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Department of Mathematics
and Computer Applications

Existence Theorems of the Solutions for the Boundary Value Problems of the Impulsive Ordinary Differential Equations

A Thesis
Submitted to the Department of Mathematics and Computer Applications, College of Science,
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بِسْمِ اللّهِ الـرَّحمنِ الـرَّحِيم

((يَـرِفَـعِ اللّهُ الَّـذيـنَ
آمَـنُوا مِـنْكُمْ و الَّـذيـنَ
أُوتُـوا الـعِلْمَ دَرَجَـاتٍ
وَاللّهُ بِـمَـا تَـعْمَـلُونَ خَبِيرٌ))

صَدَقَ أللّهُ ألـعَظِيم
سُورَة ألمُـجَادَلَـة (
أ لايــة (11))

الإهداء

إلى معلمي الأول و قرّة عيني ...

والدي العزيز

إلى القلب الكبير الذي منحني الحنان و الدفيء و الاستقرار

والى من اخصّ اللّه الجنّة تحت قدميها ...

والدتي الحنون

إلى من شدّ أزري و شاركني أفراحي و أحزاني ...

إخوتي و أخواتي

إلى الشموع التي أضاءت لي طريق العلم...

أساتذتي

إلى الذّين واكبوا معي طريق السنين بأخلاص ...

أصدقائي و زملائي الأعزاء.

نور شوقي كامل

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Noor Showqi Kamel

2008

SUPERVISORS CERTIFICATION

I certify that this thesis was prepared under my supervision at the Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

Signature:

Name: Assist. Prof. Dr. Ahlam J. Khaleel

Data: / / 2008

In view of the available recommendations, I forward this thesis for debate by the examining committee.

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Data: / / 2008

EXAMINING COMMITTEE CERTIFICATION

We certify that we have read this thesis entitled "***Existence Theorems of the Solutions for the Boundary Value Problems of the Impulsive Ordinary Differential Equations***" and as examining committee examined the student (***Noor Showqi Kamel Mohammad***) in its contents and that in our opinion, it is adequaqte as a thesis for the degree of Master of Science in Mathematics and Computer Applications.

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Abstract

The main theme of this work can be divided into three categories, which can be summarized as follows:

First, we give some definitions of impulsive differential equations with or without delays with some illustrative examples and some real life applications.

Second, we give the explicit forms of the solutions of the boundary value problems (periodic and nonperiodic) which consist of the first order linear ordinary differential equations with non-constant coefficients together with finite impulsive conditions and boundary condition (periodic and nonperiodic).

Third, we transform the boundary value problems (periodic and nonperiodic) which consists of the first order nonlinear ordinary differential equations together with finite impulsive conditions and boundary condition (periodic and nonperiodic) into equivalent integral equations. Also the existence of the solutions for the above periodic boundary value problems are discussed.

CONTENTS

INTRODUCTION

The theory of impulsive differential equations is imerging as important area of investigation, since it is a lot richer than the corresponding theory of differential equations without impulse effect. Moreover, such equations appear to represent a natural framework for mathematical modelings of several real world phenomena [Liu M. and et. al., 2007]. Such as medicine (pulstile signaling in intercellular communication) [Goldbeter A. and et. al., 1993], biology (pulse mass measles vaccination across age cohorts) [Agur Z. and et. al., G. 1993], Physics (a ball that jumping on a flat horizontal surface) [Randelovic B. and et. al., 2000] and mechanical engineering (the verge and folit escapment mechanism consist of two rigid bodies rotating on bearings) [Roup A., 2003]. However, the mathematical theory of systems with impulsive conditions has developed rather slowly, owing to the considerable difficulties of a theoretical and technical nature related to the specific character of these systems [Liu M. and et. al., 2007].

For the impulsive ordinary differential equations, there are many authors and researchers who studied these types of equations [Mil’man V. and Myshkis A. in 1960] studied the stability of impulsive ordinary differential equation, [Lakshmikantham and et al. in 1992] proved the existence and uniquness of solution of the first order non-linear impulsive ordinary differential equations by using the Banach fixed point theorem, [Liz E., 1995] studied the existence of an upper and lower solutions of the first order non-linear boundary value problems of impulsive ordinary differential equations.

[Berezansky L. and et. al., 1997] discussed the stability of the solution of the first order linear impulsive delay ordinary differential equations, [Nieto J. and et. al., 2000] studied the existence of an upper and lower solutions of the first order nonlinear impulsive ordinary differential equations with antiperiodic boundary conditions, [Benchohra M. and et. al., 2002] investigated the existence of solutions for second order impulsive delay ordinary differential equations in Banach spaces, [Dajun G. 2005] obtained the existence of solutions for a boundary value problem of n-th order nonlinear impulsive integro-differential equations, [John R. and et. al. 2006] investigated the existence and uniqueness of solution of first order boundary value problems for impulsive delay ordinary differential equations, [Abdelghani O. 2007] discussed local and global existence and uniqueness of the solutions for first order impulsive delay ordinary differential equations, [Wei D. and et. al. 2007] studied the existence of lower and upper solutions of antiperiodic boundary value problems for first order nonlinear delay ordinary differential equations, [Yu Tian et. al. 2008] studied the existence of solutions for the second order impulsive ordinary differential equations with periodic boundary conditions.

The main purpose of this work is to study the impulsive ordinary differential equations.

This study includes the existence of the solution for the periodic and nonperiodic boundary value problems which consist of the linear and nonlinear first order differential equations together with the finite impulsive conditions and periodic and nonperiodic boundary conditions.

This thesis consists of three chapters.

In chapter one, we give some definitions of impulsive ordinary differential equations.

In chapter two, we devote the existence of the solutions for the periodic boundary value problem of first order non-linear finite impulsive ordinary differential equations.

In chapter three, we give the existence of the solutions for the boundary value problem of first order non-linear finite impulsive ordinary differential equations.

Introduction:

The aim of this chapter is to study the impulsive ordinary differential equations (or ordinary differential equations with impulsive conditions) with their real life applications.

This chapter consists of three sections.

In section one, we give the definitions of the finite and infinite impulsive ordinary differential equations and their solutions with some illustrative examples. Also, the initial value problems and the boundary value problems of these types of equations are given.

In section two, we give the definitions of the finite and infinite impulsive delay ordinary differential equations (or delay ordinary differential equations with (infinite and finite) impulsive conditions) with some illustrative examples.

In section three, we give two real life applications for the impulsive ordinary differential equations namely the mechanical systems and growth of capital.

1.1 The Non-Linear Impulsive Ordinary Differential Equations:

In this section, we give the definition of the non-linear infinite impulsive ordinary differential equation and its solution. Depending on this definition we give the definition of the non-linear finite impulsive ordinary differential equation and its solution. To illustrate these definitions, we give some examples.

We start this section by the following definition.

Definition (1.1), [Lakshmikantham V. and et. al. 1989]:

The ordinary differential equation of the form:

$x'(t) = f(t, x(t)), t \in J'$, ……………..(1.1.a)

together with the infinite impulsive conditions:

$\Delta x(\tau_k) = \varphi_k(x(\tau_k)),\quad k = 0, \mp 1, \mp 2, \ldots$ ……………………………………(1.1.b)

is called the first order non-linear infinite impulsive ordinary differential equation (or the first order non-linear ordinary differential equation with infinite impulsive conditions), where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$, $x(\tau_k^+) = \lim_{t \to \tau_k^+} x(t)$, $x(\tau_k^-) = \lim_{t \to \tau_k^-} x(t)$, $\varphi_k : \Re^n \to \Re^n$ is a continuous function for each $k = 0, \mp 1, \mp 2, \ldots$, $\tau_k < \tau_{k+1}$ for each $k \in \aleph$, $\tau_k \to \infty$ when $k \longrightarrow \infty$, $\tau_k \longrightarrow -\infty$ when $k \longrightarrow -\infty$, $J \subset \Re$ is any real interval, $J' = J \setminus \{\tau_k, k = 0, \mp 1, \mp 2, \ldots\}$, $f : J \times \Re^n \to \Re^n$ is a continuous function on every set $(\tau_k, \tau_{k+1}) \times \Re^n$, $k = 0, \mp 1, \mp 2, \ldots$ and x is the unknown function that must be determined.

Next, from the above definition one can get the following definition.

Definition (1.2):

The ordinary differential equation of the form:

$x'(t) = f(t, x(t)), t \in J'$, ……………………………………………(1.2.a)

together with the finite impulsive conditions:

$\Delta x(\tau_k) = \varphi_k(x(\tau_k)),\quad k = 1, 2, \ldots, m$ ……………………………………(1.2.b) is called the first order non-linear finite impulsive ordinary differential equation (or the first order non-linear ordinary differential equation with finite impulsive conditions), where $\varphi_k : \Re^n \to \Re^n$ is a continuous function for each $k = 1,2,\ldots,m,$ $J \subset \Re$ is any real interval, $J' = J \setminus \{\tau_k, k = 1,2,...,m\}$, $f : J \times \Re^n \to \Re^n$ is a continuous function on every set $(\tau_k, \tau_{k+1}) \times \Re^n$, $k = 1,2,\ldots,m$ and x is the unknown function that must be determined.

To illustrate these definitions, we consider the following examples:

Example (1.1):

Consider the ordinary differential equation:

$x'(t) = 3x(t) + 8t^2,\ t \in J'$

together with the infinite impulsive conditions:

$\Delta x(\tau_k) = \sin(x(\tau_k)),\ k = 0,\mp1,\mp2,...$

where J is the set of all rational numbers, $\tau_k = k,\ k = 0,\mp1,\mp2,...,$ and $J' = J \setminus \{k,\ k = 0,\mp1,\mp2,...\}$. Moreover $k = \tau_k < k+1 = \tau_{k+1},$ $\forall k \in \aleph,$ $\lim\limits_{k\to\infty} \tau_k = \lim\limits_{k\to\infty} k = \infty$ and $\lim\limits_{k\to-\infty} \tau_k = \lim\limits_{k\to-\infty} k = -\infty.$

Example (1.2):

Consider the ordinary differential equation:

$x'(t) = x(t) + t,\ t \in J'$

together with the finite impulsive conditions:

$\Delta x(\tau_k) = \tan(x(\tau_k)),\ k = 1,2,...,50$

where J is the set of all rational numbers, $\tau_k = k,\ k = 1,2,\dots,50$ and $J' = J \setminus \{k,\ k = 1,2\dots,50\}$.

Example (1.3):

Consider the system of the ordinary differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -2 & 0.1 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},\ t \in J',$$

together with the infinite impulsive conditions:

$$\begin{pmatrix} \Delta x_1(\tau_k) \\ \Delta x_2(\tau_k) \end{pmatrix} = \begin{pmatrix} -1 & (2)^{k+1} \\ 5 & -0.8 \end{pmatrix} \begin{pmatrix} x_1(\tau_k) \\ x_2(\tau_k) \end{pmatrix},\ t = \tau_k$$

where $J = \Re$ and $J' = J \setminus \{3(k+1) + 0.5(2)^{k+1},\ k = 0, \mp 1 \mp 2, \dots\}$. Moreover

$[3(k+1) + 0.5(2)^{k+1}] = \tau_k < [3(k+1+1) + 0.5(2)^{k+1+1}] = \tau_{k+1}\ \ \forall k \in \aleph,$

$\lim\limits_{k\to\infty} \tau_k = \lim\limits_{k\to\infty} [3(k+1) + 0.5(2)^{k+1}] = \infty,$ and

$\lim\limits_{k\to-\infty} \tau_k = \lim\limits_{k\to-\infty} [3(k+1) + 0.5(2)^{k+1}] = -\infty.$

Example (1.4):

Consider the system of the ordinary differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},\ t \in J',$$

together with the finite impulsive conditions:

$$\begin{pmatrix} \Delta x_1(\tau_k) \\ \Delta x_2(\tau_k) \end{pmatrix} = \begin{pmatrix} -1 & 2k \\ 5k^2 & -0.2 \end{pmatrix} \begin{pmatrix} x_1(\tau_k) \\ x_2(\tau_k) \end{pmatrix},\ t = \tau_k$$

where $J = \Re$ and $J' = J \setminus \{3(k+1) + 0.5(3)^{k+1},\ k = 1,2,3\}$.

Definition (1.3), [Lakshmikantham V. and et. al. 1989]:

The solution of the first order non-linear infinite (or finite) impulsive ordinary differential equation (1.1) (or (1.2)) is a piecewise continuous function $x : J \longrightarrow \Re^n$ with piecewise continuous first derivative which satisfies equations (1.1) (or (1.2)).

Definition (1.4), [Lakshmikantham V. and et. al. 1989]:

The impulsive ordinary differential equation (1.1) (or (1.2)) together with the initial condition $x(a) = \alpha,\ \alpha \in \Re^n$ is said to be initial value problem of the first order non-linear infinite (or finite) impulsive ordinary differential equation. In this case $J = \{x | x \geq a\}$.

Definition (1.5), [Lakshmikantham V. and et. al. 1989]:

The impulsive ordinary differential equation (1.1) (or (1.2)) together with the boundary condition:

$$Mx(a) + Nx(b) = \alpha, M, N \in \Re, \alpha \in \Re^n$$

is said to be the boundary value problem of the first order non-linear infinite (or finite) impulsive ordinary differential equation. In this case $J = [a, b]$.

Remark (1.1), [Lakshmikantham V. and et. al. 1989]:

The impulsive ordinary differential equation (1.1) (or (1.2)) together with the periodic boundary condition:

$$x(a) = x(b)$$

is said to be the periodic boundary value of the first order non-linear infinite (or finite) impulsive ordinary differential equation.

Also if $x(a)=-x(b)$ then equation (1.1) (or (1.2)) together with this boundary condition is said to be the antiperiodic boundary value of the first order non-linear infinite (or finite) impulsive ordinary differential equation.

Definition (1.6), [Lakshmikantham V. and et. al. 1989]:

The ordinary differential equation of the form:

$$x^{(n)}(t)=f(t,x(t),x'(t),...,x^{(n-1)}(t)), \quad t\in J' \qquad \text{(1.3.a)}$$

together with the infinite impulsive conditions:

$$\Delta x^{(i)}(\tau_k)=\varphi_k^{(i)}(x(\tau_k)), k=0,\mp1,\mp2,..., \; i=0,1,...,n-1 \qquad \text{(1.3.b)}$$

is called the n-th order non-linear infinite impulsive ordinary differential equation (or the n-th order non-linear ordinary differential equation with infinite impulsive conditions), where $\Delta x^{(i)}(\tau_k)=x^{(i)}(\tau_k^+)-x^{(i)}(\tau_k^-)$, $x^{(i)}(\tau_k^+)=\lim_{t\to\tau_k^+} x^{(i)}(t)$, $x^{(i)}(\tau_k^-)=\lim_{t\to\tau_k^-} x^{(i)}(t)$, $i=0,1,...,n-1$, $k=0,\mp1,\mp2,...$,

$\tau_k<\tau_{k+1}$ for each $k\in\aleph$, $\tau_k\longrightarrow\infty$ when $k\longrightarrow\infty, \tau_k\longrightarrow-\infty$ when $k\longrightarrow-\infty$, $\varphi_k:\Re^n\to\Re^n$ is a differentiable function such that $\varphi_k^{(i)}$ exists for each $i=0,1,\ldots,n-1$, $J\subset\Re$ is any real interval, $J'=J\setminus\{\tau_k, k=0,\mp1,\mp2,...\}$, $f:J\times\Re^n\times\Re^n\times...\times\Re^n\to\Re^n$ is a continuous function on every set $(\tau_k,\tau_{k+1})\times\Re^n\times\ldots\times\Re^n, k=0,\mp1,\mp2,\ldots$ and x is the unknown function that must be determined.

Next, from the above definition one can get the following definition.

Definition (1.7):

The ordinary differential equation of the form:

$$x^{(n)}(t) = f(t, x(t), x'(t), ..., x^{(n-1)}(t)), \quad t \in J' \quad \text{.........(1.4.a)}$$

together with the finite impulsive conditions:

$$\Delta x^{(i)}(\tau_k) = \varphi_k^{(i)}(x(\tau_k)), k = 1,2,...,m, \; i = 0,1,...,n-1 \quad \text{.........(1.4.b)}$$

is called the n-th order non-linear finite impulsive ordinary differential equation (or the n-th order non-linear ordinary differential equation with finite impulsive conditions), where $\varphi_k : \Re^n \to \Re^n$ is a differentiable function such that $\varphi_k^{(i)}$ exists for each $i = 0,1,\dots,n-1$, $J \subset \Re$ is any real interval, $J' = J \setminus \{\tau_k, k = 1,2,...,m\}$, $f : J \times \Re^n \times \Re^n \times ... \times \Re^n \to \Re^n$, is a continuous function on every set $(\tau_k, \tau_{k+1}) \times \Re^n \times \dots \times \Re^n$, $k = 1,2,\dots,m$ and x is the unknown function that must be determined.

To illustrate these definitions, we consider the following examples:

Example (1.5):

Consider the ordinary differential equation:

$$x''(t) = x'(t) + x(t) + t, \; t \in J'$$

together with the infinite impulsive conditions:

$$\Delta x(\tau_k) = 2x(\tau_k) + 1, \;\; k = 0, \mp 1, \mp 2, ...$$

$$\Delta x'(\tau_k) = 2x'(\tau_k), \;\; k = 0, \mp 1, \mp 2, ...$$

where J is the set of all integer numbers, $\tau_k = 2k+1,\ k = 0,\mp1,\mp2,\ldots,$ $J' = J \setminus \{2k+1,\ k = 0,\mp1,\mp2,\ldots\}$. It is clear that $\tau_k = 2k+1 < 2k+1+1 = \tau_{k+1}$, $\forall\, k \in \aleph$, $\lim\limits_{k\to\infty} \tau_k = \lim\limits_{k\to\infty} 2k+1 = \infty$ and $\lim\limits_{k\to-\infty} \tau_k = \lim\limits_{k\to-\infty} 2k+1 = -\infty$.

Example (1.6):

Consider the ordinary differential equation:

$$x'''(t) = x''(t) + x'(t) + x(t) + t,\ t \in J'$$

together with the finite impulsive conditions:

$$\Delta x(\tau_k) = 2x(\tau_k) + 1,\ \ k = 1,2,\ldots,100$$

$$\Delta x'(\tau_k) = 2x'(\tau_k),\ \ k = 1,2,\ldots,100$$

$$\Delta x''(\tau_k) = 2x''(\tau_k),\ \ k = 1,2,\ldots,100$$

where J is the set of all integer numbers, $\tau_k = 2k+1,\ k = 1,2,\ldots,100,$ $J' = J \setminus \{2k+1,\ k = 1,2,\ldots,100\}$.

