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Solution of Fuzzy Network Problems Using Statistical Methods

A Thesis

Submitted to the College of Science/ Al-Nahrain University as a partial fulfillment of the requirements for the Degree of Master of Science in Mathematics.

By

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بِسْمِ الآبِ والإبْنِ وَالرُّوحِ القُدُسِ

الإلـهُ الواحِد آمِينْ

"أَّنتُم نورُ العالَمِ. لا تَخْفى مَدينَةٌ عَلى جَبَلٍ، ولا يُوقَدُ سِراجٌ وَيُوضَعُ تَحْتَ المِكيال، بَل عَلى المَنارَة، فَيُضِيءُ لِجَميعِ الَّذينَ في البَيْت. هكذا فَلْيُضِيءْ نُورُكُم لِلَّناس، لِيَرَوْا أَعمالَكُم الصَّالِحة، فَيُمَجِّدُوا أَباكُم الَّذي فِي السَّمَاواتِ."

متى 5/14-16

الاهــداء

الى الملاذ الامن والحضن الدافيء والدافع الاول ...

عائلتي الكريمة

الى الاخوة والاخوات الذين حالت الظروف دون اكمال مسيرتهم العلمية ...

ابناء وطني المهجرين والنازحين

الى من وقفوا بجانبي وساعدوني وساندوني لاتمام هذا البحث ...

زملائي واصدقائي الاعزاء

الى من اشعلوا الشموع لينيروا دربنا واعطوا من حصيلة فكرهم لنكمل عملنا ...

اساتذتي الكرام

مع شكري وامتناني ...

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Sanar Mazin Younis

September, 2014 ✍

Supervisors Certification

We, certify that this thesis entitled ***"Solution of Fuzzy Networks Problem Using Statistical Methods"*** was prepared by ***"Sanar Mazin Younis"*** under our supervision at the College of Science/ Al-Nahrain University as a partial fulfillment of the requirements for the Degree of Master of Science in Mathematics.

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Committee Certification

We, the examining committee certify that we have read this thesis entitled ***"Solution of Fuzzy Networks Problem Using Statistical Methods"*** and examined the student ***"Sanar Mazin Younis"*** in its contents and that in our opinion, it is accepted for the degree of Master of Science in Mathematics.

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Summary

This thesis developed three defuzzification approaches to convert the coefficients and the variables of the fuzzy linear programming problems (FLPP) into crisp (deterministic) linear programming problems (CLPP) and obtain the critical path with the optimal completion time for the different fuzzy network problems.

The three defuzzification approaches are based respectively on the philosophies of probability density function, ranking measures and the program evaluation and review technique (PERT).

Finally, the critical path method (CPM) has been used to compare its results with our obtained results to give more credit to our approaches.

The case study is considered from a real problem to verify our results that obtained using “Matlab2010R” software.

Nomenclatures and Notations

LP	*Linear Programming*
FLP	*Fuzzy Linear Programming*
FLPP	*Fuzzy Linear Programming Problems*
CLPP	*Crisp Linear Programming Problems*
PERT	*Program Evaluation and Review Technique*
CPM	*Critical Path Method*
AoA	*Activity on Arrow*
t_o	*Optimistic time*
t_p	*Pessimistic time*
t_m	*Most likely time*
T_E	*Earliest expected time*
T_L	*Latest allowable time*
COG	*Center of Gravity*
FM	*Fuzzy Mean*
WFM	*Weighted Fuzzy Mean*
QT	*Quality Technique*
EQT	*Extended Quality Technique*
FOM	*First of Maxima*
MOM	*Middle of Maxima*
LOM	*Last of Maxima*
RCOM	*Random Choice of Maxima*
COA	*Center of Area*
(i, j)	*Activity between the nodes* i, j
U	*Universal set*
$\tilde{A}$	*Fuzzy set, Fuzzy number*
$\tilde{\tilde{A}}$	*Fuzzy stochastic variable*

$\mu_{\tilde{A}}(x)$	*Membership of the fuzzy set* $\tilde{A}$
$\tilde{P}$	*Fuzzy Critical Path*
$\Re(\tilde{A})$	*Ranking function*
$M_X(s)$	*Mellin transform*
$E[x]$	*Expected value of the random variable* x

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Introduction

In recent years, the range of project management applications has greatly expanded. Project management concerns the scheduling and controlling of activities (tasks) in such a way that the project can be completed in a little time as possible. To ensure the project's success, the project management team must identify the stakeholders, determine and manage their needs and expectations. A project network is defined as a set of activities that must be performed according to precedence constraints stating that the activities must start after the completion of specified other activities. In the project network, the nodes represent activities and the arcs represent precedence relations. A path through a project network is one of the routes from the starting node to the ending node. The length of a path is the sum of the durations of the activities on the path. The project duration equals the length of the longest path through the project network. The longest path is called the critical path of the network. In order to specify the critical path in project networks in the traditional models, the durations of activities are represented as crisp numbers. However, the operation time for each activity is usually difficult to define and estimate precisely in a real situation.

The longest path problem is concentrate on finding the path with maximum distance, time or benefit or other variables, and it is one of the basic problems in networks and is widely applied in transportation, communication and computer network and has been studied extensively in the field of computer science, operation research, transportation engineering and so on.

The aim of the longest path problem is to find the longest path between:

(1) two given nodes of a graph,

(2) a given node to all other nodes,

(3) all pair of nodes.

The Bellman algorithm is one of the efficient algorithms used to determine the longest and/or shortest path in a crisp network.

In real problems, uncertainty cannot be avoided and usually, the arc lengths cannot be determined precisely. For instance, on road networks, for several reasons, e.g., traffic, accidents, arc lengths representing the vehicle travel time are subject to uncertainty. In these cases, deterministic values for representing the arc weights cannot be used. A typical way of expressing these uncertainties in the arc weights is to utilize probability theory. However, sometimes the probability distributions of the lengths of arc are difficult to acquire due to lack of historical data. In dealing with such case, the expert, using the fuzzy theory as a powerful tool, estimate the approximate length of the arc. Fuzzy set theory has been proposed to handle non crisp (fuzzy) parameters by generalizing the notion of membership in a set. Essentially, in a fuzzy set each element is associated with a point value selected from the unit interval [0,1], which is an arbitrary grade of truth referred to as the grade of membership in the set [1].

Many previous studies on fuzzy project management network are reviewed before. Prade (1979) first applied fuzzy set theory into the project scheduling problem. Furthermore, Dubios and Prade (1979), Chanas and Kamburowski (1981), Kaufmann and Gupta (1988), Hapke and Kaufmann (1993) and Ke and Liu (2010) discussed various types of project scheduling problems with fuzzy activity duration times. Furthermore, randomness and fuzziness may coexist in project scheduling problem. Ke and Liu (2007) proposed project scheduling models with mixed uncertainty of randomness and fuzziness using the tool of random fuzzy variable.

Linear programming (LP) is the most widely used and understood mathematical optimization technique employed by the business and industrial community. The conventional LP deals with crisp parameters. However, managerial decision making is subject to professional judgments usually based on imprecise, vague, uncertain or incomplete information (Leung, 1988).

The main objective in FLP is to find the best solution possible with imprecise, vague or uncertain. There are many sources of imprecision in FLP, for example, sometimes the coefficient variables are not known precisely, other times constraints satisfaction limits may be vague. The challenge in FLP is to construct an optimization model that can produce the optimal solution with subjective professional judgments.

The theory of fuzzy mathematical programming was first proposed by Tanaka et al. (1974) based on the fuzzy decision framework of Bellman and Zadeh, [11], to address the impreciseness and vagueness of the parameters in problems with fuzzy constraints and objective functions. Zimmermann (1978) introduced the first formulation of FLP. He constructed a crisp model of the problem and obtained its crisp results using an existing algorithm. He then used the crisp results and fuzzified the problem by considering subjective constants of admissible deviations for the goal and the constraints. Finally, he defined an equivalent crisp problem using an auxiliary variable that represented the maximization of the minimization of the deviations on the constraints. Zimmermann (1978, 1987) used Bellman and Zadeh’s, [11], interpretation that a fuzzy decision is a union of goals and constraints.

In the past decade, researchers have discussed various properties of FLP problems and proposed an assortment of models (Luhandjula, 1989). Zhang et al. (2003) proposed a FLP with fuzzy numbers for the coefficients of objective functions. They introduced a number of optimal solutions to the FLP problems and developed a number of theorems for converting the FLP problems to multi-objective optimization problems with four-objective functions. Stanciulescu (2003) proposed a FLP model with fuzzy coefficients for the objectives and the constraints. He used fuzzy decision variables with a joint membership function instead of crisp decision variables and linked the decision variables together to sum them up to a constant. He considered lower-bounded fuzzy decision variables that set up the lower bounds of the decision variables. He then generalized the method to lower–upper-bounded fuzzy decision variables that set up also the upper bounds of the decision variables. Ganesan and Veeramani (2006) proposed a FLP model with symmetric trapezoidal fuzzy numbers. They proved fuzzy analogues of some important LP theorems and obtained some interesting results which in turn led to the solution for FLP problems without converting them into crisp LP problems.

Mahdavi-Amiri and Nasseri (2006) proposed a FLP model where a linear ranking function was used to order trapezoidal fuzzy numbers. They established the dual problem of the LP problem with trapezoidal fuzzy variables and deduced some duality results to solve the FLP problem directly with the primal simplex tableau.

Zadeh et al. (2009) considered full FLP problems where all parameters and variables were triangular fuzzy numbers. They pointed out that there is no method in the literature for finding the fuzzy optimal solution of full FLP problems and proposed a new method to find the fuzzy optimal solution of full FLP problems with equality constraints. They used the concept of the symmetric triangular fuzzy numbers and introduced an approach to defuzzify a general fuzzy quantity. They first approximated the fuzzy triangular numbers to its nearest symmetric triangular numbers, with the assumption that all decision variables were symmetric triangular, then they converted every FLP model into two crisp complex LP models and used a special ranking for fuzzy numbers to transform their full FLP model into a multi-objective linear programming where all variables and parameters were crisp. Ebrahimnejad (2010) introduced a new primal-dual algorithm for solving FLP problems by using the duality results proposed by Mahdavi-Amiri and Nasseri, [21].

Kumar et al. (2011) further studied the full FLP problems with equality introduced by Hosseinzadeh Lotfi et al., [19], and proposed a new method for finding the fuzzy optimal solution in these problems.

Ebrahimnejad (2011) showed that the method proposed by Ganesan and Veermani, [17], stops in a finite number of iterations and proposed a revised version of their method that was more efficient and robust in practice. He also proved the absence of degeneracy and showed that if an FLP problem has a fuzzy feasible solution, it also has a fuzzy basic feasible solution and if an FLP problem has an optimal fuzzy solution, it also has an optimal fuzzy basic solution. Ebrahimnejad (2011) has also generalized the concept of sensitivity analysis in FLP problems by applying fuzzy simplex algorithms and using the general linear ranking functions on fuzzy numbers.

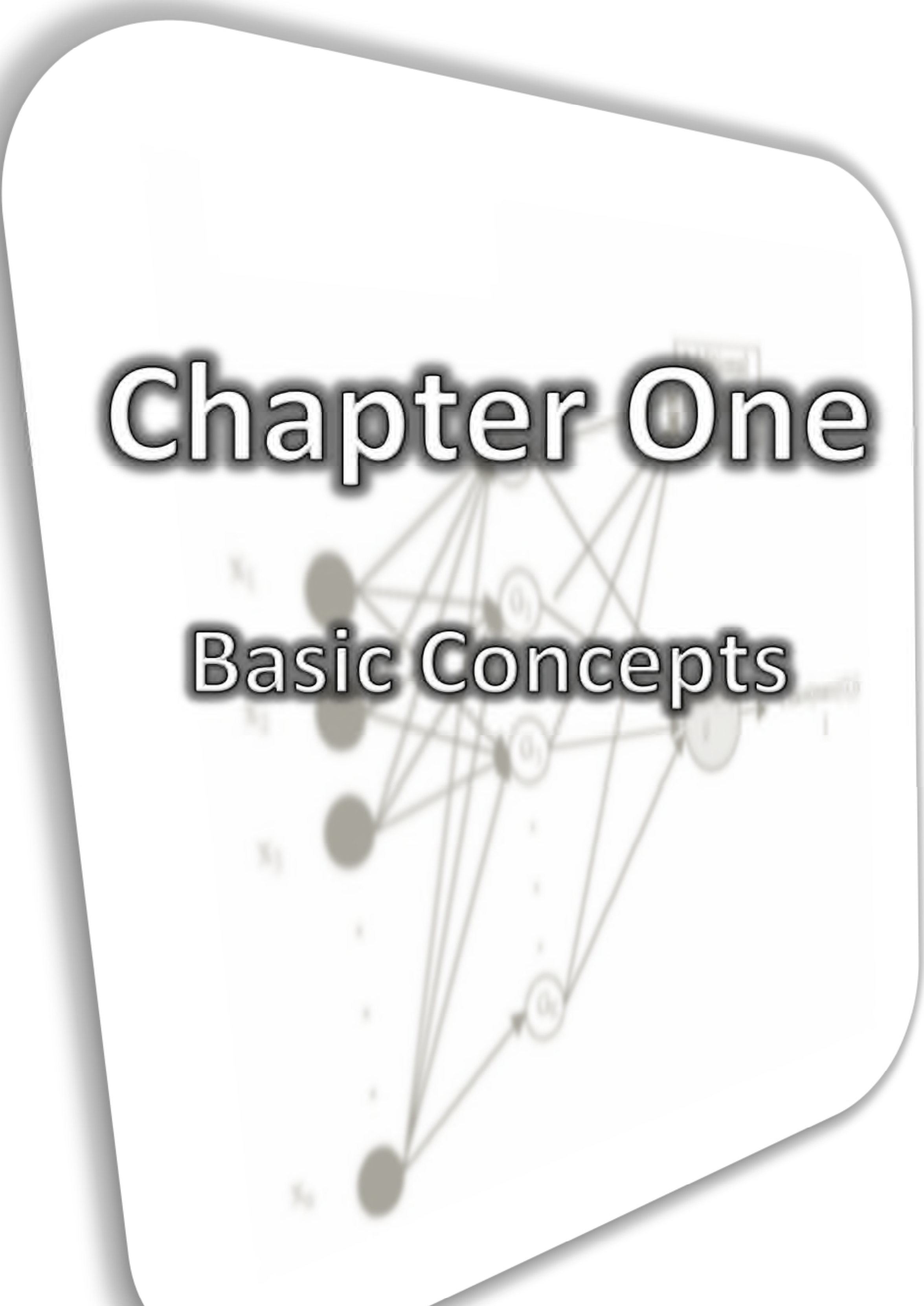
The aim of this thesis is to solve fuzzy network projects by developing three mathematical approaches using the features of probability theory.

This thesis consists of four chapters, as well as the introduction. In chapter one, the basic concepts that are needed and related to the network and linear programming problem, are presented. In chapter two, the fuzzy models related to the fuzzy set theory, fuzzy network and fuzzy linear programming problems are presented. In chapter three, some of proposed defuzzification techniques are discussed and three modified approaches are constructed based on Mellin transform, ranking method and program evaluation review technique (PERT) respectively.

Finally, in chapter four, the case study is considered and solved by our constructed methods. Our results are compared with the results obtained from the classical critical path method (CPM).

Chapter One

Basic Concepts



1 Basic Concepts

In this chapter, we are discussed a brief presentation of specific deterministic models such as Networks Projects and Linear Programming Problems, as the basic concepts which are needed in this thesis.

1.1 Networks Projects:

In this section we will present the following definitions:

Definition (1.1) Network, [24]:

The network is a flow diagram showing the sequence of operations of a process. Each individual operation is known as an activity and each meeting point or transfer stage between one activity and another is an event or node. If the activities are represented by straight lines and the events by circles, it is very simple to draw their relationships graphically, and the resulting diagram is known as the Network (Figure 1.1).

In order to show whether an activity has to be performed before or after its neighbour, arrowheads are placed on the straight lines, but it must be explained that the length or orientation of these lines is quite arbitrary. This format of network is called activity on arrow (AoA), as the activity description is written over the arrow.

It can be seen, therefore, that each activity has two nodes or events; one at the beginning and one at the end (Figure 1.2).

We can now describe the activity in two ways:

1. By its activity title (in this case A).
2. By its starting and finishing event nodes (in this case (i, j)).

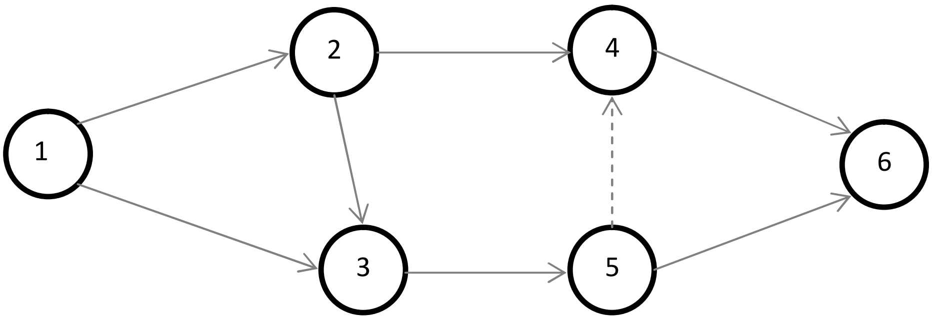


Figure (1.1) Network

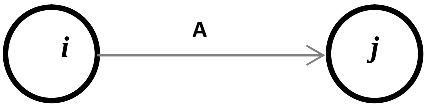


Figure (1.2) Activity

Definition (1.2) Dummy Activity, [24]:

An activity which has the duration of zero time that does not affect the logic or overall time of the project. Dummy are usually represented by dotted line arrows.

Definition (1.3) Path, [24]:

It is a series of connected activities between any two events in a network.

Definition (1.4) Critical Path, [25]:

The critical path is the longest path through a network and determines the earliest completion of project work.

A distinguishing feature of PERT is in its ability to deal with uncertainty in activity completion time. For each activity it is usually includes three time estimates:

Definition (1.5) Optimistic Time, [26]:

It is the shortest possible time in which an activity can be completed, denoted by t_o.

Definition (1.6) Pessimistic Time, [26]:

It is the best guess of the longest possible time that would be required to complete the activity, denoted by t_p.

Definition (1.7) Most Likely Time, [26]:

It represents the time that the activity would most often under normal condition, it is estimated lies between the optimistic and pessimistic time estimates, denoted by t_m.

