\beta_0-\beta_1]=0 \text{ and } [\alpha_0-\gamma_1]=0 \qquad \dots (3.58)
$$

Definition (3.2):

The characteristic polynomial of equation (3.40) is given by:

$$\rho(r) = \alpha_0\ r^k + \alpha_1\ r^{k-1} + \ldots + \alpha_k \qquad \ldots (3.59)$$

and the stochastic linear multistep method given in equation (3.40) is said to be stable if:

(i) The roots of $\rho(r)$ lie on or within the unit circle.

(ii) The roots of $\rho(r)$ on the unit circle are simple.

Remark (3.4):

To study the stability of the semi-explicit, implicit, and mixed stochastic Runge-Kutta methods, we may compare the stochastic one step method given in equation (3.43) with stochastic Runge-Kutta methods given in equation (3.3)

i.e., comparing

$$\alpha_0 y_n + \alpha_1 y_{n-1} = h\beta_0 f(t_n, y(t_n)) + h\beta_1 f(t_{n-1}, y(t_{n-1})) + g(t_{n-1}, y(t_{n-1}))\, J_1^{t_{n-1}, t_n}$$

with the corresponding SRKM

$$y_{n+1} = y_n + h[\alpha_1\ f(Y_1) + \alpha_2\ f(Y_2)] + J_1^{t_{n-1}, t_n}\,[\gamma_1\ g(Y_1) + \gamma_2\ g(Y_2)]$$

which may be rewritten as

$$y_{n+1} = y_n + h\,\phi(t_n, y(t_n), h) + J_1^{t_n, t_{n+1}}\,\varphi(t_n, y(t_n), J_1^{t_n, t_{n+1}})$$

hence we get:

$$\alpha_0 = 1,\ \ \alpha_1 = -1,\ \ \beta_0 = 0,\ \ \beta_1 = 1 \ \text{ and } \ \gamma_1 = 1$$

which satisfy the consistency conditions [see remark (3.3)] and by using definition (3.2) the stochastic Runge-Kutta methods are stable.

3.4 Illustrative Examples.

In this section some examples of autonomous SODE's are considered and will be solved using SRK schemes which considered in this work and note that, the obtain solutions for these examples are represented at average of 10000 simulated solutions by using $N(0,h)$ random number generations for wiener process W_t.

Example (3.1) [22]:

Consider the Stratonovich SDE:

$$dy_t = (y_t^2 - 1)dt + (0.1 - 0.1y_t^2) \circ dW_t \; ; y_0 = 0$$

when has the exact solution given by:

$$y_t = \frac{\exp(-2t + 0.2W_t) - 1}{\exp(-2t + 0.2W_t) + 1}$$

Following tables (3.2), (3.3), (3.4), (3.5) and (3.6) which represent the numerical solution of example (3.1) by using the SRKM's which are (Explicit (R2, PL) [7], Semi explicit, Implicit, MSRKM1, MSRKM2, MSRKM3, MSRKM4, MSRKM5 and MSRKM6) respectively:

Table (3.2)
The exact and numerical results of example (3.1) using explicit-R2 and semi- explicit methods.

ti	*Exact Solutions*	*Explicit-R2*	*Absolute Errors*	*Semi-explicit*	*Absolute Errors*
0.1	-0.09967	-0.10014	0.00047	-0.09791	0.00176
0.2	-0.1971	-0.19697	0.00013	-0.19371	0.00339
0.3	-0.29104	-0.29117	0.00013	-0.28551	0.00553
0.4	-0.3802	-0.37905	0.00115	-0.37193	0.00827
0.5	-0.46156	-0.46013	0.00143	-0.45205	0.00951
0.6	-0.53684	-0.53417	0.00267	-0.52562	0.01122
0.7	-0.60442	-0.60032	0.0041	-0.5915	0.01292
0.8	-0.66395	-0.65904	0.00491	-0.65025	0.0137
0.9	-0.71597	-0.71056	0.00541	-0.70201	0.01396
1	-0.76159	-0.75544	0.00615	-0.74749	0.0141

