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Approximate Method for Solving Fuzzy Integral Equations of Fractional Order

A Thesis
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بسم الله الرحمن الرحيم

نَرفَعُ دَرَجاتٍ مَّن نَّشَاءُ وفَوقَ كُلِّ ذِي عِلمٍ عَليمٌ

صدق الله العظيم

سورة يوسف

الآية (76)

DEDICATION

To...

All Persons who

Encouraged and

Supported Me in My Life

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I owe my gratitude to Allah and all peoples who have made this thesis possible.

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Ruaa Hisham Fadhel

Abstract

The main aim of this thesis is oriented about finding the approximate solution of fuzzy integral equations of fractional order as follows:

First studying the basic concept of the main subjects related to the work of this thesis which are so called fractional calculus and fuzzy set theory.

Second studying the existence and uniqueness of solutions of the fuzzy integral equations of fractional order.

Third finding the approximate solutions of the fuzzy integral equations of fractional order using Adomian decomposition method.

Contents

Introduction

The main subjects which deals with the work of this thesis are the so called fractional calculus and fuzzy set theory.

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables [Kilbas, 2006].

The concept of fractional calculus is popularly believed to have stemmed from a question raised in the year 1695 by Marquis de L'Hopital (1661-1704) to Gottfried wilhelm Leibniz (1646-1716)which sought the meaning of Leibniz's (currently popular) notation $\frac{d^n y}{dx^n}$ for the derivatives of order $n \in \mathbb{N}_0 = \{0,1,2,...\}$ when $n = \frac{1}{2}$ (what if $n = \frac{1}{2}$). In his reply, dated 30 September 1695, Leibniz wrote to L'Hopital as follows: "This is an apparent paradox form which, one day, useful consequences will be drown", [Kilbas, 2006].

Following L'Hopital and Leibniz's first inquisition, fractional calculus was primary a study reserved for the best minds in mathematics. Fourier, Euler and Laplace are among the many that dabbled with fractional calculus and the mathematical consequences [Nishimoto, 1991].

Many found, using their own notations and methodology, definitions that fit the concept of a non-integer order integral or derivative.

The most famous of these definitions that have been popularized in the world of fractional calculus are the Rimann-Liouville and Grunwald-Letnikov definition.

Most of the mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the 20^{th} century. However, it is in the past 100 years that the most intriguing leaps in engineering and scientific application have been found. [Nishimoto, 1991].

The mathematics has in some cases to change to meet the requirements of physical reality. Caputo reformulated the more ‘classic’ definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions to solve his fractional order differential equations [Podlubny, 1999]. Kolowankar reformulated again, the Riemann- Liouville fractional derivative in order to differentiate no-where differentiable fractional functions.

This subject of fractional calculus, devoted exclusively to the subject of fractional calculus in the book by Oldham and Spanir [Oldham, 1974] published in 1974.

Today there exist at least two international journals which are devoted almost entirely to the subject of fractional calculus: (i) Journal of fractional Calculus. (ii) Fractinal Calculus and Applied Analysis.

The second subject which deals to this work is the so called fuzzy set theory:

Zadeh had introduced fuzzy set theory in 1965, in which, Zadeh’s original definition of fuzzy sets could be given as follows “a fuzzy set is a class of objects with a continuum grades of membership. Such a set is characterized by a membership value ranging between zero and one" [Zadeh,1965].

The concept of fuzzy sets which was introduced by Zadeh [Zadeh, 1965] led to the definition of the fuzzy number and implementation in fuzzy control [Chang, 1972] and approximate reasoning problems [Zadeh, 1965], [Zadeh, 1983]. The basic arithmetic structure for fuzzy numbers was later developed by Mizumoto and Tanaka [Mizumoto, 1976], [Mizumoto, 1979], Nahmias [Nahmias, 1978], Dubios and Prade [Dubios, 1978], [Dubios,1980],[Dubios,1982] and Ralescu [Ralescu, 1979] all of which observed the fuzzy number as a location of α-levels $0 < \alpha < 1$, [Chang, 1972].

Further applications such as solving integral equations required appropriate and applicable definitions of fuzzy function and fuzzy integral of fuzzy function. The fuzzy function was introduced by Chang and Zadeh [Chang, 1972]. Later Dubois and Prade [Dubois, 1982] presented an elementary fuzzy calculus based on extension principle [Zadeh, 1965]. The concept of integration of fuzzy functions was first introduced by Dubois and Prade in 1980[Dubois and Prade, 1980]. Alternative approaches were later suggested by Goetschel and Voxman [Goetschel and Voxman, 1986], Kaleva [Kaleva, 1987], Matloka[Matloka,1987], Nanda [Nanda,1989], and others. While Goetschel and Voxman [Goetschel, 1986] and later Matloka[Matloka,1987], preferred a Riemann integral type approach, Kaleva[Kaleva, 1987] chose to define the integral of fuzzy function, using the Lebesgue type concept for integration.

One of the first applications of fuzzy integration was given by [Wu, 1991] who investigated the Fuzzy Fredholm integral equations of the second kind.

Later many authors investigated the numerical solution of fuzzy integral equation among them Babolian [Babolian, 2005] solve linear Fredholm fuzzy integral equations of the second kind by Adomian method, Jahantigh and et al., [Jahantigh and et al., 2008] proposed a numerical procedure for solving fuzzy integral equations, Allahviranloo and et al. [Allahviranloo and et al.,2010] used a Homotopy perturbation method for fuzzy voltera integral equations and Allahviranloo and et al. [Allahviranloo and et al., 2011] solved linear Fredholm fuzzy integral equations of the second kind by Modified Trapezoidal method.

The present thesis concerns with the approximate solution of fuzzy integral equations of fractional order by using Adomian decomposition method.

This thesis consists of three chapters as well as this introduction. In chapter one, the basic concepts of fractional calculus and fuzzy set theory are given. While in chapter two the existence and uniqueness theorem for fuzzy integral equations of fractional order is presented. Finally the approximate solution of fuzzy integral equations of fractional order by using Adomian decomposition method will be given in chapter three. It is important to notice that, the computer programs are coded in MATHCAD 14 computer software and the results are presented in a tabulated form.

Chapter One

1

Basic Concepts

CHAPTER ONE

Basic Concepts

1.1 Introduction:

In this chapter we presented some necessary basic concepts and notations, that needed later to define and illustrate some subjects related to the work of the thesis, which including fractional calculus and fuzzy set theory.

1.2 Fractional Calculus:

In this section some of the basic and fundamental concepts and definitions concerning to fractional calculus will be introduced for completeness purpose.

1.2.1 The Gamma Function, [Oldham,1974]:

Gamma function is one of the most important notation in fractional calculus, since it is play an important role in fractional differentiation and integration.

The gamma function $\Gamma(x)$ of a positive real x, is defined by:

$$\boldsymbol{\Gamma(x) = \int_0^\infty y^{x-1} e^{-y}\, dy}\ ,\ x > 0 \qquad (1.2)$$

Following are some of the most important properties of the gamma function:

1. $\Gamma(1) = 1$.
2. $\Gamma(x + 1) = x\Gamma(x)$.
3. $\Gamma(x + 1) = x!,\ \boldsymbol{x} \in \mathbb{N}$.
4. $\Gamma\left(\frac{1}{2} - n\right) = \frac{(-4)^n\, n!\sqrt{\pi}}{(2n)!}$.

5. $\Gamma\left(\frac{1}{2}+n\right)=\frac{(2n)!\sqrt{\pi}}{4^{n}\,n!}$.

6. $\boldsymbol{\Gamma(-x)=\frac{-\pi\csc(\pi x)}{\Gamma(x+1)},\ \ x\neq 1,2,3,\ldots}$.

7. $\Gamma(nx)=\sqrt{\frac{2\pi}{n}}\left[\frac{n^{x}}{\sqrt{2\pi}}\right]\prod_{k=0}^{n-1}\left(n+\frac{k}{n}\right)$.

1.2.2 Fractional Integral

There are many literatures introduces different definitions of fractional integrations, such as:

1***. Riemann-Liouville fractional integral, [Oldham, 1974]:***

The generalization to non-integer q of Riemann-Liouville integral can be written for suitable function f(x) ($x \in R$) as:

$$\frac{d^{q}}{dx^{q}}f(x)=\frac{1}{\Gamma(-q)}\int_{0}^{x}(x-y)^{-q-1}f(y)\,dy\,,\ q<0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(1.3)}$$

2. Wely fractional integral, [Oldham, 1974]:

The left hand fractional order integral of order $q > 0$ of a given function f is defined as:

$${}_{-\infty}D_{x}^{q}\,f(x)=\frac{1}{\Gamma(q)}\int_{-\infty}^{x}\frac{f(y)}{(x-y)^{1-q}}dy,\ x>-\infty \ldots\ldots\ldots\ldots\ldots\ldots \text{(1.4)}$$

And the right hand fractional order integral of order $q > 0$ of a given function f is defined as:

$${}_{\infty}D_{x}^{q}\,f(x)=\frac{1}{\Gamma(q)}\int_{x}^{\infty}\frac{f(y)}{(y-x)^{1-q}}dy,\ x<\infty$$

3. Abel-Riemann fractional integral, [Mittal, 2008]:

The Abel-Riemann (A-R) fractional integral of any order $q > 0$ for a function $f(t)$ with $t \in \mathbb{R}^+$ is defined as

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau)\, d\tau,\ t > 0\ q > 0 \qquad (1.5)$$

1.2.3 Fractional Derivatives:

Many literatures discussed and presented fractional derivatives of certain functions; therefore in this subsection some definitions of fractional derivatives are presented:

1. Riemann-Liouville fractional derivetive, [Oldham, 1974]:

Among the most important formulae used in fractional calculus is the Riemann-Liouville formula. For a given function f(x), $\forall\ x \in [a, b]$, the left and right hand Riemann-Liouville fractional derivatives of order $q > 0$ and m is a natural number, are given by:

$$_x D^q_{a+} f(x) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_a^x \frac{f(t)}{(x-t)^{q-m+1}} dt \qquad (1.6)$$

$$_x D^q_{b-} f(x) = \frac{(-1)^m}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_x^b \frac{f(t)}{(x-t)^{q-m+1}} dt \qquad (1.7)$$

where $m - 1 < q \le m$, $m \in \mathbb{N}$. These equations are usually named as the Riemann-Liouville fractional derivatives.

2. The Abel-Riemann fractional derivative [Mittal, 2008]:

The Abel-Riemann fractional derivative (of order $q > 0$) is defined as the left inverse of corresponding A-R fractional integral, i.e.,

$$D^q J^q = I \qquad (1.8)$$

For positive integer m such that $m - 1 < q \le m$, $m \in \mathbb{N}$

$$(D^m J^{m-q})J^q = D^m (J^{m-q} J^q) = I, \text{ i.e.,}$$

$$D^q x(t) = \begin{cases} \dfrac{1}{\Gamma(m-q)} \dfrac{d^m}{dt^m} \displaystyle\int_0^t \dfrac{x(t)}{(t-\tau)^{q+1-m}} d\tau, & m-1 < q < m \\ \dfrac{d^m}{dt^m} x(t) & q = m \end{cases} \quad \text{...... (1.9)}$$

Properties of the operator J^q and D^q can be found in [Podulbny, 1999], we mention the following:

$$J^q t^p = \frac{\Gamma(p+1)}{\Gamma(p+1+q)} t^{p+q}$$

$$D^q t^p = \frac{\Gamma(p+1)}{\Gamma(p+1-q)} t^{p-q}$$

for $t > 0$, $q \geq 0$, $p \geq -1$.

3. Caputo fractional derivative, [Caputo, 1967]:

In the late sixties an alternative definition of fractional derivatives was introduced by Caputo. Caputo and Mirandi used this definition in their work on the theory of viscoelasticity. According to Caputo's definition

$$D_*^q = J^{m-q} D^m, \text{ for } m-1 < q \leq m$$

which means that:

$$D_*^q x(t) = \begin{cases} \dfrac{1}{\Gamma(m-q)} \displaystyle\int_0^t \dfrac{x^{(m)}(\tau)}{(t-\tau)^{q+1-m}} d\tau, & m-1 < q < m \\ \dfrac{d^m}{dt^m} x(t) & q = m \end{cases}$$

The basic properties of the Caputo fractional derivative are:

1. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

2. $J^q D_*^q x(t) = x(t) - \sum_{k=0}^{m-1} x^{(k)}(0^+)\frac{t^k}{k!}$.