Program evaluation and review technique (PERT) is one of the common scheduling techniques. It assumes a Beta Probability Distribution for the time estimate the expected time for each activity which can be approximated using the following weighted average:-

Expected time $(t_e) = (t_o + 4t_m + t_p)/6.$ (1.1)

which is explain in figure (1.3)

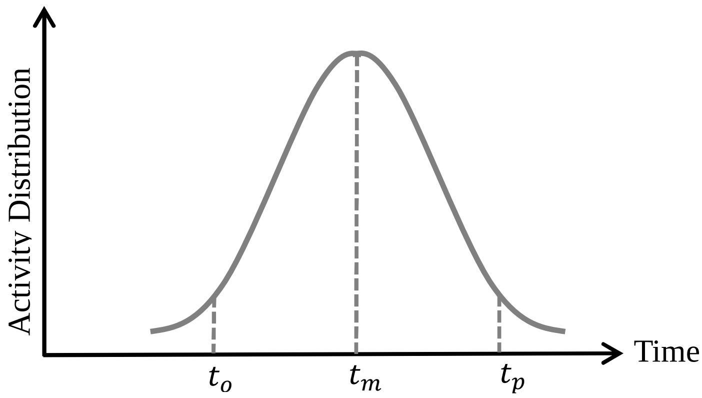


Figure (1.3): Expected Time

Definition (1.7) Earliest Expected Time, [27]:

It is the time when an event can be expected to occur, denoted by T_E. If we considered two events i and j in which i is the predecessor event and j is the successor event, and $(i \to j)$ is the activity connecting the two events, then T_E^j for a successor event j is equal to T_E^i for the predecessor event i plus the expected activity time t_E^{ij}, such that:

$$T_E^j = T_E^i + t_E^{ij} \qquad (1.2)$$

if more than one activity path leads to that event then the maximum $(T_E^i = T_E^j + t_E^{ij})$ along various activity paths.

Definition (1.8) Latest Allowable Time, [27]:

It is the time by which an event must occur, to keep the project on schedule is called the latest allowable occurrence time, denoted by symbol T_L. If we considered two events i and j in which i is the predecessor event and j is the successor event such that the latest occurrence time T_L^j be known, then the latest occurrence time T_L^i for predecessor event is given by:

$$T_L^i = T_L^j - t_E^{ij} \qquad (1.3)$$

if there are more than one successor event, the minimum of $(T_L^j - t_E^{ij})$ will be appropriate latest occurrence time T_L^i for the event i.

Since there exist only one path through the network that is the longest, the other paths must be either equal in length to or shorter than path. Therefore, there must be exist events and activities that can be completed before the time when they are actually needed.

Definition (1.9) Slack Time:

The difference between the latest and earliest times of any event is called slack. Since each activity has two events, a beginning event and an end event, it follows that there are two slacks for each activity. Thus the slack of the beginning event can be expressed as $T_{L_b} - T_{E_b}$ and called beginning slack, the slack of the end event can be expressed as $T_{L_e} - T_{E_e}$ is called end slack [24].

So far we are discussed two event times: the earliest event time T_E and the latest allowable time T_L. Since CPM networks are activity times oriented, the following activity times are useful for network computations:

1- The Earliest Starting time (*ES*).

2- The Earliest Finishing time (*EF*).

3- The Latest Starting time (*LS*).

4- The Latest Finishing time (*LF*).

If we set $T_E = 0$ for the initial event of the project, then the Forward Pass, using (1.2) to calculate the total T_E for the final event of the project. Then if we set $T_L = T_E$ on the final event of the project, then Backword Pass, using (1.3) to calculate T_L at the initial event.

1.2 Linear Programming Problem, [27]:

Programming problems in general are concerned with the use or allocation of scarce resource-labor, materials, machines, and capital-in the “best” possible manner so that costs are minimized or profits are maximized.

In using the term “best” it is implied that some choice or a set of alternative courses of actions is available for making the decision. In general, the best decision is found by solving a mathematical problem. The term linear programming merely defines a particular class of programming problems that meet the following conditions:

(1) The decision variables involved in the problem are nonnegative (i.e., positive or zero).

(2) The criterion for selecting the “best” values of the decision variables can be described by a linear function of these variables, that is, a mathematical function involving only the first powers of the variables with no cross products. The criterion function is normally referred to as the objective function.

(3) The operating rules governing the process (e.g., scarcity of resources) can be expressed as a set of linear equations or linear inequalities. This set is referred to as the constraint set.

The last two conditions are the reasons for the use of the term linear programming.

Linear programming techniques are widely used to solve a number of military, economic, industrial, and social problems. Three primary reasons for its wide use are:

(1) A large variety of problems in diverse fields can be represented or at least approximated as linear programming models.

(2) Efficient techniques for solving linear programming problems are available.

(3) Ease through which data variation (Sensitivity Analysis) can be handled through linear programming models.

The three basic steps in constructing a linear programming model are as follows:

Step I: Identify the unknown variables to be determined (decision variables), and represent them in terms of algebraic symbols.

Step II: Identify all the restrictions or constraints in the problem and express them as linear equations or inequalities which arc linear functions of the unknown variables.

Step III: Identify the objective or criterion and represent it as a linear function of the decision variables, which is to be maximized or minimized.

1.2.1 Linear Programming in Standard Form, [27]:

The linear form of linear programming problem with m constraint and n variables can be represented as follows:

$$\left.\begin{array}{ll} \text{Maximize (Minimize)} & Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n, \\ \text{Subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ & \vdots \qquad\qquad\qquad\qquad \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \\ & x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, \\ & b_1 \geq 0, b_2 \geq 0, \ldots, b_m \geq 0, m \leq n \end{array}\right\} \quad (1.4)$$

the main features of the standard form are:

1- The objective function is of maximization or minimization type.

2- All constraint are expressed as equations.

3- All variables are restricted to be nonnegative.

4- The right-hand side constant of each constraint is nonnegative.

In the matrix-vector notation, the standard linear programming problem can be expressed in a compact form as:

$$\left.\begin{array}{ll} \text{Maximize (Minimize)} & Z = cx, \\ \text{Subject to} & Ax = b, \\ & x \geq 0, \\ & b \geq 0. \end{array}\right\} \quad (1.5)$$

where A is an *(m×n)* matrix, x is an *(n×1)* column vector, b is an *(m×1)* column vector, and c is an *(1×n)* row vector.

In other words,

$$A_{(m\times n)} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, x_{(n\times 1)} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b_{(m\times 1)} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

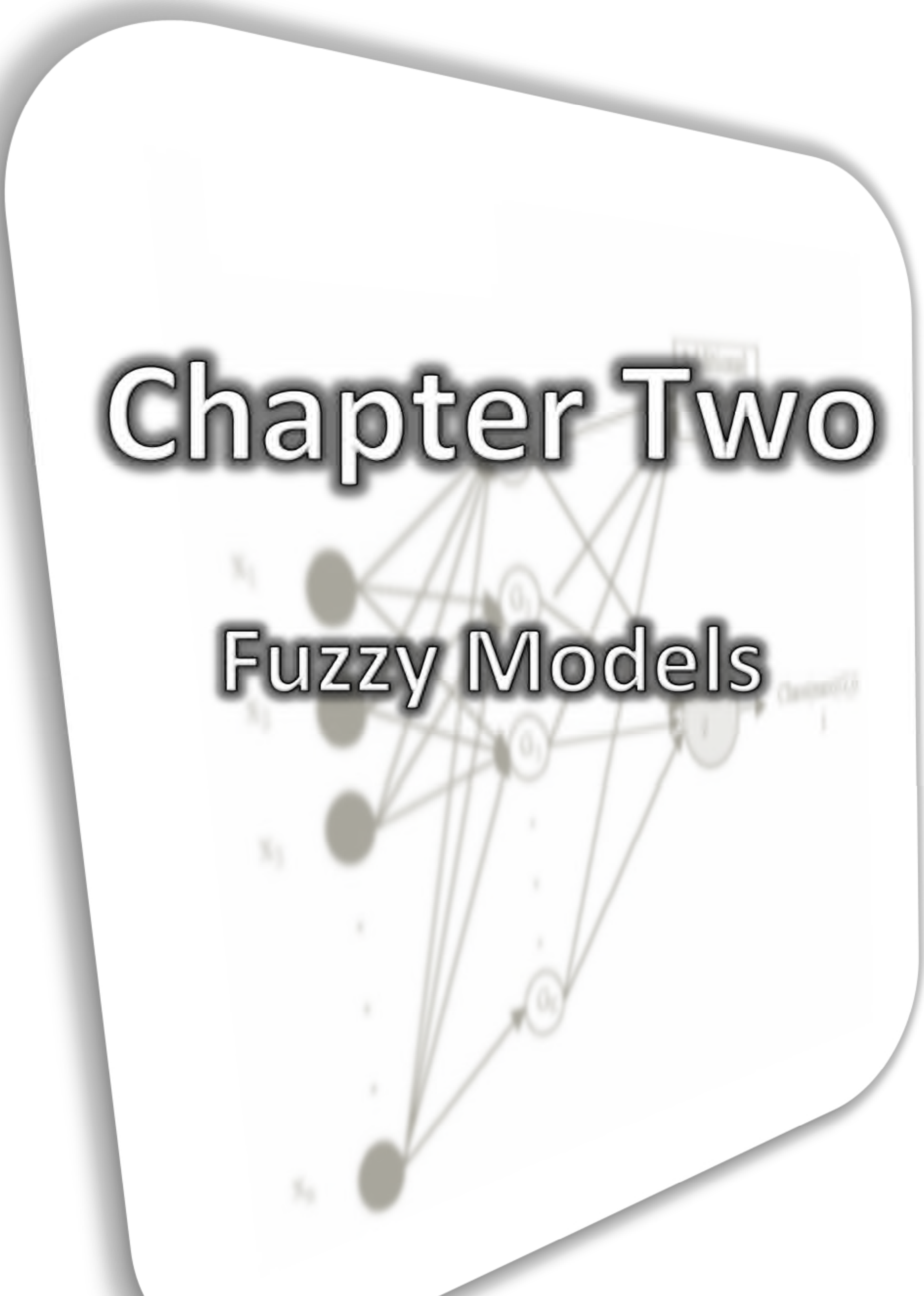
and

$$c_{(1\times n)} = (c_1, \quad c_2, \quad \cdots, \quad c_n)$$

In practice, A is called the coefficient matrix, x is the decision vector, b is the requirement vector and c is the profit (cost) vector of linear programming.

Chapter Two

Fuzzy Models



2 Fuzzy Models

In this chapter, we will present the basic concepts of fuzzy set theory, fuzzy network and fuzzy linear programming problems.

2.1 Basic Fuzzy Sets Theory, [28]:

This section deals with naive set theory when membership is no longer an all-or-nothing notion. There is no unique way to build such a theory. But, all the alternative approaches presented previously include ordinary set theory as a particular case. However Zadeh's fuzzy set theory may appear to be the most intuitive among them, although such concept as inclusion or set equality may seem too strict in this particular framework. Usually the structures embedded in fuzzy set theory are less rich than the Boolean lattice of classical set theory. Moreover, there is also some arbitrariness in the choice of the valuation set for the elements: the real interval [0,1] is most commonly used, but other choices are possible and even worth considering: these are structured set, such as fuzzy groups and convex fuzzy sets, are also presented.

Definition (2.1) Fuzzy Sets, [29],[30],[31]:

Let U be the universal set. A fuzzy set $\tilde{A}$ of U is defined by a membership function $\mu_{\tilde{A}}(x) \mapsto [0,1]$, where $\mu_{\tilde{A}}(x)$ indicates the degree of x in $\tilde{A}$ which defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & (-\infty, a_1) \\ f_1(x), & [a_1, a_2] \\ 1, & [a_2, a_3] \\ f_2(x), & [a_3, a_4] \\ 0, & (a_4, +\infty) \end{cases} \tag{2.1}$$

where a_1, a_2, a_3 and a_4 are real number, note that $f_1(x)$ and $f_2(x)$ may be linear or convex nonlinear functions.

The fuzzy sets can be expressed by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in U\}$.

Definition (2.2) Normal Fuzzy Subset, [29],[30],[31]:

A fuzzy subset $\tilde{A}$ of universal set U is normal if $sup_{x \in U} \mu_{\tilde{A}}(x) = 1$

Definition (2.3) Convex Fuzzy Subset, [29],[30],[31]:

A fuzzy subset $\tilde{A}$ of universal set U is convex iff $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq (\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{A}}(y)), \forall x, y \in U, \forall \lambda \in [0,1]$.

Definition (2.4) Fuzzy Number, [28]:

A fuzzy number $\tilde{A}$ is a convex normalized fuzzy set $\tilde{A}$ of the real line R such that:

1. It exists exactly one $x_0 \in$ R with $\mu_{\tilde{A}}(x_0) = 1$ (x_0 is called the median value of A).
2. $\mu_{\tilde{A}}(x)$ is piecewise continuous.

Definition (2.5) Triangular Fuzzy Number, [29],[30],[31]:

A triangular fuzzy number $\tilde{A}$ is a fuzzy number with a piecewise linear membership function $\mu_{\tilde{A}}$ defined by:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x < a_2 \\ 1, & x = a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 < x \leq a_3 \\ 0, & otherwise. \end{cases} \qquad (2.2)$$

which can be denoted as triplet (a_1, a_2, a_3).

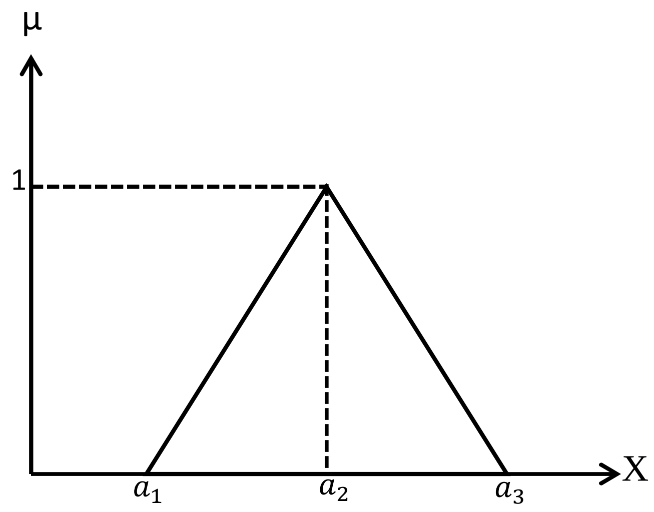


Figure (2.1): Traingular Fuzzy

Definition (2.6) Trapezoidal Fuzzy Number, [29],[30],[31]:

A trapezoidal fuzzy number $\tilde{A}$ is a fuzzy number with a membership function $\mu_{\tilde{A}}$ defined by:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1}, & a_1 \le x < a_2 \\ 1, & a_2 \le x < a_3 \\ \frac{a_4 - x}{a_4 - a_3}, & a_3 < x \le a_4 \\ 0, & otherwise. \end{cases} \tag{2.3}$$

which can be denoted as quartet (a_1, a_2, a_3, a_4).

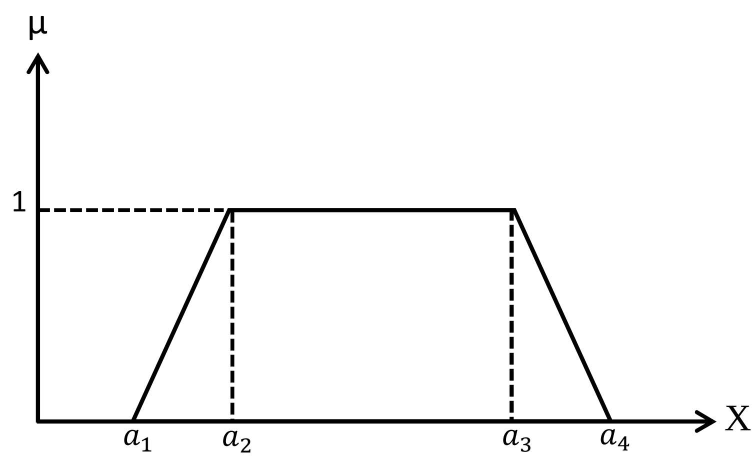


Figure (2.2): Trapezoidal Fuzzy

2.2 The Concept of a Network with Fuzzy Activity Times, [32]:

A network $S = \langle V, A, \tilde{T} \rangle$, being a project model, is given. V is a set of nodes (event) and $A \subset V \times V$ is a set of arcs (activities). The network S is a directed, compact, acyclic graph. The set $V = \{1,2,\ldots,n\}$ is labeled in such a way that the following condition holds: $(i,j) \in A \Rightarrow i < j$. By means of function $\tilde{T}$ which is defined in the following way, $\tilde{T}: A \rightarrow F(\mathbb{R}^+)$, where $F(\mathbb{R}^+)$ is the set of nonnegative fuzzy numbers. Fuzzy number $\tilde{T}(i,j) \overset{\text{def}}{=} \tilde{T}_{ij}$ determines imprecisely a duration time of activity $(i,j) \in A$. Membership function $\mu_{\tilde{T}_{ij}}$ generates a possibility distribution for the duration time of activity $(i,j) \in A$, i.e. value $\mu_{\tilde{T}_{ij}}(t_{ij})$ means a possibility degree of performance of activity (i,j) in time $t_{ij} \in \mathbb{R}^+$.

Definition (2.7) Fuzzy Critical Path, [32]:

The fuzzy set $\tilde{P}$ in set P with the membership function $\mu_{\tilde{P}}: \mathrm{P} \rightarrow [0;1]$ determined by:

$$\mu_{\tilde{P}}(p) = \sup_{\substack{t_{ij} \in \mathbb{R}^+, (i,j) \in A \\ \text{and p is critical with} \\ \text{activity times} \\ \text{equal to } t_{ij}, (i,j) \in A}} \min_{(i,j) \in A} \mu_{\tilde{T}_{ij}}(t_{ij}),\ p \in P, \tag{2.4}$$

is called the fuzzy critical path in S.

We say that a path p is fuzzy critical with the degree $\mu_{\tilde{P}}(p)$. The value $\mu_{\tilde{P}}(p)$ stands for the path degree of criticality, possibility of the criticality of path p. To put it in another way, $\mu_{\tilde{P}}$ determines a possibility distribution of the criticality of the path in the set P which is generated by possibility distributions of activities duration times $\mu_{\tilde{T}_{ij}}, (i,j) \in A$ (generated according to extension principle of Zadeh if the criticality, or lack of it, is treated as the activities duration times function in the network).

Definition (2.8) Fuzzy Critical Activity, [32]:

The fuzzy set $\tilde{A}(\tilde{E})$ in set $A(V)$ with the membership function determined by:

$$\mu_{\tilde{A}}(k,l) = \sup_{\substack{t_{ij}\in\mathbb{R}^+,(i,j)\in A \\ \text{and } (k,l) \text{ is critical with} \\ \text{activities times} \\ \text{equal to } t_{ij},(i,j)\in A}} \min_{(i,j)\in A} \mu_{\tilde{T}_{ij}}\left(t_{ij}\right),\ (k,l)\in A, \quad (2.5)$$

$$\left(\mu_{\tilde{E}}(k) = \sup_{\substack{t_{ij}\in\mathbb{R}^+,(i,j)\in A \\ \text{and k is critical with} \\ \text{activities duration times} \\ \text{equal to } t_{ij},(i,j)\in A}} \min_{(i,j)\in A} \mu_{\tilde{T}_{ij}}\left(t_{ij}\right),\ k\in V,\right) \quad (2.6)$$

is called the fuzzy critical activity (event) in S.

Also, the following theorems are stated in order to give the relations between a criticality degree of a path and criticality degrees of activities forming this path.

