Republic of Iraq
Ministry of Higher Education
and Scientific Research
Al–Nahrain University
College of Science

Least Square Method for Finding Absorbing Areas of Planar Quadratic Maps

A Thesis
Submitted to the College of Science at
Al-Nahrain University in Partial Fulfillment of
the Requirements for the Degree of
Master of Science in Mathematics and Computer Applications

by
Mohanad Nafaa Gaafer Al-Zubaidy
(B.Sc.,2006)

Supervised by

Dr. Zainab Abdulnabi Salman

September 2009 Shawwal 1430

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

قَالُواْ سُبْحَانَكَ لاَ عِلْمَ لَنَا إِلاَّ مَا عَلَّمْتَنَا إِنَّكَ أَنتَ الْعَلِيمُ الْحَكِيمُ

صدق الله العظيم

سورة البقرة

الآية (٣٢)

الأهداء

الى الشفاه التي أكثرت الدعاء لنا كلما نطقت

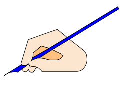
الى العين التي رأت فينا أملاً كلما نظرت

الى القلوب التي أزدادت بنا فخراً كلما نبضت

الى أمي وأبي

الى أخواني

الى من أحبهم ويحبوني



مهند

Acknowledgement

Before every thing …

Thanks to Allah for helping me to complete my thesis

I wish to express my deep appreciation and sincere thanks to my supervisor Dr. Zainab Abdul-Nabi Salman for his appreciable advices, important comments support and encouragement during the research.

It is pleasure to record my independences to the staff members of Mathematics and Computer Applications, whom gave me all facilities during my work and pursuit of my academic study.

Thanks are extended to the College of Science of Al-Nahrain University for giving me the chance to complete my postgraduate study.

Last but not least, I am deeply indebted to my family, father, mother, my sister and my brothers. And my friends Salam, Mohammed, Omar,Akram, Adel, and all my friends, and every one who helped me in my project.

Mohanad N.

September 2009

I certify that this thesis was prepared under my supervision at the College of Science of Al–Nahrain University as partial fulfillment of the requirements for the degree of Master of Science in Mathematics and Computer Applications.

Signature:

Name: Dr. Zainab Abdul-Nabi Salman

In view of the available recommendations. I forward this thesis for debate by the examination committee.

Signature:
Name: Dr. Akram M. AL-Abood
(Assist. Prof.)
Head of Mathematics and Computer Applications Department
Date: / / 2009

Examining Committee Certification

We certify that we have read this thesis entitled " Least Square Method for Finding Absorbing Areas of Planar Quadratic Maps" and as examining committee examined the student *(Mohanad Nafaa Gaafer)* in its contents and in what it
connected with, and that, in our opinion, it meets the standards of a thesis for the degree of Master of Science in Mathematics.

(Chairman)

Signature:

Name: Dr. Akram M. Al-Abood

Date: / / 2009

(Member)

Signature:

Name: Dr. Raid K. Naji

Date: / / 2009

(Member)

Signature:

Name: Dr. Shatha A. Aziz

Date: / / 2009

(Member and Supervisor)

Signature:

Name: Dr. Zeinab A. salman

Date: / / 2009

Approved by the Collage of Science

Signature

Name: Asst. Prof. Dr. LAITH ABDUL AZIZ Al-ANI

Dean of the Collage of Science

Data: / / 2 009

List of Symbols

E	Euclidean space
P^*	Fixed point
$W_l^u(p^*)$	Local unstable set of fixed point p^*
$W^u(p^*)$	global unstable set of fixed point p^*
$W_l^s(p^*)$	Local stable set of fixed point p^*
$W^s(p^*)$	global stable set of fixed point p^*
B_p	Basin of attraction
A	Closed set
T	Planar quadratic map
LC_{-1} or $J(T)$	Critical set of T
LC_i	Critical curve of rank-i of the map T
$\overline{L}C_i$	Extra critical curve
$EC(T^M)$	Critical curve of T^M
d'	Absorbing area of non-mixed type
$\tilde{d}'$	Absorbing area of mixed type
d'_a	Approximated absorbing area of non-mixed type
$\tilde{d}'_a$	Approximated absorbing area of mixed type
d''_a	Approximated invariant area (mixed or non-mixed)
d	Chaotic area
S	Invariant area
Δ	Closed area
s	Absorbing area non invariant
d''	Connected non invariant absorbing area
Δ_0	Closed subset of $\overline{R_2}$

Abstract

Planar noninvertible maps have been studied recently by several authors such as Mira [32], Gardini [17], and Cathala [9], much of their work has been concentrated on analyzing some examples and making some conclusions on the properties of the maps.

The main purpose of this thesis can be divided into three objectives:

First objective: introduce the mathematical background of the main notions and proposition on the theory of the dynamical system. Specifically we shall foucus our study on planar nonivertiable continuously differentiable maps T: $\Re^2 \rightarrow \Re^2$. Definition of critical curves and some different types of noninvertible maps related to their critical curves and some properties of critical curves are presented.

Second objective: we have studied some properties of such kind of maps in particular absorbing areas, invariant areas of such maps. Also, we give proposed algorithm to approximate the equations of the critical curves LC_i which cause find an approximated absorbing and invariant areas such as least square method.

Third objective: give some illustrative examples that use the proposed algorithm to find an approximated absorbing and invariant areas.

In our work, we have made use of the Matlab version 7.0 software to solve the discussed examples.

Contents

Introduction

What is a dynamical system? Dynamical systems are a branch of mathematics that attempts to understand processes in motion. Such processes occur in some branches of science. For example, the motion of planets is a dynamical system, one that has been studied for centuries. Some other systems are the stock market, the world's weather, and the rise and fall of populations [5].

Dynamical systems are the study of the long-term behavior of evolving systems. The modern theory of dynamical systems originated at the end of 19th century with fundamental questions concerning the stability and evolution of the solar system. Attempts to answer those questions led to the development of a rich and powerful field with applications to physics, and biology, meteorology, astronomy, economics, and other areas[5].

The basic goal of dynamical systems is to understand the eventual or asymptotic behavior of an iterative process. If this process is differential equations whose independent variable is time, then the theory attempts to predict the uptime behavior of solutions of the equation in either the distant future ($t\to\infty$) or the distant past ($t\to-\infty$). If this process is a discrete process such as the iteration of a function, then the theory hopes to understand the eventual behavior of the points x , $f(x)$, $f^2(x)$, …,$f^n(x)$ as n becames large. That is, the dynamical systems asks the some what non mathematical sounding question: where do points go and what do get there? Functions which determine dynamical systems are also called mappings, or maps, for short [11].

The complex dynamical behavior of solutions of various mathematical models has been an object of study for a number of years. Point-mappings or recurrence, are especially of interest became they appear as natural descriptions of evolutionary phenomena in physics ecology, biology and control systems [9,33&34].

A complex dynamical behavior called "chaos" is observed in mathematical models expressed in the from of recurrences with non-unique inverses. The attractive limit sets of an endomorphism are located in phase plane domains bounded by segments of critical curve and absorbing area [14& 35].

The theory of critical curves for maps of the plane provides powerful tools for locating the chief characteristic features of a discrete dynamical system in two dimensions: the location of its chaotic attractors, its basin boundaries, and the mechanisms of its bifurcations. Nowadays one begins to recognize the role played by critical curves of maps in the analysis, in the understanding and description of the bifurcations, and transition to chaotic behavior in coupled maps [4].

Critical curve permit to define the essential notions of absorbing area, and chaotic area [25], [26] [28]. Roughly speaking an absorbing area (d') is a region bounded by critical curves segments of finite rank, such that, the successive images of all points of a neighborhood U(d') enter into (d') and cannot get away after entering after a finite number of iterations. Except for some bifurcation cases, a chaotic area is an invariant absorbing area, the points of which give rise to iterated sequences (or orbits) having the property of sensitivity to initial conditions.

The term of critical curve was first introduced in 1964 by Mira who provides an entry into certain areas of current research on noninvertible maps and the role of such curve in bifurcations basin. It is a natural generalization in $\Re^2$ of the notion of critical points of one dimensional endomorphisms. Several authors have investigated and have shown the importance of critical curves in the bifurcations specially Gumowski and Mira [25, 26] and Gardini [15,16] who have developed the role of critical curves in bifurcations.

Many researchers were interested in the field of noninvertible maps due to their importance. The following are some of them:

- Gardini L. in [15] studied the global dynamics and bifurcations of a croeconomic model which showed the interactions between "good market "and" the money market" by using the role of critical curves.
- Gardini L. et al., in [24] studied the dynamics occurring in Logistic map and by use of critical curves, absorbing and invariant areas were determined inside which global bifurcation of the attracting sets (fixed points, closed invariant curves, cycles or chaotic attractors) take place. The basin of attraction of the absorbing areas are determined together with their bifurcation.
- Cathala J., in [10] examined chaotic areas and absorbing area without specifying the structures of the attractors that they contain for the map ($T: x \to ax + y, y \to bx + x^3, b = -1.9$) also he defined some bifurcations that modify the nature of the chaotic areas.
- Mira C., et al., in [31] determined dynamical properties and bifurcations for the map ($T: x \to x^2 - y^2 + \lambda + \varepsilon x, y \to 2xy - \frac{5}{2}\varepsilon y$) by using critical curves.
- Illhem D. , and boukemara I. , in [29] studied the behavior under iteration of a three parameters family of piecewise linear maps of the plane

$T: \begin{matrix} x'=1-ax-by, & y'=x & if\ x \succ 0 \\ x'=1+ax-by+c, & y'=x & if\ x \prec 0 \end{matrix}$ defined by linear functions, where a, b, c are real parameters, treateded by numerical methods and they showed that this family have several attractors .

- Brahim K. et al. , in [4] studied properties of the noninvertible maps (T_1: $x_{n+1}=1-ax_n^2$, $y_{n+1}=1-ay_n^2+b|x_n-y_n+c|$), (T_2: $x_{n+1}=1-ax_n^2+b(y_n^2-x_n^2)$, $y_{n+1}=1-ay_n^2+b(x_n^2-y_n^2)$, (T_3: $x_{n+1}=1-ax_n^2+\frac{b'}{2\pi}\sin 2\pi\ (y_n^2-x_n^2)$, $y_{n+1}=1-ay_n^2+\frac{b'}{2\pi}\sin 2\pi\ (x_n^2-y_n^2)$) which posses a chaotic attractor played by such curves in bifurcation theory are give by [7], [8], [9], [17], [18], [19], [20], [21], [22] &[23].

The aim of this thesis is to study the noninvertible planar maps $T:\Re^2\rightarrow\Re^2$ in particular approximated absorbing areas of such maps, via use Least square method to approximate the equation of the critical curves LC_i.

It is important to remark that much of the work done on planar maps concentrated on presented certain examples and pointing out certain phenomena.

The work is divided into three chapters; these chapters are organized as follows:

Chapter one introduced the mathematical background of the main notions and proposition on the theory of the dynamical system. Definition of critical curves and some different types of noninvertible maps related to their critical curves are presented, some properties relate to images & preimages are given. Also, we shall give the definition of planar quadratic maps.

Chapter two deals with a special type of planar maps, namely (z_0-z_2) maps. This chapter includes proving some properties of absorbing areas and invariant areas and some properties of a (z_0-z_2) maps and how to construct approximated absorbing areas, approximated invariant areas by using the critical curves.

In chapter three we shall illustrate the concepts of the two previous chapters by applications on some noninvertible maps. First we shall apply our proposed approach on example and compare geted area that with area that obtained by applying construction algorithm. In the other examples we shall use our proposed algorithm to determine some phenomena.

Our examples in this chapter illustrate certain phenomena that are different from the ones found in Literature.

Chapter one

Preliminaries

Introduction

It is always very useful to derive some consequences from a few bits of information. A dynamical system describe the stable of points of a given space δ (where δ may be a Euclidean space or open subset of a Eualidean space), as time passage [12],[26]. Then we can recognize two types of dynamical system: discrete dynamical system and continuous dynamical system [2].

For discrete dynamical system we can define it to be a map $g:\delta\to\delta$, where δ is an open set in E which assigns each point x in δ to $g(x)$ after one unit of time. After two units of time x will be in state $g^2(x)=g(g(x))$, after n units of time, x will in the state $g^n(x)$. Then we have discrete family $\{g^n(x) \mid n\in N\}$ where N is the natural set.

For continuous dynamical system is a mapping φ_t: $\delta\to\delta$. Where takes x into x_t, (i.e. $\varphi_t(x)=x_t$) which defines for each t. At time zero, x is at x_0, and after one unit of time x will be at x_1, two units of time later x at x_2 and so on.

The goal of this chapter is to introduce several of the basic definitions from dynamical systems, it contain four sections:

In the first section, we shall introduce such concepts: preimages, periodic points, local & global unstable set, local & global stable set. In the second section, we recall definitions of the critical curves and give some types of noninvertible maps that related the critical curves. In the third one, we give some properties of the critical curves. In last one, we shall give the definition and some properties of planar quadratic maps.

Throughout this work we shall focus our study on continuously differential maps and discrete systems, moreover, our maps are of form $\Re^2 \to \Re^2$ i.e. (planar maps).

1.1 Definitions and Notations

Definition (1.1.1):[34,p.2]

Let $T: X \to X$, x_r ,(r positive integer) is called the rank-r image of x, if $x_r = T^r x$. Similarly, x is one of the rank-r preimage of x_r.

Definition (1.1.2):[34,p.2]

Let $X = \Re^n$, then the map $T: X \to X$ will be called diffeomorphism, if it is continuously differentiable function of x, and T^{-1} exists, unique and continuously differentiable (in the case T is invertible) in the domain of definition of T. when T is such that T^{-1} may be multi-valued, or may not exist, then T will be called a noninvertible map.

Example (1.1.1):

Consider the one dimensional map T, i.e. $T: \Re \to \Re$ which is given by:

$$x' = x^2$$

T^{-1} is given by:

$$x = \pm\sqrt{x'}$$

so the rank-one preimage of a point x' is double-real value for $x' > 0$, and is not real for $x' < 0$.

Definition (1.1.3):[27,p.21]

A periodic point of period k is a point x in which the domain of T such that $T^k(x) = x$ and in addition $x, T(x), T^2(x), \ldots, T^{k-1}(x)$ are distinct.

Definition (1.1.4):[27,p.21]

The orbit of $x \in X$ is the set $\{T^k(x): k \geq 0\}$. If x is a periodic point of period k, then the orbit of x which is $\{x, T(x), T^2(x), \ldots, T^{k-1}(x)\}$ will be a periodic orbit and is called a k-cycle.

Definition (1.1.5):[23,p.6]

A cycle of order k=1 is called a fixed point of T. Every point of k-cycle is fixed point of T^k.

Definition (1.1.6):[27, p.158]

Let p be a fixed point of T. Then p is attracting if and only if there is a disk centered at p such that $T^n(v) \to p$, for every v in the disk. By contrast, p is repelling if and only if there is a disk centered at p such that $\|T(v)-T(p)\| > \|v-p\|$ for every v in the disk for which $v \neq p$.

Remark (1.1.1):[34, p.4]

A k-cycle is attracting, if all the eigenvalues of the jacobian matrix of T^k at the period point, have their modulus less than one. If at least one of the eigenvalues in modulus is larger then one the cycle is repulsive, these eigenvalues are called the multipliers of the cycle, and are denoted by S_i. A k-cycle is expanding if all the $|S_i|>1$, i=1,2,…,p. and there exists a neighborhood U of the cycle such that $|S_i|>1$, i=1,2,…,p, for any x belonging to U, this is in case all the eigenvalues are real.

Remark (1.1.2):[34, p.4]

If some of the eigenvalues of the Jacobin matrix of T^k are complex at the periodic point then a periodic point will be stable (attracting) if all the eigenvales have negative real parts otherwise it is unstable (repeller).

Definition (1.1.7):[34, p.109]

A fixed point p is called a snap-back repeller, or SBR if

a) it is expanding;

b) if in the neighborhood of p $U(p)$ there exists a point q such that $T^m(q) = p$ for some positive integer m.

Definition (1.1.8):[27,p.16]

The basin of attraction B_p of p is the set consists of all x such that $T^n(x) \to p$ as n increases without bound.

Definition (1.1.9):[4,p.247]

A non empty set A is said to be invariant by T if $T(A) = A$. The set A is a backward invariant by T if $T^{-1}(A) = A$, where T^{-1} represents all the rank-one preimage of T.

Definition (1.1.10):[23,p.7]

A closed invariant set A is an attracting set if an arbitrary small neighborhood U of A exists such that $T(U) \subset U$ and $T^n(x) \to A$ when $n \to \infty$, for any $x \in U$.

Definition (1.1.11):[23,p.7]

A closed invariant set A called is topologically transitive, if for any two open set $U, V \subset A$ a positive integer k exists such that $T^k(U) \cap V \neq \emptyset$, or equivalently a point $p \in A$ exists the orbit of which is dense in A.

Definition (1.1.12):[23,p.7]

An attractor is an attracting set which is topologically transitive.

Definition (1.1.13):[34,p.15]

Let T be a p-dimensional noninvertiable map define in $\Re^p$ and p^* a repulsive fixed point, and U be a neighborhood of p^*. The local unstable set $W_\ell^u(p^*)$ of p^* in U, and the global unstable set of p^*, $W^u(p^*)$ are given by:

$W_\ell^u(p^*)=\{x\in U : x_{-n}\in T^{-n}(x)\to p^* \text{ and } x_{-n}\in U, \forall n\}$,

$W^u(p^*)=\{x\in\Re^p : x_{-n}\in T^{-n}(x)\to p^*\}=W^u(p^*)=\bigcup_{n\geq 0} T^n(W_\ell^u(p^*))$.

Some properties of the unstable set are given in the following proposition that appeared in [34,p.15]:

Proposition(1.1.1):

(P1) $T(W^u(p^*)=W^u(p^*)$, i.e. it is invariant set.

(P2) For any map T, $T^{-1}(W^u(p^*))\supseteq W^u(p^*)$.

Note that if T is noninvertible, then $W^u(p^*)$ may not be backward invariant.

(P3) Let $V(p^*)$be a neighborhood of p^*. For any $x\in W^u(p^*)$a finite integer N exists (which depends on x) such that a rank-N perimage x_{-N} of x belongs to V and a sequence of preimages of x_{-N} exists which belongs to V and converges to p^*.

Proof: can be found in [34]

Definition (1.1.14):[34, p.16]

Let p^* be fixed point of T which may be attracting or repulsive. The local stable set of p^* in a neighborhood U, and the global stable $W_\ell^s(p^*)$, $W^s(p^*)$ are given by:

$W_\ell^s(p^*) = \{x \in U,\ x_n = T^n(x) \to p^* \text{ and } x_n \in U,\ \forall n\}$,

$W^s(p^*) = \{x \in \Re^p,\ x_n = T^n(x) \to p^*\} = \bigcup_{n \geq 0} T^{-n}(W_\ell^s(p^*))$.

Some properties of the stable set are given in the following proposition that appeared in [34,p.16]

Proposition(1.1.2):

(P1) $T^{-1}(W^s(p^*) = W^s(p^*)$.

(P2) $T\ (W^s(p^*)) \subseteq W^s(p^*)$.

(P3) Let $V(p^*)$ be a neighborhood of p^*. For any $x \in W^s(p^*)$ an integer N exists (which depends on x) such that a rank-N image x_N of x belongs to V and converge to p^*.

Proof: can be found [34,p.16]

Definition(1.1.15):[34,p.62]

The point q is said homoclinic to the non attracting fixed point p^* (or homoclinic point of p^*) if $q \in W^u(p^*) \cap W^s(p^*)$, $p^* \neq q$.

Definition(1.1.16):[34,p.62]

A point q is said to be heteroclinic from the repulsive (or expanding) fixed point p^* to the repulsive fixed point r^*, if $q \in W^u(p^*) \cap W^s(r^*)$.

1.2 Two Dimensional Noninvertible Maps: Definition of Critical Curves, Types of Noninvertible Maps

The notion of critical curve *LC* (From Ligne Critique in French) was first introduced in 1964 by Mira [13] & Gardini [15,16]. It is the two-dimesional generalization of the notion of critical point of a one-dimensional noninvertiable maps.

This section essentially concerns family of continuously differentiable, two-dimensional noninvertible maps (endomorphisms), $T:\Re^2 \to \Re^2$.

We shall start by giving the definition of the critical curve *LC*.

Definition (1.2.1):[23, p.114]

Let $T:\Re^2 \to \Re^2$ be a noninvertible map defined by:

$$T: \begin{array}{l} x' = f(x,y) \\ y' = g(x,y) \end{array}$$

Where *f* and *g* are continuously differentiable functions. The critical curve of rank-1 of *T*, denoted as *LC*, is generally the image by *T* of LC_{-1} where LC_{-1} is the set of points in which the Jacobian determinant of *T* vanishes.

i.e. $LC_{-1} = \{X \in \Re^2 : |J(X)| = 0\}$;

Critical curves of rank (*i*+1) of *T* are the image of rank-*i* of the critical curve *LC*, i.e. $LC_i = T^i(LC) = T^{i+1}(LC_{-1})$, i=0,1,2,…, $LC_0 = LC$.

