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Numerical and Approximate Methods for Solving Stochastic Integral Equations

A Thesis

Submitted to the College of Sciences / Al-Nahrain University as a partial fulfillment of the requirements for the Degree of Master of Science in Mathematics.

By

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بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

۞ رَبِّ قَدۡ ءَاتَيۡتَنِي مِنَ ٱلۡمُلۡكِ وَعَلَّمۡتَنِي مِن تَأۡوِيلِ ٱلۡأَحَادِيثِۚ
فَاطِرَ ٱلسَّمَٰوَٰتِ وَٱلۡأَرۡضِ أَنتَ وَلِيِّۦ فِي ٱلدُّنۡيَا وَٱلۡأٓخِرَةِۖ تَوَفَّنِي
مُسۡلِمٗا وَأَلۡحِقۡنِي بِٱلصَّٰلِحِينَ ﴿١٠١﴾

صدق الله العظيم

سـورة يوسـف – الآية (١٠١)

الإهداء

علّم الإسلام ...
أزكى النسب ...
فخر العرب ...

سيدي رسول الله (ﷺ)

إلى من لا يغيب رسمه ولا يُنسى اسمه

إلى من دفعني إلى العلم وبه ازداد افتخاراً

أبي

إلى من يسعد قلبي بلقياها وأخرجت روحي من روحها

أمي

إلى من هو أقرب إليّ من روحي

إلى من غمرني بحبه وعطفه , استمدُ منه قوتي وإصراري

زوجي العزيز

إلى الصرح العلمي الفتى الجبار

الذي دعمني علماً وساندني معرفةً لأصل الى ما انا عليه

استاذي المشرف

إلى من ينبضُ القلب بجسمي لهم

أحبائي

إلى كل من أسهم في خروج رسالتي إلى النور ... حباً ووفاءً

إلى بلدي الحبيب

إليهم .... جميعاً أهدي ثمرة جهدي هذا

رؤى

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Ruaa Qahtan Mohammed

September, 2014

Supervisor Certification

I, certify that this thesis entitled "**Numerical and Approximate Methods for Solving Stochastic Integral Equations"** was prepared by "**Ruaa Qahtan Mohammed**" under my supervision at the Department of Mathematics and Computer Applications / College of Science / Al-Nahrain University as a partial fulfillment of the requirements for the Degree of Master of Science in Mathematics.

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We, the examining committee certify that we have read this thesis entitled "**Numerical and Approximate Methods for Solving Stochastic Integral Equations**" and examined the student "**Ruaa Qahtan Mohammed**" in its contents and that in our opinion, it is accepted for the Degree of Master of Science in Mathematics.

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Summary

Stochastic and random integral equations are of great importance that may be used in modeling certain type of problems that contains random process and noise. Therefore, the main objectives of this thesis may be oriented as follows:

The first objective is to study the theoretical side of stochastic calculus and stochastic processes, which include the basic definitions and fundamental concepts related to this topic, such as stochastic processes, stochastic differentiation and stochastic integration the existence and uniqueness theorem.

The second objective is to compare between stochastic differential and integral equations and then provides analytical methods to evaluate the stochastic integrals.

The third objective, which is the main goal, that includes numerical and approximate methods for solving stochastic integral equations in both cases, linear and nonlinear, with some illustrative examples for each case.

Basic Notations and Abbreviations

Ω	The sample space.
$\mathcal{A}$	σ-Algebra of subsets of a sample space Ω.
$(\Omega, \mathcal{A})$	Probability measurable space.
$X(\omega)$	Random variables.
$\{X_n(\omega)\}$	Sequence of random variables.
P	Probability measure of $\mathcal{A}$.
w.p.1, p-w.p.1	P converges with probability one.
A^c	The complement of a set A in $\mathcal{A}$.
$\{A_n\}$	Sequence of events such that $\{A_n\} \subseteq \mathcal{A}$.
ω	The element of the sample space Ω.
$E(X_n^2) < \infty$	Converge in the mean square.
$X(t,w)$, $X_t(w)$	Stochastic process.
$(\Omega, \mathcal{A}, P)$	The probability sample space.
t	The parameter of time.
$\mathcal{A}_t$	Filtrartion, which is an increasing family of σ-algebra fields.
W_t	Brownian motion or Weiner process on time t.
ξ_t	The white noise process
$E(\|X_t\|^2) < \infty$	Strictly stationary.

SDE	Stochastic Differential Equation.
SIE	Stochastic Integral Equation.
B	The σ-algebra of Borel subset of $\mathbb{R}^m$.
$\tau_i^{(n)}$	The mid point of the interval $[t_i^{(n)}, t_{i+1}^{(n)}]$.
L	All expectation functions, such that $E\left(\int_0^t X_s^2\,ds\right) < \infty$.
$L_\omega^2[\alpha,\beta]$	The set of all square integrable non-anticipating functions.
π_n	Sequence of partitions, $n \in \mathbb{N}$.
a.s.	Almost sure convergence.
SODE	Stochastic Ordinary Differential Equation.
$C_g[\mathbb{R}^+, L_2(\Omega, \mathcal{A}, P)]$	The subspace of all mappings X(t,w) o X(t,w).
B	Banach space.
(B,D)	Admissible with respect to the operator.

Contents

Chapter Three: Approximate and Numerical Solutions of Stochastic Integral Equations

Introduction

In recent years, stochastic process and stochastic calculus have been applied to a wide range of scientific disciplines, such as physics, engineering and finance. Stochastic calculus concerns with a specific class of stochastic process that are stochastically integrable and are often expressed as a solution to the stochastic differential equations, [Lin, 2006].

They are typically describing the time dynamics of the evolution of a state vector, based on the (approximate) physics of the real system, together with a driving noise process. It is often represents processes not included in the model, but represented in the real system, the aim of these notes is to introduce the theory of random or stochastic integral equations of the Volterra Fredholm types and to apply the results to certain general problems in system theory. We hope to convey the manner in which such equations arise and to develop some general theory using tools of the methods of probability theory, functional analysis and topology, [Archambeau, 2007].

Due to the nondeterministic nature of phenomenon in the general areas of the engineering, biological, oceanographic and physical sciences, the mathematical descriptions of such phenomena frequently result in random or stochastic equations. These equations arise in various ways and in order to understand better the importance of developing the theory of such equations and its application, it is of interest to consider how such theory arise. Sometime, the mathematical models or equations that describe physical phenomena of the parameters or coefficients, which are specific physical interpretations, but those values are unknown, [Adomian, 1970].

As an example, we have the volume-scattering coefficient under water acoustics, the coefficient of viscosity in fluid mechanics, the coefficient of diffusion in the theory of elasticity. Many times, this unknown value is regarded as the true state of nature and is estimated by using the mean value of a set of observations obtained experimentally. The equation must be solved as a random equation, and its random solution must be obtained. Once such a solution is obtained then its statistical properties should be studied, [Ahmed, 1969].

There are many other ways in which random stochastic equations arise, stochastic differential equations appear in the study of diffusion process and Brownian motion which is studied by [Gikhman and Skorokhod, 1969]. The classical Itô random integral equations [Itô, 1946] which is a Stieltjes integral with respect to the Brownian motion process that may be found in many texts, for example in [Doob, 1953].

Integral equations with random kernels arise in random eigenvalue problems, [Bharucha-Reid, 1964]. Stochastic integral equations describe the wave propagation in random media, [Bharncha-Ried , 1968] and the total number of conversations held at a given time in telephone traffic theory, [Fortet, 1956], [Padgett and Tsokos, 1971].

In the theory of stochastic calculus, stochastic integral equations arise in describing the motion of a point in a continuous fluid in turbulent motion, [Lumley, 1962], [Padgett and Tsokos, 1971]. Integral equations were used by Bellman, Jacquez and Kalaba, in deterministic sense in the development of mathematical models for chemotherapy. However, due to the random nature of diffusion processes from the blood plasma in body tissue, the stochastic versions of these equations are more realistic and should be used, [Padgett and Toskos, 1970].

Stochastic and random equations also arise in systems theory, for so many example, [Morozan, 1966], [Morozan 1967], [Morozan 1969], [Tsokos, 1969], and [Tsokos, 1971].

Begun by Spacek A. in Czechosolvakia, there have been recent attempts by many scientists and mathematicians develop to and unify the concepts and methods of probability theory and functional analysis, [Adomian, 1970], [Anderson, 1966], [Baharucha Ried, 1964], [Baharucha Ried, 1968], [Bharucha-Reid and Arnold, 1968], and [Tsokos, 1969].

In this thesis, we study the numerical and approximate solutions of stochastic integral equations, in which a combination of methods of the numerical solution of an integral equation and stochastic integral inference including examples and programs written in Mathcad 2014 computer program are given.

This thesis includes three chapters:

In chapter one, the basic definitions belong to the concepts of stochastic calculus, stochastic processes, stochastic different and integration and the theory of existence and uniqueness of stochastic solution of a stochastic integral equation are given.

Chapter two consists of the numerical and approximate methods of providing the integral equations and stochastic integrals.

Chapter three which presents the numerical and approximate methods, namely the collocation method and the method of approximating the integrals for solving stochastic integral equations of both types, linear and nonlinear. Then we have been discussed the numerical solution of the generalized form of the stochastic integral equations.

Finally, some appendices are given in order to present the large scale results and the computer programs which are coded in Mathcad 14 computer software.

CHAPTER ONE

FUNDAMENTAL CONCEPTS

CHAPTER ONE

FUNDAMENTAL CONCEPTS

1.1 Introduction:

The basic primitive concepts and results of probability and stochastic processes needed later in this work are presented here in this chapter. Therefore this chapter consists of four sections. In section (1.2), some basic concepts related to probability theory and then to stochastic process are reviewed in several definitions. In section (1.3), and because of their basic role in stochastic process, the theory of stochastic differentiation and integration which are related to each other are given. Also, their discretization is also given for the sake of numerical and approximate solution of stochastic differential or stochastic integral equations. Finally, in section (1.4), stochastic differential and integral equations and their relationship are studied.

1.2 Basic Concepts of Stochastic Calculus:

Stochastic calculus is concerned with the study of stochastic process, which involves randomness or noise. Therefore, in this section some preliminary concepts related to such topic will be presented for completeness purpose.

Definition (1.1), [Arnold, 1974]:

The σ-algebra $\mathcal{A}$ of subsets of a sample space Ω satisfies the following:

1. $\Omega \in \mathcal{A}$.
2. If $A \in \mathcal{A}$, then $A^c = \{\omega \in \Omega \mid \omega \notin A\} \in \mathcal{A}$.
3. For any sequence $\{A_n\} \subseteq \mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ and $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$

The elements of $\mathcal{A}$ are events and the pair $(\Omega, \mathcal{A})$ is called a probability measurable space.

Sequences and their convergence plays an important role in the theory of stochastic calculus, and it is remarkable that the convergence of such sequences may be defined using different criterions and among them are given in the next three definitions:

Definition (1.2), [Kloeden, 1995]:

A sequence of random variables $\{X_n(\omega)\}$, $n \in \mathbb{N}$ converges with probability one (denoted by P-w.p.1 or w.p.1) to $X(\omega)$ if :

$$P(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1$$

This is also called almost sure convergence, (a.s.).

Definition (1.3), [Burrage, 1999]:

A sequence of random variables $\{X_n(\omega)\}$, $n\in \mathbb{N}$ such that $E(X_n^2) < \infty$, for all $n\in \mathbb{N}$ is said to be converges in the mean square to $X(\omega)$ if:

$$\lim_{n\to\infty} E\left(\|X_n - X\|^2\right) = 0$$

Definition (1.4), [Kloeden, 1995]:

A sequence of random variables $\{X_n(\omega)\}$, $n\in \mathbb{N}$ converges in probability to $X(\omega)$, if:

$$\lim_{n\to\infty} P(\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\}) = 0, \ \forall\ \varepsilon > 0$$

1.2.1 Stochastic Process, [Kloeden, 1995]:

In many physical applications, there are many processes in which the random variables depends on the space and/or time and this introductory material will be the main subject of the present section.

Definition (1.5), [Kloeden, 1995]:

A stochastic process X(t,ω), $t \in [t_0,\infty)$, $\omega \in \Omega$ is a family of random variables on a probability space $(\Omega, \mathcal{A}, P)$, which is denoted by $X_t(\omega)$ (or briefly X_t), and assumes real values and is p-measurable as a function of ω for each fixed t.

The parameter t is interpreted as a time and $X_t(.)$ represents a random variable on the probability space Ω, while $X_.(\omega)$ is called a sample path or trajectory of the stochastic process.

Now, an important class of stochastic processes is that with independent increments; that is, where the difference $X_{t+1} - X_t$ are independent for any finite strictly increasing sequence $\{t_i\} \subset [t_0,T]$, t_0, $T \in \mathbb{R}^+$ and $T > t_0$.

Definition (1.6), [Burrage, 1999]:

The stochastic process X_t be a non-anticipating, by which it is mean that the information about X_t at time t does not depend on events occurring after time t.

