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On Optimality of Stochastic Non-Linear Tracking Control Systems

A Thesis
Submitted to the Department of Mathematics and Computer Application, College of Science, Al-Nahrain University, as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By
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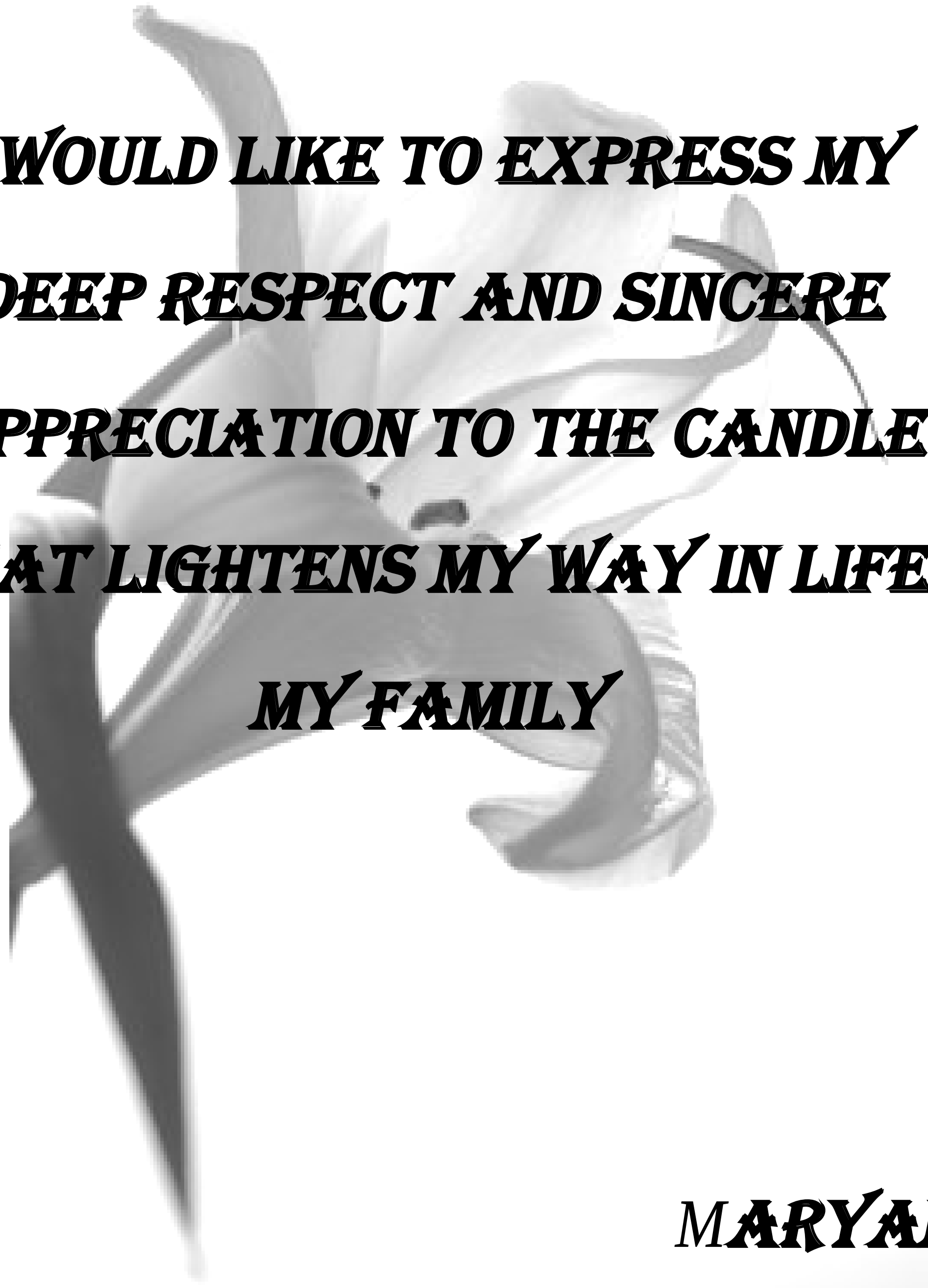
بِٱسْمِ ٱلآبِ وَٱلِابْنِ وَالرُّوحِ ٱلْقُدُسِ ٱلإِلَهِ ٱلوَاحِدِ...آمِين.

(١٣طُوبَى لِلإِنسَانِ ٱلَّذِي يَجِدُ ٱلْحِكْمَةَ وَلِلرَّجُلِ ٱلَّذِي يَنَالُ ٱلْفَهْمَ.١٤
لِأَنَّ تِجَارَتَهَا خَيْرٌ مِنْ تِجَارَةِ ٱلْفِضَّةِ وَرِبْحُهَا خَيْرٌ مِنَ ٱلذَّهَبِ
ٱلْخَالِصِ.١٥هِيَ أَثْمَنُ مِنَ ٱللَّآلِئِ وَ كُلُّ جَوَاهِرِكَ لَا تُسَاوِيهَا.)

(سفر الامثال ٢.١٣-١٥)

Dedication



I would like to express my deep respect and sincere appreciation to the candle that lightens my way in life, my family

Maryam

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October , 2008

SUPERVISORE CERTIFICATION

I certify that this thesis was prepared under my supervision at the department of mathematics and computer applications, ***College of Science, Al-Nahrain University*** as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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Examining Committee's Certification

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Abstract

The tracking problem for differential stochastic equations in the present of stochastic uncertainty of white noise, and control input have been considered.

In this work, our consideration have been focused on the case where both original dynamic state stochastic system and the desired stochastic dynamic system, are driven by white noise stochastic process.

The main aim of this work is to make the behavior of the original dynamic system following the behavior of the desired one for arbitrary controller, using tracking control system approach.

The tracking and stabilizing controller that guarantee the optimum tracking error system between the original system and the desired one have been derived and developed.

The necessary theorems for optimum tracking have been stated and proved supported with some concluding remarks. The controller can also been divided into robust one and optimal one.

The optimum controller can be obtained as a solution of some linear deterministic differential Riccati equation, while the robust one can be obtained so that some controllability properties are ensured.

The Riccati equation associated with linear stochastic optimal controller and tracking one, have also been desired and discussed.

Finally some illustration ranking for time varying system and for law order differential system to larger one, have been illustrated, with details and corresponding Riccati equation for justification of the present work.

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Introduction

At the end of sixties, much attention had been devoted to the study Linear Quadratic Gaussian of the optimal control problem of class of (LQG), (that is the optimal control problems of linear systems with quadratic cost function in the presence of Gaussian noises) either for the mathematical and algorithmic aspects or for the possible applicative capabilities. In witness of this, it is noteworthy that the special issue of the IEEE Transactions on Automatic Control of December 1971, was totally devoted to the above class of problems. Inside this issue, we want to mention in particular the interesting work (Athans 1971) [4], as well as the careful bibliographical classification work (Mendel and Gieseking 1971) [24], see (Kushner 1967, 1971) [18], [19], (Astrom 1971) [2], (Kwakernaak and Sivan 1972) [20].

There has been considerable research in the past two decades on the subject of optimal control for system regulation under various types of uncertainty. The types of uncertainty include among many others additive exogenous disturbances, lack of knowledge about the system model, and time varying dynamics, see (Basar 1995) [5].

More recently, further results have been published concerning optimal control problems (LQG), which constitute extensions of the previous general theory, as far as different particular aspects are concerned. For instance, in after (1990) a stationary regulation problem is studied in the presence of stochastic and deterministic disturbances. In (Lim et al, Moore and Faybusovich 1996) [21] a discrete time regulation problem is considered with linear constraints on the state and control variables.

In (Grimble and Hearns 1998) [15] an (LQG) stationary regulation problem is discussed, taking into account possible delays, and an application to hot strip mills is presented.

The uncertainties in the dynamical system could also be modeled as random noise. The well-known linear-quadratic-Gaussian (LQG) optimal control problem is just one example, where the uncertainty is modeled as exogenous (Gaussian) noise. The case of dynamic uncertainty (with the possibility of non-Gaussian noise) can be formulated as a minimized type optimization problem (Savkin and Petersen 1995) [32], (Uginovskii and Petersen 1999) [35]. More generally, a robust version of the (LQG) technique was discussed in (Petersen and James 1996) [28], (Petersen, James and Dupuis 2000) [29], (Uginovskii and Petersen 1999) [34], (Uginovskii and Petersen 2001) [36], where the concept of an uncertain stochastic system was introduced. A gain, the problem is of the minimized type and it involves construction of a controller which minimizes the worst-case performance, with the uncertainty system satisfying certain stochastic uncertainty constraint. One advantage of such an uncertainty description is that it allows for stochastic uncertainty inputs to depend dynamically on the uncertainty outputs, (Dia pra , Meneghini and Runnggaldier 1996) [13], (Dupuis and Ellis 1997) [14], (Ruunolfsson 1994) [31], (Uginovskii and Petersen 2001) [36].

In tradition linear quadratic regulator (LQR) theory the standard assumption that the control weighting matrix in the cost functional is strictly positive definite; is considered for example, see (Anderson and Moore 1989) [1]. In the deterministic case, this is necessary for there to exist a finite optimal cost that is achievable by a unique optimal control. This assumption means that penalty cost is associated with the control that tries to drive the system state as closely as possible to a desirable position, which is clearly a sensible assumption. Under this assumption, there is a tradeoff between the closeness of the state from the target and the size of the control, and controller has to carefully balance the two in order to achieve an overall minimum cost.

The extension of deterministic (LQR) control to the stochastic case, or the so-called linear-quadratic-Gaussian problem, design and applications, see (Athans 1971) [3], (Bensoussan 1992) [6], (Davis 1971) [12], (Wonham 1968) [37]. In the literature on the stochastic (LQR) problem, however, positive definiteness of the control weight is generally taken for granted. In such a case, there appears to be little difference between the deterministic and the stochastic (LQR) problems. Indeed, the optimal control for both of these problems is given by a linear state feedback, the feedback gain being identical in both cases and determined by the solution of a backward Riccati equation. However, recent results are given by (Chan el al and Zhou 1998) [10].

The optimal tracking problem is studied both for available and for unavailable state and reference, assuming that the last variable is the addition of a known deterministic component and of a random noise component. In both cases the optimal solution is exist, unique, and the optimal control is a suitable affine function of the current state and reference variable (when they are available) or of the corresponding optimal estimates (when they are not available). The minimum cost function is expressed in closed form. (Bruni and Iacoviello 2001) [7].

One of the main contributions of our work is to generalize optimization problem to stochastic nonlinear system in strict feedback form.

In this work the (LQG) tracking stochastic problem is studied, assuming that the state reference is generated by a linear model driven by a Gaussian white noise, with known and possibly non-zero mean value. Of course this formulation includes the particular cases of completely deterministic or completely stochastic (with zero mean) reference variables.

The optimal (LQG) tracking problem is considered either in the case of completely available state and reference or in the case of noisy measurements for the same variables. When state and reference are available, the optimal control is an affine function of the current value of the above variables.

The tracking problem for some class of non-linear equations dependent on both state and control variables is considered. The Riccati equation associated with this problem is also derived. We establish that under stabilizability and tracking conditions this Riccati equation which has a unique positive-definite solution. Using this result we find the optimal control (and the optimal cost) as well as robust control law for our tracking problem.

This thesis consists of three chapters. The first chapter deals with some basic concepts of stochastic dynamic system, the formulation of control problem, mathematical requirements, dynamical control equations, quadratic optimal regulator system, vector stochastic processes and white noise.

The second chapter is concerned with the linear tracking problem and its mathematical requirements. A sufficient theorem for optimum tracking have been stated and proved, and some simulation and conclude remarks, comments, useful mathematical facts.

In chapter three, some illustrations using closed loop controller have been presented and developed, future work, list of references, appendix have also been presented.

Chapter One

Some Basic Concepts of Stochastic Dynamic System

Chapter one presents basic concepts of some stochastic dynamic system which are needed later on. This chapter is divided into six sections, the first section discusses the formulation of control problems, while the second section is about the basic mathematical requirements for dynamic control. The third one concerns the dynamical control system. The design of control system by using linear state feedback control is the matter of section four. The vector stochastic processes is started from section five, and the last one is discussing the linear differential systems driven by white noise and the corresponding stochastic optimal linear quadratic systems.

1.1 THE FORMULATION OF CONTROL PROBLEMS [20]

The following figure represent the general class of control problems (tracking problem). Let a system, (usually called the plant) be given, with the variables as shown in Figure (1.1).

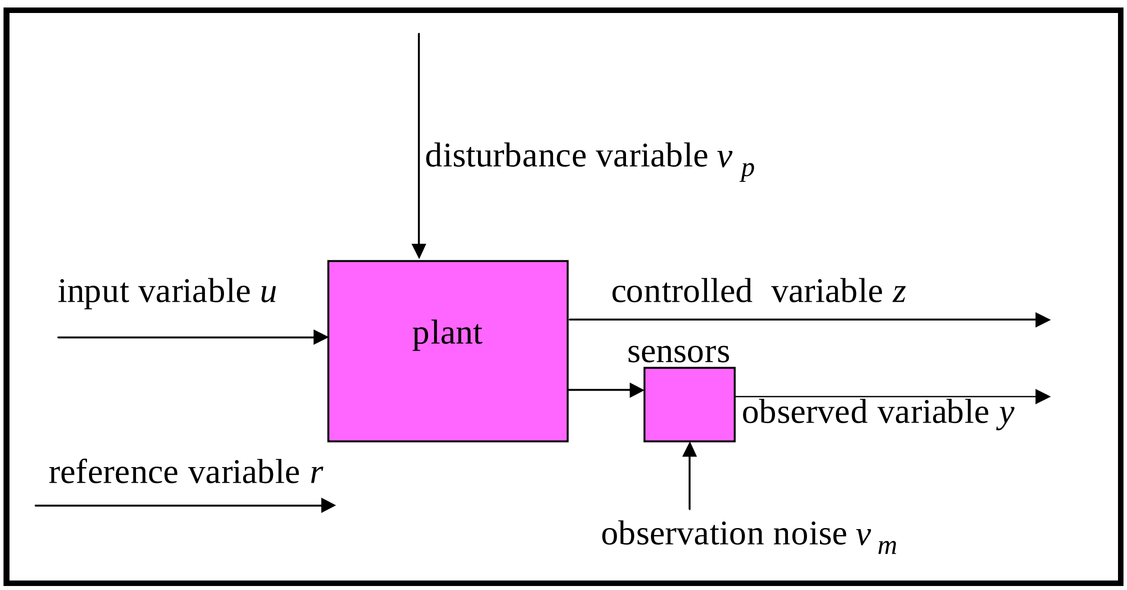

Figure (1.1): The Plant

Where:

1) A plant is a physical object to be controlled.
2) An input variable $u(t)$ (usually called control) which influences the plant and can be manipulated.
3) A disturbance variable $v_p(t)$ which influences the plant cannot be manipulated.