Definition (1.8), [Lakshmikantham V. and et. al. 1989]:

The solution of the n-th order non-linear infinite (or finite) impulsive ordinary differential equations (1.3) (or (1.4)) is a piecewise continuous function $x : J \to \Re^n$ with piecewise continuous n-th order derivatives which satisfy equations (1.3) (or (1.4)).

Remark (1.2), [Lakshmikantham V. and et. al. 1989]:

If $f(t, x, x'(t),\ldots, x^{(n-1)}(t))$ in equations (1.3) (or (1.4)) takes the form:

$$f(t,x) = a_0(t)x(t) + a_1(t)x'(t) + \cdots + a_{n-1}(t)x^{(n-1)}(t) + g(t)$$

then equations (1.3) (or (1.4)) are called the n-th order linear infinite (or finite) impulsive ordinary differential equations, where $a_i : J \to \Re^{n \times n}$ is continuous function for each $i = 0,1,\ldots,n-1$. Also, if $g(t) = 0$ in the above equation then equations (1.3) (or (1.4)) are said to be the n-th order homogeneous linear infinite (or finite) impulsive ordinary differential equations, otherwise it is nonhomogeneous.

1.2 The Non-Linear Impulsive Delay Ordinary Differential Equations:

It is known that the delay ordinary differential equations are generalization of the ordinary differential equations [Driver R. 1977]. Thus, the impulsive delay ordinary differential equation is a generalization of the impulsive ordinary differential equation. The aim of this section is to give two definitions of the finite and infinite impulsive delay ordinary differential equations with some illustrative examples.

Definition (1.9), [Lakshmikantham V. and et. al. 1989]:

The delay ordinary differential equation of the form:

$x'(t) = f(t, x(t), x(t+\theta)),\ t \in J'$ ……………………………………...(1.5.a)

together with the infinite impulsive conditions:

$\Delta x(\tau_k) = \varphi_k(x(\tau_k)),\ k = 0, \mp 1, \mp 2, \ldots$ ………..……………………………(1.5.b)

and the initial condition:

$x(t) = \Omega(t),\quad t \in [-r, 0]$ ………………………………………………..…(1.5.c)

is called the first order non-linear infinite impulsive delay ordinary differential equation (or the first order non-linear delay ordinary differential equation with infinite impulsive conditions), where $\varphi_k : \Re^n \to \Re^n$ is a continuous function for each $k = 0,\mp 1,\mp 2,...$, $\tau_k < \tau_{k+1}$, for each $k \in \aleph$, $\tau_k \longrightarrow \infty$ when $k \longrightarrow \infty$, $\tau_k \longrightarrow -\infty$ when $k \longrightarrow -\infty$, $J = [0,T] \subset \Re$, $\theta \in [-r,0]$, $J' = J \setminus \{\tau_k,\ k = 0,\mp 1,\mp 2,...\}$, $0 < r < \infty$, $f : [0,T] \times \Re^n \times D \to \Re^n$ is a continuous function on every set $(\tau_k, \tau_{k+1}) \times \Re^n \times D$, $k = 0,\mp 1,\mp 2,\ldots$, where $D = \{\psi : [-r,0] \to \Re^n : \psi$ is continuous everywhere except at finite number of points t^* at which $\psi(t^*)$ and $\varphi_k(t^{*+})$ exist and $\varphi_k(t^{*-}) = \varphi_k(t^*)\}$, $\Omega \in D$ and x is the unknown function that must be determined.

Next, from the above definition one can get the following definition.

<u>Definition (1.10):</u>

The delay ordinary differential equation of the form:

$x'(t) = f(t, x(t), x(t+\theta))$, $t \in J'$ ……………………………………………..(1.6.a)

together with the finite impulsive conditions:

$\Delta x(\tau_k) = \varphi_k(x(\tau_k))$, $k = 1,2,...,m$ …………………………………………(1.6.b)

and initial condition:

$x(t) = \Omega(t)$, $t \in [-r,0]$ ……………………………………………………(1.6.c)

is called the first order non-linear finite impulsive delay ordinary differential equation (or the first order non-linear delay ordinary differential equation with finite impulsive conditions), where $\varphi_k : \Re^n \to \Re^n$ is a continuous function for each $k = 1,2,...,m$, $J = [0,T] \subset \Re$ is any real interval, $\theta \in [-r,0]$, $J' = J \setminus \{\tau_k,\ k = 1,2,...,m\}$, $0 < r < \infty$, $f : [0,T] \times \Re^n \times D \to \Re^n$ is a continuous function on every set $(\tau_k, \tau_{k+1}) \times \Re^n \times D \to \Re^n$ $k = 1,2,\ldots m$,

$D=\{\psi:[-r,0]\to\Re^n:\psi$ is continuous everywhere except at finite number of points t^* at which $\psi(t^*)$ and $\varphi_k(t^{*+})$ exist and $\varphi_k(t^{*-})=\varphi_k(t^*)\}$, $\Omega\in D$, x is unknown function that must be determined.

To illustrate these definitions, we consider the following examples:

Example (1.7):

Consider the delay ordinary differential equation:

$x'(t)=3x(t)+x(t-1),\ t\in J'$

together with the infinite impulsive conditions:

$\Delta x(\tau_k)=\cos(x(\tau_k)),\ k=0,\mp1,\mp2,...$

and the initial condition:

$x(t)=\sin(t),\ t\in[-1,0]$

where J is the set of all rational numbers, $\tau_k=k,\quad k=0,\mp1,\mp2,...$ $J'=J\setminus\{k,\quad k=0,\mp1,\mp2,...\}$. It is clear that $k=\tau_k<k+1=\tau_{k+1}$, $\forall k\in\aleph$, $\lim\limits_{k\to\infty}\tau_k=\lim\limits_{k\to\infty}k=\infty$, and $\lim\limits_{k\to-\infty}\tau_k=\lim\limits_{k\to-\infty}k=-\infty$.

Example (1.8):

Consider the delay ordinary differential equation:

$x'(t)=\dfrac{x(t)}{3}+x(t-\dfrac{1}{2}),\ t\in J'$

together with the finite impulsive conditions:

$\Delta x(\tau_k)=2(x(\tau_k))+7,\ k=1,2,...,15$

and the initial condition:

$x(t)=8t^3,\ t\in[-1,0]$

where J is the set of all integer numbers, $\tau_k = 2k$, $k = 1,2,...,15$ and $J' = J \setminus \{2k, k = 1,2,...,15\}$.

Definition (1.11), [[Lakshmikantham V. and et. al. 1989]:

The solution of the first order non-linear infinite (or finite) impulsive delay ordinary differential equations (1.5) (or (1.6)) is a piecewise continuous function $x : J \to \Re^n$ with piecewise continuous first derivative which satisfy equations (1.5) (or (1.6).

Definition (1.12), [Lakshmikantham V. and et. al. 1989]:

The delay ordinary differential equation:

$$x^{(n)} = f(t, x(t), x'(t), \ldots, x^{(n-1)}(t), x(t+\theta), x'(t+\theta), ..., x^{(n-1)}(t+\theta)), \ t \in J' \qquad \text{......................(1.7.a)}$$

together with the infinite impulsive conditions:

$$\Delta x^{(i)}(\tau_k) = \varphi_k^{(i)}(x(\tau_k)), k = 0, \mp 1, \mp 2, ..., \quad i = 0,1,2,...n-1 \qquad \text{......................(1.7.b)}$$

and the initial condition:

$$x(t) = \Omega(t), \quad t \in [-r, 0] \qquad \text{......................(1.7.c)}$$

is called the n-th order non-linear infinite impulsive delay ordinary differential equation (or the n-th order non-linear delay ordinary differential equation with infinite impulsive conditions), where $\Delta x^{(i)}(\tau_k) = x^{(i)}(\tau_k^+) - x^{(i)}(\tau_k^-)$, $x^{(i)}(\tau_k^+) = \lim_{t \to \tau_k^+} x^{(i)}(t)$, $x^{(i)}(\tau_k^-) = \lim_{t \to \tau_k^-} x^{(i)}(t)$, $i = 0,1,...,n-1$, $k = 0, \mp 1, \mp 2, ...$, $\varphi_k : \Re^n \to \Re^n$ is a differentiable function where $\varphi_k^{(i)}$ exists for each $i = 0,1,\ldots,n-1$, $\tau_k < \tau_{k+1}$ for each $k \in \aleph$, $\tau_k \longrightarrow \infty$ when $k \longrightarrow \infty$, $\tau_k \longrightarrow -\infty$ when $k \longrightarrow -\infty$, $J \subset \Re$ is any real interval, $\theta \in [-r, 0]$,

$J' = J \setminus \{\tau_k, k = 0, \mp 1, \mp 2, ...\}$ and $f : J \times \Re^n \times \Re^n \times ... \times \Re^n \times D \times \ldots \times D \to \Re^n$ is a continuous function on every set $(\tau_k, \tau_{k+1}) \times \Re^n \times \Re^n \times \ldots \times D \times D \times \ldots \times D \to \Re^n$, $k = 0, \mp 1, \mp 2, \ldots$

$D = \{\psi : [-r, 0] \to \Re^n : \psi$ is continuous everywhere except at finite number of points t^* at which $\psi(t^*)$ and $\varphi_k(t^{*+})$ exist and $\varphi_k(t^{*-}) = \varphi_k(t^*)\}$, $\Omega \in D$ and x is the unknown function that must be determined.

Next, from the above definition one can get the following definition.

<u>Definition (1.13):</u>

The delay ordinary differential equation:

$$x^{(n)} = f(t, x(t), x'(t), \ldots, x^{(n-1)}(t), x(t+\theta), x'(t+\theta), \ldots, x^{(n-1)}(t+\theta)), t \in J' \quad \ldots\ldots(1.8.a)$$

together with the finite impulsive conditions:

$$\Delta x^{(i)}(\tau_k) = \varphi_k^{(i)}(x(\tau_k)), k = 1, 2, ..., m, i = 0, 1, 2, ... n-1 \quad \ldots\ldots(1.8.b)$$

and the initial condition:

$$x(t) = \Omega(t), \quad t \in [-r, 0] \quad \ldots\ldots(1.8.c)$$

is called the n-th order non-linear finite impulsive delay ordinary differential equation (or the n-th order non-linear delay ordinary differential equation with finite impulsive condition), where $\Delta x^{(i)}(\tau_k) = x^{(i)}(\tau_k^+) - x^{(i)}(\tau_k^-)$,

$x^{(i)}(\tau_k^+) = \lim\limits_{t \to \tau_k^+} x^{(i)}(t)$, $\quad x^{(i)}(\tau_k^-) = \lim\limits_{t \to \tau_k^-} x^{(i)}(t)$, $\quad i = 0, 1, ..., n-1, k = 1, 2, ..., m$,

$\varphi_k : \Re^n \to \Re^n$ is a differentiable function where $\varphi_k^{(i)}$ exists for each $i = 0, 1, \ldots, n-1$, $J \subset \Re$ is any real interval, $\theta \in [-r, 0]$, $J' = J \setminus \{\tau_k, k = 1, 2, ..., m\}$,

$f : J \times \Re^n \times \Re^n \times ... \times \Re^n \times D \times D \times \ldots \times D \to \Re^n$ is a continuous function on every set $(\tau_k, \tau_{k+1}) \times \Re^n \times \Re^n \times \ldots \times D \times D \times \ldots \times D \to \Re^n$ $k = 1,2,\ldots m,$ $D = \{\psi : [-r,0] \to \Re^n : \psi$ is continuous everywhere except at finite number of points t^* at which $\psi(t^*)$ and $\varphi_k(t^{*+})$ exist and $\varphi_k(t^{*-}) = \varphi_k(t^*)\}$, $\Omega \in D$ and x is the unknown function that must be determined.

Remark (1.3), [Lakshmikantham V. and et. al. 1989]:

If $x^{(n)} = f(t, x(t), x'(t), \ldots, x^{(n-1)}(t), x(t+\theta), x'(t+\theta), ..., x^{(n-1)}(t+\theta)),\ t \in J'$ in equations (1.7) (or (1.8)) takes the form:

$$f(t,x) = a_0(t)x(t+\theta) + a_1(t)x^{(1)}(t+\theta) + ... + a_{n-1}(t)x^{(n-1)}(t+\theta) + g(t)$$

then equations (1.7) (or (1.8)) are called the n-th order linear infinite (or finite) impulsive delay ordinary differential equations, where $a_i : J \to \Re^{n \times n}$ is continuous functions for each $i = 0,1,2,\ldots,n-1$. Also, if $g(t) = 0$ in the above equation then equations (1.7) (or (1.8)) are said to be homogeneous n-th order linear infinite (or finite) impulsive delay ordinary differential equation, otherwise it is nonhomogeneous.

To illustrate these definitions, we consider the following examples:

Example (1.9):

Consider the delay ordinary differential equation:

$$x''(t) = \frac{x'(t-1)}{4} + 3x(t-2),\ t \in J'$$

together with the infinite impulsive conditions:

$\Delta x(\tau_k) = 2x(\tau_k) + 1, \;\; k = 0, \mp 1, \mp 2, \ldots$

$\Delta x'(\tau_k) = 2x'(\tau_k), \;\; k = 0, \mp 1, \mp 2, \ldots$

and the initial condition:

$x(t) = \dfrac{t}{9}, \; t \in [-2, 0],$

where J is the set of all integer numbers, $\tau_k = 2k + 1, \; k = 0, \mp 1, \mp 2, \ldots,$ $J' = J \setminus \{2k + 1, \; k = 0, \mp 1, \mp 2, \ldots\}$, It is clear that $2k + 1 = \tau_k < 2k + 1 + 1 = \tau_{k+1}$, $\forall k \in \aleph, \; \lim\limits_{k \to \infty} \tau_k = \lim\limits_{k \to \infty} 2k + 1 = \infty$ and $\lim\limits_{k \to -\infty} \tau_k = \lim\limits_{k \to -\infty} 2k + 1 = -\infty.$

Example (1.10):

Consider the delay ordinary differential equation:

$x'(t) = \dfrac{x(t - 0.5)}{2}, \; t \in J'$

together with the finite impulsive conditions:

$\Delta x(\tau_k) = 2x(\tau_k) + 1, \;\; k = 1, 2, 3$

and the initial condition:

$x(t) = 3t, \; t \in [-1, 0],$

where J is the set of all rational numbers, $\tau_k = k, \; k = 1, 2, 3$ and $J' = J \setminus \{k, \; k = 1, 2, 3\}.$

Definition (1.14), [Lakshmikantham V. and et. al. 1989]:

The solution of n-th order nonlinear infinite (or finite) impulsive delay ordinary differential equations (1.7) (or (1.8)) is the function $x : J \to \Re^n$ with piecewise continuous n-th order derivatives which are satisfy equations (1.7) (or (1.8)).

1.3 Some Real Life Applications of Impulsive Ordinary Differential Equations:

The impulsive differential equations occur in many real life applications say in physics, population dynamics, ecology, biological systems, biotechnology, industrial robotic, pharmacokinetics, optimal control, etc, [Nieto J., 1997]. In this section we give some of these real life applications.

(i) The Mechanical Systems, [Joelianto E., 2001]:

Consider a model of a multi-input/multi-output mechanical systems subject to collision or shock effects accoording to:

$$M\,y'' + D\,y' + Ky = Lu + Pf \quad \text{……..…………………………………...(1.9)}$$

Here $y \in \Re^{d}$ is the system coordinate vector, $u \in \Re^{p}$ is the force vector, $M \in \Re^{d\times d}$ is the generalized positive definite inertia matrix, $D \in \Re^{d\times d}$ is the generalized structural damping matrix, $K \in \Re^{d\times d}$ is the generalized stiffness damping matrix, $L \in \Re^{d\times m}$ is the actuator force distribution matrix, $P \in \Re^{d}$ and f is the external input which is defined as

$$f(t) = \begin{cases} 0, & t \neq \tau_k \\ \Psi, & t = \tau_k \end{cases}$$

The force Ψ represent an impulsive force during collision the results in a jump discontinuity in the velocity components of the states. By define $x = [y^T \;\; y'^T]^T$, equation (1.9) can be written in the form:

$$x'(t) = \begin{bmatrix} 0 & 1 \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ M^{-1}L \end{bmatrix} u(t),\ t \neq \tau_k$$

together with the impulsive conditions:

$$x(\tau_k^+) = x(\tau_k) + \begin{bmatrix} 0 \\ M^{-1}P \end{bmatrix} \Psi, \ t = \tau_k$$

At time τ_k, the position y is continuous but the velocity y' is discontinuous.

The mathematical model given by equation (1.9) could be used in a number of applications such as to model a rigid bar hinged at one end which is subjected to an impact load at time τ_k at the other end, or a constrained manipulator.

(ii) Growth of Capital, [Jean F. and et. al., 2002]:

Consider the mathematical model of the growth of capital which can described by the following ordinary differential equation:

$$K'(t) = sF(K(t), L_0 e^{nt}), t > t_0, t \in J'$$

together with the finite impulsive conditions:

$$\Delta K(\tau_k) = J_k(K(\tau_k)), \ k = 1,2,\ldots m$$

and the initial condition:

$$K(t_0) = K_0$$

where *K(t)* is the capital, $L_0 e^{nt}$, $n > 0$ is the exponentially growing labor force, L_0 is the initial labor, $\tau_1, \tau_2, \ldots, \tau_k > 0$ are the moments, the functions J_k characterize the magnitude of the impulse effect at the times τ_k, $K(\tau_k^-)$ and $K(\tau_k^+)$ are respectively the capital level before and after the impulsive effect, s is constant and K_0 is the initial capital.

Introduction:

The aim of this chapter is to discuss the existence of the solutions of the periodic boundary value problem of the first order non-linear ordinary differential equation with finite impulsive conditions.

This chapter consists of four sections.

In section one, we give the explicit forms of the solutions for the periodic boundary value problem of the first order linear ordinary differential equations that contains only one impulsive condition.

In section two, we generalize the previous section to be valid for the same types of the periodic boundary value problem but with more than one impulsive condition.

In section three, we give an integral equation that is equivalent to the periodic boundary value problem of the first order nonlinear ordinary differential equation with finite impulsive conditions.

In section four, we give some necessary conditions for the existence of the solutions for the periodic boundary value problem of the first order nonlinear ordinary differential equation with finite impulsive conditions.