Theorem (2.1), [32]:

For any path $p \in P$ the following relation holds:

$$\mu_{\tilde{P}}(p) \le \mu_{\tilde{A}}(k,l), \text{ for each} (k,l) \in p. \quad (2.7)$$

Theorem (2.2), [32]:

For any path $p \in P$ the following relation holds:

$$\mu_{\tilde{P}}(p) \le \mu_{\tilde{E}}(k) \text{ for each } k \in p. \quad (2.8)$$

Theorem (2.3), [32]:

The following equality is true:

$$\mu_{\tilde{A}}(k,l) = \max_{p\in P(k,l)} \mu_{\tilde{P}}(p)\,, (k,l) \in A, \quad (2.9)$$

where

$$P(k,l) = \{p \mid p \in P \text{ and } (k,l) \in p\} \subseteq P.$$

Theorem (2.4), [32]:

The following equality is true:

$$\mu_{\tilde{E}}(k) = \max_{p \in P(k)} \mu_{\tilde{P}}(p), k \in V, \tag{2.10}$$

where

$$P(k) = \{p \mid p \in P \text{ and } k \in p\} \subseteq P.$$

2.3 Fuzzy Linear Programming, [33]:

The fuzzy sets theory proposed by Zadeh (1965) and further developed by Dubois and Prade (1988) is a popular method for dealing with decision problems that are formulated as linear programming models with imprecise, vague or uncertain variables and coefficients of the constraints.

In this section we introduce a fuzzy LP (FLP) problem where the decision variables, the coefficients of the constraints and resources (right-hand-side values) are fuzzy quantities. We then define the feasible and the optimal solution based on some fuzzy relations. Contrary to the classical LP problems, defined in section (1.2.1), x, A and b are the fuzzy numbers denoted by symbols with the tilde. Let $\mu_{\tilde{b}}: R \to [0,1], \mu_{\tilde{A}}: R \to [0,1], \mu_{\tilde{x}}: R \to [0,1]$ be the membership functions of the fuzzy numbers, $\tilde{b}, \tilde{A}$ and $\tilde{x}$, respectively. To define a FLP problem, the following proposition will be used:

Proposition (2.1): Let $\tilde{x} \in F(R)$ where $F(R)$ presents the set of all fuzzy subsets. Then, the fuzzy set $c\tilde{x}$ is a fuzzy number based on the extension principle.

The FLP problem associated with the standard LP problem (1.5) can be expressed as follows:

$$\left.\begin{array}{ll} \text{Minimize} & Z = c\tilde{x}, \\ \text{Subject to} & \tilde{A}\tilde{x} \geq \tilde{b}, \\ & \tilde{x} \geq 0. \end{array}\right\} \tag{2.11}$$

where $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots \tilde{x}_n)$ are fuzzy decision variables, $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \dots \tilde{b}_m)$ and $\tilde{A} = [\tilde{a}_{ij}]_{m\times n}$ represent the fuzzy parameters involved in the objective function and constraints while $c = (c_1, c_2, \dots c_n)$ are the crisp parameters in the objective function.

Definition (2.9) Feasible Solution, [33]:

The feasible solution is a set of values of the fuzzy variables $\tilde{x}$ which satisfies all of the constraints in model (2.11).

Definition (2.10) Optimal Solution, [33]:

The optimal solution for model (2.11) is $\tilde{x}^*$ if for all feasible solutions $\tilde{x}$, we have $c\tilde{x}^* \le c\tilde{x}$, where c is a parameter.

Definition (2.11), [33]:

Assuming that $\tilde{A} = (a_1, a_2, a_3, a_4)$ represents a trapezoidal fuzzy number, $\tilde{A}$ can be changed into the following crisp value:

$$A = \frac{(a_3 - a_2) + (a_4 - a_1)}{2}. \tag{2.12}$$

Next, we discuss the fuzzy basic feasible solution and the optimal solution.

Consider the FLP problem (2.11). After using (2.12), let $rank(A) = m$ and define the partition of A as $[B \;\; N]$ where B, $m \times m$, is non-singular matrix of basic variable and N, $m \times m$, is non-singular matrix of non-basic variable. Let y be the solution to $By = a_j$ where a_j is the j^{th} column of the coefficient matrix. Thus, $\tilde{x}_B = (\tilde{x}_{B_1}, \dots \tilde{x}_{B_m})^T = B^{-1}\tilde{b}$ and $\tilde{x}_N = 0$ is a solution of $A\tilde{x} = \tilde{b}$. $\tilde{x} = (\tilde{x}_B^T \;\; 0)$ is called a fuzzy basic solution (*FBS*) corresponding to the basic B when $\tilde{x}_B \geq 0$. It is valid that the FBS is feasible, therefore, the fuzzy objective value is $\tilde{z} = c_B\tilde{x}_B$ where $c_B = (c_{B_1}, \dots c_{B_m})$. Then:

$$z_j - c_j = c_B B^{-1} a_j - c_j$$

$$= c_B e_j - c_j = c_{B_i} - c_j$$

$$= c_j - c_j = 0.$$

Note that $B^{-1}a_j = e_j$ where $e_j = (0, \dots, 1, \dots, 0)^T$. If $\tilde{x} > 0$, then $\tilde{x}$ iscalled a no degenerate fuzzy basic feasible solution, and if one component of $\tilde{x}$ is zero, then is called a degenerate basic feasible solution. A fuzzy solution is optimal if and only if $z_j = c_B B^{-1} a_j \leq c_j$. In other words, the FLP problem can be rewritten as follows:

$$\left.\begin{array}{ll} \text{Minimize} & \tilde{Z} = c_B\tilde{x}_B + c_N\tilde{x}_N \\ \text{Subject to} & B\tilde{x}_B + N\tilde{x}_N = \tilde{b},\ \ \tilde{x}_B, \tilde{x}_N \geq 0. \end{array}\right\} \qquad (2.13)$$

If $\tilde{x}^* \leq (\tilde{x}_B^T\ \ x_N^T) = (B^{-1}\tilde{b}\ \ 0)$ is a fuzzy basic feasible solution, then, $z^* = c_B\tilde{x}_B = c_B B^{-1}\tilde{b}$.

Now, we can have

$$\tilde{z} = c\tilde{x} = c_B\tilde{x}_B + c_N\tilde{x}_N$$

$$= c_B B^{-1}\tilde{b} - (c_B B^{-1} N - c_N)\tilde{x}_N$$

$$= c_B B^{-1}\tilde{b} - \sum_{j=1}^{n} (c_B B^{-1} a_j - c_j)\tilde{x}_j$$

$$= c_B B^{-1}\tilde{b} - \sum_{j=1}^{n} (z_j - c_j)\tilde{x}_j$$

$$= \tilde{z}^* - \sum_{j=B_i} \left(z_j - c_j\right)\tilde{x}_j$$

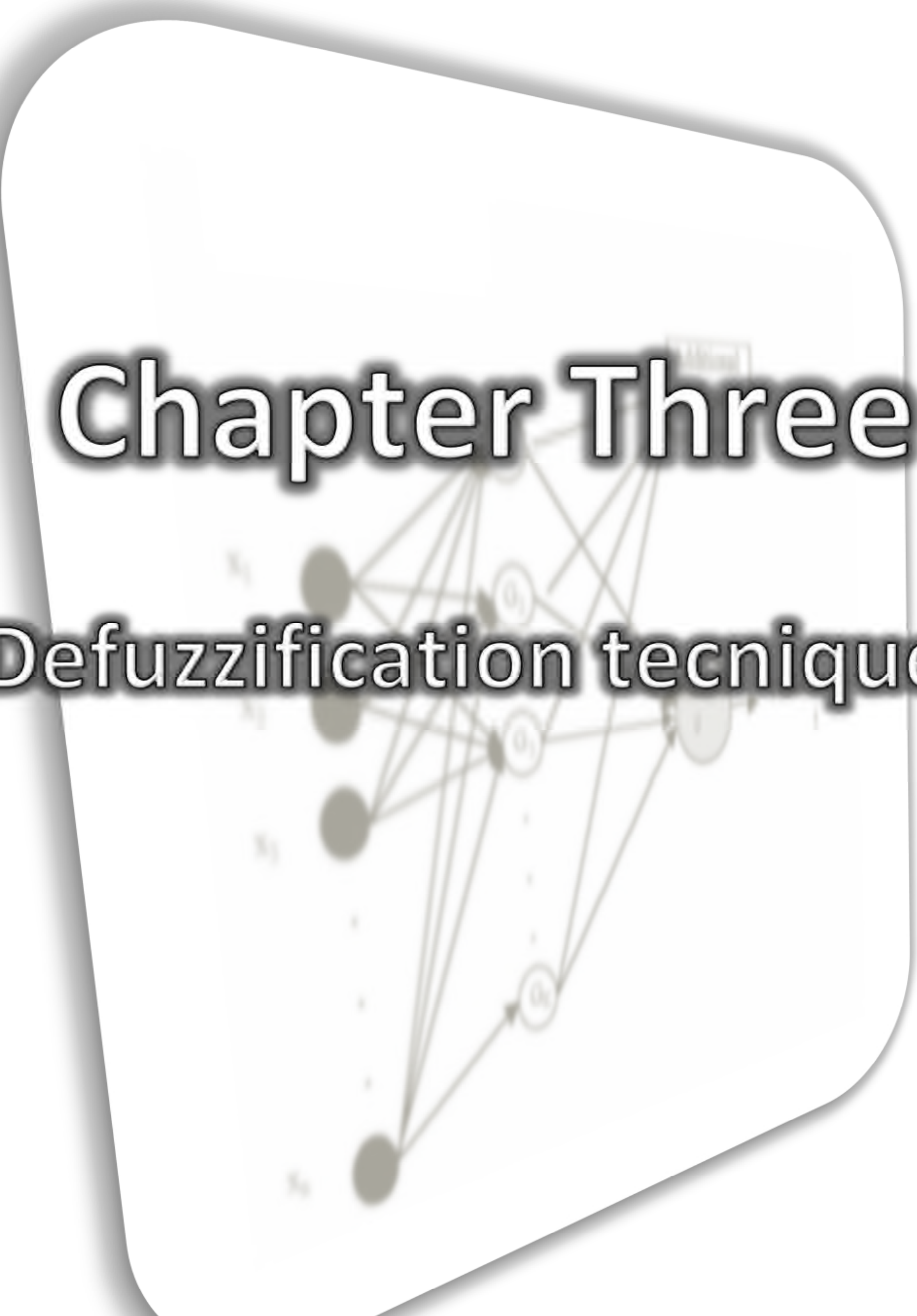
for each feasible $\tilde{x}$, z_j is smaller than or equal to c_j; therefore, $\left(z_j - c_j\right)\tilde{x}_j \leq 0$ and

$$\sum_j \left(z_j - c_j\right)\tilde{x}_j \leq 0 \rightarrow \tilde{z}^* \leq \tilde{z}.$$

That is to say, $\tilde{x}^*$ is an optimal solution.

Chapter Three

Defuzzification tecniques



3 Defuzzification Techniques

From an application point of view the following features are important: defuzzification result continuity, computational efficiency and design suitability.

Under defuzzification result continuity is considered the following feature: small changes in membership values of the output fuzzy set should not give large changes in the results of defuzzification. This feature is important in the case of fuzzy controllers, which require input-output continuity: small changes of input parameters should give small changes of output values.

Computational efficiency depends mostly on a kind and a number of operations required for obtaining the result of defuzzification.

Design suitability expresses the impact of a defuzzification technique on a software or hardware implementation and tuning of fuzzy system [34].

Definition (3.1) Defuzzification, [35]:

Deffuzification is a mapping from space of fuzzy action defined over an output universe into a space of nonfuzzy (crisp) actions.

3.1 The Overview of Defuzzification Techniques, [36],[38]:

The most often used defuzzification techniques are grouped according to the basic methods used in them: a group is made of the basic technique and of the all techniques extended from that basic technique. Extended techniques differ from the basic ones by introducing additional parameters. A fuzzy system designer defines more precisely the defuzzification process by defining those additional parameters. In the general case, defuzzification techniques can be formulated in a discrete form (using $\sum$) or in a continuous form (using $\int$). For the sake of simplicity, only discrete form is considered as follows:

3.1.1 Distribution Techniques:

The characteristic of that group of techniques is that the output fuzzy set membership function is treated as a distribution, for which the average value is evaluate. Due to that heuristic approach, the output has continuous and smooth change for the change of values of input variable in the universe of discourse. The basic technique of this group is the ***center-of- gravity*** technique, denoted by **COG** and given by the following expression.

$$y_0 = defuzzifier(\mu) = \frac{\sum_{i=1}^{N_q} \mu(y_i)\, y_i}{\sum_{i=1}^{N_q} \mu(y_i)} = cog(\mu) \tag{3.1}$$

where: N_q is the number of quantizing samples y_i, used in order to get the discrete form of the membership function $\mu(y)$ of the output fuzzy set μ. This technique is less convenient for a hardware implementation, because it requires large number of multipliers, as well as it requires passing through the whole universe of discourse of the output variable. Nevertheless, due to continuity and, often smoothness of changes of defuzzified values, this technique is used with fuzzy controllers [37].

Many best other extended techniques based on **COG** are proposed, such as ***specific*** distribution techniques. The main characteristic of the ***specific*** techniques is that the processes of aggregation, [35], and defiizzification are combined in one process, in order to improve the computational efficiency. The basic technique from that group of techniques is one referenced as the ***fuzzy mean*** **FM**. For every output fuzzy set β_i in the process of fuzzy reasoning, the degree of applicability β_i for that set is calculated. Those values, in the **FM** technique, are not used for aggregation, but are, with b_i the numerical values of output sets β_i, directly used for calculating of defuzzified value:

$$y_0 = \frac{\sum_{i=1}^{n} \beta_i\, b_i}{\sum_{i=1}^{n} \beta_i} = fm(\mu) \tag{3.2}$$

where *n* is the number of the output fuzzy sets.

Due to its computational efficiency, the **FM** technique is one of the most widely used techniques in fuzzy controllers. This technique gives relatively faster operation of the block implementing it and smaller area of its hardware implementation. It is the base for the following extended techniques: ***weighted fuzzy mean*** technique **WFM**, as well as ***quality technique*** **QT**, and ***extended quality technique*** **EQT.**

3.1.2 Maxima Techniques:

Maxima techniques give as a result of defuzzification of an element from a fuzzy set core. A fuzzy set core (denoted as ***core)*** consists of elements of a universe of discourse on which that set is defined with the highest degree of membership to the fuzzy set. As the basic representative of that group, the ***first-of-maxima*** technique **FOM**, can be considered, given by the expression (3.3):

$$y_0 = \min core(\mu) = fom(\mu) \tag{3.3}$$

Those techniques are convenient for the general fuzzy expert systems. ***Maxima*** techniques belong to the group of the fastest defuzzification techniques, because they require passing through values of the core only. According to the element with the maximal membership which is extracted as the defuzzification result, there are also the following maxima techniques: ***middle-of-maxima*** **MOM**, ***last-of-maxima*** **LOM**, and ***random-choice-of-maxima*** **RCOM**. The techniques are compatible with the max operation.

3.1.3 Area Techniques:

Area defuzzification techniques use the area under the membership function to determine the defuzzification value. ***The center-of-area*** technique **COA**, minimizes the following expression:

$$\left| \sum_{inf\,y}^{coa(\mu)} \mu(y) - \sum_{coa(\mu)}^{sup\,y} \mu(y) \right| \tag{3.4}$$

where: *inf* is the greatest lower bond, and *sup* is the least upper bound of the support of the fuzzy set μ, respectively. The expression (3.4) gives numerical value $y_0 = y_{coa(\mu)}$, which divides an area under the membership function in two (approximately) equal parts. The value $y_{coa(\mu)}$ differs from the defuzzification value obtained by the **COG** technique. The method is fast, because only simple operations are used in it, it gives continual change of defuzzification value, hence it is convenient to be used in fuzzy controllers [34].

3.1.4 Ranking Approach:

Yager (1981) proposed a procedure for ordering fuzzy sets in which a ranking index $\Re(\tilde{A})$ is calculated for the fuzzy number from its α-cut interval:

I. $A_\alpha = [a_1 + (a_2 - a_1)\alpha, a_3 - (a_3 - a_2)\alpha], \alpha \in [0,1]$ (3.5)

by considering the triangular case where $\tilde{A} = (a_1, a_2, a_3)$,

then the following formula is considered:

$$\Re(\tilde{A}) = \frac{1}{2}\left(\int_0^1 (a_1 + (a_2 - a_1)\alpha)d\alpha + \int_0^1 (a_3 - (a_3 - a_2)\alpha)d\alpha\right)$$

$$= \frac{a_1 + 2a_2 + a_3}{4}. \tag{3.6}$$

II. $A_\alpha = [a_1 + (a_2 - a_1)\alpha, a_4 - (a_4 - a_3)\alpha], \alpha \in [0,1]$ (3.7)

by considering the trapezoidal case where $\tilde{A} = (a_1, a_2, a_3, a_4)$,

then the following formula is considered:

$$\Re(\tilde{A}) = \frac{1}{2}\left(\int_0^1 (a_1 + (a_2 - a_1)\alpha)d\alpha + \int_0^1 (a_4 - (a_4 - a_3)\alpha)d\alpha\right)$$

$$= \frac{a_1 + a_2 + a_3 + a_4}{4}. \tag{3.8}$$

3.2 Proposed Defuzzification Techniques:

In the situations in which there are several output fuzzy variables, defuzzification can be considered as decision-making problem under fuzzy constraints.

Based on philosophies of probability and ranking theories we developed the following defuzzification techniques:

3.2.1 Defuzzification with Probability Density Function from Membership Function :

3.2.1.1 Fuzzy Number with Linear Membership Function:

If we consider $\tilde{A}$ is a fuzzy number with membership function defined as (2.1), where $f_1(x)$ and $f_2(x)$ are linear functions.