Remark (1.1) [26]

A disturbance is signal that tends to a diversely affect the value of the output of a system. If a disturbance is generated within the system, it is called internal while an external disturbance is generated outside the system and is an input.

4) An observed variable $y(t)$ which is measured by means of sensors is used to obtain information about the state of the plant; this observed variable is usually contaminated with observation noise $v_m(t)$.
5) A controlled variable $z(t)$ which is the variable we wish to control.
6) A reference variable $r(t)$ which represents the prescribed value of the controlled variable $z(t)$.

The tracking problem roughly is the following. For a given reference variables, find appropriate input so that the controlled variable tracks the reference variable, that is,

$$z(t) \cong r(t), \qquad t \geq t_0 \tag{1.1}$$

Where t_0 is the time at which control starts. Typically, the reference variable is not known in advance. A practical constraint is that the range of values over which the input $u(t)$ is allowed to vary is limited. Increasing this range usually involves replacement of the plant by a larger and thus more expensive one. This constraint is of major importance and prevents from obtaining systems that track perfectly. In designing tracking systems so as to satisfy (1.1). The following aspects must be taken into account.

1) The disturbance influences the plant in an unpredictable way.
2) The plant parameters may not be known precisely and may vary.
3) The initial state of the plant may not be known.
4) The observed variable may not directly give an information about the state of the plant and more may be contaminated with observation noise.

The input to the plant is to be generated by a piece of equipment that will be called the controller. We distinguish between two types of controllers: ***open-loop*** and ***closed-loop***. Open-loop controllers generate $u(t)$ on the basis of past and present values of the reference variable only (as shown in Figure (1.2)), that is

$$u(t) = f[r(t)], \qquad t_0 \le t \le t \tag{1.2}$$

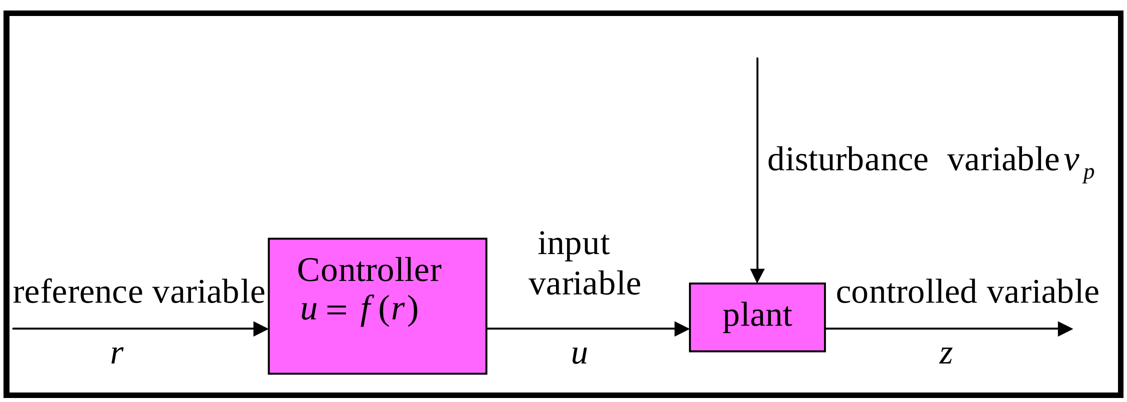

Figure (1.2): An Open-loop Control System

While the closed-loop controllers take advantage of the information about the plant that comes with observed variable; (as shown in Figure (1.3))

$$u(t) = g[r(t); y(t)], \qquad t_0 \le t \le t \tag{1.3}$$

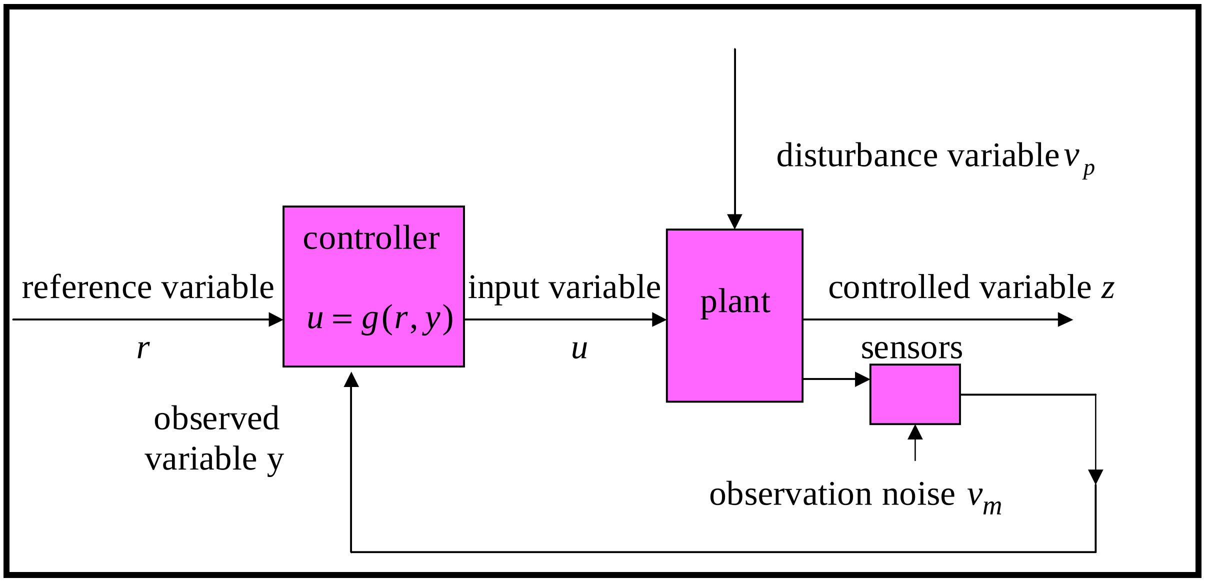

Figure (1.3): A closed –loop Control System

Remark (1.2) [20]

Neither (1.2) nor (1.3) are future values of the reference variable or the observed variable used in generating the input variable since they are unknown. The plant and the controller will be referred to as the control system. An important class of tracking problems consists of those problems where the reference variable is constant over long periods of time. In such cases it is customary to refer to the reference variable as the set point of the system and to speak of regulator problems. Here the main problem usually is to maintain the controlled variable at the set point in spite of disturbance that act upon the system.

1.2 MATHEMATICAL REQUIREMENTS

The following mathematical definitions are needed for complete understanding of the subject.

Definition (1.1) (Asymptotically Stable Matrix) [20]

The $n \times n$ constant matrix A is called asymptotically stable if all its eigenvalues have strictly negative real parts. The eigenvalues of A are the roots of the characteristic polynomial $\det(\lambda I - A)$.

Definition (1.2) (Controllable System) [30]

A system is said to be controllable at time t_0 if it is possible to find an unconstrained control vector to transfer any initial state to the origin in a finite time interval.

State mathematically, the system is controllable at t_0 if for any $x(t_0)$, there exists $u(t)$, $t_0 \le t \le t_1$ that gives $x(t_1) = 0$ $(t_1 > t_0)$.
If this statement is true for all initial time t_0 and initial states $x(t_0)$, then the system is called completely state controllable.

The following section discusses the mathematical description of a non-linear deterministic dynamical control system.

1.3 DYNAMICAL CONTROL EQUATIONS [20]

Many systems can be described by a set of simultaneous differential equations of the form

$$\dot{x}(t) = f[x(t), u(t), t)] \qquad (1.4)$$

where t is the time variable, $x(t)$ is a real n-dimensional time-varying column vector which denotes the ***state*** of the system, and $u(t)$ is a real m-dimensional column vector which indicates the ***input variable*** or ***control variable***. The function f is a real and a vector-valued function.
For many systems the choice of the state follows naturally from the physical structure, and (1.4) which will be called the ***state differential equation***,

usually follows direct from the elementary physical laws that govern the system.

Let $y(t)$ be a real l-dimensional system variable that can be observed or through which the system influences its environment. Such a variable we call an ***output variable*** of the system. It can often be expressed as

$$y(t) = g[x(t), u(t), t] \tag{1.5}$$

Equation (1.5) is called the ***output equation*** of the system.

We call systems that are described by (1.4) and (1.5), the ***finite dimensional differential system*** or for short, a ***differential system***. Equations (1.4) and (1.5) together are called the ***system equations.*** If the vector-valued function g contains u explicitly, we say that the system has a ***direct link*****.**

We are mainly concerned with the case where f and g are linear functions. We then speak of a (***finite-dimensional***) ***linear differential system***. Its state differential equation has the form

$$\dot{x}(t) = \mathrm{A}(t)x(t) + \mathrm{B}(t)u(t) \tag{1.6}$$

where $\mathrm{A}(t)$ and $\mathrm{B}(t)$ are time-varying matrices of appropriate dimensions. We call the dimension n of x the ***dimension*** of the system. The output equation for such a system takes the form

$$y(t) = C(t)x(t) + D(t)u(t) \tag{1.7}$$

If the matrices A, B, C, and D are constants, the system is time-invariant.

The linearization about a critical points or on a nominal solution of a deterministic nonlinear dynamic control system can be found in Appendix [A.1].

Theorem (1.1) [9], [16]

Consider the linear time invariant system of $\dot{x} = Ax + Bu$ where $x \in R^n$, $u \in R^m$, $A \in R^{n\times n}$ and $B \in R^{n\times m}$ are constant matrices.

The necessary and sufficient condition for the complete controllability of the system $\dot{x} = Ax + Bu$ is that $n \times nm$ matrix

$$r(A,B) = \left[B \vdots \quad AB \vdots \quad \cdots \quad \vdots (A)^{n-1}B \right]$$ has rank n.

Example (1.1)

Consider the deterministic system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e \\ h \end{bmatrix} u$$

since

$$r(B,AB) = \begin{bmatrix} e & ae+bh \\ h & ce+dh \end{bmatrix} = \begin{bmatrix} e & f \\ h & g \end{bmatrix}$$

Where $ae + bh = f$ and $ce + dh = g$

Since $\left| \begin{bmatrix} A & B \end{bmatrix} \right| = eg - fh$, if $eg - fh \neq 0$, then the system is completely state controllable.

Theorem (1.2) [9], [17]

Consider the linear state time invariant dynamical control state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with linear state feedback control

$$u(t) = -Kx(t)$$

Then, the closed-loop characteristic values (regulators poles), that is, the characteristic values of $(A - BK)$, can be arbitrarily located in the complex plane (with the restriction that complex characteristic values occur in complex conjugate pairs) by choosing K suitably ***if and only if*** the system above is completely state controllable.

1.4 QUADRATIC OPTIMAL REGULATOR SYSTEM

Due to the important application of optimal control problems of quadratic form and in this study, the following section is then necessary to be discussed.

Theorem (1.3) [25], [16]

Consider the system described by

$$\dot{x}(t) = \mathrm{A}x(t)$$

Where x is a state vector (n-vector) and A is an $n \times n$ constant non-singular matrix. A necessary and sufficient condition that the equilibrium state $x = 0$ be asymptotically stable in the large, is that given any positive definite real symmetric matrix Q, there exist a positive definite real symmetric matrix P such that:

$$\mathrm{A}^T \mathrm{P} + \mathrm{PA} = -Q$$

Remarks (1.3)

1) If the origin of a linear autonomous system $\dot{x} = \mathrm{A}x(t)$ is stable, then there exists a unique Lyapunov function for this system, of the form:

   $$V(x) = x^{\mathrm{T}}\mathrm{P}x \text{ where } \mathrm{A}^{\mathrm{T}}\mathrm{P} + \mathrm{PA} = -Q$$

   And Q is any symmetric positive definite matrix, where P is the positive definite solution of $\mathrm{A}^{\mathrm{T}}\mathrm{P} + \mathrm{PA} = -Q$. [16]

2) Equation $\dot{x}(t) = \mathrm{A}x(t)$ admits a unique real symmetric positive definite solution P for any real, symmetric positive definite matrix Q

if no two eigenvalues of A (distinct) sum to zero, i.e., $l_i + l_j \neq 0$, $(i, j = 1,2,\ldots,n)$.

Note that, if Q is positive definite, then the condition that P is positive is a sufficient condition for the stability of A, in as much as $V(x) = x^{\mathrm{T}}\mathrm{P}x$ is a Lyapunve function of the system. [16]

3) Instead of first specifying a positive-definite matrix P and examining whether or not Q is positive definite, it is convenient to specify a positive-definite matrix Q first and then examine whether or not P determine from

$$\mathrm{A}^T\mathrm{P} + \mathrm{PA} = -Q.$$

is positive definite. [26]

1.4.1 Quadratic Optimal Regulator Problems [27]

Consider the optimal control problem that, given the system equation

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t) \tag{1.8}$$

Let

$$u(t) = -\mathrm{K}x(t) \tag{1.9}$$

Where the cost function is defined as follows:

$$J = \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)dt \tag{1.10}$$

Where Q and R are positive definite.

Then one can show that (see Appendix [B.1]) the feedback control is selected as:

$$u(t) = -\mathrm{K}x(t)$$

$$= -R^{-1}\mathrm{B}_u^T\mathrm{P}x(t) \tag{1.11}$$

Such that the matrix P is the solution of the following differential Riccati equation:

$$-\dot{P}(t) = P(t)A + A^T P(t) + Q - P(t)B_u R^{-1} B_u^T P(t) \qquad (1.12)$$

The following are generalization of a random variable and stochastic dynamic system which are needed later on:

1.5 VECTOR STOCHASTIC PROCESSE [20]

We use stochastic process as mathematical models for disturbances and noise phenomena simultaneously influence a given system.