Definition (1.7), [Burrage, 1999]:

Let X_t, $t \in [a,b] \subseteq \mathbb{R}$ be a stochastic process on probability space $(\Omega, \mathcal{A}, \boldsymbol{P})$ and let $\{\mathcal{A}_t\}_{t\in[a,b]}$ be a non-decreasing family of σ-algebras of **A**, such that for each $t \in [a,b]$, X_t is $\mathcal{A}_t$-measurable. Then X_t is a martingale with respect to $\mathcal{A}_t$, if:

$E(X_{t+s} \mid \mathcal{A}_t) = X_t$, for all $s > 0$.

Definition (1.8), [Burrage,1999]:

A stochastic process W_t, $t \geq 0$, is said to be a Brownian motion or Wiener process, if:

1. $p(\{\omega \in \Omega \mid W_0(\omega) = 0\}) = 1$, i.e., $p(W_0 = 0) = 1$.
2. For $0 < t_0 < t_1 < \ldots < t_n$, the increments $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$ …, $W_{t_n} - W_{t_{n-1}}$ are independent.
3. For an arbitrary t and h > 0, is the discretization step size implies $W_{t+h} - W_t$ has a normal distribution with mean 0 and variance h.

Definition (1.9), [Arnold, 1974]:

A stochastic process X_t such that $E(|X_t|^2) < \infty$, $t \in [t_0,T]$ is said to be strictly stationary if its distribution is invariant under time displacements, i.e.,

$$F_{t_1+h,t_2+h,\ldots,t_n+h}(x_1, x_2, \ldots, x_n) = F_{t_1,t_2,\ldots,t_n}(x_1, x_2, \ldots, x_n)$$

In other words, the distribution of X_t is the same for all $t \in [t_0,T]$.

Remark (1.1), [Burrage, 1999]:

In general, a standard Wiener process has the properties that:

$W_0 = 0$ w.p.1, $E(W_t) = 0$, $Var(W_t - W_s) = t - s$

for all $0 \le s \le t$; and so the increments are stationary.

The property $E(W_sW_t) = \min\{s,t\}$ can be used to demonstrate the independence of Wiener increments. Suppose that $0 \le t_0 < \ldots < t_{i-1} < t_i < \ldots < t_{j-1} < t_j < \ldots < t_n$; then:

$$\begin{aligned} E[(W_{t_i} - W_{t_{i-1}})(W_{t_j} - W_{t_{j-1}})] &= E(W_{t_i}W_{t_j}) - E(W_{t_i}W_{t_{j-1}}) - \\ &\quad E(W_{t_{i-1}}W_{t_j}) + E(W_{t_{i-1}}W_{t_{j-1}}) \\ &= t_i - t_i - t_{i-1} + t_{i-1} = 0 \end{aligned}$$

and hence the increments $W_{t_i} - W_{t_{i-1}}$ and $W_{t_j} - W_{t_{j-1}}$ are independent.

Definition (1.10), [Burrage, 1999]:

The white noise process ξ_t is formally defined as the derivative of non anticipating Wiener process, i.e.,

$$\xi_t \, dt = dW_t$$

This derivative does not exist as a function of t in the usual sense, since a Wiener process is nowhere differentiable function.

Sometimes, this derivative is called Gaussian white noise, which is an important example of stochastic process of a purely random process.

1.3 Stochastic Differentiation and Integration:[Karatzas,1999]

A stochastic differentiation and stochastic integral are related to each other and each of them has certain advantages in the theory of stochastic calculus, therefore it is preferable to discuss each of them and give their connection in the next section with stochastic differential equations (SDE's) and stochastic integral equations (SIE's).

A sequence of node points which discretized the time interval $I = [t_0,T]$ and given by:

$$t_0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_{N_n}^{(n)} = T$$

with the property that they are refinements for increasing n and with:

$$\max_{0 \le i \le N_n - 1} \left\{ t_{i+1}^{(n)} - t_i^{(n)} \right\} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

If we define $\tau_i^{(n)} = \theta t_{i+1}^{(n)} + (1-\theta) t_i^{(n)}$, for a fixed $\theta \in [0, 1]$, then the following series of random variables is called an approximation of stochastic integral:

$$\int_{t_0}^{T} X_{\tau_i^{(n)}} dW_t = \sum_{i=0}^{N_n - 1} X_{\tau_i^{(n)}} (W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}}) \qquad \ldots(1.1)$$

converges as $n \longrightarrow \infty$ in probability if $W_{t^{(n)}}$, $t^{(n)} \ge 0$ be a Wiener process and $W_{\tau^{(n)}}$ a real-valued stochastic process (sometimes, called a stochastic function or briefly a function) with respect to the Wiener process W_t. It is necessary that X_τ and W_t are both defined on the same probability space $(\Omega, \mathcal{A}, p)$.

Now, let $\mathcal{A}_t$ be an increasing family of σ-algebra fields, which is called also filtration, for all $t \geq 0$, i.e., $\mathcal{A}_{t_1} \subset \mathcal{A}_{t_2}$ if $t_1 < t_2$, such that $\mathcal{A}_t \subset \mathcal{A}$, where $\mathcal{A}(W_s,\ 0 \leq s \leq t)$ is in $\mathcal{A}_t$ and $\mathcal{A}(W_{\lambda+t} - W_t, \lambda > 0)$ is independent of $\mathcal{A}_t$, for all $t \geq 0$. One can take, for instance, $\mathcal{A}_t = \mathcal{A}(W_s, 0 \leq s \leq t)$, [Friedman,1975]. If the filtration $(\mathcal{A}_t)_{t\in I}$ satisfy the usual conditions, i.e., $(\mathcal{A}_t)_{t\in I}$ is a right-continuous filtration (satisfies $\mathcal{A}_t = \bigcap_{\varepsilon>0} \mathcal{A}_{t+\varepsilon}$ for all $t \geq 0$) and $\mathcal{A}_0$ contains all p-negligible events in $\mathcal{A}$. Furthermore, let $(\mathcal{A}_t)_{t\in I}$ be such that W_t is a martingale of $(\mathcal{A}_t)_{t\in I}$, [Röbler, 2003].

Definition (1.11), [Krishnan,1984]:

Let X_t, $t \in I$ be a stochastic process defined on a probability space $(\Omega, \mathcal{A}, p)$ and let $(\mathcal{A}_t)_{t\in I}$ be a filtration σ-algebra. The process X_t is adapted to the family $(\mathcal{A}_t)_{t\in I}$ if X_t is $\mathcal{A}_t$-measurable for every $t \in I$, or:

$$E(X_t \mid \mathcal{A}_t) = X_t,\ t \in I$$

where $\mathcal{A}_t$ -adapted random processes and also $\mathcal{A}_t$ -measurable.

Definition (1.12), [Burrage, 1999]:

Consider a probability space $(\Omega, \mathcal{A}, P)$ with filtration $(\mathcal{A}_t)_{t\in I}$ then a nonnegative random variable $\tau(\omega)$ on $(\Omega, \mathcal{A}, P)$ is called a Markov time (or stopping time) if the event $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{A}_t$, for each $t \geq 0$.

Definition (1.13), [Röβler, 2003]:

Let $(\Omega, \mathcal{A}, P)$ be a probability space with filtration $(\mathcal{A}_t)_{t\in I}$ for $I = [0,\infty)$. The set L be the class of all $\beta\times\mathcal{A}$-measurable, $\mathcal{A}_t$-adapted processes $X_t : I \times \Omega \longrightarrow \mathbb{R}$, where β is the Borel set on I, for which:

$$E\left(\int_0^t X_s^2(\omega)\, ds\right) < \infty, \quad t > 0 \qquad \text{...(1.2)}$$

holds and the set P is the class of all $\beta\times\mathcal{A}$-measurable $\mathcal{A}_t$-adapted processes $X_t : I \times \Omega \longrightarrow \mathbb{R}$, satisfying:

$$p\left(\int_0^t X_s^2(\omega)\, ds < \infty\right) = 1, \ \forall\, t > 0 \qquad \text{... (1.3)}$$

Now, back to equation (1.1) which is converges as $n \longrightarrow \infty$ in probability if $X_{\tau_i^{(n)}} \in P$ and in the mean-square sense if $X_{\tau_i^{(n)}} \in L$, $\forall\, i = 0, 1, \ldots, N_n - 1, n \in \mathbb{N}$

However, the integral for dW_s are unlike the Riemann-Stieltjes integral, here is the selection of which makes a difference. For $\theta = 0$, which means that $\tau_i^{(n)}$ represent the left end point $t_i^{(n)}$, we have the Itô calculus. The limit of equation (1.1), denotes the first model given by:

$$\int_{t_0}^{t} X_s\, dW_s$$

and is called the Itô stochastic integral.

At Stratonovich calculus, we have to set $\theta = \frac{1}{2}$ and $\tau_i^{(n)}$ described the mid point of $[t_i^{(n)}, t_{i+1}^{(n)}]$. Now, the limit of equation (1.1) denotes the second model, which is given by:

$$\int_{t_0}^{t} X_s \circ dW_s$$

and is called the Stratonovich stochastic integral.

To determine a value for the integral $\int_a^b W_t\, dW_t$, approximate W_t by the function $\varphi_n^{(\theta)}(t)$, where:

$$\varphi_n^{(\theta)}(t) = \theta\, W_{t_{i+1}^{(n)}} + (1-\theta)\, W_{t_i^{(n)}}, \quad t_i^{(n)} \le t \le t_{i+1}^{(n)} \qquad \text{...(1.4)}$$

for $\theta \in [0,1]$, and then the integration of $\varphi_n^{(\theta)}(t)$ over [a,b] equals to the approximate stochastic integral given in equation (1.1):

$$\int_a^b \varphi_n^{(\theta)}(t)\ dW_t = \sum_{i=0}^{N_n-1} \varphi_n^{(\theta)}(t_i)\,(W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}}) \qquad \text{...(1.5)}$$

The right hand side of equation (1.5) may be written as:

$$\theta \sum_{i=0}^{N_n-1} W_{t_{i+1}^{(n)}}(W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}}) + (1-\theta)\sum_{i=0}^{N_n-1} W_{t_i^{(n)}}\ (W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}})$$

and by rearranging the terms algebraically, when $n \longrightarrow \infty$:

$$\lim_{n\to\infty} \sum_{i=0}^{N_n-1} W_{t_i^{(n)}}[W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}}] = \lim_{n\to\infty} \sum_{i=0}^{N_n-1} [W_{t_i^{(n)}} W_{t_{i+1}^{(n)}} - W^2_{t_i^{(n)}}]$$

$$= \frac{1}{2} \lim_{n\to\infty} \sum_{i=0}^{N_n-1} 2W_{t_i^{(n)}} W_{t_{i+1}^{(n)}} - 2W^2_{t_i^{(n)}} + W^2_{t_{i+1}^{(n)}} - W^2_{t_{i+1}^{(n)}}$$

$$= \frac{1}{2} \lim_{n\to\infty} \sum_{i=0}^{N_n-1} [W^2_{t_{i+1}^{(n)}} - W^2_{t_i^{(n)}} - [W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}}]^2]$$

such that:

$$\lim_{n\to\infty} \sum_{i=0}^{N_n-1} [W^2_{t_{i+1}^{(n)}} - W^2_{t_i^{(n)}}] = \lim_{n\to\infty} [W^2_{t_{N_n}^{(n)}} - W^2_{t_0^{(n)}}]$$

$$= W_b^2 - W_a^2$$

where $\lim_{n\to\infty}$ is taken as the limit in probability, then:

$$\lim_{n\to\infty} \sum_{i=0}^{N_n-1} W_{t_i^{(n)}}(W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}}) = \frac{1}{2} W_b^2 - \frac{1}{2} W_a^2 - \frac{1}{2} \lim_{n\to\infty} \sum_{i=0}^{N_n-1} [W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}}]^2$$

In a similar manner with $\sum_{i=0}^{N_n-1} W_{t_{i+1}^{(n)}}(W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}})$, such that:

$$\sum_{i=0}^{N_n-1} W_{t_{i+1}^{(n)}}(W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}}) = \frac{1}{2}W_b^2 - \frac{1}{2}W_a^2 + \frac{1}{2}\lim_{n\to\infty}\sum_{k=0}^{N_n-1}(W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}})^2$$

Thus:

$$\sum_{i=0}^{N_n-1}\varphi_n^{(\theta)}(t)(W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}}) = \theta\left(\frac{1}{2}W_b^2 - \frac{1}{2}W_a^2 + \frac{1}{2}\lim_{n\to\infty}\sum_{i=0}^{N_n-1}(W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}})^2\right) + (1-\theta)\left(\frac{1}{2}W_b^2 - \frac{1}{2}W_a^2 - \frac{1}{2}\lim_{n\to\infty}\sum_{i=0}^{N_n-1}(W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}})^2\right)$$

$$= \frac{1}{2}W_b^2 - \frac{1}{2}W_a^2 + \frac{1}{2}(2\theta-1)\lim_{n\to\infty}\sum_{i=0}^{N_n-1}(W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}})^2 \quad ...(1.6)$$

1.4 Stochastic Differential and Integral Equations [Karatzas, 1999]:

Stochastic differential equations (SDE's) incorporate white noise and however, it should be mentioned that other types of random fluctuation are possible, [Arnold, 1974]. Solution of SDE's from a very large class of stochastic process, this class includes the Brownian motion and many other stochastic processes used in stochastic modeling, [Lin, 2006].