The critical curve *LC* and the curve LC_{-1} may be made up of several branches with respect to the inverse map T^{-1}, we observed that $\Re^2$ can be subdivided into open regions Z_i ($\Re^2 = \bigcup_i \bar{z}_i$), the points of which have *i* distinct preimages of rank one. The boundaries of these regions are the branches of rank one critical curve *LC*. Then the maximum value of index *i* of Z_i (the maximum number of first rank preimages, generated by a given map) is called the map degree *N*. i.e the plane can be considered to be made up of *N* sheets joining at the branches of first rank critical curve *LC*.

Remark (1.2.1)[1]:

If T is invertiable, then it is diffiemorphism which implies det(J (T)) $\neq 0$, so T has no critical points.

1.2.1 Types of Noninvertible Maps with Critical Curves, their Symbolic Representation [34,p.127]

Noninvertible maps, giving rise to regions Z_i, i-e noninvertible maps for which a critical curve can be defined, will be classified into types related to the nature of the regions Z_i characterizing the considered map. We shall try to mention some of these types:

1) Type (Z_0-Z_2): LC is made up of only one branch separating $\Re^2$ into two regions, one Z_0 with no preimage the other Z_2 with two first rank preimages.
2) Type (Z_1-Z_3-Z_1): LC is constituted by two branches separating $\Re^2$ into three regions, one Z_3 with three first rank preimages and two Z_1 non connected regions on both sides of Z_3 with only one first rank preimage.
3) Maps of type (Z_0-Z_2-Z_4),…, or more complex types generated by the presence of regions with higher number of rank-one preimages, the branches of LC separating these regions.

Maps may exhibit another kind of complexity related to the presence of one or several cusp points on the critical curve LC. In the simplest case, a cusp point is such that three first rank preimages coincide, and the symbolic representation of maps may be refined by introducing the symbols "<", and ">" for the presence of such a point, some of these maps are:

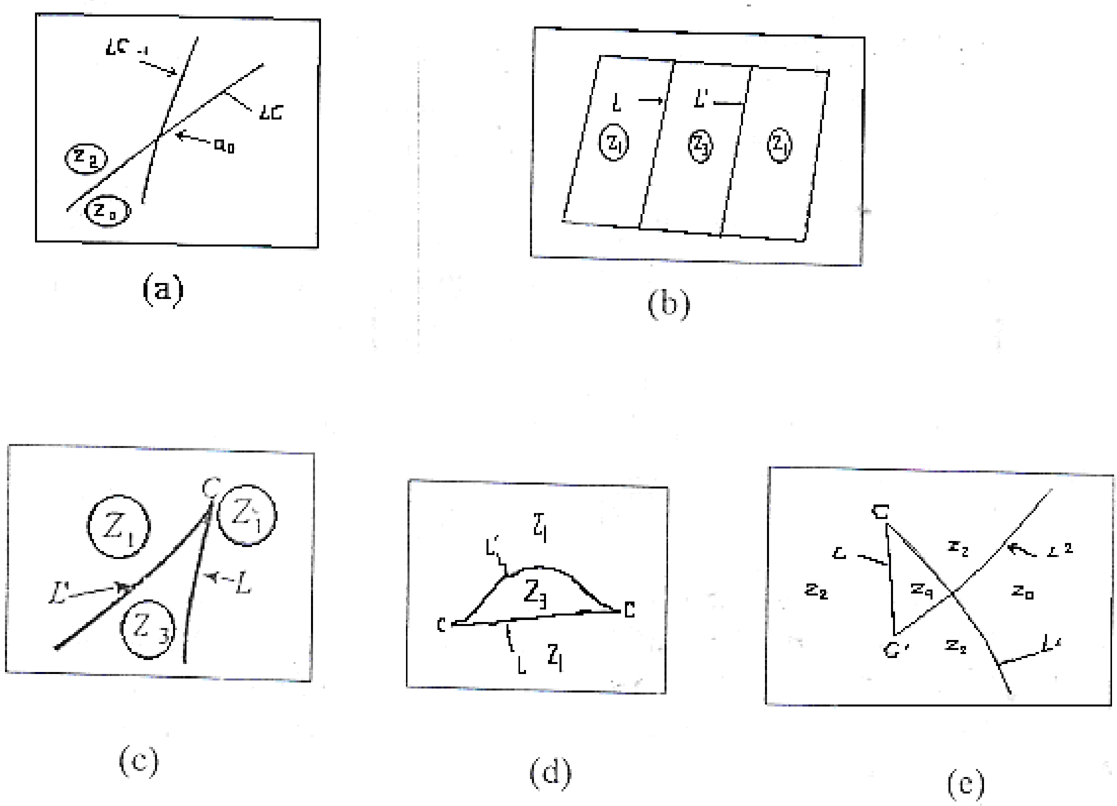
1) A map of type (Z_1<Z_3): LC has a cusp corresponding to a "cape" of Z_3 "penetrating" into Z_1.

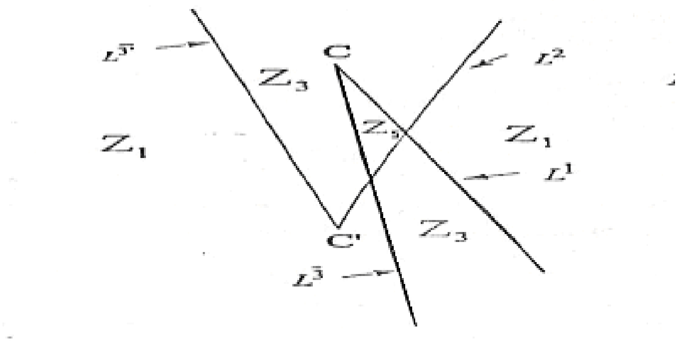
2) A map of type (Z_1<Z_3>): means that *LC* is a closed curve with two cusps forming a "lip" shape.

3) A map of type (Z_0-Z_2 <<Z_4) is such that *LC* presents two cusps each casp is a "cape" of Z_4 "penetrating" into Z_2 with a dovetail shape i.e. the two cusps have a common critical segment, and they are called a djacents.

4) A map of type (Z_1-Z_3<Z_5-Z_3>Z_1): *LC* presents two cusps not having a common segment. The minimal map degree for this type is five.

Figure (1.2.1) illustrates the above types of maps.





(f)

Fig(1.2.1)

(a) (Z_0-Z_2) map $\overline{Z}_0 \cap \overline{Z}_2 = LC$ (b) (Z_1-Z_3-Z_1)map, $LC=L\cup L'$, $\overline{Z}_1 \cap \overline{Z}_3 = L$, $\overline{Z}_3 \cap \overline{Z}_1 = L'$; (c) ($Z_1$<$Z_3$) map; $LC=L\cup L'$ cusp point ≡ c = $L \cap L'$; (d) (Z_1<Z_3>) map; LC= $L \cap L'$, $\overline{Z}_1 \cap \overline{Z}_3 = L$, $L\cap L'$=c,c'; (e) (Z_0-Z_2<<Z_4) map, $LC=L\cup L'\cup L''$, c=$L \cap L'$, c'=$L \cap L''$. (f) (Z_1-Z_3<Z_5-Z_3>Z_1) map, c=$L^1 \cap L^{\bar{3}}$, c'=$L^2 \cap L^{\bar{3}'}$, with $L^3 = L^{\bar{3}} \cup L^{\bar{3}'}$.

Note: In this work we have restricted the attention to the maps of type (Z_0-Z_2) unless otherwise stated.

1.2.2 Characterization of the Different Determinations of the Inverse Map [1]

We can define different inverses in each region Z_i, with $i>0$ (for i=0, there is no inverse that exists). Let R $_{i,j}$ be the range of one of the inverses of T defined in Z_i, j=1,2,...,i , then the corresponding inverse is:

$$T_{i,j}^{-1} : \overline{Z}_i \to \overline{R}_{i,j}$$

$$T_{1,1}^{-1} \equiv T_1^{-1}, \quad \overline{R}_{1,1} \equiv \overline{R}_1$$

Where $\overline{Z}_i, \overline{R}_{i,j}$ are the closures of Z_i, $R_{i,j}$ respectively.

The $R_{i,j}$'s are disjoint open regions bounded by arcs of LC_{-1} (the curve of rank-one merging preimages), since T possesses more than two first rank inverses, the rank-one preimages of LC consist of points at which the Jacobian of T does not vanish, these points are called extra preimages, i.e. $T^{-1}(LC) = LC_{-1} \cup \bar{L}C_{-1}$, where $\bar{L}C_{-1}$ the extra set.

Now, Let us have a little closed look at the extra set $\bar{L}C_{-1}$. Consider a branch $L \in LC$ separating the two regions Z_p, and Z_{p+2}, $p>0$. Then $p+2$ inverses, $T^{-1}_{p+2,j}(L), j = 1,2,3,\ldots,p+2$ are defined in region Z_{p+2}. Similarly p inverses, $T^{-1}_{p,j}(L)$, $j = 1,2,\ldots,p$, are defined in Z_p. If $x \in L$, let $p+1$, $p+2$ are two of a first rank preimages of x merge into $L_{-1}, T(L_{-1}) = L$ and $L_{-1} \subset T^{-1}(L)$. Thus L_{-1} is given by:

$$T^{-1}_{p+2,p+1}(L) = T^{-1}_{p+2,p+2}(L) = L_{-1}$$

The other first rank preimages, those given by $T^{-1}_{p,j}(L)$, $j=1,2,\ldots,p$, belong to the extra set $\bar{L}C_{-1}$, and are given by

$$(\bar{L}_{-1})_j = T^{-1}_{p,j}(L) \quad , j = 1,2,\ldots, p.$$

Remark(1.2.2):[1]:-

we have noticed that by the inverse function theorem the inverses of T are continuously differentiable in the interior of their domains of definition, i.e. in each region Z_i. Moreover, LC_{-1} separates the plane into regions, inside which the elements of Jacobian of T has a constant sign.

Example (1.2.1):[34]

Consider the map T defined by

$$T : \begin{array}{l} x' = y^2 + x \\ y' = ax + 1 \end{array} \quad \text{, with } a \neq 0$$

T is continuosly differentiable and noninvertible map and has type (Z_0-Z_2).

T has two inverses :

$$T^{-1} = \begin{cases} x = \dfrac{y'-1}{a} \\ y = \pm\sqrt{x' - \dfrac{y'-1}{a}} \end{cases}$$

T has a fixed point $(-\frac{1}{a},0)$. The curve LC_{-1} is given by y=0 which divides the plane $\Re^2$ into two regions R_1 with y<0, R_2 with y>0, the equation of the critical curve LC is $y=ax+1$. LC separates the plane $\Re^2$ into two regions: Z_0 with $y<ax+1$ where each point has no preimages, Z_2 with $y>ax+1$ where each point has two first rank preimages, the point of intersection of LC_{-1} and LC is $a_0=(-\frac{1}{a},0)$. We can define different inverses in region Z_2, so let $T_{2,1}^{-1}:\overline{Z}_2 \to \overline{R}_{2,1}$ be defined by

$$\begin{cases} x = \dfrac{y'-1}{a} \\ y = -\sqrt{x' - \dfrac{y'-1}{a}} \end{cases}$$

and $T_{2,2}^{-1}:\overline{Z}_2 \to \overline{R}_{2,2}$ be defined by

$$\begin{cases} x = \dfrac{y'-1}{a} \\ y = \sqrt{x' - \dfrac{y'-1}{a}} \end{cases}$$

For particular case when a=1, T has fixed point $p=(-1,0)\in LC$, Then

$$T_{2,1}^{-1}(-1,0) = T_{2,2}^{-1}(-1,0) = (-1,0) \in LC_{-1}$$

$$T_{2,1}^{-1}(0,0) = (-1,-1) \in \overline{R}_{2,1},\ T_{2,2}^{-1}(0,0) = (-1,-1) \in \overline{R}_{2,2}$$

1.2.3 Critical Set of A power of the map *T* [1].

The critical set $EC(T^m)$ of T^m [*EC* "ensemble critique" in French] can be defined by the following proposition

Proposition (1.2.1):[18]

Let T be a continuous maps

(1) If T is a map without a Z_0 region, the critical set $EC(T^m)$ of T^m, $m>1$, is given by:

$$EC(T^m) = \bigcup_{i=0}^{m-1} LC_i \text{ , } LC_0 \equiv LC \text{……………..(1.2.1)}$$

A critical curve LC_i (called critical curve of rank i) belonging to $EC(T^m)$, separates the (x,y)-plane locally into two regions, one with points having p preimage of rank m, the other with points having q preimages of rank m, $p \geq 0$, $q \geq 0$. In the general case, $q=p+2h$, $h=1,2,\ldots$

(2) When a Z_0 region exists, the critical set $EC(T^m)$ of T^m, $m>1$, is given by $EC(T^m)=LC_{m-1} \cup T^m(LC_{-2}) \cup \ldots \cup T^m(LC_{-m})$…………(1.2.2)

Where $LC_{-2}=T^{-1}(LC_{-1})$, $LC_{-3}=T^{-1}(LC_{-2})$ etc.

(3) In both case (1) and (2), $EC_{-1}(T^m)$ is defined by

$$EC_{-1}(T^m)= LC_{-1} \cup LC_{-2} \cup \ldots \cup LC_{-m}$$

Proof: can be found in [18]

1.3 Some Properties of Critical curves [34,p.138-141].

In this section we shall give some properties of critical curves, before we do this the following notations are recalled: L is a segment of the critical curve LC, L_k is a segment of the critical curve LC_k, L_{-h}, $h>0$, is a segment of the curve LC_{-h} , $\bar{L}_{-h}$ is a segment of the curve $\bar{L}C_{-h}$ of rank h extra preimages. i.e. components of $T^{-h}(LC)$ not belonging to LC_{-h}. $\bar{L}C_{k-1}$ is a non-critical (extra) curve of rank $(k-1)$ belonging to $T^{-1}(LC_k)/LC_{k-1}$. Moreover an order one contact (tangential contact) between two curves is a point of quadratic tangency of these curves**.**

The following propositions give some properties of the critical curve, will be stated without proofs, and are justified from the situations illustrated by the figures related to them, for details about proofs see [34,p.138-141].

Proposition:(1.3.1) Let $\eta_1=T(\eta)$ be the first rank image of a smooth segment η (basin boundary, critical curve,…), crossing through (transverse to) LC_{-1} in a_0.

i.e. $\eta \cap LC_{-1}=a_0$. Assume that $a_1=T(a_0)$ is not a singular point of *LC*. Then in a neighborhood of a_1, the segment η_1 is not transverse to *LC*, being located on the same side of *LC*, the one where the number of first rank preimages is the greatest. The segment η_1 and *LC* are tangent (contact of order one) in a_1. $T^k(\eta)$, $k>1$, has the same property for the point $a_k=T^k(a_0)$ and LC_{k-1}. the points a_k are non transverse contact points between $T^k(\eta)$ and LC_{k-1}, if a_k is not a singular point.

Prop.(1.3.1) will be illustrated in the following figure:

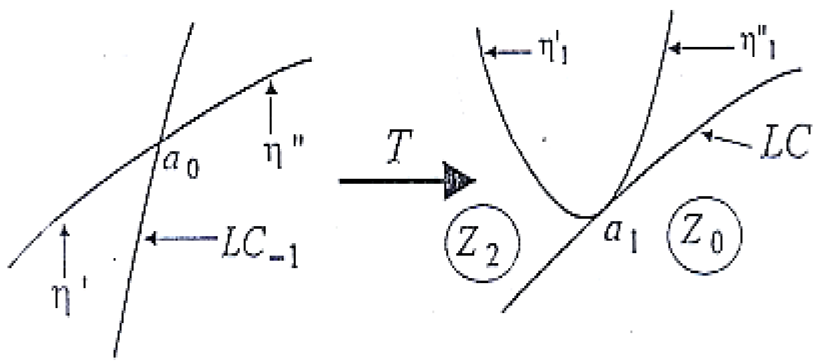
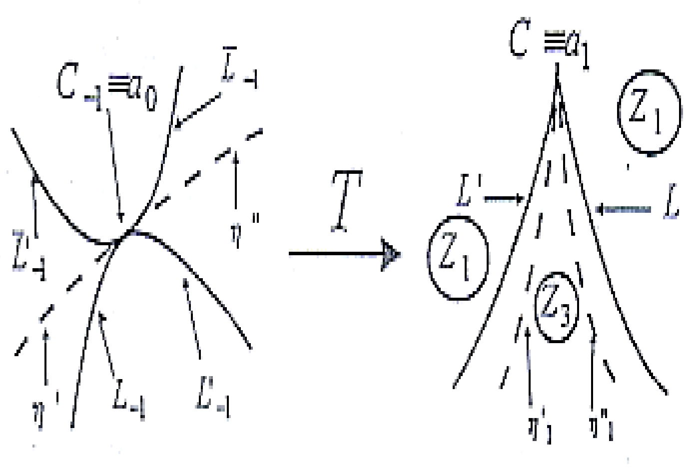


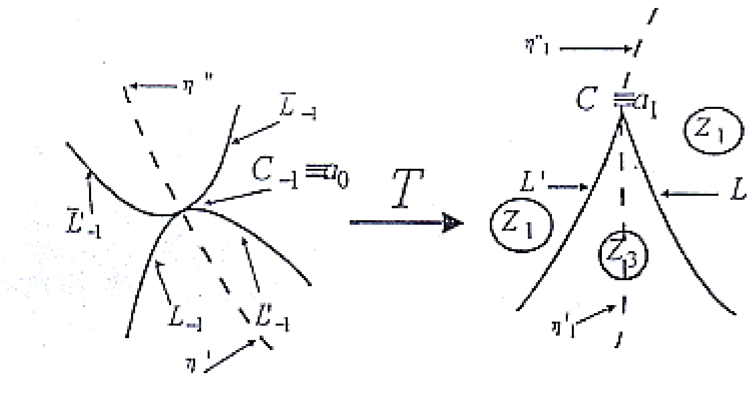
Figure (1.3.1) illustration of prop(1.3.1), $\eta=\eta'\cup\eta''$, $\eta\cap LC_{-1}=a_0$, $\eta_1=\eta_1'\cup\eta_1''$

Remarks(1.3.1):[34,p.129](a):- if $a_1=T(a_0)$ is a singular point of LC, in general this proposition does not hold, except in the particular case of figure (1.3.2a), where η, LC_{-1}, and the extra branch $\overline{L}_{-1}$ are tangent in a_0. The general case when the proposition does not hold corresponds to figure (1.3.2b).

(b) consider a parametrized curve η intersecting LC_{-1} at a_0. when T is a smooth map, by definition one of the two eigenvalues of the jacobian of T is zero a long LC_{-1}. As long as the angle θ of the curve segment η with LC_{-1} at a_0 is different from the angle made by LC_{-1} and the eigenvector V_0 associated with the zero eigenvalue, η_1 is quadratically tangent to LC_{-1} at the point a_1. When η is collinear with V_0, then η_1 forms a cusp at a_1. Figure (1.3.3) shows the three possible basic configurations of η_1.



(a)



(b)

Figure (1.3.2) (a) $\eta = LC_{-1} \cap L_{-1}$ are tangent at a_0, $a_1 = T(a_0)$ is singular point prop(1.3.1) hold.

Figure (1.3.2) (b) $LC_{-1} \cap \eta = a_0$ and $\bar{L} \cap \eta = a_0$, $a_1 = T(a_0)$ is singular point, prop(1.3.1) not hold.

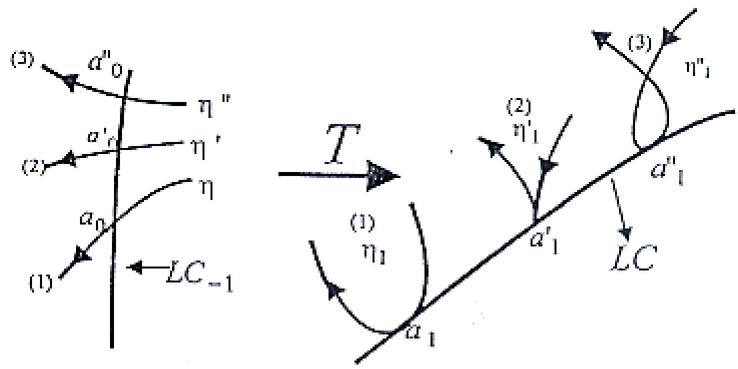


Figure (1.3.3) the three basic configurations of $\eta_1 = T(\eta)$

(c):- The folding of η_1 on the same side of LC, when a_1 is not singular is due to the fact that the two segments η', η'' of η, separated by LC_{-1}, are mapped into two different sheets of the plane. This means that two different components of the inverse map T^{-1} must be used, for obtaining the original segment η from η_1: one associated with η'_1, the other with η''_1, $\eta_1 = \eta'_1 \cup \eta''_1$. So for a ($Z_0$-$Z_2$) map, using the two components (figure 1.3.4):

$$T_1^{-1}(\eta'_1) = \eta', T_2^{-1}(\eta''_1) = \eta''$$

$$T_1^{-1}(\eta'_1) = \eta' \cup \eta''_e \ , \ \ T_2^{-1}(\eta_1) = \eta'' \cup \eta'_e$$

$$T_1^{-1}(\eta''_1) = \eta''_e, \ T_2^{-1}(\eta'_1) = \eta'_e, \ \ T^{-1}(\eta_1) = \eta \cup \eta_e \ , \ \ \eta_e = \eta'_e \cup \eta''_e$$

If $[\eta \cap \eta_e] \backslash a_0 = q_{-1} \cup q'_{-1}$, then $T(q_{-1}) = T(q'_{-1}) = q$.