Definition (1.3) (Stochastic Process) [33]

A collection $\{x(t),\ t \in T\}$ of random variables is called a stochastic process. That is, for each t in the index set T, $x(t)$ is random variable. We often interpret t as time and call $x(t)$ the state of the process at time t.

Remarks (1.4) [33]

1) If the index of T is countable set, we call $\{x(t),\ t \in T\}$ a discrete-time stochastic process.
2) If T is uncountable, we call it a continuous-time process.

Definition (1.4) (Vector Stochastic Process) [20]

The n-scalar stochastic process that $x_1(t), x_2(t), ..., x_n(t)$ which are possibly mutually dependent, $x(t) = col[x_1(t), x_2(t), ..., x_n(t)]$ is called vector stochastic process.

1.5.1 Second-Moment Analysis [8]

The first moments, or the mean of a random process are the expected values of the random variables that form the process. The second moments, or correlations, of the process are the expected values of these random variables squared and the expected values of products of pairs of these random variables. The correlation function is a function composed of the correlations within a random process. The analysis of a random process, via the specification of the mean and correlation function, is known as second-moment analysis.

The mean of the stochastic process $\{x(t),\ t \in T\}$ is the expected values of the vector stochastic process:

$$m_x(t) = E[x(t)] = \int_{-\infty}^{\infty} x f_x(x;t)dx \qquad or$$

$$E\left\{\begin{bmatrix} x_1(t) \\ \vdots \\ x_{n_x}(t) \end{bmatrix}\right\} = \begin{bmatrix} \int_{-\infty}^{\infty} x_1 f_{x_1}(x_1;t)dx_1 \\ \vdots \\ \int_{-\infty}^{\infty} x_{n_x} f_{x_{n_x}}(x_{n_x};t)dx_{n_x} \end{bmatrix} \tag{1.13}$$

Where E $[x(t)]$ is the expected value of $x(t)$. $f_x(x;t)$ is the density function of the vector stochastic process $\{x(t),\ t \in T\}$, the first integral in the equation (1.13) represents an $n-th$ order multiple integral, and $f_{x_i}(x_i;t)$ is the density function for $i-th$ element of this vector stochastic process. The expected value of the product of pair of stochastic variables is called the correlation (or the cross-correlation) of the stochastic variables. The correlation provides information on the relationship among the stochastic variables. The correlation of the stochastic process is defined as:

$$R_x(t_1,t_2) = E[x_1(t_1)x_2^T(t_2)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2^T f_{x_1,x_2}(x_1,x_2;t_1,t_2)dx_1 dx_2 \qquad (1.14)$$

The mean square value of the stochastic process is defined as:

$$\mathrm{E}[x^T(t)x(t)] = \int_{-\infty}^{\infty} x^T x f_x(x;t)dx \qquad (1.15)$$

The mean square value can be given in terms of the correlation function:

$$\mathrm{E}[x^T(t)x(t)] = tr\ [R_x(t,t)] \qquad (1.16)$$

The correlation matrix of the stochastic process is defined:

$$R_x(t,t) \overset{\Delta}{\equiv} \Sigma_x(t) = E[x(t)x^T(t)] \qquad (1.17)$$
$$= \int_{-\infty}^{\infty} xx^T f_x(x;t)dx$$

The covariance function of the stochastic process is defined:

$$\Xi_x(t_1,t_2) = \mathrm{E}\left\{ [x(t_1)-m_x(t_1)]\ [x(t_2)-m_x(t_2)]^T \right\}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{ [x(t_1)-m_x(t_1)][x(t_2)-m_x(t_2)]^T \right\} f_{x_1,x_2}(x_1,x_2;t_1,t_2)dx_1 dx_2$$
$$(1.18)$$

The covariance matrix, of the stochastic process is defined as:

$$\Xi_x(t,t) \overset{\Delta}{\equiv} \bar{\Sigma}_x(t) = \mathrm{E}\left\{ [x(t)-m_x(t)]\ [x(t)-m_x(t)]^T \right\}$$
$$= \int_{-\infty}^{\infty}\left\{ [x(t)-m_x(t)][x(t)-m_x(t)]^T \right\} f_x(x;t)dx \qquad (1.19)$$

Remarks (1.5) [20]

1) Properties of expectation operator:
   a) $E[ax+by]=aE(x)+bE(y)$ for constant a and b.
   b) $E[K]=K$ if K is constant .
2) Covariance matrix and the correlation matrix are equal when the stochastic process has zero mean. And the covariance function is equal to the correlation function when the stochastic process has zero mean.
3) A pair of vectors stochastic process is said to be uncorrelated if $\mathrm{E}[x(t_1)x^T(t_2)]=m_x(t_1)m_x^T(t_2)$.
4) A pair of stochastic process is said to be orthogonal if $\mathrm{E}[x(t_1)x^T(t_2)]=0.$
5) The correlation function $R_x(t_1,t_2)$ is also called (second-order joint moment matrix) of $x(t)$, While the correlation matrix $R_x(t,t)\overset{\Delta}{\equiv}\Sigma_x(t)$ is also called (second-order moment matrix of the process).
6) The covariance matrix $\Xi_x(t,t)\overset{\Delta}{\equiv}\overline{\Sigma}_x(t)$ is sometimes termed as the variance matrix.

Example (1.2) [8]

Consider the stochastic process given by

$$x(t)=\sum_{k=-\infty}^{\infty}u_k\,p_1(t-k)$$

Where $p_1(t)$ is the unit function and

$$p_1(t-k)=\begin{cases}1 & k\le t\le k+1\\ 0 & else\end{cases}$$

Then control samples $\{u_k\}$ can be approximated by independent stochastic variable which is uniformly distributed from $(-0.3,0.3)$ with density function

$$f(u_k)=\begin{cases}\dfrac{1}{0.6} & -0.3<u_k<0.3\\ 0 & else\end{cases}$$

And $m_{u_k}=0$ and $s_{u_k}{}^2 = 0.36$. A sample of the random signal $x(t)$ is

$$m_x(t)=\mathrm{E}\,[x(t)] = \mathrm{E}\,[\sum_{k=-\infty}^{\infty} u_k\, p_1(t-k)] = \sum_{k=-\infty}^{\infty} \mathrm{E}\,[u_k]p_1(t-k)=0$$

The correlation function is obtained by:

$$R_x(t_1,t_2)=\mathrm{E}\,[x(t_1)x(t_2)] = \mathrm{E}\,[\sum_{k=-\infty}^{\infty} u_k\, p_1(t_1-k)\sum_{l=-\infty}^{\infty} x_l\, p_1(t_2-l)]$$

$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \mathrm{E}[u_k x_l]p_1(t_1-k)p_1(t_2-l)$$

$$= \sum_{k=-\infty}^{\infty} 0.03 p_1(t_1-k)p_1(t_2-k)$$

The correlation matrix and the covariance matrix are then found:

$\Sigma_x(t)=\Xi_x(t,t)=R_x(t,t)=\sum_{k=-\infty}^{\infty} 0.03p_1^2(t-k)= 0.03$ (since the mean is zero see remark (1.5), point (2)).

Definition (1.5) (Stationary) [8]

A stochastic process $\{x(t),\ t\in T\}$ is said to be stationary process if for all n, $t_1,t_2,...,t_n$ the vector stochastic process $x(t_1),x(t_2),...,x(t_n)$ and $x(t_1+s),x(t_2+s),...,x(t_n+s)$, have the same joint distribution.

Remark (1.6) [33]

A process is a stationary such that, if choosing any fixed point as the origin, then the ensuring process has the same probability law.

Definition (1.6) (Wide-Sense Stationary) [20]

The stochastic $\{x(t),\ t \in T\}$ is called wide-sense stationary if its correlation matrix $R_x(t,t)$ is finite for all *t*, its mean $m(t)$ is constant, and its covariance function $\Xi_x(t_1,t_2)$ depends on $t_2 - t_1$ only.

Theorem (1.4) [20]

Suppose that $\{x(t),\ t \in T\}$ is a stationary stochastic process. Then its mean $m(t)$ is constant and its covariance function $\Xi_x(t_1,t_2)$ depends on $t_2 - t_1$ only.

Remark (1.7)

By theorem (1.4), we can get any stationary process is also wide-sense stationary.

Example (1.3) [8]

Consider the stochastic process given by

$$x(t) = a\cos(0.5t + q)$$

Where the random variable (a) have normal distribution, with

$$m_a = 0.63,\ s_a^2 = 0.11 \text{ and}$$

q is uniformly distributed (continuous probability distribution) such that, a and q are uncorrelated random variables.

The mean of the stochastic process $x(t)$ is computed as:

$$\mathrm{E}[x(t)] = E\ [a\cos(0.5t+q)]$$

$$= \mathrm{E}[a]\ \mathrm{E}\ [\cos(0.5t)\cos(q) - \sin(0.5t)\sin(q)]$$

$$= \mathrm{E}\ (a)\ \frac{1}{2p}\int_0^{2p} [\cos(0.5t)\cos(q) - \sin(0.5t)\sin(q)]dq$$

$$= \frac{0.63}{2p}\left(\ \cos(0.5t)\int_0^{2p}\cos(q)dq - \sin(0.5t)\int_0^{2p}\sin(q)dq\ \right)$$

$$= \frac{0.63}{2p}\left(\ \cos(0.5t)\sin(q)\Big|_0^{2p} - \sin(0.5t)(-\cos(q))\Big|_0^{2p}\ \right)$$

$$\equiv 0$$

The covariance function (which is equal to correlation function of the stochastic variable since the mean is zero, (see remark (1.5)) is then computed as follows:

$$\Xi_x(t_1,t_2) = \mathrm{E}[x(t_1)x(t_2)] = \mathrm{E}\left(\ [a\cos(0.5t_1+q)]\,[a\cos(0.5t_2+q)]\ \right) \quad (1.20)$$

Since

$$\cos(q_1+q_2) = \cos(q_1)\cos(q_2) - \sin(q_1)\sin(q_2) \quad (1.21)$$

And

$$\cos(q_1-q_2) = \cos(q_1)\cos(q_2) + \sin(q_1)\sin(q_2) \quad (1.22)$$

Now we added the equation (1.21) and equation (1.22) we obtain:

$$\frac{1}{2}\cos(q_1+q_2) + \cos(q_1-q_2) = \cos(q_1)\cos(q_2) \quad (1.23)$$

Let

$$\begin{aligned} q_1 &= 0.5t_2 \\ q_2 &= 0.5t_1 \end{aligned} \quad (1.24)$$

Substitute (1.24) into (1.20), the following have been obtained:

$$E[(x(t_1)x(t_2)] = \frac{1}{2}E[a^2][\cos(0.5t_2 + q + 0.5t_1 + q) + \cos(0.5t_2 + q - 0.5t_1 - q)]$$

$$= \frac{1}{2}E[a^2]\ E[\cos(0.5(t_1 + t_2) + 2q) + \cos(0.5(t_2 - t_1)]$$

And from (1.20), we have:

$$E[x(t_1)x(t_2)] = \frac{m_a^2 + s_a^2}{2p}\left[\int_0^{2p} \cos(0.5(t_1 + t_2)) + 2q\,]dq + \cos(0.5(t_1 - t_2))\right]$$

$$E[x(t_1)x(t_2)] = \frac{(0.63)^2 + (0.11)^2}{2p}\left[\int_0^{2p} [\cos(0.5(t_1 + t_2))\cos(2q) - \sin(0.5(t_1 + t_2))\sin(2q)]dq + \cos(0.5(t_1 - t_2))\right]$$

$$= 0.2\left[\frac{\cos(0.5(t_1 + t_2))}{2p}\int_0^{2p} \cos(2q)dq - \frac{\sin(0.5(t_1 + t_2))}{2p}\int_0^{2p} \sin(2q)dq + \cos(0.5(t_1 - t_2))\right]$$

$$E[x(t_1)x(t_2)] = 0.20\ [\cos(0.5(t_2 - t_1))].$$

So

$$\Xi_x(t_1, t_2) = R_x(t_1, t_2) = 0.20(\cos(0.5(t_2 - t_1)).$$

And the correlation matrix:

$R_x(t,t) = 0.20$ is finite.

On using the above result of the mean and covariance function we conclude that the mean is constant, its covariance function depends only on the time difference $(t_2 - t_1)$, and its correlation matrix $R_x(t,t)$ is finite so this stochastic signal is wide-sense stationary.

Definition (1.7) (Power Spectral Density Matrix) [8]

The power spectral density matrix $S_x(w)$ of a wide-sense stationary vector stochastic process is defined as the Fourier transform, if it exists, of both covariance matrix $\Xi_x(t_1 - t_2)$ of the process, that is ,

$$S_x(w) = \int_{-\infty}^{\infty} e^{-jwt}\, \Xi_x(t)\, dt \quad , \qquad \text{where} \quad j^2 = 1 \tag{1.25}$$

1.5.2 Gaussian Stochastic Process [20]

A Gaussian stochastic process is a stochastic process where for each set of instants of time $t_1, t_2, ..., t_n \geq t_0$, the n-dimensional vector $x(t_1), ..., x(t_n)$ has a Gaussian joint probability distribution. If compound covariance matrix:

$$\Xi = \begin{bmatrix} \Xi_x(t_1,t_1) & \Xi_x(t_1,t_2) & \mathbf{L} & \Xi_x(t_1,t_m) \\ \Xi_x(t_2,t_1) & \Xi_x(t_2,t_2) & \mathbf{L} & \Xi_x(t_2,t_m) \\ \mathbf{L} & \mathbf{L} & \mathbf{L} & \mathbf{L} \\ \Xi_x(t_m,t_1) & \Xi_x(t_m,t_2) & \mathbf{L} & \Xi_x(t_m,t_m) \end{bmatrix}$$

is nonsingular, and the corresponding density function can be written:

$$f(x_1, ..., x_m) = \frac{1}{(2p)^{mn} \det(\Xi)^{\frac{1}{2}}} \exp\{-\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} (x_i - m(t_i))^T \Xi^{-1} (x_j - m(t_j))\}, \tag{1.26}$$

where $m(t)$ is standing for the mean.