Table (3.3)
The exact and numerical results of example (3.1) using explicit-PL and implicit methods.

t_i	*Exact Solutions*	*Explicit-PL*	*Absolute Errors*	*Implicit*	*Absolute Errors*
0.1	-0.09967	-0.09985	0.00018	-0.09746	0.00221
0.2	-0.1971	-0.19851	0.00141	-0.19369	0.00341
0.3	-0.29104	-0.29306	0.00202	-0.28526	0.00578
0.4	-0.3802	-0.38369	0.00349	-0.37212	0.00808
0.5	-0.46156	-0.46763	0.00607	-0.45381	0.00775
0.6	-0.53684	-0.54411	0.00727	-0.52816	0.00868
0.7	-0.60442	-0.6135	0.00908	-0.59487	0.00955
0.8	-0.66395	-0.67469	0.01074	-0.65463	0.00932
0.9	-0.71597	-0.72761	0.01164	-0.70695	0.00902
1	-0.76159	-0.77367	0.01208	-0.75241	0.00918

Table (3.4)
The exact and numerical results of example (3.1) using MSRKM1 and MSRKM2 methods.

t_i	*Exact Solutions*	*MSRKM1*	*Absolute Errors*	*MSRKM2*	*Absolute Errors*
0.1	-0.09967	-0.09898	0.00069	-0.09956	0.00011
0.2	-0.1971	-0.1966	0.0005	-0.19696	0.00014
0.3	-0.29104	-0.28918	0.00186	-0.29086	0.00018
0.4	-0.3802	-0.37628	0.00392	-0.37868	0.00152
0.5	-0.46156	-0.45729	0.00427	-0.4592	0.00236
0.6	-0.53684	-0.53081	0.00603	-0.53295	0.00389
0.7	-0.60442	-0.5974	0.00702	-0.59936	0.00506
0.8	-0.66395	-0.65701	0.00694	-0.65821	0.00574
0.9	-0.71597	-0.70936	0.00661	-0.71	0.00597
1	-0.76159	-0.75437	0.00722	-0.75519	0.0064

Table (3.5)
The exact and numerical results of example (3.1) using MSRKM3 and MSRKM4 methods.

t_i	*Exact Solutions*	*MSRKM3*	*Absolute Errors*	*MSRKM4*	*Absolute Errors*
0.1	-0.09967	-0.09894	0.00073	-0.09826	0.00141
0.2	-0.1971	-0.19592	0.00118	-0.19438	0.00272
0.3	-0.29104	-0.28796	0.00308	-0.28777	0.00327
0.4	-0.3802	-0.37554	0.00466	-0.37419	0.00601
0.5	-0.46156	-0.45595	0.00561	-0.45468	0.00688
0.6	-0.53684	-0.52992	0.00692	-0.52793	0.00891
0.7	-0.60442	-0.59636	0.00806	-0.59445	0.00997
0.8	-0.66395	-0.65555	0.0084	-0.65367	0.01028
0.9	-0.71597	-0.70747	0.0085	-0.70642	0.00955
1	-0.76159	-0.75263	0.00896	-0.75205	0.00954

Table (3.6)
The exact and numerical results of example (3.1) using MSRKM5 and MSRKM6 methods.

t_i	*Exact Solutions*	*MSRKM5*	*Absolute Errors*	*MSRKM6*	*Absolute Errors*
0.1	-0.09967	-0.09832	0.00135	-0.09777	0.0019
0.2	-0.1971	-0.19386	0.00324	-0.1935	0.0036
0.3	-0.29104	-0.2856	0.00544	-0.28548	0.00556
0.4	-0.3802	-0.37246	0.00774	-0.37261	0.00759
0.5	-0.46156	-0.45248	0.00908	-0.45333	0.00823
0.6	-0.53684	-0.52605	0.01079	-0.52662	0.01022
0.7	-0.60442	-0.59251	0.01191	-0.59322	0.0112
0.8	-0.66395	-0.65152	0.01243	-0.65221	0.01174
0.9	-0.71597	-0.70381	0.01216	-0.70393	0.01204
1	-0.76159	-0.74899	0.0126	-0.74928	0.01231

