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# Stochastic Nonlinear Control Stablizability Based on Invers Optimality

A Thesis
Submitted to the Department of Mathematics and Computer Application, College of Science, Al-Nahrain University, as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

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بسم الله الرحمن الرحيم

(إِنَّا فَتَحْنَا لَكَ فَتْحًا مُّبِينًا ﴿١﴾ لِيَغْفِرَ لَكَ اللَّهُ مَا تَقَدَّمَ مِن
ذَنبِكَ وَمَا تَأَخَّرَ وَيُتِمَّ نِعْمَتَهُ عَلَيْكَ وَيَهْدِيَكَ صِرَاطًا
مُّسْتَقِيمًا ﴿٢﴾ وَيَنصُرَكَ اللَّهُ نَصْرًا عَزِيزًا ﴿٣﴾)

صدق الله العلي العظيم

(سورة الفتح/١-٣)

# الاهـــــــــــــــــــــــــداء

***الى البـــــــــــــارىء الذي خلقني***

***الى الذي اكرمــــــــــــــــــــــــني بفضله***

**الى الشفاه التي اكثرت لي الدعاء كلما نطقت**
**الى العين التي رأتني املا كلما نظرتْ**
**الى القلب الذي ازداد بي فخرا كلما نبض...**

**والــدي الحبيب**

**الى من جعل الرحمن تحت اقدامها الجنان**

**ومن اشعلت شموع عمرها لتضيء دربي بأمان**

**رمز التضحيه وعنوان الحنان...**

**امي الحبيبه**

**الى ملامح طفولتنا وعنفوان شبابنا**

**الى من طال بهم شوط الانتظـــــــــار**

**الى روافد الوفـــــــــــاء**

**الى من اشد بهم ازري في الحياة...**

**اخوتي الاعزاء**

**الى رفاق الدرب الطويل**

**اصدقائي الاوفياء**

**الى كل من يحبني واراد مودتي**

**اهدي ثمرة هذا الجهد المتواضع**

**نــــورا**

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*Noora Ali Aziz*

*October , 2008*

# SUPERVISORE CERTIFICATION

I certify that this thesis was prepared under my supervision at the department of mathematics and computer applications, ***College of Science, Al-Nahrain University*** as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

***Signature:***

***Name:*** Assist. Prof. Dr. Radhi Ali Zboon

***Date:*** /10/2008

In view of the available recommendations; I forward this thesis for debate by examination committee.

***Signature:***

***Name:*** Assist. Prof. Dr. Akram M. Al-Abood

Head of the Department of Mathematics and Computer Applications

***Date:*** /10/2008

# *Examining Committee's Certification*

We certify that we read this thesis entitled ***"Stochastic Nonlinear Control Stabilizability Based on Inverse Optimality"*** and as examining committee examined the student, (***Noora Ali Aziz***) in its contents and in what it connected with, and that is in our opinion it meet the standard of thesis for the degree of Master of Science in Mathematics and Computer Application.

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Date: **/ /2008**

# *Abstract*

The main aim of this work is focused on studying the global asymptotic stability in the probability for some class of closed-loop control system of Ito-type in the presence of system uncertainty.

Some nonlinear continuous-time Ito-dynamic stochastic system deriven by unbounded stochastic noise input have been considered, where the equilibrium point of the stochastic system is preserved even in the presence of noise .

The global asymptotic stability in probability has been developed by using stabilization controller and Lyapunov stochastic approach.

The stochastic Lyapunov function is computed to guarantee the global asymptotic stability in probability. Some resulte of estimation of exponential stability is also discussed.

The necessary theorem for finding the controller design and stability Lyapunov stochastic function have been stated and proved which are supported by some concluding remarks and illustrations.

# Contents

The design of global stabilization controller for stochastic nonlinear systems has been an active area of research in recent years ([Deng, Krstic and Williams 2001]and [Liu, Zhang and Jiang 2007]) and the references therein). Since Deng and Krstic [Deng and Krstic 1999] firstly gave a result of output-feedback stabilization, the output feedback controller design for stochastic nonlinear systems has received more intensive investigation [Liu and Zhang 2006], [Deng and Krstic 2000] and [Liu, Zhang and Pan 2003], which is because not only in general, the design of output-feedback control is more difficult and challenging than that of full state-feedback control, but also the output-feedback control is more practical in engineering. These known results are limited to the systems in output-feedback form, in which the nonlinear terms only depend on the measured output. For the deterministic systems, in [Mazence, Praly and Dayawansa 1994] counterexamples were given indicating global stabilization of the nonlinear systems in the general lower-triangular form via output feedback is usually impossible without introducing extra growth conditions on the unmeasurable state of the system. Since then, much research work has been focused on the output-feedback global stabilization of nonlinear systems under various structure or growth conditions [Jiang 2000] and [Praly and Jiang 2004]. Recently, there are some results of output-feedback control for the stochastic nonlinear systems in which nonlinear terms are dependent on the output and unmeasurable inverse dynamics or unmodeled dynamics [Liu and Zhang 2005] and [Wu, Xie and Zhang 2007].

In nonlinear systems, the stability theory is much richer than in linear systems and hence various notions of stability, such as exponential stability, global versus local stability, practical stability, and boundedness, have been introduced [Sastry 1999], [Khalil 1996] and [Kristic, Kanellakopoulos and Kokotovic 1995].

Lyapunov theory is a well-known proper mean for linear and non-linear systems analysis. The major problem of that theory which can be pointed, especially for non-linear systems is to derive a function such that it shoud satisfy the Lyapunov conditions. If such a function is derived, system stability can be guaranteed, while in this regard the designer's experiences are also desired. Although regarding this issue, there are several proposed method available while each individual may face with some particular constrains. Some general methods to determine the Lyapunov functions like,***Method of linearization around the operating point;*** where the major issue for this technique is eliminating the non-linear dynamics of system as well as procuring local stability, ***Crossofskey method;*** where in the case of large number of system states, solving the related equations and determining of conditions can be a tough job, ***Generalized Crosophsky method;*** where in this method determining of conditions are easy job, while computational works are so high and ***Variable gradient method;*** while in this method solving the equations is not so easy; whereas the results are similar to the method of linearization [Ali Akbar, Rushidi-Njad and Sadrnia 2005].

Regarding to the above issue; our attempts is ended to a simple proposed technique the so-called back-stepping methodology. This technique is a backward technique that can help one to find the Lyapunov functions. One of the advantages of this method is to prevent eliminating nonlinear dynamics of the system. In fact, back-stepping method is a modification from state feedback of the linear systems to non-linear systems by using Lyapunov theories. It seems that the origin of back-stepping theory is not precisely recognized, while some concurrent analysis with regards to this method has been done. The most important study from the literature can be addressed to some research paper of the 1980 decade. It is important to mention that the researches of Kokotowich and his colleagues have introduced this issue [Harkegard 2001]. In 1991 Kolotowich et.al. presented this idea through his published paper [Kokotovic 1992]. Kanlacupulos proposed a mathematical for designing a non-linear controller using back-stepping technique [Kanellakopoulos 1992]. Follow to these researches some years later, reasearchers such as Christic [Kristic, Kanellakopoulos and Kokotovic 1995], [Freeman and Peter 1996], and [Spultcher, Jankovic and Kokotovic 1997] published several research paper with regards to this subject. Also Kokotowich in 1990 at international IFAC symposium reviewed the progresses of back-stepping technique during 1990 decade [Kokotovic 1999].

After considering the stabilization of a specific class of stochastic nonlinear systems, we address the classical equation of when is a stabilizating (in probability) controller optimal and show that for every system with a stochastic control Lyapunov function it is possible to construct a controller which is optimal with respect to a meaningful cost functional.

After considering the stabilization of feedback stochastic systems, we present the following result: for general stochastic systems affine in the control and noise input, we design stabilizing robust for some class of a nonlinear stochastic dynamic system control [Hua Deng and Miroslav Krstic 1997].

While the current robust nonlinear control toolbox includes a number of methods for systems affine in deterministic bounded disturbance, the problem when the disturbance is unbounded stochastic noise has hardly been considered. We present a control design which achieves global asymptotic (Lyapunov) stability in probability for a class of strict-feedback nonlinear continuous-time systems deriven by white noise [Miroslav 1997].

Despite major advance in robust stabilization of deterministic nonlinear systems achieved over the last few years and reported in [Freeman and Kokotovic 1996], [Kristic, Kanellakopoulos and Kokotovic 1995] and reference therein, the stabilization problem for stochastic systems is yet to be addressed. While not as refined as their deterministic counterparts in [Khalil 1996], Lyapunov techniques for stability analysis of stochastic systems do exist, for example, the classical book of Khas'minskii [Khas 1980], [Kushner 1967]. Efforts toward (global) stochastic nonlinear systems have been initiated in the work of Florchinger [Flochinger 1993], [Flochinger 1995] who, among other things, extended the concept of control Lyapunov functions and Sontag's stabilization formula [Sontage 1989] to the stochastic setting. A breakthrough towards arriving at constructive method for stabilization of broader classes of stochastic nonlinear systems came with the result of Pan and Basar [Pan and Basar 1996], who derived a backstepping for design for strict-feedback systems motivated by a risk-senstive cost criterion [Nagai and Bellman 1996], [Runolfsson 1994], [Liu and Zhang 2005].

This thesis consists of three chapters. The first chapter deals with the basic concept of stochastic dynamic system.

In chapter two, the necessary mathematical principles concerning stochastic integration, Ito formula, existence and uniqueness of Ito SDEs, as well as solvable examples have been presented.

In chapter three, we design a backstepping control law which guarantees global asymptotic stability in probability.

Future work, concluding remarks and references are presented.

# Chapter One

# Basic Concepts of Modern Control Theory

This chapter presents the basic mathematical theory that will be needed later on, like, the concept of probability theory, stochastic process Brownian motion etc.

## *1.1 BASIC CONCEPT OF PROBABILITY THEORY [47]:*

Randomness and probability are not easy to define precisely, but we certainly recognize random events when we meet them. For example, randomness is in effect when we flip a coin, buy a lottery ticket, run a horse race. The following remarks and terminologies are needed in the following:

***Remarks (1.1) [47]:***

1. Probability theory is the mathematical study of phenomena occurring due to chance mechanism.
2. A mathematical experiment or a random experiment is that one in which the possible outcomes may be finite or infinite.
3. The collection of all elementary outcomes of a random experiment is called sample space and is denoted by $\Omega$.In set terminology the sample space is termed as the universal set, thus, the sample space $\Omega$ is a set consisting of mutually exclusive, collectively exhaustive listing of all possible outcomes of a random experiment. That is,

   $\Omega = \{w_1, w_2, ..., w_n\}$ denotes the set of all finite outcomes.

   $\Omega = \{w_1, w_2, ...\}$ denotes the set of all ccountably infinite outcomes.

   $\Omega = \{0 \le t \le T\}$ denotes the set of unccountably outcomes.

### *1.1.1 FIELDS, σ – FIELDS [47]:*

We define $\Re$ as the nonempty class of subsets drawn from the sample space $\Omega$. We say that the class $\Re$ is a field or an algebra of sets in $\Omega$ if it satisfies the following definition:

***Definition (1.1) (Field or Algebra) [47]:***

A class of a collection of subsets $A_j \subset \Omega,\ \forall j = 1,2,...,n$ denoted by $\Re$ is a field when the following conditions are satisfied:

1. If $A_i \in \Re$, then $A_i^c \in \Re$

2. If $\{A_i = 1,2,..., n\} \in \Re$, then $\bigcup_{i=1}^{n} A_i \in \Re$

***Example (1.2) [47]:***

Let Ω = R and consider a class $\Re$ of all interval of the form (a, b], that

$$(a,b] \cap (c,d] = \begin{cases} \varnothing & a<b<c<d \\ (c,d] & a<c<b<d \\ (a,d] & c<a<d<b \\ (c,d] & a<c<d<b \\ (a,b] & c<a<b<d \end{cases}$$

Clearly the class $\Re$ is closed under intersections. However,

$$(a,b]^c = (-\infty,a] \cup (b,\infty) \notin \Re$$
$$(a,b] \cup (c,d] \notin \Re \qquad if\ a<b<c<d$$

The class $\Re$ is not a filed.