Let the function f defined by $f(x) = c\mu_{\tilde{A}}(x)$ is a probability density function associated with $\tilde{A}$, where c can be obtained by the property that $\int_{-\infty}^{\infty} f(x)dx = 1$ as follows:

Case (I) By considering $\tilde{A}$ as a triangular fuzzy number where $\tilde{A} = (a_1, a_2, a_3)$, with the membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1}, & a_1 \le x < a_2 \\ 1, & x = a_2 \\ \frac{a_3 - x}{a_3 - a_2}, & a_2 < x \le a_3 \\ 0, & otherwise. \end{cases} \tag{3.9}$$

and since

$$1 = \int_{a_1}^{a_3} f(x)\, dx = \int_{a_1}^{a_3} c\mu dx$$

$$= c\left[\int_{a_1}^{a_2} \frac{x-a_1}{a_2-a_1} dx + \int_{a_2}^{a_2} 1\, dx + \int_{a_2}^{a_3} \frac{a_3-x}{a_3-a_2} dx\right]$$

$$= c\left[\frac{(a_2-a_1)}{2} + \frac{(a_3-a_2)}{2}\right]$$

$$= \frac{c}{2}[a_3-a_1]$$

then

$$c = \frac{2}{a_3-a_1} \tag{3.10}$$

Case (II) By considering $\tilde{A}$ as a trapezoidal fuzzy number where $\tilde{A} = (a_1, a_2, a_3, a_4)$, with the membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & a_1 \le x < a_2 \\ 1, & a_2 \le x \le a_3 \\ \dfrac{a_4-x}{a_4-a_3}, & a_3 < x \le a_4 \\ 0, & otherwise. \end{cases} \tag{3.11}$$

and since

$$1 = \int_{a_1}^{a_4} f(x)dx = \int_{a_1}^{a_4} c\mu dx$$

$$= c\left[\int_{a_1}^{a_2} \frac{x-a_1}{a_2-a_1} dx + \int_{a_2}^{a_3} 1\, dx + \int_{a_3}^{a_4} \frac{a_4-x}{a_4-a_3} dx\right]$$

$$= c\left[\frac{(x-a_1)^2}{2(a_2-a_1)} + (a_3-a_2) + \frac{(a_4-a_3)^2}{2(a_4-a_3)}\right]$$

$$= c\left[\frac{2}{a_4+a_3-a_2-a_1}\right]$$

then

$$c = \frac{2}{a_4 + a_3 - a_2 - a_1}. \quad (3.12)$$

3.2.1.2 Fuzzy Number with Convex Nonlinear Membership Function:

If we consider $\tilde{A}$ as a fuzzy number with membership function defined by (2.1), where $f_1(x)$ and $f_2(x)$ are convex nonlinear functions.

In this case, we may use any approximate method to linearize $f_1(x)$ and $f_2(x)$, and then we processed as in section (3.2.1.1)

Now, we are using the following transformation called Mellin Transform to find the expected value.

Definition (3.2) Mellin Transform, [39]:

The Mellin transform $M_X(s)$ of a probability density function $f(x)$, where x is positive, is defined as

$$M_X(s) = \int_0^{\infty} x^{s-1} f(x)dx. \quad (3.13)$$

whenever the integral exist.

Now, it is possible to think of the Mellin transform in terms of expected values recall that the expected value of any function $g(x)$ of the random variable X, whose distribution is $f(x)$, is given by

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)f(x)dx \quad (3.14)$$

Therefor it follows that

$$M_X(s) = E[x^{s-1}] = \int_0^{\infty} x^{s-1} f(x)dx \quad (3.15)$$

Hence

$$E[x^s] = M_X(s+1) \quad (3.16)$$

Thus, the expectation of random variable X is $E[x] = M_X(2)$.

Now, if we let $\tilde{A} = (a_1, a_2, a_3, a_4)$ an arbitrary trapizoidal fuzzy number, then the density function $f(x)$ corresponding to $\tilde{A}$ is as follows:

$$f_{\tilde{A}}(x) = \begin{cases} \dfrac{2(x-a_1)}{(a_4+a_3-a_2-a_1)(a_2-a_1)}, & a_1 \le x < a_2 \\ \dfrac{2}{a_4+a_3-a_2-a_1}, & a_2 \le x \le a_3 \\ \dfrac{2(a_4-x)}{(a_4+a_3-a_2-a_1)(a_4-a_3)}, & a_3 < x \le a_4 \\ 0, & otherwise. \end{cases} \tag{3.17}$$

The Mellin transform is the obtained by:

$$M_{\tilde{A}}(s) = \int_0^\infty x^{s-1} f_{\tilde{A}}(x)dx$$

$$= \frac{2}{(a_4+a_3-a_2-a_1)(s^2+s)}\left[\frac{(a_4^{s+1}-a_3^{s+1})}{a_4-a_3} - \frac{(a_2^{s+1}-a_1^{s+1})}{a_2-a_1}\right]$$

and

$$E[\tilde{A}] = M_{\tilde{A}}(2)$$

$$= \frac{1}{3}\left[(a_1+a_2+a_3+a_4) + \frac{a_1a_2 - a_3a_4}{a_4+a_3-a_2-a_1}\right] \tag{3.18}$$

3.2.2 Extended Ranking Method:

Based on Ranking method, we built the following approach:

3.2.2.1 Fuzzy Number with Linear Membership Function:

If we consider $\tilde{A}$ as a fuzzy number with membership function defined by (2.1), where $f_1(x)$ and $f_2(x)$ are linear functions, then our approach can be illustrated into the following two cases:

Case (I) By consider $\tilde{A}$ as a triangular fuzzy number where $\tilde{A} = (a_1, a_2, a_3)$, with the membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1}, & a_1 \le x < a_2 \\ 1, & x = a_2 \\ \frac{a_3 - x}{a_3 - a_2}, & a_2 < x \le a_3 \\ 0, & otherwise. \end{cases} \tag{3.19}$$

By setting $f_{\tilde{\tilde{A}}}(x) = c\mu_{\tilde{A}}$, where

$$c = \frac{2}{a_3 + 2a_2 - a_1}$$

then

$$f_{\tilde{\tilde{A}}}(x) = \begin{cases} \frac{2(x - a_1)}{(a_3 + 2a_2 - a_1)(a_2 - a_1)}, & a_1 \le x < a_2 \\ \frac{2}{a_3 + 2a_2 - a_1}, & x = a_2 \\ \frac{2(a_3 - x)}{(a_3 + 2a_2 - a_1)(a_3 - a_2)}, & a_2 < x \le a_3 \\ 0, & otherwise. \end{cases} \tag{3.20}$$

By setting

$$f_1(x) = \frac{2(x - a_1)}{(a_3 + 2a_2 - a_1)(a_2 - a_1)} \tag{3.21}$$

$$f_2(x) = \frac{2(a_3 - x)}{(a_3 + 2a_2 - a_1)(a_3 - a_2)} \tag{3.22}$$

then

$$E_1(\tilde{\tilde{A}}) = \int_0^1 f_1^{-1}(\mu)d\mu$$

$$= \int_0^1 \left[\frac{(a_3 + 2a_2 - a_1)(a_2 - a_1)\mu}{2} + a_1\right] d\mu$$

$$= \frac{(a_3 + 2a_2 - a_1)(a_2 - a_1)}{4} + a_1 \tag{3.23}$$

$$E_2(\tilde{\tilde{A}}) = \int_0^1 f_2^{-1}(\mu)d\mu$$

$$= \int_0^1 \left[a_3 - \frac{(a_3 - a_1)(a_3 - a_2)\mu}{2}\right] d\mu$$

$$= \ a_3 - \frac{(a_3 + 2a_2 - a_1)(a_3 - a_2)}{4} \tag{3.24}$$

where $\tilde{\tilde{A}}$ is the fuzzy stochastic variable of the fuzzy number $\tilde{A}$.

Then the expected interval of fuzzy stochastic variable $\tilde{\tilde{A}}$ can be expressed as:

$$EI(\tilde{\tilde{A}}) = [E_1(\tilde{\tilde{A}}), E_2(\tilde{\tilde{A}})] \tag{3.25}$$

and the expected value is given by:

$$EV(\tilde{\tilde{A}}) = \frac{E_1(\tilde{\tilde{A}}) + E_2(\tilde{\tilde{A}})}{2}$$

$$= \frac{(2a_2 - a_3 - a_1)(2a_2 + a_3 - a_1)}{8} + \frac{a_1 + a_3}{2} \tag{3.26}$$

Case (II) By considering $\tilde{A}$ is a trapezoidal fuzzy number where $\tilde{A} = (a_1, a_2, a_3, a_4)$, with the membership function of $\tilde{A}$ is:

$$\mu_{\tilde{A}} = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x < a_2 \\ 1, & a_2 \le x \le a_3 \\ \dfrac{a_4 - x}{a_4 - a_3}, & a_3 < x \le a_4 \\ 0, & otherwise. \end{cases} \tag{3.27}$$

By setting $f_{\tilde{\tilde{A}}}(x) = c\mu_{\tilde{A}}$, where

$$c = \frac{2}{a_4 + a_3 - a_2 - a_1}$$

then

$$f_{\tilde{\tilde{A}}}(x) = \begin{cases} \dfrac{2(x-a_1)}{(a_4+a_3-a_2-a_1)(a_2-a_1)}, & a_1 \le x < a_2 \\ \dfrac{2}{a_4+a_3-a_2-a_1}, & a_2 \le x \le a_3 \\ \dfrac{2(a_4-x)}{(a_4+a_3-a_2-a_1)(a_4-a_3)}, & a_3 < x \le a_4 \\ 0, & otherwise. \end{cases} \tag{3.28}$$

By setting

$$f_1(x) = \frac{2(x-a_1)}{(a_4+a_3-a_{2-}a_1)(a_2-a_1)} \tag{3.29}$$

$$f_2(x) = \frac{2(a_4-x)}{(a_4+a_3-a_{2-}a_1)(a_4-a_3)} \tag{3.30}$$

then

$$\begin{aligned} E_1(\tilde{\tilde{A}}) &= \int_0^1 f_1^{-1}(\mu)d\mu \\ &= \int_0^1 \left[\frac{(a_4+a_3-a_{2-}a_1)(a_2-a_1)\mu}{2} + a_1\right] d\mu \\ &= \frac{(a_4+a_3-a_{2-}a_1)(a_2-a_1)}{4} + a_1. \end{aligned} \tag{3.31}$$

$$\begin{aligned} E_2(\tilde{\tilde{A}}) &= \int_0^1 f_2^{-1}(\mu)d\mu \\ &= \int_0^1 \left[a_4 - \frac{(a_4+a_3-a_{2-}a_1)(a_4-a_3)\mu}{2}\right] d\mu \\ &= a_4 - \frac{(a_4+a_3-a_{2-}a_1)(a_4-a_3)}{4}. \end{aligned} \tag{3.32}$$

where $\tilde{\tilde{A}}$ is the fuzzy stochastic variable of the fuzzy number $\tilde{A}$.

Then the expected interval of fuzzy stochastic variable $\tilde{\tilde{A}}$ can be expressed as:

$$EI(\tilde{\tilde{A}}) = [E_1(\tilde{\tilde{A}}), E_2(\tilde{\tilde{A}})] \tag{3.33}$$

and the expected value is given by:

$$EV(\tilde{\tilde{A}}) = \frac{E_1(\tilde{\tilde{A}}) + E_2(\tilde{\tilde{A}})}{2}$$

$$= \frac{(a_4 + a_3 - a_{2-}a_1)[(a_2 - a_1) - (a_4 - a_3)}{8} + \frac{a_1 + a_4}{2} \quad (3.34)$$

3.2.2.2 Fuzzy Number with Convex Nonlinear Membership Function:

If we consider $\tilde{A}$ as a fuzzy number with membership function defined by (2.1), where $f_1(x)$ and $f_2(x)$ are convex nonlinear functions.

In this case, we must use any approximate method to linearize $f_1(x)$ and $f_2(x)$, and then we processed as in section (3.2.2.1).

3.2.3 Interval Method:

3.2.3.1 Fuzzy Number with Linear Membership Function:

Recall (3.7), we calculate the following:

1) The optimistic time $t_o = a_1 + (a_2 - a_1)\mu$,
2) The pessimistic time $t_p = a_4 - (a_4 - a_3)\mu$, and setting
3) The most likely time $t_m = (t_o + t_p)/2$.

Now, the expected time t_e which is crisp is calculated as follows:

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

$$= \frac{(a_1 + a_4) - (a_1 - a_2 - a_3 + a_4)\mu}{2} \quad (3.35)$$

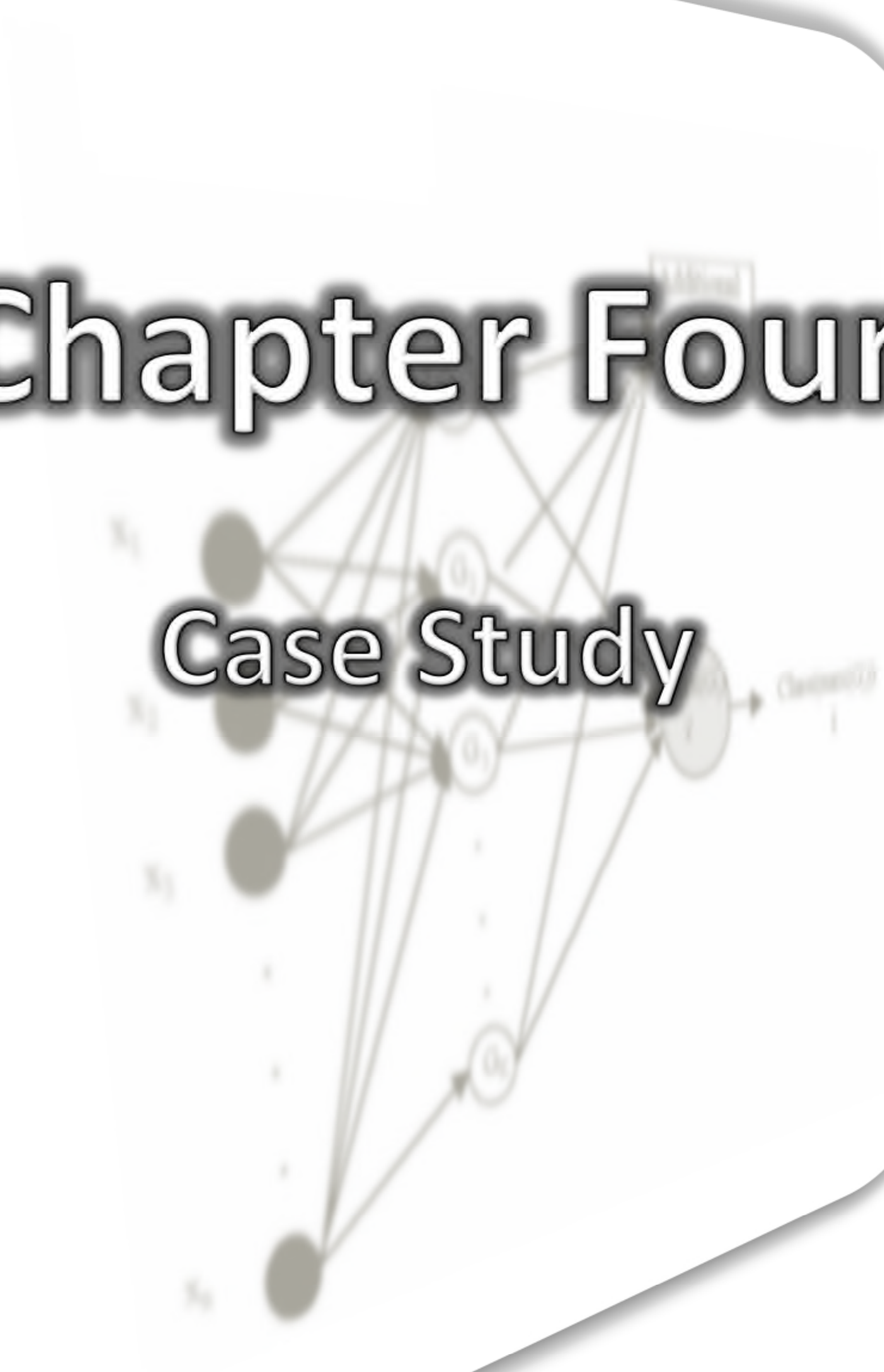
3.2.2.2 Fuzzy Number with Convex Nonlinear Membership Function:

If we consider $\tilde{A}$ as a fuzzy number with membership function defined by (2.1), where $f_1(x)$ and $f_2(x)$ are convex nonlinear functions.

In this case, we may use any approximate method to linearize $f_1(x)$ and $f_2(x)$, and then we processed as in section (3.2.3.1).

Chapter Four

Case Study



4 Case Study

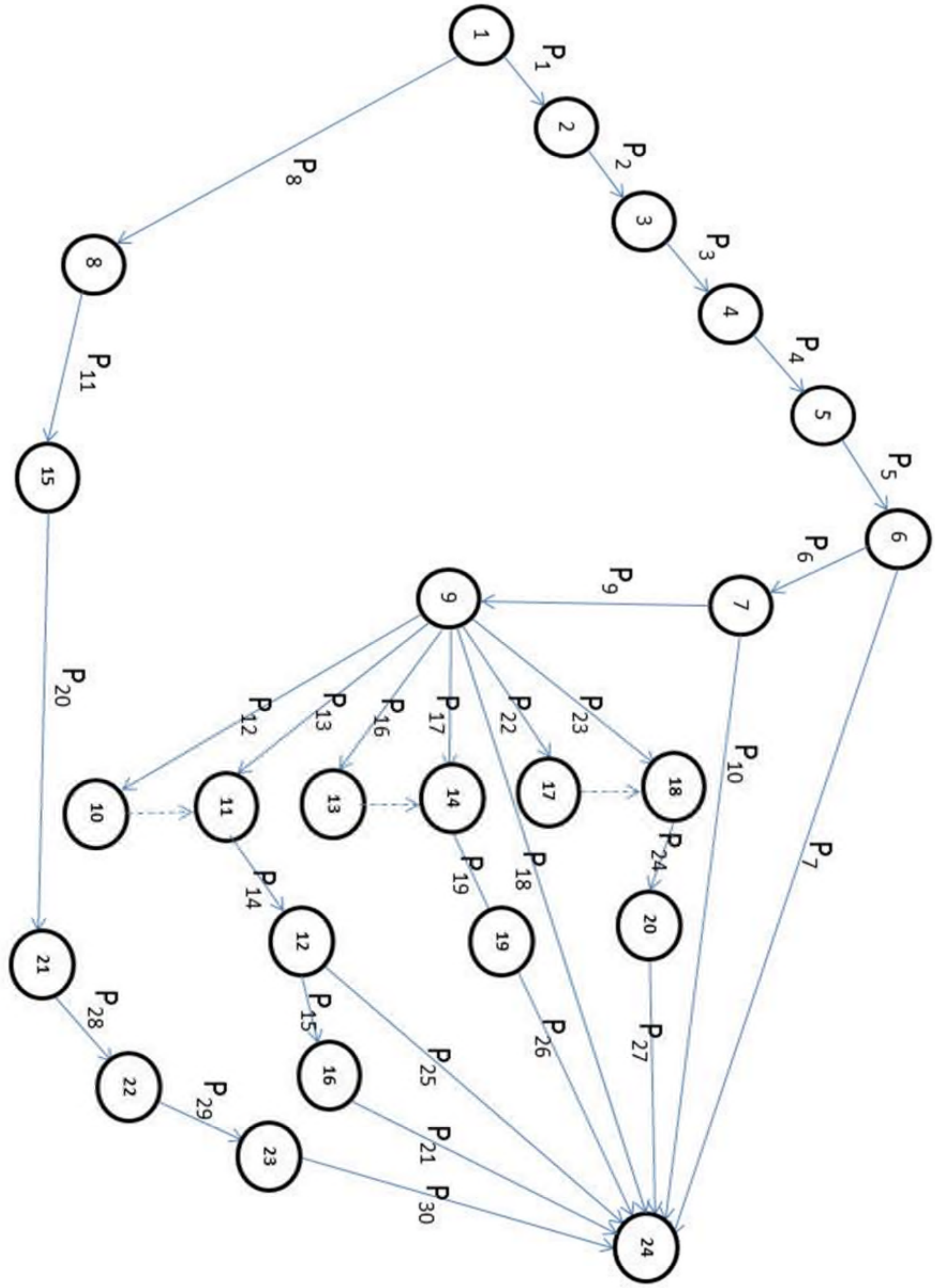
Foreign multinational companies are willing to invest in construction and building projects in Egypt, many factors help this type of projects to be developed and expanded over time in Egypt. Makro is a German company that works in hyper supermarkets, specialized in mass trade, selling food stuffs like: meats, fishes, vegetables, fruits, sugar, macaroni, rice, and many other commodities. It has many branches in several countries like Turkey, Germany, France, Italy and more than 30 other countries. A feasibility study was done by the company to enter the Egyptian market by investing 4.5 milliards dollars, and building 45 branches all over Egypt in the next five years; the estimated budgeted cost for each mall is 100 millions dollars, the first branch is planned to be built in Al-Salam City, the mall consists of one floor store of 15000 m^2 steel structure building, a 30000 m^2 parking area, and 5000 m^2 backyard for trucks maneuvering [40].