3. Caputo's fractional differentiation is a linear operator, similar to integer order differentiation

$$D_*^q[a_1 f(t) + a_2 g(t)] = a_1 D_*^q f(t) + a_2 D_*^q g(t).$$

where a_1 and a_2 are constants.

4. Gruünwald fractional derivatives, [Oldham, 1974]:

The Gruünwald derivatives of any integer order to any fraction order derivative takes the form:

$$\frac{d^q}{dx^q} f(x) = \lim_{N\to\infty}\left\{\frac{\left(\frac{x}{N}\right)^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(x - j\left(\frac{x}{N}\right)\right)\right\} \quad \text{.................(1.10)}$$

<u>***1.3 Fuzzy Sets Theory:***</u>

Fuzzy sets theory is a generalization of abstract set theory it has a wider scope of applicability than abstract set theory in solving problems that involve to some degree subjective evaluation [Kandel, 1986].

Let X be the space of objects and x be the generic element of X, a classical set A, $A \subseteq X$ is defined as the collection of elements or objects $x \in X$ such that each element x can either belong or not to the set A [Kandel, 1986]. By defining a characteristic (or membership) function for each element x in X, we can represent a classical set A by a set of order pairs (x, 0) or (x, 1), which indicates $x \in A$ or $x \notin A$, respectively [Kandel, 1986].

A fuzzy set express the degree to which an element belongs to a set. Hence, for simplicity, a membership function of a fuzzy set is allowed to have values between (0 and 1) which reflect the degree of membership of an element in the given set [Kandel, 1986].

<u>Definition (1.1):</u>

In mathematical symbols, the membership function is given by $\mu_{\tilde{A}} : X \longrightarrow [0,1]$ and the fuzzy set (denoted by $\tilde{A}$) in X is defined as a set of ordered pairs, [Zadeh, 1965]:

$$\tilde{A} = \left\{ (x, \mu_{\tilde{A}}(x)) \middle| x \in X \right\}$$

<u>1.3.1 Some Basic Concepts of Fuzzy Sets[Dubios,1980],[Zimmermann, 1985]:</u>

Let X be the space of objects and let $\tilde{A}$ be a fuzzy set in X, then one can define the following concepts:

1. The support of $\tilde{A}$ in the universal set X is a crisp set, denoted by:

$$S(\tilde{A}) = \left\{ x \middle| \mu_{\tilde{A}}(x) > 0, \forall x \in X \right\}$$

2. The core (uncleus) of a fuzzy set $\tilde{A}$ is the set of all points $x \in X$, such that:

$$\mu_{\tilde{A}}(x) = 1.$$

3. The height of a fuzzy set $\tilde{A}$ is the largest membership grade over X, i.e.,

$$hgt(\tilde{A}) = \sup_{x \in X} \mu_{\tilde{A}}(x).$$

4. The crossover point of a fuzzy set $\tilde{A}$ is the point in X, whose grade of membership in $\tilde{A}$ is 0.5.
5. Fuzzy singleton is a fuzzy set whose support is a single point in X, with $\mu_{\tilde{A}}(x) = \alpha,\ \alpha \in (0,1]$.
6. A fuzzy set $\tilde{A}$ is called normalized, if its height is 1,otherwise it is subnormal, i.e., $\text{hgt}(\tilde{A}) < 1$.

***<u>Remark (1.1) [Dubios, 1980]</u>*:**

A non-empty fuzzy set $\tilde{A}$ can always be normale by letting:

$$\mu^*_{\tilde{A}}(x) = \frac{\mu_{\tilde{A}}(x)}{\underset{x\in X}{\text{Sup}}\,\mu_{\tilde{A}}(x)}$$

7. The empty fuzzy set $\tilde{\emptyset}$ and the universal set X are fuzzy sets, where $\forall x \in X\ \ \mu_{\tilde{\emptyset}}(x) = 0$ and $\mu_X(x) = 1$, respectively.
8. If $\tilde{A}$ and $\tilde{B}$ are any two fuzzy subsets of X, then $\tilde{A} = \tilde{B}$ if and only if

$$\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x),\ \forall x \in X.$$

9. If $\tilde{A}$ and $\tilde{B}$ are any two fuzzy subsets of X, then $\tilde{A} \subseteq \tilde{B}$ if and only if

$$\mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x),\ \forall\ x \in X.$$

10. $\tilde{A}^c$ (the complement of fuzzy set $\tilde{A}$) is a fuzzy set whose membership function is defined by:

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x),\ \forall x \in X.$$

11. Given two fuzzy sets $\tilde{A}$ and $\tilde{B}$ their standard intersection $\tilde{A} \cap \tilde{B}$, and standard union $\tilde{A} \cup \tilde{B}$, are fuzzy sets and their membership functions are defined for simplicity for all x ∈ X, by the equations:

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$$

$$\mu_{\tilde{A}\cap\tilde{B}}(x)=\min\{\mu_{\tilde{A}}(x),\, \mu_{\tilde{B}}(x)\}$$

Remark (1.2) [Zimmermann, 1985]:

It is important to notice that the only law of contradiction is $A\cup A^{c}=X$ and the law of excluded middle $A\cap A^{c}=\varnothing$. Both laws are broken for the fuzzy sets, since $\tilde{A}\cup\tilde{A}^{c}\neq X$ and $\tilde{A}\cap\tilde{A}^{c}\neq\tilde{\varnothing}$, indeed $\forall\ x\in\tilde{A}$ such that $\mu_{\tilde{A}}(x)=\alpha$, then according to point (7), we have:

$$\mu_{\tilde{A}\cup\tilde{A}^{c}}(x)=\max\{\alpha,1-\alpha\}\neq 1$$

$$\mu_{\tilde{A}\cap\tilde{A}^{c}}(x)=\min\{\alpha,1-\alpha\}\neq 0$$

12. The Cartesian product of fuzzy sets is defined as follows: let $\tilde{A}_1,\tilde{A}_1,...,\tilde{A}_n$ be fuzzy sets in $\tilde{X}_1,\tilde{X}_2,...,\tilde{X}_n$.

The Cartesian product is then a fuzzy set in the product space $X_1\times X_2\times...\times X_n$ with the membership function:

$$\mu_{(\tilde{A}_1\times\tilde{A}_2\times...\times\tilde{A}_n)}(x)=\min\left\{\mu_{\tilde{A}_i}(x_i)\middle|x=(x_1,x_2,...,x_n),x_i\in X_i\right\}$$

13. The m^{th} power of a fuzzy set $\tilde{A}$ is a fuzzy set with the membership function:

$$\mu_{\tilde{R}^m}(x)=\left[\mu_{\tilde{A}}(x)\right]^m,\quad x\in X$$

14. The algebraic sum $\tilde{C}=\tilde{A}+\tilde{B}$ is defined as:

$$\tilde{C}=\left\{(x,\mu_{\tilde{A}+\tilde{B}}(x)\middle|x\in X\right\}$$

where:

$$\mu_{\tilde{A}+\tilde{B}}(x)=\mu_{\tilde{A}}(x)+\mu_{\tilde{B}}(x)-\mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x)$$

15. The bounded sum $\tilde{C}=\tilde{A}\oplus\tilde{B}$ is defined as:

$$\tilde{C} = \left\{\left(x, \mu_{\tilde{A}\oplus\tilde{B}}(x)\right) \middle| x \in X\right\}$$

where:

$$\mu_{\tilde{A}\oplus\tilde{B}}(x) = \min\left\{1, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)\right\}$$

16. The bounded difference $\tilde{C} = \tilde{A} \ \Theta \ \tilde{B}$ is defined as

$$\tilde{C} = \left\{\left(x, \mu_{\tilde{A}\Theta\tilde{B}}(x)\right) \middle| x \in X\right\}$$

where

$$\mu_{\tilde{A}\Theta\tilde{B}}(x) = \max\left\{0, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 1\right\}$$

17. The algebraic product of two fuzzy sets $\tilde{C} = \tilde{A} \odot \tilde{B}$ is defined as:

$$\tilde{C} = \left\{\left(x, \mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x)\right) \middle| x \in X\right\}$$

Example (1.1) [Zimmermann, 1985]:

Let $\tilde{A} = \{(3,0.5),(5,1),(7,0.6)\}$

and

$$\tilde{B} = \{(3,1),(5,0.6)\}$$

Then

$$\tilde{A} \times \tilde{B} = \{[(3,3),0.5],[(5,3),1],[(7,3),0.6],[(3,5),0.5],[(5,5),0.6],[(7,5),0.6]\}$$

$$\tilde{A}^2 = \{(3,0.25),(5,1),(7,0.36)\}$$

$$\tilde{A} + \tilde{B} = \{(3,1),(5,1),(7,0.6)\}$$

$$\tilde{A} \oplus \tilde{B} = \{(3,1),(5,1),(7,0.6)\}$$

$$\tilde{A}\Theta\tilde{B} = \{(3,0.5),(5,0.6)\}$$

$$\tilde{A}.\tilde{B}=\{(3,0.5),(5,0.6)\}$$

1.3.2 α-Cut Set [Georege, 1995]:

Among the basic concepts in fuzzy set theory is the concept of an α-cut or "α-level set" and its variant, a strong α-cut or "strong α-level set". Given a fuzzy set $\tilde{A}$ defined on X and any number $\alpha \in (0, 1]$, the α-cut A_α is the crisp set that contain all elements of the universal set X, whose membership grades in $\tilde{A}$ are greater than or equal to the specified value of α

$$A_\alpha=\{x\in X:\mu_{\tilde{A}}(x)\geq\alpha\},\ \forall x\in X$$

while

$$A_{\alpha^+}=\{x\in X:\mu_{\tilde{A}}(x)>\alpha\},\ \forall x\in X,$$

is called "strong α-cut"

The following properties are satisfied for all $\alpha \in (0, 1]$:

1. If $\alpha_1,\alpha_2\in(0,1]$, and $\alpha_1\leq\alpha_2$, then $A_{\alpha_1}\supseteq A_{\alpha_2}$.
2. $(\tilde{A}\cup\tilde{B})_\alpha=A_\alpha\cup B_\alpha$.
3. $(\tilde{A}\cap\tilde{B})_\alpha=A_\alpha\cap B_\alpha$
4. $(\tilde{A}\subseteq\tilde{B})_\alpha$ gives $A_\alpha\subseteq B_\alpha$.
5. $\tilde{A}=\tilde{B}$ if and only if $A_\alpha=B_\alpha$, $\forall\ \alpha\in(0,1]$.

Remarks (1.3) [George, 1995]:

1. The set of all levels $\alpha\in(0,1]$, is the image of a fuzzy set that represent distinct α-cuts of a given fuzzy set $\tilde{A}$ is called a level set of $\tilde{A}$, which is denoted by:

$$\wedge(\tilde{A})=\{\alpha|\mu_{\tilde{A}}(x)=\alpha,\ \text{for some }x\in X\}$$

2. The support of $\tilde{A}$ is exactly the same as the strong α-cut of $\tilde{A}$ for $\alpha = 0$,

$$A_{0^+} = S(\tilde{A}).$$

3. The core of $\tilde{A}$ is exactly the same as the α-cut of $\tilde{A}$ for $\alpha = 1$ [i.e., $A_1 = \text{core}(\tilde{A})$].

4. The height of $\tilde{A}$ may also be viewed as the supremum of α-cut for which $A_\alpha \neq \varnothing$.

1.3.3 Convex Fuzzy Sets [George, 1995]:

As important property of fuzzy sets defined on $\mathbb{R}^n$ (for some $n \in \mathbb{N}$) is their convexity; this property is viewed as a generalization of the classical concept of convexity of crisp sets.

Remark (1.4):

The definition of convexity for fuzzy set does not necessarily mean that the membership function of a convex fuzzy set is also convex function.

Definition (1.2) [George, 1995]:

A fuzzy set $\tilde{A}$ on $\mathbb{R}$ is convex if and only if:

$$\mu_{\tilde{A}}\left(\lambda x_1 + (1-\lambda)x_2\right) \geq \min\left\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\right\} \qquad (1.11)$$

for all $x_1, x_2 \in \mathbb{R}$, and all $\lambda \in [0,1]$.

Remark (1.5) [George, 1995]:

A_α is convex for any $\alpha \in (0, 1]$.