***Definition (1.2) (σ-Field or σ-Algebra) [47]:***

A class of a countably infinite collection of subsets $A_j \subset \Omega,\ \forall j = 1,2,...$ denoted by $\Im$ is a $s - field$ when following conditions are satisfied:

1- If $A_i \in \Im$, then $A_i^c \in \Im$.

2- If $\{A_i, i = 1,2,\ldots\} \in \Im$, then $\bigcup_{i=1}^{\infty} A_i \in \Im$.

## Remarks (1.2) [47]:

1. In general a $s$ - *field* is a *filed*, but a *filed* may not be a $s$ - *field*
2. The intersection of any nonempty but arbitrary collection of $s$ - fields in $\Omega$ is a $s$ - *field* in $\Omega$.
3. In general the arbitrary union of a collection of $s$ - *field* may not be a $s$ - *field* .

We can always construct the smallest $s$ – *field* over $\Re$ which will $\Re$ contain $\Re$ and will be denbted by $s(\Re) = \Im$.

This will always exist since $s(\Re)$ can be defined as the intersection of all $s$ – *filed* containing $\Re$.

If $s_1(\Re), s_2(\Re), \ldots$ are all $s$ – *fields* containing $\Re$, then

$$s(\Re) = \bigcap_{i=1}^{\infty} s_i(\Re)$$

Further the minimal $s$ – *filed* thus generated is unique, we shall call $s(\Re)$ the $s$ – *filed* generated by $\Re$.

## Definition (1.3) (Borel σ- Field) [47]:

The minimum $s$ - field generated by the collection of open sets of a topological space $\Omega$ is called the *Borel* $s$ – *filed* or *Borel field* .

## Remarks (1.3) [47]:

1. Members of $s$ – *filed* of (definition (1.3)) are called ***Borel sets***.
2. The *Borel* $s$ – *field* is a $s$ – *filed*, and hence each closed set is also a *Borel* set.
3. The space $(\Omega, \Im)$ thus created is called ***a measurable space***.
4. Subsets of $\Omega$ which are elements in the $s$ – *filed* are called ***events***.
5. Elements of $\Omega$ are points.

6. If $\{A_i, i=1,2,...n\}$ is a class of disjoint sets of $\Omega$ such that $\bigcup_{i=1}^{n} A_i = \Omega$ then the $\{A_i\}$ collectively exhaust $\Omega$ and the class $\{A_i\}$ is called ***a partition*** of $\Omega$ .

## *1.1.2 PROBABILITY SPACE [37], [41]:*

### *Definition (1.4) (Probability Measure) [41]:*

A probability measure is a set function $P$ defined on a $s$ - field $\Im$ of subsets of a sample space $\Omega$ such that it satisfies the following axioms of kolmogorv for any $A \in \Im$ :

1- $P(A) \geq 0$ (non negativity)

2- $P(\Omega) = 1$ (normalization)

3- $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$ ( $s$ - additively)

With $A_n \in \Im$, and $A_i$ and $A_j$ begging pairwise disjoint.

Any set function $m$ defined on a measurable space $(\Omega, \Im)$ satisfying axioms (1) and (3) is called a measure.

### *Definition (1.5) (Probability Space) [7]:*

The measure space $(\Omega, \Im, P)$ is called a probability space, which serves to describe any random experiment where:

1. $\Omega$ is a nonempty set called the sample space, whose elements are the elementary outcomes of a random experiment.
2. $\Im$ is a $s$ - field of subsets of $\Omega$.
3. P is a probability measure defined on the measurable space $(\Omega, \Im)$.

***Remarks (1.4) [47]:***

1. Let $\{A_n\}$ be a monotone decreasing sequence of events in $\Im$ such that $A_{n+1} \subset A_n$, and let $\lim_{n\to\infty} A_n = \emptyset$. Then $\lim_{n\to\infty} P(A_n) = 0$. The probability measure is said to satisfy the sequential monotone continuity at $\emptyset$.

2. Let $\{A_n\}$ be a convergent sequence of events in $\Im$, with $\lim_n A_n = A$. Then $\lim_{n\to\infty} P(A) = P(\lim_{n\to\infty} A_n) = P(A)$ the probability measure is sequentially continuous.

## ***1.2 A RANDOM VARIABLES [7], [45], [47]:***

An important class of functions are the measurable functions which are different from the measure function $m$. Whereas measure functions are set functions, measurable functions are invariably point function.

***Definition (1.6) (Measurable Function) [47]:***

Let $(\Omega_1, \Im_1)$ and $(\Omega_2, \Im_2)$ be two measurable spaces. Let g be a function with domain $E_1 \subset \Omega_1$ and range $E_2 \subset \Omega_2$

$$g : \Omega_1 \to \Omega_2$$

Then g is called an $\Im_1$-measurable function or an $\Im_1$-measurable mapping, if for every $E_2 \in \Im_2$.

$$g^{-1}(E_2) = \{w : g(w) \in E_2\} = E_1$$

Is in the $s - field$ $\Im_1$.

***Remarks (1.5) [47]:***

1. The set $E_1$ given by $g^{-1}(E_2)$ is called the ***inverse image or inverse mapping*** of $E_2$, and it is measurable set.
2. Let $g$ be a measurable mapping from $(\Omega_1, \Im_1) \to (\Omega_2, \Im_2)$. If $\Re$ is a nonempty class of subsets of $\Omega_2$, then

$$s\left(g^{-1}(\Re)\right) = g^{-1}\left(s\,(\Re)\right).$$

***Definition (1.7) (Random Variable) [7]:***

Let $(\Omega, \Im)$ be a measurable space and $(R, \Re)$ another measurable space consisting of the real line $R$ and the $s$-field of Borel sets $\Re$. Let the probability measure P be defined on $(\Omega, \Im)$. The measurable mapping $X$ from $(\Omega, \Im)$ in to $(R, \Re)$ is called ***a real-valued random variable***.

## *1.2.1 Properties of Real-Valued Random Variables [45]:*

1. Let $\{X_n, n = 1, 2, \mathbf{K}, N\}$ be a convergent sequence of real-valued random variables converging to a limit $X$.
2. Let $\{X_n, n = 1, 2, \mathbf{K}, N\}$ be a convergent sequence of real-valued random variable. Then the set on which $\{X_n\}$ convergence is measurable.

***Definition (1.8) (Absolute Continuity of Signed Measures) [47]:***

Let $(\Omega, \Im)$ be a measurable space and let $m$ and $v$ be signed measures on $\Im$. We say that $v$ is absolutely continuous with respect to $m$ if for every measurable set $A \in \Im$, $v(A) = 0$, for which $m(A) = 0$.

## 1.2.2 CONVERGENCE OF RANDOM VARIABLES [47]:

We discuss the convergence of sequences of random variables $\{X_n\}$ where the probability measure plays an important role. The pointwise convergence of any sequence $\{X_n\}$ to a limit $X$ is defined as follows.

### Definition (1.9) (Point wise Convergence)[47]:

A sequence $\{X_n\}$ converges to a limit $X$ if and only if for any $e > 0$, however small , we can find an integer $n_0$ such that

$$| X_n - X | < e \qquad \text{for every } n > n_0.$$

### Remark (1.6) [47]:

If we consider a sequence of random variables $\{X_1, X_2, \mathbf{K}, X_n, \mathbf{K}\}$ and defined a pointwise convergence to another random variable $X$ as in definition (1.9), then we must have for every $w$ – point in $\Omega$ the sequence of numbers $X_1(w), X_2(w), \mathbf{K}, X_n(w)$ converging to $X(w)$. This type of convergence is called ***everywhere convergence***.

### Definition (1.10) (Almost Sure Convergence) [32]:

A sequence of random variables $\{X_n\}$ converges almost surely (a.s), or almost certainty or strongly to $X$ if for every $w$-point not belonging to the null event $A$,

$$\lim_{n\to\infty} \left| X_n(w) - X(w) \right| = 0$$

This type of convergence is known as ***convergence with probability 1*** and is denoted by

$$X_n(w) \xrightarrow[n\to\infty]{a.s} X(w) \ \text{ or } \ X(w) = \lim_{n\to\infty} X_n(w) \quad (a.s.)$$

***Remark (1.7) [32]:***

If the limit $X$ is not known a priori, then we can define ***a mutual convergence almost surely***. The sequence $X_n$ converges mutually almost surly if $\sup_{m \geq n} \left| X_m - X_n \right| \xrightarrow[n\to\infty]{a.s.} 0$

Both definitions are equivalent.

***Remark (1.8) [32]:***

Let $A_1, \ldots, A_n, \ldots$ be events in a probability space. Then the event

$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \{ w \in \Omega \,|\, w$ belongs to infinitely many of the $A_n \}$,

is called "$A_n$ ***infinitely often***" , $A_n$ *i.o.*".

***Definition (1.11) (Convergence in Probability) [7]:***

A sequence of random variables $\{X_n\}$ converges in probability to X if for every $e > 0$, however small, $\lim_{n\to\infty} P(| X_n - X | \geq e) = 0$

Or, equivalently,

$$\lim_{n\to\infty} P(| X_n - X | < e) = 1$$

It is denoted by

$$X_n(w) \xrightarrow[n\to\infty]{l.i.p.} X(w) \quad or \quad X(w) = \underset{n\to\infty}{l.i.p.} X_n(w)$$

***Remarks (1.9) [47]:***

1- We can define mutual convergence in probability as:

$$\lim_{n\to\infty} \sup_{m\geq n} P(| X_m - X_n |\geq e ) \to 0$$

2- If a sequence of random variables $\{X_n\}$ converges almost surely to $X$ , then it converges in probability to the same limit. The converse is not true.

3- If $\{X_n\}$ converges in probability to $X$ ,then there exists a subsequence $\{X_{nk}\}$ of $\{X_n\}$ which converges almost surely to the same limit.

4- $\{X_n\}$ converges in probability if and only if it converges mutually in probability.

## 1.3 INTRODUCTION TO STOCHASTIC PROCESSES [41],[45]:

We have looked at single random variable, and finite collections of random variables $(X_1, X_2, ..., X_n)$, which we termed random vectors. However, many practical applications of probability are concerned with random processes evolving in time, or space, or both, without any limit on the time (or space) for which this may continue.

### Definition (1.13) (Stochastic Process) [7]:

A stochastic process is a collection of random variables $\{ X(t) : t \in T \}$ where t is a parameter that runs over an index set T.

In general we call t the time-parameter (or simple the time), and $T \subseteq \mathbb{R}$ . Each $X(t)$ takes values in some set $S \subseteq \mathbb{R}$ called the state space, then $X(t)$ is the state of the process at time t.

***Remarks (1.10) [7]:***

1. If the index set $T$ is a countable set, we call $X$ a discrete-time stochastic process, and if $T$ is a continuum, we call it a continuous-time process.
2. A continuous-time stochastic process $\{X(t), t \in T\}$ is said to have independent increments if for all $t_0 < t_1 < t_2 < \mathbf{K} < t_n$, the random variables $X(t_1) - X(t_0), X(t_2) - X(t_1), \mathbf{K}, X(t_n) - X(t_{n-1})$ are independent. It is said to possess stationary increments if $X(t+s) - X(t)$ has the same distribution for all t. That is, it possesses independent increments if the changes in the processes value over nonoverlapping time intervals are independent, and it possesses stationary increments if the distribution of the change in the value between any two points depends only the distance between those points.

***Definition (1.13) (Vector Stochastic Process) [45]:***

Suppose that $X_1(t), X_2(t), ..., X_n(t)$ are $n$ scalar stochastic processes which are possibly mutually dependent. Then we call

$$X(t) = [X_1(t), X_2(t), ..., X_{n(t)}]^T$$

A vector stochastic process.