4.1 Problem definition, [40]:

As being the responsible of the co-ordination between the companies working in the project and scheduling the whole project's activities, the Egyptian consultant should be so precise in scheduling the times of the activities and the whole project duration. The consultant should have interactive discussions, agreements, and decisions with the executive companies to optimize both the time and the cost of the project, any deviation in the assessment of the activitie's times will lead to extra cost and time. The activitie's duration times in the project are not deterministic and imprecise so the concept of fuzziness is employed to deal with the vague activity times. The Egyptian consultant scheduled the project into 30 activities and represented their times by fuzzy sets after asking the experts, interacting with the companies to build the membership functions used. As shown in table (4.1); the 30 activity are listed with their fuzzy operation time.

Table (4.1): Construction Project

Activity Item	Activity Description	Precedence Item	Fuzzy Operation Time (per day)
P_1	Concrete works foundation	-	(25,28,30,35)
P_2	Insulation works	P_1	(3,4,4,5)
P_3	Parking area + Roads + Landscape	P_2	(25,29,30,35)
P_4	Back filling works	P_3	(3,7,12,15)
P_5	Sub-base	P_4	(5,6,6,10)
P_6	Steel structure erection	P_5	(26,30,35,40)
P_7	Under ground drainage system	P_5	(7,10,10,13)
P_8	Water tank - civil works	-	(15,21,21,25)
P_9	Steel structure testing	P_6	(2,3,4,5)
P_{10}	Roofing works	P_6	(9,10,12,15)
P_{11}	Water tank – finishing	P_8	(6,7,8,10)
P_{12}	HVAC works - 1st fix	P_9	(12,14,14,16)
P_{13}	Fire fighting works 1st fix	P_9	(7,9,11,12)
P_{14}	Electrical system works - 1st fix	P_{12}, P_{13}	(5,6,7,10)
P_{15}	Flooring	P_{14}	(7,9,11,12)
P_{16}	HVAC work-2nd fix	P_9	(12,14,14,16)
P_{17}	Fire fighting works – 2nd fix	P_9	(7,9,11,12)
P_{18}	Cladding works	P_9	(15,24,25,30)
P_{19}	Electrical system works - 2nd fix	P_{16},P_{17}	(5,6,7,10)
P_{20}	Water tank – MEP	P_{11}	(9,11,12,14)
P_{21}	Finishing works	P_{15}	(15,18,18,20)
P_{22}	HVAC works - 3rd	P_9	(12,14,14,16)
P_{23}	Fire fighting work - 3rd fix	P_9	(7,9,11,12)
P_{24}	Electrical system works -3rd fix	P_{22},P_{23}	(5,6,7,10)
P_{25}	Plumbing works - 1st fix	P_{14}	(5,6,6,8)
P_{26}	Plumbing works – 2nd fix	P_{19}	(5,6,6,8)
P_{27}	Plumbing works - 3rd fix	P_{24}	(5,6,6,8)
P_{28}	Water tank testing	P_{20}	(1,2,2,3)
P_{29}	Testing and commissioning	P_{28}	(1,2,2,3)
P_{30}	Snag list and Initial handling	P_{29}	(5,7,7,9)



In order to solve such problem, two methods are derived in [40] to convert the fuzzy time number into crisp time number and find the optimum value for objective function and the critical path for activities by using two different linear programming models.

In this thesis, three defuzzification approaches are implemented using FLPP to solve this problem and compared these results with the results obtained by using CPM technique.

4.2 First Approach :

In this approach our problem can be expressed as the following (0-1) integer model (one objective function) with fuzzy time number, which can be written as:

Maximize

$$Z = (25,28,30,35)P_1 + (3,4,4,5)P_2 + (25,29,30,35)P_3 + (3,7,12,15)P_4 + (5,6,6,10)P_5 + (26,30,35,40)P_6 + (7,10,10,13)P_7 + (15,21,21,25)P_8 + (2,3,4,5)P_9 + (9,10,12,15)P_{10} + (6,7,8,10)P_{11} + (12,14,14,16)P_{12} + (7,9,11,12)P_{13} + (5,6,7,10)P_{14} + (7,9,11,12)P_{15} + (12,14,14,16)P_{16} + (7,9,11,12)P_{17} + (15,24,25,30)P_{18} + (5,6,7,10)P_{19} + (9,11,12,14)P_{20} + (15,18,18,20)P_{21} + (12,14,14,16)P_{22} + (7,9,11,12)P_{23} + (5,6,7,10)P_{24} + (5,6,6,8)P_{25} + (5,6,6,8)P_{26} + (5,6,6,8)P_{27} + (1,2,2,3)P_{28} + (1,2,2,3)P_{29} + (5,7,7,9)P_{30}$$

Subject to

$P_1 + P_8 = 1$

$P_1 = P_2$

$P_2 = P_3$

$P_3 = P_4$

$P_4 = P_5$

$P_5 = P_6 + P_7$

$P_6 = P_9 + P_{10}$

$P_8 = P_{11}$

$P_{11} = P_{20}$

$P_{20} = P_{28}$

$P_{28} = P_{29}$

$P_{29} = P_{30}$

$P_9 = P_{12} + P_{13} + P_{16} + P_{17} + P_{18} + P_{22} + P_{23}$

$P_{12} + P_{13} = P_{14}$

$P_{14} + P_{15} = P_{25}$

$P_{15} = P_{21}$

$P_{16} + P_{17} = P_{19}$

$P_{19} = P_{26}$

$P_{22} + P_{23} = P_{24}$

$P_{24} = P_{27}$

$P_7 + P_{10} + P_{18} + P_{21} + P_{25} + P_{26} + P_{27} + P_{30} = 1$

$P_j \geq 0, j = 1,2, \dots ,30.$

Now, by using the first method that we constructed in (3.2.1) to convert the FLPP into CLPP with crisp obtained numbers, as shown in table (4.2), we can rewrite the objective function with crisp numbers as:

Maximize

$$Z = 29.611P_1 + 4P_2 + 29.818P_3 + 9.215P_4 + 7P_5 + 32.789P_6 + 10P_7 + 20.333P_8 + 3.5P_9 + 11.583P_{10} + 7.8P_{11} + 14P_{12} + 9.714P_{13} + 7.111P_{14} + 9.714P_{15} + 14P_{16} + 9.714P_{17} + 23.208P_{18} + 7.111P_{19} + 11.5P_{20} + 17.666P_{21} + 14P_{22} + 9.714P_{23} + 7.111P_{24} + 6.333P_{25} + 6.333P_{26} + 6.333P_{27} + 2P_{28} + 2 P_{29} + 7P_{30}$$

Subject to

$P_1 + P_8 =1$

$P_1 = P_2$

$P_2 = P_3$

$P_3 = P_4$

$P_4 = P_5$

$P_5 = P_6 + P_7$

$P_6 = P_9 + P_{10}$

$P_8 = P_{11}$

$P_{11} = P_{20}$

$P_{20} = P_{28}$

$P_{28} = P_{29}$

$P_{29} = P_{30}$

$P_9 = P_{12} + P_{13} + P_{16} + P_{17} + P_{18} + P_{22} + P_{23}$

$P_{12} + P_{13} = P_{14}$

$P_{14} + P_{15} = P_{25}$

$P_{15} = P_{21}$

$P_{16} + P_{17} = P_{19}$

$P_{19} = P_{26}$

$P_{22} + P_{23} = P_{24}$

$P_{24} = P_{27}$

$P_7 + P_{10} + P_{18} + P_{21} + P_{25} + P_{26} + P_{27} + P_{30} = 1$

$P_j \geq 0, j = 1,2, \ldots ,30.$

Table (4.2): Problem Crisp Operation Time

Activity Item	Activity Description	Precedence Item	Fuzzy Operation Time (per day)	Crisp Operation Time (per day)
P_1	Concrete works foundation	-	(25,28,30,35)	29.611
P_2	Insulation works	P_1	(3,4,4,5)	4
P_3	Parking area + Roads + Landscape	P_2	(25,29,30,35)	29.818
P_4	Back filling works	P_3	(3,7,12,15)	9.215
P_5	Sub-base	P_4	(5,6,6,10)	7
P_6	Steel structure erection	P_5	(26,30,35,40)	32.789
P_7	Under ground drainage system	P_5	(7,10,10,13)	10
P_8	Water tank - civil works	-	(15,21,21,25)	20.333
P_9	Steel structure testing	P_6	(2,3,4,5)	3.5
P_{10}	Roofing works	P_6	(9,10,12,15)	11.583
P_{11}	Water tank – finishing	P_8	(6,7,8,10)	7.8
P_{12}	HVAC works - 1st fix	P_9	(12,14,14,16)	14
P_{13}	Fire fighting works 1st fix	P_9	(7,9,11,12)	9.714
P_{14}	Electrical system works - 1st fix	P_{12}, P_{13}	(5,6,7,10)	7.111
P_{15}	Flooring	P_{14}	(7,9,11,12)	9.714
P_{16}	HVAC work-2nd fix	P_9	(12,14,14,16)	14
P_{17}	Fire fighting works – 2nd fix	P_9	(7,9,11,12)	9.714
P_{18}	Cladding works	P_9	(15,24,25,30)	23.208
P_{19}	Electrical system works - 2nd fix	P_{16},P_{17}	(5,6,7,10)	7.111
P_{20}	Water tank – MEP	P_{11}	(9,11,12,14)	11.5
P_{21}	Finishing works	P_{15}	(15,18,18,20)	17.666
P_{22}	HVAC works - 3rd	P_9	(12,14,14,16)	14
P_{23}	Fire fighting work - 3rd fix	P_9	(7,9,11,12)	9.714
P_{24}	Electrical system works -3rd fix	P_{22},P_{23}	(5,6,7,10)	7.111
P_{25}	Plumbing works - 1st fix	P_{14}	(5,6,6,8)	6.333
P_{26}	Plumbing works – 2nd fix	P_{19}	(5,6,6,8)	6.333
P_{27}	Plumbing works - 3rd fix	P_{24}	(5,6,6,8)	6.333
P_{28}	Water tank testing	P_{20}	(1,2,2,3)	2
P_{29}	Testing and commissioning	P_{28}	(1,2,2,3)	2
P_{30}	Snag list and Initial handling	P_{29}	(5,7,7,9)	7

On solving CLP problem, the following feasible solutions are obtained:

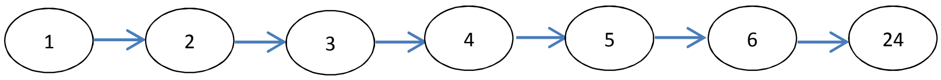
1- $P_1 = P_2 = P_3 = P_4 = P_5 = P_7 = 1$

and

$P_6 = P_9 = P_{10} = P_{12} = P_{13} = P_{16} = P_{17} = P_{18} = P_{22} = P_{23} = P_{24} = P_{27} = P_{19} = P_{26}$

$= P_{14} = P_{15} = P_{25} = P_{21} = P_8 = P_{11} = P_{20} = P_{28} = P_{29} = P_{30} = 0$

that mean the path solution is:



with path time value =89.644 days.

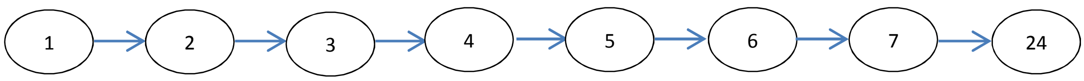
2- $P_1 = P_2 = P_3 = P_4 = P_5 = P_6 = P_{10} = 1$

and

$P_7 = P_9 = P_{12} = P_{13} = P_{16} = P_{17} = P_{18} = P_{22} = P_{23} = P_{24} = P_{27} = P_{19} = P_{26} = P_{14}$

$= P_{15} = P_{25} = P_{21} = P_8 = P_{11} = P_{20} = P_{28} = P_{29} = P_{30} = 0$

that mean the path solution is:



with path time value =124.016 days.

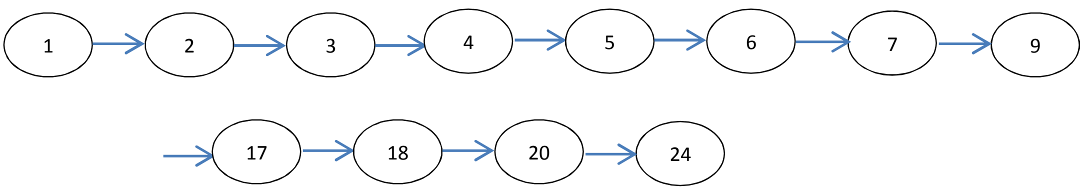
3- $P_1 = P_2 = P_3 = P_4 = P_5 = P_6 = P_9 = P_{23} = P_{22} = P_{24} = P_{27} = 1$

and

$P_7 = P_{10} = P_{18} = P_{17} = P_{19} = P_{26} = P_{16} = P_{13} = P_{14} = P_{15} = P_{25} = P_{21} = P_{12} =$

$P_8 = P_{11} = P_{20} = P_{28} = P_{29} = P_{30} = 0$

that mean the path solution is:



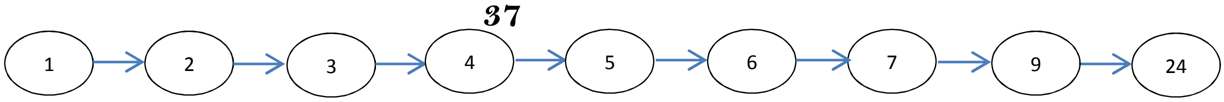
with path time value =153.091 days.

4- $P_1 = P_2 = P_3 = P_4 = P_5 = P_6 = P_9 = P_{18} = 1$

and

$P_7 = P_{10} = P_{23} = P_{22} = P_{24} = P_{27} = P_{17} = P_{19} = P_{26} = P_{16} = P_{13} = P_{14} = P_{15} =$

$P_{25} = P_{21} = P_{12} = P_8 = P_{11} = P_{20} = P_{28} = P_{29} = P_{30} = 0$



that mean the path solution is:

with path time value =139.141 days.

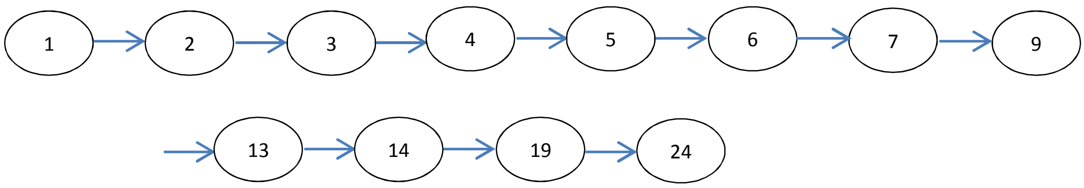
5- $P_1 = P_2 = P_3 = P_4 = P_5 = P_6 = P_9 = P_{17} = P_{16} = P_{19} = P_{26} = 1$

and

$P_7 = P_{10} = P_{23} = P_{22} = P_{24} = P_{27} = P_{18} = P_{13} = P_{12} = P_{14} = P_{15} = P_{21} = P_{25} = P_8$

$= P_{11} = P_{20} = P_{28} = P_{29} = P_{30} = 0$

that mean the path solution is:



with path time value =153.091 days.

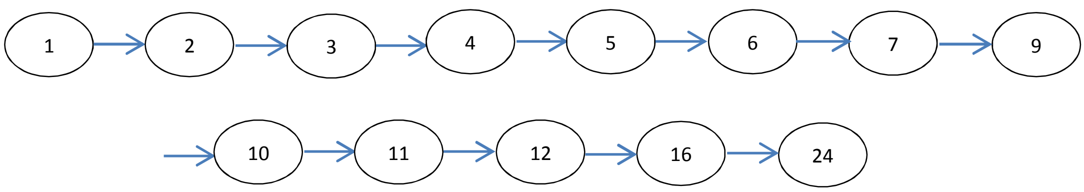
6- $P_1 = P_2 = P_3 = P_4 = P_5 = P_6 = P_9 = P_{12} = P_{14} = P_{15} = P_{21} = 1$

and

$P_7 = P_{10} = P_{13} = P_{23} = P_{22} = P_{24} = P_{27} = P_{18} = P_{17} = P_{16} = P_{19} = P_{26} = P_8 = P_{11}$

$= P_{20} = P_{28} = P_{29} = P_{25} = P_{30} = 0$

that mean the path solution is:



with path time value =164.424 days.

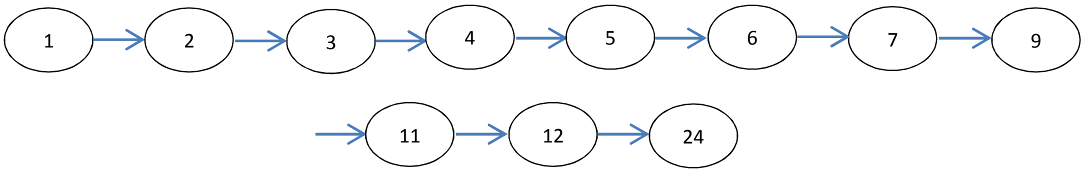
7- $P_1 = P_2 = P_3 = P_4 = P_5 = P_6 = P_9 = P_{13} = P_{14} = P_{25} = 1$

and

$P_7 = P_{10} = P_{23} = P_{22} = P_{24} = P_{27} = P_{18} = P_{17} = P_{16} = P_{15} = P_{19} = P_{26} = P_8 = P_{11}$

$= P_{20} = P_{28} = P_{29} = P_{21} = P_{30} = 0$

that mean the path solution is:



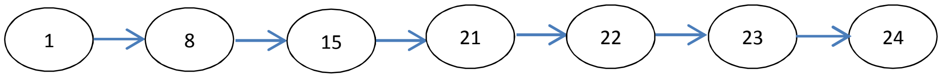
with path time value =153.091 days.