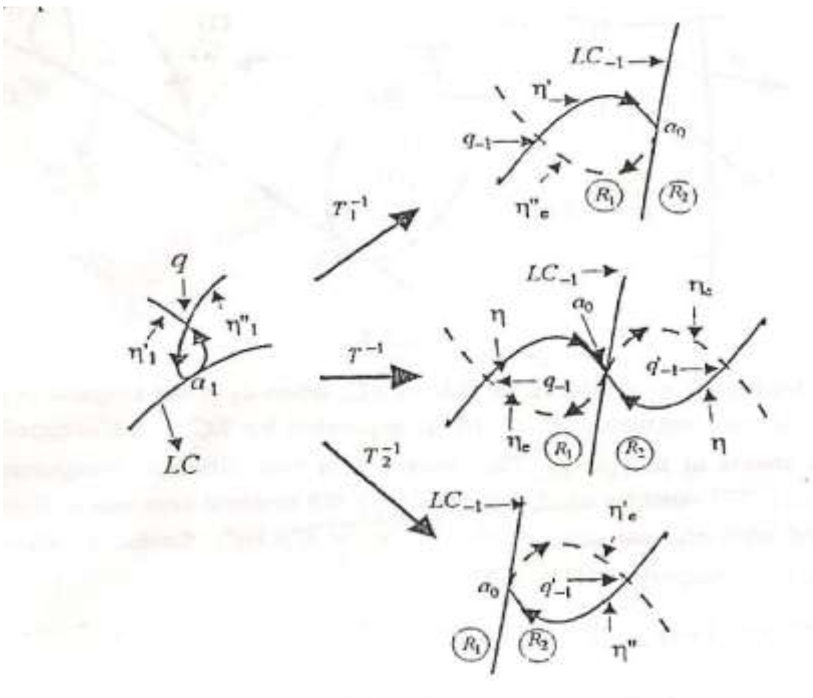


Figure (1.3.4) folding of η_1 on the same side of *LC*, a_1 is not singular point, $\eta_1=\eta_1'\cup\eta_1''$, $\eta_1=T(\eta)$, $\eta_1=\eta'\cup\eta''$

The following proposition is an extension of proposition(1.3.1)

Proposition (1.3.2): Let η be a curve segment crossing through a curve LC_i , i is a positive integer:

(1) for 1<m<i , the segment $\eta_m=T^m(\eta)$ is transverse to LC_{m-i} ,

(2) for m≥i , and in a neighborhood of $LC_{m-i}\cap\eta_m$, the segment η_m does not cross through LC_{m-i} , and is located on the LC_{m-i} side where the number of rank m-i+1 preimages is the greatest.

The following proposition is a consequence of the previous remarks

Proposition (1.3.3): let η be a curve segment crossing through LC_{-1} at a_0, and $\eta_1=T(\eta)$. Then

(1) either η_1 is tangent to *LC* (contact of order one) in a_1, $\eta_1=\eta'_1\cup\eta''_1$, $\eta'_1\cap\eta''_1=a_1\in LC$,

(2) or η_1 is the result of the superposition of η'_1, η''_1, the intersection of which have a common segment with a_1 as one of its two extremities.

(3) Or exceptionally presents a cusp point in a_1.

Proposition: (1.3.4) Let $a_0=L_{-1}\cap L_k$, $k\geq 0$. Then $T(a_0)=a_1=L\cap L_{k+1}$ is a point of tangency (contact of order 1) between L and L_{k+1}, or exceptionally a cusp point of L_{k+1}. In the neighborhood of a_1, L_{k+1} is located on the L side where the number of first rank preimages is the greatest.

proposition (1.3.4) is illustrated by figure (1.3.5)

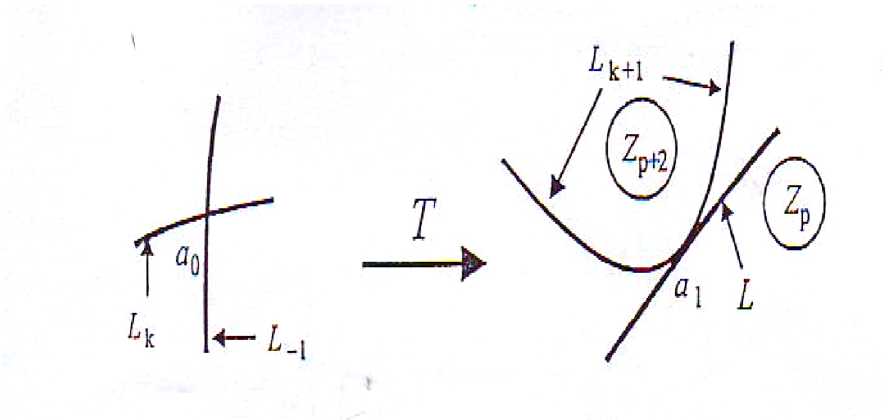


Figure (1.3.5) illustration of prop.(1.3.4) $a_0=L_{-1}\cap L_k$

Remark(1.3.2): From prop.(1.3.4) the sequence of rank n images, $n\geq 1$, $T^n(L_{-1}\cap L_k)=a_n$, $k\geq 0$, constitutes a sequence of points of tangency (contact of order 1) between L_{n-1}, and L_{n+k}, or exceptionally cusp points. In the neighborhood of a_n, L_{n+k} is located on the L_{n-1} side where the number of rank n preimages is the greatest.

Proposition(1.3.5): Let $a_{1-h}=L_{-h}\cap L_k$, h>1, $k\geq 0$, L_{-h} transverse to L_k . If $n<h$, then L_{k+n} is transverse to L_{n-h} at the point $T^n(a_{1-h})=a_{n+1-h}$. If $n\geq h$, then L_{k+n} has an order 1 contact with L_{n-h}. Exceptionally the contact may correspond to a cusp point. In a neighborhood of a_{n+1-h}, L_{k+n} is located on the L_{-h+n} side where the number of rank (n-h+1) preimages is the greatest.

Proposition(1.3.6): Let $a_h=L_h\cap L_k$, 1≤h, 1≤k, i.e. L_h transverse to L_k. Then $T^n(a_h)=L_{h+n}\cap L_{k+n}$, and L_{h+n} is transverse to L_{k+n} at $T^n(a_h)$.

Let L be a segment separating two regions where the number of first rank preimages is p, p+2, p≥0 respectively. Let $(L_{k-1})_e$ be the are with the sense given to η_e in remark (1.3.1c). The preimages of contact point a_1 between L and a segment L_k , k>0 is given by the following proposition:

Proposition(1.3.7): Let L be a segment of critical curve separating two regions Z_p, Z_{p+2}, where the number of first rank preimages is respectively p, $p+2$, p>0, and a segment L_k, k>0, transverse to L, $L\cap L_k=a_1$. The point a_1 has $p+1$ different first rank preimages:

$$T^{-1}(a_1)=a_0\bigcup_{\beta=1}^{\beta=p-1}(\bar{a}_0)_\beta$$

$$a_0=L_{-1}\cap(L_{k-1})_e \quad , \quad (\bar{a}_0)_\beta=(\bar{L}_{-1})_\beta\cap(\bar{L}_{k-1})_\beta, 1\le\beta\le p\ ,$$

In each of these points one of the considered segment is trasverse to the other.

To illustrate proposition (1.3.7), let a segment L and a segment L_k, k>0, transverse at a_1 i.e. $L\cap L_k=a_1$ and L separating Z_p & Z_{p+2} . $T^{-1}_{p,i}, i=1,...,p,$ the inverses of T in Z_p, and assume that the inverses of T in Z_{p+2} are $T^{-1}_{p+2,i}=T^{-1}_{p,i}$, i=1,…, p, plus two inverses merging for points belonging to L. Denote by T_1^{-1},T_2^{-1} these two inverses for simplifying the notation, $T_1^{-1}(L)=T_2^{-1}(L)$.

The first rank preimage of L_k generated from T_1^{-1},T_2^{-1} are only Those of the arc $L_k\cap Z_{p+2}$, which gives a segment:

$$(L_{k-1})_\ell=T_1^{-1}(L_k\cap Z_{p+2})\cup T_2^{-1}(L_k\cap Z_{p+2}),$$

Transverse to L_{-1} at the point $a_0=T_1^{-1}(a_1)=T_2^{-1}(a_1)$.Then the first rank preimage of L_k giving an arc L_{k-1} of the critical curve LC_{k-1} is one of the remaining p inverses, noted $T_{p,1}^{-1}(L_k)$ (i.e $\beta=1$). As shown in figure (1.3.7).

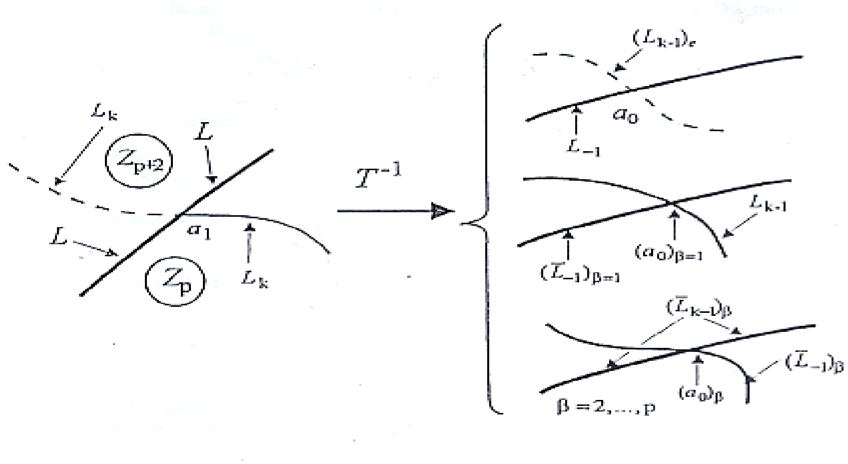


Figure (1.3.7) illustration of prop.(1.3.7)

1.4 Planar Quadratic maps

The main objective of this section is to give a brief description of the dynamics of planar quadratic maps that have a nonempty critical set bounded or not.

Definition(1.4.1):[12]

A planar quadratic map has the form

$T(x,y)=(t_1(x,y),t_2(x,y))$……..(1.4.1)

where

$t_1(x,y)=a_0x^2+a_1xy+a_2y^2+a_3x+a_4y+a$, $t_2(x,y)=b_0x^2+b_1xy+b_2y^2+b_3x+b_4y+b$

and where a,b,a's and b's are real constants.

The critical set or singular set $\Im(T)$of planar quadratic map (1.4.1) is the set:

$\Im(T)=\{x\in R^2:\det(J(T(x))=0\}$.

Clearly the critical set $\Im(T)$ is a real planar algebraic curve of order not greater than two. This set may be bounded or not, it is bounded when the following conditions are satisfied [1]

(1a) $a_0b_1-a_1b_0=a_1b_2-a_2b_1$ (we get circle) or

(1b) $a_0b_1-a_1b_0\neq a_1b_2-a_2b_1$ (we get ellipse)

(2) $a_0b_2-a_2b_0=0$

(3)$A^2+B^2-C\geq 0$, where $A=a_0b_4-a_4b_0+\frac{(a_3b_1-a_1b_3)}{4}$, $B=a_3b_2-a_2b_3-\frac{(a_4b_1-a_1b_4)}{4}$,

$C=a_3b_4-a_4b_3$.

When one of the above condition is not satisfied the critical set is unbounded.

Remarks(1.4.1)[1]:

1) The critical set $\Im(T)$ is empty when $\det(J(T))$ is constant i.e. when condition (1) is satisfied and equal zero, condition (2) is satisfied and $A=B=0$ and $C\neq 0$.

2) The standard form:

   $T(x,y)=(x^2+a_1xy-y^2+a,\ b_1xy+b)$……(1.4.2a) whose critical set is a point while the standard form

   $T(x,y)=(x^2+a_1xy-y^2+a\ ,\ b_1xy+b_3x+b_4y+b)$……..(1.4.2b) whose critical set as an ellipse.

3) The standard from:

$T(x,y)=(a_0x^2+a_1xy+a_4y+a,\ b_1xy+b)$……….(1.4.3a) gives a critical set as a parabola, while the standard form:

$T(x,y)=(a_0x^2+a_2y^2+a_1xy+a_3x+a_4y+a, b_0x^2+b_2y^2+b_3x+b_4y+b)$…(1.4.3b) gives a critical set as a hyperbaric or a straight line (we get a line in case $a_0b_2-a_2b_0=0$).

4) Each planar map T whose nonempty critical set bounded or not can be brought into standard form via an affine coordinate change.

Chapter Two

Absorbing Areas and invariant Areas of Two Dimensional Noninvertible Maps

This chapter is devoted to study the structure of absorbing areas and chaotic areas generated by noninvertible maps the plane (two-dimensional endomorphism) to their bifurcations. Here the terminology "area" is not always related to some measure. "Area" only refers to a closed and bounded set.

2.1 Definitions and General Properties

We start this section by giving definition of an absorbing area:

Definition (2.1.1): [4, p.246]

An absorbing area d' is a closed and bounded subset of $\Re^2$ satisfies:

1) T (d')$\subseteq$d'.

2) Its fronties, ∂d' is made up of finite number of segments of critical curves of LC, LC_1, LC_2,…, LC_k.

3) A neighborhood U(d') exists, such that T[U(d')]$\subset$U(d'), and any point x$\in$U(d')/d' have a finite rank image in the interior of d'.

This type of an absorbing area is called an absorbing area of non-mixed type.

From definition (2.1.1) we can conclude that an absorbing area d' is implicity associated with the existence of an attracting set belonging to d'.

We can define another type of an absorbing area as:

Definition (2.1.2):[34,p.187]

An area $\tilde{d}'\subseteq\Re^2$ is said to be absorbing of mixed type if it satisfies:

1) T($\tilde{d}'$)$\subseteq\tilde{d}'$,

2) $\tilde{d}'$ is attracting, a neighborhood U($\tilde{d}'$) exists such that T(U($\tilde{d}'$))$\subset$U($\tilde{d}'$), and almost all the points x$\in$U($\tilde{d}'$)\\$\tilde{d}'$ have a finite rank image in the interior of $\tilde{d}'$,

3) The boundary $\partial\tilde{d}'$ is made up of segments of critical curves and segments of the unstable set W^u of a saddle fixed point, or a saddle cycle (periodic point), or even segments of several an stable sets associated with different cycles.

In both definitions (2.1.1) & (2.1.2) an absorbing area may be not an invariant, i.e. T(s)$\subset$s {where s may be d' or $\tilde{d}'$} then an invariant set can be obtained as follows:

(a) either a finite integer k exists such that S= $\bigcap_{j=0}^{j=k} T^j(s)$ is an invariant absorbing area.

Or (b) S=$\bigcap_{j\geq 0} T^j(S) = S_\infty$ is a closed invariant absorbing set.

Definition (2.1.3): [34, p.189]

A non mixed chaotic area (d) is an invariant non mixed absorbing area bounded by critical segments, with chaotic dynamics in the whole area (d).

Definition (2.1.4): [34, p.189]

A mixed chaotic area ($\tilde{d}$) is an invariant mixed absorbing area, the points of which are chaotic.

2.1.1 Two Basic Propositions

In this subsection we shall give two basic propositions which are preparatory for the properties of (Z_0-Z_2) map that will be gives in the next subsection.

Proposition (2.1.1):[34, p.208]

Let A be a closed subset of the plane. Then the points internal to A which can be mapped on the boundary of T (A) belong to $A \cap LC_{-1}$.

Proof: can be found [34] .♦

Remark (2.1.2):

If $A \cap LC_{-1} = \phi$ then $\partial T(A) = T(\partial A)$, i.e. only points of the boundary of A are mapped on the boundary of T(A).

Proposition (2.1.2): [34, p.208]

Let A be closed subset of the plane. If ∂A is made up of critical segments, then also $\partial T(A)$ is made up of points of critical segments.

Proof: can be found [34] .♦

Remark (2.1.3):

The segment of the unstable set of a cycle are mapped by T into segments of the same unstable set.

The following proposition is a consequence of prop(2.1.2) with remark (2.1.3).

Proposition (2.1.3):[34, p.208]

Let s be absorbing area, mixed or not. Then also $T(s)$ is an absorbing area of the same type as s.

Proof: can be found [34] .♦

2.1.2 Properties of (Z_0-Z_2) Maps [34, p.213]

Let T be a(Z_0-Z_2) map of the plane. For such continuously differentiable maps T it is recalled that the critical curve LC separates the plane in two regions Z_0 and Z_2 such that $\overline{Z}_2 \cap \overline{Z}_0 = LC$, $\overline{Z}_2 \cup \overline{Z}_0 = \Re^2$. Like LC the curve LC_{-1} separates the plane in two open regions R_1, R_2 such that $R_1 \cap R_2 = \phi$, $\overline{R}_1 \cap \overline{R}_2 = LC_{-1}$, $\overline{R}_1 \cup \overline{R}_2 = \Re^2$. For every point $X \in Z_2$, let $T_1^{-1}(X) \in R_1, T_2^{-1}(X) \in R_2$ be the two first rank preimages of X.

The following propositions state some properties of the (Z_0-Z_2) map:

Proposition (2.1.4): Let T be a (Z_0-Z_2) map

i) $X \in Z_2 \Rightarrow T^{-1}(X) = \{ T_1^{-1}(X) \in R_1, T_2^{-1}(X) \in R_2 \}$.

ii) $X \in LC \Rightarrow T_1^{-1}(X) = T_2^{-1}(X) \in LC_{-1}$.

iii) $X \in Z_0 \Rightarrow T^n(X) \in \overline{Z}_2$, $n \geq 1$.

iv) $T(\overline{Z}_2) \subset \overline{Z}_2$.

Proof: can be found [34] .♦

Proposition (2.1.5):[34, p.212]

If Δ_0 is a closed subset of $\overline{R}_2$, bounded by critical curves segments of LC_{-1}, then:

i) $\Delta = \bigcup_{i=1}^{k} T^i(\Delta_0)$ is bounded by critical curves segments $\forall k \geq 1$;

ii) $T^n(\Delta)$ is bounded by critical curves segments $\forall n \geq 1$.

Proof: can be found [34] .♦

Corollary (2.1.1):[16]

Let Δ_0 be a bounded area whose bounded consist of arcs of critical lines and $\Delta_k = \bigcup_{i=0}^{k} T^i(\Delta_0)$. If there exists an integer m such that $T(\Delta_m) \subseteq \Delta_m$, then Δ_m is an absorbing area.

2.2 Construction of Absorbing Areas & Invariant Areas.

In this section we shall study the construction of absorbing and invariant areas for $(Z_0\text{-}Z_2)$ maps.

2.2.1 Construction Algorthirm of Absorbing Areas [24, 34; p.191& 36]

The structure of this algorithm depends on the use of the critical curves to obtain closed bounded regions (will be denoted by Δ) whose boundary consists of segments of critical curves LC_i, i=0,1,2,…,N(N is finite integer), then such area is an absorbing.

First we suppose that the first critical curve LC and the curve LC_{-1} of merging preimages are made up of only one branch, these two curves having only one point of intersection say a_0. When these two curves intersection in more than one point, one of them plays the rule of a_0.

A segment of curve will be represented by $(\alpha\ \beta)$ where α , β are the two endpoints, the point a_n represents the n^{th} iterate of a_0 i.e. $a_n=T^n(a_0)$.

Now, we are ready to describe the algorithm of construction as:

Let N be the first integer > 0, such that the segment $(a_N a_{N+1})$ of LC_N (LC_N critical curve of rank-N) intersects LC_{-1} at a point say b_0, i.e. $b_0 \in (a_N a_{N+1}) \cap LC_{-1}$.

Then, define a simply connected area Δ bounded by

$$\partial\Delta=(b_1 a_1 a_2 \ldots a_N a_{N+1} b_1)$$

where

$(b_1 a_1)$ is a segment of LC, i.e. $(b_1 a_1) \subset LC$,

(a_1a_2) is a segment of LC_1, i.e. $(a_1a_2)\subset LC_1$,

.

.

.

(a_Na_{N+1}) is a segment of LC_N, i.e. $(a_Na_{N+1})\subset LC_N$,

$(a_{N+1}b_1)$ is a segment of LC_{N+1}, i.e. $(a_{N+1}b_1)\subset LC_{N+1}$,

$b_1=T(b_0)$, $a_i=T(a_{i-1})$

So, we get Δ is an absorbing areas.

Remark (2.2.1):

Algorithm (2.2.1) may not work for certain examples, and it does not include all possible cases of absorbing areas with a finite boundary (i.e. boundary made up of a finite number of critical segments). i.e. it may happen this is not absorbing area, that will be seen in the illustrative examples.