We always assume that each, of the components of $X(t)$ takes real values, and that $t \geq t_0$, with $t_0$ given.

***Remarks (1.11) [45]:***

1. A stochastic process can be through of as a family of time functions. Each time function we call a realization of the process.
2. A stochastic process can be characterized by specifying the joint probability distributions.

$$P\{X(t_1) \le X_1, X(t_2) \le X_2, \ldots, X(t_m) \le X_m\}$$

for all real $X_1, X_{2,} \ldots, X_m$. For all $t_1, t_2, \ldots, t_m \ge t_0$ and for every natural $m$.

***Definition (1.14) (Stationary) [45]:***

A stochastic process X (t) is stationary if:

$$P\{\, X(t_1) \le X_1, \ldots, X(t_m) \le X_m \,\} = P\{\, X(t_1 + \theta) \le X_1, \ldots, X(t_m + \theta) \le X_m \,\}$$

for all $t_1, t_2, \ldots, t_m$, and all real $X_1, \ldots, X_m$. For every natural number $m$ and for all $\theta$.

***Definition (1.15) (Covariance Matrix) [45]:***

Consider a vector-valued stochastic processes $X(t)$. Then we call

$$m(t) = E\{X(t)\}$$

the ***mean*** of the processes,

$$R_x(t_1, t_2) = E\{[X(t_1) - m(t_1)][X(t_2) - m(t_2)]^T\}$$

the ***covariance matrix***, and

$$C_x(t_1, t_2) = E\{X(t_1) X^T(t_2)\}$$

The second- order joint moment matrix of $X(t)$. $R_x(t, t) = Q(t)$ is termed as the ***variance matrix***, while $C_x(t, t) = Q'(t)$ is the second-order moment matrix of the processes.

***Remark (1.12) [45]:***

Here $E$ is the expectation operator. We shall often assume that the stochastic processes under consideration has zero mean, that is, $m(t) = 0$ for all $t$; in this case the covariance matrix and the second-order joint moment matrix coincide. The joint moment matrix written out more explicitly is

$$C_x(t_1,t_2) = E\{X(t_1)X^T(t_2)\} = \begin{pmatrix} E\{X_1(t_1)X_1(t_2)\} & \cdots & E\{X_1(t_1)X_m(t_2)\} \\ E\{X_2(t_1)X_1(t_2)\} & \cdots & E\{X_2(t_1)X_m(t_2)\} \\ \vdots & \cdots & \vdots \\ E\{X_m(t_1)X_1(t_2)\} & \cdots & E\{X_m(t_1)X_m(t_2)\} \end{pmatrix}$$

Each element of $C_x(t_1,t_2)$ is a scalar joint moment function. Similarly, each element of $R_x(t_1,t_2)$ is a scalar covariance function.

***Theorem (1.1) [45]:***

Suppose that $X(t)$ is a stationary stochastic process. Then its mean $m(t)$ is constant and its covariance matrix $R_x(t_1,t_2)$ depends on $t_1 - t_2$ only.

***Definition (1.16) (Wide Sense Stationary) [47]:***

The stochastic processes $X(t)$ is called ***wide-sense stationary*** if its second order moment matrix $C_x(t,t)$ is finite for all $t$ ,its mean $m(t)$ is constant, and its covariance matrix $R_x(t_1,t_2)$ depends on $t_1 - t_2$ only.

***Remark (1.13) [45]:***

Any stationary process with finite second-order moment matrix is also wide-sense stationary.

***Example (1.4) [45]:***

Let us consider a stochastic process consisting of a sequence $\{X_1, X_2, \ldots\}$ of independent identically distributed random variables with mean $m$ and variance $s^2$.

The auto covariance $\sigma_x(h) = \sigma^2\delta_h$, where h is the lag and $\delta_h$ is the Kroneker delta.

this process is wide sense stationary.

***Example (1.5) [45]:***

Let us consider another process given by $Y_1 = Y_2 = \mathbf{K} = Y$ with mean $m$ and variance $s^2$. Here $C_x(h) = s^2$ for all $h$. This process is also wide sense stationary.

***Definition (1.18) (Brownian motion Process) [32]:***

One of the best-known processes with uncorrelated increment is that Brownian motion process, also known as the ***Wiener process*** or the ***Wiener-Lery process***. This is a process, with un correlated increments where each of the increments $X(t_2) - X(t_1)$ is a Gaussian stochastic vector with zero mean and variance matrix $(t_2 - t_1)I$, where I is the unit matrix.

A random process $W(t), t \geq 0$, is said to be a Wiener process if,

(a) $W(0) = 0$

(b) $W(t)$ is continuous in $t \geq 0$

(c) $W(t)$ has independent increments such that $W(t+s) - W(s)$ has the normal distribution $N(\mu, \sigma^2 t)$, for all $s, t \geq 0$ and some $0 < \sigma^2 < \infty$.

***Remark (1.14) [7]:***

1. If $s^2 = 1$, then $W(t)$ is said to be the standard Wiener process, we always make this assumption unless stated otherwise.
2. In fact the assumption (b) is not strictly necessary, in that one can constract (by a limiting procedure) a random process $W(t)$ that obeys (a) and (c) and is almost surely continuous. This is Wiener's theorem.

***Theorem (1.2) [41]:***

Let $W(t)$ be a Brownian motion and let $0 = t_0 < t_1 < \mathbf{L} < t_n$. Then, for any $l > 0$,

$$P\left[\max_{0\le j\le n} W(t_j) > l\right] \le 2P\left[W(t_n) > l\right],$$

$$P\left[\max_{0\le j\le n} \left|W(t_j)\right| > l\right] \le 2P\left[\left|W(t_n)\right| > l\right].$$

## *1.4 BROWNAIN MOTION IN N-DIMENTIONS:*

***Definition [41]:***

An n-dimensional processes $W(t) = (W_1(t), \mathbf{K}, W_n(t))$ is called an n-dimensional Brownian motion (or Wiener processes) if each processes $W_i(t)$ is a Brownian motion and if the $s - fields\ \Im\ (W_i(t), t \ge 0), 1 \le i \le n,$ are independent.

***Lemma (1.1) [32]:***

Suppose that $W(\cdot)$ is a one dimensional Brownian motion. Then $E(W(t)W(s)) = t \wedge s = \min\{s,t\}$ for $t \ge 0,\ s \ge 0$

**Proof**

Assume $t \ge s \ge 0.$ Then

$$E(W(t)W(s)) = E((W(s) + W(t) - W(s))W(s))$$

$$= E(W^2(s) + E((W(t) - W(s))W(s))$$

$$= s + E(W(t) - W(s))E(W(s))$$

$$= s = t \wedge s.$$

Since $W(s)$ is $N(0,s)$ and $W(t) - W(s)$ is independent of $W(s)$.

## 1.5 THE RESPONSE OF LINEAR DIFFERENTIAL SYSTEM TO WHITE NOISE [45]:

One frequently encounters in practice zero-mean scalar stochastic processes w with the property that $w(t_1)$ and $w(t_2)$ are uncorrelated even for values of $|t_1 - t_2|$that are quite small, that is,

$R_w(t_2,t_1)$ **;** $0$ for $|t_1 - t_2| > \varepsilon$where $\varepsilon$ is a "small" number.

The covariance function of such stochastic processes can be idealized as follows:

$$R_w(t_2 - t_1) = X(t_1)\delta(t_2 - t_1), \text{ where } X(t_1) \geq 0$$

Here $\delta(t_2 - t_1)$ is a delta function and $X(t_1)$ is referred to as the intensity of the process at time t. such process are called white noise process.

***Definition (1.20) (White Noise) [41]:***

Let $X(t)$ be a zero mean vector-valued stochastic process with covariance matrix

$$R_w(t_2,t_1) = X(t_1)\delta(t_2 - t_1), \text{ where } X(t_1) \geq 0.$$

the process $w(t)$ is then said to be a white noise stochastic process with intensity $X(t_1)$.

***Theorem (1.4) [45]:***

Let $w(t)$ be a vector-valued white noise process with intensity $X(t_1)$, also, let $A_1(t), A_2(t)$and $A_n(t)$ be a given time-varying matrices. Then

(a) $E\left\{\int_{t_1}^{t_2} A(t)w(t)dt\right\} = 0$

(b)
$$E\left\{\left[\int_{t_1}^{t_2} A_1(t)w(t)dt\right]^T S\left[\int_{t_3}^{t_4} A_2(t')w(t')dt'\right]\right\}$$
$$=\int_I tr\left[V(t)A_1^T(t)SA_2(t)\right]dt$$

Where I is the intersection of $[t_1,t_2]$ and $[t_3,t_4]$ and $S$ is any weighting matrix

(c)
$$E\left\{\left[\int_{t_1}^{t_2} A_1(t)w(t)dt\right]\left[\int_{t_3}^{t_4} A_2(t')w(t')dt'\right]^T\right\}$$
$$=\int_I A_1(t)V(t)A_2^T(t)dt$$

Where I is as defined before.

***<u>Remark (1.15) [45]:</u>***

A special case that is of considerable interest occurs when the processes $X(t)$ from which the white noise processes derives is the Brownian motion. The white noise processes then obtained is often referred to as Gaussian white noise.

## ***<u>1.5.1 Linear Differential System Driven by White Noise [32 [45]:</u>***

A linear differential system driven by white noise is a very convenient model for formulating and solving linear control problems that involve disturbances and noise.

In this section we obtain some of the statistical properties of the state of a linear differential system with a white noise process as an input.

In particular, we compute the mean, the covariance, Joint moment, variance and moment matrices of the state $X$ .

***Theorem (1.5) [45]:***

Suppose that $x(t)$ is the solution of

$$\dot{x} = A(t)x(t) + B(t)w(t),$$
$$x(t_0) = x_0,$$

Where $w(t)$ is white noise with intensity $V(t)$ and $x_0$ is a stochastic variable independent of $w(t)$, with mean $m_0$ and $Q_0 = E\left\{(x_0 - m_0)(x_0 - m_0)^T\right\}$ as its variance matrix. Then $x(t)$ has mean $m_x(t) = \Phi(t,t_0)m_0$

where $\Phi(t,t_0)$ is the transition matrix of the system . The covariance matrix of $x(t)$ is

$$R_x(t_1,t_2) = \Phi(t_1,t_0)Q_0\Phi^T(t_2,t_1) + \int_{t_0}^{min(t_1,t_2)} \Phi(t_1,\tau)B(\tau)V(\tau)B^T(\tau)\Phi^T(t_2,\tau)d\tau$$

The variance matrix $Q(t) = R_x(t,t)$ satisfies the matrix differential equation

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A^T(t) + B(t)V(t)B^T(t)$$
$$Q(t_0) = Q_0$$

*Furthermore,*

$$R_x = \begin{cases} Q(t_1)\Phi^T(t_2,t_1), & t_2 \ge t_1 \\ \Phi(t_1,t_2)Q(t_2) & t_1 \ge t_2 \end{cases}$$

The second-order joint moment matrix of $x(t)$ is:

$$C_x(t_1,t_2) = E\left\{x(t_1)x^T(t_2)\right\}$$
$$= \Phi(t_1,t_0)C_x(t_0,t_0)\Phi^T(t_2,t_0)$$
$$+ \int_{t_0}^{min(t_1,t_2)} \Phi(t_1,\tau)B(\tau)V(\tau)B^T(\tau)\Phi^T(t_2,\tau)d\tau$$

The moment matrix $C_x(t,t) = Q'(t)$ satisfies the matrix differential equation

$$\dot{Q}'(t) = A(t)Q'(t) + Q'(t)A^T(t) + B(t)V(t)B^T(t)$$
$$Q'(t_0) = E\{x_0 x_0^T\}$$

Finally;

$$C_x(t_1,t_2) = \begin{cases} Q'(t_1)\Phi^T(t_2,t_1), & t_2 \ge t_1 \\ \Phi(t_1,t_2)Q'(t_2), & t_1 \ge t_2 \end{cases}$$

***Example (1.6) [45]:***

We consider the first-order stochastic differential equation

$$x(t) = -\frac{1}{q} z(t) + w(t),$$

Where $w(t)$ is scalar white noise with constant intensity $m$. Let us suppose that $z(0) = z_0$, where $z_0$ is scalar stochastic variable with mean zero and variance $E(z_0^2) = s^2$. $z(t)$ has the covariance function

$$R_z(t_1,t_2) = \left(s^2 - \frac{mq}{2}\right) e^{-(t_1 - t_2)/q} + \frac{mq}{2} e^{-|t_1 - t_2|/q}, \quad t_1, t_2 \ge 0.$$

The variance matrix of the processes is

$$Q(t) = \left(s^2 - \frac{mq}{2}\right) e^{-2t/q} + \frac{mq}{2}, \quad t \ge 0.$$

## ***1.3.1 Modeling of Stochastic Processes [41]:***

Suppose that $X(t)$ is given by

$$X(t) = C(t)z(t),$$

With

$$\dot{z}(t) = A(t)z(t) + B(t)w(t),$$

Where $w(t)$ is white noise. Choosing such a representation for the stochastic processes $X$, we call the modeling of the stochastic processes $X$. The use of such models can be justified as follows.