2.1 Solutions for the Periodic Boundary Value Problems of the First Order Linear Ordinary Differential Equations with One Impulsive Condition:

In this section, we give the explicit forms for the solutions of the periodic boundary value problems which consist of the first order linear ordinary differential equations together with one impulsive condition and periodic boundary condition. This section is a modification of the facts that appeared in [Jinhai C. and et. al., 2007]. To do this first consider the periodic boundary value problem which consists of the first order linear ordinary differential equations with constant coefficients:

$x'(t) = Ax(t) + f(t),\ t \in J'$ ……………………………………………(2.1.a)

together with the impulsive condition:

$\Delta x(\tau_1) = \varphi_1(x(\tau_1))$ ……………………………………………(2.1.b)

and the periodic boundary condition:

$x(0) = x(T)$ ……………………………………………(2.1.c)

where $f:[0,T] \to \Re^n$ is a continuous functions on $t \in J' = [0,T] \setminus \{\tau_1\}$, $\varphi_1 : \Re^n \to \Re^n$ is a continuous function, $\tau_1 \in (0,T)$ and A is $n \times n$ nonzero constant matrix.

The following theorem gives the explicit form of the solution of equations (2.1).

Theorem (2.1):

(i) If *x(t)* is a solution of equations (2.1) then

$$x(t) = G(t,\tau_1)\varphi_1(x(\tau_1)) + \int_0^T G(t,s) f(s)\, ds \quad \text{……………(2.2)}$$

where G is the Green's function that takes the form:

$$G(t,s) = \frac{1}{1-e^{AT}} \begin{cases} e^{A(t-s)}, & 0 \le s \le t \le T \\ e^{A(T+t-s)}, & 0 \le t < s \le T \end{cases} \quad \text{……………(2.3)}$$

(ii) If *x(t)* satisfies equation (2.2) then *x(t)* is a solution of equations (2.1).

Proof:

(i) Consider

$$\frac{d}{dt}\left(e^{-At}x(t)\right) = -Ae^{-At}x(t) + e^{-At}x'(t)$$

Thus

$$\frac{d}{dt}\left(e^{-At}x(t)\right)=e^{-At}(x'(t)-Ax(t))$$

$$=e^{-At}f(t)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots...(2.4)$$

By integrating the above equation from τ_1 to t with $t\in(\tau_1,T]$ one can have:

$$x(t)e^{-At}=x(\tau_1^+)e^{-A\tau_1}+\int_{\tau_1}^{t}f(s)e^{-As}ds\,.$$

Another integration for equation (2.4) from 0 to τ_1 will give:

$$x(\tau_1^-)e^{-A\tau_1}=x(0)+\int_{0}^{\tau_1}f(t)e^{-At}dt\,.$$

Hence by adding the above two equations one can get:

$$x(t)e^{-At}=x(0)+x(\tau_1^+)e^{-A\tau_1}-x(\tau_1^-)e^{-A\tau_1}+\int_{0}^{\tau_1}f(t)e^{-At}dt+\int_{\tau_1}^{t}f(s)e^{-As}ds$$

$$=x(0)+\varphi_1(x(\tau_1))e^{-A\tau_1}+\int_{0}^{t}f(s)e^{-As}ds\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2.5)$$

Therefore from the above equation and by using the periodic boundary condition given by equation (2.1.c) one can have:

$$x(T)e^{-AT}=x(0)+\varphi_1(x(\tau_1))e^{-A\tau_1}+\int_{0}^{T}f(s)e^{-As}ds$$

$$=x(0)e^{-AT}$$

and this implies that

$$x(0)=\frac{e^{-A\tau_1}}{e^{-AT}-1}\varphi_1(x(\tau_1))+\frac{1}{e^{-AT}-1}\int_{0}^{T}f(s)e^{-As}ds\,.$$

By substituting the above equation in equation (2.5) one can get:

$$x(t)=\frac{e^{-A(\tau_1-t)}}{e^{-AT}-1}\varphi_1(x(\tau_1))+\frac{e^{At}}{e^{-AT}-1}\int_0^T f(s)e^{-As}ds+\varphi_1(x(\tau_1))e^{-A(\tau_1-t)}+e^{At}\int_0^t f(s)e^{-As}ds$$

$$=\frac{e^{-A(T+\tau_1-t)}}{e^{-AT}-1}\varphi_1(x(\tau_1))+\frac{e^{At}}{e^{-AT}-1}\int_0^T f(s)e^{-As}ds+e^{At}\int_0^t f(s)e^{-As}ds$$

$$=\frac{e^{-A(\tau_1-t)}}{1-e^{AT}}\varphi_1(x(\tau_1))+\frac{e^{A(T+t)}}{1-e^{AT}}\int_0^T f(s)e^{-As}ds+e^{At}\int_0^t f(s)e^{-As}ds$$

$$=\frac{e^{A(t-\tau_1)}}{1-e^{AT}}\varphi_1(x(\tau_1))+\frac{e^{At}}{1-e^{AT}}\int_0^t f(s)e^{-As}ds+\frac{e^{A(T+t)}}{1-e^{AT}}\int_t^T f(s)e^{-As}ds$$

$$=G(t,\tau_1)\varphi_1(x(\tau_1))+\int_0^T G(t,s)f(s)ds.$$

(ii) From equation (2.2) one can have:

$$x'(t)=A\frac{e^{A(t-\tau_1)}}{1-e^{AT}}\varphi_1(x(\tau_1))+A\frac{e^{At}}{1-e^{AT}}\int_0^t f(s)e^{-As}ds+\frac{e^{At}}{1-e^{AT}}f(t)e^{-At}+$$

$$A\frac{e^{A(T+t)}}{1-e^{AT}}\int_t^T f(s)e^{-As}ds-\frac{e^{A(T+t)}}{1-e^{AT}}f(t)e^{-At}$$

$$=A\left[\frac{e^{A(t-\tau_1)}}{1-e^{AT}}\varphi_1(x(\tau_1))+\frac{e^{At}}{1-e^{AT}}\int_0^t f(s)e^{-As}ds+\frac{e^{A(T+t)}}{1-e^{AT}}\int_t^T f(s)e^{-As}ds\right]+$$

$$\frac{1}{1-e^{AT}}f(t)-\frac{e^{AT}}{1-e^{AT}}f(t)$$

$$=Ax(t)+f(t)$$

This implies that *x(t)* given by equation (2.2) satisfied equation (2.1.a).

Next, by taking the limit of x that given by equation (2.2) as $t\to\tau_1^+$ one can get:

$$x(\tau_1^+) = \lim_{t\to\tau_1^+}\left[G(t,\tau_1)\varphi_1(x(\tau_1)) + \int_0^T G(t,s)\,f(s)ds\right]$$

$$= \frac{1}{1-e^{AT}}\varphi_1(x(\tau_1)) + \int_0^T G(\tau_1,s)\,f(s)ds.$$

Also, we take the limit of *x(t)* that given by equation (2.2) as $t \to \tau_1^-$ to get:

$$x(\tau_1^-) = \lim_{t\to\tau_1^-}\left[G(t,\tau_1)\varphi_1(x(\tau_1)) + \int_0^T G(t,s)\,f(s)ds\right]$$

$$= \frac{e^{AT}}{1-e^{AT}}\varphi_1(x(\tau_1)) + \int_0^T G(\tau_1,s)\,f(s)ds\,.$$

Hence

$$x(\tau_1^+) - x(\tau_1^-) = \left[\frac{1}{1-e^{AT}} - \frac{e^{AT}}{1-e^{AT}}\right]\varphi_1(x(\tau_1))$$

and this implies that

$$\Delta x(\tau_1) = \varphi_1(x(\tau_1)).$$

Therefore *x(t)* given by equation (2.2) satisfied equation (2.1.b).

Since

$$G(0,s) = \frac{1}{1-e^{AT}}e^{A(T-s)}$$

$$= G(T,s),\ 0 \le s \le T\,.$$

Thus

$$x(0) = G(0,\tau_1)\varphi_1(x(\tau_1)) + \int_0^T G(0,s)\,f(s)ds$$

$$= G(T,\tau_1)\varphi_1(x(\tau_1)) + \int_0^T G(T,s)\,f(s)ds$$

Thus

$x(0) = x(T)$.

Therefore $x(t)$ given by equation (2.2) satisfies equation (2.1.c).■

Second, consider the periodic boundary value problem which consists of the first order linear ordinary differential equation with nonconstant coefficients:

$x'(t) = A(t)x(t) + f(t), t \in J'$ ……………………………………………..……..(2.6.a)

together with the impulsive condition:

$\Delta x(\tau_1) = \varphi_1(x(\tau_1))$ ……………………………………………………..…...(2.6.b)

and the periodic boundary condition:

$x(0) = x(T)$ ……………………………………………...……………….(2.6.c)

where $f : [0,T] \to \Re^n$ is a continuous function on $t \in J' = [0,T] \setminus \{\tau_1\}$, A is $n \times n$ nonzero function matrix, $\varphi_1 : \Re^n \to \Re^n$ is a continuous function and $\tau_1 \in (0,T)$.

The following theorem gives an explicit form for the solution of equations (2.6).

Theorem (2.2):

(i) If $x(t)$ is a solution of equations (2.6) then

$$x(t) = G(t,\tau_1)\varphi_1(x(\tau_1)) + \int_0^T G(t,s) f(s)ds \quad \text{……………………………………(2.7)}$$

where G is the Green's function that takes the form:

$$G(t,s)=\frac{1}{1-e^{\int_0^T A(s)ds}}\begin{cases} e^{\int_0^t A(s)ds-\int_0^s A(y)dy}, & 0\le s\le t\le T \\ e^{\int_0^T A(s)ds+\int_0^t A(s)ds-\int_0^s A(s)ds}, & 0\le t<s\le T \end{cases} \qquad \text{..................(2.8)}$$

(ii) If *x(t)* satisfies equation (2.7) then *x(t)* is a solution of equations (2.6).

Proof:

(i) Consider

$$\frac{d}{dt}\left[e^{-\int_0^t A(s)ds}x(t)\right]=e^{-\int_0^t A(s)ds}x'(t)-e^{-\int_0^t A(s)ds}A(t)x(t)$$

$$=e^{-\int_0^t A(s)ds}\left[x'(t)-A(t)x(t)\right]$$

$$=e^{-\int_0^t A(s)ds}f(t) \qquad \text{..(2.9)}$$

By integrating the above equation from τ_1 to t with $t\in(\tau_1,T]$ one can have:

$$x(t)e^{-\int_0^t A(s)ds}=x(\tau_1^+)e^{-\int_0^{\tau_1} A(s)ds}+\int_{\tau_1}^t e^{-\int_0^s A(y)dy}f(s)ds.$$

Another integrating for equation (2.9) from 0 to τ_1 will give:

$$x(\tau_1^-)e^{-\int_0^{\tau_1} A(s)ds}=x(0)+\int_0^{\tau_1} e^{-\int_0^t A(s)ds}f(t)dt.$$

Hence by adding the above two equations one can have:

$$x(t)e^{-\int_0^t A(s)ds} = x(0) + \left[x(\tau_1^+) - x(\tau_1^-)\right]e^{-\int_0^{\tau_1} A(s)ds} + \int_0^{\tau_1} e^{-\int_0^t A(s)ds} f(t)dt + \int_{\tau_1}^t e^{-\int_0^s A(y)dy} f(s)ds$$

$$= x(0) + \varphi_1(x(\tau_1))e^{-\int_0^{\tau_1} A(s)ds} + \int_0^t e^{-\int_0^s A(y)dy} f(s)ds \quad \ldots\ldots\ldots\ldots\ldots(2.10)$$

Therefore from the above equation and by using the periodic boundary condition given by equations (2.6.c) one can have:

$$x(T)e^{-\int_0^T A(s)ds} = x(0) + \varphi_1(x(\tau_1))e^{-\int_0^{\tau_1} A(s)ds} + \int_0^T e^{-\int_0^s A(y)dy} f(s)ds$$

$$= x(0)e^{-\int_0^T A(s)ds}$$

and this implies that

$$x(0) = \frac{1}{e^{-\int_0^T A(s)ds} - 1} e^{-\int_0^{\tau_1} A(s)ds} \varphi_1(x(\tau_1)) + \frac{1}{e^{-\int_0^T A(s)ds} - 1} \int_0^T e^{-\int_0^s A(y)dy} f(s)ds .$$

By substituting the above equation in equation (2.10) one can get:

$$x(t)=\frac{1}{e^{-\int_0^T A(s)ds}-1}e^{-\int_0^{\tau_1} A(s)ds+\int_0^t A(s)ds}\varphi_1(x(\tau_1))+\frac{e^{\int_0^t A(s)ds}}{e^{-\int_0^T A(s)ds}-1}\int_0^T e^{-\int_0^s A(y)dy} f(s)ds$$

$$+\varphi_1(x(\tau_1))e^{-\int_0^{\tau_1} A(s)ds+\int_0^t A(s)ds}+e^{\int_0^t A(s)ds}\int_0^t e^{-\int_0^s A(y)dy} f(s)ds$$

$$=\frac{e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\left[e^{-\int_0^{\tau_1} A(s)ds}\varphi_1(x(\tau_1))+\int_0^t e^{-\int_0^s A(y)dy} f(s)ds+\right.$$

$$\left.e^{\int_0^T A(s)ds}\int_t^T e^{-\int_0^s A(y)dy} f(s)ds\right]$$

and by using the definition of Green's function given by equation (2.8), one can have:

$$x(t)=G(t,\tau_1)\varphi_1(x(\tau_1))+\int_0^T G(t,s)f(s)ds.$$

(ii) From equation (2.7) one can have:

$$x'(t)=\frac{1}{1-e^{\int_0^T A(s)ds}}\left[A(t)e^{-\int_0^{\tau_1} A(s)ds+\int_0^t A(s)ds}\varphi_1(x(\tau_1))+f(t)+A(t)e^{\int_0^t A(s)ds}\int_0^t e^{-\int_0^s A(y)dy} f(s)ds+\right.$$

$$\left.e^{\int_0^T A(s)ds}\left(A(t)\int_t^T e^{\int_0^t A(y)dy}e^{-\int_0^s A(y)dy} f(s)ds-f(t)\right)\right]$$

Thus

$$x'(t)=\frac{A(t)}{1-e^{\int_0^T A(s)ds}}\left[e^{-\int_0^{\tau_1} A(s)ds+\int_0^t A(s)ds}\varphi_1(x(\tau_1))+e^{\int_0^t A(s)ds}\int_0^t e^{-\int_0^s A(y)dy}f(s)ds+\right.$$

$$\left.e^{\int_0^T A(s)ds}\int_t^T e^{\int_0^t A(s)ds}e^{-\int_0^s A(y)dy}f(s)ds\right]+\frac{1-e^{\int_0^T A(s)ds}}{1-e^{\int_0^T A(s)ds}}f(t)$$

$x'(t)=A(t)x(t)+f(t).$

This implies that *x(t)* given by equation (2.7) satisfied equation (2.6.a).

Next, by taking the limit of *x(t)* that given by equation (2.7) as $t\to\tau_1^+$ one can get:

$$x(\tau_1^+)=\lim_{t\to\tau_1^+}\left[G(t,\tau_1)\varphi_1(x(\tau_1))+\int_0^T G(t,s)f(s)ds\right]$$

$$=\frac{1}{1-e^{\int_0^T A(s)ds}}\varphi_1(x(\tau_1))+\int_0^T G(\tau_1,s)f(s)ds.$$

Also, we take the limit of *x(t)* that give by equation (2.7) as $t\to\tau_1^-$ to get:

$$x(\tau_1^-)=\lim_{t\to\tau_1^-}\left[G(t,\tau_1)\varphi_1(x(\tau_1))+\int_0^T G(t,s)f(s)ds\right]$$

$$=\frac{e^{\int_0^T A(s)ds}}{1-e^{\int_0^T A(s)ds}}\varphi_1(x(\tau_1))+\int_0^T G(\tau_1,s)f(s)ds.$$

Hence

$$x(\tau_1^+) - x(\tau_1^-) = \left[\frac{1}{1 - e^{\int_0^T A(s)ds}} - \frac{e^{\int_0^T A(s)ds}}{1 - e^{\int_0^T A(s)ds}} \right] \varphi_1(x(\tau_1))$$

$$= \varphi_1(x(\tau_1))$$

and this implies that

$\Delta x(\tau_1) = \varphi_1(x(\tau_1))$.

Therefore *x(t)* given by equation (2.7) satisfied equation (2.6.b).