Remark (2.2.2)[34]

1- If there are more than one point of intersection between the segment (a_Na_{N+1}) and LC_{-1} we choose the point b_0 that is farthest from a_0.
2- When the above algorithm work, then we distinguish two possible cases for b_1:
   i) $b_1\notin(a_0a_1)\subset LC$ or equivalently $b_0\notin(a_{-1}a_0)\subset LC_{-1}$.
   ii) $b_1\in(a_0a_1)\subset LC$ or equivalently $b_0\in(a_{-1}a_0)\subset LC_{-1}$.

2.2.2 Different Kinds of Absorbing Areas (Nonmixed Ones):

Let T be a map of type a $(Z_0$-$Z_2)$, recall that LC_{-1} divides the plane $\Re^2$ into two open regions R_1, R_2 such that $R_1\cap R_2=\phi$, $\overline{R}_1\cap\overline{R}_2=LC_{-1}$, $\overline{R}_1\cup\overline{R}_2=\Re^2$.

Let φ be a fixed point of T with $\varphi \in R_2$, and LC made up of only one branch, $a_0 \in LC_{-1} \cap LC$. Then one of the following cases is possible:

(1) None of the successive images of the segment (a_0a_1) intersect LC_{-1}.

In this case we can obtain an absorbing area as follows:

In this situation one of the two inverse map T_2^{-1} gives rise to a rank-m preimage $(a_{-m}\, a_{-m+1})$ of $(a_0\ a_1)$ which intersects LC_{-1} a first time at the point say h_0 Fig. (2.2.1).

Necessarily $(a_0h_0) \supset (a_0a_1)$, thus $T^i(a_0h_0)=(a_ih_i) \supset T^i(a_0a_1)$ for $i>0$, so $h_i \in LC_i$, $h_{m-1} \in LC_{-1}$, fig(2.2.1b). Now applying construction algorithm (2.2.1) to obtain an absorbing area Δ bounded by the closed curve $(h_ma_1a_2\ldots a_mh_m) \subset R_2$ where (h_ma_1) is a segment of critical curve LC, $(a_mh_m) \subset LC_m$, $(a_ia_{i+1}) \subset LC_i$, $i=1,2,\ldots,m-1$.

Figure (2.2.1) illustrates this situation.

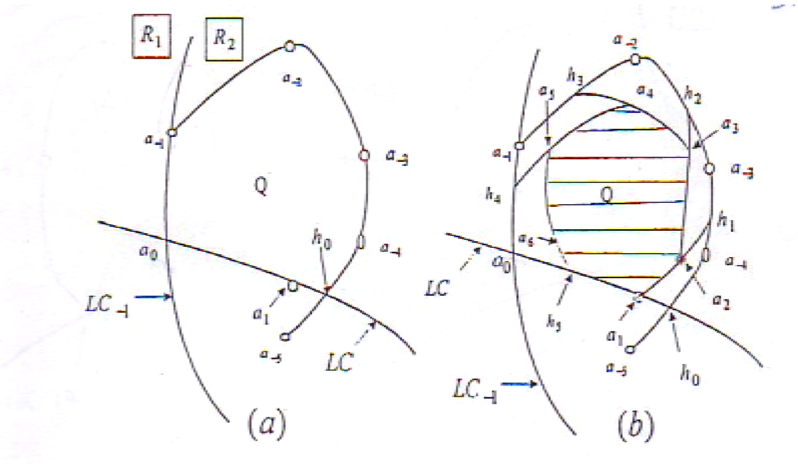


Fig.(2.2.1) Non of the successive image of the segment (a_0a_1) intersect LC_{-1}.

(2) One of the images of (a_0a_1) has a non transverse contact (order one, or order zero) with LC_{-1}.

In this case, let $b_0=(a_Na_{N+1}) \cap LC_{-1}$ be the non transverse contact point. An absorbing area Δ is defined as in the construction algorithm and the boundary $\partial\Delta=(b_1a_1a_2\ldots a_Na_{N+1}b_1) \subset \overline{R}_2$ with $\partial\Delta \cap LC_{-1}=b_0$.

(3) An image of (a_0a_1) has a transverse intersection with LC_{-1}. Let N be the least integer such that (a_Na_{N+1}) intersects LC_{-1}. Let $b_0 \in (a_Na_{N+1}) \cap LC_{-1}$ be the intersection point farthest from a_0 . So we apply the construction algorithm to obtain absorbing area, the two possible situation appear in remark (2.2.2) may occur.

Now, consider that LC is made up of two segments joining at a_0, $LC=L^{(1)} \cup L^{(2)}$, $a_0= L^{(1)} \cap L^{(2)}$. Then $LC_1= L_1^{(1)} \cup L_1^{(2)} \in \overline{Z}_2$ is folded in $a_1=T(a_0)= L_1^{(1)} \cap L_1^{(2)} \in LC$. From property (iv) of prop. (2.1.4), it follows that there exists a region S_0, bounded by a segment $L^{(1)}$ of LC and by $LC_{-1} \cap \overline{Z}_0$, such that $T(S_0)=S_1$ and any point of S_1 has its two first rank preimges in Z_0. $S_1=\overline{Z}_2 \setminus T(\overline{Z}_2)$, bounded by $L^{(1)}$ and $L_1^{(1)}$ ($L_1^{(1)}=T(L^{(1)})$), is uninteresting for the analysis of the asymptotic behavior of sequences of images with increasing rank. Thus one has the following proposition :

Proposition (2.2.1) [Elimination process]:[34]

Consider a$(Z_0\text{-}Z_2)$ map then

i) Either $T^{k+1}(\overline{Z}_2) \subseteq T(\overline{Z}_2)$ for any k>0, which preimags to define $V=\bigcap_{k \succ 0} T^k(\overline{Z}_2)$;

ii) or there exists a finite j such that $T^{j+1}(\overline{Z}_2) = T^j(\overline{Z}_2)$ which permits to define $V_j= T^j(\overline{Z}_2)$.

In both cases V and V_j are invariant absorbing regions. The boundary of V and V_j is made up of a finite number of critical curve segments.

Prop.(2.2.1) it follows that it is possible to restrict the analysis of asymptotic behaviors of a sequence of images with increasing rank, to the points of an absorbing region.

2.2.3 Determination of Invariant Areas [34, p.217]

Recall that when we apply the construction algorithm to construct an absorbing area Δ the two possible situations that appear in the remark (2.2.1) occur:

(i) $b_1 \notin (a_0 a_1) \subset LC$ or equivalently $b_0 \notin (a_{-1} a_0) \subset LC_{-1}$.

(ii) $b_1 \in (a_0 a_1) \subset LC$ or equivalently $b_0 \in (a_{-1} a_0) \subset LC_{-1}$.

The two situation are different because in case (i) $T(\Delta) \supseteq \Delta$ while in case (ii) $T(\Delta) \subset \Delta$, or $T(\Delta)$ is not comparable with Δ. i.e. $T(\Delta)$ is neither included in Δ nor includes Δ. In both cases (i) and (ii), Δ may be absorbing, or not.

In fact, in the case (i) $\partial\Delta$ intersects LC in two points b_0 and a_0, i.e. $(b_0 a_0) = LC_{-1} \cap \Delta$ and $(b_0 a_0) \supset (a_{-1} a_0)$. Then under application of T the whole boundary $\partial\Delta$ is construct again and new parts may only come from $T(\Delta \cap \overline{R}_1) = T(\delta_0) = \delta_1$, thus if $T(\Delta \cap \overline{R}_1) \subset \Delta$ then the area Δ is invariant, $T(\Delta) = \Delta$ as in fig.(2.2.2a), while if $T(\Delta \cap \overline{R}_1)$ is not include in Δ, $T(\Delta) \supset \Delta$ fig. (2.2.2b), in this case $T^{m+1}(\Delta) \supseteq T^m(\Delta)$, $\forall\ m \geq 0$. Thus, either a finite integer M exists such that $T^{M+1}(\Delta) = T^M(\Delta)$, so we get $d'' = T^M(\Delta)$ is invariant areas, or a finite M does not exist, in which case we define

$$d'_\infty \equiv \overline{\bigcup_{j=1}^{\infty} T^j(\Delta)} \qquad (2.2.1)$$

The area d'_∞ may be bounded or not. When it is bounded, it may be absorbing or not, and generally this situation denote a bifurcation resulting from the contact of the area boundary with its basin boundary [18].

In case (ii) $\partial\Delta$ intersects LC_{-1} in two points b_0 and c_0, where $c_0 \in (b_0 a_0)$ i.e. $(b_0 c_0) \subset (b_0 a_0)$ so that boundary of Δ include the segment $(b_1 a_1)$, while $T(\Delta) \cap LC = (b_1 c_1) \subset (b_1 a_1)$, from which it appears that $T(\Delta) \supseteq \Delta$ is not possible. It follows that either T(Δ) ⊂ Δ fig.(2.2.2c) or T(Δ) is not comparable with Δ fig.(2.2.2d).

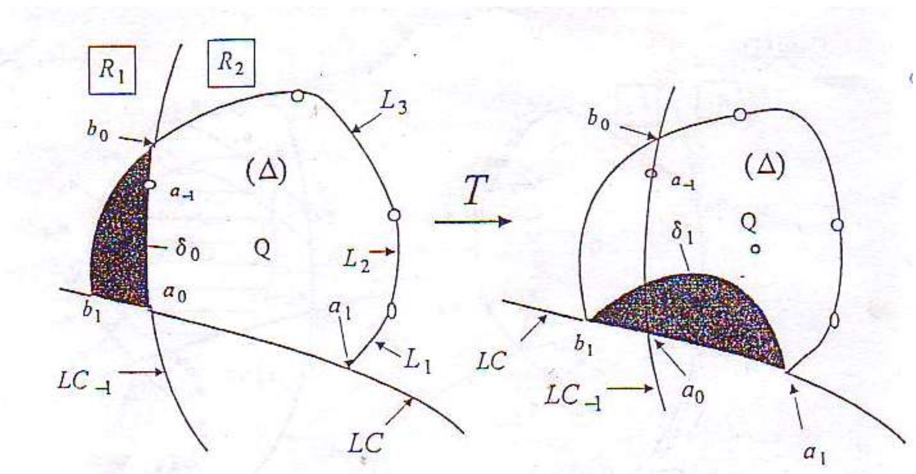
When T(Δ) ⊂ Δ then $T^{m+1}(\Delta) \subseteq T^{m}(\Delta)$, m ≥ 0, so that either a finite integer M exists such that d"=$T^{M}(\Delta)$ is invariant or a finite M does not exists, in which case we define

$$d'_{\infty} \equiv \bigcap_{j=1}^{\infty} T^{j}(\Delta) \qquad (2.2.2)$$

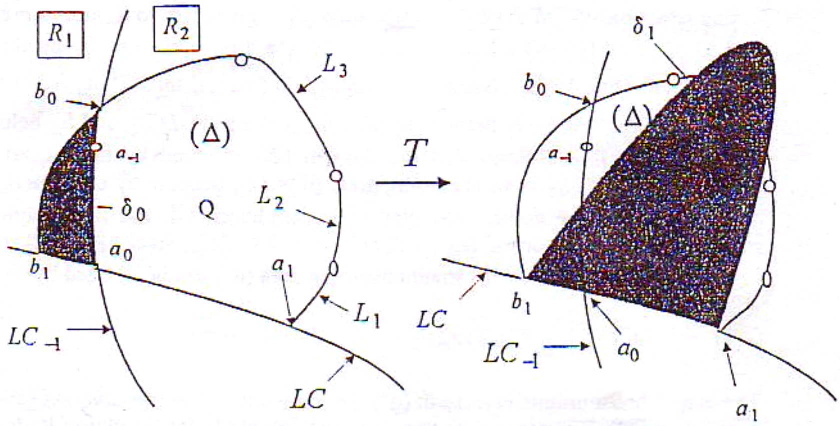
Since each area $T^{k}(\Delta)$, k ≥ 0 is absorbing, then d'_{∞} is bounded and absorbing. Case (ii) is more complex when T(Δ) is not comparable with Δ. In the simplest case a finite M such that d"=$T^{M}(\Delta)$ is invariant. However it may occur that a finite M does not exist, this situation is more complex, it is possible to define

$$A = \overline{\bigcup_{j=1}^{\infty} T^{j}(\Delta)} \qquad (2.2.3)$$

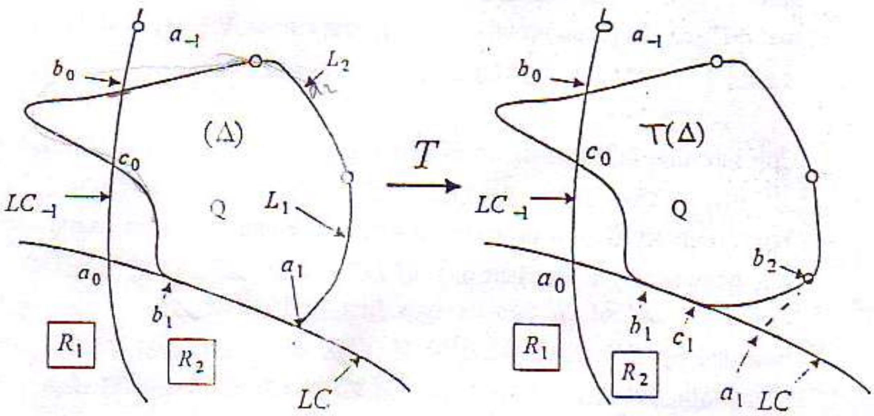
This area may be bounded or not. If it is bounded, it may be invariant or not. When it is not invariant, if there exist k such that $\bigcap_{j=1}^{\infty} T^{j}(A_{\infty}) = T^{k}(A_{\infty})$, then this intersection is an invariant area. If there is no such k then $\bigcap_{j=1}^{\infty} T^{j}(A_{\infty})$ is an invariant.



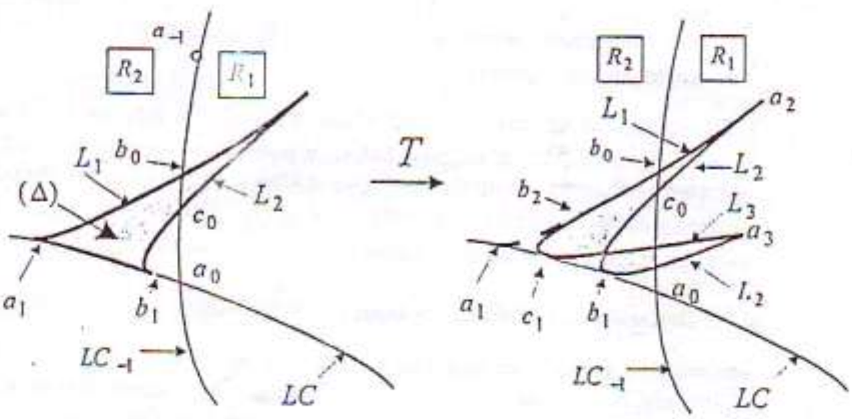
(a)


R1
R2
b0
a-1
δ0
a0
b1
(Δ)
Q
L3
L2
a1
L1
LC-1
LC
T
δ1
b0
(Δ)
a-1
b1
LC
LC-1
a0
a1

(b)


a-1
b0
L2
(Δ)
c0
LC-1
Q
L1
a0
a1
b1
R1
R2
LC
T
a-1
b0
T(Δ)
c0
LC-1
Q
b2
a0
b1
c1
R1
R2
a1
LC

(c)


a-1
R2
R1
L1
b0
(Δ)
L2
c0
a0
a1
b1
LC-1
LC
T
R2
R1
L1
a2
b0
L2
b2
c0
L3
a3
a1
c1
b1
a0
LC-1
LC

(d)

Fig (2.2.2) An absorbing area constructed by algorithm (2.2.1) (a): $b_1 \notin (a_0a_1)$, $T(\Delta)=\Delta$, (b) $b_1 \notin (a_0a_1)$, $T(\Delta)\supset\Delta$, (c) $b_1 \in (a_0a_1)$, $T(\Delta)\subset\Delta$, (d) $b_1 \in (a_0\ a_1)$, $T(\Delta)$ is not comparable with Δ.

2.3 Properties of Absorbing Areas & of Invariant Areas.

The absorbing areas (s) considered here are of either nonmixed types or mixed types invariant, or non invariant. The following propositions give some properties of the absorbing and invariant areas for a(Z_0-Z_2) maps.

Proposition (2.3.1):[34, p.220]

If a map generates a region Z_0, then absorbing area (s) is such that (s) $\cap Z_0=\emptyset$.

Proof: can be found [34] .♦

Proposition (2.3.2):[34, p.221]

Let $T(S) =S$, then any point $p\in S$ has at least one infinite sequence of preimages in S.

Remark (2.3.1):

If S is a set invariant under backward iteration of T that is $T^{-1}(S) =S$, then any point p in S has its image in S, i.e. if a set, backward invariant for T, is also forward invariant. The converse is not true.

Proposition (2.3.3):[34, p.221]

Let $T(S) = S$ then any point $p\notin S$ has all its preimages of rank 1, 2, 3,…, out of S.

Proof:- The proof is immediate.

Definition (2.3.1):[3]

The largest subset of $A \cap LC_{-1}$, mapped by T on the boundary of T(A), is denoted g_{-1}, i.e. $g_{-1} \subseteq A$ is such that $g = T(g_{-1}) \subseteq \partial T(A) \cap LC$.

Remarks (2.3.1):

1. Any other point of $A \cap LC_{-1}$ not belong to g_{-1}, when it exist, is not mapped on the boundary of T(A). This occurs when $LC \cap \partial A$ has isolated points limiting segments internal to A.
2. The definition (2.3.1) is constructive, because from the set $A \cap LC_{-1}$ the points not mapped on the boundary of T(A) are eliminated, i.e. the points mapped in the interior of T(A). We shall refer to g as " arc g" whichever is its structure.
3. In the case of an invariant area S, closed and bounded by a finite number critical segments, it will be seen that the boundary ∂S is made up of segments belonging to the images of the set g_{-1}, which will be called, as g, "generating arc of ∂S".

It is worth noting that for a generic map T one may $g_{-1} \subset \partial T(A) \cap LC$, or $g = \partial T(A) \cap LC$, this property holds for $(Z_0\text{-}Z_2)$ maps which given by the following proposition:

Proposition (2.3.4):[34, p.222]

Let T be a $(Z_0\text{-}Z_2)$ map, and A be a closed subset of the plane. Then T $(A \cap LC_{-1}) = T(A) \cap LC$.

Proof: can be found [34] .♦

Prop.(2.3.4) holds for an absorbing area. For a closed invariant set, it turns into the following proposition.

Proposition (2.3.5): Let T be a (Z_0-Z_2) map, and S be a closed invariant subset of the plane. Then $T(S \cap LC_{-1}) = S \cap LC \Rightarrow \partial S \cap LC$.

Proof: the proof is similar to the proof of prop.(2.3.4) ♦

Proposition (2.3.6):[34, p.223]

Let A be closed and $p \in \partial T(A) \backslash g$, then all the rank-one preimages of p in A must belong to ∂A.

Proof: can be found [34] . ♦

Prop.(2.3.6) holds in particular for an absorbing area, for a closed invariant area it becomes:

Proposition (2.3.7):[34, p.223]

Let S be closed, T(S) =S, and $p \in \partial S \backslash g$, then all the rank-one preimages of p in S must belong to ∂S.

Proof:

Follows by use prop.(2.3.7). ♦

The next proposition may be used to characterize more explicitly the boundary point of an invariant area S.

Proposition (2.3.8):[1]

Let S be closed, T(S) =S, and $p \in \partial S$, then

i) either finite k, $k \geq 0$ exist such that $p \in T^k(g) \subset LC_k$.

ii) or $T^{-n}(p) \cap S \subset \partial S$, $\forall n \geq 0$.

Proof: can be found [1] .♦

Proposition (2.3.9):[34, p.224]

Let S be a closed area with finite boundary. T(S) =S, and $L_k \in \partial S$ a segment of critical curve LC_k, $k \geq 0$. Then its critical preimages $L_{k-1},\dots, L_1$, $L_0 = L \subseteq g$ also belong to the boundary ∂S.

Proof: can be found [34] .♦

Prop.(2.3.9) justifies the fact that g (or equivalently g_{-1}) is called "generating arc", because all critical segments on the boundary of S belong to the images of this basic set.

Remarks (2.3.2):

(i) When prop.(2.3.9) is applied to a nonmixed invariant area S with finite boundary a finite integer M exists such that $\partial S \subset \bigcup_{i=0}^{M} T^i(g) = \bigcup_{i=1}^{M+1} T^i(g_{-1}), T^i(g)$ being a critical segment belonging to LC_i. when we applying this proposition to a mixed invariant area (i.e. its boundary is made up of a finite number of critical segments and segments of saddle unstable set) a finite integer *M* exists such that all the critical segments on ∂S belong to $\bigcup_{i=0}^{M} T^i(g) = \bigcup_{i=0}^{M} T^{i+1}(g_{-1})$, while a segment of ∂S belonging to some saddle unstable set W^u has at least infinite sequence of its preimages on W^u belonging to the boundary of S which converge toward a saddle cycle.