1. Very often practical stochastic phenomena are generated by very fast fluctuations which act upon a much slower differential system. In this case the model of white noise acting upon a differential system is very appropriate. A typical example of this situation is thermal noise in an electronic circuit.
2. As we shall see, in linear control theory almost always only the mean and covariance of the stochastic processes matter. Through the use of a linear model, it is always possible to approximate any experimentally obtained mean and covariance matrix arbitrarily closely.
3. Sometimes the stochastic processes to be modeled is a stationary processes with known power spectral density matrix. Again, one can always generate a stochastic processes by a linear differential equation driven by white noise so that it is power spectral density matrix approximates arbitrarily closely the power spectral density matrix of the original stochastic processes.

***Example (1.7) [41]:***

Suppose that the covariance function of a stochastic scalar processes $v$, which is known to be stationary, has been measured and turns out to be the exponential function

$$R_x(t_1,t_2) = s^2 e^{-|t_1-t_2|/q}$$, where $a$ and $q$ are constsnts

One can model this processes for $t \geq t_0$ as the state of a first-order differential system:

$$\dot{X}(t) = -\frac{1}{q} X(t) + w(t),$$

With $w(t)$ white noise with intensity $2s^2/q$ and where $X(t_0)$ is a stochastic variable with zero mean and variance $s^2$.

# CHAPTER TWO

# STOCHASTIC INTEGRAL AND STOCHASTIC DIFFRERNTIAL EQUATIONS

## 2.1 BASIC CONCEPT AND DEFFINITIONS [41]:

The following definitions and concepts are needed to understand some principle of this work:

***Definition (2.1) (Increasing σ-field) [41]:***

Let $(\Omega,\Im)$ be a complete measurable space and let $\{\Im_t, t\in T, T=R^+\}$ be a family of sub-$s$ -fields of $\Im$ such that for $s\le t$, $\Im_s \subset \Im_t$. Then $\{\Im_t\}$ is called an ***increasing family of sub-$s$ -fields*** on $(\Omega,\Im)$ or ***the filtration $s$ -field*** of $(\Omega,\Im)$.

$\Im_t$ is called the $s$ -field of events prior to t. If $\{X_t, t\in T\}$ is a stochastic process defined on $(\Omega,\Im,P)$ then $\Im_t$ given by

$$\Im_t = s\{X_s, s\le t, t\in T\} \tag{2.1}$$

Is increasing.

***Remark (2.1) [41]:***

Since the probability space $(\Omega,\Im,P)$ is complete, the $s$ -field $\Im$ contains all subsets of $\Omega$ having probability measure zero. We shall assume here that the filtration $s$ -field $\{\Im_t, t\in T\}$ also contains all the sets from $\Im$ having probability measure zero.

***Definition (2.2) (continuity concepts for the filtration σ -field) [41]:***

The filtration $s$ -field $\{\Im_t, t\in T, T=R^+\}$ is right continuous if

$$\Im_t = \Im_{t+} = \mathbf{I}_{t>t} \Im_t \text{ , for all } t\in T \tag{2.2}$$

and left continuous if

$$\Im_t = \Im_{t-} = s\left\{\bigcup_{t<t} \Im_t\right\} = \max_{t<t} \Im_t \tag{2.3}$$

***Definition (2.3) (Adaptation of {X_t} [41] :***

Let $\{X_t, t \in T, T = R^+\}$ be a stochastic process defined on a probability space $(\Omega, \Im, P)$ and let $\{\Im_t, t \in T, T = R^+\}$ be a filtration $s$ -field. The process $\{X_t\}$ is adapted to the family $\{\Im_t\}$ if $X_t$ is $\Im_t$ -measurable for every $t \in T$ . Or

$$E^{\Im_t} X_t = X_t \qquad t \in T$$

$\Im_t$ ***-adapted*** random process are also $\Im_t$ -measurable and nonanticipative with respect to the $s$ -field $\Im_t$ .

***Definition (2.4) (Increasing Process) [41]:***

Let $(\Omega, \Im, P)$ be a probability space, and let $\{\Im_t, t \in T\}$ be a right continuous filtration $s - field$ defined on it. A real right continuous stochastic process $\{A_t, t \in T\}$ is an ***increasing process*** with respect to the family $\{\Im_t, t \in T\}$ if:

1. $A_0 = 0$
2. $A_t$ is $\Im_t$ -measurable
3. $A_s \le A_t$ for $s \le t$ (a.s)
4. $EA_t < \infty$ for all $t \in T$

***Remark (2.2) [41]:***

If $T = R^+$ and $EA_\infty < \infty$, then the increasing process is integrable and if $T = [a,b]$ then the definition implies that the process is integrable. In order work we assume that the process $\{A_t, t \in T\}$ is integrable.

***Definition (2.5) (Predictable Process) [41]:***

An integrable increasing process is ***predictable*** (also called natural) if for all $t \in T$,

$$E\int_0^t Y_s dA_s = E\int_0^t Y_{s-} dA_s \tag{2.4}$$

for any nonnegative bounded right continuous $\Im_t$-martingale $\{Y_t, t \in T\}$.

As a consequence of the definition we have the following proposition.

***Proposition (1):***

Let $\{A_t, t \in T\}$ be an integrable increasing process. Then $A_t$ is predictable if and only if

$$E\int_0^\infty Y_s dA_s = E\int_0^\infty Y_{s-} dA_s \tag{2.5}$$

for any nonnegative bounded right continuous $\Im_t$-martingale $Y_t$.

***Proposition of Increasing Predictable Process (2):***

If $X_t$ is any $\Im_t$-sub martingale, the $\Im_t$-increasing predictable process $\{A_t, t \in T\}$ can be found as a weak limit (Meyer's weak limit) by

$$A_t = \lim_{h \downarrow 0} \int_0^t \frac{E^{\Im_s}(X_{s+h} - X_s)}{h} ds$$

That is,

$$A_t = \int_0^t E^{\Im_s} X_s ds \tag{2.6}$$

Or

$$dA_t = E^{\Im_t} dX_t \tag{2.7}$$

***Remarks (2.3) [41]:***

All predictable processes may not be increasing. In fact, from equation (2.5) and (2.6) we can conclude that: all continuous $\Im_t$-adapted processes are predictable, not necessarily increasing. (A continuous $\Im_t$-adapted martingale $X_t$ may not be given by equation (2.5) and (2.6) since $E^{\Im_s}(X_{s+h} - X_s) = 0, h \geq 0.$)

***Definition (2.6) (Continuous Martingale) [41]:***

Let $(\Omega, \Im, P)$ be a probability space, and let $\{\Im_t, t \in T\}$ be a filtration $s$-field defined on it and $\{X_t, t \in T\}$ a real-valued stochastic process adapted to $\{\Im_t, t \in T\}$. Then the process $\{X_t, t \in T\}$ is a ***martingale*** with respect to the family $\{\Im_t, t \in T\}$ if

1. $E|X_t| < \infty$.
2. For all $s, t \in T$ and $s \leq t$,

$$E^{\Im_s} X_t = X_s \quad \text{(a .s.)} \tag{2.8}$$

$X_t$ is a ***submartingale*** if

$$E^{\Im_s} X_t \geq X_s \quad \text{(a.s.)}$$

and $X_t$ is a ***supermartingale*** if

$$E^{\Im_s} X_t \leq X_s \quad \text{(a.s.)}$$

***Remarks (2.4) [41]:***

1. Since $E^{\Im_s} X_t = X_s$ for a martingale, we have for $s \leq t$,

$$EE^{\Im_s} X_t = EX_s \Rightarrow EX_t = EX_s$$

As a consequence if $X_0$ is the initial value and if $X_t$ is a martingale, then

$$EX_t = EX_0 \text{ for all } t \in T.$$

2. For $s \le t, E^{\Im_s} X_t$ is increasing for a submartingale and decreasing for a supermartingale.

***Definition (2.7) (Right Continuous Martingales) [41]:***

The martingale (submartingale) $\{X_t, \Im_t, t \in T\}$ is right continuous if

1. The sample paths of $X_t$ are right continuous almost surely.
2. The filtration $s$ -field $\{\Im_t, t \in T\}$ is a right continuous, that is,

$$\Im_t = \bigcap_{s>t} \Im_s = \Im_{t+} \qquad t \in T .$$

***Example (2.1) [41]:***

Let $Z$ be any integrable random process defined on $(\Omega, \Im, P)$, and let $\{\Im_t, t \in T\}$ be the filtration $s$ -field. Then the stochastic process

$$X_t = E^{\Im_t} Z$$

is a martingale because for $s \le t$ we can write

$$E^{\Im_s} X_t = E^{\Im_s} E^{\Im_t} Z = X_s$$

***Example (2.2) [41]:***

Let $\{X_t, t \in T\}$ be integrable stochastic process, adapted to $\{\Im_t, t \in T\}$, with independent increments, that is, for $s \le t$, $X_t - X_s$ is independent of the $s$ -field $\Im_t$. Then the process $\{X_t - EX_t, t \in T\}$ is an $\Im_t$-martingale, since

$$E^{\Im_s}(X_t - EX_t) = E^{\Im_s}(X_t - EX_t - X_s + EX_s + X_s - EX_s)$$

$$= E^{\Im_s}(X_s - EX_s) + E^{\Im_s}(X_t - X_s - EX_t + EX_s)$$

$$= E^{\Im_s}(X_s - EX_s)$$

Because of independent increaments.

***Definition (2.8) [41]:***

Let $\{X_t, t \in T\}$ be a square integrable martingale belonging to the family of martingales$\{M_t, \Im_t, t \in T\}$. The quadratic variance process of the $L^2$-martingale $\{X_t, t \in T\}$ is defined as:

$$\langle X, X \rangle_t = \lim_{n \to \infty} \sum_{v=0}^{N(n)-1} E^{\Im_{t_v}} (X_{t_{v+1}^{(n)}} - X_{t_v^{(n)}})^2 .$$

## 2.2 Stochastic Integral [47]:

We defined on a probability space $(\Omega, \Im, P)$ a simple stochastic integral of the form $\int_{-\infty}^{\infty} g(t) dZ_t$ where $g(t)$ is a function of $t$ only and $Z_t$ was a processes of orthogonal increments corresponding to a white noise processes $X_t$. A generalization of this simple stochastic integral is quantity of the form

$$I_t(j) = \int_0^t j(w,s) dW_s \qquad t \in R^+$$

(2.9)

Where $R^+$ is the positive real line and $W_t$ is a Brownian motion processes defined on a complete probability space $(\Omega, \Im, P)$ satisfying the following usual conditions with $s \geq 0$:

$$EW_t = 0$$

$$E^{\Im_t} W_{t+s} = W_t \tag{2.10}$$

$$E^{\Im_t} (W_{t+s} - W_t)^2 = s^2 s$$

In equation (2.9) the integrand $j$ depends upon $w$, and since $W_t$ is neither differentiable nor of bounded variation, the integral $I_t(j)$ has to be defined properly. The integral when $j$ is independent of $w$ has been defined that the orthogonal increment property of $W_t$. For the stochastic integral $I_t(j)$ to be properly defined, the integrand $j(w,t)$ has to satisfy the following conditions:

1. If $\mathbf{l}$ is the $s$ – field of Borel sets on the positive real line, then $j$ is jointly measurable in the product $s$ – field $\Im \otimes \mathbf{l}$,

$$j \in \Im \otimes \mathbf{l} \qquad (2.11a)$$

2. If $\{\Im_t, t \in T, T = R^+\}$ is a right continuous filtration $s$ -field of the probability space, then for each $t \in T$, $j(w,t)$ is adapted to $\Im_t$,

$$j(w,t) \in \Im_t \qquad (2.11b)$$

As already seen, such functions are called nonanticipative with respect to the family $\{\Im_t, t \in T\}$.