Note that this process is completely characterized by its mean and covariance matrix, thus a Gaussian process is stationary ***if and only if*** it is wide-sense stationary.

1.5.3 Process with Uncorrelated Increments [20]

A process $\{x(t),\ t \in T\}$ with uncorrelated increments can have the following important properties which are needing in stochastic dynamic system.

1) For any sequence of instants t_1, t_2, t_3, and with $t_0 \le t_1 \le t_2 \le t_3 \le t_4$, the increments $x(t_2) - x(t_1)$ and $x(t_4) - x(t_3)$ have zero means and are uncorrelated, that is,

$$\left.\begin{aligned} &E\{x(t_2) - x(t_1)\} = E\{x(t_4) - x(t_3)\} = 0 \\ &E\{[x(t_2) - x(t_1)][x(t_4) - x(t_3)]^T\} = 0 \end{aligned}\right\} \tag{1.27}$$

where the initial value is given by

$$x(t_0) = 0 \tag{1.28}$$

The mean of such a process is easily determined:

$$\begin{aligned} m(t) &= E\{x(t)\} \\ &= 0\ , \quad t \ge t_0 \end{aligned} \tag{1.29}$$

2) Suppose for the moment that $t_2 \ge t_1$. Then we have for the covariance function

$$\begin{aligned} \Xi_x(t_1, t_2) &= E\{x(t_1) x^T(t_2)\} \\ &= E\{[x(t_1) - x(t_0)][x(t_2) - x(t_1) + x(t_1) - x(t_0)]^T\} \\ &= E\{[x(t_1) - x(t_0)][x(t_1) - x(t_0)]^T\} \\ \Xi_x(t_1, t_2) &= E\{x(t_1) x^T(t_1)\} \\ &= \overline{\Sigma}_x(t_1) \qquad t_2 \ge t_1 \ge t_0 \end{aligned} \tag{1.30}$$

where

$$\overline{\Sigma}_x(t) = E\{x(t) x^T(t)\} \tag{1.31}$$

is the variance matrix of the process. Similarly,

$$\Xi_x(t_1,t_2) = \bar{\Sigma}_x(t_2) \quad \text{for } t_1 \geq t_2 \geq t_0 \tag{1.32}$$

3) Let us consider the variance matrix of the process. We can write for $t_2 \geq t_1 \geq t_0$:

$$\bar{\Sigma}_x(t_2) = E\{x(t_2)x^T(t_2)\}$$

$$= E\{[x(t_2) - x(t_1) + x(t_1) - x(t_0)][x(t_2) - x(t_1) + x(t_1) - x(t_0)]^T\}$$

$$= E\{[x(t_2) - x(t_1)][x(t_2) - x(t_1)]^T\} + \bar{\Sigma}_x(t_1) \tag{1.33}$$

$\bar{\Sigma}_x(t)$ is monotonically nondecreasing matrix function of t in the sense that

$$\bar{\Sigma}_x(t_2) \geq \bar{\Sigma}_x(t_1) \qquad \text{for all } t_2 \geq t_1 \geq t_0$$

Hence, if A and B are two symmetric real matrices, the notation $\mathrm{A} \geq \mathrm{B}$ implies that the matrix $\mathrm{A} - \mathrm{B}$ is nonnegative-definite. Assuming that the matrix function $\bar{\Sigma}_x(t)$ is absolutely continuous, that is we can write

$$\bar{\Sigma}_x(t) = \int_0^t S(t)dt \tag{1.34}$$

Where $S(t)$ is a nonnegative-definite symmetric matrix function. It then following from (1.33) that the variance matrix of the increment $x(t_2) - x(t_1)$ is given by

$$E\{[x(t_2) - x(t_1)][x(t_2) - x(t_1)]^T\} = \bar{\Sigma}_x(t_2) - \bar{\Sigma}_x(t_1) = \int_0^t S(t)dt$$

Comparing (1.30) and (1.32), we see that if (1.34) holds the covariance matrix of the process can be expressed as:

$$\Xi_x(t_1,t_2) = \int_{t_0}^{\min(t_1,t_2)} S(t)dt \tag{1.35}$$

Remarks (1.8) [20]

1) The process discussed in (1.5.3), is a process with uncorrelated increments where each of the increments $x(t_2)-x(t_1)$ is a Gaussian stochastic vector with zero mean and variance matrix $(t_2-t_1)I$, where I is the unite matrix.
2) Such a process with uncorrelated increments is usually called ***the Brownian motion process.***

1.6 WHITE NOISE [8], [20]

One frequently encounters in practice zero-mean scalar stochastic process x with the property that $x(t_1)$ and $x(t_2)$ are uncorrelated even for values of $|t_2-t_1|$ that are quite small, that is, $R_x(t_1,t_2)\cong 0$ for $|t_2-t_1|>e$, Where e is "small" number. The correlation function of such stochastic process can be idealized as follows $R_x(t_1,t_2)=S_x(t_1)d(t_2-t_1)$, $S_x(t_1)\geq 0$. $d(t_2-t_1)$ is delta function (see Appendix [A.3]) and $S_x(t)$ is referred to as intensity of the process at time *t*. Such process are called ***white noise process.***

Definition (1.8) (White Noise) [20]

The stochastic process $x(t)$ is called white noise with intensity $S_x(t)\geq 0$, if its mean vector-valued stochastic process equal to zero and correlation function $R_x(t_1,t_2)=S_x(t_1)d(t_2-t_1)$.

Remark (1.9) [8]

In the case in which the intensity of the white noise process is constant, the process is wide-sense stationary and its power spectral density matrix formally, taking the Fourier transform of $S_x d(t)$, that wide-sense stationary white noise has the power spectral density matrix $S_x(w)=S_x$.

Theorem (1.5) [20]

Let $\{x(t),\ t \in T\}$ be a vector-valued stochastic process. Then if $W(t)$ is a symmetric matrix, $E\{[x^T(t)\ W(t)\ x(t)]\} = tr[W(t)\ R_x(t,t)]$, where $R_x(t,t)$ is the second-order joint moment matrix (correlation matrix) of the process.

Theorem (1.6) [20]

Let $x(t)$ be a vector-valued white noise process with intensity $S_x(t)$. Also, let $A_1(t), A_2(t)$ and $A(t)$ be given time-varying matrices. Then

a) $E[\int_{t_1}^{t_2} A(t)x(t)dt] = 0$.

b) $E\{[\int_{t_1}^{t_2} A_1(t)x(t)dt][\int_{t_3}^{t_4} A_2(t')x(t')dt']^T\} = \int_{\bar{T}} A_1(t)S_x(t)A_2^T(t)\, dt$.

where $\bar{T}$ is the intersection of $[t_1,t_2]$ and $[t_3,t_4]$ and W is any symmetric matrix.

c) $E\{[\int_{t_1}^{t_2} A_1(t)x(t)dt]^T\ W\ [\int_{t_3}^{t_4} A_2(t')x(t')dt']\} = \int_{\bar{T}} tr[S_x(t)A_1^T(t)W\ A_2(t)]\, dt$.

where $\bar{T}$ is as defined before.

The following remark state the relationship between white noise stochastic process and uncorrelated increment process.

Remark (1.10) [20]

White noise can be expressed as the derivative of process with uncorrelated increments. (see Appendix [B.2] for details).

1.6.1 Linear Differential Systems Driven by White Noise [8], [20]

We present here a linear differential system driven by white noise which is a very convenient model for formulating and solving stochastic linear control problems that involve disturbances and noise. In this work, some of the statistical properties of the state of a linear differential system with a white noise process as input have been discussed.

Theorem (1.7) [20], [22]

Consider the stochastic differential equation

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}x(t)$$
$$x(t_0) = x_0,$$

where A and B are constant and $x(t)$ is white noise with constant intensity S_x. Then if A is asymptotically stable matrix and $t_0 \to -\infty$ or $t \to \infty$, the variance matrix of $x(t)$ tends to the nonnegative-definite constant matrix

$$\Sigma_x(\infty) = \int_0^\infty e^{\mathrm{A}t}\mathrm{B}S_x\mathrm{B}^T e^{\mathrm{A}^T t}dt,$$

which is the unique solution of the matrix equation

$$0 = \mathrm{A}\Sigma_x(\infty) + \Sigma_x(\infty)\mathrm{A}^T + \mathrm{B}S_x\mathrm{B}^T.$$

The matrix equations of this form are sometimes known as ***Lyapunov equations***.

1.6.2 Linear System with White Noise Inputs [8], [23]

Consider a linear system with a random initial state and a stationary white noise inputs $x(t)$:

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}x(t) \tag{1.36 a}$$

$$y(t) = Cx(t). \tag{1.36 b}$$

The white noise input has a known correlation function:

$$R_x(t) = E[x(t)x^T(t+t)] = S_x d(t) \tag{1.37}$$

The mean of the initial condition and the mean of white noise are assumed to be zero.

$$m_x(0) = 0 \tag{1.38}$$

The correlation matrix of the initial condition is assumed to be known:

$$R_x(0) = E\,[x(0)x^T(0)] \tag{1.39}$$

The initial conditions are assumed to be uncorrelated with the input:

$$E[x(0)x^T(t)] = E\,[x(0)]\;E\,[x^T(t)] = 0 \quad \text{, for all } t \ge 0. \tag{1.40}$$

The initial state should be uncorrelated with inputs occurring after the initial time.

The solution of the non-homogeneous state equation (1.36) is given by (see in Appendix [A.2.1]):

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-t)}Bx(t)\,dt \tag{1.41}$$

The mean of the state in (1.36) is found by taking the expected value of the solution of the state equation.

$$m_x(t) = E\,[e^{At}x(0) + \int_0^t e^{A(t-t)}Bx(t)\,dt\,]$$

$$= e^{At}E[x(0)] + \int_0^t e^{A(t-t)}BE[x(t)]\,dt = 0 \tag{1.42}$$

The mean of the output is also zero:

$$m_y(t) = E[Cx(t)] = CE[x(t)] = Cm_x(t) = 0 \tag{1.43}$$

1.6.3 The Output Correlation Function [8]

The state correlation function in (1.36) is found by substituting the solution of the state equation into the definition of the correlation function equation (1.14) we obtain:

$$R_x(t_1,t_2) = E[x(t_1)x^T(t_2)]$$

$$= E\left[\left\{ e^{\mathrm{A}t_1}x(0) + \int_0^{t_1} e^{\mathrm{A}(t_1-t)}\mathrm{B}x(t)dt \right\} \cdot \left\{ e^{\mathrm{A}t_2}x(0) + \int_0^{t_2} e^{\mathrm{A}(t_2-s)}\mathrm{B}x(s)ds \right\}^T \right] \tag{1.44}$$

$$= E\left[\left\{ e^{\mathrm{A}t_1}x(0) + \int_0^{t_1} e^{\mathrm{A}(t_1-t)}\mathrm{B}x(t)dt \right\} \cdot \left\{ e^{\mathrm{A}^T t_2}x^T(0) + \int_0^{t_2} e^{\mathrm{A}^T (t_2-s)}\mathrm{B}^T x^T(s)ds \right\} \right]$$

$$R_x(t_1,t_2) = e^{\mathrm{A}t_1}E[x(0)x^T(0)]e^{\mathrm{A}^T t_2} + \int_0^{t_2} e^{\mathrm{A}t_1}E[x(0)x^T(s)]\mathrm{B}^T e^{\mathrm{A}^T (t_2-s)}ds + \int_0^{t_1} e^{\mathrm{A}(t_1-t)}\mathrm{B}E[x(t)x^T(0)]e^{\mathrm{A}^T t_2}dt + \int_0^{t_1}\int_0^{t_2} e^{\mathrm{A}(t_1-t)}\mathrm{B}E[x(t)x^T(s)]\mathrm{B}^T e^{\mathrm{A}^T (t_2-s)}dsdt$$

$$R_x(t_1,t_2) = e^{\mathrm{A}t_1}\Sigma_x(0)e^{\mathrm{A}^T t_2} + \int_0^{t_1}\int_0^{t_2} e^{\mathrm{A}(t_1-t)}\mathrm{B}S_x d(t-s)\mathrm{B}^T e^{\mathrm{A}^T (t_2-s)}dsdt \tag{1.45}$$

$$R_x(t_1,t_2) = e^{\mathrm{A}t_1}\Sigma_x(0)e^{\mathrm{A}^T t_2} + \int_0^{\min(t_1,t_2)} e^{\mathrm{A}(t_1-s)}\mathrm{B}S_x\mathrm{B}^T e^{\mathrm{A}^T (t_2-s)}ds \tag{1.46}$$

The transition from (1.44) to (1.45), using the properties of expectation operator (see remark (1.5)), equation (1.17), (1.37), and the equation (1.46) is then obtained. (for details, one can see in Appendix [A.3]).