1.3.4 Fuzzy Number [Zimmermann, 1985], [Kandel, 1986]:

A fuzzy number $\tilde{M}$ is a convex normalized fuzzy set $\tilde{M}$ of the real line $\mathbb{R}$, such that:

1. There exist exactly one $x_0 \in \mathbb{R}$, with $\mu_{\tilde{M}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{M}$).
2. $\mu_{\tilde{M}}(x)$ is piecewise continuous.

Definition (1.3) [Dubois, 1980]:

A fuzzy number $\tilde{M}$ is called positive (negative) if its membership function is such that $\mu_{\tilde{M}}(x) = 0, \forall x < 0 \ (\forall x > 0)$.

Definition (1.4) [Dubois, 1987]:

A fuzzy number $\tilde{M}$ is of LR-type if there exists functions L (called the left function), R (called the right function), such that $L(x) \le \mu_{\tilde{M}}(x) \le R(x)$, $\forall x \in X$ and scalars a > 0, b > 0 with:

$$\mu_{\tilde{M}}(x) = \begin{cases} L\left(\dfrac{m-x}{a}\right), & \text{for } x \le m \\ R\left(\dfrac{x-m}{b}\right), & \text{for } x \ge m \end{cases}$$

m is a real number called the mean value of $\tilde{M}$, a and b are called the left and right spreads of m respectively. Symbolically $\tilde{M}$ is denoted by $(m, a, b)_{LR}$.

Remark (1.6) [Nguyem, 2000]:

In fact, fuzzy number is fuzzy interval, the only difference is that fuzzy number contain the value 1 at only one place while a fuzzy interval can have several value of 1 on many places, (see Fig. (1.1) and Fig (1.2)).

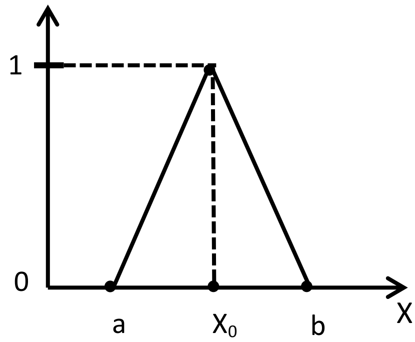

Fig. (1.1) the triangular membership function

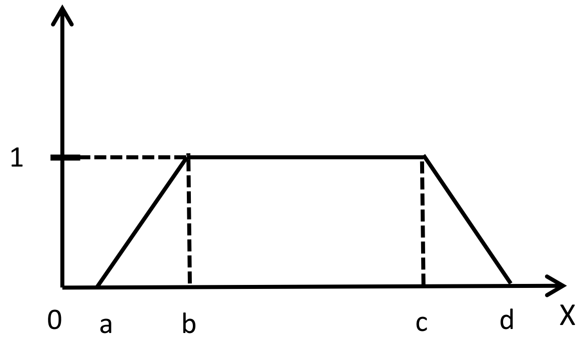

Fig. (1.2) the trapezoidal membership function

For fig. (1.1):

$$\mu_A(x)=\begin{cases}0 & ,0<x\le a\\ x-1 & ,a\le x\le b\\ -\frac{1}{2}x+2 & ,b\le x\le c\\ 0 & ,x\ge c\end{cases}$$

For fig. (1.2):

$$\mu_A(x)=\begin{cases}0 & ,x\le a\\ \frac{x-a}{b-a} & ,a\le x\le b\\ 1 & ,b\le x\le c\\ \frac{c-x}{d-c} & ,c\le x\le d\\ 0 & ,x\ge d\end{cases}$$

Now, in applications, the representation of a fuzzy number in terms of its membership function is so difficult to use, therefore two approaches are given for representing the fuzzy number in terms of its α-level set, as in the following remarks:

Remark (1.7) [Mohammed, 2010]:

A fuzzy number $\tilde{M}$ may be uniquely represented in terms of its α-level set, as the following closed intervals of the real line:

$$M_\alpha = [m - \sqrt{1-\alpha}, m + \sqrt{1-\alpha}] \quad \text{or} \quad M_\alpha = [\alpha m, \frac{1}{\alpha} m]$$

where m is the mean value of $\tilde{M}$ and $\alpha \in (0, 1]$. This fuzzy number may be written as $M_\alpha = [\underline{M}, \overline{M}]$, where $\underline{M}$ refers to the lower bound of M_α and $\overline{M}$ to the upper bound of M_α.

Remark (1.8) [Mohammed, 2010]:

Similar to the second approach given in remark (1.7), one can fuzzyfy any crisp or nonfuzzy function f, by letting:

$\underline{f}(x) = \alpha f(x), \overline{f}(x) = \frac{1}{\alpha} f(x), \forall x \in X, \alpha \in (0,1]$ and hence the fuzzy function $\tilde{f}$ in terms of its α-levels is given by $f_\alpha = [\underline{f}, \overline{f}]$.

1.3.5 The Extension Principle of Fuzzy Sets [Zimmermann, 1985]:

One of the most basic concepts of fuzzy set theory, which can be used to generalize crisp mathematical concepts to fuzzy sets, is the extension principle.

Let X be a Cartesian product of universes $X_1, X_2 \ldots, X_r$ and $\tilde{A}_1, \tilde{A}_2 \ldots, \tilde{A}_r$ be r-fuzzy sets in $X_1, X_2 \ldots, X_r$, respectively, f is a mapping from X to a universe Y $y = f(x_1, x_2 \ldots, x_r)$. Then the extension principle allows us to define a fuzzy set $\tilde{B}$ in Y by:

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y = f(x_1, x_2 \ldots, x_r), (x_1, x_2 \ldots, x_r) \in X\}$$

where

$$\mu_{\tilde{B}}(y)=\begin{cases}\sup\limits_{(x_1,x_2,\ldots,x_r)\in f^{-1}(y)} \min\{\mu_{\tilde{A}_1}(x_1),\ldots,\mu_{\tilde{A}_r}(x_r)\}, & \text{if } f^{-1}(y)\neq\varnothing \\ 0, & \text{otherwise}\end{cases}$$

where f^{-1} is the inverse image of f.

For r = 1, the extension principle, of course, reduces to:

$$\tilde{B}=f(\tilde{A})=\{(y,\mu_{\tilde{B}}(y)) \mid y=f(x), x\in X\}$$

where:

$$\mu_{\tilde{B}}(y)=\begin{cases}\sup\limits_{x\in f^{-1}(y)} \mu_{\tilde{A}}(x), & \text{if } f^{-1}(y)\neq\varnothing \\ 0, & \text{otherwise}\end{cases}$$

Example (1.2) [Zimmermann, 1985]:

Let $\tilde{A}=\{(-1,0.5),(0,0.8),(1,1),(2,0.4)\}$ and $f(x)=x^2$

Then by applying the extension principle we obtain:

$$\tilde{B}=f(\tilde{A})=\{(0,0.8),(1,1),(4,0.4)\}$$

Fig (1.4) illustrates the relationship

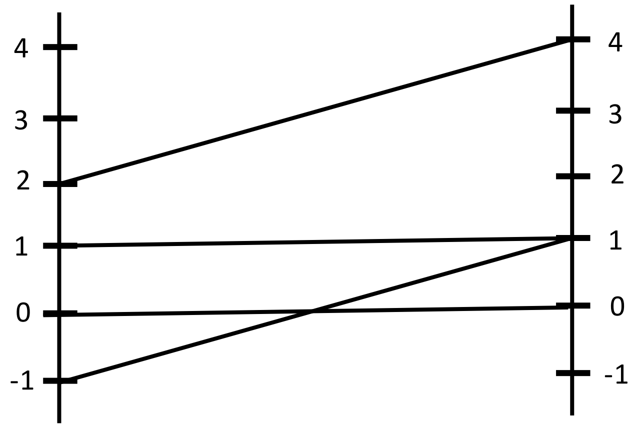

Fig. (1.4)

The relationship of example (1.2)

1.3.6 Fuzzy Integral Equations:

As we know, the main aim of this thesis is to find the approximate solution of fuzzy integral equations of fractional order, therefore it is important here to give a short summary about the meaning of fuzzy integral equation first and then we will treat the approximate solution of fuzzy integral equations of fractional order in chapter two and three respectively.

The integral equations which are discussed in this section are the Fredholm equations of the second kind.

The Fredholm integral equation of the second kind is [Hochstadt, 1973]

$$F(t) = f(t) + \beta\int_a^b K(s,t)F(s)\,ds \text{ ....}(1.12)$$

where $\beta > 0$, K(s, t) is an arbitrary kernel function over the square $a \le s, t \le b$ and f(t) is a function of $t : a \le t \le b$. If f(t) is a crisp function then the solutions of eq. (1.12) are crisp as well. However, if f(t) is a fuzzy function these equations may only possess fuzzy solutions. Sufficient conditions for the existence of a unique solution to the fuzzy Fredholm integral equation of the second kind, i.e., to eq. (1.12) where f(t) is a fuzzy function, are given in [Wu, 1990].

Now, we introduce parametric form of a Fredholm integral equation of the second kind with respect to section (1.3.4)

Let $(\underline{f}(t,r),\overline{f}(t,r))$ and $(\underline{u}(t,r),\overline{u}(t,r)), 0 \le r \le 1$ and $t \in [a, b]$ are parametric form of f(t) and u(t), respectively then, parametric form of Fredholm integral equation of the second kind is as follows:

$$\left.\begin{aligned}\underline{u}(t,r) &= \underline{f}(t,r)+\beta\int_a^b v_1\left(s,t,\underline{u}(s,r),\overline{u}(s,r)\right)ds,\\ \overline{u}(t,r) &= \overline{f}(t,r)+\beta\int_a^b v_2\left(s,t,\underline{u}(s,r),\overline{u}(s,r)\right)ds\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(1.13)$$

where $$v_1\left(s,t,\underline{u}(s,r),\overline{u}(s,r)\right)=\begin{cases}K(s,t)\underline{u}(s,r), & K(s,r)\geq 0\\ K(s,t)\overline{u}(s,r), & K(s,r)<0\end{cases}$$

and $$v_2\left(s,t,\underline{u}(s,r),\overline{u}(s,r)\right)=\begin{cases}K(s,t)\overline{u}(s,r), & K(s,r)\geq 0\\ K(s,t)\underline{u}(s,r), & K(s,r)<0\end{cases}$$

for each $0 \leq r \leq 1$ and $a \leq t \leq b$. We can see that (1.13) is a system of linear Fredholm integral equations in crisp case for each $0 \leq r \leq 1$ and $a \leq t \leq b$.

The work of this thesis is concerned with the approximate solution of fuzzy integral equations of fractional order given by the following form:

$$\tilde{y}(t)=\tilde{f}(t)+I^q g(t,\tilde{y}(t)) \ , \quad t\geq 0.$$

Where $\tilde{f}$ is assumed to be fuzzy function and $0 < q \leq 1$.

Studying the existence and uniqueness of such equations will be treated in chapter two while chapter three will be oriented towards the approximate solution of such equations using Adomian decomposition method.

Chapter Two

2

The Existence And Uniqueness Theorem For Fuzzy Integral Equation Of Fractional Order

CHAPTER TWO

The Existence and Uniqueness Theorem for Fuzzy Integral Equations of Fractional Order

2.1 Introduction

This chapter concerned with the existence and the uniqueness of the solution of fuzzy integral equations of fractional order. Also, some necessary definitions and theorems related to the prove of the existence and uniqueness theorem are included in order to make this chapter of self contained as possible.

2.2 Preliminaries:

Let $P_k(\mathbb{R}^n)$ denote the collection of all nonempty compact convex subset of $\mathbb{R}^n$ and define the addition and scalar multiplication in $P_k(\mathbb{R}^n)$ as usual. Let I = [0,1] ⊆ R be a compact interval and let E^n be defined as follows:

$$E^n = \left\{u : R^n \to [0,1] : u \text{ satisfies (i)} - \text{(iv) below}\right\},$$

where

(i) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex,

(iii) u is upper semicontinuous,

(iv) $[u]^0$ = closure of $\{x \in R^n : \mu_A(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$ denote $[u]^\alpha = \left\{x \in R^n : \mu_A(x) > \alpha\right\}$ is compact. Then from (i)-(iv) it follows that the α-level set $[u]^\alpha \in P_k(\mathbb{R}^n)$ for all $0 \leq \alpha \leq 1$.

let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Housdorff metric:

$$H_d(A,B)=\max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

where

$$d(b,A)=\inf\{d(b,a): a\in A\}$$

Its clear that $(P_k(\mathbb{R}^n),d)$ is a complete metric space [Kisielewicz, 1991]

<u>Remarks (2.1) [Benchohra, 2008]:</u>

1. Suppose that the closure of $\{x\in\mathbb{R}^n : \mu_A(x)>0\}$, which is denoted by A_0 is compact since A_0 is the smallest closed set containing $\{x\in\mathbb{R}^n : \mu_A(x)>0\}$.