3. For each $t \in T, j(w,t)$ satisfying

$$\int_0^t E\left|j(w,s)\right|^2 ds < \infty \quad \text{(a.s.)} \qquad (2.11c)$$

This condition can be weakened into

$$\int_0^t \left|j(w,s)\right|^2 ds < \infty \quad \text{(a.s.)}$$

4. $j(w,t)$ belongs to a class of left continuous functions.

The processes $j(w,t)$ under the four conditions given above is a predictable processes with respect to the filtration $s$ – field $\{\Im_t, t \in T, T = R^+\}$.

***Definition (2.9) (Simple Processes) [41]:***

A function $g(w,t)$ is called ***simple*** if, for the partitions $0 \le t_0 \le t_1 \le \mathbf{L} \le t_n = b$ of the interval $T=[0,b]$, it can be represented in the form

$$g(t,w)=\sum_{v=0}^{n-1} g_v(w) I_{(t_v,t_{v+1})}(t) \tag{2.12}$$

Where the $g_v$ are $\Im_{t_v}$ $-$measurable.

Since the simple function $g(w,t)$ is left continuous and adapted to the filtration $s-$field$\{\Im_t, t\in T, T=[0,b]\}$. Therefore it is a predictable process satisfying conditions (2.11). It is not essential that the function $g(w,t)$ be defined as left continuous. The left continuity and hence predictability plays a crucial role in the definition of the stochastic integral.

***Remark (2.5) [41]:***

Let $\{j(w,t), t\in T\}$ be a random processes satisfying conditions (2.11). Then there exist a sequence of simple processes $\{j_n(w,t), n=0,1,2,\mathbf{K}, t\in T\}$ satisfying conditions (2.11) such that

$$\lim_{n\to\infty} \int_T E\left|j(w,t)-j_n(w,t)\right|^2 dt \to 0$$

***Definition (2.10) (Stochastic Integral) [32]:***

Let $\{\Omega, \Im, P\}$ be a complete probability space, and let $\{\Im_t, t\in T, T=[0,b]\}$ be a filtration $s-$field. Let $\{W_t, t\in T\}$ be a Brownian motion martingale adapted to $\Im_t$.

1. If $\{j(w,t), t\in T\}$ is a simple processes of definition 2.1 given by

$$j(w,t) = \sum_{v=0}^{N(n)-1} j_v(w) I_{(t_v{}^{(n)}, t_{v+1}{}^{(n)})}$$

For partitions $0 \le t_0{}^{(n)} < \ldots < t_{N(n)}{}^{(n)} = t,$ then the stochastic integral is defined by

$$I_t(j) = \int_0^t j(w,s)dW_s = \sum_{v=0}^{N(n)-1} j_v(w)[W_{t_{v+1}{}^{(n)}} - W_{t_v{}^{(n)}}] \quad (2.13)$$

2. If $\{j(w,t), t \in T\}$ is a general processes satisfying conditions (2.11), then there exists a sequences of simple functions $\{j_n(w,t), n = 0,1,\ldots, t \in T\}$ approximating $j(w,t)$ in the quadratic mean. In this case the stochastic integral is defined as:

$$I_t(j) = \int_0^t j(w,s)dW_s = \underset{n\to\infty}{l.i.q.m.} \int_0^t j_n(w,s)dW_s$$

$$= \underset{n\to\infty}{l.i.q.m.} I_t(j_n) \quad (2.14a)$$

Thus the sequences of random variables $I_t(j_n) = \int_0^t j_n(w,s)dW_s$ converges in the quadratic mean to the random variable $I_t(j) = \int_0^t j(w,s)dW_s$, which is called the stochastic integral of the function $j(w,s)$ relative to the Brownian motion martingale $\{W_t, \Im_t, t \in T\}$. The limiting value (to within stochastic equivalence) of the integral $I_t(j_n)$ is independent of the choice of the approximating sequence $\{j_n\}$.

3. Under the weakened condition we define the approximating sequence $\{j_n(w,t), n = 0,1,\ldots, t \in T\}$ as

$$j_n(w,t)=\begin{cases} j(w,t) & \int_0^t |j_n(w,t)|dt < n \quad (a.s.) \\ 0 & otherwise \end{cases}$$

So that $\{j_n(w,t)\}$ converges in probability to $j(w,t)$ as $n\to\infty$.

In this case the stochastic integral is defined as

$$I_t(j)=\int_0^t j(w,s)dW_s$$

$$=\underset{n\to\infty}{l.i.p.}\, I_t(j_n) \qquad (2.14b)$$

## *2.2.1 PROPERTIES OF STOCHASTIC INTEGRALS [41]:*

The stochastic integral $I_t(j)$ as defined above satisfies the following basic properties:

1. $E\int_0^t j(w,s)dW_s = 0 \qquad t\in T$ (2.15)

2. $I_t(aj_1+bj_2)=aI_t(j_1)+bI_t(j_2) \qquad a,b$ are constants (2.16)

3. $\int_0^t j(s,w)dW_s = \int_0^t j(s,w)dW_s + \int_t^t j(s,w)dW_s$ (2.17)

4. $I_t(j)$ is progressively measurable for $t\in T$ and $j(t,w)$ satisfying conditions (2.11). In particular, for each $t, I_t(j)$ is $\Im_t$-measurable.

## *Example (2.3) [32]:*

Let $W(\cdot)$ be a 1-dimentional Wiener process, then $W(\cdot)$ is a Martingale.

To see this, we write $W(t)=U\left(W(s)|0\le s\le t\right)$ and let $t\ge s$, then

$$E\left(W(t)\middle|W(s)\right) = E\left(W(t)-W(s)\middle|W(s) + E\left(W(s)\middle|W(s)\right)\right)$$

$$= E\left(W(t)-W(s)\right) + W(s) = W(s) \qquad \text{(a.s.)}$$

***Lemma (2.1) [32]:***

Suppose $X(\cdot)$ is a real-valued martingale and $\Phi:$ ¡ $\to$ ¡ is convex, then if $E\left(\left|\Phi\left(X(t)\right)\right|\right) < \infty$ for all $t \geq 0$, $\Phi\left(X(\cdot)\right)$ is a sub-martingale.

***2.2.2 STOCHASTIC INTEGRAL AS A MARTINGALE [7]:***

Let $(\Omega, \Im, P)$ be a complete probability space, and let $\{\Im_t, t \in T\}$ be a right continuous filtration $s$ – field. Let $\{W_t, t \in T, T = [a,b]\}$ be a Brownian motion process. Let $j$ be a function satisfying conditions(2.11), namely,

1. $j(w,t) \in \Im \otimes \mathbf{l}$
2. $j(w,t) \in \Im_t$ for each $t$
3. $\int_a^b E\left|j(w,t)\right|^2 dt < \infty$
4. $j(w,t)$ belongs to the class of left continuous functions then the stochastic integral

$$X_t(j) = \int_a^t j(w,s)\,dW_s(w) \qquad t \in T \tag{2.18}$$

is an $\Im_t$-martingale satisfying the martingale property

$$E^{\Im_t} X_t = X_s \qquad s \leq t, \quad t \in T$$

***Example (2.4) [47]:***

Let an integral $I_t(W)$ be given by

$$I_t(W) = \int_0^t W_t(w) dW_t(w) \tag{2.19}$$

Where $W_t(w)$ is a Brownian motion process. The integrand $W_t(w)$ can be satisfy conditions (2.11), and $I_t(W)$ is a stochastic integral. Therefore $I_t(W)$ must be an $\Im_t$-martingale by proporties of (2.1.2).

If $I_t(W)$ is treated as an ordinary integral, then

$$I_t(W) = \int_0^t W_t \, dW_t = \frac{W_t^2}{2} \tag{2.20}$$

We now check whether $\frac{W_t^2}{2}$ is a martingale.

$$E^{\Im_t} \frac{W_t^2}{2} = \frac{1}{2} E^{\Im_t} (W_s^2 + W_t^2 - W_s^2)$$

$$= \frac{1}{2} W_t^2 + \frac{1}{2}(t-s)$$

Which is not a martingale. Therefore the conventional rule for integrations not applicable here.

## ***2.3 ITO PROCESS (GENERALIZED STOCHASTIC INTEGRAL) [7]:***

In the stochastic integral given by equation (2.9) the integration was carried out with respect to the Brownian motion $W_t$. The stochastic integral with respect to $W_t$ was carefully defined in equations (2.13),(2.14a) and (2.14b). However, in many nonlinear filtering problems the integration may have to be carried out not with respect to the Brownian motion process but with respect to an Ito process. We now define the Ito process under the weakened condition $\int_0^t |j(w,s)|^2 ds < \infty$ (a.s.).

***Definition (2.11) (Ito Process) [32]:***

Let $(\Omega, \Im, P)$ be a complete probability space, with $\{\Im_t, t \in T\}$ being a right continuous filtration $s-$field defined on it. Let $\{W_t, \Im_t, t \in T\}$ be a Brownian motion process. The continuous random process $\{X_t, \Im_t, t \in T\}$ is called an ***Ito process*** (relative to Brownian motion process $\{W_t, \Im_t, t \in T\}$ if there exist two nonanticipative $\Im_t$-measurable random processes $a_t(w)$ and $b_t(w)$, satisfying for each $t \in T$

$$\int_0^t |a_s(w)| ds < \infty \quad \text{(a.s.)} \tag{2.21}$$

$$\int_0^t |b_s(w)|^2 ds < \infty \quad \text{(a.s)} \tag{2.22}$$

With $b_t(w)$ being left continuous, and if, with probability 1, $X_t(w)$ satisfies the equation

$$X_t(w) = X_0(w) + \int_0^t a_s(w) ds + \int_0^t b_s(w) dW_s \tag{2.23}$$

The process is a basic stochastic differential equation which is discussed in its generality later. The existence of the stochastic integral $\int_0^t b_s(w) dW_s$ has been established under the weakened condition in definition (2.10). The existence of the integral $\int a_s(w) ds$ is guaranteed by condition (2.21). The left continuity condition for $a_t(w)$ is not necessary for the definition of this integral. Thus the integral equation (2.23) is well defined.

Equation (2.23) can also be given in a stochastic differential equation representation as

$$dX_t(w) = a_t(w)dt + b_t(w)dW_t \qquad t \in T, \quad X_0(w) \qquad (2.24)$$

However, equation (2.24), in general, is not equivalent to

$$\frac{dX_t}{dt} = a_t(w) + b_t(w)\frac{dW_t}{dt} \qquad t \in T \qquad (2.25)$$

Where $dW_t/dt$ is a white noise process.