A system of SDE's which arise when a random noise is introduced into ordering differential equations, [Klebaner, 2005]:

Consider the SDE:

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dW_t, \quad X_t(w_0) = X_{t_0} \quad ...(1.7)$$

where $f : I\times\mathbb{R} \longrightarrow \mathbb{R}$, $g : I\times\mathbb{R} \longrightarrow \mathbb{R}$ be a Borel-measurable functions, f is called the drift function and g the diffusion function.

A solution X_t of the SDE (1.7) must also satisfy equation (1.7) when it is written as a SIE of the form:

$$X_t = X_{t_0} + \int_{t_0}^{t} f(s, X_s)\, ds + \int_{t_0}^{t} g(s, X_s)\ dW_s \qquad ...(1.8)$$

However, the second integral given in equation (1.8) cannot be defined in the usual sense, where W_s is the Wiener process. The variance of the Wiener process satisfies $Var(W_t) = t$, and so this increases as time increases even thought the mean stays at 0. Because of this, typical sample paths of n Wiener process attain larger values in magnitude as time progresses, and consequently the sample paths of the Wiener process are not bounded, hence the second integral in equation (1.8) cannot be considered as a Riemann-Stieltjes integral.

Note that, more general processes which has the martingale property can be used in place of W_s, but in this thesis only Wiener process will be used in the formulation given in equation (1.8) of SIE. Also, note that there is only single given scalar Wiener process, so the SDE is then represented by rewriting the SIE (1.8) as:

$$X_t = X_{t_0} + \int_{t_0}^{t} f(s, X_s)\, ds + \int_{t_0}^{t} g(s, X_s) * dW_s \qquad ...(1.9)$$

or

$$X_t = X_{t_0} + \int_{t_0}^{t} f(s, X_s)\ ds + \int_{t_0}^{t} g(s, X_s) \circ dW_s \qquad ...(1.10)$$

Where $\int_{t_0}^{t} g(s, X_s) * dW_s$ refers to either Itô stochastic integral $\int_{t_0}^{t} g(s, X_s) \circ dW_s$, such that the first integral in equation (1.9) is pathwise Lebseque-integrable since the paths of the Wiener process are almost sure of unbounded variation, we cannot interpret the second integral in equation (1.9) in the sense of a pathwise Riemann-Stieltijes integral.

1.5 The Existness and Unqinneces Theorem, [Fridman,1975]:

Consider the SDE:

$$dX_t = f(X_t, t)\, dt + g(X_t, t)\, dW_t \qquad \dots(1.11)$$

with initial condition:

$$X_{t_0} = X_0 \qquad \dots(1.12)$$

where $f(X_t, t)$ and $g(X_t, t)$ are measurable functions satisfies :

$$\left.\begin{aligned} &\|f(X, t) - f(\bar{X}, t)\| \le K^*\|X - \bar{X}\| \\ &\|g(X, t) - g(\bar{X}, t)\| \le K^*\|X - \bar{X}\| \\ &\|f(X, t)\| \le K(1 + \|X\|) \\ &\|g(X, t)\| \le K(1 + \|X\|) \end{aligned}\right\} \qquad \dots(1.13)$$

Where K^*, K are constants.

Hence, to find the equivalent SIE, integrate both sides of eq. (1.11) and use the initial condition (1.12)

$$X_t = X_{t_0} + \int_0^t f(X_s, s)\, ds + \int_0^t g(X_s, s)\, dW_s \qquad \dots(1.14)$$

and hence an iterated sequence of solutions of the resulting integral equation may be evaluated as follows:

$$\left.\begin{aligned} X_{t_1} &= X_0 + \int_0^t f(X_{s_0}, s)\, ds + \int_0^t g(X_{s_0}, s)\, dW_s \\ X_{t_2} &= X_0 + \int_0^t f(X_1(s), s)\, ds + \int_0^t g(X_{s_1}, s)\, dW_s \\ &\vdots \\ X_{t_{m+1}} &= X_0 + \int_0^t f(X_{s_m}, s)\, ds + \int_0^t g(X_{s_m}, s)\, dW_{s3} \end{aligned}\right\} \qquad \dots(1.15)$$

Theorem (1.1) (The Existence Theorem).

Suppose that $f(X_t, t)$, $g(X_t, t)$ are measurable functions in $(\Omega, \mathcal{A}, P)$ and $f(X_t, t)$, $g(X_t, t)$ satisfies the equation(1.13). Let X_0 be any n-dimensional random vector independent of $\mathcal{A}_t$, $0 \leq t \leq T$, such that $E|x_0|^2 < \infty$. Then there exist a unique solution of equation (1.11), and equation (1.12) in $L^2_\omega[0, T]$.

Proof: See [Jassim, 2009].

Chapter Two

Approximate Methods for Solving Integral Equations and Approximation of Stochastic Integrals

CHAPTER TWO

APPROXIMATE METHODS FOR SOLVING INTEGRAL EQUATIONS AND APPROXIMATION OF STOCHASTIC INTEGRALS

2.1 Introduction:

In general, for stochastic deterministic integral equations, the approximation of the solution and/or integrals plays a fundamental role in the numerical solution of such type of problems. Therefore, this chapter consists of four sections. In section (2.2), two elementary methods used for solving integral equations are discussed, namely the collocation method and the method of approximation of the integral. In section (2.3), we use the theory of stochastic integrals to approximate some easy stochastic integrals analytically that will be included in some types of SIE’s in the next chapter. While in section (2.4), more complicated stochastic integrals are considered, which will set the basis for the general methods for solving SIE’s.

2.2 Approximate Methods for Solving Integral Equations:

In this section, the concerne will be on the approximate and numerical methods for solving Fredholm and Volterra integral equations, in which two approaches will be considered.

2.2.1 The Collocation Method, [Chambers,1976]:

The collocation method is one of the approximate methods that may be used to solve Fredholm integral equations of the first and second kinds, and also can be used to find an approximate solution for nonlinear integral equations.

To illustrate this method, consider the first kind linear Fredholm integral equation:

$$f(t) = \lambda \int_a^b K(t,x)y(x)\,dx, \quad t \in [0,T] \qquad \ldots(2.1)$$

and let:

$$R(\phi,t) = \lambda \int_a^b K(t,x)\phi(x)\,dx - f(t) \qquad \ldots(2.2)$$

so, if $\phi(t)$ is the exact solution of equation (2.1), then $R(\phi,x) \equiv 0$.

Suppose, now for approximation that:

$$\phi(t) = \sum_{k=1}^{m} a_k B_k(t) \qquad \ldots(2.3)$$

where $B_k(t)$, $k = 1, 2, \ldots, m$; $m \in \mathbb{N}$ are any linearly independent set of known functions and a_k, $k = 1, 2, \ldots, m$; are constants to be determined.

Substituting equation (2.3) in to equation (2.2), then the problem is reduced to the problem of finding the values of a_k, $k = 1, 2, \ldots, m$; that make $R(\phi,t) \cong 0$ at a number of selected points within the domain of definition of the integral equation. Thus:

$$R(\phi,t) = \sum_{k=1}^{m} \left\{ a_k \lambda \int_a^b K(t,x)B_k(x)\,dx \right\} - f(x) \qquad \ldots(2.4)$$

and hence:

$$\sum_{k=1}^{m} \left\{ a_k \lambda \int_a^b K(t_k,x)B_k(x)\,dx \right\} - f(t_k) \approx 0 \qquad \ldots(2.5)$$

Where t_k, $k = 1, 2, \ldots, m$; are some selected grid points in the domain of definition of the integral equation.

Now, by solving the above system of linear algebraic equations for a_k, $k = 1, 2, \ldots, m$; we will get an approximate solution for the integral equation, and for the nonlinear integral equation one must solve the resulting system of nonlinear algebraic equations.

2.2.2 Approximation of the Integral, [Arfken, 1978]:

This method is also one of the approximate methods that may be used to solve both Fredholm and Volterra integral equations and sometimes it can be used to solve nonlinear integral equations. In this method, the integral operator is approximated by a sum of N+1 terms where $N \in \mathbb{N}$. So, as a result, the integral equation is reduced to a set of N+1 linear or nonlinear algebraic equations.

The basis for this method is that it is possible to approximate the following integral:

$$\int_a^b p(x)\,dx = \sum_{k=0}^{N} c_k p_k(x_k) \qquad \ldots(2.6)$$

where c_k, $\forall\ k = 0,1,.....,m$ are the weighting coefficients associated with the selected points x_k, k = 0, 1, …, N; that are called the roots of the integral approximation.

For simplicity, we will take the trapezoidal rule, and consider the linear Volterra integral equation of the second kind:

$$y(t) = f(t) + \int_a^t K(t,x)y(x)\,dx\,,\ x \in [t_0,T]$$

and by dividing the interval of integration (a,t) into N-equal subintervals, the following discretized equation is obtained:

$$\int_a^t K(t,x)y(x)\,dx \approx \frac{h}{2}\{K(t,t_0)y(t_0) + 2K(t,t_1)y(t_1) + \ldots + 2K(t,t_{N-1})y(t_{N-1}) + K(t,t_N)y(t_N)\} \qquad \ldots(2.7)$$

where $h = \dfrac{T - t_0}{N}$, $t_i = a + ih$, i = 0, 1, …, N; so:

$$y(t) = f(t) + \frac{h}{2}\{K(t,t_0)y(t_0) + 2K(t,t_1)y(t_1) + \ldots + 2K(t,t_{N-1})y(t_{N-1}) + K(t,t_N)y(t_N)\} \qquad \ldots(2.8)$$

Now, consider N+1 samples of y(t), namely:

$$y(t_0) = f(t_0)$$

$$y(t_i) = f(t_i) + \frac{h}{2}\{K(t_i,t_0)y(t_0) + 2K(t_i,t_1)y(t_1) + \ldots + 2K(t_i,t_{N-1})y(t_{N-1}) + K(t_i,t_N)y(t_N)\} \quad \ldots(2.9)$$

which are N linear algebraic equations in $y(t_i)$, $i = 1, 2, \ldots, N$; which have to be solved to find the numerical solution of the integral equation.

2.2.3 Other Numerical and Approximate Methods:

There are so many other approximate and numerical methods that may be used to solve integral equations. These methods may be classified according to the type of the integral equation and the kernel of the equation.

The following is a list of some of these methods:

1. Least square method, [Al-Shather, 1999].
2. Iterated solution of Fredholm integral equations with symmetric kernel by quasi-Newton method, [Sadhen, 1981].
3. The solution of the integral equations with symmetric kernel, [Jerri, 1985].
4. Iterative approximates the characteristic function, [Hildbrand, 1965].
5. Linear and nonlinear programming methods to solve integral equations, [Delves, 1973].

2.3 Certain Types of Stochastic Integrals:

In this section, we shall consider the stochastic integral and its properties from a more mathematical perspective at some time extending the definition to wider class of integrals. For this, it is supposed that the a probability space $(\Omega,\mathcal{A},P)$, a Wiener process $W = \{W_t: t \geq 0\}$ and an increasing family $\{\mathcal{A}_t: t \geq 0\}$ of sub σ-algebras of $\mathcal{A}$, such that W_t is $\mathcal{A}_t$ -measurable with $E(W_t|\mathcal{A}_0) = 0$ and $E(W_t - W_s|\mathcal{A}_s) = 0$ w.p.1, for all $0 \leq s \leq t$, [Kloeden, 1995].

Definition (2.1), [Friedman, 1975]:

A stochastic process X_t defined on [a,b] is called a step function if there exist a partition $a = t_0 < t_1 < \ldots < t_N = b$ of [a,b], such that $X_t = X_{t_i}$ if $t_i < t \leq t_{i+1}$, i = 0, 1, …, N – 1.

Lemma (2.1), [Friedman, 1975]:

Let $f \in L^2_\omega[\alpha,\beta]$, where $L^2_\omega[\alpha,\beta]$ is the space of all functions f, such that $\int\limits_\alpha^\beta |f(t)|^2\, dt < \infty$

1. There exist a sequence of continuous functions g_n in $L^2_\omega[\alpha,\beta]$, such that:

$$\lim_{n\to\infty} \int\limits_\alpha^\beta |f(t) - g_n(t)|^2\, dt = 0 \text{ a.s.} \qquad \ldots(2.10)$$

2. There exists a sequence of step functions f_n in $L^2_\omega[\alpha,\beta]$, such that:

$$\lim_{n\to\infty} \int\limits_\alpha^\beta |f(t) - f_n(t)|^2\, dt = 0 \text{ a.s.} \qquad \ldots(2.11)$$

Definition (2.2), [Friedman, 1975]:

Let X_t be a step function in $L^2_\omega[\alpha,\beta]$, say:

$X_t = X_{t_i}$, if $t_i < t < t_{i+1}$, $0 \leq i \leq N-1$

where $a = t_0 < t_1 < \ldots < t_N = b$. The random variable:

$$\sum_{k=0}^{N-1} X_{t_k}\left[X_{t_{k+1}} - X_{t_k}\right]$$

is denoted by:

$$\int\limits_a^b X_t\, dW_t$$

and is called the stochastic integral with respect to the Brownian motion W_t, which is also called the Itô integral.