2. $A_\alpha \in P_k(\mathbb{R}^n)$ for all $0\le\alpha\le 1$.

Now suppose if $g:\mathbb{R}^n\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a function, then according to the Zadeh's extension principle [Zadeh, 1965] we can extend g to $E^n\times E^n \longrightarrow E^n$ by the equation:

$$g(\tilde{A},\tilde{B})(z)=\sup_{z=g(x,y)} \min\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(y)\}$$

where g is any relation between $\tilde{A}$ and $\tilde{B}$.

Define $\hat{0}:\mathbb{R}^n \longrightarrow [0,1]$ by:

$$\hat{0}=\begin{cases}1, & \text{if } t=0\\ 0, & \text{otherwise}\end{cases}$$

and let $D:E^n\times E^n \longrightarrow [0,\infty)$ be defined by:

$$D(\tilde{A},\tilde{B}) = \sup_{0\le\alpha\le1} H_d(A_\alpha, B_\alpha)$$

Where d is the Housdorff metric define in $P_k(\mathbb{R}^n)$

Then (E^n, D) is a complete metric space [Fadhel, 1997].

Remark(2.1) [Kaleva, 1987]:

1. $D(\tilde{A}+\tilde{Z}, \tilde{B}+\tilde{Z}) = D(\tilde{A},\tilde{B})$ for $\tilde{A},\tilde{B},\tilde{Z} \in E^n$

2. $D(\lambda\tilde{A}, \lambda\tilde{B}) = |\lambda| D(\tilde{A},\tilde{B})$ for all $\tilde{A},\tilde{B} \in E^n, \lambda \in \mathbb{R}$.

Now, the following definitions and theorems concerning integrability properties for the set-valued mapping of a real variable whose values are in (E^n, D) [Kaleva, 1987], [Puri, 1986]:

Definition (2.1) [Park,1999]:

Suppose $T=[c,d]\subset\mathbb{R}$ be compact interval, then a mapping $F: T \longrightarrow E^n$ is called levelwise continuous at $t_0 \in T$ if the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is continuous at $t = t_0$ with respect to the Housdorff metric d for all $\alpha \in [0, 1]$.

Definition (2.2) [Park, 1999]:

Suppose that $T=[c,d]\subset\mathbb{R}$ be a compact interval, then a mapping $F: T \longrightarrow E^n$ is strongly measurable if for all $\alpha \in [0, 1]$, the set-valued mapping $F_\alpha: T \longrightarrow P_k(\mathbb{R}^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable functions.

Definition (2.3) [Park, 1999]:

A mapping $F:T \longrightarrow E^n$ is called integrably bounded if there exists an integrable function h such that $\|y\| \le h(t)$, for all $y \in F_0(t)$, where $T=[c,d] \subset \mathbb{R}$ be a compact interval.

Definition (2.4) [Park,1999]:

Let $F:T \longrightarrow E^n$. The integral of F over T denoted by $\int_T F(t)$ or $\int_c^d F(t)$, is defined levelwise for all $0 < \alpha \le 1$, by:

$$\left(\int_c^d F(t)\right)_\alpha = \int_c^d F_\alpha(t)\,dt$$

$$= \left\{ \int_T f(t)\,dt \,\middle|\, f : T \to \mathbb{R}^n \text{is Lebesegue measurable selection for } F_\alpha \right\}$$

Theorem (2.1) [Park, 1999]:

If $F:T \longrightarrow E^n$ is strongly measurable and integrably bounded, then F is integrable.

Theorem (2.2) [Park, 1999]:

If $F:T \longrightarrow E^n$ is levelwise continuous, then it is strongly measurable.

Theorem (2.3) [Park, 1999]:

If $F:T \longrightarrow E^n$ is levelwise continuous then it is integrable.

Theorem (2.4) [Park, 1999]:

Let $F:T \longrightarrow E^n$ be integrable and $c \in T$, then :

$$\int_{t_0}^{t_0+p} F(t)\,dt = \int_{t_0}^{c} F(t)\,dt + \int_{c}^{t_0+p} F(t)\,dt,\quad c \in [t_0, t_0+p],\ p>0$$

Theorem (2.5) [Park, 1999]:

Let $F, G : T \longrightarrow E^n$ be integrable, and $\lambda \in \mathbb{R}$, then:

(i) $\int_T \left(F(t)+G(t)\right)dt = \int_T F(t)\,dt + \int_T G(t)\,dt$.

(ii) $\int_T \lambda F(t)\,dt = \lambda \int_T F(t)\,dt$.

(iii) $\left(D(F,G)\right)(t)$ is integrable.

(iv) $D\left(\int_T F(t)\,dt, \int_T G(t)\,dt\right) \le \int_T D\left(F,G\right)(t)\,dt$.

2.3 The Existence and Uniqueness Theorem [Benchohra, 2008]:

In this subsection, the existence and uniqueness of the following equation:

$$\tilde{y}(t) = \tilde{f}(t) + I^q g(t, \tilde{y}(t)),\quad t \in [0,T], \ldots \quad (2.1)$$

where $\tilde{f} : [0,T] \longrightarrow E^n$ and $\tilde{g} : [0,T] \times E^n \longrightarrow E^n$. Will be proved by assuming that the following assumptions are satisfied:

Let L and T be positive numbers:

(1) $\tilde{f} : [0,T] \longrightarrow E^n$ is continuous and bounded.

(2) $\tilde{g} : [0,T] \times E^n \longrightarrow E^n$ is continuous and satisfies the Lipschitz condition, i.e.,

$$D(g(t,\tilde{y}_2(t)), g(t,\tilde{y}_1(t))) \le L\, D(\tilde{y}_2(t), \tilde{y}_1(t)),\quad t \in [0,T],$$

where $\tilde{y}_i : [0,T] \longrightarrow E^n, \quad i = 1,2.$

(3) $g(t,\hat{0})$ is bounded on [0, T]

Theorem (2.6):

Let the assumptions (1) − (3) be satisfied. If:

$$T < \left(\frac{\Gamma(q+1)}{L} \right)^{1/q}$$

then eq.(2.1) has a unique solution $\tilde{y}$ on [0, T] and the successive iterations

$$\left. \begin{aligned} &\tilde{y}_0(t) = \tilde{f}(t) \ , \\ &\tilde{y}_{n+1}(t) = \tilde{f}(t) + I^q g(t, \tilde{y}_n(t)) \ , \quad n = 0,1,2, \ldots \end{aligned} \right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (2.2)$$

are uniformly convergent to $\tilde{y}$ on [0,T].

Proof: First we prove that $\tilde{y}_n$ are bounded on [0,T]. We have $\tilde{y}_0(t) = \tilde{f}(t)$ is bounded, which comes from (1). Assume that $\tilde{y}_{n-1}(t)$ is bounded. From (2) we have:

$$\begin{aligned} D(\tilde{y}_n(t), \hat{0}) &= D(\tilde{f}(t) + I^q g(t, \tilde{y}_{n-1}(t)), \hat{0}) \\ &\le D(\tilde{f}(t), \hat{0}) + D(I^q g(t, \tilde{y}_{n-1}(t)), \hat{0}) \\ &\le D(\tilde{f}(t), \hat{0}) + \frac{1}{\Gamma(q)} \int_0^t D\left(\frac{g(s, \tilde{y}_{n-1}(s))}{(t-s)^{1-q}}, \hat{0} \right) ds \\ &\le D(\tilde{f}(t), \hat{0}) + \frac{1}{\Gamma(q)} \sup_{0 \le t \le T} D(g(t, \tilde{y}_{n-1}(t)), \hat{0}) \int_0^t \frac{ds}{(t-s)^{1-q}} \end{aligned}$$

But

$$D(g(t, \tilde{y}_{n-1}(t)), \hat{0}) \le D(g(t, \tilde{y}_{n-1}(t)), g(t, \hat{0})) + D(g(t, \hat{0}), \hat{0})$$

$$\leq L\,D(\tilde{y}_{n-1}(t),\hat{0}) + D(g(t,\hat{0}),\hat{0})$$

So

$$D(\tilde{y}_n(t),\hat{0}) \leq D(\tilde{f}(t),\hat{0}) + \frac{T^q}{\Gamma(q+1)} \sup_{0\leq t\leq T} [L\,D(\tilde{y}_{n-1}(t),\hat{0}) + D(g(t,\hat{0}),\hat{0})]$$

$$\leq D(\tilde{f}(t),\hat{0}) + \sup_{0\leq t\leq T} D(\tilde{y}_{n-1}(t),\hat{0}) + \frac{T^q}{\Gamma(q+1)} \sup_{0\leq t\leq T} D(g(t,\hat{0}),\hat{0})$$

This proves that $\tilde{y}_n$ is bounded $\forall n \in \mathbb{N}$. Therefore, $\{\tilde{y}_n\}$ is a sequence of bounded functions on [0, T].

Second, we prove that $\tilde{y}_n$ are continuous on [0, T]. For $0 \leq t \leq \tau \leq T$, we have:

$$D(\tilde{y}_n(t),\tilde{y}_n(\tau)) \leq D(\tilde{f}(t),\tilde{f}(\tau)) + \frac{1}{\Gamma(q)} D\left(\int_0^t \frac{g(s,\tilde{y}_{n-1}(s))}{(t-s)^{1-q}}ds, \int_0^\tau \frac{g(s,\tilde{y}_{n-1}(s))}{(\tau-s)^{1-q}}ds\right)$$

$$\leq D(\tilde{f}(t),\tilde{f}(\tau)) + \frac{1}{\Gamma(q)} D\left(\int_0^t \frac{g(s,\tilde{y}_{n-1}(s))}{(t-s)^{1-q}}ds, \int_0^t \frac{g(s,\tilde{y}_{n-1}(s))}{(\tau-s)^{1-q}}ds\right) + \frac{1}{\Gamma(q)} D\left(\int_t^\tau \frac{g(s,\tilde{y}_{n-1}(s))}{(\tau-s)^{1-q}}ds, \hat{0}\right)$$

$$\leq D(\tilde{f}(t),\tilde{f}(\tau)) + \frac{1}{\Gamma(q)} \int_0^t D\left(\frac{g(s,\tilde{y}_{n-1}(s))}{(t-s)^{1-q}}, \frac{g(s,\tilde{y}_{n-1}(s))}{(\tau-s)^{1-q}}\right)ds + \frac{1}{\Gamma(q)} \int_t^\tau D\left(\frac{g(s,\tilde{y}_{n-1}(s))}{(\tau-s)^{1-q}}, \hat{0}\right)ds$$

$$\leq D(\tilde{f}(t),\tilde{f}(\tau)) + \frac{1}{\Gamma(q)} \sup_{0\leq t\leq T} D(g(t,\tilde{y}_{n-1}(t),\hat{0}) \int_0^t |(t-s)^{q-1} - (\tau-s)^{q-1}|ds + \frac{1}{\Gamma(q)} \sup_{0\leq t\leq T} D(g(t,\tilde{y}_{n-1}(t),\hat{0}) \int_t^\tau \frac{ds}{(\tau-s)^{1-q}}$$

$$\leq D(\tilde{f}(t),\tilde{f}(\tau)) + \frac{1}{\Gamma(q+1)}[|t-\tau|^q - |t^q - \tau^q|] \sup_{0\leq t\leq T} D(g(t,\tilde{y}_{n-1}(t),\hat{0})$$

$$+\frac{1}{\Gamma(q+1)}|t-\tau|^q \sup_{0\leq t\leq T} D(g(t,\tilde{y}_{n-1}(t),\hat{0})$$

$$\leq D(\tilde{f}(t),\tilde{f}(\tau)) + \frac{1}{\Gamma(q+1)}[2|t-\tau|^q - |t^q - \tau^q|]$$

$$\sup_{0\leq t\leq T} D(g(t,\tilde{y}_{n-1}(t),\hat{0})$$

$$\leq D(\tilde{f}(t),\tilde{f}(\tau)) + \frac{1}{\Gamma(q+1)}[2|t-\tau|^q - |t^q - \tau^q|]$$

$$\sup_{0\leq t\leq T} [L\,D(g(t,\tilde{y}_{n-1}(t)),\hat{0}) + D(g(t,\hat{0}),\hat{0})].$$

The last inequality, by symmetry, is valid for all t, $\tau \in$ [0, T] regardless whether or not $t \leq \tau$. Thus $D(\tilde{y}_n(t),\tilde{y}_n(\tau)) \longrightarrow 0$ as $t \longrightarrow \tau$. Therefore, the sequence $\{\tilde{y}_n\}$ is continuous on [0,T].