8- $P_8 = P_{11} = P_{20} = P_{28} = P_{29} = P_{30} = 1$

and

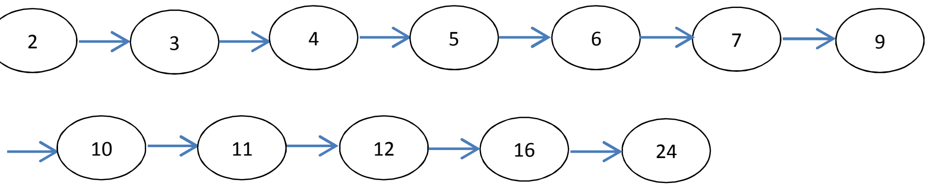
$P_1 = P_2 = P_3 = P_4 = P_5 = P_6 = P_7 = P_9 = P_{10} = P_{23} = P_{22} = P_{24} = P_{27} = P_{18} = P_{17} = P_{19} = P_{26} = P_{16} = P_{13} = P_{12} = P_{14} = P_{15} = P_{25} = P_{21} = 0$

that mean the path solution is:



with path time value =50.633 days.

There for from the above results the optimal path solution (critical path) is:



which has the maximum time value = 164.424 days.

4.3 Second Approach:

In this approach the extended ranking method is implemented to transform the problem fuzzy data into a crisp values as follows:

The problem can be illustrated using the following standard model with fuzzy time numbers in the right-hand-side of all constraints, which can be written as:

Minimize

$$Z = t_{24} - t_1$$

Subject to

$$t_2 - t_1 \geq P_1$$

$$t_3 - t_2 \geq P_2$$

$$t_4 - t_3 \geq P_3$$

$$t_5 - t_4 \geq P_4$$

$$t_6 - t_5 \geq P_5$$

$$t_7 - t_6 \geq P_6$$

$$t_{24} - t_6 \geq P_7$$

$$t_8 - t_1 \geq P_8$$

$$t_9 - t_7 \geq P_9$$

$$t_{24} - t_7 \geq P_{10}$$

$$t_{15} - t_8 \geq P_{11}$$

$$t_{10} - t_9 \geq P_{12}$$

$$t_{11} - t_9 \geq P_{13}$$

$$t_{12} - t_{11} \geq P_{14}$$

$$t_{16} - t_{12} \geq P_{15}$$

$$t_{13} - t_9 \geq P_{16}$$

$$t_{14} - t_9 \geq P_{17}$$

$$t_{24} - t_9 \geq P_{18}$$

$$t_{19} - t_{14} \geq P_{19}$$

$$t_{21} - t_{15} \geq P_{20}$$

$$t_{24} - t_{16} \geq P_{21}$$

$$t_{17} - t_9 \geq P_{22}$$

$$t_{18} - t_9 \geq P_{23}$$

$$t_{20} - t_{18} \geq P_{24}$$

$$t_{24} - t_{12} \geq P_{25}$$

$$t_{24} - t_{19} \geq P_{26}$$

$$t_{24} - t_{20} \geq P_{27}$$

$$t_{22} - t_{21} \geq P_{28}$$

$t_{23} - t_{22} \geq P_{29}$

$t_{24} - t_{23} \geq P_{30}$

where $t_i \geq 0, i = 1,2,3, \ldots ,24$, which represent the events (nodes) of the network problem and P_j, j=1,2,…,30, is the activity fuzzy time number.

Using (3.34) to convert the 30 fuzzy time activities into crisp numbers, as shown in table (4.3), the standard model can be rewritten into a deterministic model as follows:

Minimize

$Z = t_{24} - t_1$

Subject to

$t_2 - t_1 \geq 7$

$t_3 - t_2 \geq 4$

$t_4 - t_3 \geq 28.625$

$t_5 - t_4 \geq 11.125$

$t_6 - t_5 \geq 5.625$

$t_7 - t_6 \geq 30.625$

$t_{24} - t_6 \geq 10$

$t_8 - t_1 \geq 22.5$

$t_9 - t_7 \geq 3.5$

$t_{24} - t_7 \geq 10$

$t_{15} - t_8 \geq 7.375$

$t_{10} - t_9 \geq 14$

$t_{11} - t_9 \geq 10.375$

$t_{12} - t_{11} \geq 6$

$t_{16} - t_{12} \geq 10.375$

$t_{13} - t_9 \geq 14$

$t_{14} - t_9 \geq 10.375$

$t_{24} - t_9 \geq 30.5$

$t_{19} - t_{14} \geq 6$

$t_{21} - t_{15} \geq 10.75$

$t_{24} - t_{16} \geq 16.875$

$t_{17} - t_{9} \geq 14$

$t_{18} - t_{9} \geq 10.375$

$t_{20} - t_{18} \geq 6$

$t_{24} - t_{12} \geq 6.125$

$t_{24} - t_{19} \geq 6.125$

$t_{24} - t_{20} \geq 6.125$

$t_{22} - t_{21} \geq 2$

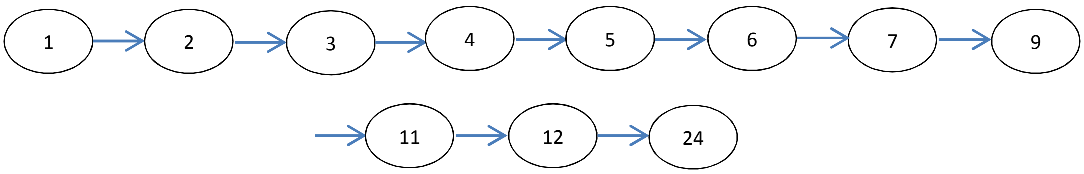
$t_{23} - t_{22} \geq 2$

$t_{24} - t_{23} \geq 7$

where $t_i \geq 0, \; i = 1,2,3, \dots ,24$

The critical activities are determined and the optimal value of the objective function is calculated using "Matlab R2010b" software. The results are shown in the table (4.4).

It's clear that from the table (4.4) the optimal path solution (critical path) is:



which has the minimum time value = 154.125 days.

Table (4.3): Problem Crisp Operation Time

Activity Item	Activity Description	Precedence Item	Fuzzy Operation Time (per day)	Crisp Operation Time (per day)
P_1	Concrete works foundation	-	(25,28,30,35)	27
P_2	Insulation works	P_1	(3,4,4,5)	4
P_3	Parking area + Roads + Landscape	P_2	(25,29,30,35)	28.625
P_4	Back filling works	P_3	(3,7,12,15)	11.125
P_5	Sub-base	P_4	(5,6,6,10)	5.625
P_6	Steel structure erection	P_5	(26,30,35,40)	30.625
P_7	Under ground drainage system	P_5	(7,10,10,13)	10
P_8	Water tank - civil works	-	(15,21,21,25)	22.5
P_9	Steel structure testing	P_6	(2,3,4,5)	3.5
P_{10}	Roofing works	P_6	(9,10,12,15)	10
P_{11}	Water tank – finishing	P_8	(6,7,8,10)	7.375
P_{12}	HVAC works - 1st fix	P_9	(12,14,14,16)	14
P_{13}	Fire fighting works 1st fix	P_9	(7,9,11,12)	10.375
P_{14}	Electrical system works - 1st fix	P_{12}, P_{13}	(5,6,7,10)	6
P_{15}	Flooring	P_{14}	(7,9,11,12)	10.375
P_{16}	HVAC work-2nd fix	P_9	(12,14,14,16)	14
P_{17}	Fire fighting works – 2nd fix	P_9	(7,9,11,12)	10.375
P_{18}	Cladding works	P_9	(15,24,25,30)	30.5
P_{19}	Electrical system works - 2nd fix	P_{16},P_{17}	(5,6,7,10)	6
P_{20}	Water tank – MEP	P_{11}	(9,11,12,14)	10.75
P_{21}	Finishing works	P_{15}	(15,18,18,20)	16.875
P_{22}	HVAC works - 3rd	P_9	(12,14,14,16)	14
P_{23}	Fire fighting work - 3rd fix	P_9	(7,9,11,12)	10.375
P_{24}	Electrical system works -3rd fix	P_{22},P_{23}	(5,6,7,10)	6
P_{25}	Plumbing works - 1st fix	P_{14}	(5,6,6,8)	6.125
P_{26}	Plumbing works – 2nd fix	P_{19}	(5,6,6,8)	6.125
P_{27}	Plumbing works - 3rd fix	P_{24}	(5,6,6,8)	6.125
P_{28}	Water tank testing	P_{20}	(1,2,2,3)	2
P_{29}	Testing and commissioning	P_{28}	(1,2,2,3)	2
P_{30}	Snag list and Initial handling	P_{29}	(5,7,7,9)	7

Table (4.4) :Implementing results for the second approach

Time of nodes	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24		
	68.98	95.98	99.98	128.61	139.73	145.36	175.98	108.68	179.48	296.37	189.86	195.86	296.37	196.90	132.85	206.23	296.37	196.90	210.46	210.46	160.49	179.85	199.99	223.11		
	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	Min Z = 154.125	
$P_1=t_2-t_1$	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	27	=27
$P_2=t_3-t_2$	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	=4
$P_3=t_4-t_3$	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	28.625	=28.625
$P_4=t_5-t_4$	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11.125	=11.125
$P_5=t_6-t_5$	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5.625	=5.625
$P_6=t_7-t_6$	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	30.625	=30.625
$P_7=t_{24}-t_6$	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	77.75	≥10
$P_8=t_8-t_1$	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	39.6963	≥22.5
$P_9=t_9-t_7$	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3.5	=3.5
$P_{10}=t_{24}-t_7$	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	47.125	≥10
$P_{11}=t_{15}-t_8$	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	24.176	≥7.375
$P_{12}=t_{10}-t_9$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	116.891	≥14
$P_{13}=t_{11}-t_9$	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	10.375	=10.375
$P_{14}=t_{12}-t_{11}$	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	6	=6
$P_{15}=t_{16}-t_{12}$	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	0	0	0	0	0	0	0	10.375	=10.375
$P_{16}=t_{13}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	116.891	≥14
$P_{17}=t_{14}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	17.42	≥10.375
$P_{18}=t_{24}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	43.625	≥30.5
$P_{19}=t_{19}-t_{14}$	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0	13.554	≥6
$P_{20}=t_{21}-t_{15}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	27.64	≥10.75
$P_{21}=t_{24}-t_{16}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	16.875	=16.875
$P_{22}=t_{17}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	116.89	≥14
$P_{23}=t_{18}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	17.42	≥10.375
$P_{24}=t_{20}-t_{18}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	13.554	≥6
$P_{25}=t_{24}-t_{12}$	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	1	27.25	≥6.125
$P_{26}=t_{24}-t_{19}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	1	12.649	≥6.125
$P_{27}=t_{24}-t_{20}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	12.649	≥6.125
$P_{28}=t_{22}-t_{21}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	19.358	≥2
$P_{29}=t_{23}-t_{22}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	20.136	≥2
$P_{30}=t_{24-23}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	23.117	≥7

4.4 Third Approach:

In this approach, the activities expected times t_e for each μ-cut are calculated using (3.35), and the optimal critical path is obtained for each $\mu = 0,0.1,0.25,0.5,0.75,1$ by implementing the following standard linear programming model for each μ-cut value:

Minimize

$$Z = t_{24} - t_1$$

Subject to

$$t_2 - t_1 \geq t_{e1}$$

$$t_3 - t_2 \geq t_{e2}$$

$$t_4 - t_3 \geq t_{e3}$$

$$t_5 - t_4 \geq t_{e4}$$

$$t_6 - t_5 \geq t_{e5}$$

$$t_7 - t_6 \geq t_{e6}$$

$$t_{24} - t_6 \geq t_{e7}$$

$$t_8 - t_1 \geq t_{e8}$$

$$t_9 - t_7 \geq t_{e9}$$

$$t_{24} - t_7 \geq t_{e10}$$

$$t_{15} - t_8 \geq t_{e11}$$

$$t_{10} - t_9 \geq t_{e12}$$

$$t_{11} - t_9 \geq t_{e13}$$

$$t_{12} - t_{11} \geq t_{e14}$$

$$t_{16} - t_{12} \geq t_{e15}$$

$$t_{13} - t_9 \geq t_{e16}$$

$$t_{14} - t_9 \geq t_{e17}$$

$$t_{24} - t_9 \geq t_{e18}$$

$$t_{19} - t_{14} \geq t_{e19}$$

$$t_{21} - t_{15} \geq t_{e20}$$

$t_{24} - t_{16} \geq t_{e21}$

$t_{17} - t_9 \geq t_{e22}$

$t_{18} - t_9 \geq t_{e23}$

$t_{20} - t_{18} \geq t_{e24}$

$t_{24} - t_{12} \geq t_{e25}$

$t_{24} - t_{19} \geq t_{e26}$

$t_{24} - t_{20} \geq t_{e27}$

$t_{22} - t_{21} \geq t_{e28}$

$t_{23} - t_{22} \geq t_{e29}$

$t_{24} - t_{23} \geq t_{e30}$

$t_i \geq 0, i = 1,2,3, \dots ,24.$

where t_{ej}, $j = 1,2, \dots ,30$ is the expected time that obtained using (3.35).

The critical path activities are determined and the optimal value of the objective function is calculated for each value of μ-cut, using "Matlab R2010b" software. The results are shown in the following tables:

Table (4.5): problem data at $\mu = 0$

Activity Item	Fuzzy Operation Time (in day)	Optimistic Time t_o	Pessimistic Time t_p	Most Likely Time t_m	Expected Time t_e
P_1	(25,28,30,35)	25	35	30	30
P_2	(3,4,4,5)	3	5	4	4
P_3	(25,29,30,35)	25	35	30	30
P_4	(3,7,12,15)	3	15	9	9
P_5	(5,6,6,10)	5	10	7.5	7.5
P_6	(26,30,35,40)	26	40	33	33
P_7	(7,10,10,13)	7	13	10	10
P_8	(15,21,21,25)	15	25	20	20
P_9	(2,3,4,5)	2	5	3.5	3.5
P_{10}	(9,10,12,15)	9	15	12	12
P_{11}	(6,7,8,10)	6	10	8	8
P_{12}	(12,14,14,16)	12	16	14	14
P_{13}	(7,9,11,12)	7	12	9.5	9.5
P_{14}	(5,6,7,10)	5	10	7.5	7.5
P_{15}	(7,9,11,12)	7	12	9.5	9.5
P_{16}	(12,14,14,16)	12	16	14	14
P_{17}	(7,9,11,12)	7	12	9.5	9.5
P_{18}	(15,24,25,30)	15	30	22.5	22.5
P_{19}	(5,6,7,10)	5	10	7.5	7.5
P_{20}	(9,11,12,14)	9	14	11.5	11.5
P_{21}	(15,18,18,20)	15	20	17.5	17.5
P_{22}	(12,14,14,16)	12	16	14	14
P_{23}	(7,9,11,12)	7	12	9.5	9.5
P_{24}	(5,6,7,10)	5	10	7.5	7.5
P_{25}	(5,6,6,8)	5	8	6.5	6.5
P_{26}	(5,6,6,8)	5	8	6.5	6.5
P_{27}	(5,6,6,8)	5	8	6.5	6.5
P_{28}	(1,2,2,3)	1	3	2	2
P_{29}	(1,2,2,3)	1	3	2	2
P_{30}	(5,7,7,9)	5	9	7	7

Table (4.6): Implementing results at $\mu = 0$

Time of nodes	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24		
	68.87	98.87	102.87	132.87	141.87	149.37	182.37	106.98	185.87	302.65	195.37	202.87	302.65	202.34	133.05	212.37	302.65	202.34	217.31	217.31	162.81	183.58	205.24	229.87		
	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	Min Z = 161	
$P_1=t_2-t_1$	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	30	=30
$P_2=t_3-t_2$	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	=4
$P_3=t_4-t_3$	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	30	=30
$P_4=t_5-t_4$	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9	=9
$P_5=t_6-t_5$	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7.5	=7.5
$P_6=t_7-t_6$	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	33	=33
$P_7=t_{24}-t_6$	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	80.5	≥10
$P_8=t_8-t_1$	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	38.1119	≥20
$P_9=t_9-t_7$	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3.5	=3.5
$P_{10}=t_{24}-t_7$	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	47.5	≥12
$P_{11}=t_{15}-t_8$	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	26.0675	≥8
$P_{12}=t_{10}-t_9$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	116.778	≥14
$P_{13}=t_{11}-t_9$	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	9.5	=9.5
$P_{14}=t_{12}-t_{11}$	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	7.5	=7.5
$P_{15}=t_{16}-t_{12}$	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	0	0	0	0	0	0	0	9.5	=9.5
$P_{16}=t_{13}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	116.778	≥14
$P_{17}=t_{14}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	16.4741	≥9.5
$P_{18}=t_{24}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	44	≥22.5
$P_{19}=t_{19}-t_{14}$	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0	14.9697	≥7.5
$P_{20}=t_{21}-t_{15}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	29.7626	≥11.5
$P_{21}=t_{24}-t_{16}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	17.5	=17.5
$P_{22}=t_{17}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	116.778	≥14
$P_{23}=t_{18}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	16.4741	≥9.5
$P_{24}=t_{20}-t_{18}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	14.9697	≥7.5
$P_{25}=t_{24}-t_{12}$	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	1	27	≥6.5
$P_{26}=t_{24}-t_{19}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	1	12.5562	≥6.5
$P_{27}=t_{24}-t_{20}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	12.5562	≥6.5
$P_{28}=t_{22}-t_{21}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	20.7674	≥2
$P_{29}=t_{23}-t_{22}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	21.6584	≥2
$P_{30}=t_{24-23}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	24.6322	≥7

Table (4.7): problem data at $\mu = 0.1$

Activity Item	Fuzzy Operation Time (in day)	Optimistic Time t_o	Pessimistic Time t_p	Most Likely Time t_m	Expected Time t_e
P_1	(25,28,30,35)	25.3	34.3	29.9	29.9
P_2	(3,4,4,5)	3.1	4.9	4	4
P_3	(25,29,30,35)	25.4	34.5	29.95	29.95
P_4	(3,7,12,15)	3.4	14.7	9.05	9.05
P_5	(5,6,6,10)	5.1	9.6	7.35	7.35
P_6	(26,30,35,40)	26.4	39.5	32.95	32.95
P_7	(7,10,10,13)	7.3	12.7	10	10
P_8	(15,21,21,25)	15.6	24.6	20.1	20.1
P_9	(2,3,4,5)	2.1	4.9	3.5	3.5
P_{10}	(9,10,12,15)	9.1	14.7	11.9	11.9
P_{11}	(6,7,8,10)	6.1	9.8	7.95	7.95
P_{12}	(12,14,14,16)	12.2	15.8	14	14
P_{13}	(7,9,11,12)	7.2	11.9	9.55	9.55
P_{14}	(5,6,7,10)	5.1	9.7	7.4	7.4
P_{15}	(7,9,11,12)	7.2	11.9	9.55	9.55
P_{16}	(12,14,14,16)	12.2	15.8	14	14
P_{17}	(7,9,11,12)	7.2	11.9	9.55	9.55
P_{18}	(15,24,25,30)	15.9	29.5	22.7	22.7
P_{19}	(5,6,7,10)	5.1	9.7	7.4	7.4
P_{20}	(9,11,12,14)	9.2	13.8	11.5	11.5
P_{21}	(15,18,18,20)	15.3	19.8	17.55	17.55
P_{22}	(12,14,14,16)	12.2	15.8	14	14
P_{23}	(7,9,11,12)	7.2	11.9	9.55	9.55
P_{24}	(5,6,7,10)	5.1	9.7	7.4	7.4
P_{25}	(5,6,6,8)	5.1	7.8	6.45	6.45
P_{26}	(5,6,6,8)	5.1	7.8	6.45	6.45
P_{27}	(5,6,6,8)	5.1	7.8	6.45	6.45
P_{28}	(1,2,2,3)	1.1	2.9	2	2
P_{29}	(1,2,2,3)	1.1	2.9	2	2
P_{30}	(5,7,7,9)	5.2	8.8	7	7