Theorem (2.1), [Friedman, 1975]:

Let X_t and X_{t_n} be in $L^2_\omega[\alpha,\beta]$ and suppose that:

$$\int_a^b \left|X_{t_n} - X_t\right|^2 dt \xrightarrow{p} 0 \quad \text{as} \quad n \longrightarrow \infty \qquad \ldots(2.12)$$

Then:

$$\int_a^b X_{t_n}\, dt \xrightarrow{p} \int_a^b X_t\, dt \text{ as } n \longrightarrow \infty \qquad \ldots(2.13)$$

where $\xrightarrow{p}$ refers that the converge is in probability.

Lemma (2.2), [Friedman, 1975]:

If $X_t \in L^2_\omega[\alpha,\beta]$ and X_t is continuous, then for any sequence π_n of partitions $a = t_{n,0} < t_{n,1} < \ldots < t_{n,m} = b$ of [a,b] with $|\pi_n| \longrightarrow 0$

$$\sum_{k=0}^{m_n-1} X_{t_{n,k}} \left[W_{t_{n,k+1}} - W_{t_{n,k}}\right] \xrightarrow{p} \int_a^b X_t\, dW_t \quad \text{as} \quad n \longrightarrow \infty \qquad \ldots(2.14)$$

Theorem (2.2), [Friedman, 1975]:

If X_t is a step function in $L^2_\omega[\alpha,\beta]$, where $L^2_\omega[\alpha,\beta]$, and W_t is the Brownian motion, then:

$$E\left(\int_a^b X_t\, dW_t\right) = 0 \qquad \ldots(2.15)$$

$$E\left(\left|\int_a^b X_t\, dW_t\right|^2\right) = E\left(\int_a^b X_t^2\, dt\right) \qquad \ldots(2.16)$$

2.3.1 The Simples of Stochastic Integrals:

Let us start with the easiest possible stochastic integral of the form:

$$\int_a^b dW_t \qquad \ldots(2.17)$$

This type of stochastic integral is with respect to W_t, which is more exactly represents, the Wiener process. The Itô integral of the function which is always equals 1, because this is a definite integral.

This work chose any set of time-points t_i, $i = 0, 1, \ldots, N - 1$; we like, and treat 1 as an elementary function with those descritized times as its break-points. Then using our definition of the Itô integral for elementary function:

$$\int_a^b dW_t = \sum_{j=0}^{N-1} W_{t_{j+1}} - W_{t_j}$$

$$= W_b - W_a \qquad \ldots(2.18)$$

Following are some fundamental and necessary concepts in the theory of stochastic integrals that will be used in the solution of SIE's.

2.3.2 More Complicated Stochastic Integrals.

Another type of simple stochastic integrals which is more complicated than the stochastic integral given subsection (2.3.1) which arises from the following SODE:

$$dX_t = W_t \, dW_t \qquad \ldots(2.19)$$

where the behavior of W_t is governed by the rules of the Brownian motion. Integrating both sides, one may get:

$$X_t = \int_0^t W_s \, dW_s \qquad \ldots(2.20)$$

where it is assumed for simplicity that $X_0 = 0$.

In the Itô integral, equation (2.20) is approximated with the following stochastic sum:

$$X_t = \sum_{j=0}^{N-1} W_j (W_{j+1} - W_j) \qquad \ldots(2.21)$$

where it is assumed that $t = N\Delta t$, $\Delta t = \dfrac{b-a}{N}$ and $W_N = W_{(t_N)}$, with $W_0 = W(0)$, which is just the discrete form of the variance of a random variable with zero mean, i.e.,

$$E(\Delta W_j) = 0,\ Var(\Delta W_j) = W(\Delta W_j^2) \qquad \dots(2.22)$$

Equation (2.21) may be rewritten as:

$$X_t = \frac{1}{2}\sum_{j=0}^{N-1} W_{j+1}^2 - W_j^2 - (W_{j+1}^2 - 2W_{j+1}W_j + W_j^2)$$

$$= \frac{1}{2}\sum_{j=0}^{N-1}\left[W_{j+1}^2 - W_j^2 - (W_{j+1} - W_j)^2\right]$$

$$= \frac{1}{2}\left\{\left[(W_1^2 - W_0^2) + (W_2^2 - W_1^2) + \dots + (W_N^2 - W_{N-1}^2)\right] - \sum_{j=0}^{N-1}(W_{j+1} - W_j)^2\right\}$$

$$= \frac{1}{2}\left[(W_N^2 - W_0^2) - \sum_{j=0}^{N-1}(W_{j+1} - W_j)^2\right]$$

$$= \frac{1}{2}(W_N^2 - W_0^2) - \frac{1}{2}\sum_{j=0}^{N-1}\Delta W_j^2$$

The sum can be written as:

$$\sum_{j=0}^{N-1}\Delta W_j^2 = N\left(\frac{1}{N}\sum_{j=0}^{N-1}\Delta W_j^2\right) \qquad \dots(2.23)$$

Since it is known that $\Delta W_j = W_{j+1} - W_j$ is normally distributed with mean 0 and variance Δt, because it governs the jump for Brownian motion, then as $N \longrightarrow \infty$ and by equation (2.22):

$$Var(\Delta W) = E(\Delta W^2) = \frac{1}{N}\sum_{j=0}^{N-1}\Delta W_j^2 = \Delta t \qquad \dots(2.24)$$

So that the approximation to the equation (2.20) becomes:

$$X_t = \frac{1}{2}(W_N^2 - W_0^2) - \frac{1}{2}N\Delta t \qquad \dots(2.25)$$

From the definition of the Brownian motion, $W_0 = W(0) = 0$ and after substituting for $W_N = W(t_N)$ and $N\Delta t = t$, we have:

$$X_t = \frac{1}{2}W_N^2 - \frac{1}{2}t \qquad \dots(2.26)$$

In the Stranovich integral, equation (2.20) is approximated with midpoint rules as:

$$X_t = \sum_{j=0}^{N-1} W\left(\frac{t_j + t_{j+1}}{2}\right)(W_{j+1} - W_j) \qquad \dots(2.27)$$

where the value of $W\left(\frac{t_{j+1}}{2}\right)$ is approximated by:

$$W\left(\frac{t_{j+1}}{2}\right) = \frac{1}{2}(W_j + W_{j+1}) + C_j \text{ , } j=0,1,\dots\dots,N-1$$

where C_j, j = 0,1,…..,N-1 must be determined so that the above approximation still satisfies the values of Brownian motion. Then:

$$X_t = \frac{1}{2}W_t^2 \qquad \dots(2.28)$$

Theorem (2.3), [Friedman, 1975]:

Let X_t be a Weiner process and $\pi_n = \{t_1^{(n)}, t_2^{(n)}, \dots, t_{N_n}^{(n)}\}$ be a sequence of partitions of the finite closed interval [a,b] with $|\pi_n| \longrightarrow 0$ as $n \longrightarrow \infty$. Let:

$$S_n = \sum_{k=1}^{N_n}\left(X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)}\right)^2$$

Then $S_n \longrightarrow b - a$ in the mean.

2.4 Other Types of Stochastic Integrals, [Al-Afif, 2012]

We shall give some remarks concerning the development the discussed certain types of stochastic integrals, in which it is pointed out the essential difference between them. Historically, in 1930 N. Weiner introduced an integrals of the form:

$$\int_a^b g(t)\, dW_t$$

where g(t) is a deterministic real-valued function and $\{W_t : t \in [a,b]\}$ is a scalar Brownian motion process. Auther [Itô, 1944] generalized the integral include those cases were the integrand is random. That is he obtained an integrals of the form:

$$\int_0^t g(s, W_s)\, dW_s\ ,\ t \in [0,1].$$

which is referred to as the Itô stochastic integral or simply the stochastic integral.

Let $\{W_t : t \in [a,b]\}$ be a scalar Brownian process. In this subsection, we shall be concerned with the integral:

$$\int_a^b g(s, W_s)\, dW_s\ ,\ a < b \qquad \ldots(2.29)$$

For a fairly general class of functions g. As it is well known, almost all the sample functions of Brownian motion process are of unbounded variation and hence the integral (2.29) cannot be defined as an ordinary Stieltjes integral.

First, equation (2.29) will be defined for the class of step functions, that is, the function g is rewritten in the form:

$$g(t,W_t) = \begin{cases} 0, & t < a \\ g_i(W_t), & t_i \le t \le t_{i+1} \\ 0, & t > b \end{cases} \qquad \ldots(2.30)$$

where $a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b$, $g_i(W_t)$ are measurable with respect to the σ-algebra $\mathcal{A}_t$, and:

$$E\{|g_i(W_t)|^2\} < \infty$$

for such function, define the Itô integral by:

$$\int_a^b g(t, W_t)\, dW_t = \sum_{i=0}^{N-1} g_i(W_t)(W_{t_{i+1}} - W_{t_i}) \qquad \ldots(2.31)$$

where it is supposed that $g(t,W_t)$ is any function satisfying the following conditions:

1. $g(t,W_t)$ is a P-measurable function from $[a,b]\times\Omega$ into $\mathbb{R}^+$, assuming the usual Lebsegue measure on $\mathbb{R}^+$.
2. For each $t \in [a,b]$, $g(t,W_t)$ is measurable with respect to σ-algebra $\mathcal{A}_t$,is the smallest σ-algebra on Ω, such that W_s, $s \leq t$; is measurable.
3. $\int_{-\infty}^{\infty} E\,|\,g(t, W_t)\,|^2\, dt < \infty$.

In view of equation (2.30) it is evident that the classes of step functions satisfy the above conditions.

For the function $g(t,W_t)$ satisfying last conditions 1-3, their norm will be defined as follows:

$$\|g(t,W_t)\| = \left\{\int_a^b E(|\,g(t, W_t)\,|^2)\, dt\right\}^{1/2} \qquad \ldots(2.32)$$

For this case Doob has shown the following [Doob, 1953]:

1. $g(t,W_t)$ can approximated in the mean-square sense by a sequence of step functions $\{g_n(t,W_t)\}$, that is:

   $\|g(t,W_t) - g_n(t,W_t)\| \longrightarrow 0$ as $n \longrightarrow \infty$
2. The sequence of integrals:

   $$\int_a^b g(t, W_t)\, dW_t$$

   possesses a mean square limit.

CHAPTER THREE

APPROXIMATE AND NUMERICAL SOLUTIONS OF STOCHASTIC INTEGRAL EQUATIONS

CHAPTER THREE

APPROXIMATE AND NUMERICAL SOLUTIONS OF STOCHASTIC INTEGRAL EQUATIONS

3.1 Introduction:

To introduce the concepts and main issues concerning the time discrete approximation of the solution of SIE's, we shall concern in this chapter with the collocation method and approximating the integrals for solving SIE's in linear and nonlinear cases and for all the cases of stochastic integrals that are defined and discussed in chapter two.

This chapter consists of five sections. In section (3.2), either numerical or approximate solutions of certain types of SIE's are considered. In section (3.3), the general form of SIE's and its approximate and numerical solutions, as well as a modified approach, using the same methods used in section (3.2) will be considered. In section (3.4), nonlinear SIE's are given and discussing its approximate and numerical solutions. Finally, in section (3.5), a special type of SIE's is considered, which is called random integral equations.

3.2 Solution of Special Types of Linear Stochastic Integral Equations:

This section concerns with the numerical and approximate solution of linear SIE's using two approaches, the first one is an approximation method which is based on the collocation method and the second approach is based on the numerical method by approximating the integrals using certain numerical integration methods, which is for simplicity here is the trapezoidal rule.