### 2.3.1 A stochastic Integral over an Ito Process [47]:

Let $\{X_t, t \in T, T = [a,b]\}$ be an Ito processes as given in definition (2.11) .Let $j_t(w)$ be a nonanticipative process satisfying condition (2.11). The stochastic integral over the Ito process $X_t$ is given by

$$I_t(j) = \int_a^t j_t(w)dX_t(w) \qquad t \in T \qquad (2.26)$$

Or

$$I_t(j) = \int_a^t j_t(w)a_t(w)dt + \int_a^t j_t(w)b_t(w)dW_t \qquad (2.27)$$

and for both integrals to exist it is sufficient that

$$\int_T |j_t(w)a_t(w)|dt < \infty \qquad \text{(a.s.)}$$

$$\int_T |j_t(w)b_t(w)|^2 dt < \infty \qquad \text{(a.s.)}$$

The stochastic integral $I_t(j)$ over the Ito processes $X_t$ can also be well defined analogous to definition (2.10). By remark (2.5) an approximating sequence of simple functions $\{j_{nt}(w), t \in T\}$ can be found such that

$$\underset{n\to\infty}{l.i.p.}\int_T\left[|a_t(w)||j_t(w)-j_{nt}(w)|+|b_t(w)|^2|j_t(w)-j_{nt}(w)|^2\right]dt \to 0 \qquad (2.28)$$

Then the stochastic integral of equation (2.26) is the limit in probability of the integral sums $I_t(j_n)$ given by

$$I_t(j_n)=\sum_{v=0}^{N(n)-1} j_{vt_v(n)}(w)(X_{t^{(n)}_{(v+1)}}+X_{t^{(n)}_v}) \qquad (2.29)$$

For the partitions

$$a=t_0^{(n)}<t_1^{(n)}<t_2^{(n)}<\mathbf{L}<t^{(n)}_{N(n)}=t$$

Or

$$I_t(j)=\underset{n\to\infty}{l.i.p.}I_t(j_n)$$

***Remarks (2.6) [47]:***

1. The stochastic integral over a Wiener processes is a martingale whereas the stochastic integral over the Ito process is not a martingale but a continuous semi martingale.
2. If the sufficiency condition $\int_0^t |j_s(w)b_s(w)|^2 ds<\infty$ is replaced by the strong condition

$$\int_0^t E|j_s(w)b_s(w)|^2 ds<\infty$$

Then we have quadratic mean convergence of the integral sums $I_t(j_n)$

$$I_t(j)=\underset{n\to\infty}{l.i.q.m.}I_t(j_n)$$

3. Under suitable assumptions the stochastic differential equation given by equations (2.23), (2.24) may be generalized by replacing the Brownian motion process $W_t$ by a general martingale $M_t$.

## *2.3.2 ITO FORMULA [32]:*

We discuss the Ito rule applied to stochastic integral. First we discuss the rule for the scalar case and extended it to vector situations.

Let $\{X_t, \Im_t, t \in T\}$ be an Ito process as defined by equation (2.24). Let $y(t,\cdot)$ be a measurable function with continuous first and second partial derivatives. Then sentation, popularly known as the Ito formula or Ito rule.

## *Theorem (2.1) [47]:*

Let the function $y(t,x)$ be continuous and have bounded continuous partial derivatives $\partial y/\partial t, \partial y/\partial x$, and $\partial^2 y \big/ \partial x^2$. Let $\{X_t, \Im_t, t \in T\}$ be an Ito process having the stochastic differential equation representation

$$dX_t(w) = a_t(w)dt + b_t(w)dW_t \qquad t \in T, \quad X_0(w) \tag{2.30}$$

Then the process $Y_t(w) = y(t, X_t(w))$ also admits stochastic differential equation representation given by

$$\begin{aligned} dY_t(w) &= \frac{\partial y(t,X_t)}{\partial t}dt + \frac{\partial y(t,X_t)}{\partial x}dX_t \\ &\quad + \frac{1}{2}s^2 \frac{\partial^2 y(t,X_t)}{\partial x^2} b_t^2 dt \end{aligned} \qquad t \in T, \quad Y_a \tag{2.31}$$

Where $s^2$ is the variance parameter associated with $W_t$.

By substituting equation (2.30) into equation (2.31), the Ito rule can be given in an alternate form as

$$\begin{aligned} dY_t(w) &= \frac{\partial y(t,X_t)}{\partial t}dt + \frac{\partial y(t,X_t)}{\partial x}a_t(w)dt + \frac{\partial y(t,X_t)}{\partial x}b_t(w)dW_t \\ &\quad + \frac{1}{2}s^2 \frac{\partial^2 y(t,X_t)}{\partial x^2} b_t^2(w)dt \end{aligned} \qquad t \in T, Y_a$$

(2.24)

**Remarks (2.7) [47]:**

1. If $Y_t = y\,(t, X_t)$ and $X_t$ is a deterministic process, then the differential for $dY_t$ will be

$$dY_t = \frac{\partial y}{\partial t}dt + \frac{\partial y}{\partial x}dX_t \qquad (2.33)$$

Since the higher order terms containing $dt^2$ and $dX_t^2$ and above are negligible.

2. Since $W_t$ is a Brownian motion process, the quadratic term $dW_t^2$ is not negligible and it is of order $dt$.
3. The term $\frac{1}{2}s^2(\partial^2 y / \partial x^2)b_t^2$ in equation (2.31) is the additional term because $dW_t^2$ can not be neglected.
4. If $W_t$ is a deterministic process, $s^2 = 0$ and equation (2.31) degenerates to equation (2.33).
5. The presence of the extra term $\frac{1}{2}s^2(\partial^2 y / \partial x^2)b_t^2 dt$ prevents us from using ordinary calculus, as given by equation (2.33) for stochastic differentials.

**Example (2.5) [47]:**

We back to example (2.4) and use the Ito formula to show that the stochastic integral $I_t(W) = \int_0^t W_t\,(w)dW_t\,(w)$ is indeed a martingale. Let us now assume

$$Y_t = y\,(W_t) = \frac{W_t^2}{2} \qquad (2.34)$$

And fined the differential representation of $Y_t$ using Ito's rule. The Ito process $X_t$ associated with the Brownian motion process $W_t$ is $W_t$ itself,

Or

$$dX_t = dW_t$$

So that $a_t = 0$ and $b_t = 1$ in equation (2.30).

Applying Ito's rule to $Y_t$, we have from equation (2.24)

$$dY_t = 0 \cdot dt + \frac{\partial y}{\partial w} \cdot 0 \cdot dt + \frac{\partial y}{dw} \cdot 1 \cdot dW_t + \frac{1}{2} s^2 \frac{\partial^2 y}{\partial w^2} \cdot 1 \cdot dt$$

Or

$$dY_t = W_t dW_t + \frac{1}{2} s^2 dt$$

Integrating,

$$Y_t = \int_0^t W_t \, dW_t + \frac{1}{2} s^2 t$$

Hence

$$\int_0^t W_t \, dW_t = Y_t - \frac{1}{2} s^2 t = \frac{1}{2}(W_t^2 - s^2 t) \tag{2.35}$$

It is very clear that $\frac{1}{2}(W_t^2 - s^2 t)$ is indeed a martingale since $s^2 t$ is the adapted continuous predictable increasing process $< W, W >_t$ associated with the martingale $W_t$.

Where $< W, W >_t$ is the quadratic variance process of $L^2$-martingale $\{W_t, t \in T\}$.

### ***2.3.3 VECTOR FORMULATION OF ITO'S RULE [47]:***

We can enunciate the vector form of Ito's rule. Let $\{X_t, \Im_t, t \in T\}$ be an n-vector Ito process, having the stochastic differential equation

$$dX_t(w) = a_t(w)dt + B_t(w)dW_t(w) \tag{2.36}$$

Where $a_t^T = \{a_{1t}(w) \quad a_{2t}(w) \ldots \quad a_{nt}(w)\}$ and $B_t(w)$ is an $n \times m$ matrix of functions given by

$$\{\|b_{ijt}(w)\|, i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, m\}$$

$W_t(w)$ is an m-dimensional independent vector Brownian motion process with variance parameter $s^2$. The function $a_{it}(w)$ and $b_{ijt}(w)$ satisfy conditions similar to definition (2.11) with

$$\int_0^t |a_{is}(w)| ds < \infty \qquad i = 1, 2, \ldots, n \quad (a.s.)$$

$$\int_0^t |b_{ijs}(w)|^2 ds < \infty \qquad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m \quad (a.s.)$$

Let another scalar process $Y_t(w)$ be defined by $y(t, X_t(w))$ where $y(t, \cdot)$ is a measurable function.

***Theorem (2.2) [47]:***

Let the function $y(t, x_1, x_2, \ldots, x_n)$ be continuous and have bounded continuous partial derivatives

$$\frac{\partial y}{\partial t}, \quad \frac{\partial y}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)^T y$$

Then the process $Y_t(w) = y(t, X_t(w))$ has a stochastic differential representation given by

$$dY_t(w) = \frac{\partial y\,(t, X_t)}{\partial t}dt + \left(\frac{\partial}{\partial x}\right)^T y\,(t, X_t)dX_t + \frac{1}{2}s^{\,T} B_t^T \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)^T y\,(t, X_t)B_t s\,dt$$

$t \in T\ ,\ Y_a(w)$ (2.37)

Or substituting for $dX_t(w)$ from equation (2.36), we obtain

$$dY_t(w) = \frac{\partial y\,(t, X_t)}{\partial t}dt + \left(\frac{\partial}{\partial x}\right)^T y\,(t, X_t)a_t(w)dt + \left(\frac{\partial}{\partial x}\right)^T y\,(t, X_t)B_t(w)dW_t(w)$$
$$+\frac{1}{2}s^{\,2} B_t^T \left(\frac{\partial}{\partial x}\right)^T y\,(X_t, t)B_t s\,dt$$

$t \in T,\ Y_a(w)$ (2.38)

***Remarks (2.8 [41]):***

***Special Cases***. Let us now consider some special cases where the Ito rule given by (2.38) can be simplified.

1. The Brownian motion process $W_t(w)$ is a scalar process $W_t(w)$. In this case the matrix of functions $B_t(w)$ becomes an n-dimensional vector $b_t(w)$, and the variance parameter $s^{\,2}$ becomes $s^{\,2}$. We using these simplifications, equation (2.38) becomes

$$dY_t(w) = \frac{\partial y\,(t, X_t)}{\partial t}dt + \left(\frac{\partial}{\partial x}\right)^T y\,(t, X_t)a_t(w)dt + \left(\frac{\partial}{\partial x}\right)^T y\,(t, X_t)b_t(w)dW_t(w)$$
$$+\frac{1}{2}s^{\,2} b_t^T(w)\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)^T y\,(t, X_t)b_t(w)dt \qquad t \in T,\ Y_a(w)$$

2. Let $Y_t = X_{1t}X_{2t}$, where $X_{1t}$ and $X_{2t}$ are Ito process satisfying the stochastic differential equations

$$dX_{1t} = a_{1t}dt + b_{1t}dW_t$$
$$dX_{2t} = a_{2t}dt + b_{2t}dW_t \qquad (2.39)$$

then, we using Ito rule,

$$dY_t = X_{2t}dX_{1t} + X_{1t}dX_{2t} + \frac{1}{2}s^{2} \cdot 2b_{1t}b_{2t}dt \qquad (2.40)$$

We use equation (2.39) in equation (2.40), we obtain

$$dY_t = X_{2t}(a_{1t}dt + b_{1t}dW_t) + X_{1t}(a_{2t}dt + b_{2t}dW_t) + s^{2} \cdot b_{1t}b_{2t}dt \qquad (2.41)$$

3. Let $Y_t = e^{X_t}$, and $X_t$ is the Ito process satisfying the stochastic differential equation

$$dX_t = -\frac{1}{2}s^{2}g_t^2dt + g_t dW_t \qquad (2.42)$$

Here

$$a_t = -\frac{s^{2}g_t^2}{2} \qquad b_t = g_t$$

We applying the Ito's formula, equation (2.24), to $Y_t = e^{X_t}$ ,yields

$$dY_t = 0 \cdot dt + \frac{\partial e^{X_t}}{\partial x} \cdot -\frac{s^{2}g_t^2}{2}dt + \frac{\partial e^{X_t}}{\partial x}g_t dW_t + \frac{1}{2}s^{2}\frac{\partial^2 e^{X_t}}{\partial x^2}g_t{}^2 dt$$