Table (4.8): Implementing results at $\mu = 0.1$

Time of nodes	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24		
	68.86	98.76	102.76	132.71	141.76	149.11	182.06	107.04	185.56	302.34	195.11	202.51	302.34	202.13	133.01	212.06	302.34	202.13	217.06	217.06	162.72	183.44	205.04	229.61		
	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	Min Z = 160.75	
$P_1=t_2-t_1$	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	30.1	≥29.9
$P_2=t_3-t_2$	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	=4
$P_3=t_4-t_3$	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	29.95	=29.95
$P_4=t_5-t_4$	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9.05	=9.05
$P_5=t_6-t_5$	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7.35	=7.35
$P_6=t_7-t_6$	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	32.95	=32.95
$P_7=t_{24}-t_6$	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	80.5	≥10
$P_8=t_8-t_1$	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	38.181	≥20.1
$P_9=t_9-t_7$	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3.5	=3.5
$P_{10}=t_{24}-t_7$	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	47.55	≥11.9
$P_{11}=t_{15}-t_8$	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	25.969	≥7.95
$P_{12}=t_{10}-t_9$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	116.779	≥14
$P_{13}=t_{11}-t_9$	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	9.55	=9.55
$P_{14}=t_{12}-t_{11}$	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	7.4	=7.4
$P_{15}=t_{16}-t_{12}$	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	0	0	0	0	0	0	0	9.55	=9.55
$P_{16}=t_{13}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	116.779	≥14
$P_{17}=t_{14}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	16.572	≥9.55
$P_{18}=t_{24}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	44.05	≥22.7
$P_{19}=t_{19}-t_{14}$	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0	14.921	≥7.4
$P_{20}=t_{21}-t_{15}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	29.711	≥11.5
$P_{21}=t_{24}-t_{16}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	17.55	=17.55
$P_{22}=t_{17}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	116.779	≥14
$P_{23}=t_{18}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	16.572	≥9.55
$P_{24}=t_{20}-t_{18}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	14.921	≥7.4
$P_{25}=t_{24}-t_{12}$	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	1	27.1	≥6.45
$P_{26}=t_{24}-t_{19}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	1	12.556	≥6.45
$P_{27}=t_{24}-t_{20}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	12.556	≥6.45
$P_{28}=t_{22}-t_{21}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	20.713	≥2
$P_{29}=t_{23}-t_{22}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	21.599	≥2
$P_{30}=t_{24}-_{23}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	24.575	≥7

Table (4.9): problem data at $\mu = 0.25$

Activity Item	Fuzzy Operation Time (in day)	Optimistic Time t_o	Pessimistic Time t_p	Most Likely Time t_m	Expected Time t_e
P_1	(25,28,30,35)	25.75	33.75	29.75	29.75
P_2	(3,4,4,5)	3.25	4.75	4	4
P_3	(25,29,30,35)	26	33.75	29.875	29.875
P_4	(3,7,12,15)	4	14.25	9.125	9.125
P_5	(5,6,6,10)	5.25	9	7.125	7.125
P_6	(26,30,35,40)	27	38.75	32.875	32.875
P_7	(7,10,10,13)	7.75	12.25	10	10
P_8	(15,21,21,25)	16.5	24	20.25	20.25
P_9	(2,3,4,5)	2.25	4.75	3.5	3.5
P_{10}	(9,10,12,15)	9.25	14.25	11.75	11.75
P_{11}	(6,7,8,10)	6.25	9.5	7.875	7.875
P_{12}	(12,14,14,16)	12.5	15.5	14	14
P_{13}	(7,9,11,12)	7.5	11.75	9.625	9.625
P_{14}	(5,6,7,10)	5.25	9.25	7.25	7.25
P_{15}	(7,9,11,12)	7.5	11.75	9.625	9.625
P_{16}	(12,14,14,16)	12.5	15.5	14	14
P_{17}	(7,9,11,12)	7.5	11.75	9.625	9.625
P_{18}	(15,24,25,30)	17.25	28.75	23	23
P_{19}	(5,6,7,10)	5.25	9.25	7.25	7.25
P_{20}	(9,11,12,14)	9.5	13.5	11.5	11.5
P_{21}	(15,18,18,20)	15.75	19.5	17.625	17.625
P_{22}	(12,14,14,16)	12.5	15.5	14	14
P_{23}	(7,9,11,12)	7.5	11.75	9.625	9.625
P_{24}	(5,6,7,10)	5.25	9.25	7.25	7.25
P_{25}	(5,6,6,8)	5.25	7.5	6.375	6.375
P_{26}	(5,6,6,8)	5.25	7.5	6.375	6.375
P_{27}	(5,6,6,8)	5.25	7.5	6.375	6.375
P_{28}	(1,2,2,3)	1.25	2.75	2	2
P_{29}	(1,2,2,3)	1.25	2.75	2	2
P_{30}	(5,7,7,9)	5.5	8.5	7	7

Table (4.10): Implementing results at $\mu = 0.25$

Time of nodes	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24		
	68.87	98.62	102.62	132.5	141.62	148.75	181.62	107.16	185.12	301.90	194.75	202.00	301.90	201.71	132.98	211.62	301.90	201.71	216.42	216.42	162.61	183.25	204.76	229.25		
	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	Min Z = 160.375	
$P_{1}=t_{2}-t_{1}$	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	29.75	=29.75
$P_{2}=t_{3}-t_{2}$	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	=4
$P_{3}=t_{4}-t_{3}$	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	29.875	=29.875
$P_{4}=t_{5}-t_{4}$	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9.125	=9.125
$P_{5}=t_{6}-t_{5}$	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7.125	=7.125
$P_{6}=t_{7}-t_{6}$	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	32.875	=32.875
$P_{7}=t_{24}-t_{6}$	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	80.5	≥10
$P_{8}=t_{8}-t_{1}$	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	38.2840	≥20.25
$P_{9}=t_{9}-t_{7}$	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3.5	=3.5
$P_{10}=t_{24}-t_{7}$	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	47.625	≥11.75
$P_{11}=t_{15}-t_{8}$	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	25.8241	≥7.875
$P_{12}=t_{10}-t_{9}$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	116.778	≥14
$P_{13}=t_{11}-t_{9}$	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	9.625	=9.625
$P_{14}=t_{12}-t_{11}$	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	7.25	=7.25
$P_{15}=t_{16}-t_{12}$	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	0	0	0	0	0	0	0	9.625	=9.625
$P_{16}=t_{13}-t_{9}$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	116.778	≥14
$P_{17}=t_{14}-t_{9}$	0	0	0	0	0	0	0	0	-1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	16.584	≥9.625
$P_{18}=t_{24}-t_{9}$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	44.125	≥23
$P_{19}=t_{19}-t_{14}$	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0	14.712	≥7.25
$P_{20}=t_{21}-t_{15}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	29.632	≥11.5
$P_{21}=t_{24}-t_{16}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	17.625	=17.625
$P_{22}=t_{17}-t_{9}$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	116.778	≥14
$P_{23}=t_{18}-t_{9}$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	16.584	≥9.625
$P_{24}=t_{20}-t_{18}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	14.712	≥7.25
$P_{25}=t_{24}-t_{12}$	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	1	27.25	≥6.375
$P_{26}=t_{24}-t_{19}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	1	12.827	≥6.375
$P_{27}=t_{24}-t_{20}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	12.827	≥6.375
$P_{28}=t_{22}-t_{21}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	20.631	≥2
$P_{29}=t_{23}-t_{22}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	21.512	≥2
$P_{30}=t_{24}-_{23}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	24.491	≥7

Table (4.11): problem data at $\mu = 0.5$

Activity Item	Fuzzy Operation Time (in day)	Optimistic Time t_o	Pessimistic Time t_p	Most Likely Time t_m	Expected Time t_e
P_1	(25,28,30,35)	26.5	32.5	29.5	29.5
P_2	(3,4,4,5)	3.5	4.5	4	4
P_3	(25,29,30,35)	27	32.5	29.75	29.75
P_4	(3,7,12,15)	5	13.5	9.25	9.25
P_5	(5,6,6,10)	5.5	8	6.75	6.75
P_6	(26,30,35,40)	28	37.5	32.75	32.75
P_7	(7,10,10,13)	8.5	11.5	10	10
P_8	(15,21,21,25)	18	23	20.5	20.5
P_9	(2,3,4,5)	2.5	4.5	3.5	3.5
P_{10}	(9,10,12,15)	9.5	13.5	11.5	11.5
P_{11}	(6,7,8,10)	6.5	9	7.75	7.75
P_{12}	(12,14,14,16)	13	15	14	14
P_{13}	(7,9,11,12)	8	11.5	9.75	9.75
P_{14}	(5,6,7,10)	5.5	8.5	7	7
P_{15}	(7,9,11,12)	8	11.5	9.75	9.75
P_{16}	(12,14,14,16)	13	15	14	14
P_{17}	(7,9,11,12)	8	11.5	9.75	9.75
P_{18}	(15,24,25,30)	19.5	27.5	23.5	23.5
P_{19}	(5,6,7,10)	5.5	8.5	7	7
P_{20}	(9,11,12,14)	10	13	11.5	11.5
P_{21}	(15,18,18,20)	16.5	19	17.75	17.75
P_{22}	(12,14,14,16)	13	15	14	14
P_{23}	(7,9,11,12)	8	11.5	9.75	9.75
P_{24}	(5,6,7,10)	5.5	8.5	7	7
P_{25}	(5,6,6,8)	5.5	7	6.25	6.25
P_{26}	(5,6,6,8)	5.5	7	6.25	6.25
P_{27}	(5,6,6,8)	5.5	7	6.25	6.25
P_{28}	(1,2,2,3)	1.5	2.5	2	2
P_{29}	(1,2,2,3)	1.5	2.5	2	2
P_{30}	(5,7,7,9)	6	6	7	7

Table (4.12): Implementing results at $\mu = 0.5$

Time of nodes	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24		
	68.86	98.36	102.36	132.11	141.36	148.11	180.86	107.32	184.36	301.14	194.11	201.11	301.14	201.18	132.89	210.86	301.14	201.18	215.77	21.77	162.40	182.90	204.26	228.61		
	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	Min Z = 159.75	
$P_1=t_2-t_1$	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	29.5	=29.5
$P_2=t_3-t_2$	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	=4
$P_3=t_4-t_3$	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	29.75	=29.75
$P_4=t_5-t_4$	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9.25	=9.25
$P_5=t_6-t_5$	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6.75	=6.75
$P_6=t_7-t_6$	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	32.75	=32.75
$P_7=t_{24}-t_6$	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	80.5	≥10
$P_8=t_8-t_1$	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	38.4551	≥20.5
$P_9=t_9-t_7$	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3.5	=3.5
$P_{10}=t_{24}-t_7$	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	47.75	≥11.5
$P_{11}=t_{15}-t_8$	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	25.5773	≥7.75
$P_{12}=t_{10}-t_9$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	116.778	≥14
$P_{13}=t_{11}-t_9$	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	9.75	=9.75
$P_{14}=t_{12}-t_{11}$	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	7.0696	≥7
$P_{15}=t_{16}-t_{12}$	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	0	0	0	0	0	0	0	9.6804	≥9.75
$P_{16}=t_{13}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	116.778	≥14
$P_{17}=t_{14}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	16.8196	≥9.75
$P_{18}=t_{24}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	44.25	≥23.5
$P_{19}=t_{19}-t_{14}$	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0	14.5889	≥7
$P_{20}=t_{21}-t_{15}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	29.5055	≥11.5
$P_{21}=t_{24}-t_{16}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	17.75	=17.75
$P_{22}=t_{17}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	116.778	≥14
$P_{23}=t_{18}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	16.8196	≥9.75
$P_{24}=t_{20}-t_{18}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	14.5899	≥7
$P_{25}=t_{24}-t_{12}$	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	1	27.4304	≥6.25
$P_{26}=t_{24}-t_{19}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	1	12.8405	≥6.25
$P_{27}=t_{24}-t_{20}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	12.8405	≥6.25
$P_{28}=t_{22}-t_{21}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	20.4962	≥2
$P_{29}=t_{23}-t_{22}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	21.3659	≥2
$P_{30}=t_{24}-_{23}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	24.35	≥7

Table (4.13): problem data at $\mu = 0.75$

Activity Item	Fuzzy Operation Time (in day)	Optimistic Time t_o	Pessimistic Time t_p	Most Likely Time t_m	Expected Time t_e
P_1	(25,28,30,35)	27.25	31.25	29.25	29.25
P_2	(3,4,4,5)	3.75	4.25	4	4
P_3	(25,29,30,35)	27	32.5	29.75	29.75
P_4	(3,7,12,15)	6	12.75	9.375	9.375
P_5	(5,6,6,10)	5.75	7	6.375	6.375
P_6	(26,30,35,40)	29	36.25	32.625	32.625
P_7	(7,10,10,13)	9.25	10.75	10	10
P_8	(15,21,21,25)	19.5	22	20.75	20.75
P_9	(2,3,4,5)	2.75	4.25	3.5	3.5
P_{10}	(9,10,12,15)	9.75	12.75	11.25	11.25
P_{11}	(6,7,8,10)	6.75	8.5	7.625	7.625
P_{12}	(12,14,14,16)	13.5	14.5	14	14
P_{13}	(7,9,11,12)	8.5	11.25	9.875	9.875
P_{14}	(5,6,7,10)	5.75	7.75	6.75	6.75
P_{15}	(7,9,11,12)	8.5	11.25	9.875	9.875
P_{16}	(12,14,14,16)	13.5	14.5	14	14
P_{17}	(7,9,11,12)	8.5	11.25	9.875	9.875
P_{18}	(15,24,25,30)	21.75	26.25	24	24
P_{19}	(5,6,7,10)	5.75	7.75	6.75	6.75
P_{20}	(9,11,12,14)	10.5	12.5	11.5	11.5
P_{21}	(15,18,18,20)	17.25	18.5	17.875	17.875
P_{22}	(12,14,14,16)	13.5	14.5	14	14
P_{23}	(7,9,11,12)	8.5	11.25	9.875	9.875
P_{24}	(5,6,7,10)	5.75	7.75	6.75	6.75
P_{25}	(5,6,6,8)	5.75	6.5	6.125	6.125
P_{26}	(5,6,6,8)	5.75	6.5	6.125	6.125
P_{27}	(5,6,6,8)	5.75	6.5	6.125	6.125
P_{28}	(1,2,2,3)	1.75	2.25	2	2
P_{29}	(1,2,2,3)	1.75	2.25	2	2
P_{30}	(5,7,7,9)	6.5	7.5	7	7

Table (4.14): Implementing results at $\mu = 0.75$

Time of nodes	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24		
	68.84	98.09	102.09	131.84	141.21	147.59	180.21	107.48	183.71	300.50	193.59	200.34	300.50	200.78	132.83	210.21	300.50	200.78	215.26	215.26	162.23	182.61	203.86	228.09		
	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	Min Z = 159.25	
P_1=t_2-t_1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	29.25	=29.25
P_2=t_3-t_2	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	=4
P_3=t_4-t_3	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	29.75	=29.75
P_4=t_5-t_4	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9.375	=9.375
P_5=t_6-t_5	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6.375	=6.375
P_6=t_7-t_6	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	32.625	=32.625
P_7=t_{24}-t_6	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	80.5	≥10
P_8=t_8-t_1	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	38.6404	≥20.75
P_9=t_9-t_7	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3.5	=3.5
P_{10}=t_{24}-t_7	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	47.875	≥11.25
P_{11}=t_{15}-t_8	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	25.3514	≥7.625
P_{12}=t_{10}-t_9	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	116.788	≥14
P_{13}=t_{11}-t_9	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	9.875	=9.875
P_{14}=t_{12}-t_{11}	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	6.75	=6.75
P_{15}=t_{16}-t_{12}	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	0	0	0	0	0	0	0	9.875	=9.875
P_{16}=t_{13}-t_9	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	116.788	≥14
P_{17}=t_{14}-t_9	0	0	0	0	0	0	0	0	-1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	17.067	≥9.875
P_{18}=t_{24}-t_9	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	44.375	≥24
P_{19}=t_{19}-t_{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0	14.4735	≥6.75
P_{20}=t_{21}-t_{15}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	29.3973	≥11.5
P_{21}=t_{24}-t_{16}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	17.875	=17.875
P_{22}=t_{17}-t_9	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	116.778	≥14
P_{23}=t_{18}-t_9	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	17.067	≥9.875
P_{24}=t_{20}-t_{18}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	14.4735	≥6.75
P_{25}=t_{24}-t_{12}	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	1	27.7500	≥6.125
P_{26}=t_{24}-t_{19}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	1	12.8345	≥6.125
P_{27}=t_{24}-t_{20}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	12.8345	≥6.125
P_{28}=t_{22}-t_{21}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	20.3825	≥2
P_{29}=t_{23}-t_{22}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	21.2444	≥2
P_{30}=t_{24-23}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	24.2340	≥7

Table (4.15): problem data at $\mu = 1$

Activity Item	Fuzzy Operation Time (in day)	Optimistic Time t_o	Pessimistic Time t_p	Most Likely Time t_m	Expected Time t_e
P_1	(25,28,30,35)	28	30	29	29
P_2	(3,4,4,5)	4	4	4	4
P_3	(25,29,30,35)	29	30	29.5	29.5
P_4	(3,7,12,15)	7	12	9.5	9.5
P_5	(5,6,6,10)	6	6	6	6
P_6	(26,30,35,40)	30	35	32.5	32.5
P_7	(7,10,10,13)	10	10	10	10
P_8	(15,21,21,25)	21	21	21	21
P_9	(2,3,4,5)	3	4	3.5	3.5
P_{10}	(9,10,12,15)	10	12	11	11
P_{11}	(6,7,8,10)	7	8	7.5	7.5
P_{12}	(12,14,14,16)	14	14	14	14
P_{13}	(7,9,11,12)	9	11	10	10
P_{14}	(5,6,7,10)	6	7	6.5	6.5
P_{15}	(7,9,11,12)	9	11	10	10
P_{16}	(12,14,14,16)	14	14	14	14
P_{17}	(7,9,11,12)	9	11	10	10
P_{18}	(15,24,25,30)	24	25	24.5	24.5
P_{19}	(5,6,7,10)	6	7	6.5	6.5
P_{20}	(9,11,12,14)	11	12	11.5	11.5
P_{21}	(15,18,18,20)	18	18	18	18
P_{22}	(12,14,14,16)	14	14	14	14
P_{23}	(7,9,11,12)	9	11	10	10
P_{24}	(5,6,7,10)	6	7	6.5	6.5
P_{25}	(5,6,6,8)	6	6	6	6
P_{26}	(5,6,6,8)	6	6	6	6
P_{27}	(5,6,6,8)	6	6	6	6
P_{28}	(1,2,2,3)	2	2	2	2
P_{29}	(1,2,2,3)	2	2	2	2
P_{30}	(5,7,7,9)	7	7	7	7