3.2.1 The Collocation Method For Solving Linear Stochastic Integral Equations:

As it is said previously, this method is one of the easiest and earliest approximate methods to solve SIE's. To illustrate this method, consider the second kind linear SIE:

$$X_t = X_{t_0} + \int_{t_0}^{t} K(t,s) X_s\, ds + \int_{t_0}^{t} dW_s \ , \ t_0 \in \mathbb{R} \qquad \ldots(3.1)$$

where W_t is the Wiener process with mean zero and variance h, K is an integrable function. Then applying equation (2.11), the second integral equals to $W_t - W_{t_0}$ and therefore equation (3.1) will be reduced to the following easiest form:

$$X_t = X_{t_0} + \int_{t_0}^{t} K(t,s) X_s\, ds + W_t - W_{t_0}$$

$$= \left(X_{t_0} + W_t - W_{t_0}\right) + \int_{t_0}^{t} K(t,s) X(s)\, ds \qquad \ldots(3.2)$$

and hence equation (3.2) may be written as:

$$X_t = g(t, W_t) + \int_{t_0}^{t} K(t,s) X_s\, ds \qquad \ldots(3.3)$$

where:

$$g(t, W_t) = X_{t_0} + W_t - W_{t_0}$$

Suppose for approximation that:

$$X_t = X_{t_0} + \sum_{k=1}^{m} c_k B_k(t) \qquad \ldots(3.4)$$

where $B_k(t)$, k = 1,2,…,m; $m \in \mathbb{N}$ are any linearly independent set of known functions satisfying $B_k(t_0) = 0$ and c_k, k = 1,2,…,m; are constants to be determined. Substituting equation (3.4) in to equation (3.3), yields to:

$$X_{t_0} + \sum_{k=1}^{m} c_k B_k(t) = g(t,W_t) + \int_{t_0}^{t} K(t,s)\left[X_{t_0} + \sum_{k=1}^{m} c_k B_k(s) \right] ds$$

or equivalently:

$$\sum_{k=1}^{m} c_k B_k(t) - \sum_{k=1}^{m} c_k \int_{t_0}^{t} K(t,s) B_k(s)\, ds - \int_{t_0}^{t} K(t,s) X_{t_0}\, ds = g(t,W_t) - X_{t_0}$$

and hence:

$$\sum_{k=1}^{m} c_k [B_k(t) - h_k(t)] - \int_{t_0}^{t} K(t,s) X_{t_0}\, ds = g(t,W_t) - X_{t_0} \qquad \ldots(3.5)$$

where:

$$h_k(t) = \int_{t_0}^{t} K(t,s) B_k(s)\, ds$$

and upon evaluating equation (3.5) at m-distinct discritzed points $t_1, t_2, \ldots, t_m$ in [a,b], we get a linear system of algebraic equations $Ac = b$ in $c_1, c_2, \ldots, c_m$ that can be solved to get the solution of the linear SIE given by equation (3.1), where:

$$A = \begin{bmatrix} B_1(t_1) - h_1(t_1) & B_2(t_1) - h_2(t_1) & \cdots & B_m(t_1) - h_m(t_1) \\ B_1(t_2) - h_1(t_2) & B_2(t_2) - h_2(t_2) & \cdots & B_m(t_2) - h_m(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ B_1(t_m) - h_1(t_m) & B_2(t_m) - h_2(t_m) & \cdots & B_m(t_m) - h_m(t_m) \end{bmatrix}$$

$c = [c_1 \ \ c_2 \ \ \ldots \ \ c_m]^T$,

$b = [g(t_1, W_{t_1}) - X_{t_0} \ \ g(t_2, W_{t_2}) - X_{t_0} \ \ \ldots \ \ g(t_m, W_{t_m}) - X_{t_0}]^T$

As an illustration, consider the following example:

Example (3.1):

Consider the following linear SIE:

$$X_t = 1 + \int_0^t (t+s) X_s\, ds + \int_0^t dW_s$$

and in order to use the collocation method to approximate the solution we start by letting $B_i(t)= t^i$, $i = 1,2,\dots,m$; and if $m = 10$ then the approximate solution will be of the form:

$$X_t = 1 + \sum_{i=1}^{10} c_i t^i$$

and hence

$$1+\sum_{i=0}^{10} c_i t^i = (1 + W_{10} - W_0) + \int_0^t (t+s)[1+\sum_{i=0}^{10} c_i s^i]ds$$

or equivalently:

$$\sum_{i=1}^{10} c_i[t^i - \int_0^t (t+s)s^i ds] - \int_0^t (t+s)ds = W_{10} - W_0$$

and hence the following linear system is obtained $Ac = b$, where:

$$A = \begin{bmatrix} 0.084 & -5.06\times10^{-3} & -0.014 & -0.015 & -0.015 & -0.015 & -0.015 & -0.015 & -0.015 & -0.015 \\ 0.133 & -0.021 & -0.052 & -0.058 & -0.06 & -0.06 & -0.06 & -0.06 & -0.06 & -0.06 \\ 0.142 & -0.05 & -0.109 & -0.127 & -0.133 & -0.134 & -0.135 & -0.135 & -0.135 & -0.135 \\ 0.107 & -0.095 & -0.181 & -0.216 & -0.23 & -0.236 & -0.238 & -0.239 & -0.24 & -0.24 \\ 0.021 & -0.161 & -0.264 & -0.318 & -0.346 & -0.36 & -0.368 & -0.371 & -0.373 & -0.374 \\ -0.12 & -0.256 & -0.359 & -0.428 & -0.471 & -0.498 & -0.514 & -0.524 & -0.531 & -0.534 \\ -0.321 & -0.385 & -0.468 & -0.538 & -0.592 & -0.633 & -0.662 & -0.683 & -0.698 & -0.709 \\ -0.587 & -0.559 & -0.595 & -0.647 & -0.697 & -0.743 & -0.782 & -0.815 & -0.842 & -0.865 \\ -0.923 & -0.788 & -0.752 & -0.754 & -0.773 & -0.799 & -0.828 & -0.858 & -0.887 & -0.916 \\ -1.333 & -1.083 & -0.95 & -0.867 & -0.81 & -0.768 & -0.736 & -0.711 & -0.691 & -0.674 \end{bmatrix}$$

$$b = [0.867\ \ 0.867\ \ 0.867\ \ 0.867\ \ 0.867\ \ 0.867\ \ 0.867\ \ 0.867\ \ 0.867\ \ 0.867]^T$$

and upon solving this linear system, we get the solution:

$$c = [25.358\ \ -303.58\ \ 2.00\times10^3\ \ -8.13\times10^3\ \ 2.15\times10^4\ \ -3.75\times10^4\ \ 4.31\times10^4\ \ -3.13\times10^4\ \ 1.30\times10^4\ \ -2.37\times10^3]^T$$

Hence the solution:

$$X_t = 1 + [(25.352)t^1\ (303.58)t^2\ (2.00\ 103)t^3\ (8.13\ 103)t^4\ (2.15\ 104)t^5\ (3.75\ 104)t^6\ (4.31\ 104)t^7\ (3.13\ 104)t^8\ (1.30\ 104)t^9\ (2.37\ 103)t^{10}]$$

In applications, another type of linear SIE's it's be encountered, which is more complicated to be evaluated where this complexity is due to the stochastic integral part. To illustrate such type of equations, consider the linear SIE:

$$X_t = X_{t_0} + \int_{t_0}^{t} K(t,s)X_s\,ds + \int_{t_0}^{t} W_s\,dW_s \qquad \ldots(3.6)$$

where K is an integrable function, $t_0 \in \mathbb{R}$ and W_t is the Weiner process. Then applying equation (2.28) on the second integral, which is equal to $\frac{1}{2}\left(W_t^2 - W_{t_0}^2\right)$ and therefore equation (3.6) is given by:

$$\begin{aligned} X_t &= X_{t_0} + \int_{t_0}^{t} K(t,s)X_s\,ds + \frac{1}{2}\left(W_t^2 - W_{t_0}^2\right) \\ &= \left[X_{t_0} + \frac{1}{2}\left(W_t^2 - W_{t_0}^2\right)\right] + \int_{t_0}^{t} K(t,s)X_s\,ds \qquad \ldots(3.7) \end{aligned}$$

and hence equation (3.7) may be written as:

$$X_t = g(t,W_t) + \int_{t_0}^{t} K(t,s)X_s\,ds \qquad \ldots(3.8)$$

where:

$$g(t,W_t) = X_{t_0} + \frac{1}{2}\left(W_t^2 - W_{t_0}^2\right)$$

By applying the collocation method for solving the linear SIE, we get a linear system of algebraic equations in $c_1,c_2,\ldots,c_m$ that may solved to get the solution of the linear SIE given by equation (3.3).

The resulting linear system may be derived as follows:

$$X_{t_0} + \sum_{k=1}^{m} c_k B_k(t) = g(t,W_t) + \int_{t_0}^{t} K(t,s)\left[X_{t_0} + \sum_{k=1}^{m} c_k B_k(s)\right]ds$$

Hence, the related linear system:

$$\sum_{k=1}^{m} c_k\left[B_k(t) - h_k(t)\right] - \int_{t_0}^{t} K(t,s)X_{t_0}\,ds = g(t,W_t) - X_{t_0}$$

where:

$$h_k(t) = \int_{t_0}^{t} K(t,s)B_k(s)ds$$

which may be written in matrix form as Ac = b, where:

$$A = \begin{bmatrix} B_1(t_1)-h_1(t_1) & B_2(t_1)-h_2(t_1) & \cdots & B_m(t_1)-h_m(t_1) \\ B_1(t_2)-h_1(t_2) & B_2(t_2)-h_2(t_2) & \cdots & B_m(t_2)-h_m(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ B_1(t_m)-h_1(t_m) & B_2(t_m)-h_2(t_m) & \cdots & B_m(t_m)-h_m(t_m) \end{bmatrix}$$

$$c = [c_1 \;\; c_2 \;\; \ldots \;\; c_m]^T,$$

$$b = [g(t_1, W_{t_1})-X_{t_0} \;\; g(t_2, W_{t_2})-X_{t_0} \;\; \ldots \;\; g(t_m, W_{t_m})-X_{t_0}]^T$$

As an illustration, consider the following example:

Example (3.2):

Consider the following linear SIE:

$$X_t = 1 + \int_0^t (t^2+2s)X_s\,ds + \int_0^t W_s\,dW_s$$

Hence in order to use the collocation method to approximate the solution, we let $B_k(t) = t^k$, i = 1,2,…,m; and if m = 10 with approximate solution:

$$X_t = 1 + \sum_{k=1}^{m} c_k B_k(t)$$

and hance

$$1 + \sum_{k=1}^{10} c_k t^k = (1 + \frac{1}{2}[W^2{}_t - W^2{}_{t_0}]) + \int_0^t (t^2+2s)[1 + \sum_{k=1}^{10} c_k s^k]ds$$

or equivalently:

$$\sum_{k=1}^{10} c_k[t^k - \int_0^t (t^2+2s)s^k ds] - \int_0^t (t^2+2s)ds = \frac{1}{2}[W^2{}_t - W^2{}_{t_0}]$$

and hence the following linear system of algebraic equation Ac = b is obtained, where:

$$A = \begin{bmatrix} 1.077 & 0.988 & 0.979 & 0.978 & 0.978 & 0.978 & 0.978 & 0.978 & 0.978 & 0.978 \\ 1.098 & 0.943 & 0.912 & 0.906 & 0.904 & 0.904 & 0.904 & 0.904 & 0.904 & 0.904 \\ 1.044 & 0.851 & 0.792 & 0.774 & 0.768 & 0.767 & 0.766 & 0.766 & 0.766 & 0.766 \\ 0.897 & 0.696 & 0.611 & 0.576 & 0.562 & 0.556 & 0.554 & 0.553 & 0.552 & 0.552 \\ 0.635 & 0.458 & 0.359 & 0.306 & 0.278 & 0.264 & 0.257 & 0.254 & 0.252 & 0.251 \\ 0.239 & 0.117 & 0.021 & -0.044 & -0.085 & -0.111 & -0.127 & -0.137 & -0.143 & -0.146 \\ -0.315 & -0.352 & -0.42 & -0.482 & -0.531 & -0.569 & -0.596 & -0.616 & -0.631 & -0.641 \\ -1.05 & -0.978 & -0.989 & -1.024 & -1.064 & -1.103 & -1.138 & -1.167 & -1.192 & -1.213 \\ -1.992 & -1.793 & -1.718 & -1.695 & -1.696 & -1.71 & -1.729 & -1.752 & -1.776 & -1.8 \\ -3.167 & -2.833 & -2.65 & -2.533 & -2.452 & -2.393 & -2.347 & -2.311 & -2.282 & -2.258 \end{bmatrix}$$

$$b = [1.001 \;\; 1.002 \;\; 1.001 \;\; 1.005 \;\; 1.014 \;\; 1 \;\; 1 \;\; 1.002 \;\; 1.024 \;\; 1.003]^T$$

and upon solving this linear system, we get the solution:

$$c = \begin{bmatrix} 96.321 & -1.23\times10^3 & 8.71\times10^3 & -3.81\times10^4 & 1.08\times10^5 \\ -2.03\times10^5 & 2.49\times10^5 & -1.92\times10^3 & 8.44\times10^4 & -1.61\times10^4 \end{bmatrix}$$

Hence the solution:

$$\begin{aligned} X_t = 1 + [&(96.321)t^1 \; (-1.23\times10^3)t^2 \; (8.71\times10^3)t^3 \; (-3.81\times10^4)t^4 \\ &(1.08\times10^5)t^5 \; (-2.03\times10^5)t^6 \; (2.49\times10^5)t^7 \; (-1.92\times10^3)t^8 \\ &(8.44\times10^4)t^9 \;\; (-1.61\times10^4)t^{10}] \end{aligned}$$

3.2.2 Approximation of Integrals Method for Solving Linear Stochastic Integral Equations:

The two special types of linear SIE's discussed in subsection (3.2.1) may be solved using the method of approximating the integrals based on the trapezoidal rule, i.e., solving the linear SIE's (3.3) and (3.8) using numerical integration methods, as follows:

For the first type of the integral equations given by equation (3.1), which may be written equivalently as equation (3.3), namely:

$$X_t = g(t, W_t) + \int_{t_0}^{t} K(t,s) X_s \, ds \;,\;\; t \in [t_0, T] \;,\;\; T \in \mathbb{N}$$

where:

$$g(t, W_t) = X_{t_0} + W_t - W_{t_0}$$

and upon using the trapezoidal rule, we get:

$$X_t = g(t, W_t) + \frac{h}{2}\{K(t, t_0)X_{t_0} + 2K(t, t_1)X_{t_1} + \ldots +$$
$$2K(t, t_{N-1})X_{t_{N-1}} + K(t, t_N)X_{t_N}\}$$

where $h = \dfrac{T - t_0}{N}$, $t_i = t_0 + ih$, $i = 0,1,\ldots,N$; and $N \in \mathbb{N}$.