Since

$$\frac{\partial e^{X_t}}{\partial x} = \frac{\partial^2 e^{X_t}}{\partial x^2} = Y_t$$

We have

$$dY_t = Y_t g_t dW_t \qquad (2.43)$$

4. Let $\{W_t, \Im_t, t \in T\}$ be a Brownian motion process. Let $j_t(\cdot)$ be a function satisfying conditions (2.11). Then according to proposition of stochastic integral as a martingale, $X_t = \int_0^t j_t(w)dW_t$ is an $\Im_t$ -

martingale. Let $Y_t = X_t^2$. Then the Ito process $X_t$ which is a martingale is given by the stochastic differential

$$dX_t = j_t(w)dW_t$$

Where $a_t = 0$ and $b_t = j_t$, By Ito formula, equation (2.24),

$$dY_t = 0 \cdot dt + 2X_t \cdot 0 \cdot dt + 2X_t \cdot j_t \cdot dW_t + \frac{1}{2} s^2 \cdot 2 \cdot j_t^2 dt$$

$$= 2X_t j_t dW_t + s^2 j_t^2 dt \qquad (2.44)$$

Or in the integral form,

$$Y_t = X_t^2 = Y_0 + 2\int_0^t j_t(w) X_t \, dW_t$$

Since the initial condition $Y_0 = 0$,

$$\frac{1}{2}\left[ X_t^2 - s^2 \int_0^t j_t^2(w)dt \right] = \int_0^t j_t(w) X_t(w) dW_t \qquad (2.45)$$

Is a martingale because the stochastic integral $\int_0^t j_t(w) X_t(w) dW_t$ is a martingale assuming $\int_0^t E|j_t X_t|^2 dt < \infty$, hence invoking the uniqueness of the Doob-Meyer decomposition theorem, we have

$$\langle X, X \rangle_t = s^2 \int_0^t j_t^2(w)dt \qquad (2.46)$$

As an $\Im_t$-adapted increasing predictable process associated with the martingale $X_t$.

### ***2.3.4 APPLICATIONS OF ITO'S FORMULA [41]:***

Ito's formula will become a standard tool in the sequel.

**Lemma (2.2) [41]:**

If $f \in L_w^p[a,b]$ for some $p \geq 1$, then there exists a sequence of step functions $f_n$ in $L_w^p[a,b]$ such that

$$\lim_{n\to\infty} \int_a^b \left| f(t) - f_n(t) \right|^p dt \qquad a.s.$$

**Corollary (2.1 [41]):**

If $f \in M_w^{2m}[0,T]$ where $m$ is a positive integer, then

$$E\left\{ \sup_{0\leq t\leq T} \left| \int_0^t f(s)dw(s) \right|^{2m} \right\} \leq C_m T^{m-1} E\int_0^T \left| f(t) \right|^{2m} dt$$

Where $C_m = \left[ 4m^3 / (2m-1) \right]^m$.

## 2.4 Stochastic Integrals and Differentials in N Dimensions [32]:

Let $w(t) = (w_1(t), \mathbf{K}, w_n(t))$ be an n-dimensional Brownian motion. Let $\Im_t$ $(t \geq 0)$ be an increasing family of $s-$fields such that $w(t)$ is $\Im_t$ measurable and $\Im(w(l+t) - w(t), l \geq 0)$ is independent of $\Im_t$, for any $t \geq 0$.

We shall say that a matrix of functions belongs to $L_w^p[a,b]$ (or to $M_w^p[a,b]$) if each of its elements belongs to $L_w^p[a,b]$ (or to $M_w^p[a,b]$).

Let $b=(b_{ij})$ be a $m\times n$ matrix that belongs to $L_w^2[a,b]$. The stochastic integral $\int_a^b b(t)dw(t)$ is a $m\times n$ matrix that belongs to $L_w^2[a,b]$. The stochastic integral $\int_a^b b(t)dw(t)$ is an m-vector defined by

$$\int_a^b b(t)dw(t)=\left\{\sum_{i=1}^{n}\int_a^b b_{ij}(t)dw_j(t)\right\}_{i=1,\mathbf{K},m}$$

If substitute $a=\int_{t_1}^{t_2} fdw_i$ , $b=\int_{t_1}^{t_2} gdw_i$ in the identity $4ab=(a+b)^2-(a-b)^2$,

We find that:

$$E\int_{t_1}^{t_2} f(t)dw_i(t)\int_{t_1}^{t_2} g(t)dw_i(t)=E\int_{t_1}^{t_2} f(t)g(t)dt \tag{2.47}$$

Provided $f$ and $g$ belong to $M_w^2[t_1,t_2]$.

We also have

$$E\int_{t_1}^{t_2} f(t)dw_i(t)\int_{t_1}^{t_2} g(t)dw_i(t)=0 \quad \text{If } i\neq j\,, \tag{2.48}$$

Because the integrals are independent and with zero expectation.

We are using (2.47), (2.48) we see that if $b=(b_{ij})$ is a $m\times n$ matrix in $M_w^2[t_1,t_2]$, then

$$E\left|\int_{t_1}^{t_2} b(t)dw\right|^2=E\int_{t_1}^{t_2}|b(t)|^2dt \tag{2.49}$$

Where

$$|b|^2 = \sum_{i=1}^{m}\sum_{i=1}^{n}(b_{i\,j})^2 \ .$$

***Definition (2.12) [41]:***

Let $x(t)$ be an m-dimensional process for $0 \le t_1 \le T$, and suppose that, for any $0 \le t_1 \le t_2 \le T$,

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} a(t)dt + \int_{t_1}^{t_2} b(t)dw(t)$$

Where $a = (a_1, \mathbf{K}, a_m)$ and the $m \times n$ matrix $b = (b_{i\,j})$ belong to $L_w^1[0,T]$ and $L_w^2[0,T]$ respectively. Then we say that $x(t)$ has a stochastic differential $dx(t)$ given by

$$dx(t) = a(t)dt + b(t)dw(t)$$

## ***2.5 A GENERAL EXISTENCE AND UNIQUENESS THEOREM [47]:***

We start with a useful calculus lemma:

***GRONWALL'S lemma (2.3) [32]:***

Let $f$ and $f$ be nonnegative, continuous functions defined for $0 \le t \le T$, and let $C_0 \ge 0$ denote a constant. If

$$f(t) \le C_0 + \int_0^t f f ds \ , \quad \text{for all } 0 \le t \le T,$$

Then

$$f(t) \le C_0 e^{\int_0^t f ds} \ , \quad \text{for all } 0 \le t \le T.$$

### *2.5.1 Existence and Uniqueness Theorem [32]:*

Suppose that $b:\mathbb{R}^n\times[0,T]\to\mathbb{R}^n$ and $B:\mathbb{R}^n\times[0,T]\to\mathrm{M}^{m\times n}$ are continuous and satisfy the following conditions:

a) $|b(x,t)-b(\hat{x},t)|\le L|x-\hat{x}|$

, for all $0\le t\le T$, $x,\hat{x}\in\mathbb{R}^n$

$|B(x,t)-B(\hat{x},t)|\le L|x-\hat{x}|$

b) $|b(x,t)|\le L(1+|x|)$

, for all $0\le t\le T$, $x\in\mathbb{R}^n$,

$|B(x,t)|\le L(1+|x|)$

for some constant $L$.

Let $X_0$ be any $\mathbb{R}^n$-valued random variable such that

c) $E(|X_0|^2)<\infty$

and

d) $X_0$ is independent of $W^+(0)$,

Where $W(\cdot)$ is a given $m$-dimentionsinal Brownian motion.

Then there exists a unique solution $X\in L_n^2(0,T)$ of the stochastic differential equation:

(SDE) $$\begin{cases} dX=b(X,t)dt+B(X,t)dW & (0\le t\le T) \\ X(0)=X_0 \end{cases}$$

***Remarks (2.9) [32]:***

1. "Unique" means that if $X, \hat{X} \in L_n^2(0,T)$ with continuous sample paths almost surly, and both solve (SDE), then

   $P(X(t) = \hat{X}(t)$, for all $0 \le t \le T) = 1.$

2. Hypotheses (a) says that b and B are uniformly Lipchitz conditions in the variable $x$. We notice also that hypothesis (b) actually follows (a).

***Proof [32]:***

1. ***Uniqueness***. Suppose $X$ and $\hat{X}$ are solutions. Then for all $0 \le t \le T$,

$$X(t) - \hat{X}(t) = \int_0^t b(X,s) - b(\hat{X},s)ds + \int_0^t B(X,s) - B(\hat{X},s)dW$$

Since $(a+b)^2 \le 2a^2 + 2b^2$, we can estimate

$$E(\left|X(t) - \hat{X}(t)\right|^2) \le 2E\left(\left|\int_0^t b(X,s) - b(\hat{X},s)ds\right|^2\right) +$$

$$2E\left(\left|\int_0^t B(X,s) - B(\hat{X},s)dW\right|^2\right)$$

The Cauchy Schwarz inequality implies that

$$\left|\int_0^t fds\right|^2 \le t\int_0^t |f|^2 ds$$

For any $t > 0$ and $f : [0,t] \to \mathbb{R}^n$. we use this estimate

$$E\left(E\left(\left|\int_0^t b(X,s) - b(\hat{X},s)ds\right|^2\right)\right) \le TE\left(\left|\int_0^t b(X,s) - b(\hat{X},s)\right|^2 ds\right)$$

$$\le L^2 T \int_0^t E(\left|X - \hat{X}\right|^2)ds.$$

Furthermore

$$E\left(\left|\int_0^t B(X,s)-B(\hat{X},s)dW\right|^2\right)=E\left(\int_0^t \left|B(X,s)-B(\hat{X},s)\right|^2 ds\right)$$

$$\leq L^2\int_0^t E(X-\hat{X})ds.$$

Therefore for some appropriate constant $C$ we have

$$E(\left|X(t)-\hat{X}(t)\right|^2)\leq C\int_0^t E(X-\hat{X})ds,$$

Provided $0\leq t\leq T$. If we now set $f(t)=E(\left|X(t)-\hat{X}(t)\right|^2)$, then the foregoing reads

$$f(t)\leq C\int_0^t f(s)ds \quad \text{For all } 0\leq t\leq T.$$

Therefore Gromwell's Lemma, with $C_0=0$, implies $f\equiv 0$. Thus $X(t)=\hat{X}(t)$ a.s. for all $0\leq t\leq T$, and so $X(r)=\hat{X}(r)$ for all rational $0\leq r\leq T$, except for some set of probability zero. As $X$ and $\hat{X}$ have continuous sample paths almost surely,

$$P\left(\max_{0\leq t\leq T}\left|X(t)-\hat{X}(t)>0\right|\right)=1.$$

2. ***Existence.*** We will utilize the iterative scheme introduced. We defined

$$\begin{cases} X^0(t)=X_0 \\ X^{n+1}(t)=X_0+\int_0^t b(X^n(s),s)ds+\int_0^t B(X^n(s),s)dW \end{cases}$$

For $n=0,1,\ldots$ and $0\leq t\leq T$. We defined also

$$d^n(t) = E(\left|X^{n+1}(t) - X^n(t)\right|^2).$$

We claim that

$$d^n(t) \le \frac{(MT)^{n+1}}{(n+1)} \text{ For all } n = 0,1,\mathbf{K},\ 0 \le t \le T$$

For some constant $M$, depending on $L,T$ and $X_0$. We indeed $n=0$, we have

$$d^0(t) = E(\left|X^1(t) - X^0(t)\right|^2)$$

$$= E\left(\left|\int_0^t b(X_0,s)ds + \int_0^t B(X_0,s)dW\right|^2\right)$$

$$\le 2E\left(\left|\int_0^t L(1+|X_0|)ds\right|^2\right) + 2E\left(\int_0^t L^2(1+|X_0|^2 ds\right)$$

$$\le tM$$

For some large enough constant $M$. This confirms the claim for $n \equiv 0$.