Since

$$G(0,s) = \frac{1}{1 - e^{\int_0^T A(s)ds T}} e^{\int_0^T A(s)ds - \int_0^s A(s)ds}$$

$$= G(T,s),\ 0 \le s \le T .$$

Thus

$$x(0) = G(0,\tau_1)\varphi_1(x(\tau_1)) + \int_0^T G(0,s) f(s)ds$$

$$= G(T,\tau_1)\varphi_1(x(\tau_1)) + \int_0^T G(T,s) f(s)ds$$

$$= x(T) .$$

Therefore *x(t)* given by equation (2.7) satisfied equation (2.6.c).■

Next, to illustrate the previous theorems, we consider the following examples:

Example (2.1):

Consider the periodic boundary value problem which consists of the first order linear ordinary differential equation with constant coefficient:

$$x'(t) = 2\,x(t) + 3t^2, t \in [0,1] \setminus \left\{\frac{1}{2}\right\}$$

together with the impulsive condition:

$$\Delta x\left(\frac{1}{2}\right) = x\left(\frac{1}{2}\right)$$

and the periodic boundary condition:

$$x(0) = x(1)$$

then by using theorem (2.1), the solution of the above periodic boundary value problem can be written as:

$$x(t) = \frac{e^{2(t-\frac{1}{2})}}{1-e^2}\,x\left(\frac{1}{2}\right) + \frac{e^{2t}}{1-e^2}\int_0^t 3s^2 e^{-2s}ds + \frac{e^{2(1+t)}}{1-e^2}\int_t^1 3s^2 e^{-2s}ds$$

$$= \frac{e^{(2t-1)}}{1-e^2}\,x\left(\frac{1}{2}\right) + \frac{3e^{2t}}{1-e^2}\left[\frac{-s^2e^{-2s}}{2} - \frac{se^{-2s}}{2} - \frac{e^{-2s}}{4}\right]_0^t +$$

$$\frac{3e^{2(1+t)}}{1-e^2}\left[\frac{-s^2e^{-2s}}{2} - \frac{se^{-2s}}{2} - \frac{e^{-2s}}{4}\right]_t^1$$

$$= \frac{e^{(2t-1)}}{1-e^2}\,x\left(\frac{1}{2}\right) + \frac{3e^{2t}}{1-e^2}\left[\frac{-t^2e^{-2t}}{2} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4} + \frac{1}{4}\right] +$$

$$\frac{3e^{2(1+t)}}{1-e^2}\left[\frac{-e^{-2}}{2} - \frac{e^{-2}}{2} - \frac{e^{-2}}{4} + \frac{t^2e^{-2t}}{2} + \frac{te^{-2t}}{2} + \frac{e^{-2t}}{4}\right]$$

Thus

$$x(t)=\frac{e^{(2t-1)}}{1-e^2}x\left(\frac{1}{2}\right)-\frac{3t^2}{2(1-e^2)}-\frac{3t}{2(1-e^2)}-\frac{3}{4(1-e^2)}+\frac{3e^{2t}}{4(1-e^2)}-\frac{3e^{2t}}{2(1-e^2)}-$$
$$\frac{3e^{2t}}{2(1-e^2)}-\frac{3e^{2t}}{4(1-e^2)}+\frac{3t^2e^2}{2(1-e^2)}+\frac{3te^2}{2(1-e^2)}+\frac{3e^2}{4(1-e^2)}$$
$$=\frac{e^{(2t-1)}}{1-e^2}x\left(\frac{1}{2}\right)-\frac{3t^2}{2(1-e^2)}-\frac{3t}{2(1-e^2)}-\frac{3}{4(1-e^2)}-\frac{3e^{2t}}{(1-e^2)}+\frac{3t^2e^2}{2(1-e^2)}+\frac{3te^2}{2(1-e^2)}+\frac{3e^2}{4(1-e^2)}.$$

Example (2.2):

Consider the periodic boundary value problem which consists of the first order linear ordinary differential equation with nonconstant coefficient:

$$x'(t)=\sin(t)x(t)+\sin t,\ t\in[0,2]\setminus\{1\}$$

together with the impulsive condition:

$$\Delta x(1)=\cos(x(1))$$

and the periodic boundary condition:

$$x(0)=x(2)$$

Then by using theorem (2.2) the solution of the above periodic boundary value problem can be written as:

$$x(t)=\frac{e^{\int_0^t \sin(s)ds}}{1-e^{\int_0^2 \sin(s)ds}}\left[e^{-\int_0^1 \sin(s)ds}\cos(x(1))+\int_0^t e^{-\int_0^s \sin(y)dy}\sin(s)ds+e^{\int_0^2 \sin(s)ds}\int_t^2 e^{-\int_0^s \sin(y)dy}\sin(s)ds\right]$$
$$=\frac{e^{1-\cos(t)}}{1-e^{1-\cos(2)}}\left[e^{\cos(1)-1}\cos(x(1))-\left[e^{\cos(s)-1}\right]_0^t+e^{1-\cos(2)}\left[-e^{\cos(s)-1}\right]_t^2\right]$$

Thus

$$x(t)=\frac{e^{1-\cos(t)}}{1-e^{1-\cos(2)}}\left[e^{\cos(1)-1}\cos(x(1))-e^{\cos(t)-1}+1+e^{1-\cos(2)}\left[-e^{\cos(2)-1}+e^{\cos(t)-1}\right]\right].$$

2.2 Solutions for the Periodic Boundary Value Problems of the First Order Linear Ordinary Differential Equations with Finite Impulsive Conditions:

In this section, we give the explicit forms for the solutions of the periodic boundary value problems which consist of the first order linear ordinary differential equations together with finite impulsive conditions and periodic boundary condition. This section is a modification of the previous section . To do this first consider the periodic boundary value problem which consists of the first order linear ordinary differential equation with constant coefficients:

$x'(t)=Ax(t)+f(t), t\in J'$ ……………………………………………….(2.11.a)

together with the finite impulsive conditions:

$\Delta x(\tau_k)=\varphi_k(x(\tau_k)),\ k=1,2,...,m$ ……………………………………….(2.11.b)

and the periodic boundary condition:

$x(0)=x(T)$ ………………………………………………………………..(2.11.c)

where $f:[0,T]\to\Re^n$ is a continuous function on $t\in J'=[0,T]\setminus\{\tau_1,\tau_2,\ldots,\tau_m\}$ $\varphi_k:\Re^n\to\Re^n$ is a continuous function for each $k=1,2,\ldots,m$, $\tau_k\in(0,T), k=1,2,\ldots,m$, $\tau_1<\tau_2<\ldots<\tau_m$ and A is $n\times n$ nonzero constant matrix.

The following theorem gives the explicit form of the solution of equations (2.11).

Theorem (2.3):

(i) If *x(t)* is a solution of equations (2.11) then

$$x(t)=\sum_{k=1}^{m}G(t,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(t,s)f(s)ds.\qquad(2.12)$$

where G is the Green's function defined by equation (2.3).

(ii) If *x(t)* satisfies equation (2.12) then *x(t)* is a solution of equations (2.11),

Proof:

(i) Consider

$$\frac{d}{dt}\left(e^{-At}x(t)\right)=e^{-At}(x'(t)-Ax(t))$$

$$=e^{-At}f(t)\qquad(2.13)$$

By integrating the above equation from τ_m to t with $t\in(\tau_m,T]$ one can have:

$$x(t)e^{-At}=x(\tau_m^+)e^{-A\tau_m}+\int_{\tau_m}^{t}e^{-As}f(s)ds.$$

Another integration for equation (2.13) from τ_{m-1} to τ_m will give:

$$x(\tau_m^-)e^{-A\tau_m}=x(\tau_{m-1}^+)e^{-A\tau_{m-1}}+\int_{\tau_{m-1}}^{\tau_m}f(s)e^{-As}ds.$$

Also integration for equation (2.13) from τ_{m-2} to τ_{m-1} will give:

$$x(\tau_{m-1}^-)e^{-A\tau_{m-1}}=x(\tau_{m-2}^+)e^{-A\tau_{m-2}}+\int_{\tau_{m-2}}^{\tau_{m-1}}f(s)e^{-As}ds$$

we continue in this manner to get:

$$x(\tau_2^-)e^{-A\tau_2}=x(\tau_1^+)e^{-A\tau_1}+\int_{\tau_1}^{\tau_2}f(s)e^{-As}ds.$$

Another integration for equation (2.13) from 0 to τ_1 will give

$$x(\tau_1^-)e^{-A\tau_1} = x(0) + \int_0^{\tau_1} f(t)e^{-At}dt.$$

Hence by adding the above equations one can get:

$$x(t)e^{-At} = x(0) + x(\tau_m^+)e^{-A\tau_m} + x(\tau_{m-1}^+)e^{-A\tau_{m-1}} + \cdots + x(\tau_1^+)e^{-A\tau_1} - x(\tau_m^-)e^{-A\tau_m} - x(\tau_{m-1}^-)e^{-A\tau_{m-1}} - \cdots - x(\tau_1^-)e^{-A\tau_1} + \int_0^{\tau_1} e^{-As} f(s)ds + \int_{\tau_1}^{\tau_2} e^{-As} f(s)ds + \cdots + \int_{\tau_{m-1}}^{\tau_m} e^{-As} f(s)ds + \int_{\tau_m}^{t} e^{-As} f(s)ds.$$

$$= x(0) + \sum_{k=1}^{m} \varphi_k(x(\tau_k))e^{-A\tau_k} + \int_0^t e^{-As} f(s)ds \quad \text{.................................(2.14)}$$

Therefore from the above equation and by using the periodic boundary condition given by equation (2.11.c) one can have:

$$x(T)e^{-AT} = x(0) + \sum_{k=1}^{m} \varphi_k(x(\tau_k))e^{-A\tau_k} + \int_0^T e^{-As} f(s)ds$$

$$= x(0)e^{-AT}$$

and this implies that

$$x(0) = \frac{1}{e^{-AT}-1}\sum_{k=1}^{m} \varphi_k(x(\tau_k))e^{-A\tau_k} + \frac{1}{e^{-AT}-1}\int_0^T f(s)e^{-As}ds$$

By substituting the above equation in equation (2.14) one can get:

$$x(t)=\frac{e^{At}}{e^{-AT}-1}\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-A\tau_k}+\frac{e^{At}}{e^{-AT}-1}\int_0^T f(s)e^{-As}ds+$$

$$e^{At}\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-A\tau_k}+e^{At}\int_0^t e^{-As}f(s)ds$$

$$=\sum_{k=1}^{m}\frac{e^{-A(\tau_k-t)}}{1-e^{AT}}\varphi_k(x(\tau_k))+\frac{e^{At}}{1-e^{AT}}\int_0^t f(s)e^{-As}ds+\frac{e^{A(T+t)}}{1-e^{AT}}\int_t^T f(s)e^{-As}ds$$

$$=\sum_{k=1}^{m}G(t,\tau_k)\varphi_k(x(\tau_k)+\int_0^T G(t,s)f(s)ds.$$

(ii) From equation (2.12) one can have:

$$x'(t)=A\sum_{k=1}^{m}\frac{e^{-A(\tau_k-t)}}{1-e^{AT}}\varphi_k(x(\tau_k))+A\frac{e^{At}}{1-e^{AT}}\int_0^t f(s)e^{-As}ds+\frac{1}{1-e^{AT}}f(t)+$$

$$A\frac{e^{A(T+t)}}{1-e^{AT}}\int_t^T f(s)e^{-As}ds-\frac{e^{AT}}{1-e^{AT}}f(t)$$

$$=Ax(t)+f(t).$$

This implies that *x(t)* given by equation (2.12) satisfied equation (2.11.a).

Next, by taking the limits of *x(t)* that given by equation (2.12) as $t\to\tau_i^+$ and $t\to\tau_i^-$ one can get:

$$x(\tau_i^+)=\lim_{t\to\tau_i^+}\left[\sum_{k=1}^{m}G(t,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(t,s)f(s)ds\right]$$

$$=\frac{1}{1-e^{AT}}\varphi_i(x(\tau_i))+\sum_{\substack{k=1\\k\neq i}}^{m}\frac{e^{A(T+\tau_i-\tau_k)}}{1-e^{AT}}\varphi_k(x(\tau_k))+\int_0^T G(\tau_i,s)f(s)ds,\ i=1,2,\ldots,m$$

and

$$x(\tau_i^-) = \lim_{t\to\tau_i^-}\left[\sum_{k=1}^{m} G(t,\tau_k)\varphi_k(x(\tau_k)) + \int_0^T G(t,s)f(s)ds\right]$$

$$= \frac{e^{AT}}{1-e^{AT}}\varphi_i(x(\tau_i)) + \sum_{\substack{k=1\\k\neq i}}^{m} \frac{e^{A(T+\tau_i-\tau_k)}}{1-e^{AT}}\varphi_k(x(\tau_k)) + \int_0^T G(\tau_i,s)f(s)ds,\ i=1,2,\ldots,m$$

Hence

$$x(\tau_i^+) - x(\tau_i^-) = \left(\frac{1-e^{AT}}{1-e^{AT}}\right)\varphi_i(x(\tau_i)) = \varphi_i(x(\tau_i)),\ i=1,2,\ldots,m$$

and this implies that

$\Delta x(\tau_i) = \varphi_i(x(\tau_i)),\ i=1,2,\ldots m.$

Therefore *x(t)* given by equation (2.12) satisfied equation (2.11.b).

Since

$$G(0,s) = \frac{1}{1-e^{AT}}e^{A(T-s)}$$

$$= G(T,s),\ 0\le s\le T.$$

Therefore

$$x(0) = \sum_{k=1}^{m} G(0,\tau_k)\varphi_k(x(\tau_k)) + \int_0^T G(0,s)f(s)ds$$

$$= \sum_{k=1}^{m} G(T,\tau_k)\varphi_k(x(\tau_k)) + \int_0^T G(T,s)f(s)ds$$

$$= x(T).$$

Therefore *x(t)* given by equation (2.12) satisfied equation (2.11.c).■

Second, consider the periodic boundary value problem which consists of the first order linear ordinary differential equation with nonconstant coefficients:

$x'(t) = A(t)x(t) + f(t), t \in J'$ ……………………………………………(2.15.a)

together with the finite impulsive conditions:

$\Delta x(\tau_k) = \varphi_k(x(\tau_k)), k = 1,2,\ldots,m$ ……………………………………………(2.15.b)

and the periodic boundary condition:

$x(0) = x(T)$ ……………………………………………………………(2.15.c)

where $f : [0,T] \to \Re^n$ is a continuous function on $t \in J' = [0,T] \setminus \{\tau_1, \tau_2, \ldots, \tau_m\}$, A is $n \times n$ nonzero function matrix, $\varphi_k : \Re^n \to \Re^n$, is a continuous function for each $k = 1,2,\ldots,m$, $\tau_k \in (0,T)$, $k = 1,2,\ldots,m$ and $\tau_1 < \tau_2 < \ldots < \tau_m$.

The following theorem gives an explicit form of the solution of equations (2.15).

<u>Theorem (2.4):</u>

(i) If *x(t)* is a solution of equations (2.15) then

$$x(t) = \sum_{k=1}^{m} G(t,\tau_k)\varphi_k(x(\tau_k)) + \int_0^T G(t,s) f(s) ds \quad \ldots\ldots(2.16)$$

where G is the Green's function defined by equation (2.8).

(ii) If *x(t)* satisfies equations (2.16) then *x(t)* is a solution of equations (2.15),

<u>Proof:</u>

(i) Consider

$$\frac{d}{dt}\left[e^{-\int_0^t A(s)ds}x(t)\right]=e^{-\int_0^t A(s)ds}f(t)$$

By integrating the above equation from τ_m to t with $t\in(\tau_m,T]$, from τ_{m-1} to τ_m, from τ_{m-2} to τ_{m-1},..., from τ_2 to τ_1 and from 0 to τ_1 and by adding the resulting equations one can have:

$$x(t)e^{-\int_0^t A(s)ds}=x(0)+\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\int_0^{\tau_k} A(s)ds}+\int_0^t e^{-\int_0^s A(y)dy}f(s)ds \text{ ....(2.17)}$$

Therefore from the above equation and by using the periodic boundary condition given by equation (2.15.c) one can have:

$$\begin{aligned}x(T)e^{-\int_0^T A(s)ds}&=x(0)+\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\int_0^{\tau_k} A(s)ds}+\int_0^T e^{-\int_0^s A(y)dy}f(s)ds\\&=x(0)e^{-\int_0^T A(s)ds}\end{aligned}$$

and this implies that

$$x(0)=\frac{1}{e^{-\int_0^T A(s)ds}-1}\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\int_0^{\tau_k} A(s)ds}+\frac{1}{e^{-\int_0^T A(s)ds}-1}\int_0^T e^{-\int_0^s A(y)dy}f(s)ds.$$

By substituting the above equation in equation (2.17) one can get:

$$x(t)=\frac{e^{\int_0^t A(s)ds}}{e^{-\int_0^T A(s)ds}-1}\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\int_0^{\tau_k} A(s)ds}+\frac{e^{\int_0^t A(s)ds}}{e^{-\int_0^T A(s)ds}-1}\int_0^T e^{-\int_0^s A(y)dy} f(s)ds+$$

$$e^{\int_0^t A(s)ds}\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\int_0^{\tau_k} A(s)ds}+e^{\int_0^t A(s)ds}\int_0^t e^{-\int_0^s A(y)dy} f(s)ds$$

$$=\frac{e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\int_0^{\tau_k} A(s)ds}+\frac{e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\int_0^t e^{-\int_0^s A(y)dy} f(s)ds+$$

$$\frac{e^{\int_0^T A(s)ds}e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\int_t^T e^{-\int_0^s A(y)dy} f(s)ds$$

$$=\sum_{k=1}^{m}G(t,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(t,s)f(s)ds\,.$$

(ii) From equation (2.16) one can have:

$$x'(t)=\frac{e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}A(t)\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\int_0^{\tau_k} A(s)ds+}+A(t)\frac{e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\int_0^t e^{-\int_0^s A(y)dy} f(s)ds+$$

$$\frac{1}{1-e^{\int_0^T A(s)ds}}f(t)+A(t)\frac{e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}e^{\int_0^T A(s)ds}\int_t^T e^{-\int_0^s A(y)dy} f(s)ds-\frac{e^{\int_0^T A(s)ds}}{1-e^{\int_0^T A(s)ds}}f(t)$$

Thus

$$x'(t)=A(t)\left[\frac{e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\int_0^{\tau_k} A(s)ds}+\frac{e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\int_0^t e^{-\int_0^s A(y)dy} f(s)ds+\right.$$
$$\left.\frac{e^{\int_0^t A(s)ds}\, e^{\int_0^T A(s)ds}}{1-e^{\int_0^T A(s)ds}}\int_t^T e^{-\int_0^s A(s)ds} f(s)ds\right]+\left[\frac{1-e^{\int_0^T A(s)ds}}{1-e^{\int_0^T A(s)ds}}\right]f(t)$$
$$=A(t)x(t)+f(t).$$

This implies that *x(t)* given by equation (2.16) satisfied equation (2.15.a).

Next, by taking the limits of *x(t)* that given by equation (2.16) as $t\to\tau_i^+$ and $t\to\tau_i^-$ one can get:

$$x(\tau_i^+)=\lim_{t\to\tau_i^+}\left[\sum_{k=1}^{m}G(t,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(t,s)f(s)ds\right]$$
$$=\frac{1}{1-e^{\int_0^T A(s)ds}}\varphi_i(x(\tau_i))+\sum_{\substack{k=1\\k\neq i}}^{m}\frac{e^{\int_0^T A(s)ds+\int_0^{\tau_i} A(s)ds-\int_0^{\tau_k} A(s)ds}}{1-e^{\int_0^T A(s)ds}}\varphi_k(x(\tau_k))+$$
$$\int_0^T G(\tau_i,s)f(s)ds,\ i=1,2,\ldots,m$$

and

$$x(\tau_i^-) = \lim_{t \to \tau_i^-} \left[\sum_{k=1}^{m} G(t,\tau_k)\varphi_k(x(\tau_k)) + \int_0^T G(t,s) f(s)ds \right]$$

$$= \frac{e^{\int_0^T A(s)ds}}{1-e^{\int_0^T A(s)ds}} \varphi_i(x(\tau_i)) + \sum_{\substack{k=1\\k\neq i}}^{m} \frac{e^{\int_0^T A(s)ds + \int_0^{\tau_i} A(s)ds - \int_0^{\tau_k} A(s)ds}}{1-e^{\int_0^T A(s)ds}} \varphi_k(x(\tau_k)) + \int_0^T G(\tau_i,s) f(s)ds,\ i=1,2,\ldots,m.$$

Hence

$\Delta x(\tau_i) = \varphi_i(x(\tau_i)),\ i=1,2,\ldots,m$.