Now, consider N + 1 samples of X_t

$$X_{t_i} = g(t_i, W_t) + \frac{h}{2}\{K(t_i, t_0)X_{t_0} + 2K(t_i, t_1)X_{t_1} + \ldots + K(t_i, t_N)X_{t_N}\} \qquad \ldots(3.9)$$

and upon evaluating equation (3.9) at N equations in X_{t_i}, $i = 1,2,\ldots,N$; and $t_1, t_2, \ldots, t_N$ in $[t_0, T]$, a linear system of algebraic equations is obtained and may be solved to find the approximate solution of the linear SIE (3.9).

The following linear system $AX_t = b$ is obtained, where:

$$A = \begin{bmatrix} 1 - hK(t_1, t_1) & 0 & \cdots & 0 \\ \frac{-h}{2}K(t_2, t_1) & 1 - hK(t_2, t_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-h}{2}K(t_N, t_1) & -hK(t_N, t_2) & \cdots & 1 - hK(t_N, t_N) \end{bmatrix}$$

$X_t = [X_{t_1} \; X_{t_2} \; \ldots \; X_{t_N}]^T$ and $b = [g(t_1, W_{t_1}) \; g(t_2, W_{t_2}) \; \ldots \; g(t_m, W_{t_m})]^T$

Also, for the second type of integral equation given in equation (3.6), which is equivalent to equation (3.8), namely:

$$X_t = g(t,W_t) + \int_{t_0}^{t} K(t,s)X_s\, ds$$

where:

$$g(t,W_t) = X_{t_0} + \frac{1}{2}\left(W_t^2 - W_{t_0}^2\right)$$

and upon using the trapezoidal rule, the following discredited equation is obtained:

$$X_t = g(t,W_t) + \frac{h}{2}\{K(t,t_0)X_{t_0} + 2K(t,t_1)X_{t_1} + \ldots + 2K(t,t_{N-1})X_{t_{N-1}} + K(t,t_N)X_{t_N}\} \qquad \ldots(3.10)$$

and therefore similarly a linear system of algebraic equations in X_{t_i}, $i = 1,2,\ldots,N$; is obtained that can be solved to find approximate solution of the SIE (3.10).

Example (3.3):

Consider the following linear SIE:

$$X_t = 1 + \int_0^t sX_s\, ds + \int_0^t dW_s\ ,\ t \in [1,T]$$

Solving the linear SIE's using numerical integration methods. Thus we can rewrite the integral equation as:

$$X_t = 1 + W_t - W_{t_0} + \int_0^t s.X_s\, ds$$

and upon using the trapezoidal rule, we get:

$$X_t = 1 + W_t - W_{t_0} + \frac{h}{2}\{t_0X_{t_0} + 2t_1X_{t_1} + \ldots + 2t_{N-1}X_{t_{N-1}} + t_NX_{t_N}\}$$

where $h = 0.1$, $t_i = ih$, $i = 1,2,\ldots,N$ and $N = 10$.

Now, consider N samples of X_t, namely:

$$X_{t_i} = 1 + W_{t_i} - W_{t_0} + \frac{h}{2}\{ t_0 X_{t_0} + 2t_1 X_{t_1} + \ldots + 2t_{N-1} X_{t_{N-1}} + t_N X_{t_N},$$

$$i = 1,2,\ldots,10$$

and upon evaluating 10 equations in X_{t_i}, i = 1,2,..,10, we get a linear system AX_t = b of algebraic equations that may be solved to find the approximate solution of the linear SIE, where:

$$A = \begin{bmatrix} 0.99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.01 & 0.98 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.015 & -0.015 & 0.97 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.02 & -0.02 & -0.02 & 0.96 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.025 & -0.025 & -0.025 & -0.025 & 0.95 & 0 & 0 & 0 & 0 & 0 \\ -0.03 & -0.03 & -0.03 & -0.03 & -0.03 & 9.40\times10^{-1} & 0 & 0 & 0 & 0 \\ -0.035 & -0.035 & -0.035 & -0.035 & -0.035 & -3.50\times10^{-2} & 0.93 & 0 & 0 & 0 \\ -0.04 & -0.04 & -0.04 & -0.04 & -0.04 & -4.00\times10^{-2} & -0.04 & 0.92 & 0 & 0 \\ -0.045 & -0.045 & -0.045 & -0.045 & -0.045 & -0.045 & -0.045 & -0.045 & 0.91 & 0 \\ -0.05 & -0.05 & -0.05 & -0.05 & -0.05 & -0.05 & -0.05 & -0.05 & -0.05 & 0.9 \end{bmatrix}$$

$$b = [1.056 \ 1.086 \ 1.007 \ 0.936 \ 0.976 \ 1 \ 0.784 \ 1.01 \ 0.924 \ 1.102]^T$$

which is based upon using 1000 trials for the Weiner process, we get the following results:

$$X_t = [\ 1.0711.129 \ 1.086 \ 1.062 \ 1.165 \ 1.269 \ 1.133 \ 1.481 \ 1.524 \ 1.881]^T$$

Example (3.4):

Consider the following linear SIE:

$$X_t = 1 + \int_0^t (s+1) X_s \, ds + \int_0^t W_s \, dW_s \ , \ t \in [0,1]$$

Hence by the approximation method of integrals for solving linear SIE's based on the trapezoidal rule, Thus:

$$X_t = g(t,W_t) + \frac{h}{2}\{ (t_0+1) X_{t_0} + 2(t_1+1) X_{t_1} + \ldots + 2(t_{N-1}+1)$$

$$X_{t_{N-1}} + (t_N+1) X_{t_N} \}$$

where h = 0.1, t_i = ih, i = 1,2,…,N and N = 10.

Now, consider N samples of X_t, namely:

$$X_{t_i} = g(t_i, W_{t_i}) + \frac{h}{2}\{(t_0+1)\, X_{t_0} + 2(t_1+1)\, X_{t_1} + \ldots + 2(t_{N-1}+1)\, X_{t_{N-1}} + (t_N+1)\, X_{t_N}\}, \quad i = 1,2,\ldots,10$$

and upon evaluating 10 equations in X_{t_i}, i = 1,2,..,10, we get a linear system of algebraic equations $AX_t = b$ that may be solved to find the approximate solution of the linear SIE, where:

$$A = \begin{bmatrix} 0.89 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.06 & 0.88 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.065 & -0.065 & 0.87 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.07 & -0.07 & -0.07 & 0.86 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.075 & -0.075 & -0.075 & -0.075 & 0.85 & 0 & 0 & 0 & 0 & 0 \\ -0.08 & -0.08 & -0.08 & -0.08 & -0.08 & 0.84 & 0 & 0 & 0 & 0 \\ -0.085 & -0.085 & -0.085 & -0.085 & -0.085 & -0.085 & 0.83 & 0 & 0 & 0 \\ -0.09 & -0.09 & -0.09 & -0.09 & -0.09 & -0.09 & -0.09 & 0.82 & 0 & 0 \\ -0.095 & -0.095 & -0.095 & -0.095 & -0.095 & -0.095 & -0.095 & -0.095 & 0.81 & 0 \\ -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & 0.8 \end{bmatrix}$$

$$b = [1.001 \;\; 1.002 \;\; 1.001 \;\; 1.005 \;\; 1.014 \;\; 1 \;\; 1 \;\; 1.002 \;\; 1.024 \;\; 1.003]^T$$

which is based upon using 1000 trials for the Weiner process, we get the following results:

$$X_t = [1.193 \;\; 1.296 \;\; 1.42 \;\; 1.577 \;\; 1.775 \;\; 1.988 \;\; 2.266 \;\; 2.607 \;\; 3.051 \;\; 3.54]^T$$

3.3 The General Form of Stochastic Linear and Nonlinear Stochastic Integral Equations:

The general form of a scalar linear SDE is given by:

$$dX_t = (f_1(t)X_t + f_2(t))dt + (g_1(t)X_t + g_2(t))dW_t \qquad \ldots(3.11)$$

and the equivalent linear SIE is given by:

$$X_t = X_{t_0} + \int_{t_0}^{t} (f_1(s)X_s + f_2(s))\,ds + \int_{t_0}^{t} (g_1(s)X_s + g_2(s))\,dW_s \qquad \ldots(3.12)$$

where the coefficients f_1, f_2, g_1 and g_2 are specified functions of the time t or constants, provided that they are Lebsegue measurable and bounded on an interval $0 < t < T$, ensuring the existence of a unique solution X_t on $0 \leq t_0 \leq t \leq T$ and each $\mathcal{A}_t$-measurable initial value X_{t_0} corresponding to a given Wiener process $\{W_t, t \geq 0\}$ and associated family of σ-algebras $\{\mathcal{A}_t, t \geq 0\}$. When the coefficients are all constants, then equation (3.11) is autonomous and its solution, which exist for all t, $t_0 \geq 0$; are homogeneous Markov process. In this case, it suffices to consider $t_0 = 0$ and when $f_2(t) = g_2(t) = 0$ and, then the linear SIE (3.12) will be reduced to:

$$X_t = X_{t_0} + \int_{t_0}^{t} f_1(s)X_s\, ds + \int_{t_0}^{t} g_1(s)X_s\, dW_s \qquad \ldots(3.13)$$

Equation (3.11) and hence equation (3.12) may be generalized, as it is illustrated next.

Definition (3.1), [Arnold, 1974]:

The SDE:

$dX_t = f(t,X_t)\, dt + g(t,X_t)\, dW_t,\ t \in [t_0,T]$

for the d-dimensional process X_t is said to be linear, if the functions f and g are linear functions of $X_t \in \mathbb{R}^d$ on $[t_0,T]\times\mathbb{R}^d$, in other words if:

$f(t,X_t) = A(t)X_t + a(t)$

$g(t,X_t) = [B_1(t)X_t+b_1(t), B_2(t)X_t+b_2(t), \ldots, B_n(t)X_t+b_n(t)]$

where A(t) and $B_k(t)$, k = 1,2…,n; are d×d-real valued matrices, a(t) and $b_k(t)$ are $\mathbb{R}^d$ vector valued functions, a linear SDE has the form:

$$dX_t = (A(t)X_t + a(t))\, dt + \sum_{i=1}^{m}\left(B_i(t)Xt + b_i(t)\right)dW_t^i \qquad \ldots(3.14)$$

where $W_t = [\,W_t^1 \quad W_t^2\ \ldots\ W_t^m\,]^T$ and is said to be homogeneous if $a(t) = b_1(t) = \ldots = b_m(t) = 0$. Also, it is said to be linear in the narrow sense if $B_1(t) = B_2(t) = \ldots = B_m(t) = 0$.

Theorem (3.1), [Arnold, 1974]:

Equation (3.14) has the solution:

$$X_t = \Phi_t \left(c + \int_{t_0}^{t} \Phi_s^{-1} \left(a(s) - \sum_{i=1}^{m} B_i(s) b_i(s) \right) ds + \sum_{i=1}^{m} \int_{t_0}^{t} \Phi_s^{-1} b_i(s)\, dW_s^i \right)$$

where $c = X_{t_0}$ and:

$$\Phi_t = \exp\left[\int_{t_0}^{t} \left(A(s) - \sum_{i=1}^{m} \frac{B_i^2(s)}{2} \right) ds + \sum_{i=1}^{m} \int_{t_0}^{t} B_i(s)\, dW_s^i \right]$$

is the solution of the homogeneous equation related to equation (3.14):

$$d\Phi_t = A(t)\Phi_t\, dt + \sum_{i=1}^{m} B_i(t)\Phi_t\, dW_t^i$$

with initial value $\Phi_{t_0} = I$.

Remark (3.1):

1. In equation (3.14) if $d = m = 1$, then equation (3.14) will be reduced to:

$$dX_t = (A(t)X_t + a(t))dt + (B(t)X_t + b(t))dW_t \qquad \ldots(3.15)$$

2. When $a(t) = 0$ and $b(t) = 0$, then the SDE given in equation (3.15) reduced to the following homogeneous linear SDE:

$$dX_t = A(t)X_t\, dt + B(t)X_t\, dW_t \qquad \ldots(3.16)$$

Integrating both sides of equation (3.16), the following LSIE is obtained:

$$X_t = X_{t_0} + \int_{t_0}^{t} A(s)X_s\, ds + \int_{t_0}^{t} B(s)X_s\, dW_s \qquad \ldots(3.17)$$

by theorem (3.1) and equation (3.17), the solution is given by:

$$X_t = \Phi_t c \qquad \ldots(3.18)$$

where $c = X_{t_0}$ and

$$\Phi_t = \exp\left[\int_{t_0}^{t} \left(A(s) - \frac{B^2(s)}{2} \right) ds + \int_{t_0}^{t} B(s)\, dW_s \right] \qquad \ldots(3.19)$$

with initial value $X_{t_0} = c$.

Now to study the solution of nonlinear SIE's of the form:

$$X_t = h(t,X_t)+\int_0^t K_1(t,s,W_s)f_1(s,X_s)\,ds+\int_0^t K_2(t,s,W_s)f_2(s,X_s)\,dW_s \qquad \ldots(3.20)$$

in which sufficient conditions are given to ensure the existence and uniqueness of a random solution, which are:

(i) The supporting set of a complete probability measure space $(\Omega,\mathcal{A},P)$ with $\mathcal{A}$ being the σ-algebra and P is the probability measure.