The output correlation function can be simply computed from the state correlation function equation (1.46) as follows:

$$R_s(t_1,t_2) = E[s(t_1)s^T(t_2)] = C\,E[x(t_1)x^T(t_2)]C^T = CR_x(t_1,t_2)C^T \tag{1.47}$$

Remarks (1.11) [8]

1) The state correlation matrix can be obtained from the state correlation function:

$$\Sigma_x(t) = R_x(t,t) = e^{At}\Sigma_x(0)e^{A^Tt} + \int_0^t e^{A(t-s_1)}BS_xB^Te^{A^T(t-s_1)}ds_1$$

Performing the change of variable $s = t - s_1$, we obtain:

$$\Sigma_x(t) = e^{At}\Sigma_x(0)e^{A^Tt} + \int_0^t e^{As}BS_xB^Te^{A^Ts}ds \tag{1.48}$$

2) The state correlation matrix can also be computed as the solution of a differential equation. This differential equation is obtained by differentiating equation (1.48) to get:

$$\dot{\Sigma}_x(t) = Ae^{At}\Sigma_x(0)e^{A^Tt} + e^{At}\Sigma_x(0)A^Te^{A^Tt} + \frac{d}{dt}\left\{\int_0^t e^{A(t-s)}BS_xB^Te^{A^T(t-s)}dt\right. \tag{1.49}$$

By using simple calculations, and from equation (1.48), one gets:

$$\dot{\Sigma}_x(t) = A\left\{ e^{At}\Sigma_x(0)e^{A^Tt} + \int_0^t e^{At}BS_xB^Te^{A^Tt}dt \right\} +$$

$$A^T\left\{ e^{At}\Sigma_x(0)e^{A^Tt} + \int_0^t e^{At}BS_xB^Te^{A^Tt}dt \right\} + BS_xB^T \tag{1.50}$$

From equation (1.48) and (1.50), we have

$$\dot{\Sigma}_x(t) = \mathrm{A}\,\Sigma_x(t) + \mathrm{A}^T\,\Sigma_x(t) + \mathrm{B}S_x\mathrm{B}^T \tag{1.51}$$

3) The steady-state correlation matrix of (1.48) is given by:

$$\Sigma_x(\infty) = \lim_{t\to\infty}\left\{ e^{\mathrm{A}t}\Sigma_x(0)e^{\mathrm{A}^T t} + \int_0^t e^{\mathrm{A}s}\mathrm{B}S_x\mathrm{B}^T e^{\mathrm{A}^T s}ds \right\} \tag{1.52}$$

For a stable system, equation (1.52) approaches

$$\Sigma_x(\infty) = \lim_{t\to\infty}\int_0^t e^{\mathrm{A}s}\mathrm{B}S_x\mathrm{B}^T e^{\mathrm{A}^T s}ds \overset{\Delta}{\equiv} \int_0^\infty e^{\mathrm{A}s}\mathrm{B}S_x\mathrm{B}^T e^{\mathrm{A}^T s}ds \tag{1.53}$$

The steady-state correlation matrix of the state can also be found from the matrix differential equation (1.51) provided that the system is time-invariant and stable. In steady-state, the derivative of the correlation matrix equal to zero then:

$$\mathrm{A}\Sigma_x(\infty) + \Sigma_x(\infty)\mathrm{A}^T + \mathrm{B}S_x\mathrm{B}^T = 0 \tag{1.54}$$

It is quite helpful to realize that this algebraic matrix equation in $\Sigma_x(\infty)$ has a unique solution. Matrix equation in (1.54) is also known ***Lyapunov equation***.

The following illustration for linear stochastic process driven by white noise has been considered:

Example (1.4) [8]

Consider a linear system with a random initial state and a stationary white noise inputs $x(t)$ define by

$$\dot{x}(t) = -10x(t) + 9x(t)$$
$$y(t) = x(t)$$

Where $x(t)$ is the measurement noise is modeled as white noise with correlation functions:

$R_x(t) = 0.01d(t)$, $d(t)$ is delta function .

The initial condition on the state is uniformly distributed between $(-p,p)$ which yields the mean is assumed to be zero and correlation matrix (variance):

$$\Sigma_x(0)=\frac{p^2}{3}=3.29$$

The state correlation function is given by equation (1.46):

$$R_x(t_1,t_2)=3.29e^{-10(t_1+t_2)}+\int_0^{\min(t_1,t_2)}0.81e^{-10(t_1+t_2)}e^{20s}ds$$

$$=3.29e^{-10(t_1+t_2)}+0.04e^{-10(t_1+t_2)}e^{20\min(t_1,t_2)}$$

The output correlation function equals the state correlation function because $C=1$.

The state and the output correlation matrix can be computed using (1.48):

$$\Sigma_x(t)=\Sigma_s(t)=R_x(t,t)=0.04+3.29e^{-20t}$$

The steady-state correlation matrix of the state can be computed using (1.53):

$$\Sigma_x(\infty)=\int_0^{\infty}0.81e^{-20s}ds=\frac{0.81}{20}e^{-20s}\Big|_0^{\infty}=0.04$$

Or the steady-state correlation matrix of the state can also found the equation (1.54):

$$-10\Sigma_x(\infty)-10\Sigma_x(\infty)+0.81=0 \quad \Rightarrow \Sigma_x(\infty)=0.04$$

1.6.4 Quadratic Integral Expressions [20]

Consider the linear stochastic differential system driven by white noise

$$\dot{x}(t)=A(t)x(t)+B(t)x(t) \tag{1.55}$$

where $x(t)$ is $n\times 1$ vector is said white noise stochastic process with intensity $S_x(t)$, $x(t)$ is $n\times 1$ vector $x\in R^n$, $A(t)$ is $n\times n$ matrix, $A(t)\in R^{n\times n}$, $B(t)$ is $n\times m$ matrix, $B(t)\in R^{n\times m}$ and where the initial state $x(t_0)$ is assumed to be a stochastic variable with $E\left\{x(t_0)x^T(t_0)\right\}=\Sigma_{x_0}$. We employ quadratic expression of the form

$$E\{\ x^T(t_1)\mathrm{P}_1x(t_1)+\int_{t_0}^{t_1}x^T(t)R(t)x(t)dt\ \}\tag{1.56}$$

where $R(t)$ is a symmetric nonnegative-definite weighting matrix for all $t_0 \le t \le t_1$ and P_1 is symmetric and nonnegative-definite. For the solution of the linear stochastic differential equation (1.55), (see in Appendix [B.3.1] for details):

$$x(t)=\Phi(t,t_0)x(t_0)+\int_{t_0}^{t}\Phi(t,t)\mathrm{B}(t)x(t)dt\tag{1.57}$$

Now substituting equation (1.57) into (1.56), and using the theorem (1.6), one have that:

$$E\{\ x^T(t_1)\mathrm{P}_1x(t_1)+\int_{t_0}^{t_1}x^T(t)R(t)x(t)dt\ \}=$$

$$tr\Bigg\{\ [\int_{t_0}^{t_1}\Phi^T(t,t_0)R(t)\Phi(t,t_0)dt+\Phi^T(t_1,t_0)\mathrm{P}_1\Phi(t_1,t_0)\ ]\Sigma_{x_0}+$$

$$\int_{t_0}^{t_1}[\ \int_{t_0}^{t}S_x(t)\mathrm{B}^T(t)\Phi^T(t,t)R(t)\Phi(t,t)\mathrm{B}(t)dt\ ]dt+$$

$$\int_{t_0}^{t_1}S_x(t)\mathrm{B}^T(t)\Phi^T(t_1,t)\mathrm{P}_1\Phi(t_1,t)\mathrm{B}(t)dt\ \Bigg\}$$

(1.58)

The second and third part of (1.58) can be obtained on using the fact that $tr(MN)=tr(NM)$ for compatible dimensions (see in Appendix [B.4]),

$$tr\{\int_{t_0}^{t_1}[\int_{t_0}^{t} S_x(t)B^T(t)\Phi^T(t,t)R(t)\Phi(t,t)B(t)dt\;]dt+$$

$$\int_{t_0}^{t_1} S_x(t)B^T(t)\Phi^T(t_1,t)P_1\Phi(t_1,t)B(t)dt\}$$

$$=tr\{\int_{t_0}^{t_1}B(t)S_x(t)B^T(t)[\int_{t}^{t_1}\Phi^T(t,t)R(t)\Phi(t,t)dt+$$

$$\Phi^T(t_1,t)P_1\Phi(t_1,t)]dt\} \tag{1.59}$$

On substitution (1.59) into (1.58), we have that:

$$E\{x^T(t_1)P_1x(t_1)+\int_{t_0}^{t_1}x^T(t)R(t)x(t)dt\}=tr\{P(t_0)\Sigma_{x_0}+\int_{t_0}^{t_1}B(t)S_x(t)B^T(t)P(t)dt\} \tag{1.60}$$

where the symmetric matrix $P(t)$ is defined:

$$P(t)=\Phi^T(t_1,t)P_1\Phi(t_1,t)+\int_{t}^{t_1}\Phi^T(t,t)R(t)\Phi(t,t)dt \tag{1.61}$$

By using theorem (B.1), see in Appendix [B].

One can show that $P(t)$ is the solution of the following differential equation:

$$-\dot{P}(t)=A^T(t)P(t)+P(t)A(t)+R(t) \tag{1.62}$$

Setting $t=t_1$ in (1.61) we obtain:

$$P(t_1)=P_1$$

Theorem (1.8) [20]

Consider the linear stochastic differential system driven by white noise:

$$\dot{x}(t) = \mathrm{A}(t)x(t) + \mathrm{B}(t)x(t)$$

where $x(t)$ is white noise with intensity $S_x(t)$ and $x(t_0) = x_0$ is a stochastic variable with $E\left\{x_0 x_0^T\right\} = \Sigma_{x_0}$. Let $R(t)$ be symmetric and nonnegative-definite for $t_0 \le t \le t_1$ and P_1 constant, symmetric, and nonnegative-definite. Then:

$$E\{\ x^T(t_1)\mathrm{P}_1 x(t_1) + \int_{t_0}^{t_1} x^T(t)R(t)x(t)dt\ \} = tr\left\{\mathrm{P}(t_0)\Sigma_{x_0} + \int_{t_0}^{t_1} \mathrm{B}(t)S_x(t)\mathrm{B}^T(t)\mathrm{P}(t)dt\right\}$$

where $\mathrm{P}(t)$ is symmetric nonnegative-definite matrix

$$\mathrm{P}(t) = \Phi^T(t_1,t)\mathrm{P}_1\Phi(t_1,t) + \int_t^{t_1} \Phi^T(t,t)R(t)\Phi(t,t)dt$$

$\Phi(t,t_0)$ is the transition matrix of the system $\dot{x}(t) = \mathrm{A}(t)x(t)$.

$\mathrm{P}(t)$ satisfies the matrix differential equation

$$-\dot{\mathrm{P}}(t) = \mathrm{A}^T(t)\mathrm{P}(t) + \mathrm{P}(t)\mathrm{A}(t) + R(t)$$

with terminal condition

$$\mathrm{P}(t_1) = \mathrm{P}_1 .$$

Chapter Two

The Non-Linear Stochastic Control Problem

The linear tracking problems is one of the main problem of the control theory. The aim of its asymptotic solution is to find a regulator for a given controlled system which will guarantee the asymptotical zero value of control error for deterministic case and asymptotical zero accepted value of control error with a limited variance in the stochastic case for system initial conditions and all considered tracking signal and disturbance signals.

It is known that (LQG) optimal regulator copies complexity (order of system) of given controlled system. Therefore, the regulator is very complex for a large-scale controlled system although the simpler regulator can be used without substantial deterioration of control quality in many cases.

In this chapter, large scalar Linear Quadratic Gaussian (LQG) have been developed and the tracking (stabilizing) controller that guarantee the optimum tracking between the origin system and the desired one, have been developed. The necessary theorems for optimum tracking have been stated and proved. The robust controller as well as optimum one are designed. The necessary concluding remarks are also presented.

2.1 MATHEMATICAL CONTROL EQUATIONS

The following are some necessary mathematical principals that will be needed later on this our work.

Definition (2.1) [20]

Consider the system described by

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}_u u(t) + x(t)$$

with initial state $x(t_0) = x_0$, $z(t) = D(t)x(t)$.

$x(t)$ is white noise with intensity $S_x(t) > 0$. The initial state x_0 is a stochastic variable, independent of the white noise $x(t)$ with $E\{x_0 x_0{}^T\} = \Sigma_{x_0}$,

Consider the criterion

$$\min_u J = E\{x^T(t_f)\mathrm{H}x(t_f) + \int_0^{t_f} [x^T(t)Qx(t) + u^T(t)Ru(t)]dt\}$$

where Q and R are positive-definite symmetric matrices for $0 \le t \le t_f$ and H is nonnegative-definite symmetric matrix. Then the problem of determining for each *t*, $0 \le t \le t_f$, the input $u(t)$ as a function of all information from the past such that the criterion is minimized is called ***the stochastic linear optimal regulator problem.***

2.1.1 Optimization and Control [8]

A brief summary of the mathematics of optimization is given before addressing the derivation of the linear quadratic regulator. Differentiation is the primary tool for optimization in elementary calculus texts. Therefore, it is reasonable to assume that optimal control can be found by taking the derivative of the cost function with respect to the control input. This approach is reasonable, but requires the differentiation of a real scalar cost function with respect to the control input, which is a function of time. Optimization in this case can be accomplished by using a generalization of differential called the variation.

2.1.2 Variations [8]

The real scalar function of a scalar $J(x)$ has a local minimum at x^* if and only if

$$J(x^*, dx) \overset{\Delta}{\equiv} J(x^* + dx) \geq J(x^*) \qquad (2.1)$$

for all dx sufficiently small; that is, the magnitude of dx is less than some positive number e . An equivalent statement that:

$$\Delta J(x^*, dx) \overset{\Delta}{\equiv} J(x^* + dx) - J(x^*) \geq 0$$

for all dx sufficiently small. The term $\Delta J(x^*, dx)$ is called the increment of $J(x)$. Expanding $J(x^* + dx)$ in a Taylor series around the point x^*, the optimality condition equation (2.1) can be written:

$$\Delta J(x^*, dx) = J(x^* + dx) - J(x^*)$$

$$= \frac{dJ(x^*)}{dx} dx + \frac{d^2 J(x^*)}{dx^2} dx^2 + H.O.T. \geq 0 \qquad (2.2)$$

Note that the term in $\Delta J(x^*, dx)$ is linear in dx which is the differential of $J(x)$, and the coefficient multiplying dx, in this term, is the derivative of $J(x)$. When dealing with a functional, dx is called the variation of x, and the term in the increment dx is called the variation of $J(x)$ and is denoted by $dJ(x^*, dx)$ (the linear part of $J(x^* + dx)$. The variation of $J(x)$ is a generalization of the differential and can be applied to the optimization of a functional. Equation (2.2) can be used to develop necessary conditions for optimality. In the limit as dx approaches zero, the terms dx^2, dx^3, and so on become arbitrary small compared to dx. A necessary condition for x^* to be local minimum is that

The variation of $J(x)$ is zero at x^* for all dx.