(ii) The extra preimages of the critical segments $L_k,\dots, L_1, L_0$ (i.e. non critical preimages of critical segments) on the boundary of S cannot belong to the interior of *S*. Moreover when *S* is a non-mixed area, the extra preimages cannot belong to the boundary of *S*, thus the extra preimages of such segment must be out of *S*, except at most isolated points on ∂S which may be critical points.

Proposition (2.3.10):[34,p.226]

Let S be an invariant area with a finite boundary. Then ∂S is not be an invariant.

Proof: can be found [34] .♦

2.4 Bifurcation

The term bifurcation generally refers to something " splitting a point " with general a system involving a parameter, it refers to change in the character of the solution as the parameter is changed continuously.

At the beginning of this section we shall give the definition of bifurcation followed by some types of bifurcation that be interest in our work, illustrative example will be given in the chapter three.

Definition (2.4.1):[27]

Consider the system $x_{n+1}=f_{\lambda}(x_n)$; $x\in\Re^n$, $\lambda\in\Re^k$ (2.4.1)

One is especially concerned how the phase portrait of (2.4.1) changes as λ varies. A vale λ_0 where there is basic structural change in this phase portrait is called a bifurcation point.

2.4.1 Some Types of Bifurcations

(1) Contact bifurcations:[18,19&31]

This basic bifurcations results from the contact of a basin boundary with a critical curve segment not belonging to a chaotic area boundary, also it occurs when a critical curve belonging to a chaotic area boundary. Such a bifurcation leads either to the chaotic area destruction, or to sudden and important modification of this area. Even if S is an absorbing area, the contact bifurcation may occur.

(2) Bifurcation of non smoothness points on boundaries of invariant areas: [34, p.234]

Consider a smooth map T, area called Δ constructed by the algorithm (2.2.1), and there is an integer m such that $S=T^m(\Delta)$ is an invariant area with $S \cap LC_{-1} \neq \phi$, being finite i.e. ∂S may correspond to one of the following cases:

Case1: Before and after the bifurcation the contact between ∂S and $T(\partial S)$ on LC at an endpoint of the generating segment g is smooth. At the bifurcation the contact on LC in not smooth, due to cusp point of LC_1.

Case 2: A point of non smoothness of the S boundary may also born when a self intersection of a critical arc L_k.

Case 3: A point of non smoothness of the S boundary may also created at p when two critical segments of different ranks are intersect i.e. $p \in LC_k \cap LC_j$, $k \neq j$**,** p is called an angular point.

When the boundary of S is smooth at p, p is said to be to be an ordinary point of ∂S.

2.5 The Proposed Algorithm

For a generic map, we have not a general procedure to construct the starting setΔ. The general criterion consists in selecting suitable segment on LC_{-1}, and with few of their images to get a closed area Δ bounded by a finite number of critical segments belonging to LC_k, $0 \leq k \leq N$. Generally the so obtained area Δ is not absorbing, and an invariant absorbing area is obtained after a finite number of iterations, (d")=$T^M(\Delta)$, $M \geq 0$ is the first integer such that $T^{M+1}(\Delta)=T^M(\Delta)$.The same criteria done for the (Z_0-Z_2) maps. Some maps, if we compute the equation of the critical curve LC_i directly from applying T on LC_{i-1}, required many computations. Therefore we shall use some approximation methods like the least square method to approximate the equation of the critical curve LC_i . To do this, choose suitable number of points ($x_{i-1, j}$, $y_{i-1, j}$), j=1,…,m , for some $m \in N$ which belongs to LC_{i-1} and then find the images of these points, i.e. for $T(x_{i-1,j}, y_{i-1,j}) = (x_{i,j}, y_{i,j})$, j=1,…,m we get points $(x_{i,j}, y_{i,j})$ which belong to LC_i.

Next, we approximate the equation of the critical curve LC_i that passes through the points $(x_{i,j}, y_{i,j})$ via the least square method and draw up it by continuing in this manner until we get closed bounded area which may be absorbing or not.

Chapter Three

Illustrative Examples

In this chapter we will illustrate the concepts defined in the first two chapters by several considered examples and some observation will be made on the dynamics of the maps, particularly on the absorbing areas.

We will use the proposed algorithm (2.5) to construct a closed bounded area Δ, since this area may be not absorbing so we shall verify that a closed Δ is an absorbing numerically.

It is worth nothing that results presented here were essentially obtained via a numerical method, but guided by fundamental considerations stated in chapter two and using the critical curve tools.

In all examples we shall use approximated method (least square method) to approximate the equation of the critical curves LC_i , i=1,…,n with aid of Matlab version 7.0 Software for numerical computations and for plotting figures.

In the first example we shall make a compression between the results that obtained by applying the construction algorithm (2.2.1) and the result that will be obtained by applying the proposed algorithm (2.5).

In the other examples we shall use the proposed algorithm (2.5) to determine a closed & bounded area Δ which may be absorbing or not, invariant or not for some specific values of the parameters.

3.1 Examples of Absorbing Area

In this section, w shall give some examples that illustrate some phenomena on absorbing areas.

Example (3.1.1): Consider the map *T* defined by

$$T: \begin{array}{l} x' = ax + y \\ y' = x^2 + b \end{array} , \qquad (3.1)$$

T is continuously differentiable and noninvertible map whose inverses are

$$T^{-1}: \begin{array}{l} x = \pm\sqrt{y' - b} \\ y = x' \pm a\sqrt{y' - b} \end{array}$$

T has two fixed points $(\frac{1-a\pm\sqrt{(1-a)^2-4b}}{2},(1-a)[\frac{1-a\pm\sqrt{(1-a)^2-4b}}{2}])$ if $a \geq 2\sqrt{b}+1 \; and \, b \geq 0$.

The curve LC_{-1} is given by x=0, the equation of the critical curve LC, is given by y=b. Recall that LC divides the plane into two regions: Z_0 satisfies $y<b$ where each point has no preimages and Z_2 with $y>b$ where each point has two first rank preimages. LC_{-1} divides the plane into two regions R_1,R_2. R_1 is the region $x< 0$, R_2 with $x >0$.

ALwa'li Z., in [1], studied some properties of the map (3.1) for specific values of the parameters *a*& *b* by use the constructing algorithm (2.2.1) and show numerically this area is an invariant absorbing area for some values of *a*& *b*.

Here, we shall use approximated method (least square method) to approximate the equations of the critical curves LC_i and plotting these equations until we get closed bounded area whose boundaries are segment of the critical curves LC_i, and then we compare the results with results that will be get by applying construction algorithm (2.2.1) with values of *a,b* different from one that appear in [1].

Now we shall take some values of the parameters *a* and *b* and we use the proposed algorithm (2.5) to study the dynamical behavior of the map (3.1).

For a=-1.5, b=-1.5, the fixed points of T are:

P_1=(-0.5,-1.25) with eigenvalues λ_1 =-0.75+0.6614i and λ_2=-0.75-0.6614i therefore P_1 is a stable fixed point .

P_2= (3,-7.5) with eigenvalues λ_1 = 1.8117 and λ_2=1.8117 therefore P_2 is an unstable fixed point.

To calculate the equation of LC_1 we choose the points [(-0.1,-1.5),(0.02,-1.5),(0.1,-1.5),(0.2,-1.5),(0.3,-1.5)] which belong to LC and by substituting these points in equation (3.1), we have $T(x,y)\in LC_1$. By using least square method, the equation of LC_1 is approximated by $y=0.4444x^2+1.3333x-0.5$. In the same manner we approximate the equation of the critical curve LC_2, by choosing the points [(-1.5,-1.5), (-1, -1.3889), (0,-0.5), (0.2 ,-0.2156)] that belong to LC_1, we get $y=0.2884x^2+1.7143x-0.6971$ which is the approximated equation of LC_2. When we draw up the critical curves LC, LC_1 & LC_2 we get the closed area (Δ) whose boundary $\partial\Delta=(b_1a_1a_2b_1)$. This area $\Delta=d'_a$ is an approximated absorbing area since it satisfies the conditions of the definition of an absorbing area, as is shown in the figure (3.1.1a).

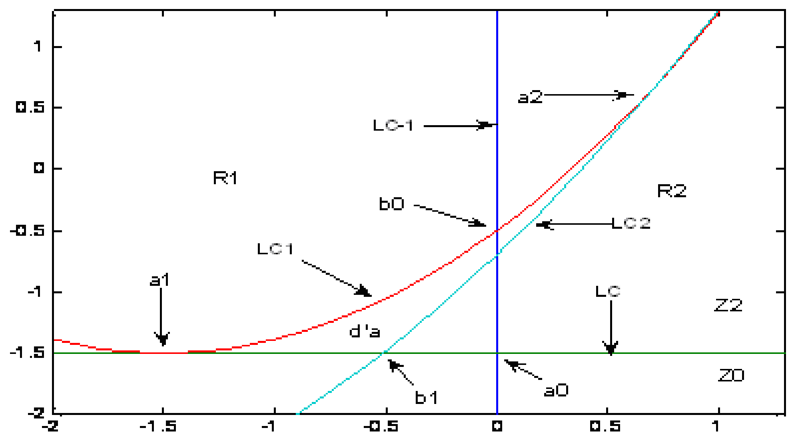


Figure (3.1.1a): Map (3.1) with

a=-1.5, b=-1.5, the approximated absorbing area d'_a by using the proposed algorithm (2.5)

$a_0=(0,-1.5)\in LC_{-1}\cap LC$, $b_0=(0,-0.5)\in LC_{-1}\cap LC_1$, $b_1=(-0.518,-1.5)\in LC\cap LC_2$, $a_1=(-1.4989,-1.5)\in LC\cap LC_1$ and $a_2=(0.7465,0.7441)\in LC_1\cap LC_2$

In fact, by construction $\partial\Delta$ consists of critical curves of finite rank, i.e. $\partial\Delta$ consists of the segment L, L_1& L_2 which are the segments of the critical curves LC, LC_1& LC_2 respectively. Numerical computations show that the successive iterates of any points which either belong to Δ or to $U(\Delta)/\Delta$, enter Δ after a finite number of iteration and can not get away after entering. Since $T(\Delta)\subset\Delta$, therefore $T^{m+1}(\Delta)\subset T^{m}(\Delta)$, $\forall$ $m\geq 0$ and we have found M=4 which satisfies $T^{M}(\Delta)=T^{M+1}(\Delta)$, So $d''_a=T^{M}(\Delta)$ is an approximated invariant absorbing area as shown in figure (3.2.1b).

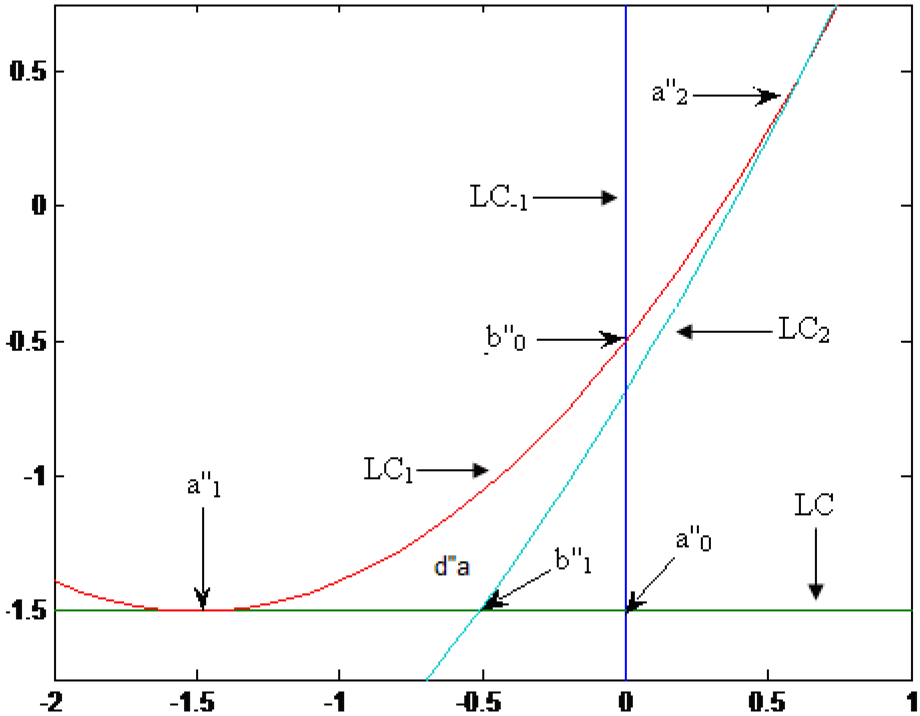


Figure (3.1.1b): Map (3.1) with

a=-1.5, b=-1.5, the approximated invariant area d"$_a$ by using the proposed algorithm (2.5)

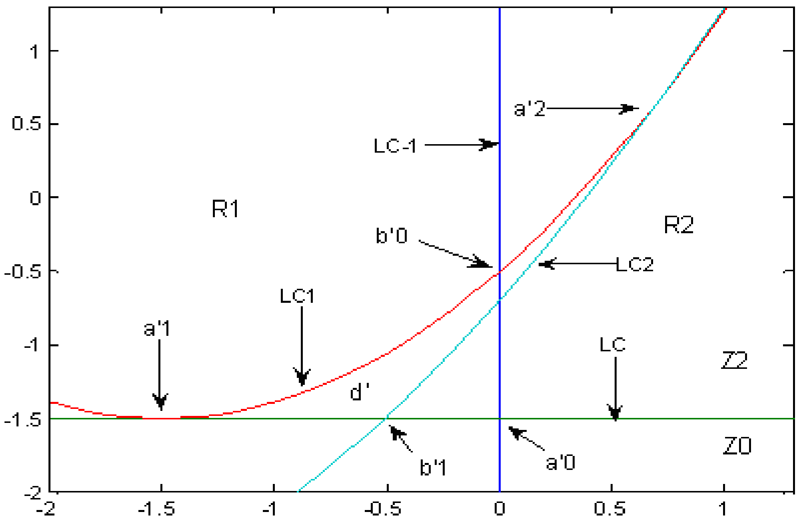
Now if we apply construction algorithm (2.2.1) with a'_0=(0,-1.5)$\in LC_{-1}\cap LC$. We look for integer N such that LC_N intersects LC_{-1}, so we shall find that $b'_0\in LC_{-1}\cap LC_1$ i.e. N=1, b'_0=(0,-0.5), and b'_1=(-0.5,-1.5)$\in LC\cap LC_2$ and a'_1=(-1.5,-1.5)$\in LC\cap LC_1$ and a'_2=(0.75,0.75) $\in LC_1\cap LC_2$. We construct the closed area Δ whose boundary $\partial\Delta=(b'_1a'_1a'_2b'_1)$. Before we represent Δ in a figure let us compute the equations of LC_1 and LC_2. The equation of LC is y=b, by substituting this in map (3.1) we get:

$$y=\left(\frac{x-b}{a}\right)^2+b,$$

which is the equation of LC_1 so it must put a≠0 while when we use our proposed algorithm we did not need this condition. Compute the equation of LC_2 by substituting equation of LC_1 in map (3.1) we get:

$$x=\pm a\sqrt{y-b}+\left(\frac{\pm\sqrt{y-b}-b}{a}\right)^2+b.$$

We notice that in fig (3.1.1c) $b'_1 \in (a'_0\ a'_1)$, therefore $T^{m+1}(\Delta)=T^m(\Delta)$, $\forall\ m \geq 0$ and have found M=4 which satisfies $T^M(\Delta)=T^{M+1}(\Delta)$. So $d''=T^M(\Delta)$ is the invariant absorbing area.



Figure(3.1.1c): Map (3.1) with

a=-1.5, b=-1.5, the absorbing area d'

by using construction algorithm (2.2.1), a'_0=(0,-1.5), b'_0=(0,-0.5),

b'_1=(-0.5,-1.5), a'_1=(-1.5,-1.5), a'_2=(0.75,0.75)

From figure (3.1.1a)& figure (3.1.1c) we notice that the boundary points of d'_a ; b_1,a_1,a_2 is closed to the boundary points of d', b'_1,a'_1,a'_2 respectively. We conclude that d'_a coincide with d' and two figures are conformable.

Now, we shall take other values of the parameter a and b, to see the dynamics of the map (3.1), we shall take a=-1.5, b=-1.75.

P_1=(-0.57,-1.4251), and P_2=(3.07,7.675) are two fixed point of T. Numerical computations show that P_1 is a stable fixed while P_2 an expanding fixed point. As act erstwhile, we choose some suitable points $(x_{0,j}, y_{0,j})$, j=1,…,5 that belong to the curve LC, we choose[(-0.5,-1.75), (-0.2,-1.75), (-0.1,-1.75), (-0.01,-1.75), (0.02,-1.75)] to get $y=0.4444x^2+1.5556x-0.3889$ which is the approximated equation of LC_1, and by choosing another points $(x'_{1,j},y'_{1,j})\in LC_1$ i.e. we choose [(-2.5,-1.5004), (-1.75,1.7502), (-0.05,-0.4656), (0,-0.3889), (0.2,-0.06)] to get $y=0.0415x^2+2.5492x-0.7993$ which is the approximated equation of LC_2. Drawing up the curves LC_{-1}, LC, LC_1&LC_2 produces a closed area as shown in figure (3.1.2) and a numerical computation shows that this area is absorbing $\Delta=d'_a$. We found that M=3 which satisfies $T^M(\Delta) = T^{M+1}(\Delta)$, so $d''_a=T^m(\Delta)$ is the approximated invariant absorbing area.

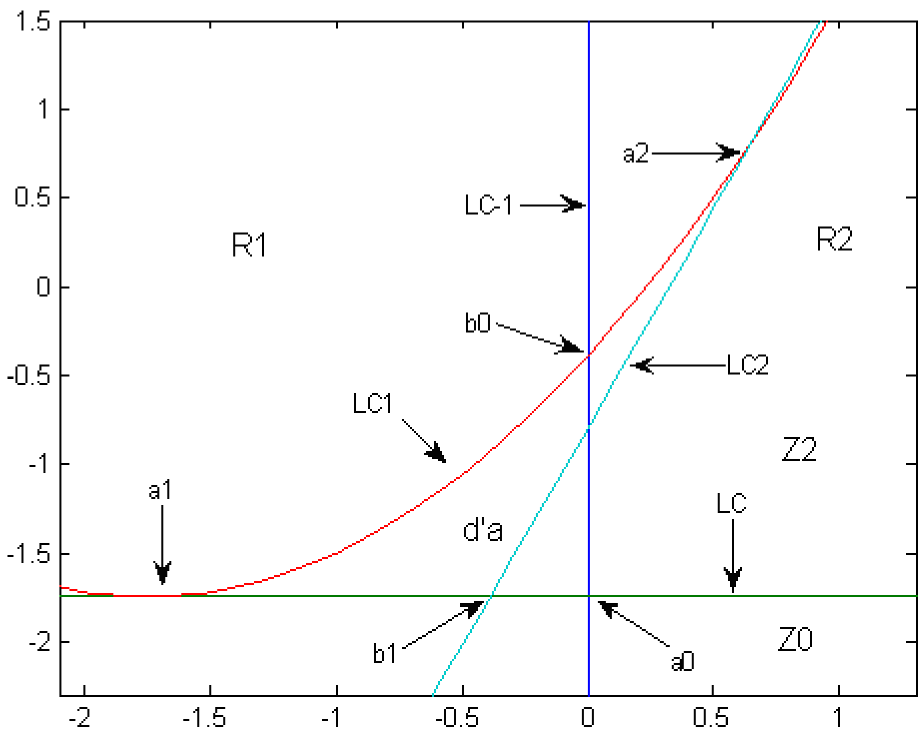


Figure (3.1.2): Map (3.1) with

a=-1.5, b=-1.75, the approximate absorbing area d'_a

a'_0=(0,-1.75), a'_1=(-1.749,-1.75), a'_2=(0.6521,0.8535),b'_0=(0,-0.3889), b'_1=(-0.3908,-1.75)

Now, if we apply algorithm (2.2.1) with a_0=(0,-1.75)$\in LC_{-1} \cap LC$. We find that N=1, construct closed area d'_a whose boundary $\partial\Delta=(b_1a_1a_2b_1)$ and is an absorbing area as shown in figure (3.1.2'). We have found M=3 whish satisfies $T^M(\Delta)=T^{M+1}(\Delta)$, So d''=$T^M(\Delta)$ is the invariant absorbing area.