Example (3.2) [7]:

Consider the Stratonovich SDE:

$$dy_t = 0.5(1 - y_t^2) \circ dW_t \; ; y_0 = 0.5$$

when has the exact solution given by:

$$y_t = \tanh(0.5W_t + \tanh^{-1}(y_0))$$

Following tables (3.7), (3.8), (3.9), (3.10) and (3.11), which represent the numerical solution of example (3.2) by using the SRKM's which are (Explicit (R2, PL) [7], Semi explicit, Implicit, MSRKM1, MSRKM2, MSRKM3, MSRKM4, MSRKM5 and MSRKM6) respectively:

Table (3.7)
The exact and numerical results of example (3.2) using explicit-R2 and semi- explicit methods.

t_i	*Exact Solutions*	*Explicit-R2*	*Absolute Errors*	*Semi-explicit*	*Absolute Errors*
0.1	0.5	0.49111	0.00889	0.49044	0.00956
0.2	0.49325	0.47348	0.01977	0.47138	0.02187
0.3	0.48932	0.45821	0.0311	0.45618	0.03313
0.4	0.49094	0.44476	0.04619	0.44249	0.04845
0.5	0.4907	0.4304	0.06029	0.43371	0.05698
0.6	0.48821	0.41688	0.07133	0.41832	0.06989
0.7	0.49051	0.40603	0.08448	0.40522	0.0853
0.8	0.48943	0.39812	0.09132	0.39865	0.09078
0.9	0.49223	0.3898	0.10243	0.3896	0.10263
1	0.4919	0.38295	0.10895	0.38128	0.11062

Table (3.8)
The exact and numerical results of example (3.2) using explicit-PL and implicit methods.

t_i	*Exact Solutions*	*Explicit-PL*	*Absolute Errors*	*Implicit*	*Absolute Errors*
0.1	0.5	0.47982	0.02018	0.49115	0.00885
0.2	0.49325	0.46562	0.02763	0.46819	0.02506
0.3	0.48932	0.44859	0.04073	0.45686	0.03246
0.4	0.49094	0.43376	0.05718	0.44114	0.0498
0.5	0.4907	0.41841	0.07228	0.44154	0.04916
0.6	0.48821	0.40511	0.0831	0.42856	0.05965
0.7	0.49051	0.39605	0.09446	0.42646	0.06405
0.8	0.48943	0.386	0.10344	0.42904	0.0604
0.9	0.49223	0.37713	0.1151	0.41659	0.07565
1	0.4919	0.37111	0.12079	0.50281	0.01092

Table (3.9)
The exact and numerical results of example (3.2) using MSRKM1 and MSRKM2 methods.

t_i	*Exact Solutions*	*MSRKM1*	*Absolute Errors*	*MSRKM2*	*Absolute Errors*
0.1	0.5	0.48936	0.01064	0.49734	0.00266
0.2	0.49325	0.47325	0.02	0.48127	0.01199
0.3	0.48932	0.45934	0.02998	0.46935	0.01997
0.4	0.49094	0.44684	0.0441	0.45352	0.03742
0.5	0.4907	0.43429	0.05641	0.44555	0.04514
0.6	0.48821	0.4232	0.065	0.4393	0.0489
0.7	0.49051	0.41264	0.07787	0.43196	0.05855
0.8	0.48943	0.40343	0.086	0.42664	0.0628
0.9	0.49223	0.39537	0.09686	0.42583	0.0664
1	0.4919	0.38538	0.10652	0.4044	0.0875

Table (3.10)
The exact and numerical results of example (3.2) using MSRKM3 and MSRKM4 methods.