2.1.3 Lagrange Multipliers [8]

The optimal control problem is a constrained minimization problem; that is, the minimization of the cost function is subject to constraints on the state and the control. Lagrange multipliers provide a method of converting a constrained minimization problem into an unconstrained minimization problem of higher order. Optimization can then be performed using the above necessary condition. Consider the problem of minimizing $J(x)$, where x is a vector, such that

$$c(x) = 0 \tag{2.3}$$

Equation (2.3) specifies a surface in the space of x. Necessary conditions for optimality of J at a point x^* are that x^* satisfies (2.3) and that the directional derivative of J at x^* equal zero in all directions along the surface. This second condition is satisfied if the gradient of J is normal to the surface at x^*. Note that the gradient of $c(x)$ is normal to the surface at all points, including x^*. Therefore, the second condition is satisfied if the gradient of J is parallel to the gradient of $c(x)$ at x^*, or equivalently,

$$\frac{dJ(x^*)}{dx} + p\frac{dc(x^*)}{dx} = 0 \tag{2.4}$$

for some scalar *p*. equation (2.3) and (2.4) form a set of necessary conditions for a solution of the constrained optimization problem. The necessary condition for optimality, (2.3) and (2.4), can be generated as the solution to an unconstrained optimization problem with the following cost function:

$$J_a(x, p) = J(x) + pc(x) \tag{2.5}$$

Taking the gradient of $J_a(x,p)$ with respect to x obtains (2.4), and taking the derivative of J_a with respect to p obtain (2.3), the parameter p is called Lagrange multiplier. The procedure of solving the constrained optimization problem for J by solving the unconstrained optimization problem for J_a is called the method of ***Lagrange multipliers***. This method is also applicable to the optimal control problem which involves the constrained minimization of a functional.

2.2 THE LINEAR QUADRATIC REGULATOR [8]

The linear quadratic regulator (LQR) is an optimal control problem where the state equation of the plant is linear, the cost function is quadratic, and the test conditions consist of initial conditions on the state and no disturbance inputs. The plant state equation is:

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}_u u(t) \tag{2.6}$$

where $x \in R^n$, $\mathrm{A} \in R^{n\times n}$ constant matrix, $\mathrm{B} \in R^{n\times m}$ constant matrix, $u(t) \in R^m$ the control input.

The reference output is:

$$y(t) = C_y x(t) \tag{2.7}$$

The cost function is:

$$\min J(x(t),u(t)) = \frac{1}{2} y^T(t_f) H_y y(t_f) + \frac{1}{2}\int_0^{t_f} [y^T(t) Q_y y(t) + u^T(t) R u(t)] dt \tag{2.8}$$

Where H_y, Q_y, and R are positive definite. The cost is evaluated subject to the initial condition $x(0) = x_0$, and no disturbance input. The cost function (2.8) is frequently written directly in terms of the state and the control:

$$\min J(x(t),u(t)) = \frac{1}{2}x^T(t_f)Hx(t_f) + \frac{1}{2}\int_0^{t_f}[x^T(t)Qx(t) + u^T(t)Ru(t)]dt \qquad (2.9)$$

Where $H = C_y^T H_y C_y$, $Q = C_y^T Q_y C_y$

Note that H and Q are positive semidefinite. This optimal control problem is a constrained optimization problem, with the cost being a functional of both $u(t)$ and $x(t)$ and the state equation providing a family of constraint equations. It is assumed that there are no other constraints on the state or control.

2.2.1 (Hamiltonian Equations) [8]

The optimal control problem, posed in (2.6) and (2.9), can be converted to an unconstrained optimization problem of higher dimension by application of Lagrange multipliers. An augmented cost function is constructed by adding a constant times each of the constraints to the cost function. The state equation represents a family of constraint equations. These constraints can be appended to the cost function by the addition of an integral:

$$J_a(x(t),u(t),p(t)) = J(x(t),u(t)) + \int_0^{t_f} p^T(t)[\mathrm{A}x(t) + \mathrm{B}_u u(t) - \dot{x}(t)]dt$$

where $p(t)$ is the family of Lagrange multipliers (see remark (2.1)), one at each point in time in the interval from 0 to t_f. The augmented cost is then:

$$J_a(x(t),u(t),p(t)) = \frac{1}{2}x^T(t_f)Hx(t_f) + \int_0^{t_f}\{\frac{1}{2}x^T(t)Qx(t) + \frac{1}{2}u^T(t)Ru(t) +$$

$$p^T(t)[\mathrm{A}x(t) + \mathrm{B}_u u(t) - \dot{x}(t)]\}dt$$

This augmented cost function is a function of the state $x(t)$, the control $u(t)$, and the Lagrange multiplier $p(t)$. The Lagrange multiplier is often referred to as the "costate" in optimal control applications. The optimal control is found by forming the increment of J_a with respect to the state, the control, and the costate:

$$\Delta \hat{J}_a(x,u,p,dx,du,dp) \overset{\Delta}{\equiv} \Delta J_a(x,u,p)$$

$$= J_a(x+dx,u+du,p+dp) - J_a(x,u,p)$$

$$= \frac{1}{2}[x(t_f)+dx(t_f)]^T H[x(t_f)+dx(t_f)] +$$

$$\int_0^{t_f} \{\frac{1}{2}(x+dx)^T Q(x+dx) + \frac{1}{2}(u+du)^T R(u+du) +$$

$$(p+dp)^T[\mathrm{A}(x+dx)+\mathrm{B}_u(u+du)-(\dot{x}+d\dot{x})]\}dt -$$

$$\frac{1}{2}x^T(t_f)Hx(t_f) - \int_0^{t_f}\{\frac{1}{2}x^T(t)Qx(t)+\frac{1}{2}u^T(t)Ru(t)+$$

$$p^T[\mathrm{A}x+\mathrm{B}_u u-\dot{x}]\}dt + \mathrm{H.O.T.} \qquad (2.10)$$

The time index has been deleted, except at the final time. Note that the variation of $\dot{x}(t)$ results from taking the derivative of the variation of $x(t)$. Expanding this expression and grouping terms obtain:

$$\Delta \hat{J}_a = \frac{1}{2}dx^T(t_f)Hdx(t_f) + \int_0^{t_f}\{\frac{1}{2}dx^T Qdx + \frac{1}{2}du^T Rdu +$$

$$dp^T(\mathrm{A}dx+\mathrm{B}_u du - d\dot{x})\}dt +$$

$$x^T(t_f)Hdx(t_f) + \int_0^{t_f}\{x^T Qdx + u^T Rdu + dp^T(\mathrm{A}x+\mathrm{B}_u u-\dot{x})\} +$$

$$p^T[\mathrm{A}dx+\mathrm{B}_u du - d\dot{x}]\}dt + \mathrm{H.O.T.} \qquad (2.11)$$

A necessary condition for the trajectory $x(t)$, $p(t)$, and $u(t)$ to be a minimum is that the variation of J_a equals zero:

$$dJ_a(x,u,p,dx,du,dp) = x^T(t_f)Hdx(t_f) + \int_0^{t_f}\{(x^TQ + p^T\mathrm{A})dx + (u^TR + p^T\mathrm{B}_u)du + dp^T(\mathrm{A}x + \mathrm{B}_u u - \dot{x}) - p^T d\dot{x}\}dt = 0 \qquad (2.12)$$

The last term in equation (2.12) involves $d\dot{x}(t)$, which is a function of $dx(t)$ and therefore not an independent variable. The term $d\dot{x}(t)$ can be eliminated from (2.12) using integration by parts:

$$\int_0^{t_f} p^T(t)d\dot{x}(t)dt = p^T(t_f)dx(t_f) - p^T(0)dx(0) - \int_0^{t_f}\dot{p}^T(t)dx(t)dt \qquad (2.13)$$

The initial condition on the state is fixed, so $dx(0) = 0$. Substituting (2.13) into (2.12), and grouping terms obtains the necessary condition for optimality:

$$dJ_a(x,u,p,dx,du,dp) = [x^T(t_f)H - p^T(t_f)]dx(t_f) + \int_0^{t_f}\{(x^TQ + p^T\mathrm{A} + \dot{p}^T)dx + (u^TR + p^T\mathrm{B}_u)du + dp^T(\mathrm{A}x + \mathrm{B}_u u - \dot{x})dt = 0$$

Since the variation $dx(t_f)$, dx, du, and dp are all arbitrary, the only way this expression can equal zero is:

$$p^T(t_f) = x^T(t_f)H \qquad (2.14)$$

$$\dot{p}^T(t) = -x^T(t)Q - p^T(t)\mathrm{A} \qquad (2.15)$$

$$u^T(t)R + p^T(t)\mathrm{B}_u = 0 \Rightarrow \qquad (2.16a)$$

$$u(t) = -R^{-1}\mathrm{B}_u^T p(t) \qquad (2.16\text{ b})$$

where the inverse is guaranteed to exist since R is positive definite.

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}_u u(t) \qquad (2.17)$$

Eliminating $u(t)$ from (2.15) and (2.17), and combining the resulting equations into a single state equation obtains:

$$\begin{bmatrix} \dot{x}(t) \\ \cdots \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & \vdots & -B_u R^{-1} B_u^T \\ \cdots & & \cdots\cdots\cdots \\ -Q & \vdots & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ \cdots \\ p(t) \end{bmatrix} = H \begin{bmatrix} x(t) \\ \cdots \\ p(t) \end{bmatrix} \tag{2.18}$$

The equation (2.18) is referred to as the Hamiltonian system, and the state matrix H is called the Hamiltonian. The Hamiltonian system (along with the initial and final values) represents a set of necessary conditions for the control to minimize the cost function. These equations are also sufficient; that is, the solution of the Hamiltonian system is the unique control that minimizes the cost function.

The optimal control depends on the costate, which can be found by solving the homogeneous state equation (2.18) subject to the initial and final conditions.

$$x(0) = x_0 \tag{2.19a}$$

$$p(t_f) = Hx(t_f) \tag{2.19 b}$$

The general solution of the state equation (2.18), at the final time, given initial condition at time *t* is:

$$\begin{bmatrix} x(t_f) \\ \cdots \\ p(t_f) \end{bmatrix} = e^{H(t_f - t)} \begin{bmatrix} x(t) \\ \cdots \\ p(t) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t_f - t) & \vdots & \Phi_{12}(t_f - t) \\ \cdots & & \cdots\cdots\cdots \\ \Phi_{21}(t_f - t) & \vdots & \Phi_{22}(t_f - t) \end{bmatrix} \begin{bmatrix} x(t) \\ \cdots \\ p(t) \end{bmatrix} \tag{2.20}$$

Note that each submatrix in the state-transition matrix is computed from the entire Hamiltonian matrix in general. Substituting equation (2.19b) into equation (2.20), we obtain:

$$\begin{bmatrix} x(t_f) \\ \cdots \\ Hx(t_f) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t_f - t) & \vdots & \Phi_{12}(t_f - t) \\ \cdots & & \cdots\cdots\cdots \\ \Phi_{21}(t_f - t) & \vdots & \Phi_{22}(t_f - t) \end{bmatrix} \begin{bmatrix} x(t) \\ \cdots \\ p(t) \end{bmatrix} \tag{2.21}$$

where the unknowns are $x(t)$, $p(t)$, and $x(t_f)$. Eliminating $x(t_f)$ from the equation (2.21) by Substituting the first equation within (2.21) into the second, we have:

$$H\{\Phi_{11}(t_f - t)x(t) + \Phi_{12}(t_f - t)p(t)\} = \Phi_{21}(t_f - t)x(t) + \Phi_{22}(t_f - t)p(t)$$

The costate can then be found from the state:

$$\begin{aligned} p(t) &= \{\Phi_{22}(t_f - t) - H\,\Phi_{12}(t_f - t)\}^{-1}\{H\,\Phi_{11}(t_f - t) - \Phi_{21}(t_f - t)\}x(t) \\ &= \mathrm{P}(t)x(t) \end{aligned} \tag{2.22}$$

where $\mathrm{P}(t)$ is the matrix of proportionality between the costate and the state. This the matrix is fully specified by the state-transition matrix of the Hamiltonian system, since the inverse in (2.22) exists at all times between the initial time and final time [11]. The optimal control is found from equation (2.16):

$$u(t) = -R^{-1}\mathrm{B}_u^T \mathrm{P}(t)x(t) = -K(t)x(t) \tag{2.23}$$

where $K(t)$ is called the optimal feedback gain matrix. The optimal control is linear, time-varying state feedback, in general, where the optimal feedback gain matrix can be found from the state-transition matrix of the Hamiltonian.

2.2.2 (The Optimal Feedback Solution in the Riccati Equation Form) [8]

The determination of the state-transition matrix for the Hamiltonian system is often a very tedious process. An alternative method of finding the optimal feedback gain matrix utilizes a nonlinear matrix differential equation, known as the Riccati equation. The Riccati equation has only final conditions and can, therefore, be solved backward in time using any numerical integration method.

A linear relationship between the costate and the state is given by (2.22):

$$p(t) = \mathrm{P}(t)x(t)$$

The solution of the optimal control problem can be reduced to finding the matrix $\mathrm{P}(t)$, since the optimal control is given by equation (2.23):

$u(t) = -R^{-1}\mathrm{B}_u^T\mathrm{P}(t)x(t) = -K(t)x(t)$, and $x(t)$ is assumed to be measured perfectly.

A differential equation for $\mathrm{P}(t)$ can be generated by taking the derivative of (2.22):

$$\dot{p}(t) = \dot{\mathrm{P}}(t)x(t) + \mathrm{P}(t)\dot{x}(t)$$

Substituting for $\dot{p}(t)$ and $\dot{x}(t)$ from (2.18) we obtain:

$$-Qx(t) - \mathrm{A}^T p(t) = \dot{\mathrm{P}}(t)x(t) + \mathrm{P}(t)\{\mathrm{A}x(t) - B_u R^{-1} B_u^T p(t)\}$$

Substituting for $p(t)$ using (2.22) and rearranging, we have:

$$\{\dot{\mathrm{P}}(t) + \mathrm{P}(t)\mathrm{A} + \mathrm{A}^T\mathrm{P}(t) + Q - \mathrm{P}(t)B_u R^{-1} B_u^T \mathrm{P}(t)\}x(t) = 0$$

This equation is valid for an arbitrary state $x(t)$ (resulting from an arbitrary initial condition x_0), which implies that $\mathrm{P}(t)$ must satisfy

$$-\dot{\mathrm{P}}(t) = \mathrm{P}(t)\mathrm{A} + \mathrm{A}^T\mathrm{P}(t) + Q - \mathrm{P}(t)\mathrm{B}_u R^{-1}\mathrm{B}_u^T\mathrm{P}(t) \qquad (2.24)$$

This differential equation is known as ***the Riccati equation***. The value of $\mathrm{P}(t)$ corresponding to the optimal trajectory is found by solving (2.24) backward in time from the final condition, which is given by (2.19 b):

$$\mathrm{P}(t_f) = H \qquad (2.25)$$

Where H is symmetric.