Next we assume the claim is valid for some $n-1$. Then

$$d^n(t) = E(\left|X^{n+1}(t) - X^n(t)\right|^2)$$

$$= E\left(\left|\int_0^t b(X^n,s) - b(X^{n-1},s)ds + \int_0^t B(X^n,s) - B(X^{n-1},s)dW\right|^2\right)$$

$$\le 2TL^2E\left(\int_0^t \left|X^n - X^{n-1}\right|^2 ds\right) + 2L^2E\left(\int_0^t \left|X^n - X^{n-1}\right|^2 ds\right)$$

$$\le 2L^2(1+T)\int_0^t \frac{M^n s^n}{n} \qquad \text{(By the induction hypothesis)}$$

$$\leq \frac{M^{n+1} t^{n+1}}{(n+1)},$$

We choose $M \geq 2L^2(1+T)$. This proves the claim.

3. Now we note

$$\max_{0\leq t\leq T}\left|X^{n+1}(t)-X^{n}(t)\right|^2 \leq 2TL^2\int_0^T\left|X^{n}-X^{n-1}\right|^2 ds + 2\max_{0\leq t\leq T}\left|\int_0^t B(X^{n},s)-B(X^{n-1},s)dW\right|^2$$

Consequently the martingale inequality implies

$$E\left(\max_{0\leq t\leq T}\left|X^{n+1}(t)-X^{n}(t)\right|^2\right) \leq 2TL^2\int_0^T E\left|X^{n}-X^{n-1}\right|^2 ds + 8L^2\int_0^T E\left|X^{n}-X^{n-1}\right|^2 ds$$

$$\leq C\frac{(MT)^n}{n}$$

4. The Borel-Cantelli Lemma thus implies, since

$$P\left(\max_{0\leq t\leq T}\left|X^{n+1}(t)-X^{n}(t)\right| > \frac{1}{2^n}\right) \leq 2^{2n} E\left(\max_{0\leq t\leq T}\left|X^{n+1}(t)-X^{n}(t)\right|^2\right) \leq 2^{2n}\frac{(MT)^n}{n}$$

And

$$\sum_{n=1}^{\infty} 2^{2n}\frac{(MT)^n}{n} < \infty.$$

Thus

$$P\left(\max_{0\leq t\leq T}\left|X^{n+1}(t)-X^{n}(t)\right| > \frac{1}{2^n} i.o.\right) = 0$$

In light of this, for almost every $w$

$$X^n = X^0 + \sum_{j=0}^{n-1}(X^{j+1}-X^{j})$$

Converges uniformly on $[0,T]$ to a process $X(\cdot)$. We pass to limits in the definition of $X^{n+1}(\cdot)$, to prove

$$X(t) = X_0 + \int_0^t b(X,s)ds + \int_0^t B(X,s)dW \quad \text{for } 0 \le t \le T.$$

That

$$\text{(SDE)} \qquad \begin{cases} dX = b(X,t)dt + B(X,t)dW \\ X(0) = X_0 \end{cases}$$

For times $0 \le t \le T$.

5. We must still show $X(\cdot) \in L_n^2(0,T)$. We have

$$E(\left|X^{n+1}(t)\right|^2) \le CE(\left|X_0\right|^2) + CE\left(\left|\int_0^t b(X^n,s)ds\right|^2\right) + CE\left(\left|\int_0^t B(X^n,s)dW\right|^2\right)$$

$$\le C(1 + E(\left|X_0\right|^2)) + C\int_0^t E(\left|X^n\right|^2 ds,$$

Where, as usual, "$C$" denotes various constants. By induction, therefore,

$$E(\left|X^{n+1}(t)\right|^2) \le \left[C + C^2 + \mathbf{L} + C^{n+2}\frac{t^{n+1}}{(n+1)}\right](1 + E(\left|X_0\right|^2)).$$

Consequently

$$E(\left|X^{n+1}(t)\right|^2) \le C(1 + E(\left|X_0\right|^2))e^{Ct}$$

Let $n \to \infty$

$$E(\left|X(t)\right|^2) \le C(1 + E(\left|X_0\right|^2))e^{Ct} \quad \text{for all } 0 \le t \le T;$$

and so $X \in L_n^2(0,T)$.

***Definition (2.13) [32]:***

A linear SDE is called homogeneous if $c \equiv E \equiv 0$, for $0 \leq t \leq T$. It is called linear in the narrow sense if $F \equiv 0$.

***Remark (2.10):***

If

$$\sup_{0 \leq t \leq T} [|c(t)| + |D(t)| + |E(t) + |F(t)||] < \infty,$$

Then $b$ and $B$ satisfy the hypotheses the Existence and Uniqueness Theorem. Thus the linear SDE

$$\begin{cases} dX = (c(t) + D(t)X)dt + (E(t) + F(t)X)dW \\ X(0) = X_0 \end{cases}$$

Has a unique solution, provided $E(|X_0|^2) < \infty$, and $X_0$ is independent of $W^+(0)$.

***Example (2.6) [32]:***

Consider first the linear stochastic differential equation

$$\begin{cases} dX = d(t)Xdt + f(t)dW \\ X(0) = X_0 \end{cases} \tag{2.51}$$

For $m = n = 1$. We will try to fined a solution having the product form

$$X(t) = X_1(t)X_2(t),$$

Where

$$\begin{cases} dX_1 = f(t)X_1 dW \\ X_1(0) = X_0 \end{cases} \tag{2.52}$$

and

$$\begin{cases} dX_2 = A(t)dt + B(t)dW \\ X_2(0) = 1, \end{cases} \tag{2.53}$$

Where the functions $A$ and $B$ are be selected. Then

$$\begin{aligned} dX &= d(X_1X_2) \\ &= X_1dX_2 + X_2dX_1 + f(t)X_1B(t)dt \\ &= f(t)XdW + (X_1dX_2 + f(t)X_1B(t)dt), \end{aligned}$$

According to (2.52). Now we try to choose $A, B$ so that

$$dX_2 + f(t)B(t)dt = d(t)X_2dt.$$

For this, $B \equiv 0$ and $A(t) = d(t)X_2(t)$ will work. Thus (2.53) reads

$$\begin{cases} dX_2 = d(t)X_2dt \\ X_2(0) = 1. \end{cases}$$

This is non-random: $X_2(t) = e^{\int_0^t d(s)ds}$. Since the solution of (2.52) is

$$X_1(t) = X_0e^{\int_0^t f(s)dW - \frac{1}{2}\int_0^t f^2(s)ds},$$

We conclude that

$$X(t) = X_1(t)X_2(t) = X_0e^{\int_0^t f(s)dW + \int_0^t d(s) - \frac{1}{2}f^2(s)ds},$$

A formula noted earlier.

***<u>Example (2.7) [32]:</u>***

Consider next the general equation

$$\begin{cases} dX = (c(t) + d(t)X)dt + (e(t) + f(t)X)dW \\ X(0) = X_0, \end{cases} \tag{2.54}$$

again for $m = n = 1$. As above, we try for a solution of the form

$$X(t) = X_1(t)X_2(t),$$

Where now

$$\begin{cases} dX_1 = d(t)X_1dt + f(t)X_1dW \\ X_1(0) = 1 \end{cases} \tag{2.55}$$

And

$$\begin{cases} dX_2 = A(t)d(t) + B(t)dW \\ X_2(0) = X_0, \end{cases} \tag{2.56}$$

The functions $A, B$ to be chosen. Then

$$\begin{aligned} dX &= X_2dX_1 + X_1dX_2 + f(t)X_1B(t)dt \\ &= d(t)Xdt + f(t)XdW + X_1(A(t)dt + B(t)dW) + f(t)X_1B(t)dt \end{aligned}$$

We now require

$$X_1(A(t)dt + B(t)dW) + f(t)X_1B(t)dt = c(t)dt + e(t)dW$$

And this identity will hold if we take

$$\begin{cases} A(t) = [c(t) - f(t)e(t)](X_1(t))^{-1} \\ B(t) = e(t)(X_1(t))^{-1} \end{cases}$$

Observe that since $X_1(t) = e^{\int_0^t fdW + \int_0^t d - \frac{1}{2} f^2 ds}$, we have $X_1(t) > 0$ almost surely. Consequently

$$X_2(t) = X_0 + \int_0^t [c(s) - f(s)e(s)](X_1(s))^{-1}ds + \int_0^t e(s)(X_1(s))^{-1}dW$$

Employing this and the expressio above for $X_1$, we arrive at the formula, a special case :

$$\begin{aligned} X(t) &= X_1(t)X_2(t) \\ &= \exp\left( \int_0^t d(s) - \frac{1}{2} f^2(s)ds + \int_0^t f(s)dW \right) \end{aligned}$$

$$\times\left( X_0+\int_0^t \exp\left(-\int_0^r d(r)-\frac{1}{2}f^2(r)dr-\int_0^s f(r)dW\right)(c(s)-e(s)f(s))ds\right.$$

$$\left.+\int_0^t \exp\left(-\int_0^s d(r)-\frac{1}{2}f^2(r)dr-\int_0^s f(r)dW\right)e(s)dW.\right)$$

# CHAPTER THREE

# STOCHASTIC NONLINEAR STABILIZATION

# *Concluding remarks*

1. The reader should be familiar with stochastic process, Ito-stochastic differential equation, control system in presence of stochastic uncertainty, as well as, Lyapunov stochastic function approach.
2. The stochastic Lyapunov function approach for guarantee the global asymptotically stability in the probability can be considered as a very good and direct approach to overcome the difficulties of unstability of some Ito-stochastic differential equations. The necessary background for this approach, the reader should be familiar with Yong’s inequality, Ito-chain rule, as well as some restrictions on the stochastic dynamic system.
3. The difficulties of this approach are coming by availability of stochastic integration in the presence of Brownian motion and undifferentibility of stochastic noise. So due to this difficulties, the Ito-stochastic differential equation is proved in order to study this subject.

# *Future Work*

1. Numerical solution of Ito-stochastic control system in the presence of stochastic uncertainty may be considered, supported by some real life applications.
2. The connection between the optimal stochastic control and the robust one with the present work may be discussed, whether the state space is a available for measurement or not.

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# المستخلــص


يركز الهدف الاساس لهذا العمل على دراسة الاستقراريه العظمى المحاذيه في الاحتماليه لبعض اصناف الانظمه الديناميكيه نوع-Ito مغلقة المسار بوجود خاصية كون النظام متغير ضمن قيود (System uncertainty).

لقد نوقش بعض الانظمه الديناميكيه نوع Itoالمستمره غير الخطيه والمؤثره بضوضاء عشوائيه متغيره مع الزمن غير متقيده.

لقد تم تطوير الاستقراريه العظمى المحاذيه في الاحتماليه باستخدام مسيطر مستقر ودالة ليابانوف العشوائيه.

لقد تم حساب دالة ليابونوف العشوائيه للنظام المقترح لضمان الاستقراريه المحاذيه العظمى في الاحتماليه للنظام.

تم عرض وبرهان بعض النظريات الاساسيه الخاصه بايجاد المسيطر العشوائي ودالة ليابونوف العشوائيه الضامنه للاستقراريه مدعمه ببعض الاستنتاجات والتطبيقات.

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جــامعة النهــــريــن
كلية العلوم
قسم الرياضيات وتطبيقات الحاسوب

# قابلية الاستقرارية في نظام سيطرة غير خطي متغير العشوائية باستعمال الامثلية المعكوسة

رسالة
مقدمة إلى قسم الرياضيات وتطبيقات الحاسوب،كليةالعلوم، جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم في الرياضيات

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