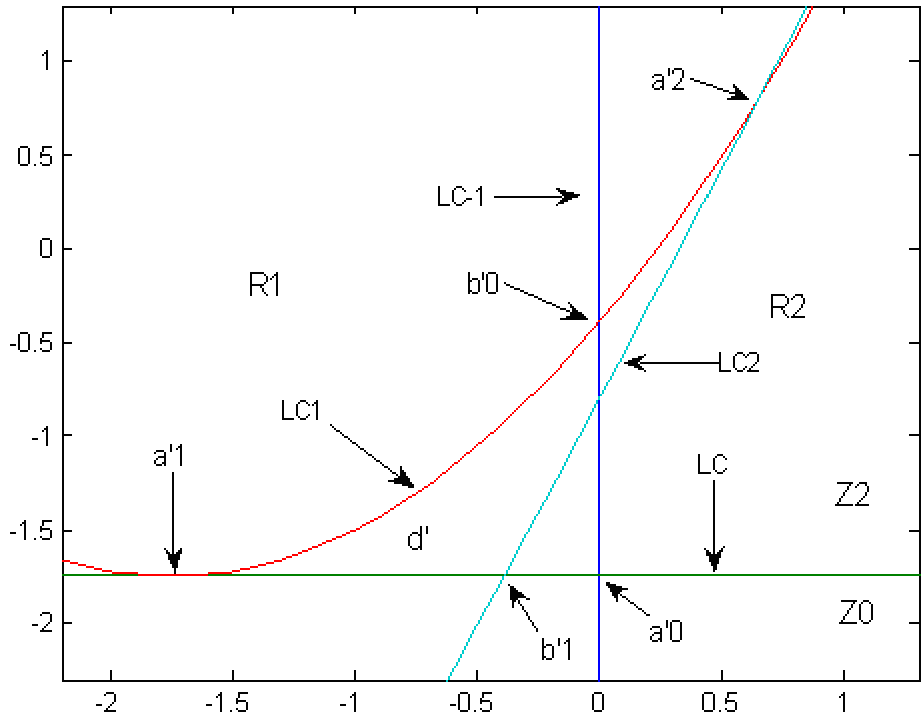


Figure (3.1.2'): Map (3.1) with

a=-1.5, b=-1.75, the absorbing area d'

by using construction algorithm (2.2.1)

a_1=(-1.75,-1.75), a_2=(0.875,1.3125), b_0=(0,-0.3889), b_1=(-0.3889,-1.75)

From figure (3.1.2)& figure (3.1.2') we notice that there is a simple difference between the boundary points b_1, a_1 and b'_1, a'_1 respectively and the boundary point a_2 is far away from the boundary point a'_2. We conclude that d'_a is of smaller shape then d'.

Again we shall take another values of a and b and apply the proposed algorithm (2.5). For the particular case a=1.5, b=0, the approximated equations of LC_1, LC_2 and LC_3 respectively are:

$y=0.4444x^2$

$y=0.6119x^2-0.66461x$

$y=-0.2159x^2+2.8293x+2.025$

Drawing up the critical curves LC_i , i=0,…,3, we get a closed area Δ. This area $\Delta=d'_a$ is an absorbing area as is shown in the figure (3.1.3).

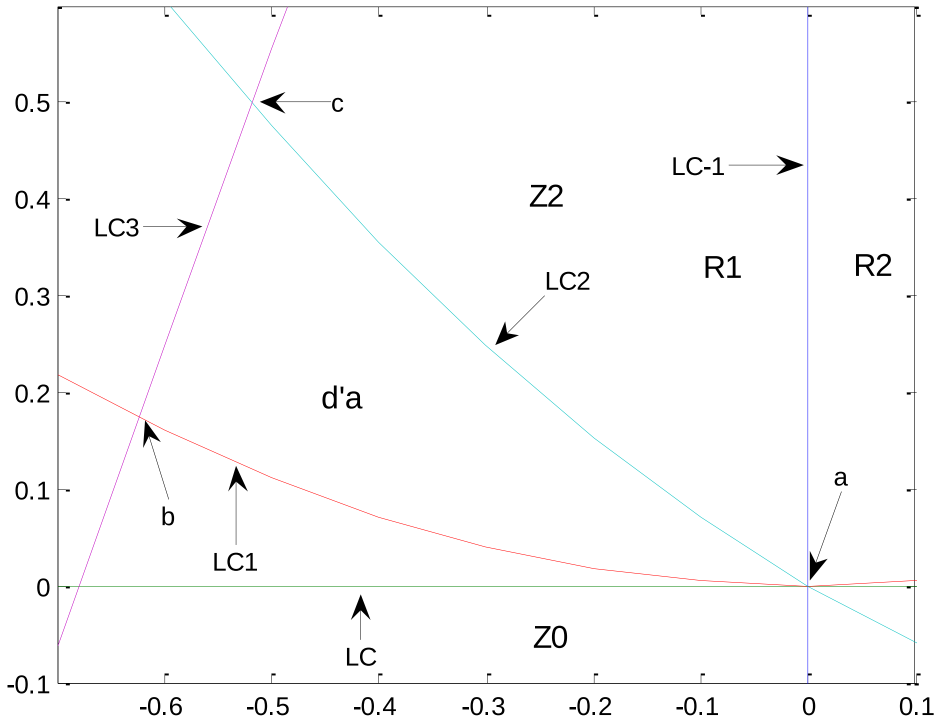


Figure (3.1.3): Map (3.1) with

a=1.5, b=0, d'_a by using proposed algorithm (2.5)

$a=(0,0)\in LC_{-1}\cap LC\cap LC_1\cap LC_2$, $b=(-0.6243,0.174)\in LC_1\cap LC_3$ and $c=(-0.5183\cap 0.5002)\in LC_2\cap LC_3$.

This area Δ=d'$_a$ is an absorbing area since it satisfies the conditions of the definition of an absorbing area. We have found M=3 whish satisfies $T^M(\Delta)=T^{M+1}(\Delta)$, So $d_a''=T^M(\Delta)$ is the approximated invariant absorbing area.

Also, we have notice that one of the eigenvalues of the jacobian of T at the fixed point P_1=(0,0) has zero eigenvalue along LC_{-1}. λ_1=1.5 & λ_2=0, (P_1 is a saddle fixed point) therefore according to remark (1.3.1b) LC_1 and LC_{-1} are quadraticlly tangent at point a_1=(0,0).

When use the constructing algorithm (2.2.1) we shall get a point a_0=(0,0). i.e. an absorbing area is just the point a_0=(0,0).

Example (3.1.2):

Consider the map *T* defined by

$$T: \begin{array}{l} x' = a - by^2 - x^2 \\ y' = bx + a \end{array} \quad \text{, with } b \neq 0, \qquad (3.2)$$

T is continuously differentiable and noninvertible map whose inverses are

$$T^{-1}: \begin{array}{l} x = (y'-a)/b \\ y = \pm\sqrt{(a - x' - [(y'-a)/b]^2 + a)/b} \end{array}$$

T has two fixed points ($\dfrac{-(1+2ab^2) \pm \sqrt{(1+2ab^2)^2 + 4a(1-ab)(b^3+1)}}{2(b^3+1)}, bx + a$)

The equation of LC_{-1} is y=0, the equation of LC is $y=\pm b\sqrt{a-x}+a$, $x \le a$. LC has two branches LC' & LC'', i.e. we write LC' as $y=b\sqrt{a-x}+a$, and the equation of LC'' as $y=-b\sqrt{a-x}+a$, LC_{-1} divides the plane into two regions R_1,R_2, R_1 is the region y<0, R_2 with y>0.

We notice that LC consists of two branches LC' & LC'' as shown in figure (3.2.1) and the region Z_0 penetrating by Z_2, so we conclude that this map of type (Z_0<Z_2), as shown in figure (3.2.1).

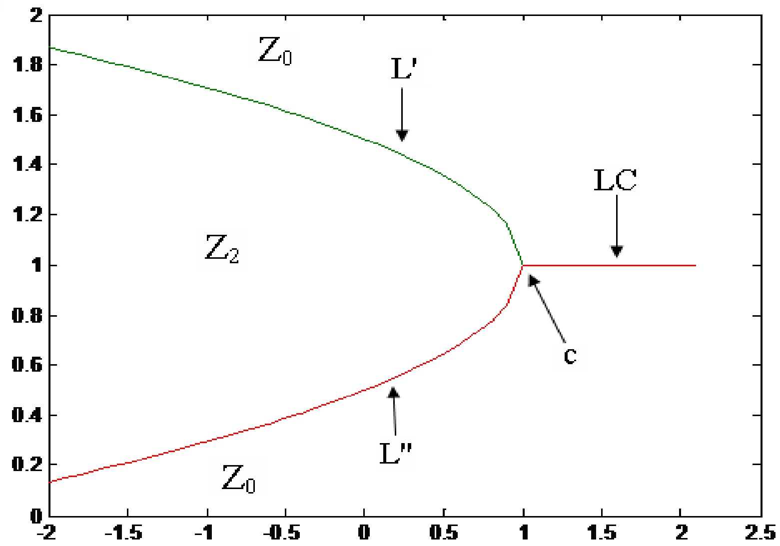


Figure (3.2.1)) (Z_0<Z_2) map; $LC=L\cup L'$, c = $L' \cap L''$;

Now, compute the equation of the critical curve LC_1, by substituting the equation of LC in map (3.2) we get:

$x = a - b[\pm\sqrt{a-(\frac{y-a}{b})+a}]^2 - (\frac{y-a}{b})$, which is the equation of LC_1.

We write LC'_1 as $y = a - b[\sqrt{a-(\frac{y-a}{b})+a^2}] - (\frac{y-a}{b})$ and LC''_1 as $y = a - b[-\sqrt{a-(\frac{y-a}{b})+a^2}] - (\frac{y-a}{b})$. As mentioned before, we shall use least square method to approximate the equations of the critical curves $LC'_1, LC''_1,\ LC'_2 \& LC''_2$.

Now we shall take some values of the parameters a and b to study the dynamical behavior of the map (3.2).

For a=1, b=0.5, the fixed points of T are:

P_1=(0.2716,1.1381) with eigenvalues λ_1 = -0.2761-0.702i and λ_2=-0.2761+0.702i therefore P_1 is an stable fixed point.

P_2= (-1.6095, 0.1953) with eigenvalues λ_1 = 0.0306 and λ_2= 3.4365 therefore P_2 is a repulsive fixed point. As example act erstwhile choose the points [(-0.25, 1.5590), (-0.19, 1.5454), (-0.1, 1.5244), (0.11, 1.4717), (-0.19, 1.5454)] that belong to LC' by using least square method we get $y=1.3032x^2+1.2935x+1.1338$, which is the approximated equation of LC_1'. Choose the points [(-0.5,0.3876), (0.5,0.6464), (0.55, 0.6646), (0.9,0.8419), (0.99,0.95)] that belong LC", we approximate the equation of LC''_1 by $y=-1.2618x^2-0.1783x+1.583$, and by choosing another points $(x'_{1,j}, y'_{1,j})$, $(x''_{1,j}, y''_{1,j})$, j=1,…,5, that belong to LC'_1, LC''_2 respectively we get $y=-0.0338x^2+0.3371x+0.55594$, and $y=0.5496x^2-1.3816x+1.6999$ which are the approximated equation of LC'_2 LC''_2 respectively.

So in figure(3.2.2) the closed area d'_a is shown. This area $\Delta=d'_a$ is an approximated absorbing area since it satisfies the conditions of the definition of an absorbing area, since $T(\Delta)\subset\Delta$, $T^{m+1}(\Delta)\subset T^m(\Delta)$, $\forall\ m\geq 0$, we notice that a,b,c & g are non-smooth boundary points. The approximated invariant area is obtained after two iterations $d''_a = T^M(\Delta)$ as shown in figure (3.2.3) which gives rise to four of non-smooth boundary points a', b', c'and g'.

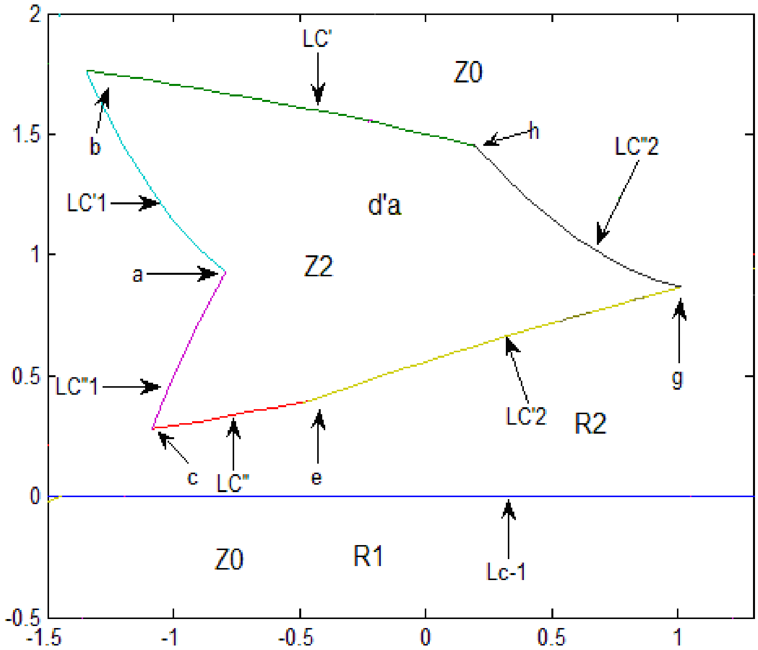


Figure (3.2.2): Map (3.2) with

a=1, b=0.5, the approximated absorbing area d'_a by using the proposed algorithm (2.5),

a=(-0.7938,0.9285),b=(-1.3503,1.7665),c=(-1.09,0.2772),

e=(-0.4676,0.395), g=(1.0106,0.8655), h=(0.1982,1.4477)

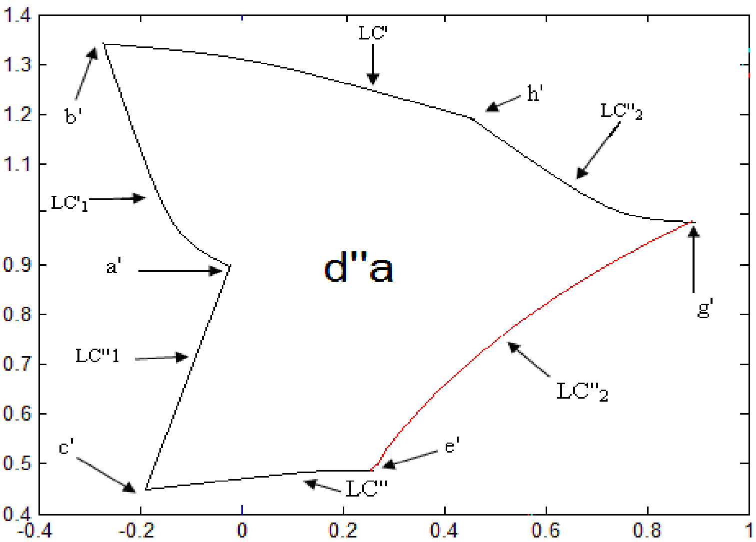


Figure (3.2.3): Map (3.2) with

a=1, b=0.5, the approximated invariant area d''_a by using the proposed algorithm (2.5).

Now, we shall take other values of the parameter *a* and *b*, we take *a*=0.5, *b*=0.5. P_1= (0.2457, 0.6228), and P_2= (-1.3568, -0.1784) are two fixed point of *T*. Numerical computation show that P_1 is a stable fixed while P_2 an expanding fixed point. As act erstwhile we choose some suitable points $(x'_{0,j}, y'_{0,j}) \in LC'$, j=1,…,5, that belongs to the curve LC', we get $y=1.5164x^2+0.8503x+0.3757$which is the approximated equation of LC'_1, and by choosing another points $(x''_{0,j}, y''_{0,j}) \in LC''$, j=1,…,5, we get $y=-0.7704x^2-0.0328x+0.7703$ which is the approximated equation of LC''_1, and by choosing another points $(x'_{0,j}, y'_{0,j})$, $(x''_{0,j}, y''_{0,j})$, j=1,…,5 that belong to LC'_1, LC''_1respectively we get $y=-0.0662x^2-0.8382x+0.0802$, and $y=-0.2695x^2-0.5565x+0.9146$ which are the approximated equations of LC'_2, LC''_2

respectively. Drawing up the curves LC_{-1}, $LC\ '$, $LC\ ''$, LC'_1, LC''_1, LC'_2 & LC''_2 produces a closed area as shown in figure (3.2.4) and a numerical computations shows that this area is absorbing $\Delta = d'_a$ which have to four of non-smooth boundary points a, b, c and g. The approximated invariant area is obtained after two iterations $d''_a = T^M(\Delta)$.

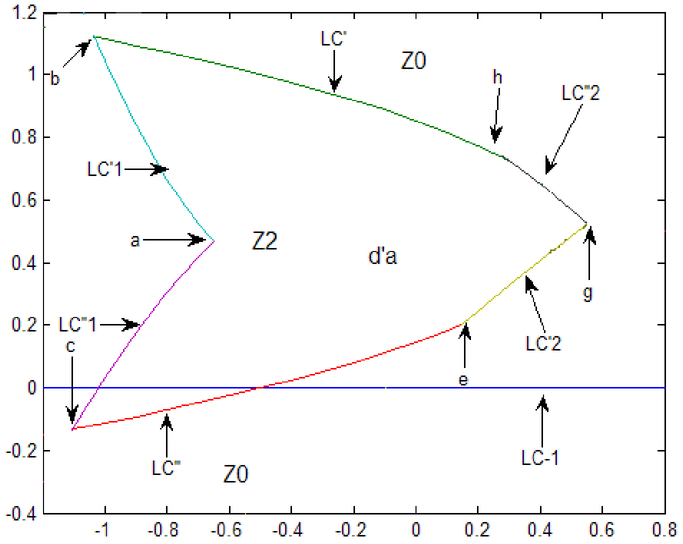


Figure (3.2.4): Map (3.2) with *a*=0.5, *b*=0.5, the approximated absorbing area d'_a by using the proposed algorithm (2.5),

a= (-0.648, 0.466), *b*= (-1.032, 1.119), *c*= (-1.104,-0.1335),

e= (0.1518, 0.2055), *g*= (0.5533, 0.5234), *h*= (0.2989, 0.72424)

For a=0, b=1, the fixed points of T are: P_1= (0,0) with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 0$ therefore P_1 is an stable fixed point.P_2= (-0.5,-0.5) with eigenvalues $\lambda_1 = -0.618$ and $\lambda_2 = 1.618$ therefore P_1 is an repulsive fixed point. $a_0 = LC_{-1} \cap LC$=(0, 0) when we choose the points [(-4.5,-0.618), (0.09,0.1798), (0.15,0.2042), (0.45,0.3882), (0.5,0.5)] that belong to LC ', we get

$y=0.0307x^2+0.4204x+0.0774$, which is the approximated equation of LC'_1, and choosing the points [(-6.6,-1.569),(-5.056,-1.2486), (-3.4,-0.8439), (-2.09,-0.4457), (-0.195,0.5584)] that belong to LC'', we get $y=0.0032x^2+0.0826x+0.9528$ which is the approximated equation of LC''_1, as is shown in figure (3.2.5).

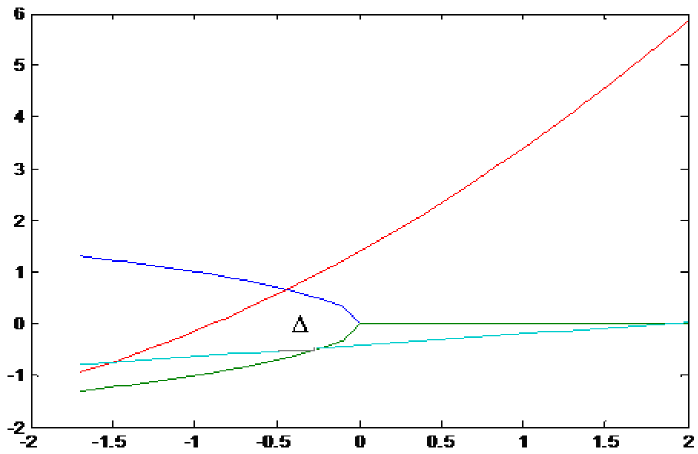


Figure (3.2.5): map (3.2) with

a=0, b=1, the closed area Δ is not absorbing

So in figure (3.2.5) the closed area Δ is shown. This area Δ is not absorbing area, since it is not satisfy the conditions of the definition of an absorbing area. For a=1, b=0.1, T has two fixed points: P_1=(0.5669,1.0567), and P_2= (-1.5859,0.8414). Numerical computations show that P_1 is an repulsive fixed while P_2 is an repulsive fixed point. Figure (3.2.6) represents the approximated absorbing area constructed by algorithm (2.5) and the approximated equations of $LC'_1, LC''_1, LC'_2 \& LC''_2$ are:

$y=0.1525x^2+0.0253x+0.8807$,

$y=-0.123x^2+0.026x+1.0885$,

$y=-0.0201x^2-0.1354x+1.1196$,

y=0.4677x^2-0.6607x+1.1603 respectively.

figure (3.2.5) represents the approximated absorbing area constructed by algorithm (2.5), which have to five of non-smooth boundary points a, b, c, e & g.

The invariant area is obtained after two iteration $d''_a=T^M(\Delta)$.