Table (4.16): Implementing results at $\mu = 1$

Time of nodes	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24		
	68.83	97.83	101.83	131.33	140.83	146.83	179.33	107.63	182.83	299.61	192.83	199.33	299.61	200.15	132.71	209.33	299.61	200.15	214.51	214.51	161.96	182.19	203.26	227.33		
	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	Min Z = 158.5	
$P_1=t_2-t_1$	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	29	=29
$P_2=t_3-t_2$	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	=4
$P_3=t_4-t_3$	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	29.5	=29.5
$P_4=t_5-t_4$	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9.5	=9.5
$P_5=t_6-t_5$	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	=6
$P_6=t_7-t_6$	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	32.5	=32.5
$P_7=t_{24}-t_6$	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	80.5	≥10
$P_8=t_8-t_1$	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	38.7925	≥21
$P_9=t_9-t_7$	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3.5	=3.5
$P_{10}=t_{24}-t_7$	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	48	≥11
$P_{11}=t_{15}-t_8$	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	25.088	≥7.5
$P_{12}=t_{10}-t_9$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	116.774	≥14
$P_{13}=t_{11}-t_9$	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	10	=10
$P_{14}=t_{12}-t_{11}$	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	6.5	=6.5
$P_{15}=t_{16}-t_{12}$	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	0	0	0	0	0	0	0	10	=10
$P_{16}=t_{13}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	116.774	≥14
$P_{17}=t_{14}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	17.3175	≥10
$P_{18}=t_{24}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	44.5	≥24.5
$P_{19}=t_{19}-t_{14}$	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0	14.3578	≥6.5
$P_{20}=t_{21}-t_{15}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	29.2493	≥11.5
$P_{21}=t_{24}-t_{16}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	18	=18
$P_{22}=t_{17}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	116.774	≥14
$P_{23}=t_{18}-t_9$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	18	≥10
$P_{24}=t_{20}-t_{18}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	14.3578	≥6.5
$P_{25}=t_{24}-t_{12}$	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	1	28	≥6
$P_{26}=t_{24}-t_{19}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	1	12.8247	≥6
$P_{27}=t_{24}-t_{20}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	12.8247	≥6
$P_{28}=t_{22}-t_{21}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	20.2261	≥2
$P_{29}=t_{23}-t_{22}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	21.0746	≥2
$P_{30}=t_{24}-_{23}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	24.0695	≥7

Table (4.17): The summary results for different μ-cut values of the third approach

α-cut value	Critical paths	duration	Criticality state
0	$t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow t_4 \rightarrow t_5 \rightarrow t_6 \rightarrow t_7 \rightarrow t_9 \rightarrow t_{11} \rightarrow t_{12} \rightarrow t_{16} \rightarrow t_{24}$	161	strong
0.1	$t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow t_4 \rightarrow t_5 \rightarrow t_6 \rightarrow t_7 \rightarrow t_9 \rightarrow t_{11} \rightarrow t_{12} \rightarrow t_{16} \rightarrow t_{24}$	160.95	weak
0.25	$t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow t_4 \rightarrow t_5 \rightarrow t_6 \rightarrow t_7 \rightarrow t_9 \rightarrow t_{11} \rightarrow t_{12} \rightarrow t_{16} \rightarrow t_{24}$	160.375	Strong
0.5	$t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow t_4 \rightarrow t_5 \rightarrow t_6 \rightarrow t_7 \rightarrow t_9 \rightarrow t_{11} \rightarrow t_{12} \rightarrow t_{16} \rightarrow t_{24}$	159.75	Weak
0.75	$t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow t_4 \rightarrow t_5 \rightarrow t_6 \rightarrow t_7 \rightarrow t_9 \rightarrow t_{11} \rightarrow t_{12} \rightarrow t_{16} \rightarrow t_{24}$	159.25	Strong
1	$t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow t_4 \rightarrow t_5 \rightarrow t_6 \rightarrow t_7 \rightarrow t_9 \rightarrow t_{11} \rightarrow t_{12} \rightarrow t_{16} \rightarrow t_{24}$	158.5	Strong

Now, we present a hybrid approach which consists of “PERT” and “CPM”.

4.5 Hybrid Approach, [4]:

For more satisfaction to our results, we will implement CPM which can be explaining as follows:

For each activity (i, j) in the project network, considering the crisp activity time t_{ij} that calculated using (3.35) for each value of the µ-cut in section (4.4).

Let ES_i and LF_i be the earliest start time event i, and latest finish time event i, respectively. Let D_j be a set of events obtained from event i and $i < j$.

We then obtain ES_j using the following equations:

$ES_i = max_{i \in D_j}[ES_i + t_{ij}]$ and $ES_1 = LS_1 = 0$.

Similarly, let H_i be a set of events obtained from event i and $i < j$.

We obtain LF_i using the following equations:

$LF_i = min_{j \in H_i}[LF_j - t_{ij}]$ and $LF_n = EF_n$.

The interval $[ES_i, LF_j]$ is the time during which the activity (i, j) must be completed. When the earliest event time and the latest event time have been obtained, we can calculate the total slack on each node. For activity (i, j) in a project network, the slack T_{ij} of each node can be computed as follows:

$T_{ij} = LF_j - ES_i - t_{ij}$.

In the following tables (4.16) - (4.21) the earliest event time, the latest event time and the slack of each node are obtained by using the above equations and the critical events are identified corresponding to their zeros values of the slack time for each value of μ-cut.

Table (4.18): Critical Event with μ = 0

Nodes	Earliest Time T_E	Latest Time T_L	Slack Time
t_1	0	0	0
t_2	30	30	0
t_3	34	34	0
t_4	64	64	0
t_5	73	73	0
t_6	80.5	80.5	0
t_7	113.5	113.5	0
t_8	20	135	110
t_9	117	117	0
t_{10}	131	131	0
t_{11}	131	131	0
t_{12}	138.5	138.5	0
t_{13}	131	151.5	20.5
t_{14}	131	151.5	20.5
t_{15}	28	143	115
t_{16}	148	148	0
t_{17}	131	151.5	20.5
t_{18}	131	151.5	20.5
t_{19}	138.5	159	20.5
t_{20}	138.5	159	20.5
t_{21}	39.5	154.5	115
t_{22}	41.5	156.5	115
t_{23}	43.5	158.5	115
t_{24}	165.5	165.5	0

Table (4.19): Critical Event with μ = 0.1

Nodes	Earliest Time T_E	Latest Time T_L	Slack Time
t_1	0	0	0
t_2	29.9	29.9	0
t_3	33.9	33.9	0
t_4	63.85	63.85	0
t_5	72.9	72.9	0
t_6	80.25	80.25	0
t_7	113.2	113.2	0
t_8	20.1	134.75	114.65
t_9	116.7	116.7	0
t_{10}	130.7	130.7	0
t_{11}	130.7	130.7	0
t_{12}	138.1	138.1	0
t_{13}	130.7	151.35	20.65
t_{14}	130.7	151.35	20.65
t_{15}	28.05	142.7	114.65
t_{16}	147.25	147.25	0
t_{17}	130.7	151.35	20.65
t_{18}	130.7	151.35	20.65
t_{19}	138.1	158.75	20.65
t_{20}	138.1	158.75	20.65
t_{21}	39.55	154.2	114.65
t_{22}	41.55	156.2	114.65
t_{23}	43.55	158.2	114.65
t_{24}	165.2	165.2	0

Table (4.20): Critical Event with μ = 0.25

Nodes	Earliest Time T_E	Latest Time T_L	Slack Time
t_1	0	0	0
t_2	29.75	29.75	0
t_3	33.75	33.75	0
t_4	63.625	63.625	0
t_5	72.75	72.75	0
t_6	79.875	79.875	0
t_7	112.75	112.75	0
t_8	20.25	133.825	113.575
t_9	116.25	116.25	0
t_{10}	130.25	130.25	0
t_{11}	130.25	130.25	0
t_{12}	137.5	137.5	0
t_{13}	130.25	150.875	20.625
t_{14}	130.25	150.875	20.625
t_{15}	28.125	141.7	113.575
t_{16}	147.125	147.125	0
t_{17}	130.25	150.875	20.625
t_{18}	130.25	150.875	20.625
t_{19}	137.5	158.125	20.625
t_{20}	137.5	158.125	20.625
t_{21}	39.625	153.2	113.575
t_{22}	41.625	155.2	113.575
t_{23}	43.625	157.2	113.575
t_{24}	164.75	164.75	0

Table (4.21): Critical Event with μ = 0.5

Nodes	Earliest Time T_E	Latest Time T_L	Slack Time
t_1	0	0	0
t_2	29.5	29.5	0
t_3	33.5	33.5	0
t_4	63.25	63.25	0
t_5	72.5	72.5	0
t_6	79.25	79.25	0
t_7	112	112	0
t_8	20.5	133.75	113.25
t_9	115.5	115.5	0
t_{10}	129.5	129.5	0
t_{11}	129.5	129.5	0
t_{12}	136.5	136.5	0
t_{13}	129.5	150.75	21.25
t_{14}	129.5	150.75	21.25
t_{15}	28.25	141.5	113.25
t_{16}	146.25	146.25	0
t_{17}	129.5	150.75	21.25
t_{18}	129.5	150.75	21.25
t_{19}	136.5	157.75	21.25
t_{20}	136.5	157.75	21.25
t_{21}	39.75	153	113.25
t_{22}	41.75	155	113.25
t_{23}	43.75	157	113.25
t_{24}	164	164	0

Table (4.22): Critical Event with μ = 0.75

Nodes	Earliest Time T_E	Latest Time T_L	Slack Time
t_1	0	0	0
t_2	29.25	29.25	0
t_3	33.25	33.25	0
t_4	63	63	0
t_5	72.375	72.375	0
t_6	78.75	78.75	0
t_7	113.375	113.375	0
t_8	20.75	133.25	112.5
t_9	114.875	114.875	0
t_{10}	128.875	128.875	0
t_{11}	128.875	128.875	0
t_{12}	135.625	135.625	0
t_{13}	128.875	150.5	21.625
t_{14}	128.875	150.5	21.625
t_{15}	28.375	140.875	112.5
t_{16}	145.5	145.5	0
t_{17}	128.875	150.5	21.625
t_{18}	128.875	150.5	21.625
t_{19}	135.625	157.25	21.625
t_{20}	135.625	157.25	21.625
t_{21}	39.875	152.375	112.5
t_{22}	41.875	154.375	112.5
t_{23}	43.875	156.375	112.5
t_{24}	163.375	163.375	0

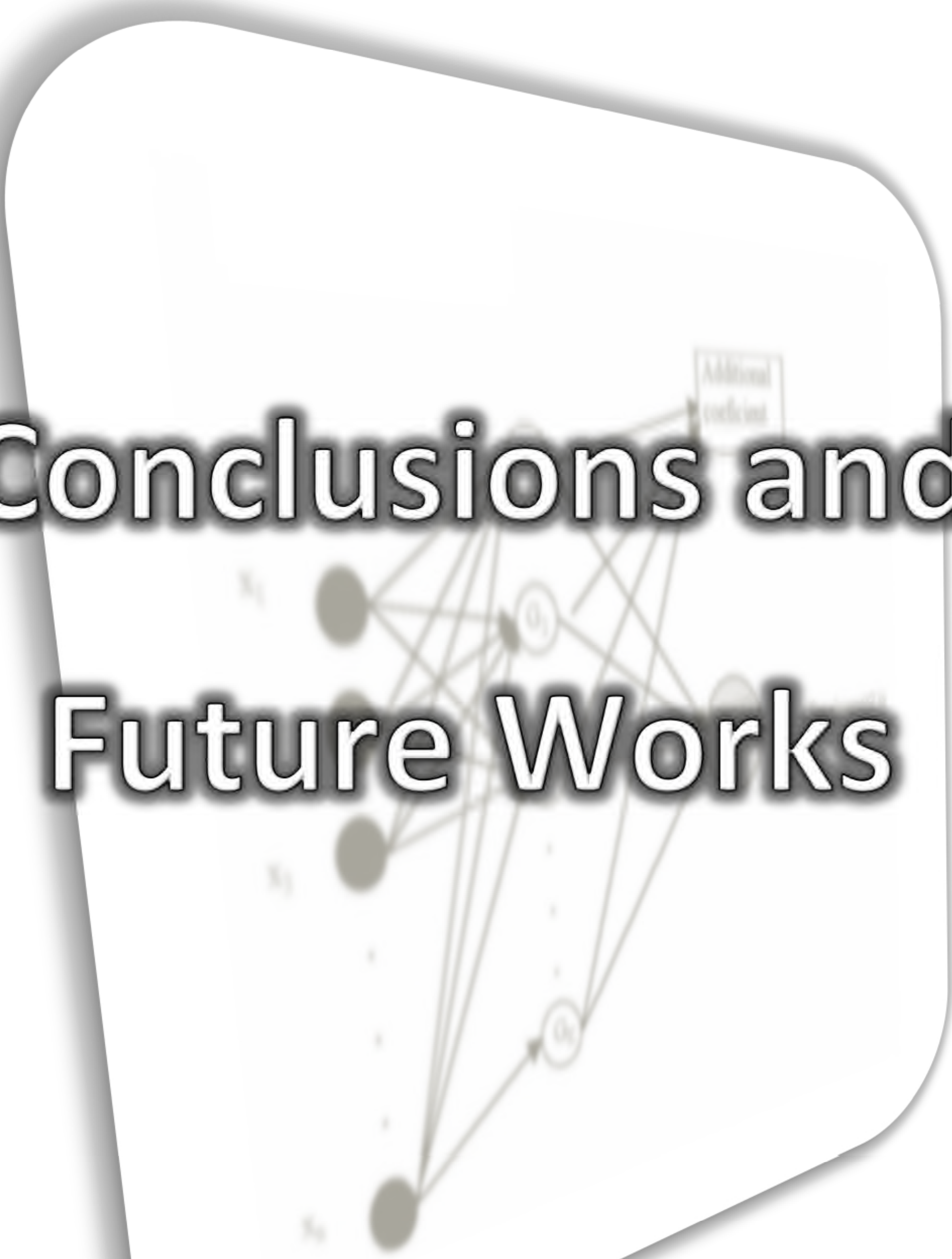
Table (4.23): Critical Event with μ = 1

Nodes	Earliest Time T_E	Latest Time T_L	Slack Time
t_1	0	0	0
t_2	29	29	0
t_3	33	33	0
t_4	62.5	62.5	0
t_5	72	72	0
t_6	78	78	0
t_7	110.5	110.5	0
t_8	21	132.5	111.5
t_9	114	114	0
t_{10}	128	128	0
t_{11}	128	128	0
t_{12}	134.5	134.5	0
t_{13}	128	150	22
t_{14}	128	150	22
t_{15}	28.5	140	111.5
t_{16}	144.5	144.5	0
t_{17}	128	150	22
t_{18}	128	150	22
t_{19}	134.5	156.5	22
t_{20}	134.5	156.5	22
t_{21}	40	151.5	111.5
t_{22}	42	153.5	111.5
t_{23}	44	155.5	111.5
t_{24}	162.5	162.5	0

Table (4.24): The summary results from tables (4.18)-(4.23)

μ-cut value	Earliest Expected Time T_E	Latest Allowable Time T_L	Slack Time	Critical Path
0	165.5	165.5	0	$t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_9 \to t_{10} \to t_{11} \to t_{12} \to t_{16} \to t_{24}$
0.1	165.2	165.2	0	$t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_9 \to t_{10} \to t_{11} \to t_{12} \to t_{16} \to t_{24}$
0.25	164.75	164.75	0	$t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_9 \to t_{10} \to t_{11} \to t_{12} \to t_{16} \to t_{24}$
0.5	164	164	0	$t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_9 \to t_{10} \to t_{11} \to t_{12} \to t_{16} \to t_{24}$
0.75	163.375	163.375	0	$t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_9 \to t_{10} \to t_{11} \to t_{12} \to t_{16} \to t_{24}$
1	162.5	162.5	0	$t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_9 \to t_{10} \to t_{11} \to t_{12} \to t_{16} \to t_{24}$

Conclusions and Future Works



Conclusions and Future Works

Over the past few decades, researchers have proposed many FLP models with different levels of sophistication. However, many of these models have limited real-world applications because of their methodological complexities and flexible assumptions.

In contrast, our proposed approaches in this study are straight forward and flexible. The managerial of the proposed approaches are their applicability to a wide range of real-word problems such as performance evaluation.

From the obtained results we can conclude the following:

1- The defuzzification techniques are possible to be implementing or solving fuzzy network problems.

2- The implementing of standard crisp model identifying the required critical path when μ-cut equal one.

3- The weak and strong critical paths can be identified.

4- The four approaches give us the same optimal critical path and different time values of the objective function.

5- Our computation results had been shown identically to the results in [40] corresponding to each μ-cut values which are considered.

For future research we are suggested to concentrate on the comparison of results obtained with those that might be obtained with other methods. In addition, we plan to extend the FLP approach proposed here to deal with fuzzy nonlinear optimization problems with multiple objectives where the vagueness or impreciseness appears in all the components of the optimization problem such as the objectives, constraints and coefficients.

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الملخص

في هذه الرسالة تم تطوير ثلاثة طرق تتضمن تحويل المسألة الضبابية (fuzzy) الى مسألة محددة المعالم (deterministic) للحصول على المسار الحرج الذي يمثل وقت الانجازالامثل لمختلف مسائل الشبكة الضبابية من خلال تحويل معاملات ومتغيرات مسائل النمذجة الضبابية الخطية (FLPP) الى مسائل النمذجة الغير ضبابية الخطية (CLPP).

الطرق الثلاثة لتحويل المسائل الضبابية الى مسائل محددة تعتمد فلسفة دالة الكثافة الاحتمالية، درجة القياس وتقنية تقييم ومراجعة المشروع (PERT).

اخيرا، تم استخدام طريقة المسار الحرج (CPM) لمقارنة نتائجها مع نتائج الطرق الثلاثة لاظهار قوة هذه الطرق ودقتها.

قد تم اعتماد حالة دراسية لاثبات صحة النتائج المستخرجة باستخدام نظام "Matlab2010R".

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حل مسائل الشبكة الضبابية باستخدام الطرق الاحصائية

رسالة
مقدمة إلى كلية العلوم/ جامعة النهرين
كجزء من متطلبات نيل درجة الماجستير في علوم الرياضيات

من قبل
سنار مازن يونس
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