Therefore *x(t)* given by equation (2.16) satisfied equation (2.15.b).

Since

$$G(0,s) = \frac{1}{1-e^{\int_0^T A(s)ds}} e^{\int_0^T A(s)ds - \int_0^s A(y)dy}$$

$$= G(T,s),\ 0 \le s \le T.$$

Therefore

$$x(0) = \sum_{k=1}^{m} G(0,\tau_k)\varphi_k(x(\tau_k)) + \int_0^T G(0,s) f(s)ds$$

$$= \sum_{k=1}^{m} G(T,\tau_k)\varphi_k(x(\tau_k)) + \int_0^T G(T,s) f(s)ds$$

$$= x(T).$$

Therefore *x(t)* given by equation (2.16) satisfied equation (2.15.c).■

Next, to illustrate the previous theorems consider the following examples:

Example (2.3):

Consider the periodic boundary value problem which consists of the first order linear ordinary differential equation with constant coefficient:

$$x'(t) = x(t) + t,\ t \in [0,1] \setminus \left\{\frac{1}{4}, \frac{1}{3}, \frac{3}{4}\right\}$$

together with the three impulsive conditions:

$$\Delta x\left(\frac{1}{2}\right) = x\left(\frac{1}{4}\right)$$

$$\Delta x\left(\frac{1}{3}\right) = 2x\left(\frac{1}{3}\right)$$

$$\Delta x\left(\frac{3}{4}\right) = 3x\left(\frac{3}{4}\right)$$

and the periodic boundary condition:

$$x(0) = x(1)$$

then by using theorem (2.3), the solution of the above periodic boundary value problem can be written as:

$$x(t) = \frac{e^{(t-\frac{1}{4})}}{1-e} x\left(\frac{1}{4}\right) + 2\frac{e^{(t-\frac{1}{3})}}{1-e} x\left(\frac{1}{3}\right) + 3\frac{e^{(t-\frac{3}{4})}}{1-e} x\left(\frac{3}{4}\right) + \frac{e^{t}}{1-e}\int_{0}^{t} s e^{-s} ds + \frac{e^{(1+t)}}{1-e}\int_{t}^{1} s e^{-s} ds$$

$$= \frac{e^{(t-\frac{1}{4})}}{1-e} x\left(\frac{1}{4}\right) + 2\frac{e^{(t-\frac{1}{3})}}{1-e} x\left(\frac{1}{3}\right) + 3\frac{e^{(t-\frac{3}{4})}}{1-e} x\left(\frac{3}{4}\right) + \frac{e^{t}}{1-e}\left[-e^{-s}(s+1)\right]_{0}^{t} + \frac{e^{(1+t)}}{1-e}\left[-e^{-s}(s+1)\right]_{t}^{1}$$

$$= \frac{e^{(t-\frac{1}{4})}}{1-e} x\left(\frac{1}{4}\right) + 2\frac{e^{(t-\frac{1}{3})}}{1-e} x\left(\frac{1}{3}\right) + 3\frac{e^{(t-\frac{3}{4})}}{1-e} x\left(\frac{3}{4}\right) + \frac{e^{t}}{1-e}\left[-e^{-t}(t+1)+1\right] + \frac{e^{(1+t)}}{1-e}\left[-2e^{-1}+e^{-t}(t+1)\right]$$

$$= \frac{e^{(t-\frac{1}{4})}}{1-e} x\left(\frac{1}{4}\right) + 2\frac{e^{(t-\frac{1}{3})}}{1-e} x\left(\frac{1}{3}\right) + 3\frac{e^{(t-\frac{3}{4})}}{1-e} x\left(\frac{3}{4}\right) - \frac{(t+1)}{1-e} + \frac{e^{t}}{1-e} - \frac{2e^{t}}{1-e} + \frac{e^{1}(t+1)}{1-e}.$$

Example (2.4):

Consider the periodic boundary value problem which consists of the first order linear ordinary differential equation with nonconstant coefficient:

$$x'(t) = \cos(t)x(t) + \cos(t),\ t \in [0,1] \setminus \left\{\frac{1}{4}, \frac{1}{2}\right\}$$

together with two impulsive conditions:

$$\Delta x\left(\frac{1}{4}\right) = x\left(\frac{1}{4}\right)$$

$$\Delta x\left(\frac{1}{2}\right) = x\left(\frac{1}{2}\right)$$

and the periodic boundary condition:

$$x(0) = x(1)$$

then by using theorem (2.4), the solution of the above periodic boundary value problem can be written as:

$$x(t) = \frac{e^{\int_0^t \cos(s)ds}}{1 - e^{\int_0^1 \cos(s)ds}} \left[x\left(\frac{1}{4}\right) e^{-\int_0^{\frac{1}{4}} \cos(s)ds} + x\left(\frac{1}{2}\right) e^{-\int_0^{\frac{1}{2}} \cos(s)ds} \right] + \frac{e^{\int_0^t \cos(s)ds}}{1 - e^{\int_0^1 \cos(s)ds}} \int_0^t e^{-\int_0^s \cos(y)dy} \cos(s)ds +$$

$$\frac{e^{\int_0^1 \cos(s)ds}\, e^{\int_0^t \cos(s)ds}}{1 - e^{\int_0^1 \cos(s)ds}} \int_t^1 e^{-\int_0^s \cos(y)dy} \cos(s)ds$$

$$= \frac{e^{\sin(t)}}{1 - e^{\sin(1)}} \left[x\left(\frac{1}{4}\right) e^{-\sin(\frac{1}{4})} + x\left(\frac{1}{2}\right) e^{-\sin(\frac{1}{2})} \right] + \frac{e^{\sin(t)}}{1 - e^{\sin(1)}} \int_0^t e^{-\sin(s)} \cos(s)ds +$$

$$\frac{e^{\sin(1)} e^{\sin(t)}}{1 - e^{\sin(1)}} \int_t^1 e^{-\sin(s)} \cos(s)ds$$

Thus

$$x(t)=\frac{e^{\sin(t)}}{1-e^{\sin(1)}}\left[x\left(\frac{1}{4}\right)e^{-\sin(\frac{1}{4})}+x\left(\frac{1}{2}\right)e^{-\sin(\frac{1}{2})}\right]+\frac{e^{\sin(t)}}{1-e^{\sin(1)}}\left[-e^{-\sin(s)}\right]_0^t+$$

$$\frac{e^{\sin(1)}e^{\sin(t)}}{1-e^{\sin(1)}}\left[-e^{-\sin(s)}\right]_t^1$$

$$=\frac{e^{\sin(t)}}{1-e^{\sin(1)}}\left[x\left(\frac{1}{4}\right)e^{-\sin(\frac{1}{4})}+x\left(\frac{1}{2}\right)e^{-\sin(\frac{1}{2})}\right]+\frac{e^{\sin(t)}}{1-e^{\sin(1)}}\left[-e^{-\sin(t)}+1\right]+$$

$$\frac{e^{\sin(1)}e^{\sin(t)}}{1-e^{\sin(1)}}\left[-e^{-\sin(1)}+e^{-\sin(t)}\right]$$

$$=\frac{e^{\sin(t)}}{1-e^{\sin(1)}}\left[x\left(\frac{1}{4}\right)e^{-\sin(\frac{1}{4})}+x\left(\frac{1}{2}\right)e^{-\sin(\frac{1}{2})}\right]+\frac{e^{\sin(t)}-1}{1-e^{\sin(1)}}+\frac{-e^{\sin(t)}+e^{\sin(1)}}{1-e^{\sin(1)}}.$$

2.3 Solutions for the Periodic Boundary Value Problems of the First Order Nonlinear Ordinary Differential Equations with Finite Impulsive Conditions:

In this section, we transform the periodic boundary value problems which consist of first order nonlinear ordinary differential equation together with finite impulsive conditions and periodic boundary condition into integral equations. This section is a modification and a correction of the facts that appeared in [Jinhai C. and et. al., 2007]. To do this, first, consider the periodic boundary value problem which consists of the first order nonlinear ordinary differential equations:

$x'(t)=f(t,x),\ t\in J'$ ……..…………………………….……………….(2.18.a)

together with the impulsive condition:

$\Delta x(\tau_1) = \varphi_1(x(\tau_1))$ ....(2.18.b)

and the periodic boundary condition:

$x(0) = x(T)$ ....(2.18.c)

where $f:[0,T]\times\Re^n \to \Re^n$ is a continuous function on $t \in J' = [0,T]\setminus\{\tau_1\}$, $\varphi_1 : \Re^n \to \Re^n$ is a continuous function and $\tau_1 \in (0,T)$.

The following theorem shows that any solution of equations (2.18) must satisfy an integral equation and conversely.

Theorem (2.5):

(i) If *x(t)* is a solution of equations (2.18) then *x(t)* satisfy the integral equation:

$$x(t) = G(t,\tau_1)\varphi_1(x(\tau_1)) + \int_0^T G(t,s)[f(s,x(s) - x(s)]ds \qquad (2.19)$$

where G is the Green's function defined by equation (2.3) in case $A=I$.

(ii) If *x(t)* satisfies equations (2.19) then *x(t)* is a solution of equations (2.18).

Proof:

(i) Consider

$$\frac{d}{dt}\left[e^{-t}x(t)\right] = [x'(t) - x(t)]e^{-t}$$

$$= [f(t,x(t)) - x(t)]e^{-t} \qquad (2.20)$$

By integrating the above equation from τ_1 to t with $t \in (\tau_1, T]$ one can have:

$$x(t)e^{-t} = x(\tau_1^+)e^{-\tau_1} + \int_{\tau_1}^{t} [f(s,x(s)) - x(s)]e^{-s}\,ds\,.$$

Another integration for equation (2.20) from 0 to τ_1 one can have:

$$x(\tau_1^-)e^{-\tau_1} = x(0) + \int_0^{\tau_1} [f(t, x(t) - x(t)]e^{-t}\,dt.$$

Hence by adding the above two equations one can get:

$$x(t)e^{-t} = x(0) + \varphi_1(x(\tau_1))e^{-\tau_1} + \int_0^t [f(s, x(s) - x(s)]e^{-s}\,ds \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.21)$$

Therefore from the above equation and by using the periodic boundary condition given by equation (2.18.c) one can have:

$$x(T)e^{-T} = x(0) + \varphi_1(x(\tau_1))e^{-\tau_1} + \int_0^T [f(s, x(s)) - x(s)]e^{-s}ds$$

$$= x(0)e^{-T}$$

and this implies that

$$x(0) = \frac{e^{-\tau_1}}{e^{-T}-1}\varphi_1(x(\tau_1)) + \frac{1}{e^{-T}-1}\int_0^T [f(s, x(s)) - x(s)]e^{-s}ds$$

By substituting the above equation in equation (2.21) one can get:

$$x(t) = \frac{1}{e^{-T}-1}e^{-\tau_1+t}\varphi_1(x(\tau_1)) + \frac{e^t}{e^{-T}-1}\int_0^T [f(s, x(s)) - x(s)]e^{-s}ds +$$

$$\varphi_1(x(\tau_1))e^{-\tau_1+t} + e^t\int_0^t [f(s, x(s)) - x(s)]e^{-s}ds$$

$$= \frac{1}{1-e^T}e^{t-\tau_1}\varphi_1(x(\tau_1)) + \frac{1}{1-e^T}\int_0^t e^{t-s}[f(s, x(s)) - x(s)]ds +$$

$$\frac{e^{t+T}}{1-e^T}\int_t^T [f(s, x(s)) - x(s)]e^{-s}\,ds$$

$$= G(t, \tau_1)\varphi_1(x(\tau_1)) + \int_0^T G(t, s)[f(s, x(s)) - x(s)]ds$$

(ii) From equation (2.19) one can have:

$$x'(t)=\frac{1}{1-e^{T}}e^{t-\tau_1}\varphi_1(x(\tau_1))+\frac{e^{t}}{1-e^{T}}\int_0^t\left[f(s,x(s))-x(s)\right]e^{-s}ds+\frac{1}{1-e^{T}}\left[f(t,x(t))-x(t)\right]+$$
$$\frac{e^{t+T}}{1-e^{T}}\int_t^T[f(s,x(s))-x(s)]e^{-s}ds-\frac{e^{T}}{1-e^{T}}\left[f(t,x(t))-x(t)\right]$$
$$=x(t)+f(t,x(t))-x(t)$$
$$=f(t,x(t)).$$

This implies that *x(t)* given by equation (2.19) satisfied equation (2.18.a).

Next, by taking the limits of x that given by equation (2.19) as $t\to\tau_1^{+}$ and $t\to\tau_1^{-}$ one can get:

$$x(\tau_1^{+})=\lim_{t\to\tau_1^{+}}\left[G(t,\tau_1)\varphi_1(x(\tau_1))+\int_0^T G(t,s)\left[f(s,x(s))-x(s)\right]ds\right]$$
$$=\frac{1}{1-e^{T}}\varphi_1(x(\tau_1))+\int_0^T G(\tau_1,s)\left[f(s,x(s))-x(s)\right]ds$$

and

$$x(\tau_1^{-})=\lim_{t\to\tau_1^{-}}\left[G(t,\tau_1)\varphi_1(x(\tau_1))+\int_0^T G(t,s)\left[f(s,x(s))-x(s)\right]ds\right]$$
$$=\frac{e^{T}}{1-e^{T}}\varphi_1(x(\tau_1))+\int_0^T G(\tau_1,s)\left[f(s,x(s))-x(s)\right]ds.$$

Hence

$$\Delta x(\tau_1)=\left(\frac{1-e^{T}}{1-e^{T}}\right)\varphi_1(x(\tau_1))$$
$$=\varphi_1(x(\tau_1)).$$

Therefore *x(t)* given by equation (2.19) satisfied equation (2.18.b).

Since

$$G(0,s)=\frac{1}{1-e^{T}}e^{T-s}$$
$$=G(T,s),\ 0\le s\le T$$

Thus

$$x(0)=G(0,\tau_1)\varphi_1(x(\tau_1))+\int_0^T G(0,s)\left[f(s,x(s))-x(s)\right]ds$$
$$=G(T,\tau_1)\varphi_1(x(\tau_1))+\int_0^T G(T,s)\left[f(s,x(s))-x(s)\right]ds$$
$$=x(T).$$

Therefore *x(t)* given by equation (2.19) satisfied equation (2.18.c).■

Next, consider the periodic boundary value problem which consists of the first order nonlinear ordinary differential equation:

$x'(t)=f(t,x),\ t\in J'$ ……………………..……...………………..…………(2.22.a)

together with the finite impulsive conditions:

$\Delta x(\tau_k)=\varphi_k(x(\tau_k)),\ k=1,2,\ldots,m$ ……………..……………………..……(2.22.b)

and periodic boundary condition:

$x(0)=x(T)$ ……………………………………………………………..(2.22.c)

where $f:[0,T]\times\Re^n\to\Re^n$ is a continuous function on $t\in J'=[0,T]\setminus\{\tau_1,\tau_2,\ldots,\tau_m\}$, $\varphi_k:\Re^n\to\Re^n$ is a continuous functions for each $k=1,2,\ldots,m$, $\tau_k\in(0,T),\ k=1,2,\ldots,m$ and $\tau_1<\tau_2<\ldots<\tau_m$.

The following theorem shows that any solution of equations (2.22) must satisfy an integral equation and conversely.

Theorem (2.6):

(i) If $x(t)$ is a solution of equations (2.22) then

$$x(t)=\sum_{k=1}^{m}G(t,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(t,s)[f(s,x(s))-x(s)]ds \quad\text{......................(2.23)}$$

where G is the Green's function defined by equation(2.3) in case $A=I$.

(ii) If $x(t)$ satisfies equation (2.23) then $x(t)$ is a solution of equations (2.22).

Proof:

(i) Consider

$$\frac{d}{dt}\left[e^{-t}x(t)\right]=e^{-t}\left[x'(t)-x(t)\right]$$

$$=e^{-t}\left[f(t,x(t))-x(t)\right] \quad\text{..(2.24)}$$

By integrating the above equation from τ_m to t with $t\in(\tau_m,T]$ one can have:

$$x(t)e^{-t}=x(\tau_m^+)e^{-\tau_m}+\int_{\tau_m}^{t}e^{-s}\left[f(s,x(s))-x(s)\right]ds\,.$$

Another integration for equation (2.24) from τ_{m-1} to τ_m will give:

$$x(\tau_m^-)e^{-\tau_m}=x(\tau_{m-1}^+)e^{-\tau_{m-1}}+\int_{\tau_{m-1}}^{\tau_m}e^{-t}\left[f(t,x(t))-x(t)\right]dt.$$

Also integration for equation (2.24) from τ_{m-2} to τ_{m-1} will give:

$$x(\tau_{m-1}^-)e^{-\tau_{m-1}}=x(\tau_{m-2}^+)e^{-\tau_{m-2}}+\int_{\tau_{m-2}}^{\tau_m-1}e^{-t}\left[f(t,x(t))-x(t)\right]dt$$

we continue in this manner to get:

$$x(\tau_2^-)e^{-\tau_2}=x(\tau_1^+)e^{-\tau_1}+\int_{\tau_1}^{\tau_2}e^{-t}\left[f(t,x(t))-x(t)\right]dt.$$

Another integration for equation (2.24) from 0 to τ_1 will give:

$$x(\tau_1^-)e^{-\tau_1} = x(0) + \int_0^{\tau_1} e^{-t}\left[f(t,x(t)) - x(t)\right]dt$$

Hence by adding the above equations one can have:

$$x(t)e^{-t} = x(0) + \sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\tau_k} + \int_0^t e^{-s}[f(s,x(s) - x(s)]ds \text{ ....(2.25)}$$

Therefore from the above equation and by using the periodic boundary condition given by equation (2.22.c) one can have:

$$x(T)e^{-T} = x(0) + \sum_{k=1}^{m} e^{-\tau_k}\varphi_k(x(\tau_k)) + \int_0^T e^{-s}\left[f(s,x(s)) - x(s)\right]ds$$

$$= x(0)e^{-T}$$

and this implies that

$$x(0) = \frac{1}{e^{-T}-1}\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\tau_k} + \frac{1}{e^{-T}-1}\int_0^T e^{-s}\left[f(s,x(s)) - x(s)\right]ds\,.$$

By substituting the above equation in equation (2.25) one can get:

$$x(t) = \sum_{k=1}^{m}\frac{1}{e^{-T}-1}e^{-\tau_k+t}\varphi_k(x(\tau_k)) + \frac{e^t}{e^{-T}-1}\int_0^T e^{-s}\left[f(s,x(s)) - x(s)\right]ds +$$

$$\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\tau_k+t} + e^t\int_0^t e^{-s}\left[f(s,x(s)) - x(s)\right]ds$$

$$= \sum_{k=1}^{m}\frac{e^{t-\tau_k}}{1-e^T}\varphi_k(x(\tau_k)) + \frac{e^t}{1-e^T}\int_0^t e^{-s}\left[f(s,x(s)) - x(s)\right]ds +$$

$$\frac{e^{T+t}}{1-e^T}\int_t^T e^{-s}\left[f(s,x(s)) - x(s)\right]ds$$

Thus

$$x(t)=\sum_{k=1}^{m}G(t,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(t,s)[f(s,x(s))-x(s)]ds$$

(ii) From equation (2.23) one can have:

$$x'(t)=\sum_{k=1}^{m}\frac{1}{1-e^T}e^{(t-\tau_k)}\varphi_k(x(\tau_k))+\frac{1}{1-e^T}\left[\int_0^t e^{(t-s)}[f(s,x(s))-x(s)]ds+\right.$$
$$\left.\int_t^T e^{(T+t-s)}[f(s,x(s))-x(s)]ds\right]+\frac{1}{1-e^T}[f(t,x(t))-x(t)]-\frac{e^T}{1-e^T}[f(t,x(t))-x(t)]$$
$$=x(t)+f(t,x(t))-x(t).$$
$$=f(t,x(t)).$$

This implies that *x(t)* given by equation (2.23) satisfied equation (2.22.a).