For $n \geq 1$, we have:

$$D(\tilde{y}_{n+1}(t),\tilde{y}_n(t)) = \frac{1}{\Gamma(q)} D\left(\int_0^t \frac{g(s,\tilde{y}_n(s))}{(t-s)^{1-q}}\,ds, \int_0^t \frac{g(s,\tilde{y}_{n-1}(s))}{(t-s)^{1-q}}\,ds\right)$$

$$\leq \frac{1}{\Gamma(q)} \int_0^t D\left(\frac{g(s,\tilde{y}_n(s))}{(t-s)^{1-q}}, \frac{g(s,\tilde{y}_{n-1}(s))}{(t-s)^{1-q}}\right) ds$$

$$\leq \frac{1}{\Gamma(q)} \int_0^t D(g(s,\tilde{y}_n(s)), g(s,\tilde{y}_{n-1}(s))) \frac{ds}{(t-s)^{1-q}}$$

$$\leq \frac{1}{\Gamma(q)} \sup_{0\leq t\leq T} D(g(t,\tilde{y}_n(t)), g(t,\tilde{y}_{n-1}(t))) \int_0^t \frac{ds}{(t-s)^{1-q}}$$

$$\leq \frac{LT^q}{\Gamma(q+1)} \sup_{0\leq t\leq T} D(\tilde{y}_n(t),\tilde{y}_{n-1}(t))$$

$$\leq \left(\frac{LT^q}{\Gamma(q+1)}\right)^2 \sup_{0\leq t\leq T} D(\tilde{y}_{n-1}(t),\tilde{y}_{n-2}(t))$$

$$\vdots$$

$$\leq \left(\frac{LT^q}{\Gamma(q+1)}\right)^n \sup_{0\leq t\leq T} D(\tilde{y}_1(t), \tilde{y}_0(t)) \quad \text{.....................} \quad (2.3)$$

But:

$$D(\tilde{y}_1(t), \tilde{y}_0(t)) = \frac{1}{\Gamma(q)} D\left(\int_0^t \frac{g(s, \tilde{f}(s))}{(t-s)^{1-q}} ds, \hat{0}\right)$$

$$\leq \frac{1}{\Gamma(q)} \int_0^t D\left(\frac{g(s, \tilde{f}(s))}{(t-s)^{1-q}}, \hat{0}\right) ds$$

$$\leq \frac{1}{\Gamma(q)} \sup_{0\leq t\leq T} D(g(t, \tilde{f}(t)), \hat{0}) \int_0^t \frac{ds}{(t-s)^{1-q}}$$

Thus:

$$\sup_{0\leq t\leq T} D(\tilde{y}_1(t), \tilde{y}_0(t)) \leq \frac{LT^q}{\Gamma(q+1)} [LM + N] = R$$

where:

$$M = \sup_{0\leq t\leq T} D(\tilde{f}(t), \hat{0}) \qquad \text{and} \qquad N = \sup_{0\leq t\leq T} D(g(t, \hat{0}), \hat{0})$$

Therefore (2.3) will takes the form:

$$D(\tilde{y}_{n+1}(t), \tilde{y}_n(t)) \leq R\left(\frac{LT^q}{\Gamma(q+1)}\right)^n \quad \text{..} \quad (2.4)$$

Next, we show that for each t $\in$ [0, T] the sequence $\{\tilde{y}_n\}$ is a Cauchy sequence in E^n

Let r, s be integers such that s > r and t $\in$ [0, T]. Then, by using (2.4), we have:

$$D(\tilde{y}_s(t), \tilde{y}_r(t)) \leq D(\tilde{y}_s(t), \tilde{y}_{s-1}(t)) + D(\tilde{y}_{s-1}(t), \tilde{y}_{s-2}(t)) + \ldots + D(\tilde{y}_{r+1}(t), \tilde{y}_r(t))$$

$$\leq R\left(\frac{LT^q}{\Gamma(q+1)}\right)^{s-1} + R\left(\frac{LT^q}{\Gamma(q+1)}\right)^{s-2} + \ldots + R\left(\frac{LT^q}{\Gamma(q+1)}\right)^{r}$$

$$= R\left(\frac{LT^q}{\Gamma(q+1)}\right)^{s-1}\left[1 + \frac{\Gamma(q+1)}{LT^q} + \left(\frac{\Gamma(q+1)}{LT^q}\right)^2 + \ldots + \left(\frac{\Gamma(q+1)}{LT^q}\right)^{s-r-1}\right]$$

$$= R\left(\frac{LT^q}{\Gamma(q+1)}\right)^{s-1}\left[\frac{1-\left(\dfrac{\Gamma(q+1)}{LT^q}\right)^{s-r}}{1-\dfrac{\Gamma(q+1)}{LT^q}}\right]$$

The right hand side of the last inequality tends to zero as r, s $\longrightarrow \infty$. This implies that $\{\tilde{y}_n(t)\}$ is a Cauchy sequence. Consequently, the sequence $\{\tilde{y}_n(t)\}$ is convergent, since the metric space (E^n, D) is complete.

If we denote $\tilde{y}(t) = \lim\limits_{n\to\infty} \tilde{y}_n(t)$. Then $\tilde{y}(t)$ satisfies (2.1), it is continuous and bounded on [0, T].

To prove the uniqueness, let $\tilde{x}(t)$ be any other continuous solution of (2.1) on [0, T]. Then:

$$\tilde{x}(t) = \tilde{f}(t) + I^q g(t, \tilde{x}(t)), \quad t \geq 0,$$

Now, for $n \geq 1$, we have:

$$
\begin{aligned}
D(\tilde{x}(t),\tilde{y}_n(t)) &= D(I^q g(t,\tilde{x}(t)), I^q g(t,\tilde{y}_{n-1}(t))) \\
&\le \frac{1}{\Gamma(q)}\int_0^t D\left(\frac{g(s,\tilde{x}(s))}{(t-s)^{1-q}},\frac{g(s,\tilde{y}_n(s))}{(t-s)^{1-q}}\right)ds \\
&\le \frac{1}{\Gamma(q)}\int_0^t D(g(s,\tilde{x}(s)),g(s,\tilde{y}_n(s)))\frac{ds}{(t-s)^{1-q}} \\
&\le \frac{1}{\Gamma(q)}\sup_{0\le t\le T}(g(t,\tilde{x}(t)),g(t,\tilde{y}_n(t)))\int_0^t \frac{ds}{(t-s)^{1-q}} \\
&\le \frac{LT^q}{\Gamma(q+1)}\sup_{0\le t\le T} D(\tilde{x}(t),\tilde{y}_n(t)) \\
&\qquad \vdots \\
&\le \left(\frac{LT^q}{\Gamma(q+1)}\right)^n \sup_{0\le t\le T} D(\tilde{x}(t),\tilde{y}_0(t))
\end{aligned}
$$

since $\frac{LT^q}{\Gamma(q+1)} < 1$

$$\lim_{n\to\infty}\tilde{y}_n(t)=\tilde{x}(t)=\tilde{y}(t),\quad t\in[0,T]$$

This complete the proof.

Chapter Three

3

AdomianDecomposition Method for Solving Fuzzy Integral Equations Of Fractional Order

CHAPTER THREE

Adomian Decomposition Method for Solving Fuzzy Integral Equations of Fractional Order

3.1 Introduction:

In this chapter, we present the application of Adomian Decomposition method (ADM) for solving Fuzzy integral equations of fractional order

3.2 Historical Background of the Adomian Decomposition Method:

Most phenomena's that arise in real world problems are describe by nonlinear differential and integral equations. However, most of the methods developed in mathematics are usually used in solving linear differential and integral equations.

In recent years, the decomposition method has emerged as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, differential, integral, integro–differential.

The convergence of this method has investigated by Cherruault and cooperators. In [Cherruault, 1989], Cherruault proposed a new definition of the method and then he insisted that it will become possible to prove the convergence of the decomposition method. In [Cherruault, 1993], Cherruault and Adomian proposed a new convergence proof of Adomian method based on properties of convergence series.

In this method, the solution is considered as the sum of an infinite series, rapidly converging to an accurate solution. In [Abbaoui, 2001], Abbaoui et al. proposed a new approach of decomposition which is obtained in a more natural way in the classical representation. Lesnic [Lesnic, 2002] investigated convergence of Adomian's method to periodic temperature fields in heat conductors. The advantage of this method is that it provides a direct scheme for solving the problem without any need for linearization or discritization. Essentially, the method provides a systematic computational procedure for equations of physical significance.

El–Sayed and Kaya proposed Adomian Decomposition method (ADM) to approximate the numerical and analytical solution of system of two–dimensional Burger's equations with initial conditions in [El–Sayed, 2004], and the advantages of this work is that the decomposition method reduces the computational work and improves with regards to its accuracy and rapid convergence. The nonlinear solution of one–dimensional nonlinear Burgers equation and convergence of decomposition method is proved as [Inc, 2005], in [Celik and et al., 2006] applied Adomian Decomposition method (ADM) to obtain the approximate solution for the differential algebraic equations system and the results obtained by this method indicate a high degree of accuracy through the comparison with the analytic solutions. In [Hosseini, 2006 a], [Hosseini, 2006 b] standard and modified Adomian Decomposition method are applied to solve non–linear differential algebraic equations. While, the error analysis of Adomian series solution to a class of nonlinear differential equation, where as numerical experiments show that Adomian solution using this formula converges faster is discussed in [El–Kala, 2007]. Also, a new discrete Adomian Decomposition method (ADM) to approximate the theoretical solution of discrete nonlinear Schrodinger equations is presented in [Bratsos, 2008], where this examined for plane waves and single solution waves in case of continuous, semi discrete and fully discrete Schrodinger equations. Momani, [Momani, 2008] presented numerical study of system of fractional differential equations by Adomian Decomposition method.

3.3 The Adomian Decomposition Method [ ]:

To introduce the basic idea of the ADM, we consider the operator equation Fu = G, where F represents a general nonlinear ordinary differential operator and G is a given function. The linear part of F can be decomposed as:

$$Lu + Ru + Nu = G \quad \text{.......} \quad (3.1)$$

where, N is a nonlinear operator, L is the highest–order derivative which is assumed to be invertible, R is a linear differential operator of order less than L and G is the nonhomogeneous term.

The method is based by applying the operator L^{-1} formally to the expression

$$Lu = G – Ru – Nu \quad \text{.......} \quad (3.2)$$

so by using the given conditions, we obtain:

$$u = h + L^{-1}G - L^{-1}Ru - L^{-1}Nu \quad \text{.......} \quad (3.3)$$

where, h is the solution of the homogeneous equation Lu = 0, with the initial–boundary conditions. The problem now is the decomposition of the nonlinear term Nu. To do this, Adomian developed a very elegant technique as follows:

The Adomian technique consists of approximating the solution of (3.1) as an infinite series:

$$u = \sum_{n=0}^{\infty} u_n$$

and decomposing the nonlinear term Nu as $f(u) = Nu = \sum_{n=0}^{\infty} A_n$ where A_n are the so called Adomian polynomials of $u_0, u_1, ..., u_n$ that are the terms of the analytical expansion of Nu, where $u = \sum_{i=0}^{\infty} \lambda^i u_i$, around $\lambda = 0$. That is:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} f\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]\Bigg|_{\lambda=0} \quad \text{...} (3.4)$$

The Adomian polynomials are not unique and can be generated from the Taylor expansion of f(u) about the first component u_0, i.e.,

$$f(u) = \sum_{n=0}^{\infty} \frac{f^{(n)}(u_0)}{n!}(u - u_0)^n$$

In [Adomian, 1995], Adomian's polynomials are arranged to have the form:

$$\left.\begin{aligned} A_0 &= f(u_0) \\ A_1 &= u_1 f'(u_0) \\ A_2 &= u_2 f'(u_0) + \frac{u_1^2}{2} f''(u_0) \\ A_3 &= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f'''(u_0) \\ &\vdots \end{aligned}\right\} \quad \text{.....................................} (3.5)$$

Now, we parameterize eq.(3.3) in the form:

$$u = h + L^{-1}G - \lambda L^{-1}Ru - \lambda L^{-1}Nu \quad \text{..} (3.6)$$

where, λ is just an identifier for collecting terms in a suitable way such that u_n depends on $u_0, u_1, \ldots, u_n$ and we will later set $\lambda = 1$.

$$\sum_{n=0}^{\infty} \lambda^n u_n = h + L^{-1}G - \lambda L^{-1}R \sum_{n=0}^{\infty} \lambda^n u_n - \lambda L^{-1} \sum_{n=0}^{\infty} \lambda^n A_n \quad \text{................} (3.7)$$

Equating the coefficients of equal powers of λ, we obtain:

$$\left.\begin{aligned} u_0 &= h + L^{-1}G \\ u_1 &= -L^{-1}(Ru_0) - L^{-1}(A_0) \\ u_2 &= -L^{-1}(Ru_1) - L^{-1}(A_1) \\ &\vdots \end{aligned}\right\} \qquad (3.8)$$

and in general

$$u_n = -L^{-1}(Ru_{n-1}) - L^{-1}(A_{n-1}),\ n = 1, 2, 3, \ldots$$

Finally, an N–term that approximates the solution is given by:

$$\phi_N(x) = \sum_{n=0}^{N-1} u_n(x),\ \ N = 1, 2, 3, \ldots$$

and the exact solution is $u(x) = \lim_{N \to \infty} \phi_N$.