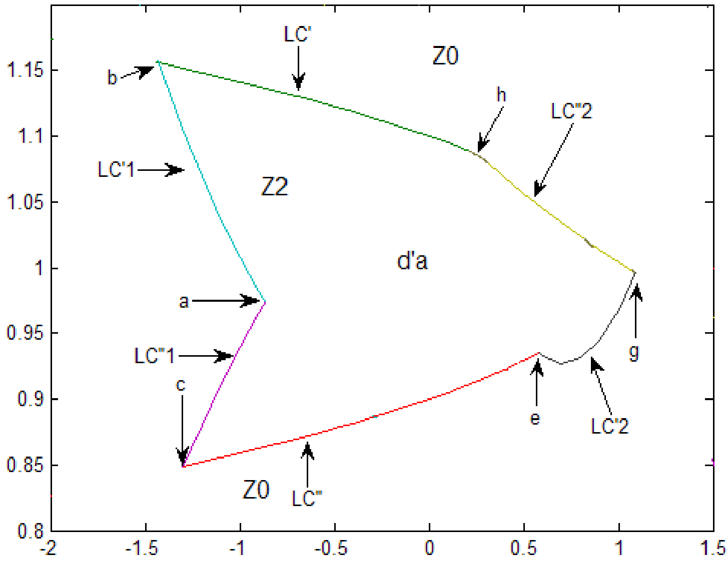


Figure (3.2.6): Map (3.2) with

a=1, b=0.1, the approximated absorbing area d'_a by using the proposed algorithm (2.5),

a= (-0.866,0.9735), b=(-1.428,1.159), c=(-1.295,0.8485),

e=(0.5797,0.9352), g=(1.0892,0.996), h=(0.2574,1.0861).

For a=0.8, b=0.3, P_1=(0.3929,0.9179), and P_2= (-1.5068, 0.3480) are two fixed point of T. Also, numerical computation show that P_1 is a stable fixed while P_2 an repulsive fixed point, the approximated equations of LC'_1, LC''_1, LC'_2 and LC''_2 respectively are :

y= -0.4842x^2-0.1288x+1.0213,

y=0.5868x^2+0.185x+0.4324,

y=-1.7262x^2-2.8886x+1.9643,

y=0.2539x^2+0.0936x+0.5007, respectively.

Drawing up critical curves LC_0, LC'_i , LC''_i ,i=1,2, we get a closed area d'_a which have to four of non-smooth boundary points a, b, c, d & g as is shown in the figure (3.2.7).

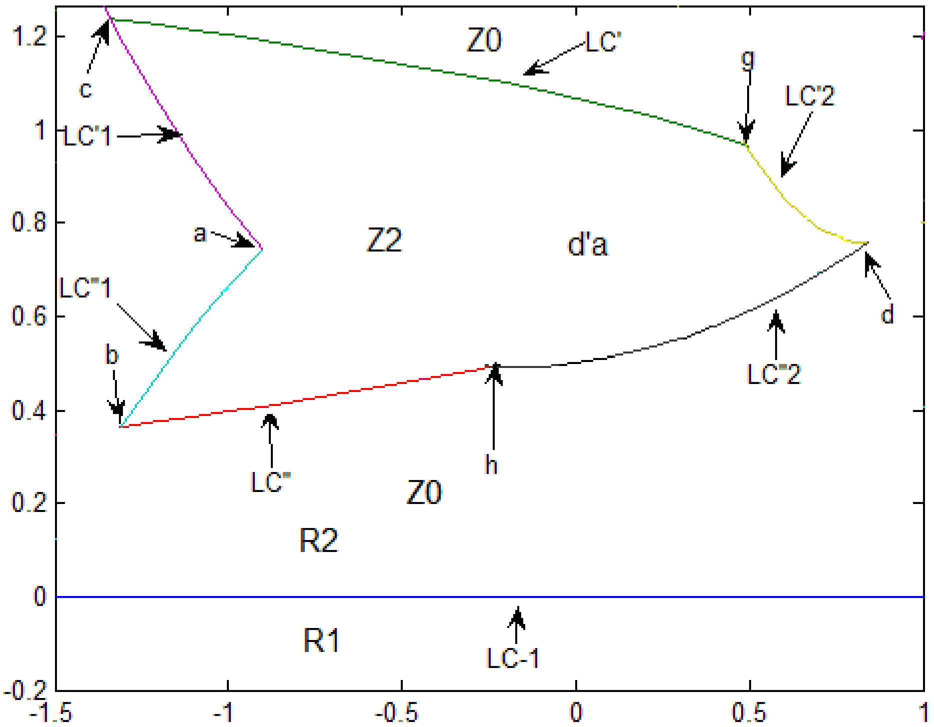


Figure (3.2.7): Map (3.2) with a=0.8, b=0.3, the approximated absorbing area d'_a by using the proposed algorithm (2.5),

a=(-0.9022,0.7433), b)=(-1.3048,0.365), c=(-1.339, 1.239

d=(0.7673,0.7226), g=(0.6111,0.9844), h=(-0.244,0.4935).

Example (3.1.3): Consider the map T defined by

$$T: \begin{array}{l} x' = ax^2 + y^2 \\ y' = bx \end{array} \quad , \quad b \neq 0 \qquad (3.3)$$

T is continuously differentiable and noninvertible map whose inverses are

$$T^{-1}: \begin{array}{l} x = y'/b \\ y = \pm\sqrt{x' - \frac{a}{b^2} y'^2} \end{array}$$

T has two fixed point (0,0), $(\frac{1}{a+b^2}, bx)$.

The equation LC_{-1} is given by y=0, the equation LC is given by $x=(a/b^2)y^2$. LC_{-1} divides the plane into two regions R_1, R_2. R_1 is the region $y<0$, R_2 with $y>0$. To compute the equation of LC_1, substitute equation of LC in the map (3.3) we get

$$x = \frac{a}{b} y^2 + \frac{b}{a} y$$ Which is the equation of LC_1.(we need to put a≠0)

to compute the equation of the critical curve LC_2, substitute equation of LC_1 in map (3.3) we get:

$$x = a[\frac{a}{b}(\frac{\frac{b^2}{a} \pm \sqrt{(\frac{b^2}{a})^2 + 4ay}}{2a})^2 - \frac{b}{a}(\frac{\frac{b^2}{a} \pm \sqrt{(\frac{b^2}{a})^2 + 4ay}}{2a})]^2 + (\frac{\frac{b^2}{a} \pm \sqrt{(\frac{b^2}{a})^2 + 4ay}}{2a}),$$ which is the equation of LC_2. we note that the difficulty of drawing up the equation of LC_2 , So we shall use least square method to approximating the equation of the critical curve LC_2.

Now we shall take some values of the parameters a and b to study the dynamical behavior of the map (3.3).

For a=1, b=1, the fixed point of T are:

P_1= (0,0) with eigenvalues λ_1 = 0 and λ_2= 0 therefore P_1 is an attracting fixed point.

P_2=(0.5,0.5) with eigenvalues λ_1 = 1.61805 and λ_2= -0.61805 therefore P_2 is a saddle fixed point .

We choose the points [(-6.25,32.8125), (0.3584,0.28), (31.11,-6.1), (31.6725, -6.15), (32.24,-6.2)] which belong to LC_1 then $T(x,y)\in LC_2$, by using least square method we can approximate the equation of LC_2 as $y=0.0443x+0.3493$. When the proposed algorithm(2.5) we get closed area whose boundaries are segments of the critical curves LC, LC_1 & LC_2 i.e. $\partial d=(a_0\ a_1\ a_2)$. Numerical computation shows this region divided into two regions d_1, d_2 as is shown in figure (3.3.1), the first region d_1 whose boundaries $\partial d_1=(a_0\ a\ b\ a_1)$ where a=1, b=1 in this region the iterates of each point enter d_1 and never it get a way after entering and the second region is d_2 whose boundary $\partial d_2=(a\ b\ a_2)$, moreover we note that $T^2(d_2)$, enter the region $d_3=\{0.1984\leq x\leq 0.3846, 0.355\leq y\leq 0.512\}$and $T(d_3)$ entered region d_1 and never get away after entering.

Another phenomena that be mentioned when we try to find M that satisfy $T^M(d_1)=T^{M+1}(d_1)$ we find that $T^M(d_1)$ approach to the fixed point P_1 of when M=23 i.e. $T^{23}(d_1)\rightarrow(0,0)$ i.e. the invariant area is just a point.

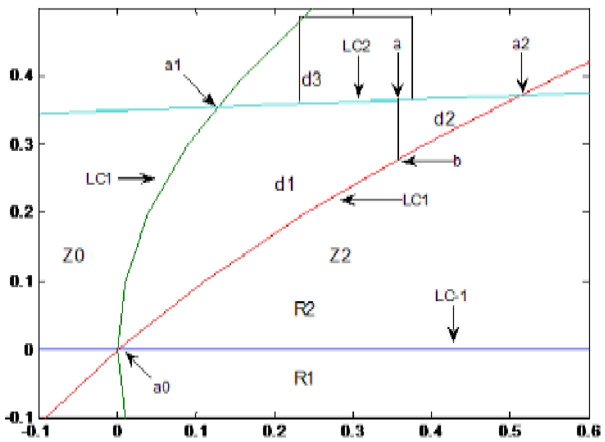


Figure (3.3.1): Map (3.3) with a=1, b=1, d'_a constructed by the proposed algorithm (2.5),
$a_0=(0,0)\in LC_{-1}\cap LC\cap LC_1$, $a_1=(0.128,0.355)\in LC\cap LC_2$,
$a_2=(0.512,0.372)\in LC_1\cap LC_2$, a=(0.355,0.366), b=(0.355,0.2755)

For a=1, b=2, the fixed points of T are:

P_1= (0,0) with eigenvalues λ_1 = 0 and λ_2= 0 therefore P_1 is an attracting fixed point.

P_2=(0.2,0.4) with eigenvalues λ_1 = 1.4806 and λ_2= -1.0806 therefore P_2 is an expanding fixed point .

Now to compute the equation of LC_2 again we subsitute the points [(-0.4688,-0.25), (-0.38,-0.2), (0.0201,0.01), (0.205,0.1), (0.3112,0.15)] that belong to LC_1, we use least squares method we get the equation $y=-25.9616x^2+2.6883x+0.2099$, which is the equation of LC_2. As is shown in figure (3.3.2)

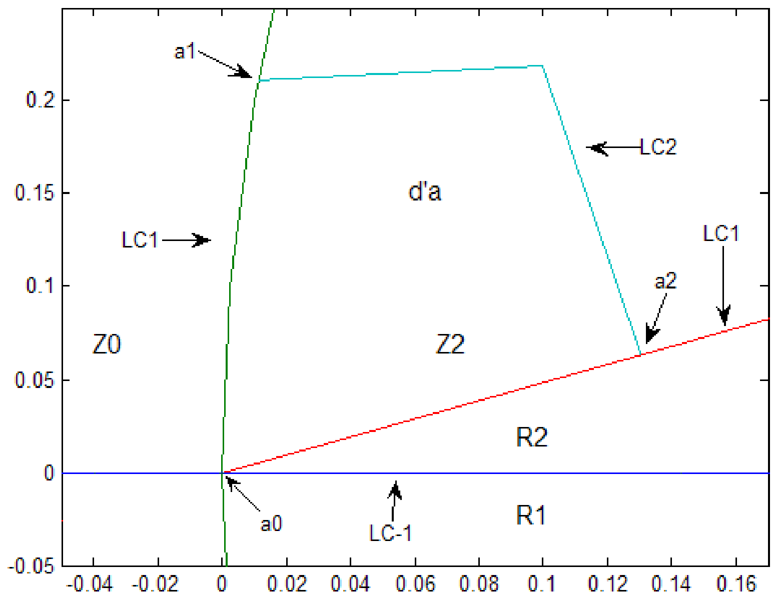


Figure (3.3.2): Map (3.3) with a=1, b=2, d'a constructed by the proposed algorithm (2.5),

a_0=(0,0)$\in LC_{-1}\cap LC\cap LC_1$, a_1=(0.1305,0.0636)$\in LC\cap LC_2$,

a_2=(0.0114,0.2109)$\in LC_1\cap LC_2$.

So in figure (3.3.2) the closed area d'_a is shown. This $\Delta=d'_a$ is an approximated absorbing area since it satisfies the conditions of the definition of an absorbing area.

We have found M=5 whish satisfies $T^M(\Delta)=T^{M+1}(\Delta)$, So $d''_a=T^M(\Delta)$ is an approximated invariant absorbing area.

Again we shall take another values of a and b and apply the proposed algorithm (2.5). For the particular case a=2, b=1, the approximated equation of LC_2 is: $y=0.0102x+0.2786$.

Drawing up critical curves LC_i , i=0,1,2, we get a closed area d'_a as is shown in the figure (3.3.3).

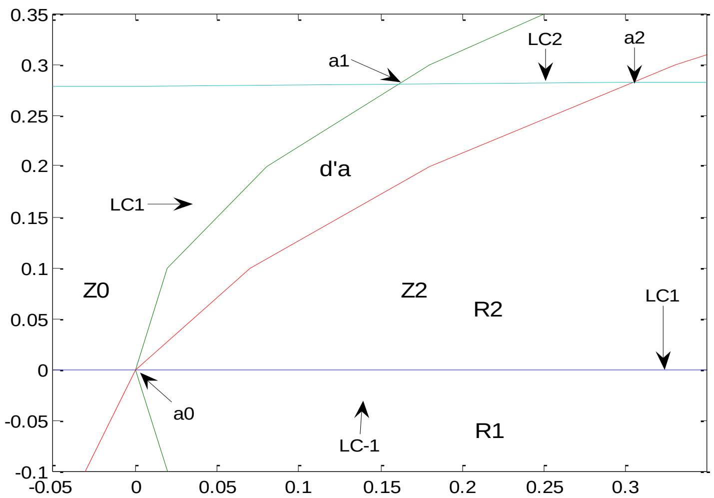


Figure (3.3.3): Map (3.3) with a=2, b=1, d'_a constructed by the proposed algorithm (2.5),

$a_0=(0,0)\in LC_{-1}\cap LC\cap LC_1$, $a_1=(0.16,0.2802)\in LC\cap LC_2$,

$a_2=(0.3025,0.2816)\in LC_1\cap LC_2$.

So in figure (3.3.3) the closed area d'_a is shown. This $\Delta=d'_a$ is an absorbing area since it satisfies the conditions of the definition of an absorbing area.

We have found M=5 whish satisfies $T^M(\Delta)=T^{M+1}(\Delta)$, So $d''_a=T^M(\Delta)$ is the invariant absorbing area.

For a=1, b=-1, the fixed points of T are:

P_1= (0,0) with eigenvalues λ_1 = 0 and λ_2= 0 therefore P_1 is an attracting fixed point.

P_2=(0.5,-0.5) with eigenvalues λ_1 = 1.61805 and λ_2= -0.61805 therefore P_2 is an repulsive fixed point .

The curve LC_{-1} is given by y=0, the equation of the critical curve LC is given by $x=y^2$.The equation of critical curve LC_1 is given by $x=-y(y+1)$ as is shown in figure (3.3.4).

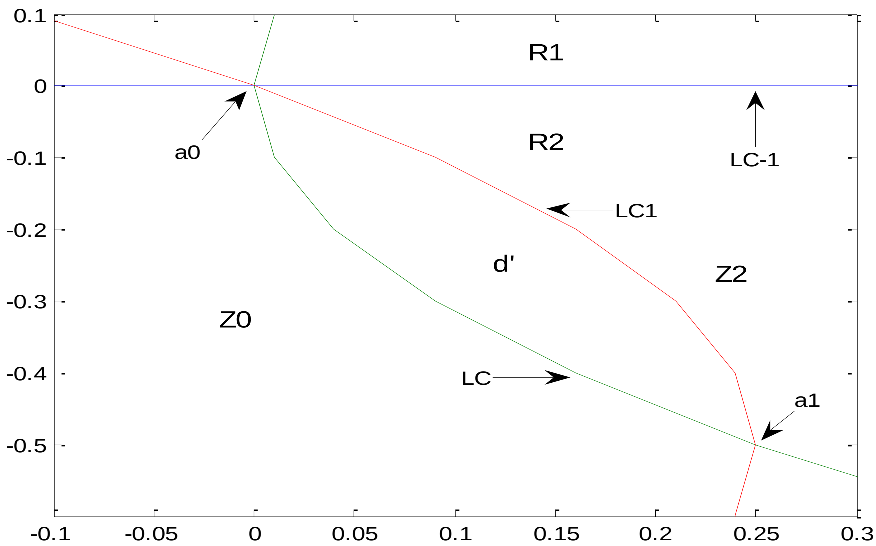


Figure (3.3.4): Map(3.3) with a=1, b=-1, d'_a is approximated absorbing area, $a_0=(0,0)\in LC_{-1}\cap LC\cap LC_1$, a_1= $(0.25,-0.5)\in LC\cap LC_1$.

So in figure (3.3.4). the closed area d'_a is shown. This $\Delta=d'_a$ is an absorbing area since it satisfies the conditions of the definition of an absorbing area.

We have found M=5 whish satisfies $T^M(d'_a)=T^{M+1}(d'_a)$, So $d''=T^M(d'_a)$ is the invariant absorbing area.

Example (3.1.4): Consider the map T defined by

$$T: \begin{array}{l} x' = ax^2 + y^2 \\ y' = bx + a \end{array} \quad , \quad b \neq 0 \qquad (3.4)$$

T is continuously differentiable and noninvertible map whose inverses are

$$T^{-1}: \begin{array}{l} x = (y' - a)/b \\ y = \pm\sqrt{x' - \frac{a}{b^2}(y' - a)^2} \end{array}$$

T has two fixed points ($\frac{-(2ab-1) \pm \sqrt{(2ab-1) - 4(a+b^2)a^2}}{2(a+b^2)}, bx + a]$).

The equation of LC_{-1}, LC given by y=0, $y = \pm b\sqrt{\frac{x}{a}} + a,\ a \neq 0$, respectively.

Now we shall take some values of the parameters a and b to study the dynamical behavior of the map (3.4).

For a=0.1, b=0.5, the fixed points of T are:

P_1= (0.3136, 0.2568) with eigenvalues λ_1 = 0.5391 and λ_2= -0.4764 therefore P_1 is an attracting fixed point.

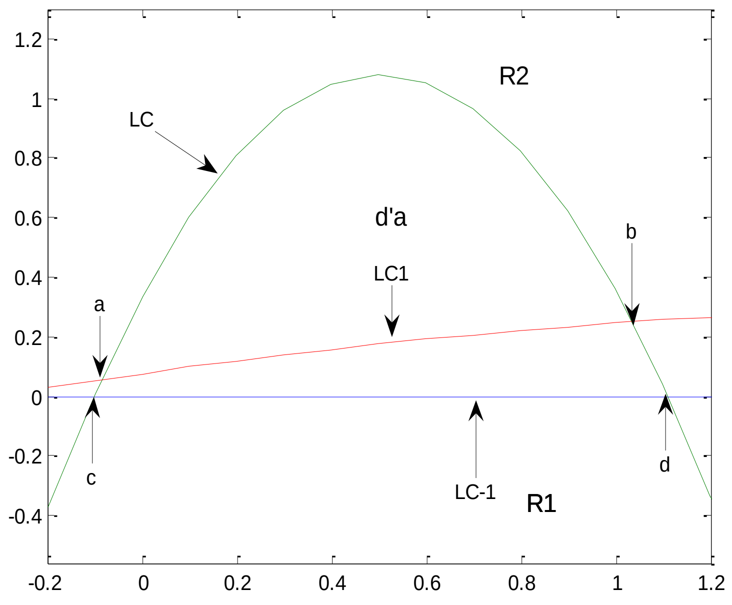
P_2= (0.0112, 0.1056) with eigenvalues λ_1 = 0.6349 and λ_2= -0.6343 therefore P_2 is an attracting fixed point.

Now, if we apply the algorithm (2.2.1) we get a closed bounded area which is not absorbing, while applying algorithm (2.5) gives an approximated absorbing area d'_a that shown in figure (3.4.1) and the approximated equations of LC, LC_1 are:

y=-2.9294x^2+2.9559x+0.3345,

y=-0.0548x^2+0.2235x+0.0763, respectively.

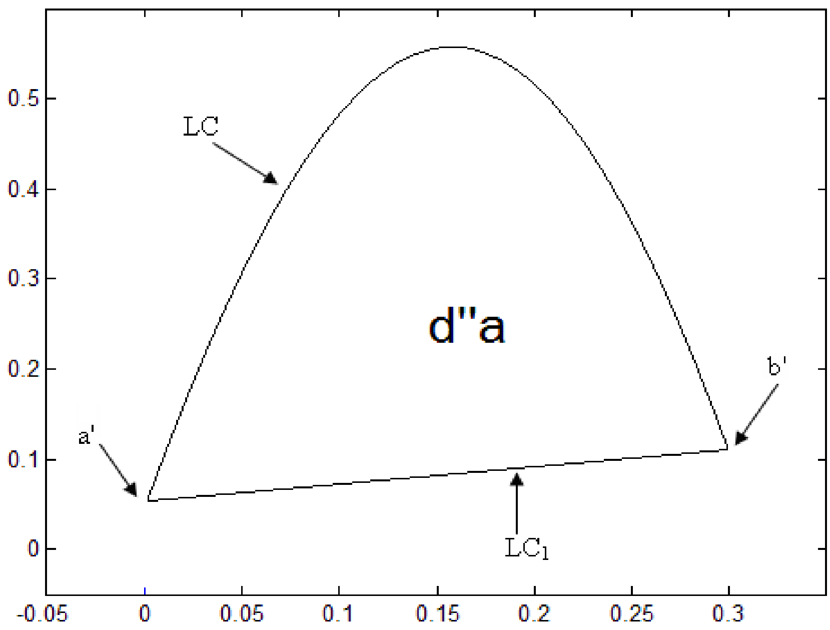
We have founded M=2 which satisfies $T^M(\Delta)=T^{M+1}(\Delta)$, so $d''_a= T^M(\Delta)$ is the invariant absorbing area as shown in figure (3.4.2).



a=0.1,b=0.5

Figure (3.4.1) d'_a constructed by the proposed algorithm(2.5).

a= (-0.0855,0.057)∈ $LC \cap LC_1$, b =(1.035,0.25)∈ $LC \cap LC_1$, c=(-0.103,0)∈$LC_{-1} \cap LC$, d=(1.111,0)∈$LC_{-1} \cap LC$.



a=0.1,b=0.5

Figure (3.4.2) d''_a constructed by the proposed algorithm(2.5).