Next, by taking the limits of *x(t)* that given by equation (2.23) as $t\to\tau_k^+$ and $t\to\tau_k^-$ one can get:

$$x(\tau_i^+)=\lim_{t\to\tau_i^+}\left[\sum_{k=1}^{m}G(t,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(t,s)[f(s,x(s))-x(s)]ds\right]$$
$$=\frac{1}{1-e^T}\varphi_i(x(\tau_i))+\sum_{\substack{k=1\\k\neq i}}^{m}\frac{e^{T+\tau_i+\tau_k}}{1-e^T}\varphi_k(x(\tau_k))+\int_0^T G(\tau_i,s)[f(s,x(s))-x(s)]ds,\ i=1,2,\ldots,m$$

and

$$x(\tau_i^-)=\lim_{t\to\tau_i^-}\left[\sum_{k=1}^{m}G(t,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(t,s)[f(s,x(s))-x(s)]ds\right]$$

Thus

$$x(\tau_i^-)=\frac{e^T}{1-e^T}\varphi_i(x(\tau_i))+\sum_{\substack{k=1\\k\neq i}}^{m}\frac{e^{T+\tau_i+\tau_k}}{1-e^T}\varphi_k(x(\tau_k))+\int_0^T G(\tau_i,s)[f(s,x(s))-x(s)]ds,\ i=1,2,\ldots,m.$$

Hence

$$\Delta x(\tau_i)=\varphi_i(x(\tau_i)),\ i=1,2,\ldots,m.$$

Therefore *x(t)* given by equation (2.23) satisfied equation (2.22.b).

Since

$$\begin{aligned}G(0,s)&=\frac{1}{1-e^T}e^{(T-s)}\\&=G(T,s),\ 0\le s\le T.\end{aligned}$$

Therefore

$$\begin{aligned}x(0)&=\sum_{k=1}^{m}G(0,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(0,s)f(s)ds\\&=\sum_{k=1}^{m}G(T,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(T,s)f(s)ds\\&=x(T).\end{aligned}$$

Therefore *x(t)* given by equation (2.23) satisfied equation (2.22.c).■

Next, to illustrate the previous theorems, we consider the following examples:

Example (2.5):

Consider the periodic boundary value problem which consists of the first order nonlinear ordinary differential equation with one impulsive condition:

$$x'(t)=\cos(x(t)),\ t\in[0,1]\setminus\left\{\frac{1}{2}\right\}$$

together with the impulsive condition:

$$\Delta x\left(\frac{1}{2}\right)=\cot\left(x\left(\frac{1}{2}\right)\right)$$

and the periodic boundary condition:

$$x(0)=x(1)$$

then by using theorem (2.5) the solution of the above periodic boundary value problem satisfy the integral equation:

$$x(t)=G\left(t,\frac{1}{2}\right)\cot\left(x\left(\frac{1}{2}\right)\right)+\int_0^1 G(t,s)\left[\cos(x(s))-x(s)\right]ds$$

$$=\frac{e^{t-\frac{1}{2}}}{1-e}\cot\left(x\left(\frac{1}{2}\right)\right)+\frac{e^t}{1-e}\int_0^t e^{-s}\left[\cos(x(s))-x(s)\right]ds+\frac{e^{t+1}}{1-e}\int_t^1 e^{-s}\left[\cos(x(s))-x(s)\right]ds.$$

Example (2.6):

Consider the periodic boundary value problem which consists of the first order nonlinear ordinary differential equation with tow impulsive condition:

$$x'(t)=\sec(x(t)),\qquad t\in[0,2]\setminus\left\{\frac{1}{2},\frac{2}{3}\right\}$$

together with the two impulsive conditions:

$$\Delta x\left(\frac{1}{2}\right)=\cot\left(x\left(\frac{1}{2}\right)\right)$$

$$\Delta x\left(\frac{2}{3}\right)=\tan\left(x\left(\frac{2}{3}\right)\right)$$

and the periodic boundary condition:

$$x(0)=x(2)$$

then by theorem (2.6) the solution of the above periodic boundary value problem satisfy the integral equation:

$$x(t) = G\left(t,\frac{1}{2}\right)\cot\left(x\left(\frac{1}{2}\right)\right) + G\left(t,\frac{2}{3}\right)\tan\left(x\left(\frac{2}{3}\right)\right) + \int_0^2 G(t,s)\left[\sec(x(s)) - x(s)\right]ds$$

$$= \frac{e^{t-\frac{1}{2}}}{1-e^2}\cot\left(x\left(\frac{1}{2}\right)\right) + \frac{e^{t-\frac{2}{3}}}{1-e^2}\cot\left(x\left(\frac{2}{3}\right)\right) + \frac{e^t}{1-e^2}\int_0^t e^{-s}\left[\sec(x(s)) - x(s)\right]ds +$$

$$\frac{e^{t+1}}{1-e^2}\int_t^2 e^{-s}\left[\sec(x(s)) - x(s)\right]ds.$$

2.4 Existence of the Solutions of the Periodic Boundary Value Problems of the First Order Nonlinear Ordinary Differential Equations with Finite Impulsive Conditions:

In this section, we present the existence theorem for the solutions of the nonlinear periodic boundary value problems which consist of first order nonlinear ordinary differential equation together with finite impulsive conditions and periodic boundary condition. This section is a modification and a correction of the facts that appeared in [Jinhai C. and et. al., 2007]. To do this, consider the nonlinear periodic boundary value problems given by equations (2.18).

Before we give the existence theorem for the solutions of equations (2.18) we need the following remark and lemmas.

Remark (2.1), [Jinhai C. and et. al., 2007]:

Let $f:[0,T]\times\Re^n \to \Re^n$ and $\varphi_1:\Re^n \to \Re^n$ be continuous functions. Let $T: PC([0,T],\Re^n) \to PC([0,T],\Re^n)$ be the mapping defined by

$$Tx(t) = G(t,\tau_1)\varphi_1(x(\tau_1)) + \int_0^T G(t,s)[f(s,x(s)) - x(s)]ds, \quad \text{……..……………..(2.26)}$$

where $PC([0,T],\Re^n)=\{x:[0,T]\to\Re^n, x\in C([0,T]\setminus\{\tau_1\},\Re^n)$, x is left continuous at τ_1, the right hand limit $x(\tau_1^+)$ exists$\}$ with the norm $\|x\|_{PC}=\sup\limits_{t\in[0,T]}\|x(t)\|$, and G is the Green's function defined by equation (2.3). If T has a fixed point, then this fixed point is also a solution to equations (2.18).

Lemma (2.1), [Jinhai C. and et. al., 2007]:

Let $f:[0,T]\times\Re^n\to\Re^n$ and $\varphi_1:\Re^n\to\Re^n$ be continuous functions. Then T defined by equation (2.26) is a compact map.

Lemma (2.2), (Schaefer), [Jinhai C. and et. al., 2007]:

Let X be a normed space and $T:X\to X$ be a compact mapping. If the set

$S=\{u\in X: u=\lambda Tu$ for some $\lambda\in[0,1]\}$

is bounded then T has at least one fixed point.

Next, the following lemma is used in the proof of the existence for the solution of equation (2.24). This lemma appeared in [Jinhai C. and et. al., 2007] without proof. Here we give its proof.

Lemma (2.3):

If *x(t)* satisfies the following integral equation:

$$x(t)=\lambda\left[G(t,\tau_1)\varphi_1(x(\tau_1))+\int_0^T G(t,s)\left[f(s,x(s))-x(s)\right]ds\right]\text{......................}(2.27)$$

then *x(t)* is a solution of the periodic boundary value problem which consists of the first order nonlinear ordinary differential equation:

$x'(t)-x(t)=\lambda\left[f(t,x(t))-x(t)\right],$ ……………………………………...…..(2.28.a)

together with the impulsive condition:

$\Delta x(\tau_1)=\lambda\,\varphi_1(x(\tau_1))$ ……………………………………………………(2.28.b)

and the periodic boundary condition:

$x(0)=x(T)$ ……………………………………………………………..(2.28.c)

Proof:

If $\lambda=0,$ then $x(t)=0$. It is clear that this solution satisfy equations (2.28). So, assume that $\lambda\in(0,1]$. By differentiating equation (2.27) with respect to *x* and by using the definition of the Green function G given by equation (2.3) one can get:

$$x'(t)=\lambda\left[\frac{1}{1-e^T}e^{-(\tau_1-t)}\varphi_1(x(\tau_1))+\frac{1}{1-e^T}e^t\int_0^t e^{-s}\left[f(s,x(s))-x(s)\right]ds+\frac{1}{1-e^T}e^{T+t}\int_t^T e^{-s}\left[f(s,x(s))-x(s)\right]ds\right]+\lambda\left[\frac{1}{1-e^T}\left[f(t,x(t))-x(t)\right]-\frac{e^T}{1-e^T}\left[f(t,x(t))-x(t)\right]\right]$$

But $\lambda\neq 0$ then

$$x'(t)=\lambda\frac{1}{\lambda}x(t)+\lambda\left[f(t,x(t))-x(t)\right]$$

and hence

$$x'(t)-x(t)=\lambda\left[f(t,x(t))-x(t)\right]$$

This implies that *x(t)* given by equation (2.27) satisfy equation (2.28.a).

Next, to verify equation (2.28.b) one must take the limits of *x(t)* given by equation (2.27) as $t\to\tau_1^+$ and $t\to\tau_1^-$ to obtain:

$$x(\tau_1^+) = \lim_{t \to \tau_1^+} \left[\lambda \left[G(t,\tau_1)\varphi_1(x(\tau_1)) + \int_0^T G(t,s)\left[f(s,x(s)) - x(s) \right] ds \right] \right]$$

$$= \lambda \left[\frac{1}{1-e^T} \varphi_1(x(\tau_1)) + \int_0^T G(\tau_1,s)\left[f(s,x(s)) - x(s) \right] ds \right]$$

and

$$x(\tau_1^-) = \lim_{t \to \tau_1^-} \left[\lambda \left[G(t,\tau_1)\varphi_1(x(\tau_1)) + \int_0^T G(t,s)\left[f(s,x(s)) - x(s) \right] ds \right] \right]$$

$$= \lambda \left[\frac{e^T}{1-e^T} \varphi_1(x(\tau_1)) + \int_0^T G(\tau_1,s)\left[f(s,x(s)) - x(s) \right] ds \right].$$

Therefore

$$\Delta x(\tau_1) = \lambda \varphi_1(x(\tau_1)).$$

and this implies that *x(t)* given by equation (2.27) satisfied equation (2.28.b).

Since

$$G(0,s) = G(T,s),\ 0 \le s \le T$$

then

$$x(0) = \lambda \left[G(0,\tau_1)\varphi_1(x(\tau_1)) + \int_0^T G(0,s)\left[f(s,x(s)) - x(s) \right] ds \right]$$

$$= \lambda \left[G(T,\tau_1)\varphi_1(x(\tau_1)) + \int_0^T G(T,s)\left[f(s,x(s)) - x(s) \right] ds \right]$$

$$= x(T)$$

and this implies that *x(t)* given by equation (2.27) satisfied equation (2.28.c).■

Now, we are ready to present the existence theorem for the solutions of equations (2.18).

Theorem (2.7):

Let $f:[0,T]\times\Re^n\to\Re^n$ and $\varphi_1:\Re^n\to\Re^n$ be continuous functions. If there exist non-negative constants α, K, β, L such that:

$$\|f(t,p)-p\|\le 2\alpha\langle p, f(t,p)\rangle + K, \quad \text{for all} \quad (t,p)\in[0,T]\setminus\{\tau_1\}\times\Re^n$$

$$\|\varphi_1(q)\|\le\beta\|q\|+L \qquad \text{for all } q\in\Re^n$$

$$\beta e^T < e^T - 1$$

where $\|.\|$ is the usual Euclidean norm and $\langle .,.\rangle$ will be the Euclidean inner product, $\langle .,.\rangle = \|.\|^2$. Then equations (2.18) have at least one solution.

Proof:

From equation (2.3) and in case $A=I$ one can deduce that

$$\gamma = \sup_{(t,s)\in[0,T]\times[0,T]} |G(t,s)| \le \frac{e^T}{e^T-1}.$$

Let T be the operator that is defined by equation (2.26).

Thus by using remark (2.1) the fixed points of T will be solutions of equations (2.18). From lemma (2.1) we know that T is a compact map. In order to show that T has at least one fixed point, we use Shaefer's lemma by showing that all solutions to:

$x(t)=\lambda Tx(t),\ \lambda\in[0,1]$

are bounded, with the bound being independent of λ. With this, let *x(t)* be a solution of the above equation. Then by using lemma (2.3) *x(t)* is a solution of equations (2.28). For each $t\in[0,T]$, one can get:

$$\|x(t)\| = \lambda\|Tx(t)\|$$

$$\leq \gamma\left\{\int_0^T \lambda\| f(s,x(s)) - x(s)\| ds + \lambda\|\varphi_1(x(\tau_1))\|\right\}$$

$$\leq \gamma\left\{\int_0^T [2\alpha\langle x(s), \lambda f(s,x(s))\rangle + \lambda K\,]ds + \beta\|x(\tau_1)\| + L\right\}$$

$$\leq \gamma\left\{\int_0^T [2\alpha\langle x(s), \lambda f(s,x(s)) + (1-\lambda)x(s)\rangle + K]ds + \beta\|x(\tau_1)\| + L\right\}$$

$$= \gamma\left\{\int_0^T [2\alpha\langle x(s), x'(s)\rangle + K\,]ds + \beta\|x(\tau_1)\| + L\right\}$$

$$= \gamma\left\{\int_0^T \left[\alpha\frac{d}{ds}\left(\|x(s)\|^2\right) + K\right]ds + \beta\|x(\tau_1)\| + L\right\}$$

$$= \gamma\left\{\alpha\left(\|x(T)\|^2 - \|x(0)\|^2\right) + KT + \beta\|x(\tau_1)\| + L\right\}$$

$$= \gamma\{KT + \beta\|x(\tau_1)\| + L\}$$

$$\leq \frac{e^T}{e^T - 1}\{KT + \beta\|x(\tau_1)\| + L\}.$$

Thus, taking the supremum of the above inequality one can have:

$$\sup_{t\in[0,T]}\|x(t)\| \leq \frac{[KT + L]e^T}{(1-\beta)e^T - 1}.$$

Thus we see that the bound on all solutions of equation (2.18) is independent of λ. Therefore by using Schaefer's lemma, T given by equation (2.26) has at least one fixed point and hence equation (2.18) has at least one solution.■

Second, we generalize the previous theorem to be valid for periodic boundary value problem given by equations (2.22). But before that we need the following remark and lemmas.

Remark (2.2), [Jinhai C. and et. al., 2007]:

Let $f:[0,T]\times\Re^n\to\Re^n$ and $\varphi_k:\Re^n\to\Re^n$, $k=1,2,\ldots,m$ be continuous functions, consider $T:PC([0,T],\Re^n)\to PC([0,T],\Re^n)$ be the mapping defined by:

$$Tx(t)=\sum_{k=1}^{m}G(t,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(t,s)[f(s,x(s))-x(s)]ds \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.29)$$

where G is the Green's function that is defined by equation (2.3) where *A*=*I*. If T has a fixed point , then this fixed point is also a solution to the equations (2.22).

Lemma (2.4), [Jinhai C. and et. al., 2007]:

Let $f:[0,T]\times\Re^n\to\Re^n$ and $\varphi_k:\Re^n\to\Re^n$, $k=1,2,\ldots,m$ both be continuous functions. Then *T* defined by equation (2.29) is a compact map.

Next, the following lemma is used in the proof of the existence for solutions of equation (2.22). This lemma is a generalization of lemma (2.3).

Lemma (2.5):

If *x(t)* is a solution of the following equation:

$$x(t)=\lambda\left[\sum_{k=1}^{m}G(t,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(t,s)\big[f(s,x(s))-x(s)\big]ds\right],k=1,2,\ldots,m$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.30)$$

then *x(t)* is a solution of the periodic boundary value problem which consists of first order nonlinear ordinary differential equation:

$x'(t) - x(t) = \lambda\left[f(t, x(t)) - x(t)\right]$ ……………………..…………………...(2.31.a)

together with the finite impulsive conditions:

$\Delta x(\tau_k) = \lambda \sum_{k=1}^{m} \varphi_k(x(\tau_k)), k = 1, 2, \ldots, m$ ………………………………...(2.31.b)

and the periodic boundary condition:

$x(0) = x(T)$ ………………………………………………………..(2.31.c)

Proof:

If $\lambda = 0$, then $x(t) = 0$. It is clear that this solution satisfy equations (2.31). So, assume that $\lambda \in (0,1]$.