t_i	*Exact Solutions*	*MSRKM3*	*Absolute Errors*	*MSRKM4*	*Absolute Errors*
0.1	0.5	0.4881	0.0119	0.49181	0.00819
0.2	0.49325	0.46943	0.02382	0.48201	0.01124
0.3	0.48932	0.4554	0.03392	0.47284	0.01648
0.4	0.49094	0.44231	0.04863	0.45815	0.0328
0.5	0.4907	0.42657	0.06413	0.45052	0.04018
0.6	0.48821	0.41359	0.07462	0.42643	0.06177
0.7	0.49051	0.40483	0.08568	0.41396	0.07655
0.8	0.48943	0.39607	0.09336	0.41445	0.07499
0.9	0.49223	0.38857	0.10366	0.40106	0.09117
1	0.4919	0.38455	0.10735	0.35367	0.13823

Table (3.11)
The exact and numerical results of example (3.2) using MSRKM5 and MSRKM6 methods.

t_i	*Exact Solutions*	*MSRKM5*	*Absolute Errors*	*MSRKM6*	*Absolute Errors*
0.1	0.5	0.48922	0.01078	0.49024	0.00976
0.2	0.49325	0.47172	0.02153	0.47349	0.01976
0.3	0.48932	0.47446	0.01485	0.45665	0.03267
0.4	0.49094	0.45834	0.03261	0.44236	0.04858
0.5	0.4907	0.44485	0.04584	0.43074	0.05996
0.6	0.48821	0.43895	0.04926	0.41892	0.06928
0.7	0.49051	0.43377	0.05674	0.40741	0.08311
0.8	0.48943	0.45068	0.03875	0.39888	0.09055
0.9	0.49223	0.44454	0.0477	0.3895	0.10273
1	0.4919	0.48618	0.00572	0.38177	0.11013

3.5 Stochastic Variable Step Size Method of Solving SODE's:

In this thesis, we gave some models of stochastic Rung-Kutta methods, those methods considered in chapter two and three with fixed step size h. As it is known from the usual methods of numerical analysis, the step size is fixed in that methods during the approach of solution, but still there are some methods may be used to reduce the local truncation error, and among such methods is the variable step size method.

In this section, the numerical solution of SODE's will be found using variable step size methods when may be considered as a new approach in this topic, where the considered SODE's is given by:

$$dy_t = f(y_t)\,dt + g(y_t) \circ dW_t \text{ ; with } y_{t_0} = y_0 \qquad \dots (3.60)$$

In all fixed step-size methods the local truncation error will depends on step size h and on the numerical method used. But, in variable step-size methods, we shall find the numerical solution y_{t_f} for the SODE given in equation (3.60), that is accurate to within a specified tolerance ε.

Therefore, it turns out for reasonable effective estimates of the step-size, it is required to attain a specified local truncation error (tolerance) ε. The variable step-size method which will be considered here, is based upon comparison between the estimates of the one and two steps of the numerical value of y_t at some time obtained by the numerical method with local truncation error term that is of the form Ch^p, where C is unknown constant and p is the order of the method. Suppose that we started with the initial condition y_{t_0} with step-size h using certain SRKM to find the solution $y^{(1)}_{t_0+h}$ and $y^{(2)}_{t_0+h}$ using the step-size h and $\frac{h}{2}$, respectively. Let:

$$E_{est.} = \| y^{(1)}_{t_0+h} - y^{(2)}_{t_0+h} \| \qquad \dots (3.61)$$

And here if $E_{est.} \le \varepsilon$, then there is no problem and one may consider $y^{(2)}_{t_0+h}$ as the solution at $t_0 + h$. Otherwise if $E_{est.} > \varepsilon$, then one can to find another estimation of the step- size say h_{new} . If this approximation was accepted then

this value of h_{new} will be used as the new value of h in the next step; if not, then it will be used as an old h and repeat similarly as above.

A common question may arise, which is how to find h_{new} ?. In this work, a new criterion has been developed for estimating the local truncation error, which control the step- size. The problem of error estimation is the most important problem that impact the user while using variable step-size method.

Theorem(3.1):

Suppose the $y^{(1)}_{t_0+h}$ and $y^{(2)}_{t_0+h}$ are the numerical solution the SODE given in equation (3.60) using certain SRKM with step sizes h and $\frac{h}{2}$, respectively; If ε is the tolerance and $E_{est.} = \| y^{(1)}_{t_0+h} - y^{(2)}_{t_0+h} \|$, then (the new value of the step size)

$$h_{new} = \frac{3h_{old}\,\varepsilon}{2E_{est.}} \qquad \dots (3.62)$$

where h_{old} refers to the old value of the step size.