(ii) X_t is the unknown random process.

(iii) $h(t, X_t)$ is a map from $\mathbb{R}^+ \times \mathbb{R}$ into $\mathbb{R}$.

(iv) $K_1(t,s,W_t)$ and $K_2(t,s,W_t)$ are the scalar stochastic kernels which are random valued functions defined for $0 \le s \le t < \infty$.

(v) $f_1(s,X_t)$ and $f_2(s,X_t)$ are maps from $\mathbb{R}^+ \times \mathbb{R}$ into $\mathbb{R}$.

(vi) $t \in \mathbb{R}^+$ and W_t is a stochastic process, [Dominik, 1984].

Also, as it is known there are two basic classes of SIE's currently under study, namely probabilistic and deterministic. Those integral equations involving Itô type stochastic integrals and those which can be considered as probabilistic analogue of classical deterministic integral equations, whose formulation involves only the Lebsegue integral, [Tsokos, 1973].

With respect to the process W_t, it is assumed that for each $t \in \mathbb{R}^+$, a minimal σ-algebra $\mathcal{A}_t \subset \mathcal{A}$ is defined such that W_t is measurable with respect to $\mathcal{A}_t$.

Furthermore, we shall assume that [Doob, 1953]:

(i) The process $\{W_t, \mathcal{A}_t, t \in \mathbb{R}^+\}$ is a real martingale.

(ii) There is a continuous monotone non-decreasing function F(t), $t \in \mathbb{R}^+$, such that for s < t.

$$E|W_t - W_s|^2 = E\{|W_t - W_s|^2 \mid A\} = F(t) - F(s)$$

Remark (3.2):

The same approaches followed previously in this chapter to find the numerical and approximate solution for solving linear SIE's may be used to find the solution of nonlinear SIE's, with the main difference that is the obtained linear system of algebraic equations will be nonlinear and hence may be solved using the standard methods for solving nonlinear algebraic equations.

As an illustration, consider the following example:

Example (3.5):

Consider the following nonlinear SIE:

$$X_t = X_{t_0} + \int_0^t (X_s^2 - 1)\,ds + \int_0^t \frac{1 - X_s^2}{10}\,dW_s \ , \ t \in [0,1]$$

Then upon using the numerical integration method based on trapezoidal and Itô stochastic integral, we have the related form:

$$X_t = X_{t_0} + \frac{h}{2}\{(X^2{}_{t_0} - 1) + 2(X^2{}_{t1} - 1) + \ldots + 2(X^2{}_{t_{N-1}} - 1) + (X^2{}_{t_N} - 1)\} + \frac{1 - X_{t_0}^2}{10}$$

$$\Delta W_{t_0} + \frac{1 - X_{t_1}^2}{10}\ \Delta W_{t_1} + \ldots + \frac{1 - X_{t_{N-1}}^2}{10}\ \Delta W_{t_{N-1}}$$

where h is the discretization step size, $t \in [a,b]$, N=10. Therefore at the discretized points $t_0, t_1, \ldots, t_N$, we have:

$$X_{t_i} - X_{t_0} - \frac{h}{2}\{(X^2{}_{t_0} - 1) + 2(X^2{}_{t1} - 1) + \ldots + 2(X^2{}_{t_{N-1}} - 1) + (X^2{}_{t_N} - 1)\} -$$

$$\left\{ \frac{1 - X_{t_0}^2}{10}\ \Delta W_{t_0} + \frac{1 - X_{t_1}^2}{10}\ \Delta W_{t_1} + \ldots + \frac{1 - X_{t_{N-1}}^2}{10} \Delta W_{t_{N-1}} \right\} = 0$$

and letting for each i = 1,2,…,N:

$$H_i(X_{t_0}, X_{t_1}, \ldots, X_{t_N}) = X_{t_i} - X_{t_0} - \frac{h}{2}\{(X^2_{t_0}\text{-}1)\text{+}2(X^2_{t1}\text{-}1)\text{+}\ldots\text{+}$$

$$\text{+}2(X^2_{t_{N\text{-}1}}\text{-}1)\text{+}(X^2_{t_N}\text{-}1)\} - \left\{\frac{1-X^2_{t_0}}{10}\Delta W_{t_0} + \right.$$

$$\left.\frac{1-X^2_{t_1}}{10}\Delta W_{t_1} + \ldots + \frac{1-X^2_{t_{N-1}}}{10}\Delta W_{t_{N-1}}\right\} = 0$$

Then solution of the last system of nonlinear algebraic equations may be obtained by minimizing the objective function:

$$H(X_{t_0}, X_{t_1}, \ldots, X_{t_N}) = \sum_{i=0}^{N} H_i^2(X_{t_0}, X_{t_1}, \ldots, X_{t_N}) \qquad \ldots(3.21)$$

The obtained results for minimizing equation (3.21) starting with the initial solution:

$X_{0.1} = 0.1$, $X_{0.2} = 0.2$, $X_{0.3} = 0.3$, $X_{0.4} = 0.4$, $X_{0.5} = 0.5$, $X_{0.6} = 0.6$,

$X_{0.7} = 0.7$, $X_{0.8} = 0.8$, $X_{0.9} = 0.9$, $X_1 = 1$

with 1000 trail for the Wiener process, are given by:

$$X_t = [0.981 \quad 0.972 \quad 0.97 \quad 0.957 \quad 0.951 \quad 0.939 \quad 0.92 \quad 0.901 \quad 0.881 \quad 0.853 \quad 0.837]^T$$

3.4 Second Kind Volterra Stochastic Integral Equations

Random Volterra integral equations may be considered as a special type of stochastic Volterra integral equations of the second kind, which has the form:

$$X_t = h(t,W_t) + \int_0^t K(t,s,W_t)f(s,X_s)\,ds\,,\ t \geq 0. \qquad \ldots(3.22)$$

that is formulated in Hilbert space over the supporting set of a complete probability measure space is $(\Omega, \mathcal{A}, P)$. A discrete version of the above random integral equation is given by:

$$X_{t_n} = h_n(W_t) + \sum_{j=1}^{n} c_{n,j}(W_{t_j}) f_j(X_{t_j}), \; n = 1, 2, \ldots \qquad \ldots(3.23)$$

In this section, we shall study the second kind random integral equation of Volterra type in the discretized form given in equation (3.23), where:

(i) The supporting set of a complete probability measure space$(\Omega, \mathcal{A}, P)$.

(ii) X_t is the unknown random variable for each $t \geq 0$.

(iii) $h(t,W_t)$ is the random free term defined for each $t \geq 0$.

(iv) $K(t,s,W_t)$ is the random kernel defined for $0 \leq s \leq t < \infty$.

(v) $f(t,X_t)$ is a scalar function for each $t \geq 0$ and scalar X_t.

Equation (3.23) is useful in obtaining an approximation to the random solution of equation (3.22) by electronic digital computation. Also, equation (3.23) provides a description of physical systems, which yield observations or outputs only at discrete terms, [Padgett, 1973].

Remark (3.3), [Padgett, 1973]:

1. We shall make the following assumptions in regarding the random functions in (3.22). The random solution X_t and the stochastic free term $h(t,W_t)$ are functions of $t \in \mathbb{R}^+$ with values in $L_2(\Omega, \mathcal{A}, P)$. The function $f(t,X_t)$ will also be a function of $t \in \mathbb{R}^+$ with values in $L_2(\Omega, \mathcal{A}, P)$ under certain conditions. The stochastic kernel $K(t,s,W_t)$ for each $0 \leq s \leq t < \infty$ is in the space $L_\infty(\Omega, \mathcal{A}, P)$; that is, $K(t,s,W_t)$ is an essentially bounded function with respect to p. Hence, the product of $K(t,s,W_t)$ and $f(s,X_t)$ will always be in $L_2(\Omega, \mathcal{A}, P)$.
2. The following assumptions are made with respect to the random functions in the stochastic discrete equation (3.23). The random solution X_{t_n} and the stochastic free term $h_n(W_t)$ are functions of $n \in \mathbb{N}$ with values in the space $L_2(\Omega, \mathcal{A}, P)$. For each value of $n \in \mathbb{N}$, $f_n(X_{t_n})$ is in

$L_2(\Omega,\mathcal{A},P)$, and for each value of X_{t_n} $f_n(X_{t_n})$ is a scalar. For each value of n and j in $\mathbb{N}$, $1 \le j \le n$, $c_{n,j}(W)$ is in the space $L_\infty(\Omega,\mathcal{A},P)$; that is, $c_{n,j}(W)$ is bounded in the ordinary sense except perhaps on a set probability zero for each n and j, $1 \le j \le n$.

Now, two cases of random integral equations will be considered, as follows:

Case I:

In order to solve equation (3.22) using the collocation method, we let:

$$X_t = \sum_{i=1}^{N} c_i B_i(t, W_t) \qquad \ldots(3.24)$$

Then substituting equation (3.24) in equation (3.22), yields to:

$$\sum_{i=1}^{N} c_i B_i(t, W_t) = h(t,W_t) + \int_0^t K(t,s,W_t) f\left(s, \sum_{i=1}^{N} c_i B_i(s, W_t)\right) ds$$

and hence:

$$\sum_{i=1}^{N} c_i B_i(t, W_t) - g(t,W_t) = h(t,W_t) \qquad \ldots(3.25)$$

where:

$$g(t,W_t) = \int_0^t K(t,s,W_t) f\left(s, \sum_{i=1}^{N} c_i B_i(s, W_t)\right) ds$$

Case II:

To illustrate this approach, consider the SIE:

$$X_t = h(t,W_t) + \int_0^t K(t,s,W_s) f(s,X_s)\, ds,\ t \ge 0$$

and upon taking the trapezoidal rule, then:

$$X_t = h(t,W_t) + \frac{h}{2}\{K(t,s_0,W_{t_0}) f(s_0, X_t) + 2K(t,s_1,W_{t_1}) f(s_1, X_t) + \ldots +$$
$$2K(t,s_{N-1},W_{t_{N-1}}) f(s_{N-1}, X_t) + K(t,s_N,W_{t_N}) f(s_N, X_t)\}$$

where $h = \dfrac{b-a}{N}$, $N \in \mathbb{N}$, $t \in [a,b]$.

Now, consider N + 1 samples of X_t namely

$$X_{t_i} = h(t_i,W_{t_i}) + \frac{h}{2}\{K(t_i,s_0,W_{t_0})f(s_0,X_{t_i})+2K(t_i,s_1,W_{t_1})f(s_1,X_{t_i})$$

$+\ldots+$

$$2K(t_i,s_{N-1},W_{t_{N-1}})f(s_{N-1},X_{t_i}) + K(t_i,s_N,W_{t_N})f(s_N,X_{t_i})\}$$

and hence:

$$X_{t_i} - h(t_i,W_{t_i}) - \frac{h}{2}\{K(t_i,s_0,W_{t_0})f(s_0,X_{t_i})+2K(t_i,s_1,W_{t_1})f(s_1,X_{t_i}) +\ldots+$$

$$2K(t_i,s_{N-1},W_{t_{N-1}})\, f(s_{N-1},X_{t_i}) + K(t_i,s_N,W_{t_N})f(s_N,X_{t_i})\} = 0$$

Also, if f is lines then the last system will be liner which may be solved easily, but if f is nonlinear then by letting for each i = 0,1,…,N:

$$H_i(X_{t_0},X_{t_1},\ldots,X_{t_N}) = X_{t_i} - h(t_i,W_{t_i}) - \frac{h}{2}\{K(t_i,s_0,W_{t_0})\, f(s_0,X_{t_i}) +$$

$$2K(t_i,s_1,W_{t_1})\, f(s_1,X_{t_i}) + \ldots + 2K(t_i,s_{N-1},W_{t_{N-1}})\, f(s_{N-1},X_{t_i}) +$$

$$K(t_i,s_N,W_{t_N})f(s_N,X_{t_i})\}$$

Then the solution of the last system of nonlinear algebraic equations may be obtained by minimizing the objective function:

$$H(X_{t_0},X_{t_1},\ldots,X_{t_N}) = \sum_{i=0}^{N} H_i^2(X_{t_0},X_{t_1},\ldots,X_{t_N}) \qquad \ldots(3.26)$$

Example (3.6) (Case I):

Consider the following linear random integral equation:

$$X_t = t.W - \tfrac{1}{2}t^5W^2 + \int_0^t t.s\ X_s\, ds,\ t\in[0,1]$$

and upon taking the collocation method and let:

$$X_t = \sum_{i=0}^{10} c_i (t^i W^i)$$

Then

$$\sum_{i=0}^{10} c_i (t^i W^i) - \left[t.W - \tfrac{1}{2} t^5 W^2 \right] - \int_0^t t.s \left[\sum_{i=0}^{10} c_i (s^i W^i) \right] ds = 0$$

$$\sum_{i=0}^{10} c_i (t^i W^i) - \int_0^t t.s \left(\sum_{i=0}^{10} c_i (s^i W^i) \right) ds - (t.W - \tfrac{1}{2} t^5 W^2) = 0$$

which may be rewritten equivalent as

$$\sum_{i=0}^{10} c_i \left[t^i .W^i - t.W^i \int_0^t s^{i+1} ds \right] - (t.W - \tfrac{1}{2} t^5 W^2) = 0$$

$$\sum_{i=1}^{10} c_i W^i \left[t^i - \tfrac{1}{i+2} t^{i+3} \right] - (t.W - \tfrac{1}{2} t^5 W^2) = 0$$

and letting

$$H_i(c_1, c_2, ..., c_{10}) = \sum_{i=1}^{10} c_i W^i \ t^i (1 - \tfrac{1}{i+2} t^3) - (t.W - \tfrac{1}{2} t^5 W^2), i = 1, 2, ..., 10$$