By differentiating equation (2.30) with respect to *x* and by using the definition of the Green function *G* given by equation (2.3) one can get:

$$x'(t) = \lambda\left[\frac{1}{1-e^T}\sum_{k=1}^{m} e^{-(\tau_k - t)}\varphi_k(x(\tau_k)) + \frac{1}{1-e^T}e^t\int_0^t e^{-s}\left[f(s, x(s)) - x(s)\right]ds + \frac{1}{1-e^T}e^{T+t}\int_t^T e^{-s}\left[f(s, x(s)) - x(s)\right]ds\right] + \lambda\left[\frac{1}{1-e^T}\left[f(t, x(t)) - x(t)\right] - \frac{e^T}{1-e^T}\left[f(t, x(t)) - x(t)\right]\right]$$

But $\lambda \neq 0$ then

$$x'(t) = \lambda\frac{1}{\lambda}x(t) + \lambda\left[f(t, x(t)) - x(t)\right]$$

and hence

$$x'(t) - x(t) = \lambda\left[f(t, x(t)) - x(t)\right]$$

This implies that *x(t)* given by equation (2.30) satisfies equation (2.31.a).

Next, to verify equation (2.31.b) one must take the limits of x given by equation (2.30) as $t \to \tau_k^+$ and $t \to \tau_k^-$ to obtain:

$$x(\tau_i^+) = \lim_{t \to \tau_i^+} \left[\lambda \left[\sum_{k=1}^{m} G(t,\tau_k))\varphi_k(x(\tau_k)) + \int_0^T G(t,s)[f(s,x(s)) - x(s)]ds \right]\right]$$

$$= \lambda \frac{1}{1-e^T} \varphi_i(x(\tau_i)) + \lambda \sum_{\substack{k=1 \\ k \neq i}}^{m} \frac{e^{T+\tau_i+\tau_k}}{1-e^T} \varphi_k(x(\tau_k)) + \lambda \int_0^T G(\tau_i,s)[f(s,x(s)) - x(s)]ds,\ i = 1,2,\ldots,m$$

and

$$x(\tau_i^-) = \lim_{t \to \tau_i^-} \left[\lambda \left[\sum_{k=1}^{m} G(t,\tau_k))\varphi_k(x(\tau_k)) + \int_0^T G(t,s)[f(s,x(s)) - x(s)]ds \right]\right]$$

$$= \lambda \frac{e^T}{1-e^T} \varphi_i(x(\tau_i)) + \lambda \sum_{\substack{k=1 \\ k \neq i}}^{m} \frac{e^{T+\tau_i+\tau_k}}{1-e^T} \varphi_k(x(\tau_k)) + \lambda \int_0^T G(\tau_i,s)[f(s,x(s)) - x(s)]ds,\ i = 1,2,\ldots,m$$

Therefore

$$\Delta x(\tau_i) = \lambda \varphi_i(x(\tau_i)),\ i = 1,2,\ldots,m$$

and this implies that *x(t)* given by equation (2.30) satisfied equation (2.31.b).

Since

$$G(0,s) = G(T,s),\ 0 \le s \le T\ .$$

Then

$$x(0) = \lambda \left[\int_0^T G(0,s)[f(s,x(s)) - x(s)]ds + \sum_{k=1}^{m} G(0,\tau_k)\varphi_k(x(\tau_k)) \right], k = 1,2,\ldots,m$$

$$= \lambda \left[\int_0^T G(T,s)[f(s,x(s)) - x(s)]ds + \sum_{k=1}^{m} G(T,\tau_k)) \right],\ k = 1,2,\ldots,m$$

$$= \lambda \left[\frac{1}{\lambda} x(T) \right]$$

Then

$x(0) = x(T)$. ■

and this implies that *x(t)* given by equation (2.30) satisfied equation (2.31.c).

Now, we are ready to present the existence theorem of the solutions of equations (2.22).

Theorem (2.8):

Let $f:[0,T]\times\Re^n \to \Re^n$ and $\varphi_k:\Re^n \to \Re^n$, $k=1,2,\ldots,m$ be continuous functions. If there exist nonnegative constants α, R, β, L such that:

$\| f(t,p) - p \| \le 2\alpha\langle p, f(t,p)\rangle + R$, for all $(t,p)\in[0,T]\setminus\{\tau_k\}\times\Re^n$, $k=1,2,\ldots,m$

$\|\varphi_k(q)\| \le \beta_k \|q\| + L_k, k=1,2,\ldots,m$ for all $q\in\Re^n$

$$\sum_{k=1}^{m}\beta_k e^T < e^T - 1$$

then equations (2.22) have at least one solution.

Proof:

From equation (2.3) one can deduce that

$$\gamma = \sup_{(t,s)\in[0,T]\times[0,T]} |G(t,s)| \le \frac{e^T}{e^T - 1}.$$

Let T be the operator that is defined by equation (2.29).

Thus by using remark (2.2) the fixed points of T will be the solutions of equations (2.22). From lemma (2.4) we know that T is a compact map. In order to show that T has at least one fixed point, we use Shaefer's lemma by showing that all solutions to

$x(t) = \lambda Tx(t),\ \lambda \in [0,1],$

are bounded, with the bound being independent of λ. With this, let *x(t)* be a solution of the above equation. Then by using lemma (2.5) *x(t)* is a solution of equations (2.31).

For each $t \in [0,T]$, one can get:

$$\|x(t)\| = \lambda\|Tx(t)\|$$

$$\le \gamma\left\{\int_0^T \lambda\| f(s,x(s)) - x(s)\|ds + \lambda\sum_{k=1}^{m}\|\varphi_k(x(\tau_k))\|\right\}$$

$$\le \gamma\left\{\int_0^T \left[2\alpha\langle x(s), \lambda f(s,x(s))\rangle + \lambda R\right]ds + \sum_{k=1}^{m}\left(\beta_k\|x(\tau_k)\| + L_k\right)\right\}$$

$$\le \gamma\left\{\int_0^T \left[2\alpha\langle x(s), \lambda f(s,x(s)) + (1-\lambda)x(s)\rangle + R\right]ds + \sum_{k=1}^{m}\left(\beta_k\|x(\tau_k)\| + L_k\right)\right\}$$

$$= \gamma\left\{\int_0^T \left[2\alpha\langle x(s), x'(s)\rangle + R\ \right]ds + \sum_{k=1}^{m}\left(\beta_k\|x(\tau_k)\| + L_k\right)\right\}$$

$$= \gamma\left\{\int_0^T \alpha\frac{d}{ds}\left(\|x(s)\|^2\right) + R\,ds + \sum_{k=1}^{m}\left(\beta_k\|x(\tau_k)\| + L_k\right)\right\}$$

$$= \gamma\left\{\alpha\left(\|x(T)\|^2 - \|x(0)\|^2\right) + RT + \sum_{k=1}^{m}\left(\beta_k\|x(\tau_k)\| + L_k\right)\right\}$$

$$= \gamma\left\{RT + \sum_{k=1}^{m}\left(\beta_k\|x(\tau_k)\| + L_k\right)\right\}$$

$$\le \frac{e^T}{e^T - 1}\left\{RT + \sum_{k=1}^{m}\left(\beta_k\|x(\tau_k)\| + L_k\right)\right\}.$$

Thus, taking the supremum of the above inequality one can have:

$$\sup_{t\in[0,T]} \|x(t)\| \le \frac{[RT + \sum_{k=1}^{m} L_k]e^T}{\left[1 - \sum_{k=1}^{m} \beta_k\right]e^T - 1}.$$

Thus we see that the bound on all solution of equation (2.22) is independent of λ. Therefore by using Schaefer's lemma, *T* given by equation (2.29) has at least one fixed point and hence equation (2.22) has at least one solution.

Introduction:

The aim of this chapter is to discuss the existence of the solutions of the boundary value problem of the first order non-linear impulsive ordinary differential equation with finite impulsive conditions.

This chapter consists of two sections.

In section one, we give the explicit forms of the solutions for the boundary value problem of the first order linear ordinary differential equations that contains finite impulsive condition.

In section two, we give an integral equation that is equivalent to the periodic boundary value problem of the first order nonlinear ordinary differential equation with finite impulsive conditions.

3.1 Solutions for the Boundary Value Problems of the First Order Linear Ordinary Differential Equations with Finite Impulsive Conditions:

Recall that Nieto J. and et al. in 2000 gave explicit forms of the solutions of the boundary value problems which consist of the first order linear ordinary differential equation together with the boundary condition. This section is a modification of the facts that appeared in [Nieto J. and et. al. 2000] and it gives explicit forms for the solutions of the boundary value problems which consist of the first order linear ordinary differential equations together with finite impulsive condition and boundary condition. To do this, consider the boundary value problem which consists of the first order linear ordinary differential equation with nonconstant coefficients:

$x'(t) = A(t)x(t) + f(t), t \in J'$ ……………………………………………(3.1.a)

together with the finite impulsive conditions:

$\Delta x(\tau_k) = \varphi_k(x(\tau_k)), k = 1, 2, \ldots, m$ ……………………………………(3.1.b)

and the boundary condition:

$x(0) = x(T) + \lambda$ ....(3.1.c)

where $f:[0,T] \to \Re^n$ is a continuous function on $t \in J' = [0,T] \setminus \{\tau_1, \tau_2, \ldots, \tau_m\}$, $\lambda \in \Re$, A is $n \times n$ nonzero function matrix, $\varphi_k : \Re^n \to \Re^n$, is a continuous function for each $k = 1,2,\ldots,m$, $\tau_k \in (0,T)$, $k = 1,2,\ldots,m$ and $\tau_1 < \tau_2 < \ldots < \tau_m$.

The following theorem gives an explicit form for the solution of equations (3.1).

Theorem (3.1):

(i) If *x(t)* is a solution of equations (3.1) then

$$x(t) = \sum_{k=1}^{m} G(t,\tau_k)\varphi_k(x(\tau_k)) + \int_0^T G(t,s) f(s)ds + \frac{\lambda e^{\int_0^t A(s)ds}}{1 - e^{\int_0^T A(s)ds}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.2)$$

where G is the Green's function defined by equation (2.8).

(ii) If *x(t)* satisfies equation (3.2) then *x(t)* is a solution of equations (3.1).

Proof:

(i) Consider equation (2.17). Therefore from equation (2.17) and by using the boundary condition given by equation (3.1.c) one can have

$$x(T)e^{-\int_0^T A(s)ds} = x(0) + \sum_{k=1}^{m} \varphi_k(x(\tau_k))e^{-\int_0^{\tau_k} A(s)ds} + \int_0^T e^{-\int_0^s A(y)dy} f(s)ds$$

$$= [x(0) - \lambda]e^{-\int_0^T A(s)ds}$$

and this implies that

$$x(0)=\frac{1}{e^{-\int_0^T A(s)ds}-1}\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\int_0^{\tau_k} A(s)ds}+\frac{1}{e^{-\int_0^T A(s)ds}-1}\int_0^T e^{-\int_0^s A(y)dy} f(s)ds+\frac{\lambda e^{-\int_0^T A(s)ds}}{e^{-\int_0^T A(s)ds}-1}.$$

By substituting the above equation in equation (2.17) one can get:

$$x(t)=\frac{e^{\int_0^t A(s)ds}}{e^{-\int_0^T A(s)ds}-1}\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\int_0^{\tau_k} A(s)ds}+\frac{e^{\int_0^t A(s)ds}}{e^{-\int_0^T A(s)ds}-1}\int_0^T e^{-\int_0^s A(y)dy} f(s)ds+$$

$$e^{\int_0^t A(s)ds}\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\int_0^{\tau_k} A(s)ds}+e^{\int_0^t A(s)ds}\int_0^t e^{-\int_0^s A(y)dy} f(s)ds$$

$$+\frac{\lambda e^{-\int_0^T A(y)dy}\,e^{\int_0^t A(y)dy}}{e^{-\int_0^T A(s)ds}-1}$$

Thus

$$x(t)=\frac{e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\int_0^{\tau_k} A(s)ds}+\frac{e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\int_0^t e^{-\int_0^s A(y)dy} f(s)ds+$$

$$\frac{e^{\int_0^T A(s)ds}\,e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\int_t^T e^{-\int_0^s A(y)dy} f(s)ds+\frac{\lambda e^{\int_0^t A(y)dy}}{1-e^{\int_0^T A(s)ds}}$$

$$=\sum_{k=1}^{m}G(t,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(t,s)f(s)ds+\frac{\lambda e^{\int_0^t A(y)dy}}{1-e^{\int_0^T A(y)dy}}$$

(ii) From equation (3.2) one can have:

$$x'(t)=A(t)\frac{e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\int_0^{\tau_k} A(s)ds}+A(t)\frac{e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\int_0^t e^{-\int_0^s A(y)dy} f(s)ds+$$

$$A(t)\frac{e^{\int_0^T A(s)ds}\,e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\int_t^T e^{-\int_0^s A(y)dy} f(s)ds+A(t)\frac{\lambda e^{\int_0^t A(y)dy}}{1-e^{\int_0^T A(y)dy}}+\frac{e^{\int_0^t A(s)ds}\,e^{-\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}f(t)-$$

$$\frac{e^{\int_0^T A(s)ds}}{1-e^{\int_0^T A(s)ds}}f(t)$$

Thus

$$x'(t)=A(t)\left[\frac{e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\int_0^{\tau_k}A(s)ds}+\frac{e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\int_0^t e^{-\int_0^s A(y)dy}f(s)ds+\right.$$

$$\left.\frac{e^{\int_0^T A(s)ds}\,e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\int_t^T e^{-\int_0^s A(y)dy}f(s)ds+\frac{\lambda e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\right]+\left[\frac{1-e^{\int_0^T A(s)ds}}{1-e^{\int_0^T A(s)ds}}\right]f(t)$$

$$=A(t)x(t)+f(t).$$

This implies that *x(t)* given by equation (3.2) satisfied equation (3.1.a).

Next, by taking the limits of *x(t)* that given by equation (3.2) as $t\to\tau_i^+$ and $t\to\tau_i^-$ one can get:

$$x(\tau_i^+)=\lim_{t\to\tau_i^+}\left[\sum_{k=1}^{m}G(t,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(t,s)f(s)ds+\frac{\lambda e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\right]$$

$$=\frac{1}{1-e^{\int_0^T A(s)ds}}\varphi_i(x(\tau_i))+\sum_{\substack{k=1\\k\neq i}}^{m}\frac{e^{\int_0^T A(s)ds+\int_0^{\tau_i}A(s)ds-\int_0^{\tau_k}A(s)ds}}{1-e^{\int_0^T A(s)ds}}\varphi_k(x(\tau_k))+$$

$$\int_0^T G(\tau_i,s)f(s)ds+\frac{\lambda e^{\int_0^{\tau_i}A(s)ds}}{1-e^{\int_0^T A(s)ds}},\ i=1,2,\ldots,m$$

and

$$x(\tau_i^-) = \lim_{t\to\tau_i^-}\left[\sum_{k=1}^{m} G(t,\tau_k)\varphi_k(x(\tau_k)) + \int_0^T G(t,s)f(s)ds + \frac{\lambda e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}}\right]$$

$$= \frac{e^{\int_0^T A(s)ds}}{1-e^{\int_0^T A(s)ds}}\varphi_i(x(\tau_i)) + \sum_{\substack{k=1\\k\neq i}}^{m}\frac{e^{\int_0^T A(s)ds+\int_0^{\tau_i} A(s)ds-\int_0^{\tau_k} A(s)ds}}{1-e^{\int_0^T A(s)ds}}\varphi_k(x(\tau_k)) +$$

$$\int_0^T G(\tau_i,s)f(s)ds + \frac{\lambda e^{\int_0^t A(s)ds}}{1-e^{\int_0^T A(s)ds}},\ i=1,2,\ldots,m.$$

Hence

$\Delta x(\tau_i) = \varphi_i(x(\tau_i)),\ i=1,2,\ldots,m$.

Therefore *x(t)* given by equation (3.2) satisfied equation (3.1.b).

Since

$$G(0,s) = \frac{1}{1-e^{\int_0^T A(s)ds}} e^{\int_0^T A(s)ds - \int_0^s A(y)dy}$$

$$= G(T,s),\ 0\le s\le T.$$

Therefore

$$x(T)+\lambda = \sum_{k=1}^{m} G(T,\tau_k) + \varphi_k(x(\tau_k)) + \int_0^T G(T,s)f(s)ds + \frac{\lambda e^{\int_0^T A(s)ds}}{1-e^{\int_0^T A(s)ds}} + \lambda$$

Thus

$$x(T)+\lambda=\sum_{k=1}^{m}G(T,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(T,s)f(s)ds+\frac{\lambda}{1-e^{\int_0^T A(s)ds}}$$

$$=\sum_{k=1}^{m}G(0,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(0,s)f(s)ds+\frac{\lambda}{1-e^{\int_0^T A(s)ds}}$$

$$=x(0)$$

and this implies that *x(t)* given by equation (3.2) satisfied equation (3.1.c).■

Next, the proof of the following corollary is easy thus we omitted it.

Corollary (3.1):

Consider the boundary value problem which consists of the first order linear ordinary differential equations with constant coefficients:

$x'(t)=Ax(t)+f(t),\ t\in J'$ ……………………………………………..….(3.3.a)

together with the finite impulsive conditions:

$\Delta x(\tau_k)=\varphi_k(x(\tau_k)),\ k=1,2,\ldots,m$ ………...………………………………...(3.3.b)

and the boundary condition:

$x(0)=x(T)+\lambda$ …………...…………………………………………..(3.3.c)

where $f:[0,T]\to\Re^n$ is a continuous functions on $t\in J'=[0,T]\setminus\{\tau_1,\tau_2,\ldots,\tau_m\}$, $\lambda\in\Re,\ \varphi_k:\Re^n\to\Re^n,\ k=1,2,\ldots,m$ is a continuous function, $\tau_k\in(0,T)$ and A is nonzero $n\times n$ constant matrix.

(i) If *x(t)* is a solution of equations (3.3) then

$$x(t)=\sum_{k=1}^{m}G(t,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(t,s)f(s)\,ds+\frac{\lambda e^{At}}{1-e^{AT}} \quad \ldots\ldots\ldots\ldots(3.4)$$

where G is the Green's function given by equation (2.3).

(ii) If $x(t)$ satisfies equation (3.4) then $x(t)$ is a solution of equations (3.3).