3.4 The Application of ADM for Solving Fuzzy Integral Equations of Fractional Order:

In this section ADM will be applied to find the solution of the fuzzy integral equation of fractional order and to do this, first we shall consider the linear case, i.e., the fuzzy integral equation of fractional order of the form:

$$\tilde{y}(t) = \tilde{f}(t) + J^q \tilde{y}(t),\ t \in [0, T],\ f : [0, T] \longrightarrow E^n$$

or equivalently:

$$\tilde{y}(t) = \tilde{f}(t) + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \tilde{y}(s)\, ds \qquad (3.9)$$

where $0 < q \le 1, \tilde{f}(t)$ is assumed to be fuzzy function which may be represented as $\tilde{f} = [\underline{f}, \overline{f}]$, and therefore the solution of equation (3.9) will be a fuzzy solution which may be given by the form $y=[\underline{y},\overline{y}]$ where $\underline{y}$ represent the solution of the equation:

$$\underline{y}(t)=\underline{f}(t)+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\underline{y}(s)\,ds \quad\text{..}(3.10)$$

while $\overline{y}$ is the solution of the following equation:

$$\overline{y}(t)=\overline{f}(t)+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\overline{y}(s)\,ds \quad\text{..}(3.11)$$

which are called the lower and upper solutions of eq.(3.9), respectively.

Adomian's method defines the solution $\underline{y}(t)$ by the series:

$$\underline{y}(t)=\sum_{n=0}^{\infty}\underline{y}_n(t) \quad\text{...}(3.12)$$

Hence from (3.10) we obtain that:

$$\underline{y}_0(t)=\underline{f}(t),$$

$$\underline{y}_1(t)=J^q\underline{y}_0(t)=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\underline{y}_0(s)\,ds,$$

$$\vdots$$

$$\underline{y}_n(t)=J^q\underline{y}_{n-1}(t)=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\underline{y}_{n-1}(s)\,ds$$

where the components will be determined recursively.

Similarly, Adomian's method defines the upper solution $\overline{y}(t)$ by the series:

$$\overline{y}(t)=\sum_{n=0}^{\infty}\overline{y}_n(t) \quad\text{..}(3.13)$$

Thus from (3.11), we obtain that:

$$\overline{y}_0(t) = \overline{f},$$

$$\overline{y}_1(t) = J^q \overline{y}_0(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \overline{y}_0(s)\,ds$$

$$\vdots$$

$$\overline{y}_n(t) = J^q \overline{y}_{n-1}(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \overline{y}_{n-1}(s)\,ds$$

Also, the components will be determined, recursively.

Second the (ADM) may be used to solve the nonlinear fuzzy integral equation of fractional order of the form:

$$\tilde{y}(t) = \tilde{f}(t) + \frac{1}{\Gamma(q)} \int_0^t \frac{N(\tilde{y}(s))}{(t-s)^{1-q}} ds, t \in [0,T],\ \tilde{f} : [0,T] \longrightarrow E^n \text{ ....(3.14)}$$

which may be considered as a special case of eq.(2.1), where $0 < q \le 1$, $\tilde{f}(t)$ is assumed to be fuzzy function and then the solution $\tilde{y}(t)$ will be a fuzzy solution given by the form $y = [\underline{y}, \overline{y}]$, where $\underline{y}$ represent the solution of the equation:

$$\underline{y}(t) = \underline{f} + \frac{1}{\Gamma(q)} \int_0^t \frac{N(\underline{y}(s))}{(t-s)^{1-q}} ds \text{ ....(3.15)}$$

While $\overline{y}$ will be the solution of the equation:

$$\overline{y}(t) = \overline{f} + \frac{1}{\Gamma(q)} \int_0^t \frac{N(\overline{y}(s))}{(t-s)^{1-q}} ds \text{ ....(3.16)}$$

The nonlinear terms $N(\underline{y}(s))$ and $N(\overline{y}(s))$ are Lipschitzian with:

$$| N(\underline{y}_1) - N(\underline{y}_2) | \le L | \underline{y}_1 - \underline{y}_2 |$$

and

$$|N(\overline{y}_1)-N(\overline{y}_2)|\le L|\overline{y}_1-\overline{y}_2|$$

and may be decomposed in the form:

$$N(\underline{y})=\sum_{n=0}^{\infty}\underline{A}_n(t) \quad \text{......(3.17)}$$

$$N(\overline{y})=\sum_{n=0}^{\infty}\overline{A}_n(t) \quad \text{......(3.18)}$$

where $\underline{A}_n$ and $\overline{A}_n$ are the Adomian polynomials given by:

$$\underline{A}_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{i=0}^{\infty}\lambda^i\underline{y}_i\right)\right]\Bigg|_{\lambda=0} \quad \text{......(3.19)}$$

and

$$\overline{A}_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{i=0}^{\infty}\lambda^i\overline{y}_i\right)\right]\Bigg|_{\lambda=0} \quad \text{......(3.20)}$$

Then $N(\underline{y}(t))$ and $N(\overline{y}(t))$ will be a functions of $\lambda,\underline{y}_0,\underline{y}_1,\ldots,\overline{y}_0,\overline{y}_1,\ldots$ respectively.

Now substituting (3.17) and (3.18) into (3.15) and (3.16), yields to:

$$\underline{y}(t)=\underline{f}(t)+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\left(\sum_{n=0}^{\infty}\underline{A}_n(s)\right)ds \quad \text{......(3.21)}$$

and

$$\overline{y}(t)=\overline{f}(t)+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\left(\sum_{n=0}^{\infty}\overline{A}_n(s)\right)ds \quad \text{......(3.22)}$$

The components $\underline{y}_0,\underline{y}_1,\ldots$ and $\overline{y}_0,\overline{y}_1,\ldots$ are determined recursively by:

$$\left.\begin{aligned} &\underline{y}_0 = \underline{f}, \\ &\underline{y}_{k+1} = J^q \underline{A}_k \quad ,k=0,1,2,\ldots \end{aligned}\right\} \qquad (3.23)$$

Hence the Adomian's method defines the lower solution $\underline{y}$ by the series

$$\underline{y} = \sum_{i=0}^{\infty} \underline{y}_i \qquad (3.24)$$

And for the upper case, we have

$$\left.\begin{aligned} &\overline{y}_0 = \overline{f}, \\ &\overline{y}_{k+1} = J^q \overline{A}_k \quad ,k=0,1,2,\ldots \end{aligned}\right\} \qquad (3.25)$$

By the same manner the Adomian's method defines the upper solution $\overline{y}$ by

$$\overline{y} = \sum_{i=0}^{\infty} \overline{y}_i \qquad (3.26)$$

3.4.1 Modification of Adomian's Polynomials [El-Kala, 2007]:

Here, we will use the modified version proposed by El-Kala, [El-Kala, 2007] in order to improve the approximate solution given by the Adomian's method where the Adomian polynomials are given by:

$$\begin{aligned} &A_0^* = N(y_0) \\ &A_1^* = y_1 N^{(1)}(y_0) + \frac{y_1^2}{2!} N^{(2)}(y_0) + \frac{y_1^3}{3!} N^{(3)}(y_0) + \ldots \\ &A_2^* = y_2 N^{(1)}(y_0) + \frac{1}{2!}\left(y_2^2 + 2y_1y_2\right) N^{(2)}(y_0) + \\ &\qquad \frac{1}{3!}\left(3y_1^2 + 3y_1y_2^2 + y_3^2\right) N^{(3)}(y_0) + \ldots \end{aligned}$$

$$A_3^* = y_3 N^{(1)}(y_0) + \frac{1}{2!}\left(y_3^{\,2} + 3y_1y_3 + 2y_2y_3\right)N^{(2)}(y_0) +$$
$$\frac{1}{3!}\left(y_3^{\,3} + 3y_3^{\,2}(y_1 + y_2) + 3y_3(y_1 + y_2)^2\right)N^{(3)}(y_0) + \ldots$$
$$\vdots$$

Define the partial sum $S_n = \sum_{i=0}^{n} y_i$, from the modified polynomials, then one can write:

$$A_0^* = N(y_0) = N(s_0)$$
$$A_0^* + A_1^* = N(y_0) + y_1 N^{(1)}(y_0) + \frac{y_1^{\,2}}{2!}N^{(2)}(y_0) + \frac{y_1^{\,3}}{3!}N^{(3)}(y_0) + \ldots$$
$$= N(y_0 + y_1)$$
$$= N(s_1)$$

Similarly:

$$A_0^* + A_1^* + A_2^* = N(y_0 + y_1 + y_2)$$
$$= N(s_2)$$

And by induction the following sum is obtained:

$$\sum_{i=0}^{n} A_i^*(y_0, y_1, \ldots, y_i) = N(s_n)$$

Therefore, in general:

$$A_n^* = N(s_n) - \sum_{i=0}^{n-1} A_i^* \quad \ldots\ldots(3.27)$$

Hence one can define the lower Adomian polynomials by the following form:

$$\underline{A}_0^* = N(\underline{y}_0)$$

$$\underline{A}_1^* = \underline{y}_1 N^{(1)}(\underline{y}_0) + \frac{\underline{y}_1^{\,2}}{2!} N^{(2)}(\underline{y}_0) + \frac{\underline{y}_1^{\,3}}{3!} N^{(3)}(\underline{y}_0) + \dots$$

$$\underline{A}_2^* = \underline{y}_2 N^{(1)}(\underline{y}_0) + \frac{1}{2!}\left(\underline{y}_2^{\,2} + 2\underline{y}_1\underline{y}_2\right) N^{(2)}(\underline{y}_0) + \frac{1}{3!}\left(3\underline{y}_1^{\,2} + 3\underline{y}_1\underline{y}_2^{\,2} + \underline{y}_3^{\,2}\right) N^{(3)}(\underline{y}_0) + \dots$$

$$\underline{A}_3^* = \underline{y}_3 N^{(1)}(\underline{y}_0) + \frac{1}{2!}\left(\underline{y}_3^{\,2} + 3\underline{y}_1\underline{y}_3 + 2\underline{y}_2\underline{y}_3\right) N^{(2)}(\underline{y}_0) + \frac{1}{3!}\left(\underline{y}_3^{\,3} + 3\underline{y}_3^{\,2}(\underline{y}_1 + \underline{y}_2) + 3\underline{y}_3(\underline{y}_1 + \underline{y}_2)^2\right) N^{(3)}(\underline{y}_0) + \dots$$

$$\vdots$$

and define the lower partial sum $\underline{S}_n = \sum_{i=0}^{n} \underline{y}_i$ from the lower polynomials, then one can write:

$$\underline{A}_0^* = N(\underline{y}_0) = N(\underline{s}_0)$$

$$\begin{aligned}\underline{A}_0^* + \underline{A}_1^* &= N(\underline{y}_0) + \underline{y}_1 N^{(1)}(\underline{y}_0) + \frac{\underline{y}_1^{\,2}}{2!} N^{(2)}(\underline{y}_0) + \frac{\underline{y}_1^{\,3}}{3!} N^{(3)}(\underline{y}_0) + \dots \\ &= N(\underline{y}_0 + \underline{y}_1) \\ &= N(\underline{s}_1)\end{aligned}$$

By the same process

$$\begin{aligned}\underline{A}_0^* + \underline{A}_1^* + \underline{A}_2^* &= N(\underline{y}_0 + \underline{y}_1 + \underline{y}_2) \\ &= N(\underline{s}_2)\end{aligned}$$

and by induction, the following sum is obtained:

$$\sum_{i=0}^{n} \underline{A}_i^*(\underline{y}_0, \underline{y}_1, \dots, \underline{y}_i) = N(\underline{s}_n)$$

Therefore, in general:

$$\underline{A}_n^{\,*} = N(\underline{s}_n) - \sum_{i=0}^{n-1} \underline{A}_i^{\,*} \quad \text{.......}(3.28)$$

Hence the approximate solutions of equations (3.15) using the modified Adomian's polynomials are given by:

$$\left.\begin{aligned} &\underline{y}_0 = \underline{f}, \\ &\underline{y}_{k+1} = J^q \underline{A}^{*}_{k},\ k = 0,1,2,\ldots \end{aligned}\right\} \quad \text{.......}(3.29)$$

Thus the Adomian's method defines the lower solution $\underline{y}$ by the series

$$\underline{y} = \sum_{i=0}^{\infty} \underline{y}_i \quad \text{.......}(3.30)$$

and similarly for the upper case, the solution $\overline{y}$ will be given by the series:

$$\overline{y} = \sum_{i=0}^{\infty} \overline{y}_i \quad \text{.......}(3.31)$$

3.4.2 Convergence Analysis:

In chapter two, we prove the existence and uniqueness theorem of fuzzy integral equations of fractional order. Now in this section, the convergence of the series solutions (3.30) and (3.31) of equations (3.15) and (3.16) are also proved.