Then the solution of the last system of linear algabric equation may be obtained by the minimizing objectine function:

$$H(c_1, c_2, ..., c_{10}) = \sum_{i=1}^{10} H_i^2 (c_1, c_2, ..., c_{10})$$

and the following result are obtained:

c_1=1.00038, c_2=57.7402, $c_3 = 1.34\times10^{-3}$, $c_4 = 5.18\times10^{-4}$, $c_5 = 1.37\times10^{-5}$, $c_6 = 0.6$, $c_7 = 1.48\times10^{-3}$, $c_8 = 156.807$, $c_9 = 0.9$, $c_{10} = 1$

hence the solution

$$\begin{aligned} X_t &= (1.00038)tW + (57,7402)(tW)^2 + (1,34 \times 10^{-3})(tW)^3 + \\ &(5.18 \times 10^{-4})(tW)^4 + (1.37 \times 10^{-5})(tW)^5 + (0.6)(tW)^6 + \\ &(1.48 \times 10^{-3})(tW)^7 + (156.807)(tW)^8 + (0.9)(tW)^9 + (tW)^{10} \end{aligned}$$

Example (3.7) (Case II):

Consider the following linear Volterra random integral equation:

$$X_t = t.W - \tfrac{1}{2}t^5W^2 + \int_0^t 2t.s.W.X_s \, ds, \quad t \in [0,1]$$

and upon using the trapezoidal rule, then:

$$X_t = t.W - \tfrac{1}{2}t^5W^2 + \frac{h}{2}\{2t_0s_0W_0X_{t_0}) + 4t_1.s_1^2.W_1.X_{t_1} + ... +$$
$$2(2t_9.s_9^2.W_9.X_{t_9}) + 2t_{10}.s_{10}^2.W_{10}.X_{t_{10}}\}$$

where $h = \dfrac{b-a}{N}$, $t_i = ih$, $i = 0,1,2,...,N$, $N = 10$, $a = 0$ and $b = 1$

Now, consider N+1 samples of X_t. namely:

and letting for each i=0,1,2,…,10

$$Hi(X_{t_0}, X_{t_1}, ..., X_{t_{10}}) = X_{t_i} - (t_i.W_i - \tfrac{1}{2}t_i^5.W_i^2) - \frac{h}{2}\{2t_i, s_0^2.W_0.X_{t_0} +$$
$$4t_i.s_1^2.W_1.X_{t_1} + ... + 2t_i.s_{10}^2.W_{10}.X_{t_{10}}\} = 0$$

Then the solution of the last system of linear algabric equation may be obtained by minimizing the objective function:

$$H(X_{t_0}, X_{t_1}, ..., X_{t_{10}}) = \sum_{i=0}^{10} Hi^2(X_{t_0}, X_{t_1}, ..., X_{t_{10}})$$

and the following result are obtained:

$X_{t_0} = 0$, $X_{t_1} = -4.3896797625\times10^{-3}$, $X_{t_2} = -0.0135873391$,

$X_{t_3} = -0.0141961494$, $X_{t_4} = -0.0390535228$, $X_{t_5} = -0.0843213914$,

$X_{t_6} = 3.018436367\times10^{-3}$, $X_{t_7} = -8.0425849784\times10^{-3}$, $X_{t_8} = 0.0446755358$,

$X_{t_9} = 0.1898116651$, $X_{t_{10}} = 0.0850251144$.

Example (3.8) (Case II):

Consider the following nonlinear Volterra random integral equation:

$$X_t = t.W - \tfrac{1}{4}t^5W^2 + \int_0^t t.s.W^2.X^2{}_t ds,\ t \in [0,1]$$

and upon using the trapezoidal rule, then:

$$X_t = t.W - \tfrac{1}{4}t^5W^2 + \frac{h}{2}\{(t_0.s_0.W^2{}_0 \cdot X_{t_0}) + 2t_1.s_1^2.W^2{}_1.X_{t_1} + ... +$$
$$2(2t_9.s_9^2.W_9.X_{t_9}) + t_{10}.s_{10}^2.W^2{}_{10}.X_{t_{10}}\}$$

where $h = \dfrac{b-a}{N},\ t_i = ih,\ i = 0,1,2,...,N,\ N = 10,\ a = 0$ and $b = 1$

Now, consider N+1 samples of X_t. namely:

$$X_{t_i} - (t_i.W_i - \tfrac{1}{4}t_i^5.W_i^2) - \frac{h}{2}\{t_i.s_0^2.W^2{}_0.X_{t_0} + 2t_i.s_1^2.W^2{}_1.X_{t_1} + ... +$$
$$4t_i.s_9^2.W_9.X_{t_9} + t_i.s_{10}^2.W^2{}_{10}.X_{t_{10}}\} = 0$$

and letting for each i=0,1,2,…,10

$$Hi(X_{t_0}, X_{t_1}, ..., X_{t_{10}}) = X_{t_i} - (t_i.W_i - \tfrac{1}{4}t_i^5.W_i^2) - \frac{h}{2}\{2t_i.s_0^2.W^2{}_0.X_{t_0}$$
$$+ 2t_i.s_1^2.W^2{}_1.X_{t_1} + ... + t_i.s_{10}^2.W^2{}_{10}.X_{t_{10}}\} = 0$$

Then the solution of the last system of linear algabric equation may be obtained by minimizing the objective function:

$$H(X_{t_0}, X_{t_1}, ..., X_{t_{10}}) = \sum_{i=0}^{10} Hi^2(X_{t_0}, X_{t_1}, ..., X_{t_{10}})$$

and the following result are obtained:

X_{t_0}=0, X_{t_1}=-4.39×10^{-3}, X_{t_2}=-0.013589462, X_{t_3}=-0.014199166,

X_{t_4}=0.038083829, X_{t_5}=-0.08450377, X_{t_6}=2.61×10^{-3}, X_{t_7}=-8.45×10^{-3},

X_{t_8}=0.044263214 , X_{t_9}=0.190298792, $X_{t_{10}}$=0.079495249

CONCLUSIONS AND FUTURE WORK

Conclusions and Future Work

From the present work of this thesis, we may conclude that the obtained results for solving SIE's by using the trapezoidal rule is more accurate than those results obtained by using the collocation method.

In addition, the nonlinear programming method is the more simplest approach than the numerical or approximate method that may be used to solve SIE's.

Also from the present study, some recommendations for future work may be drawn:

1. Using the present approaches of this thesis to solve SIE's with multi-Weiner process, for linear and nonlinear cases.
2. Using other numerical and approximate methods to solve SIE's, such as the least square method, iterative methods, spline methods, variational iteration method, etc.
3. Studying and solving stochastic integro-differential equations.

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APPENDIX

(COMPUTER PROGRAMS)

The method for solving example (3.1) using collocation method of linear stochastic integral equation, block for evaluating the result 1000 times

```
sol := for k ∈ 1 .. NN
         w2 ← rnorm(1, 0, h)
         Q ← w2 − 0
         g ← yt0 + Q
         for i ∈ 1 .. N
            for j ∈ 1 .. N
               A_{i,j} ← a(i·0.1, j)
            b_j ← g_{1,1}
         sol ← A^{−1}·b
         soll^⟨k⟩ ← sol
```

avarege of the results

$$av_{kk} := \frac{\sum_{ii=1}^{NN} sol_{kk,ii}}{NN}$$

av=

	1
1	-1.055
2	29.807
3	-340.151
4	2.10E+03
5	-7.80E+03
6	1.83E+04
7	-2.73E+04
8	2.51E+04
9	-1.30E+04
10	2.88E+03

The method for solving example (3.2) using collocation method of linear stochastic integral equation , block for evaluating the result 1000 times

```
sso := for k ∈ 1 .. NN
         w2 ← rnorm(1, 0, h)
         Q ← (1/2)·(w2^2 − 0)
         g ← yt0 + Q
         for i ∈ 1 .. 10
            for j ∈ 1 .. 10
               A[i,j] ← a(i·0.1, j)
            b[j] ← g[1,1]
         sol ← A^(−1)·b
         ssoo^<k> ← sol
```

avarege of the results

$$av_{kk} := \frac{\displaystyle\sum_{i=1}^{NN} sso_{kk,i}}{NN}$$

av =

	1
1	**-1.159**
2	**32.259**
3	**-365.705**
4	**2.24E+03**
5	**-8.32E+03**
6	**1.95E+04**
7	**-2.90E+04**
8	**2.66E+04**
9	**-1.37E+04**
10	**3.04E+03**

The method for solving example (3.3) using approximation method for solving stochastic integral equation ,block for evaluating the result 1000 times

```
sool := | for k ∈ 1 .. NN
        |     | w2 ← rnorm(1, 0, h)
        |     | w0 ← 0
        |     | Q ← w2 − w0
        |     | g ← x0 + Q
        |     | for j ∈ 1 .. N
        |     |     | for i ∈ 1 .. N
        |     |     |     A_{i,j} ← | p_{i,j} ← (−h/2)·f(t_i)  if i > j
        |     |     |               | p_{i,j} ← 1 − h·f(t_i)  if i = j
        |     |     |               | p_{i,j} ← 0  if i < j
        |     |     |               | p_{i,j} ← 1  if i = 1 ∧ j = 1
        |     |     |               | p_{i,j}
        |     |     | b_j ← g_{1,1}
        |     | sol ← A^{−1}·b
        |     | sool^{⟨k⟩} ← sol
        | sool
```

avarege of the results

$$av_i := \frac{\displaystyle\sum_{jj=1}^{NN} sool_{i,jj}}{NN}$$

av =

	1
1	1.004
2	1.034
3	1.066
4	1.11
5	1.167
6	1.24
7	1.329
8	1.437
9	1.567
10	1.724

The method for solving example (3.4) using approximation method for solving stochastic integral equation ,block for evaluating the result 1000 times .

```
sool := | for k ∈ 1..NN
        |   w2 ← rnorm(1,0,h)
        |   Q ← (1/2)·(w2^2 − 0)
        |   g ← x0 + Q
        |   for j ∈ 1..N
        |     for i ∈ 1..N
        |       A_{i,j} ← | p_{i,j} ← (−h/2)·f(t_i)  if i > j
        |                 | p_{i,j} ← 1 − h·f(t_i)  if i = j
        |                 | p_{i,j} ← 0  if i < j
        |                 | p_{i,j} ← 1  if i = 1 ∧ j = 1
        |                 | p_{i,j}
        |     b_j ← g_{1,1}
        |   sol ← A^{−1}·b
        |   sool^{⟨k⟩} ← sol
        | sool
```

avarege of the results

$$av_i := \frac{\sum_{jj=1}^{NN} sool_{i,jj}}{NN}$$

av =

	1
1	1.005
2	1.21
3	1.321
4	1.456
5	1.623
6	1.826
7	2.075
8	2.38
9	2.753
10	3.212

المستخلص

تمتلك المعادلات التصادفية التكاملية (stochastic integral equations) والمعادلات التكاملية العشوائية (random integral equations) اهمية كبيرة والتي يمكن ان تستخدم في نمذجة انواع معينة من المشاكل التي تتضمن تغيرات عشوائية (random process) وضوضاء (noise). ولذلك فأن الأهداف الرئيسية لهذه الرسالة كما يلي :

الهدف الاول : هو لدراسة الجانب النظري لموضوع حساب التفاضل والتكامل التصادفي (Stochastic calculus)، والذي يتضمن دراسة التعاريف والمفاهيم الأساسية في هذا الموضوع بالأضافة الى مبرهنة وجود و وحدانية الحلول (The existence and uniqueness theorem).

الهدف الثاني : هو اجراء مقارنة بين المعادلات التفاضلية التصادفية (Stochastic differential equations) والمعادلات التكاملية التصادفية (Stochastic integral equations) ومن ثم إعطاء بعض الطرائق التحليلية لأيجاد التكامل التصادفي.

الهدف الثالث : الذي يعتبر هدفنا الرئيسي، يتضمن دراسة الطرائق العددية والتقريبية لحل المعادلات التكاملية التصادفية الخطية والغير خطية.مع بعض الامثلة التوضيحية لكل حالة.

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم
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طرائق عددية و تقريبية لحل المعادلات التكاملية التصادفية

رسالة
مقدمة إلى كلية العلوم / جامعة النهرين
كجزء من متطلبات نيل درجة الماجستير
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من قبل
رؤى قحطان محمد
(بكالوريوس جامعة النهرين ، ٢٠٠٦)

إشراف
أ. م. د. فاضل صبحي فاضل

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