Next, to illustrate the previous theorem and its corollary, we consider the following examples:

Example (3.1):

Consider the boundary value problem which consists of the first order linear ordinary differential equation with nonconstant coefficients:

$x'(t)=tx(t)+t,\ t\in[0,4]\setminus\{1,2,3\}$

together with three impulsive conditions:

$\Delta x(1)=x(1)$

$\Delta x(2)=x(2)$

$\Delta x(3)=x(3)$

and the boundary condition:

$x(0)=x(4)+0.5$

then by using theorem (3.1), the solution of the above boundary value problem can be written as:

$$x(t)=\frac{e^{\int_0^t s\,ds}}{1-e^{\int_0^4 s\,ds}}x(1)e^{-\int_0^1 s\,ds}+\frac{e^{\int_0^t s\,ds}}{1-e^{\int_0^4 s\,ds}}x(2)e^{-\int_0^2 s\,ds}+\frac{e^{\int_0^t s\,ds}}{1-e^{\int_0^4 s\,ds}}x(3)e^{-\int_0^3 s\,ds}+$$

$$\frac{e^{\frac{t^2}{2}}}{1-e^8}\int_0^t e^{-\frac{s^2}{2}}s\,ds+\frac{e^{8+\frac{t^2}{2}}}{1-e^8}\int_t^4 e^{-\frac{s^2}{2}}s\,ds+\frac{0.5e^{\frac{t^2}{2}}}{1-e^8}$$

Thus

$$x(t)=\frac{e^{\frac{t^2}{2}}}{1-e^{\int_0^4 s\,ds}}x(1)e^{-\frac{1}{2}}+\frac{e^{\frac{t^2}{2}}}{1-e^{\int_0^4 s\,ds}}x(2)e^{-2}+\frac{e^{\frac{t^2}{2}}}{1-e^{\int_0^4 s\,ds}}x(3)e^{-\frac{9}{2}}+$$

$$\frac{e^{\frac{t^2}{2}}}{1-e^8}\left[-e^{-\frac{s^2}{2}}\right]_0^t+\frac{e^{8+\frac{t^2}{2}}}{1-e^8}\left[-e^{-\frac{s^2}{2}}\right]_t^4+\frac{0.5e^{\frac{t^2}{2}}}{1-e^8}$$

$$=\frac{e^{\frac{t^2}{2}}}{1-e^8}x(1)e^{-\frac{1}{2}}+\frac{e^{\frac{t^2}{2}}}{1-e^8}x(2)e^{-2}+\frac{e^{\frac{t^2}{2}}}{1-e^8}x(3)e^{-\frac{9}{2}}+$$

$$\frac{e^{\frac{t^2}{2}}}{1-e^8}\left[-e^{-\frac{t^2}{2}}+1\right]+\frac{e^{8+\frac{t^2}{2}}}{1-e^8}\left[-e^{-8}+e^{-\frac{t^2}{2}}\right]+\frac{0.5e^{\frac{t^2}{2}}}{1-e^8}.$$

Example(3.2):

Consider the boundary value problem which consists of the first order linear ordinary differential equation with constant coefficients:

$$x'(t)=3x(t)+t,\ t\in[0,1]\setminus\left\{\frac{1}{2},\frac{2}{3}\right\}$$

together with two impulsive conditions:

$$\Delta x\left(\frac{1}{2}\right)=2x\left(\frac{1}{2}\right)$$

$$\Delta x\left(\frac{2}{3}\right)=x\left(\frac{2}{3}\right)$$

and the boundary condition:

$$x(0)=x(1)+0.25$$

then by using corollary (3.1), the solution of the above boundary value problem can be written as:

$$x(t)=\frac{e^{3t}}{1-e^{3}}\left[2x\left(\frac{1}{2}\right)e^{-\frac{3}{2}}+x\left(\frac{2}{3}\right)e^{-2}\right]+\frac{3e^{3t}}{1-e^{3}}\int_{0}^{t}e^{-3s}sds+\frac{3e^{3(1+t)}}{1-e^{3}}\int_{t}^{1}e^{-3s}sds+\frac{0.25e^{3t}}{1-e^{3}}$$

$$=\frac{e^{3t}}{1-e^{3}}\left[2x\left(\frac{1}{2}\right)e^{-\frac{3}{2}}+x\left(\frac{2}{3}\right)e^{-2}\right]-\frac{3e^{3t}}{2(1-e^{3})}\left[-\frac{se^{-3s}}{3}-\frac{e^{-3s}}{9}\right]_{0}^{t}-$$

$$\frac{3e^{3(1+t)}}{2(1-e^{3})}\left[-\frac{se^{-3s}}{3}-\frac{e^{-3s}}{9}\right]_{t}^{1}+\frac{0.25e^{3t}}{1-e^{3}}$$

$$=\frac{e^{3t}}{1-e^{3}}\left[2x\left(\frac{1}{2}\right)e^{-\frac{3}{2}}+x\left(\frac{2}{3}\right)e^{-2}\right]-\frac{3e^{3t}}{2(1-e^{3})}\left[-\frac{te^{-3t}}{3}-\frac{e^{-3t}}{9}+\frac{1}{9}\right]-$$

$$\frac{3e^{3(1+t)}}{2(1-e^{3})}\left[-\frac{e^{-3}}{3}-\frac{e^{-3}}{9}+\frac{te^{-3t}}{3}+\frac{e^{-3t}}{9}\right]+\frac{0.25e^{3t}}{1-e^{3}}$$

3.2 Solutions for the Boundary Value Problems of the First Order Nonlinear Ordinary Differential Equations with Finite Impulsive Conditions:

In this section, we transform the boundary value problems which consist of first order nonlinear ordinary differential equation together with finite impulsive conditions and boundary condition into integral equations.

This section is a modification of the previous section. To do this, consider the boundary value problem which consists of the first order nonlinear ordinary differential equation:

$x'(t)=f(t,x),\ t\in J'$ ……………………………………………………(3.5.a)

together with the finite impulsive conditions:

$\Delta x(\tau_k)=\varphi_k(x(\tau_k)),\ k=1,2,\ldots,m$ ……………………………………………(3.5.b)

and boundary condition:

$x(0) = x(T) + \lambda$ ……………………………………………………………..(3.5.c)

where $f : [0,T] \times \Re^n \to \Re^n$ is a continuous function on $t \in J' = [0,T] \setminus \{\tau_1, \tau_2, \ldots, \tau_m\}$, $\lambda \in \Re$, $\varphi_k : \Re^n \to \Re^n$ is a continuous function for each $k = 1,2,\ldots,m$, $\tau_k \in (0,T)$, $k = 1,2,\ldots,m$ and $\tau_1 < \tau_2 < \ldots < \tau_m$.

The following theorem shows that any solution of equations (3.5) must satisfy an integral equation and conversely.

Theorem (3.2):

(i) If *x(t)* is a solution of equations (3.5) then

$$x(t) = \sum_{k=1}^{m} G(t,\tau_k)\varphi_k(x(\tau_k)) + \int_0^T G(t,s)[f(s,x(s)) - x(s)]ds + \frac{\lambda e^t}{1-e^T} \quad \text{.............(3.6)}$$

where *G* is the Green's function defined by equation (2.3) in case *A*=*I*.

(ii) If *x(t)* satisfies equation (3.6) then *x(t)* is a solution of equations (3.5).

Proof:

(i) Consider equation (2.25). Therefore from this equation and by using the boundary condition given by equation (3.5.c) one can have:

$$x(T)e^{-T} = x(0) + \sum_{k=1}^{m} e^{-\tau_k}\varphi_k(x(\tau_k)) + \int_0^T e^{-s}[f(s,x(s)) - x(s)]ds$$

$$= [x(0) - \lambda]e^{-T}$$

and this implies that

$$x(0) = \frac{1}{e^{-T}-1}\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\tau_k} + \frac{1}{e^{-T}-1}\int_0^T e^{-s}[f(s,x(s)) - x(s)]ds + \frac{\lambda}{1-e^T}$$

By substituting the above equation in equation (2.25) one can get:

$$x(t)=\sum_{k=1}^{m}\frac{1}{e^{-T}-1}e^{-\tau_k+t}\varphi_k(x(\tau_k))+\sum_{k=1}^{m}\varphi_k(x(\tau_k))e^{-\tau_k+t}+$$

$$\frac{e^{t}}{e^{-T}-1}\int_0^T e^{-s}[f(s,x(s))-x(s)]ds+e^{t}\int_0^t e^{-s}[f(s,x(s))-x(s)]ds+\frac{\lambda e^{t}}{1-e^{T}}$$

$$=\sum_{k=1}^{m}\frac{e^{-T}e^{t}e^{-\tau_k}}{e^{-T}-1}\varphi_k(x(\tau_k))+\frac{e^{-T}e^{t}}{e^{-T}-1}\int_0^t e^{-s}[f(s,x(s))-x(s)]ds+$$

$$\frac{e^{t}}{e^{-T}-1}\int_t^T e^{-s}[f(s,x(s))-x(s)]ds+\frac{\lambda e^{t}}{1-e^{T}}$$

$$=\sum_{k=1}^{m}\frac{e^{t-\tau_k}}{1-e^{T}}\varphi_k(x(\tau_k))+\frac{e^{t}}{1-e^{-T}}\int_0^t e^{-s}[f(s,x(s))-x(s)]ds+$$

$$\frac{e^{T+t}}{1-e^{T}}\int_t^T e^{-s}[f(s,x(s))-x(s)]ds+\frac{\lambda e^{t}}{1-e^{T}}$$

$$=\sum_{k=1}^{m}G(t,\tau_k)\varphi_k(x(\tau_k))+\int_0^T G(t,s)[f(s,x(s))-x(s)]ds+\frac{\lambda e^{t}}{1-e^{T}}.$$

(ii) From equation (3.6) one can have:

$$x'(t)=\sum_{k=1}^{m}\frac{1}{1-e^{T}}e^{(t-\tau_k)}\varphi_k(x(\tau_k))+\frac{1}{1-e^{T}}[f(t,x(t))-x(t)]+$$

$$\frac{1}{1-e^{T}}\left[\int_0^t e^{(t-s)}[f(s,x(s))-x(s)]ds+\int_t^T e^{(T+t-s)}[f(s,x(s))-x(s)]ds\right]-$$

$$\frac{e^{T}}{1-e^{T}}[f(t,x(t))-x(t)]+\frac{\lambda e^{t}}{1-e^{T}}$$

$$=x(t)+f(t,x(t))-x(t)$$

$$=f(t,x(t)).$$

This implies that *x(t)* given by equation (3.6) satisfies equation (3.5.a).

Next, by taking the limits of *x(t)* that given by equation (3.6) as $t \to \tau_i^+$ and $t \to \tau_i^-$ one can get:

$$x(\tau_i^+) = \lim_{t\to\tau_i^+}\left[\sum_{k=1}^{m} G(t,\tau_k)\varphi_k(x(\tau_k)) + \int_0^T G(t,s)f(s,x(s))ds + \frac{\lambda e^t}{1-e^T}\right]$$

$$= \frac{1}{1-e^T}\varphi_i(x(\tau_i)) + \sum_{\substack{k=1\\k\neq i}}^{m}\frac{e^{T+\tau_i+\tau_k}}{1-e^T}\varphi_k(x(\tau_k)) +$$

$$\int_0^T G(\tau_i,s)f(s,x(s))ds + \frac{\lambda e^{\tau_i}}{1-e^T},\ i=1,2,\ldots,m$$

and

$$x(\tau_i^-) = \lim_{t\to\tau_i^-}\left[\sum_{k=1}^{m} G(t,\tau_k)\varphi_k(x(\tau_k)) + \int_0^T G(t,s)f(s,x(s))ds + \frac{\lambda e^t}{1-e^T}\right]$$

$$= \frac{e^T}{1-e^T}\varphi_i(x(\tau_i)) + \sum_{\substack{k=1\\k\neq i}}^{m}\frac{e^{T+\tau_i+\tau_k}}{1-e^T}\varphi_k(x(\tau_k)) +$$

$$\int_0^T G(\tau_i,s)f(s,x(s))ds + \frac{\lambda e^{\tau_i}}{1-e^T},\ i=1,2,\ldots,m$$

Hence

$$\Delta x(\tau_i) = \varphi_i(x(\tau_i)),\ i=1,2,\ldots,m$$

Therefore *x(t)* given by equation (3.6) satisfied equation (3.5.b).

Since

$$G(0,s) = \frac{1}{1-e^T}e^{T-s}$$

$$= G(T,s),\ 0 \le s \le T.$$

Therefore

$$x(T)+\lambda=\sum_{k=1}^{m}G(T,\tau_k)+\varphi_k(x(\tau_k))+\int_0^T G(T,s)f(s,x(s))ds+\frac{\lambda e^T}{1-e^T}+\lambda$$

$$=\sum_{k=1}^{m}G(0,\tau_k)+\varphi_k(x(\tau_k))+\int_0^T G(0,s)f(s,x(s))ds+\frac{\lambda}{1-e^T}$$

$$= x(0)$$

and this implies that *x(t)* given by equation (3.6) satisfied equation (3.5.c).■

Next, the proof of the following corollary is easy so we omitted it.

Corollary (3.2):

Consider the boundary value problem which consists of the first order nonlinear ordinary differential equations:

$x'(t)=f(t,x),\ t\in J'$ ………..……………………………….…………(3.7.a)

together with the impulsive condition:

$\Delta x(\tau_1)=\varphi_1(x(\tau_1))$ ………………………………………………...(3.7.b)

and the boundary condition:

$x(0)=x(T)+\lambda$ ..……………………………………………………..(3.7.c)

where $f:[0,T]\times\Re^n\to\Re^n$ is a continuous function on $t\in J'=[0,T]\setminus\{\tau_1\}$, $\lambda\in\Re$, $\varphi_1:\Re^n\to\Re^n$ is a continuous function and $\tau_1\in(0,T)$.

(i) If *x(t)* is a solution of equations (3.7) then *x(t)* satisfy the integral equation:

$$x(t)=G(t,\tau_1)\varphi_1(x(\tau_1))+\int_0^T G(t,s)[f(s,x(s)-x(s)]ds+\frac{\lambda e^t}{1-e^T} \quad \text{……………...(3.8)}$$

where G is the Green's function defined by equation (2.3) in case *A=1*.

(ii) If *x(t)* satisfies equations (3.8) then *x(t)* is a solution of equations (3.7).

Next, to illustrate the previous theorem and its corollary, we consider the following examples:

Example (3.3):

Consider the boundary value problem which consists of the first order nonlinear ordinary differential equation:

$x'(t) = \cos(x(t)),\ t \in [0,3] \setminus \{1,2\}$

together with two impulsive conditions:

$\Delta x(1) = 2x(1)$

$\Delta x(2) = 7x(2)$

together with the boundary condition:

$x(0) = x(3) + 0.3$

then by using theorem (3.2), the solution of the above boundary value problem can be written as:

$$x(t) = 2\frac{e^{t-1}}{1-e^3}x(1) + 7\frac{e^{t-2}}{1-e^3}x(2) + \frac{e^t}{1-e^3}\int_0^t e^{-s}\left[\cos(x(s)) - x(s)\right]ds + \frac{e^{(3+t)}}{1-e^3}\int_t^3 e^{-s}\left[\cos(x(s)) - x(s)\right]ds + \frac{0.3e^t}{1-e^3}.$$

Example (3.4):

Consider the boundary value problem which consists of the first order nonlinear ordinary differential equation:

$x'(t) = \sin(x(t)),\ t \in [0,5] \setminus \{1\}$

together with the impulsive condition:

$\Delta x(1) = 8x(1)$

and the boundary value condition:

$x(0) = x(5) + 0.7$

then by using corollary (3.2), the solution of the above boundary value problem written as:

$$x(t) = 8\frac{e^{(t-1)}}{1-e^5}x(1) + \frac{e^t}{1-e^5}\int_0^t e^{-s}\left(\sin x(t) - x(t)\right)ds + \frac{e^{(5+t)}}{1-e^5}\int_t^5 e^{-s}\left(\sin x(t) - x(t)\right)ds .$$

Conclusions and Recommendations

From the present study, we can conclude the following:

1. If $x(\tau_k^+) = x(\tau_k^-), k = 0, \mp 1, \mp 2, \ldots$ and $J' = J$ in definition (1.6) then the n-th order impulsive ordinary differential equation reduces to the n-th order ordinary differential equation.
2. The first order impulsive delay ordinary differential equations with single delay can be also extended to the n-th order impulsive delay ordinary differential equations with multiple delays
3. Theorem (2.4) and theorem (2.6) can be obtained from theorem (3.1) and (3.2) by letting $\lambda = 0$.
4. Every periodic and nonperiodic boundary value problem which consist of the first order linear ordinary differential equation together with finite impulsive conditions has a unique solution.
5. All the previous theorems can be easily modified for the anti-periodic boundary value problems of the first order non-linear finite impulsive ordinary differential equations.

Also, for future work, we can recommened the introduction of the following open problems:

1. Give method for solving the impulsive differential equations.
2. Study the impulsive partial differential equations.
3. Discuss the existence of the solutions of the impulsive ordinary differential equation with infinite impulsive conditions.
4. Devote the impulsive integro-differential equations.

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المستخلص

الهدف الرئيسي لهذا العمل يمكن تقسيمه إلى ثلاثة محاور، و التي يمكن تلخيصها كما يلي:
أولاً، قمنا بإعطاء بعض تعاريف للمعادلات التفاضلية الدفعيّة و المعادلات التفاضلية الدفعية التباطؤية مع بعض الأمثلة التوضيحية و بعض التطبيقات الحياتية.

ثانياً، قمنا بإعطاء الأشكال الصريحة لحلول مسائل القيم الحدودية (الدورية و ألا دورية) التي تتكون من معادلات تفاضلية دفعيّة اعتيادية خطية ذات الرتبة الأولى ذات المعاملات الثابتة و المتغيرة مع عدد منتهي من الشروط الدفعيّة و الشرط الحدودي (الدوري و ألا دوري).

ثالثاً، قمنا بتحويل المسائل ذات القيم الحدودية الدورية و الغير دورية التي تتضمن المعادلات التفاضلية الدفعيّة الاعتيادية اللا ًخطية ذات الرتبة الأولى مصحوبة بشروط دفعيّة منتهية و شرط حدودي دوري إلى معادلات تكاملية مكافئة. كما قمنا بمناقشة وجود الحلول لهذه الأنواع من المسائل.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة النهرين

كلية العلوم

قسم الرياضيات وتطبيقات الحاسوب

نظريات وجود الحلول لمسائل القيم الحدودية للمعادلات التفاضلية الاعتيادية الدفعيّة

رسالة

مقدمة الى قسم الرياضيات وتطبيقات الحاسوب، كلية العلوم، جامعة النهرين

وهي جزء من متطلبات نيل درجة ماجستير علوم في الرياضيات

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بإشراف

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شوّال ١٤٢٩

تشرين الأول ٢٠٠٨