Theorem(3.1):

The series solutions (3.30) and (3.31) of equations (3.15) and (3.16) respectively using ADM converges whenever $0 < k < 1$, where $k = \dfrac{LT^q}{\Gamma(q+1)}$.

Proof: Let $\underline{s}_n$ and $\underline{s}_m$ be an arbitrary partial sums with n ≥ m, and to prove that $\{\underline{s}_n\}$ is a Cauchy sequence in the Banach space B = (C[I],‖ . ‖) of all continuous functions of I. Therefore:

$$\|\underline{s}_n - \underline{s}_m\| = \max_{t\in I} |\underline{s}_n - \underline{s}_m|$$

$$= \max_{t\in I} \left| \sum_{i=m+1}^{n} \underline{y}_i(t) \right|$$

$$= \max_{t\in I} \left| \sum_{i=m+1}^{n} J^q \underline{A}^*_{i-1}(t) \right|$$

$$= \max_{t\in I} \left| \sum_{i=m+1}^{n} \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \underline{A}^*_{i-1}(s)\, ds \right) \right|$$

$$= \max_{t\in I} \left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sum_{i=m+1}^{n} \underline{A}^*_{i-1}(s)\, ds \right|$$

$$= \max_{t\in I} \left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sum_{i=m}^{n-1} \underline{A}^*_{i}(s)\, ds \right|$$

but we have:

$$\sum_{i=m}^{n-1} \underline{A}^*_i = N(\underline{s}_{n-1}) - N(\underline{s}_{m-1})$$

Then:

$$\|\underline{s}_n - \underline{s}_m\| = \max_{t\in I} \left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[N(\underline{s}_{n-1}) - N(\underline{s}_{m-1}) \right] ds \right|$$

$$\le \max_{t\in I} \left| N(\underline{s}_{n-1}) - N(\underline{s}_{m-1}) \right| \left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\, ds \right|$$

$$\|\underline{s}_n - \underline{s}_m\| \le \frac{LT^q}{\Gamma(q+1)} \max_{t\in I} |\underline{s}_{n-1} - \underline{s}_{m-1}|$$

$$\le \frac{LT^q}{\Gamma(q+1)} \| \underline{s}_{n-1} - \underline{s}_{m-1} \|$$

Hence:

$$\|\underline{s}_n - \underline{s}_m\| \le k\| \underline{s}_{n-1} - \underline{s}_{m-1}\|, \text{ where } k = \frac{LT^q}{\Gamma(q+1)}$$

Let $n = m + 1$, then:

$$\|\underline{s}_{m+1} - \underline{s}_m\| \le k\|\underline{s}_m - \underline{s}_{m-1}\|$$

$$\le k^2\|\underline{s}_{m-1} - \underline{s}_{m-2}\|$$

$$\vdots$$

$$\le k^m\|\underline{s}_1 - \underline{s}_0\|$$

and from the triangle inequality:

$$\|\underline{s}_n - \underline{s}_m\| \le \|\underline{s}_{m+1} - \underline{s}_m\| + \|\underline{s}_{m+2} - \underline{s}_{m+1}\| + \ldots + \|\underline{s}_n - \underline{s}_{n-1}\|$$

$$\le (k^m + k^{m+1} + \ldots + k^{n-1})\|\underline{s}_1 - \underline{s}_0\|$$

$$\le \frac{k^m}{1-k}\|\underline{y}_1\| \ , \ k \ne 1$$

and then:

$$\|\underline{s}_n - \underline{s}_m\| \le \frac{k^m}{1-k} \max_{t\in I} |\underline{y}_1|$$

But $|\underline{y}_1| < \infty$, so as $m \longrightarrow \infty$, then $\|\underline{s}_n - \underline{s}_m\| \longrightarrow 0$

So $\{\underline{s}_n\}$ is a Cauchy sequence in B, and therefore the series $\sum_{i=0}^{\infty}\underline{y}_i(t)$ converges.

Similarly as we do in the lower case, we get $\sum_{i=0}^{\infty}\overline{y}_i(t)$ converges and this complete the proof of the theorem.

<u>Numerical Examples:</u>

<u>Example (3.1):</u>

Consider the linear fuzzy integral equation of fractional order

$$\tilde{y}(t) = \tilde{f}(t) + J^q \tilde{y}(t), \ t \in [0,1]$$

where q will be chosen to be $q = \frac{1}{2}$.

In this case the fuzzy function $\tilde{f}$ will be given as $\tilde{f} = [\underline{f}, \overline{f}]$,

where $\underline{f} = \beta\left(e^{2t} - \frac{2\sqrt{t}}{\sqrt{\pi}} - \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} - \frac{32t^{\frac{5}{2}}}{15\sqrt{\pi}} - \frac{128t^{\frac{7}{2}}}{105\sqrt{\pi}}\right)$, and

$$\overline{f} = \frac{1}{\beta}\left(e^{2t} - \frac{2\sqrt{t}}{\sqrt{\pi}} - \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} - \frac{32t^{\frac{5}{2}}}{15\sqrt{\pi}} - \frac{128t^{\frac{7}{2}}}{105\sqrt{\pi}}\right), \quad 0 < \beta \le 1$$

According to the Adomian decomposition method the fuzzy solution will be given as $y = [\underline{y}, \overline{y}]$ where $\underline{y} = \sum_{i=0}^{\infty}\underline{y}_i$ and $\overline{y} = \sum_{i=0}^{\infty}\overline{y}_i$, the components $\underline{y}_i$ and $\overline{y}_i$, $i = 0,1,2,\ldots$; will be determined as follows:

$$\underline{y}_0 = \beta\left(e^{2t} - \frac{2\sqrt{t}}{\sqrt{\pi}} - \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} - \frac{32t^{\frac{5}{2}}}{15\sqrt{\pi}} - \frac{128t^{\frac{7}{2}}}{105\sqrt{\pi}} \right)$$

$$\underline{y}_1 = J^{\frac{1}{2}}\underline{y}_0$$

$$\underline{y}_2 = J^{\frac{1}{2}}\underline{y}_1$$

$$\vdots$$

Similarly:

$$\overline{y}_0 = \frac{1}{\beta}\left(e^{2t} - \frac{2\sqrt{t}}{\sqrt{\pi}} - \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} - \frac{32t^{\frac{5}{2}}}{15\sqrt{\pi}} - \frac{128t^{\frac{7}{2}}}{105\sqrt{\pi}} \right)$$

$$\overline{y}_1 = J^{\frac{1}{2}}\overline{y}_0$$

$$\overline{y}_2 = J^{\frac{1}{2}}\overline{y}_1$$

$$\vdots$$

Following up to 10 terms tables (3.1) and (3.2) represent the lower and upper solution of example (3.1) for different values of β with a comparison with the exact solution at $\beta = 1$ which is $y(t) = e^{2t}$.

Table (3.1)

Lower solution when $q = 0.5$.

x_i	β				Exact solution $\beta = 1$
	0.25	0.5	0.75	1	
0.1	$3.054021345\text{x}10^{-1}$	$6.10804269\text{x}10^{-1}$	$9.162064034\text{x}10^{-1}$	1.221608538	1.221402758
0.2	$3.734213668\text{x}10^{-1}$	$7.468427336\text{x}10^{-1}$	1.1202641	1.493685467	1.491824698
0.3	$4.572815298\text{x}10^{-1}$	$9.145630595\text{x}10^{-1}$	1.371844589	1.829126119	1.8221188
0.4	$5.609855689\text{x}10^{-1}$	1.121971138	1.682956707	2.243942276	2.225540928
0.5	$6.894725418\text{x}10^{-1}$	1.378945084	2.068417625	2.757890167	2.718281828
0.6	$8.488043331\text{x}10^{-1}$	1.697608666	2.546412999	3.395217332	3.320116923
0.7	1.046387845	2.092775691	3.139163536	4.185551382	4.055199967
0.8	1.291240538	2.582481077	3.873721615	5.164962154	4.953032424
0.9	1.594309066	3.188618132	4.782927197	6.377236263	6.049647464
1	1.968852934	3.937705867	5.906558801	7.875411735	7.389056099

Table (3.2)

Upper solution when $q = 0.5$.

x_i	β				Exact solution $\beta = 1$
	0.25	0.5	0.75	1	
0.1	4.886434152	2.443217076	1.628811384	1.221608538	1.221402758
0.2	5.974741869	2.987370935	1.991580623	1.493685467	1.491824698
0.3	7.316504476	3.658252238	2.438834825	1.829126119	1.8221188
0.4	8.975769103	4.487884551	2.991923034	2.243942276	2.225540928
0.5	10.103156067	5.515780334	3.67718689	2.757890167	2.718281828
0.6	10.358086933	6.790434665	4.526956443	3.395217332	3.320116923
0.7	10.674220553	8.371102763	5.580735175	4.185551382	4.055199967
0.8	20.065984861	10.032992431	6.886616205	5.164962154	4.953032424
0.9	20.550894505	10.275447253	8.502981684	6.3772362	6.049647464
1	30.150164694	10.575082347	10.050054898	7.875411735	7.389056099

Example(3.2)

Consider the nonlinear fuzzy integral equation of fractional order

$$\tilde{y}(t) = \tilde{f}(t) + J^{q}\tilde{y}^{2}(t),\ t \in [0,1]$$

where q will be chosen to be $q = \frac{1}{2}$

In this case the fuzzy function $\tilde{f}$ will be given as $\tilde{f} = [\underline{f}, \overline{f}]$,

where $\underline{f} = \beta\left(t - \frac{2}{\Gamma\left(\frac{7}{2}\right)} t^{\frac{5}{2}}\right)$ and $\overline{f} = \frac{1}{\beta}\left(t - \frac{2}{\Gamma\left(\frac{7}{2}\right)} t^{\frac{5}{2}}\right)$, $0 < \beta \le 1$

According to the Adomian decomposition method the fuzzy solution will be given as $y = [\underline{y}, \overline{y}]$ where $\underline{y} = \sum_{i=0}^{\infty} \underline{y}_i$ and $\overline{y} = \sum_{i=0}^{\infty} \overline{y}_i$, the components $\underline{y}_i$ and $\overline{y}_i$, $i = 0,1,2,\ldots$; will be determined as follows:

$$\underline{y}_0 = \beta\left(t - \frac{2}{\Gamma\left(\frac{7}{2}\right)} t^{\frac{5}{2}}\right)$$

$$\underline{y}_1 = J^{\frac{1}{2}}\underline{A}_0 = J^{\frac{1}{2}}\underline{y}_0{}^{2}$$

$$\underline{y}_2 = J^{\frac{1}{2}}\underline{A}_1 = J^{\frac{1}{2}}\left(2\underline{y}_0\underline{y}_1\right)$$

$$\vdots$$

$$\underline{y}_{n+1} = J^{\frac{1}{2}}\underline{A}_n,\ \text{where}\ \underline{A}_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}\lambda^i \underline{y}_i\right)^2\right]\Bigg|_{\lambda=0}$$

Similarly:

$$\bar{y}_0 = \frac{1}{\beta}\left(t - \frac{2}{\Gamma\left(\frac{7}{2}\right)} t^{\frac{5}{2}}\right)$$

$$\bar{y}_1 = J^{\frac{1}{2}} \bar{A}_0 = J^{\frac{1}{2}} \bar{y}_0^{\,2}$$

$$\bar{y}_2 = J^{\frac{1}{2}} \bar{A}_1 = J^{\frac{1}{2}} \left(2\bar{y}_0 \bar{y}_1\right)$$

$$\vdots$$

$$\bar{y}_{n+1} = J^{\frac{1}{2}} \bar{A}_n, \text{ where } \bar{A}_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty} \lambda^i \bar{y}_i\right)^2\right]\Bigg|_{\lambda=0}$$

Following up to 4 terms tables (3.3) and (3.4) represent the lower and upper solution of example (3.2) for different values of β with a comparison with the exact solution at β = 1, which is y(t) = t.