Proof:

Suppose Y is the actual solution at t_0+h, by taking expectation to the both sides of equation (3.61) yields:

$$E(E_{est.}) = E(\| y^{(1)}_{t_0+h} - y^{(2)}_{t_0+h} \|) = E(\| y^{(1)}_{t_0+h} - y^{(2)}_{t_0+h} + Y - Y \|)$$

$$E_{est.} \le E(\| y^{(1)}_{t_0+h} - X_T \| + \| y^{(2)}_{t_0+h} - X_T \|)$$

$$= E(\| y^{(1)}_{t_0+h} - X_T \|) + E(\| y^{(2)}_{t_0+h} - X_T \|)$$

$$\le Ch + C(\frac{h}{2}) = C(h + \frac{h}{2}) = \frac{3}{2}Ch$$

$$E_{est.} \le \frac{3}{2}Ch$$

also yields to $C = \frac{2E_{est.}}{3h}$

since, $\varepsilon = Ch_{new} = \frac{2E_{est.}}{3h_{old}} h_{new}$

and so:

$$h_{new} = \frac{3h_{old}\,\varepsilon}{2E_{est.}}$$

■

Example (3.3):

Resolving example (3.2) using stochastic variable step size method in explicit case (R2), so the following table (3.12), represent the numerical solution of the example (3.2).

Table (3.12)
The exact and numerical results of example (3.2) using stochastic variable step size method in explicit case (R2).

t_i	*Exact Solution*	*Numerical Solution*	*Absolute Errors*
0.1	0.49993	0.50827	0.00834
0.2	0.49998	0.49602	0.00396
0.3	0.49995	0.50329	0.00334
0.4	0.50001	0.49538	0.00463
0.5	0.50001	0.49621	0.00381
0.6	0.49987	0.49421	0.00565
0.7	0.49999	0.50607	0.00608
0.8	0.50006	0.49669	0.00337
0.9	0.50001	0.49688	0.00313
1	0.50002	0.49559	0.00443

Conclusions and Recommendation

From the present study, one may conclude the following conclusion:

1. Variable step size methods improve the accuracy of the results, but it required more calculation which is increase the consuming time.
2. Implicit and mixed methods have some difficulties in programming since a nonlinear system must be solved at each step.

Also from the present study the following conclusion may be drown:

1. Study the stability of implicit SRKM's using the concept of mean square stability.
2. Deriving 3-stages SRKM's for solving SODE's.
3. Solving system of SODE's using the considered schemes followed in this thesis.
4. Applying variable step size methods for solving SODE's based on implicit SRKM's.
5. Introducing variable order methods for solving SODE's.

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ألمستخلص

الهدف الرئيسي لهذه الرسالة ينقاد الى ثلاث اتجاهات رئيسيه:

الهدف الاول: هو لدراسة و استعراض المبادئ الاساسية في الحساب التصادفي (Stochastic Calculus) أضافة الى دراسة المعادلات التفاضلية الاعتيادية التصادفية (Stochastic Ordinary Differential Equations) .

ألهدف الثاني: هو لدراسة طرائق رانج-كوتا التصادفية الصريحة (Explicit Runge-Kutta Method)، ومن ثم إعمام هذه الطرائق الى طرائق شبه صريحة (Semi-Explicit Runge-Kutta Method)، طرائق ضمنية (Implicit Runge-Kutta Method) وطرائق مختلطة (Mixed Schemes) ومن ثم دراسة استقرارية هذه الطرق عدديا.

ألهدف الثالث: هو عرض طريقة متغيرة الخطوات (Variable Step Size Method) لحل المعادلات التفاضلية التصادفية والتي أعطت نتائج ذات دقة أعلى.

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حلول المعادلات التفاضلية التصادفية الاعتيادية بأستخدام طرائق رانج-كوتا متغيرة الخطوات

رسالة
مقدمة إلى كلية العلوم – جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

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