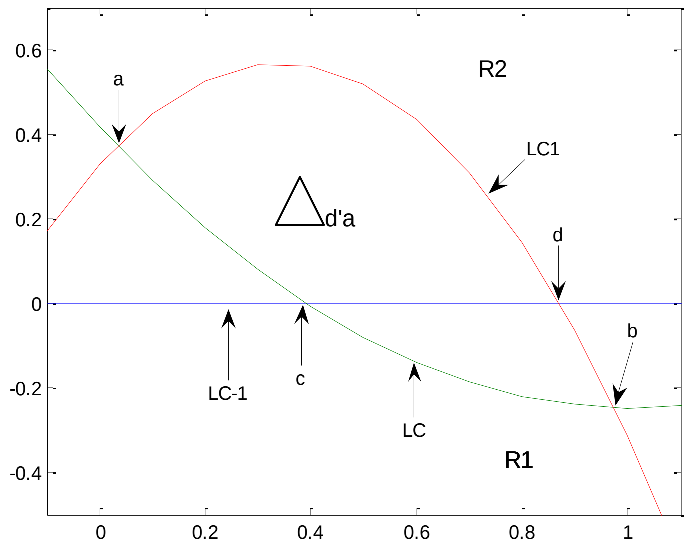
Again we take other values of the parameter a and b, i.e. a=0.3, b=0.3.

P_1= (0.3021,0.3906), and P_2=(0.1162, 0.3349) are two fixed points of T. Numerical computation show that P_1 is an attracting fixed while P_2 an attracting fixed point, the approximated equations of LC and LC_1, are :

$y=0.6581x^2-1.3226x+0.417$,

$y=-2.0373x^2+1.3947x+0.3306$, respectively

by proposed algorithm (2.5) we get a closed area Δ as is shown in the figure (3.4.3) which is an absorbing area, i.e. $d'_a=\Delta$.



a=0.3,b=0.3

Figure (3.4.3) d'_a constructed by the proposed algorithm (2.5).

a=(0.035,0.37)∈ $LC \cap LC_1$, b=(0.973,-0.245)∈ $LC \cap LC_1$,

c=(0.392,0)∈$LC_{-1} \cap LC$, d=(0.87,0)∈$LC_{-1} \cap LC$.

We note that $\overline{R}_1 \cap \Delta$ is the region whose boundary is b c d and $T(\overline{R}_1 \cap \Delta) \subset \Delta$ then T(Δ)=Δ=d''_a is invariant.

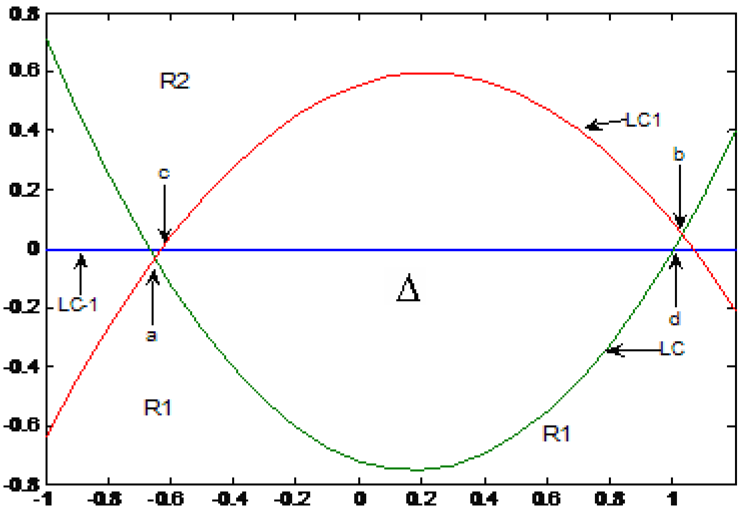
Again we take other values of the parameter a and b, i.e. a=0.-0.5, b=0.5.

P_1= (-0.3851,-0.6926), and P_2=(0.1623,-0.4189) are two fixed points of T. Numerical computation show that P_1 is an attracting fixed while P_2 an attracting fixed point, the approximated equations of LC and LC_1, are :

$y=1.0743x^2-0.3621x-0.7226$,

$y=-0.8313x^2+0.3655x+0.5567$, respectively

by proposed algorithm (2.5) we get a closed area Δ as is shown in the figure (3.4.4) which is an absorbing area, i.e. d'_a =Δ.



a=-0.5,b=0.5

Figure (3.4.4) d'_a constructed by the proposed algorithm(2.5).

a=(0.65,-0.034)∈ $LC \cap LC_1$, b=(1.031,0.048)∈ $LC \cap LC_1$, c=(-0.625,0)∈$LC_{-1} \cap LC$, d=(1.005,0)∈$LC_{-1} \cap LC$.

We note that $\overline{R}_1 \cap \Delta$ is the region whose boundary is b c d and $T(\overline{R}_1 \cap \Delta) \subset \Delta$ then T(Δ)=Δ= d''_a is invariant.

3.2 Conclusions & for future work

From the present study, we can conclude the following:

(1) In example (3.1.1) when we make a comparison between the construction algorithm (2.2.1) and the proposed algorithm (2.5) we see that for some values of the parameter a and b, in both algorithm we get the same invariant absorbing area but for the specific case when a=1.5, b=0 if we apply the proposed we get area which is absorbing as is shown in figure (3.1.3) while when we apply the construction algorithm we get just a point.

Also, for such case we notice that the Jacobian of *T* has zero eigenvalue at the fixed point P_1=(0,0).

(2) In example (3.1.2) we notice that this map of type ($Z_0<Z_2$) and there are points of non-smoothness in the boundary of the approximated absorbing area as shown in figure (3.2.2) and when we take another value of the parameter b, i.e. b=0.1, a new point of non-smoothness on the boundary of the absorbing area d'_a is born which is e as shown in figure (3.2.6).

(3) In example (3.1.3) for a=1,b=1 the Jacobian of T has two zero eigenvalues at the fixed point P_1=(0,0) and when we apply algorithm (2.5) gives a closed area as is shown in figure (3.2.1) and this closed area can be divided into two regions d_1, d_2 one of them is absorbing which is d_1 while d_2 is not, but $T^3(d_2)$ enter into d_1.

(4) We notice that if we apply the constraction algorithm (2.2.1) we get area that will be not absorbing while when we use the proposed algorithm (2.5) we get absorbing area.

Also, for future work, our recommendations are:

(1) Use another method to approximate the equation of the critical curve LC_i.

(2) Studying noninvertible maps with degree greater than two and make general observations.

References

1- Al-WA'Li Z., 2005., "**On Absorbing Areas Of Planar Quadratic Maps**", ph. D. thesis Al-Nahrain University.

2- Beck A. and Lewin M., 1974., "**Continuous Flow n the Plane**", Berln Hdlbrg, Nuw York.

3- Bischi G.I. & Gardini L., 1999., "**Role of Invariant and Minmal Absorbing Areas in Chaos Synchronization**", Physical Review , p.5710-5719.

4- Brahim, K., Mohamed, L. S. & Iihem, D. 2007., "**Critical Curves and 2D Coupled Maps**", Elec. Energ., **20**(2), p. 245- 254.

5- Brin M., & Stuck G., 2002., "**Introduction to Dynamical Systems**",Printed in the United States of America.

6- Burden, Richard L., 1984., "**Numerical Analysis**", Prindle. Weber & Schmidt Boson.

7- Cathala J.C., 1983., "**Absorptive Areas and Chaotic Areas in Two-dimensional Endomorphism**", Nonlinear Analysis,7, p.147-160.

8- Cathala J.C., 1984., "**On the Bifurcation between a Chaotic Area of T^k and a Chaotic Area of T**", Proc. of Int. Symp. on Iteration Theory, Lochau (Austria, 28 Sept.1 Oct. 1984). Lecture Notes in Math., 1163, Springer, Berlin 1985, 14-22.

9- Cathala J.C., 1987., "**Bifurcation Occurring in Absorptive Areas and Chaotic Areas**", Int. J. Systems Sci., **18**(2), 339-349.

10- Cathala J.C., 1990., **"Multi Connected Chaotic Areas in Second Order Endomorphisms"**, Int. J. Systems Sci., **21**(5), p.863-887.

11- Devany, R. 1989., **"An Introduction to Chaotic Dynamical Systems**", Second Edition, Addision- Wesely.

12- Friedland S. & Milnor J., 1989., "**Dynamical Properties of Planar Polynomial Automorphisms**", Ergod. Th. & Dynam. Sys. P.67-99.

13- Frouzakis, C., Gardini, L., Keverkidis, I., Millerioux, G. and Mira, C. (1997)., "**On Some Properties of Invariant Sets of Two-Dimensional Noninvertible Maps**", International journal of bifurcation and chaos, **7**(6), p. 1167- 1194.

14- Foroni I. & Gardini L., 2003., "**Homoclinic Bifurcation in Heterogeneous Market Models**", Solitions & Fractals, p. 747-760.

15-Gardini L., 1991., "**On The Global Bifurcation of Two-Dimensional Endomorphisms by use of Critical Lines**", Non Linear Analysis T., M. & A., **18**(4), p.361-399**.**

16-Gardini L., 1991a., "**Global Analysis and Bifurcation in Two-Dimensional Endomorphisms by use of Critical Lines**", Proceedings of ECIT (European conference on Iteration Theory) Lisbona, Portugal, Portugal, Sept., 91, p.114-125.

17- Gardini L., 1992., "**Homoclinic Orbits of Saddles in Two-Dimensional Eendomorphisms**", Proceedings of ECIT (European conference on Iteration Theory) Batschuns, Austria.

18- Gardini L., 1992a., "**Absorbing Areas and Their Bifurcations**", Internal report of the Istituto di Scienze Economiche Universita' di Urbino.

19-Gardini L., J. C. Cathala, C. Mria, 1992., "**Contact Bifurcations of Absorbing and Chaotic areas in Two-Dimensional Endmomrphisms**", Proceedings of ECIT (European conference on Iteration Theory) Batschuns, Austria, Sept. 92. World Scientific Ed.

20-Gardini L., C. Mria, D. Fournier Prunaret, 1992a., "**Properties of Invariant Areas in Two-Dimensional Endomorphisms**", Proceedings of ECIT (European conference on Iteration Theory) Batschuns, Austria. World Scientific Ed.

21-Gardini L., 1993., "**On a model of Financial Crisis. Critical Lines as New Tools of Global Analysis**", In Nonlinear Dynamics in Economics and Social Sciences, Gori, Geronazzo, Galetti eds., Springer-Verlag, N-Y.

22-Gardini L., 1994., "**Homoclinic Bifurcations in n-Dimensional Endomorphisms Due to Expanding Periodic Points**", Non linear Analysis T., M. & A., **23**(8), p.1039-1089.

23- Gardini L., Abraham R., Fournier Prunaret D., and Record R. J., 1994., "**A Double Logistic Map**", Int. J. of Bif. and Chaos, **4**(1), p.145-176.

24- Gardini L., C. Mira, 1995., "**Properties of Noninvertible Maps**", Internal note of the "Istituto di Scienze Ecoomiche ",Universita degli di Urbino (Italia).

25- Gumowski I. & Mria C., 1978., "**Bifurcation Detabilisant une Solution Chaotique d'un Endomorphisme du 2nd Order**", Comptes Rendus Aead. Sc. Paris, Serie A, **286**, serie A, p.427-471.

26-Gumowski I. & Mria C., 1980(a)., "**Recurrences and Discrete Dynamic Systems**", Lecture notes in Mathematics, Springer.

27-Gulick, D. 1992., "**Encounter with Chaos**", Mc. Graw – Hill, Inc.

28- Kawakami H. & Kobayashi 1979., "**Compute Experiments on Chaotic Solutions of x(t+2)-ax(t+1) x^2(t)=b**", Bull. Fac. of Engin., Tokushima Univerity, **16**, p.29-46.

29- Illhem D., & Boukemara I., 2007., "**Dynamics of a Three Parameters Family of Piecewise Maps**", Ser.: Elec. Energ. **20**(1), p.85-92.

30-Mira C., 1990., "**Systemes Asservis Non Lineaires**", Ed. Hermes, Paris, 5 pages.

31- Mira C.& Narayaninsamy T., 1993., "**On Two Behaviours of Two-Dimensional Endomorphisms : Role of the Critical Curves**", International journal of Bifurcation and Chaos, **3**(1), p.187-194.

32- Mira C., Fournier-Prunaret D., Gardini L., Karakami H., Cathala J.C., 1994., "**Basin Bifurcations of Two-Dimensional Noninvertible Maps. Fractalization of Basins**", International journal of Bifurcation and Chaos, **4(**2**)**, p.347-381.

33-Mira C. & Rauzy C., 1995., "**Fractal Agreated of Basin Islands in Two-Dimensional Quadratic Non Invertible Maps**", Int. J. of Bif. and Chaos, **5**(4), p.991-1019.

34-Mira, C., Gardini, L., Barugola, A. and Cathala, J.C. 1996., "**Chaotic Dynamics in Two–Dimensional Noninvertible Maps**", World scientific series on Nonlinear science, series editor: Lean O. Chua.

35-Nien C., 1998., "**The Dynamic of Planar Quardatic Maps with Nonempty Bounded Critical Set**", International journal of Bifurcation and Chaos, p.95-105.

36-Ruette S., 2003., "**Chaos for Continuous Interval Maps**", relationship between the various sorts of chaos, Universite Paris-Sud, http://www.math.u-pusd.fr/ruette/.

APPENDICES

Appendix (A)

Definition (A1):[27, p.84]

Let T be an interval, and suppose that $T:\Im\to\Im$. Then T has sensitive dependence on initial conditions at x. or just sensitive dependence at x if there is an $\varepsilon>0$ such that for each $\delta>0$, there is a y in T and a positive integer n such that

$|x-y| < \delta$ and $|T^{[n]}(x) - T^{[n]}(y)| > \in$

If T has sensitive dependence on initial conditions at each x in $\Im$, we say that T has sensitive dependence on initial conditions on $\Im$, or that f has sensitive dependence on $\Im$, or that f has sensitive dependence.

Definition(A2):[27,p.96]

A subset A of the interval J is dense in J if A intersects every nonempty open subinterval of J.

Definition (A3):[27, p.41]

Let A and B be closed sets in $\Re^2$ and let f: $A\to A$ and g: $B\to B$ be two maps, f and g are said to be topological conjugate if there exists a homeomorphism h: $A\to B$ such that $h\circ f=g\circ h$. The homeomorphism h is called a topological conjagacy. In this case, we write $f\approx_h g$.

Theorm (A1):[27,p.40]

Suppose that J is a closed interval and f:J→J. Then f is transitive if and only if there is x in J whose orbit is dense in J.

Theorm (A2):[27, p.41]

Let $f \approx_h g$ then

1) $h \circ f^n = g^n \circ h$ for n=1, 2,3,…

2) if x is a periodic point for of an period n, then h(x) is a periodic point of g of period n.

3) If f has a dense set of periodic points, so does g.

Theorm (A3):[27, p.42]

Let $f \approx_h g$, if f is transitive then g is transitive, too

Definition (A4):[27, p.54]

Let $\Im$ be a bounded interval, and $T:\Im \to \Im$ continuously differentiable on $\Im$. Fixed x in $\Im$, and let λ(x) be defined by

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \, In|(T^{[n]})`(x)|$$

Provided that the limit exists, λ(x) is called the Lyaunor exponent of T at x. If λ(x) is defined, then the common value of λ(x) is denoted by λ, and is the Lyapunor exponent of T.

Definition (A5):[27, p.55]

A function T is chaotic if it satisfies at least one of the:

i) T has a positive Lyapunov exponent at each point in its domain that is not eventually periodic.

ii) T has sensitive dependence on initial conditions on its domain.

Definition (A5):[27 ,p,77]

A function f on an internal $\Im$ is strongly chaotic if:

i) f is chaotic.

ii) f has a dense set of periodic points.

iii) f is transitive.

The Inverse Function Theorom:[11, p.172]

The $T:\Re^2 \to \Re^2$. suppose $T(0)=0$ and $J(T(0))$ is an invertible matrix. Then there exists a neighborhood U of 0 and a C^∞ map $G:U\to\Re^2$ such that $T\circ G(X)=X$ for all $X\in U$.

Definition (A6)[1]

Let (X,d) be a metric space and let $S \subset X$ and $\varepsilon \geq 0$. Then the neighborhood of S of radius ε is the set:

$N_\varepsilon(S)=\{x \in X : d(x,s) < \varepsilon$ for some $s \in S\}$

Definition(A7):[27,p.159]

Let $T:\Re^2 \to \Re^2$ be a map and let p be a fixed point of T with eigenvalues λ and μ such that $|\lambda|<1$ and $|\mu|>1$, then p is called a saddle point.

Appendix(B)

[1] Al-Sa'idi N., "On the Mulit-Fuzzy Fractal space " ph.D. thesis, Al-Nahrain University, 2002

Approximation of the Equation of the Critical Curve LC_i By Using Least Square Method

Enter your value n, m where n is the nuber of iterations and m is the number of points that belong to LCi-1

Choose suitable points $(x_{i-1,j}, y_{i-1,j}) \in LC_{i-1}$, $j=1,2,\dots,m$

Define the map $T(x,y)=(f(x,y) ; g(x,y))$

Evaluate $T(x_{i-1,j}, y_{i-1,j}) = (x_{i,j}, y_{i,j}) \in LC_i$

```
for i=1:m
   x=xi ; y=yi ;
   g1(i)=eval(g);
   f1(i)=eval(f);
end
```

Using the Least Square Method to Approximate the Equation of the Critical Curve LC_i

```
for j=1:n+1
  for k=1:n+1
  A(j,k)=0; D(j)=0;
    for i=1:m
    A(j,k)=A(j,k)+(f1(i))^(j+k-2);
    D(j)=D(j)+g1(i)*(f1(i))^(j-1);
    end;
  end;
end;
b=((inv(A))*D')'
LCi=(b(3))*power(x,2)+(b(2))*x+(b(1));
plot(x,LCi)
```

الخلاصة

تطبيقات المستوي الغير قابلة للانعكاس درست في الاونه الأخيرة من قبل العديد من الباحثين Cathala[9], Gardini[17], Mira[25] الذين تضمن عملهم بناء بعض الأمثلة وإعطاء استنتاجاتهم عليها مع بعض الصفات.

الغرض الرئيسي لهذه الرساله يمكن تقسيمه الى ثلاثه أهداف:

يتضمن الهدف الأول دراسة الخلفية الرياضية للنظم الدينامية حيث تتضمن الدراسه التعاريف الاساسيه والمفاهيم الاساسيه المتعلقه بهذا الموضوع وكذلك تعريف المنحنيات الحرجه LC_i وبعض الأنواع المختلفة للدوال الغير قابل للانعكاس بمنحنياتهم الحرجة مع ذكربعض الخواص لهذه المنحنيات. الهدف الثاني هو دراسه بعض الخواص لهذا النوع من الدوال الغير قابله للانعكاس بشكل خاص المناطق الماصة والمناطق الغير متغيره وعرض طريقه مقترحه لايجاد هكذا مناطق بواسطه استخدام بعض الطرق التقريبه مثل طريقه المربعات الصغرى لتقريب معادله المنحنيات الحرجهLC_i. الهدف الثالث هو اعطاء بعض الأمثلة التطبيقيه للطريقة المقترحه لإيجاد تقريب للمساحات الماصة والانعكاسية.

في عملنا استخدمنا برنامج Matlab v.7.0 في حل الأمثلة المعطاة .

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم

طريقة التربيعات الصغرى لإيجاد المساحات الماصة لدوال المستوي التربيعية

رسالة
مقدمة الى كلية العلوم في جامعة النهرين وهي جزء من متطلبات نيل درجة
ماجستير علوم في الرياضيات وتطبيقات الحاسب

أعداد
مهند نافع جعفر الزبيدي
(بكالوريوس ٢٠٠٦)

اشراف
د. زينب عبد النبي سلمان

شوال 1430 ايلول 2009