Table (3.3)

Lower solution when $q = 0.5$.

x_i	β				Exact solution $\beta = 1$
	0.3	0.5	0.75	1	
0.1	$3.290609477 \times 10^{-2}$	$4.951687563 \times 10^{-2}$	$7.463482429 \times 10^{-2}$	$9.999999736 \times 10^{-2}$	1×10^{-1}
0.2	$6.420461733 \times 10^{-2}$	$9.718898904 \times 10^{-2}$	$1.478432665 \times 10^{-1}$	$1.999991143 \times 10^{-1}$	2×10^{-1}
0.3	$9.305225394 \times 10^{-2}$	$1.41964901 \times 10^{-1}$	$2.187035494 \times 10^{-1}$	$2.999748624 \times 10^{-1}$	3×10^{-1}
0.4	$1.186704145 \times 10^{-1}$	$1.827722216 \times 10^{-1}$	$2.86110187 \times 10^{-1}$	$3.997433391 \times 10^{-1}$	4×10^{-1}
0.5	$1.402526862 \times 10^{-1}$	$2.183818232 \times 10^{-1}$	$3.485263377 \times 10^{-1}$	$4.985156975 \times 10^{-1}$	5×10^{-1}
0.6	$1.569404931 \times 10^{-1}$	$2.473459868 \times 10^{-1}$	$4.037151632 \times 10^{-1}$	$5.940494904 \times 10^{-1}$	6×10^{-1}
0.7	$1.678266973 \times 10^{-1}$	$2.679807144 \times 10^{-1}$	$4.484926796 \times 10^{-1}$	$6.815862216 \times 10^{-1}$	7×10^{-1}
0.8	$1.719837748 \times 10^{-1}$	$2.784119675 \times 10^{-1}$	$4.786547735 \times 10^{-1}$	7.5311518×10^{-1}	8×10^{-1}
0.9	$1.685175568 \times 10^{-1}$	$2.767103273 \times 10^{-1}$	$4.892599032 \times 10^{-1}$	$7.976668968 \times 10^{-1}$	9×10^{-1}
1	$1.566448654 \times 10^{-1}$	$2.611222517 \times 10^{-1}$	$4.753729427 \times 10^{-1}$	$8.031211895 \times 10^{-1}$	1

Table (3.4)

Upper solution when $q = 0.5$.

x_i	β				Exact solution $\beta = 1$
	0.3	0.5	0.75	1	
0.1	3.125775193 $\times10^{-1}$	2.040544356 $\times10^{-1}$	1.342150497 $\times10^{-1}$	9.999999736 $\times10^{-2}$	1 $\times10^{-1}$
0.2	6.868570097 $\times10^{-1}$	4.259793554 $\times10^{-1}$	2.720643419 $\times10^{-1}$	1.999991143 $\times10^{-1}$	2 $\times10^{-1}$
0.3	1.217055368	6.849675977 $\times10^{-1}$	4.16482437 $\times10^{-1}$	2.999748624 $\times10^{-1}$	3 $\times10^{-1}$
0.4	2.097941817	1.012756279 $\times10^{-1}$	5.710338932 $\times10^{-1}$	3.997433391 $\times10^{-1}$	4 $\times10^{-1}$
0.5	3.642617476	1.452614991	7.382356157 $\times10^{-1}$	4.985156975 $\times10^{-1}$	5 $\times10^{-1}$
0.6	6.205096295	2.041114972	9.142527416 $\times10^{-1}$	5.940494904 $\times10^{-1}$	6 $\times10^{-1}$
0.7	9.996695805	2.759492178	1.078570097	6.815862216 $\times10^{-1}$	7 $\times10^{-1}$
0.8	14.86658717	3.418282368	1.176708413	7.5311518 $\times10^{-1}$	8 $\times10^{-1}$
0.9	20.17714369	3.359297548	1.089798861	7.976668968 $\times10^{-1}$	9 $\times10^{-1}$
1	24.89370642	6.648447026 $\times10^{-1}$	5.718377498 $\times10^{-1}$	8.031211895 $\times10^{-1}$	1

And if we apply the modified Adomian's method for this case, thus we have

$$\underline{y}_0 = \beta\left(t - \frac{2}{\Gamma\left(\frac{7}{2}\right)} t^{\frac{5}{2}}\right)$$

$$\underline{y}_1 = J^{\frac{1}{2}} \underline{A}^*_0 = J^{\frac{1}{2}} \underline{y}_0{}^2$$

$$\underline{y}_2 = J^{\frac{1}{2}} \underline{A}^*_1 = J^{\frac{1}{2}} \left(\underline{y}_1{}^2 + 2\underline{y}_0\underline{y}_1\right)$$

$$\underline{y}_3 = J^{\frac{1}{2}} \underline{A}^*_2 = J^{\frac{1}{2}} \left(\underline{y}_2{}^2 + 2\underline{y}_0\underline{y}_2 + 2\underline{y}_1\underline{y}_2\right)$$

$$\vdots$$

Similarly:

$$\overline{y}_0 = \frac{1}{\beta}\left(t - \frac{2}{\Gamma\left(\frac{7}{2}\right)} t^{\frac{5}{2}} \right)$$

$$\overline{y}_1 = J^{\frac{1}{2}} \overline{A}^*_0 = J^{\frac{1}{2}} \overline{y}_0^{\,2}$$

$$\overline{y}_2 = J^{\frac{1}{2}} \overline{A}^*_1 = J^{\frac{1}{2}} \left(\overline{y}_1^{\,2} + 2\overline{y}_0 \overline{y}_1 \right)$$

$$\overline{y}_3 = J^{\frac{1}{2}} \overline{A}^*_2 = J^{\frac{1}{2}} \left(\overline{y}_2^{\,2} + 2\overline{y}_0 \overline{y}_2 + 2\overline{y}_1 \overline{y}_2 \right)$$

$$\vdots$$

Following up to 4 terms tables (3.5) and (3.6) represent the modified lower and upper solution of example (3.2) respectively for different values of β with a comparison with the exact solution at $\beta = 1$ which is $y(t) = t$.

Table (3.5)

Modified lower solution when $q = 0.5$.

x_i	β				Exact solution $\beta = 1$
	0.3	0.5	5	1	
0.1	2.959667348 x10^{-2}	4.951687566x10^{-2}	7.463482463x10^{-2}	9.999999924x10^{-2}	1x10^{-1}
0.2	5.768014891 x10^{-2}	9.718899864x10^{-2}	1.478433776x10^{-1}	1.999997311x10^{-1}	2x10^{-1}
0.3	8.346979683 x10^{-2}	1.419651679x10^{-1}	2.18706661 x10^{-1}	2.999918369x10^{-1}	3x10^{-1}
0.4	1.062594473 x10^{-1}	1.827749277x10^{-1}	2.861416736x10^{-1}	3.99909685x10^{-1}	4x10^{-1}
0.5	1.253326666 x10^{-1}	2.183978493x10^{-1}	3.487087583x10^{-1}	4.994274128x10^{-1}	5x10^{-1}
0.6	1.39942898 x10^{-1}	2.474153424x10^{-1}	4.04463405x10^{-1}	5.974569848x10^{-1}	6x10^{-1}
0.7	1.493209879 x10^{-1}	2.682305126x10^{-1}	4.50946509x10^{-1}	6.911999598x10^{-1}	7x10^{-1}
0.8	1.52709941 x10^{-1}	2.792211776x10^{-1}	4.856312503x10^{-1}	7.747426955x10^{-1}	8x10^{-1}
0.9	1.494361204 x10^{-1}	2.79160551x10^{-1}	5.07465012x10^{-1}	8.374869749x10^{-1}	9x10^{-1}
1	1.390322156 x10^{-1}	2.68136558x10^{-1}	5.206116656x10^{-1}	8.632416353ex10^{-1}	1

Table (3.6)

Modified upper solution when $q = 0.5$.

x_i	β				Exact solution $\beta = 1$
	0.3	0.5	0.75	1	
0.1	$3.125697069 \times 10^{-1}$	$2.040545586 \times 10^{-1}$	$1.342150631 \times 10^{-1}$	$9.999999924 \times 10^{-2}$	1×10^{-1}
0.2	$6.860596158 \times 10^{-1}$	$4.260215159 \times 10^{-1}$	$2.720688962 \times 10^{-1}$	$1.999997311 \times 10^{-1}$	2×10^{-1}
0.3	1.207217828	$6.861995885 \times 10^{-1}$	$4.166138901 \times 10^{-1}$	$2.999918369 \times 10^{-1}$	3×10^{-1}
0.4	2.050911	1.02580653	$5.724015623 \times 10^{-1}$	$3.99909685 \times 10^{-1}$	4×10^{-1}
0.5	3.523931905	1.531592323	$7.463022874 \times 10^{-1}$	$4.994274128 \times 10^{-1}$	5×10^{-1}
0.6	6.048066118	2.376020454	$9.472305179 \times 10^{-1}$	$5.974569848 \times 10^{-1}$	6×10^{-1}
0.7	9.940316271	3.871499805	1.182533354	$6.911999598 \times 10^{-1}$	7×10^{-1}
0.8	14.70141535	6.517031707	1.44600423	$7.747426955 \times 10^{-1}$	8×10^{-1}
0.9	17.30000609	10.096973075	1.687379805	$8.374869749 \times 10^{-1}$	9×10^{-1}
1	8.621431358	10.778413579	1.744288777	$8.632416353 \times 10^{-1}$	1

Conclutions and Recommendations

Conclusions and Recommendations

From the present study, one can conclude the following:

1. Exact solution of fuzzy integral equation of fractional order may be in sometimes so difficult to be evaluated, especially in nonlinear case.
2. The Adomian decomposition method gave us an acceptable solution to the fuzzy integral equation of fractional order although we are take a summation of 10 terms in the linear case and 4 terms for the nonlinear case.

Also, we may recommend the following problems for future work:

1. Using other approximate methods for solving fuzzy integral equations of fractional order such as the homotopy analysis method, the homotopy perturbation method, the variational iteration method and the differential transform method.
2. Studying the approximate solution of fuzzy fredholm-voltera integral equations of fractional order.
3. Studying the existence and uniqueness of fuzzy stochastic integral equations of fractional order.

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الخلاصه

الهدف الرئيسي لهذه الرساله يدور حول ايجاد الحلول التقريبيه للمعادلات التكامليه الضبابيه ذات الرتب الكسريه وكالتالي:

اولاً دراسة المفاهيم الاساسيه للمواضيع الرئيسيه المتعلقه بعمل هذه الرساله واللذين هما الحسبان الكسوري ونظرية المجموعات الضبابيه.

ثانيا دراسة وجود و وحدانية الحل للمعادلات التكامليه الضبابيه ذات الرتب الكسريه.

ثالثا ايجاد الحلول التقريبيه للمعادلات التكامليه الضبابيه ذات الرتب الكسريه باستخدام طريقة أدومين للتجزئه.

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
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طريقة تقريبية لحل المعادلات التكاملية الضبابية ذات الرتب الكسرية

رسالة

مقدمة إلى كلية العلوم ـ جامعة النهرين وهي جزء من متطلبات نيل درجة ماجستير في علوم الرياضيات

مِن قِبَل

رؤى هشام فاضل

(بكالوريوس علوم رياضيات، جامعة النهرين، 2009)

باشراف

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