The optimal control is then found by using (2.23). the solution of the Riccati equation yields a unique optimal control that minimizes the cost function.

Remarks (2.1) [8]

1) The Riccati equation was generated by solving for the matrix of proportionality between the state and costate in the Hamiltonian equations. It turns out that the Riccati solution has significance other than as tool for computing the optimal feedback gain matrix.

   Consider the plant state equation from (2.6):

$$\mathrm{A}x(t) + \mathrm{B}_u u(t) - \dot{x}(t) = 0$$

The optimal cost is, therefore, unaffected by the addition of any product of terms, which includes the state constraint equation. Note that the optimal cost depends only on the initial state, since the control and state trajectories are specified as those obtaining the optimal cost.

$$J(x(0)) = \frac{1}{2}x^T(t_f)H\,x(t_f) + \int_0^{t_f}\{\frac{1}{2}x^T Q\,x + \frac{1}{2}u^T R\,u + \frac{1}{2}p^T[\mathrm{A}x + \mathrm{B}_u u - \dot{x}]\}dt \tag{2.26}$$

From (2.15) and (2.16 a):

$$p^T(t)\mathrm{A} = -\dot{p}^T(t) - x^T(t)Q$$

$$p^T(t)\mathrm{B}_u = -u^T(t)R$$

Substituting these equations into (2.26) we obtain:

$$J(x(0)) = \frac{1}{2}x^T(t_f)Hx(t_f) - \frac{1}{2}\int_0^{t_f}\{\dot{p}^T x + p^T\dot{x}\}dt$$

$$= \frac{1}{2}x^T(t_f)Hx(t_f) - \frac{1}{2}\int_0^{t_f}\frac{d(p^T x)}{dt}dt$$

$$= \frac{1}{2}x^T(t_f)Hx(t_f) - \frac{1}{2}p^T(t_f)x(t_f) + \frac{1}{2}p^T(0)x(0)$$

From equation (2.22) and (2.25), we have:

$$J(x(0)) = \frac{1}{2}x^T(t_f)Hx(t_f) - \frac{1}{2}x^T(t_f)P^T(t_f)x(t_f) + \frac{1}{2}x^T(0)P^T(0)x(0)$$

$$J(x(0)) = \frac{1}{2}x^T(0)P(0)x(0) \qquad (2.26)$$

2) The Riccati solution can be used to generate the cost associated with the optimal control.
3) The result (2.26) is derived by noting that for the optimal trajectory, the state equation is satisfied at each point in time.

2.3 SOME OPTIMAL TRACKING PROBLEMS

In the following section, some optimal tracking problems have been considered. Some theorems have been developed for nonlinear stochastic dynamic systems.

2.3.1 problem formulation1

Consider the non-linear dynamical control system:

$$\dot{x}(t) = Ax(t) + B_u u(t) \qquad (2.27a)$$

Where $x \in R^n$. $x(0)$ is a random variable, $A \in R^{n\times n}$ constant matrix, $B \in R^{n\times m}$ constant matrix, $u(t) \in R^m$ the control input.

The controlled variable is defined by:

$$z(t) = D(t)x(t)\ ,\ z(t) \in R^p\ ,\ D(t) \in R^{p\times n}\ . \qquad (2.27b)$$

Let the target dynamic space:

$$\dot{x}_r(t) = A_r x_r(t) + B_{x_r} x_r(t) \qquad (2.27c)$$

Where $x_r(t) \in R^n$, $x_r(0)$ is random variable, $A_r \in R^{n\times n}$ constant matrix, $B_{x_r} \in R^{n\times r}, x_r(t) \in R^r$ (white noise stochastic process with intensity $S_{x_r}(t) > 0$).

The output of target dynamic system:

$$z_r(t) = D_r(t)x_r(t),\ z_r(t) \in R^p,\ D_r(t) \in R^{p\times n} \qquad (2.27d)$$

The following performance index is defined such that:

$$\min_u J = \frac{1}{2}E\left\{ x^T(t_f)Hx(t_f) + \int_0^{t_f} x^T(t)Qx(t) + u^T(t)Ru(t)dt \right\} \qquad (2.28)$$

The aim is to find an optimal control $u(t)$ so that the controlled variable can follow as closely as possible the output of the tracking model.

<u>Remark (2.2)</u>

To obtain an optimal tracking system, we consider:

$$\min_u J = \frac{1}{2}E\left\{ x^T(t_f)Hx(t_f) + \int_0^{t_f}\left\{ [z(t)-z_r(t)]^T Q[z(t)-z_r(t)] + u^T(t)Ru(t) \right\}dt \right\}$$

(2.29)

One can define the following:

$\tilde{z}(t) = z(t) - z_r(t)$

The (2.29), can be expressed in terms of the augmented state $x(t)$ as follows:

$$\min_u J = \frac{1}{2}E\left\{ x^T(t_f)Hx(t_f) + \int_0^{t_f} \tilde{z}^T(t)Q^*\tilde{z}(t) + u^T(t)Ru(t)dt \right\}$$

Where $\tilde{z}(t) = [D(t), -D_r(t)]\tilde{x}(t)$ (2.30)

The following theorem studies and proves the necessary and sufficient condition for the existence of such controller that present of in problem formulation 1.

Theorem (2.1)

Consider the non-linear stochastic dynamical optimal control defined in problem formulation 1:

Assuming that:

I. The initial conditions are zero-mean random vectors with the following correlation matrices (The second-order moment matrix):

$$E[x(0)x^T(0)] = \Sigma_x(0) \text{, and } E\{x(0)\} = \mathbf{0}$$

$$E[x_r(0)x_r^T(0)] = \Sigma_{x_r}(0) \text{, and } E\{x_r(0)\} = \mathbf{0}$$

The pair $(\mathrm{A}, \mathrm{B}_u)$ of a non-linear dynamical control system is controllable [see theorem (1.1)].

II. A_r is stable matrix.

III. $E[x(0)x_r^T(0)] = \mathbf{0}$

IV. $E[x_r(0)x^T(0)] = \mathbf{0}$

V. $E[x(0)x_r^T(t)] = \mathbf{0}$

VI. $E[x_r(0)x_r^T(t)] = \mathbf{0}$

VII. $E[x_r(t)x^T(0)] = \mathbf{0}$

VIII. $E[x_r(t)x_r^T(0)] = \mathbf{0}$

IX. $R, Q,$ and H are positive-definite, symmetric, matrices.

Then:

a. The optimal control law is obtained as:

$$u(t) = -K\tilde{x}(t)$$

Furthermore, it can be found by partitioning the Riccati equation $P_{11}(t)$, $P_{12}(t)$, and $P_{22}(t)$ are the solution of the matrix differential equations if possible:

$$-\dot{P}_{11}(t) = P_{11}(t)A + A^T P_{11}(t) + D^T(t)Q^* D(t) - P_{11}(t)B_u R^{-1} B_u^T P_{11}(t);$$

$$P_{11}(t_f) = H_{11}$$

$$-\dot{P}_{12}(t) = P_{12}(t)A_r + A^T P_{12}(t) - D^T(t)Q^* D_r(t) - P_{11}(t)B_u R^{-1} B_u^T P_{12}(t)$$

$$P_{12}(t_f) = H_{12}$$

$$-\dot{P}_{22}(t) = P_{22}(t)A_r + A_r^T P_{22}(t) + D_r^T(t)Q^* D_r(t) - P_{12}^T(t)B_u R^{-1} B_u^T P_{12}(t);$$

$$P_{22}(t_f) = H_{22}$$

And

$$K = [K_1, -K_2] = [R^{-1} B_u^T P_{11}(t), -R^{-1} B_u^T P_{12}(t)]$$

b. The optimal cost value is obtained to be:

$$J_{SR} = \frac{1}{2} tr\left\{ P(0)\Sigma_{\tilde{x}}(0) + \int_0^{t_f} P(t)\tilde{B}_x \tilde{S}_x \tilde{B}_x^T dt \right\}$$

Where

$$P(0) = \begin{bmatrix} P_{11}(0) & P_{12}(0) \\ P_{12}^T(0) & P_{22}(0) \end{bmatrix},\ \Sigma_{\tilde{x}}(0) = \begin{bmatrix} E[x(0)x^T(0)] & E[x(0)x_r^T(0)] \\ E[x_r(0)x^T(0)] & E[x_r(0)x_r^T(0)] \end{bmatrix},$$

$$\tilde{B}_x \tilde{B}_x^T = \begin{bmatrix} 0 \\ B_{x_r} \end{bmatrix} \begin{bmatrix} 0 & B_{x_r}^T \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & B_{x_r} B_{x_r}^T \end{bmatrix},\ P(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}^T(t) & P_{22}(t) \end{bmatrix}$$

$$\tilde{S}_x = E\left\{ \begin{bmatrix} 0 \\ x_r(t) \end{bmatrix} \begin{bmatrix} 0 & x_r^T(s) \end{bmatrix} \right\} = \begin{bmatrix} 0 & 0 \\ 0 & E[x_r(t)x_r^T(s)] \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & S_{x_r} d(t-s) \end{bmatrix}$$

c. The correlation function, the correlation matrix as well as the mean, vector of $x(t)$, are found to be:

$$R_{xx}(t_1,t_2) = \Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{11}^T(t_2;K) + \Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{12}^T(t_2;K) + \int_0^{\min(t_1,t_2)} \Phi_{12}(t_1,s;K)B_{x_r}S_{x_r}B_{x_r}^T\Phi_{12}^T(t_2,s;K)ds$$

$$R_{xx_r}(t_1,t_2) = \Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K) + \Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K) + \int_0^{\min(t_1,t_2)} \Phi_{12}(t_1,s;K)B_{x_r}S_{x_r}B_{x_r}^T\Phi_{22}^T(t_2,s;K)ds$$

$$R_{x_rx_r}(t_1,t_2) = \Phi_{21}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K) + \Phi_{22}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K) + \int_0^{\min(t_1,t_2)} \Phi_{22}(t_1,s;K)B_{x_r}S_{x_r}B_{x_r}^T\Phi_{22}^T(t_2,s;K)ds$$

$$\Sigma_{xx}(t) = \Phi_{11}(t;K)\Sigma_x(0)\Phi_{11}^T(t;K) + \Phi_{12}(t;K)\Sigma_{x_r}(0)\Phi_{12}^T(t;K) + \int_0^t \Phi_{12}(t,s;K)B_{x_r}S_{x_r}B_{x_r}^T\Phi_{12}^T(t,s;K)ds$$

$$\Sigma_{xx_r} = \Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K) + \Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K) + \int_0^t \Phi_{12}(t_1,s;K)B_{x_r}S_{x_r}B_{x_r}^T\Phi_{22}^T(t_2,s;K)ds$$

$$\Sigma_{x_rx_r}(t) = \Phi_{21}(t;K)\Sigma_x(0)\Phi_{21}^T(t;K) + \Phi_{22}(t;K)\Sigma_{x_r}(0)\Phi_{22}^T(t;K) + \int_0^t \Phi_{22}(t,s;K)B_{x_r}S_{x_r}B_{x_r}^T\Phi_{22}^T(t,s;K)ds$$

And

$$\begin{bmatrix} E[x(t)] \\ \\ E[x_r(t)] \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t;K)E[x(0)] + \Phi_{12}(t;K)E[x_r(0)] + \int_0^t \Phi_{12}(t,t;K)\mathrm{B}_{x_r} E[x_r(t)]dt \\ \\ \Phi_{21}(t;K)E[x(0)] + \Phi_{22}(t;K)E[x_r(0)] + \int_0^t \Phi_{22}(t,t;K)\mathrm{B}_{x_r} E[x_r(t)]dt \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}$$

Where $e^{(\tilde{A}-\tilde{B}K)t} = \Phi(t;K) = \begin{bmatrix} \Phi_{11}(t;K) & \Phi_{12}(t;K) \\ \Phi_{21}(t;K) & \Phi_{22}(t;K) \end{bmatrix}$

Proof

Define the augmented state between state space $x(t)$ and target state space $x_r(t)$:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_r(t) \end{bmatrix} = \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{A}_r \end{bmatrix} \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} + \begin{bmatrix} \mathrm{B}_u \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \mathrm{B}_{x_r}(t)x_r(t) \end{bmatrix} \qquad (2.31)$$

Let $\tilde{x}(t) = col[x(t), x_r(t)]$, $\tilde{x}(t) \in R^{2n\times 1}$

similarly $\tilde{z}(t) = col[z(t), z_r(t)]$, $\tilde{z}(t) \in R^{2P\times 1}$ And from (2.27b), (2.27d), and (2.30), we have:

$$\tilde{z}(t) = \left[D(t), -D_r(t)\right]\tilde{x}(t)$$

Let

$$u(t) = -K\tilde{x}(t) \qquad (2.32)$$

From (2.31), we have:

$$\dot{\tilde{x}}(t) = (\tilde{A} - \tilde{B}_u K)\tilde{x}(t) + \tilde{B}_x \tilde{x}(t) \qquad (2.33)$$

Let $\Phi(t;K) = (\tilde{A} - \tilde{B}_u K)$, where $\Phi(t;K)$ is $n \times n$ nonsingular (transition or fundamental matrix satisfying the following matrix differential equation

$$\dot{\Phi}(t;K) = (\tilde{A} - \tilde{B}_u K)\ \Phi(t;K), \text{ and } \Phi(0,0) = I.$$

To solve the system $\dot{\tilde{x}}(t) = (\tilde{A} - \tilde{B}_u K)\tilde{x}(t) + \tilde{B}_x \tilde{x}(t)$ (see Appendix [B.3.1]), we obtain:

$$\tilde{x}(t) = \Phi(t;K)\tilde{x}(0) + \int_0^t \Phi(t,t;K)\tilde{B}_x \tilde{x}(t)dt \qquad (2.34)$$

Set $z(t) \cong z_r(t)$, in optimal manner, and define:

$\tilde{z}(t) = z(t) - z_r(t)$,

And substituting (2.27b), (2.27d), into (2.29), one gets:

$$\min J(\mathrm{K}) = \frac{1}{2} E \left\{ \tilde{x}^T(t_f) \begin{bmatrix} \mathrm{H}_{11} & \vdots & \mathrm{H}_{12} \\ \cdots & \cdots & \cdots \\ \mathrm{H}_{21} & \vdots & \mathrm{H}_{22} \end{bmatrix} \tilde{x}(t_f) + \right.$$

$$\int_0^{t_f} \tilde{x}^T(t) \begin{bmatrix} D^T(t) \\ -D_r^T(t) \end{bmatrix} \begin{bmatrix} Q^* & \vdots & Q^* \\ \cdots & \cdots & \cdots \\ Q^* & \vdots & Q^* \end{bmatrix} \cdot$$

$$\left[D(t), \quad -D_r(t) \right] \tilde{x}(t) + \tilde{u}^T(t) R \tilde{u}(t) dt$$

Where $\mathrm{H}_{ij}, Q^* \in R^{p \times p}$

From (2.32), one get:

$$\min J(\mathrm{K}) = \frac{1}{2} E \left\{ \tilde{x}^T(t_f) \begin{bmatrix} \mathrm{H}_{11} & \vdots & \mathrm{H}_{12} \\ \cdots & \cdots & \cdots \\ \mathrm{H}_{21} & \vdots & \mathrm{H}_{22} \end{bmatrix} \tilde{x}(t_f) + \right.$$

$$\int_0^{t_f} \tilde{x}^T(t) \left\{ \begin{bmatrix} D^T(t) \\ -D_r^T(t) \end{bmatrix} \begin{bmatrix} Q^* & \vdots & Q^* \\ \cdots & \cdots & \cdots \\ Q^* & \vdots & Q^* \end{bmatrix} \left[D(t), \quad -D_r(t) \right] + \right.$$

$$\left. \begin{bmatrix} K^T RK & \vdots & K^T RK \\ \cdots & \cdots & \cdots \\ K^T RK & \vdots & K^T RK \end{bmatrix} \right\} \tilde{x}(t)$$

(2.35)

Let us define a suitable $2n \times 2n$ matrices as follows:

$$\mathrm{H} \overset{\Delta}{\equiv} \begin{bmatrix} \mathrm{H}_{11} & \vdots & \mathrm{H}_{12} \\ \cdots & \cdots & \cdots \\ \mathrm{H}_{21} & \vdots & \mathrm{H}_{22} \end{bmatrix} \tag{2.36}$$

$$Q(t) = \begin{bmatrix} D^T(t) \\ -D_r^T(t) \end{bmatrix} \begin{bmatrix} Q^* & \vdots & Q^* \\ \cdots & \cdots & \cdots \\ Q^* & \vdots & Q^* \end{bmatrix} \begin{bmatrix} D(t), & -D_r(t) \end{bmatrix}$$

$$Q(t) \overset{\Delta}{\equiv} \begin{bmatrix} D^T(t)Q^*D(t) & \vdots & -D^T(t)Q^*D_r(t) \\ \cdots & \cdots & \cdots \\ -D_r^T(t)Q^*D(t) & \vdots & D_r^T(t)Q^*D_r(t) \end{bmatrix} \tag{2.37}$$

And from (2.35), (2.37), we let:

$$Q_R(t;K) = \begin{bmatrix} D^T(t)Q^*D(t) & \vdots & -D^T(t)Q^*D_r(t) \\ \cdots & \cdots & \cdots \\ -D_r^T(t)Q^*D(t) & \vdots & D_r^T(t)Q^*D_r(t) \end{bmatrix} + \begin{bmatrix} K^T RK & \vdots & K^T RK \\ \cdots & \cdots & \cdots \\ K^T RK & \vdots & K^T RK \end{bmatrix} \tag{2.38}$$

On substituting equations (2.36), (2.37) and (2.38) into equation (2.35) we obtain:

$$\min J_{SR}(K) = \frac{1}{2} E\left\{ \tilde{x}^T(t_f)\mathrm{H}\tilde{x}(t_f) + \int_0^{t_f} \tilde{x}^T(t)Q_R(t;K)\tilde{x}(t)dt \right\} \tag{2.39}$$

Using the result of theorem (1.5) in the section (1.6), then (2.39) becomes:

$$\min J_{SR}(\mathrm{K}) = \frac{1}{2} tr\mathrm{H}E[\tilde{x}(t_f)\tilde{x}^T(t_f)] + \frac{1}{2}\int_0^{t_f} trQ_R(t;K)E[\tilde{x}(t)\tilde{x}^T(t)]dt \tag{2.40}$$

Where:

$$E[\tilde{x}(t_f)\tilde{x}^T(t_f)] = \begin{bmatrix} E[x(t_f)x^T(t_f)] & E[x(t_f)x_r^T(t_f)] \\ E[x_r(t_f)x^T(t_f)] & E[x_r(t_f)x_r^T(t_f)] \end{bmatrix} \overset{\Delta}{\equiv} \Sigma_{\tilde{x}}(t_f) \qquad (2.41)$$

And similarly

$$E[\tilde{x}(t)\tilde{x}^T(t)] = \begin{bmatrix} E[x(t)x^T(t)] & E[x(t)x_r^T(t)] \\ E[x_r(t)x^T(t)] & E[x_r(t)x_r^T(t)] \end{bmatrix} \overset{\Delta}{\equiv} \Sigma_{\tilde{x}}(t) \qquad (2.42)$$

By assumptions V , and VI:

$E[x(t)x_r^T(t)] = 0$, similarly $E[x(t_f)x_r^T(t_f)] = 0$

$E[x_r(t)x^T(t)] = 0$, similarly $E[x_r(t_f)x^T(t_f)] = 0$

And from (2.39), we have:

$$\min J_{SR}(K) = \frac{1}{2} tr \left\{ H\Sigma_{\tilde{x}}(t_f;K) + \int_0^{t_f} trQ_R(t;K)\Sigma_{\tilde{x}}(t;K)dt \right\} \qquad (2.43)$$

Expanding that state correlation matrix, using the methods in subsection (1.6.3) of remarks (1.11) in chapter one, we have, for arbitrary K , that:

$$\Sigma_{\tilde{x}}(t,K) = \Phi(t;K)\Sigma_{\tilde{x}}(0)\Phi^T(t;K) + \int_0^t \Phi(t,t;K)\tilde{B}_x\tilde{S}_x\tilde{B}_x^T\Phi^T(t,t;K)dt \qquad (2.44)$$

On using (2.44) into (2.43), one can get:

$$\min J_{SR}(K) = \frac{1}{2} tr \Bigg\{ \; H\Phi(t_f;K)\Sigma_{\tilde{x}}(0) + \Phi^T(t_f;K) +$$

$$\int_0^{t_f} H\Phi(t_f,t;K)\tilde{B}_x \tilde{S}_x \tilde{B}_x^T \Phi^T(t_f,t;K)dt \; \Bigg\} +$$

$$\frac{1}{2} tr \Bigg\{ \int_0^{t_f} Q_R(t;K)\Phi(t;K)\Sigma_{\tilde{x}}(0)\Phi^T(t;K)dt +$$

$$\int_0^{t_f}\int_0^{t} Q_R(t;K)\Phi(t,t;K)\tilde{B}_x \tilde{S}_x \tilde{B}_x^T \Phi^T(t,t;K)dtdt \; \Bigg\}$$

(2.45)

Rearranging terms and interchanging the order of integration in the fourth term in (2.45), we have:

$$\min J_{SR}(K) = \frac{1}{2} tr \Bigg\{ \; H\Phi(t_f;K)\Sigma_{\tilde{x}}(0) + \Phi^T(t_f;K) +$$

$$\int_0^{t_f} Q_R(t;K)\Phi(t;K)\Sigma_{\tilde{x}}(0)\Phi^T(t;K)dt \; \Bigg\} +$$

$$\int_0^{t_f} \frac{1}{2} tr \Bigg\{ \; H\Phi(t_f,t;K)\tilde{B}_x \tilde{S}_x \tilde{B}_x^T \Phi^T(t_f,t;K) +$$

$$\int_t^{t} Q_R(t;K)\Phi(t,t;K)\tilde{B}_x \tilde{S}_x \tilde{B}_x^T \Phi^T(t,t;K)dt \; \Bigg\} dt$$

(2.46)

The cost function of the linear quadratic regulator with an initial condition at time t_0 can be expanded using the solution of (2.33) and the fact that the trace (see Appendix [B .4]).

$$J_{LQR}(x_0,t_0,K)=\frac{1}{2}tr\Big\{ \mathrm{H}\Phi(t_f,t_0;K)\tilde{x}(t_0)\tilde{x}^T(t_0)\Phi^T(t_f,t_0;K)+$$

$$\int_{t_0}^{t_f} Q_R(t;K)\Phi(t,t_0;K)\tilde{x}(t_0)\tilde{x}^T(t_0)\Phi^T(t,t_0;K)dt \Big\}+$$

$$\int_{t_0}^{t_f}\frac{1}{2}tr\Big\{ \mathrm{H}\Phi(t_f,t;K)\tilde{B}_x\tilde{S}_x\tilde{B}_x^T\Phi(t_f,t;K)+$$

$$\int_{t}^{t} Q_R(t;K)\Phi(t,t;K)\tilde{B}_x\tilde{S}_x\tilde{B}_x^T\Phi^T(t,t;K)dt \Big\}dt$$

(2.47)

The stochastic regulator cost can be formalized by expanding $\Sigma_{\tilde{x}}(0)$ and $\tilde{S}_x$ in equation (2.47), using their singular value decompositions, (see Appendix [B.5]):

$$J_{SR}(K)=\sum_{j=1}^{n_x} s_j(\Sigma_{\tilde{x}}(0))\frac{1}{2}tr\Big\{ \mathrm{H}\Phi(t_f;K)U_j(\Sigma_{\tilde{x}}(0))U_j^T(\Sigma_{\tilde{x}}(0))\Phi^T(t_f;K)+$$

$$\int_{0}^{t_f} Q_R(t;K)\Phi(t;K)U_j(\Sigma_{\tilde{x}}(0))U_j^T(\Sigma_{\tilde{x}}(0))\Phi^T(t;K)dt \Big\}+$$

$$\int_{0}^{t_f}\sum_{K=1}^{n_u} s_j(\tilde{S}_x)\frac{1}{2}tr\Big\{ \mathrm{H}\Phi(t_f,t;K)\tilde{B}_x U_j(\tilde{S}_x)U_j^T(\tilde{S}_x)\tilde{B}_x^T\Phi^T(t_f,t;K)+$$

$$\int_{t}^{t} Q_R(t;K)\Phi(t,t;K)\tilde{B}_x U_j(\tilde{S}_x)U_j^T(\tilde{S}_x)\tilde{B}_x^T\Phi^T(t,t;K)dt \Big\}dt$$

(2.48)

Set:

$$\Sigma_{\tilde{x}}(0)=\sum_{j=1}^{n_x} s_j(\Sigma_{\tilde{x}}(0))U_j(\Sigma_{\tilde{x}}(0))U_j^T(\Sigma_{\tilde{x}}(0)),\text{ and}$$

$$\tilde{S}_x=\sum_{j=1}^{n_u} s_j(\tilde{S}_x)\,U_j(\tilde{S}_x)\,U_j^T(\tilde{S}_x)$$

singular vectors decomposition (see Appendix [B.5]).

Note that the right and left singular vectors are equal for symmetric matrices. The stochastic regulator cost is then a weighted of LQR cost functions, while on using trace operator see Appendix [B.4]:

$$J_{SR}(K)=\frac{1}{2}\left\{ \sum_{j=1}^{n_x} s_j(\Sigma_{\tilde{x}}(0))J_{LQR}(U_j(\Sigma_{\tilde{x}}(0)),0,K)+\int_0^{t_f}\sum_{j=1}^{n_u} s_j(\tilde{S}_x)J_{LQR}(\tilde{B}_xU_j(\tilde{S}_x),t,K)dt \right. \tag{2.49}$$

On using (2.26), to obtain:

$$J_{SR}=\frac{1}{2}\left\{ \sum_{j=1}^{n} s_j(\Sigma_{\tilde{x}}(0))U_j^T(\Sigma_{\tilde{x}}(0))\mathrm{P}(0)U_j(\Sigma_{\tilde{x}}(0))+ \right.$$

$$\int_0^{t_f}\sum_{j=1}^{m} s_j(\tilde{S}_x)U_j^T(\tilde{S}_x)\tilde{B}_x\mathrm{P}(t)\tilde{B}_x^TU_j(\tilde{S}_x)dt$$

Using trace operator :

$$J_{SR}=\frac{1}{2}tr\left\{ \mathrm{P}(0)\Sigma_{\tilde{x}}(0)+\int_0^{t_f}\mathrm{P}(t)\tilde{B}_x\tilde{S}_x\tilde{B}_x^T dt \right\} \tag{2.50}$$

$$=\frac{1}{2}tr\left\{ \begin{bmatrix}\mathrm{P}_{11}(0) & \mathrm{P}_{12}(0)\\ \mathrm{P}_{12}^T(0) & \mathrm{P}_{22}(0)\end{bmatrix}\begin{bmatrix}\Sigma_{xx}(0) & \Sigma_{xx_r}(0)\\ \Sigma_{x_rx}(0) & \Sigma_{x_rx_r}(0)\end{bmatrix}+\right.$$

$$\left.\int_0^{t_f}\begin{bmatrix}\mathrm{P}_{11}(t) & \mathrm{P}_{12}(t)\\ \mathrm{P}_{12}^T(t) & \mathrm{P}_{22}(t)\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & \mathrm{B}_{x_r}S_{x_r}\mathrm{B}_{x_r}^T\end{bmatrix}dt\right\}$$

where all the weights are non-negative. From R is positive-definite and $(\tilde{A},\tilde{B}_u)$ are controllable, the Hamiltonian approach (see subsection (2.2.1)) can be applied to compute $K=(R^{-1}\mathrm{B}_u^T\mathrm{P}_{11}(t),-R^{-1}\mathrm{B}_u^T\mathrm{P}_{12}(t))$ Furthermore, we partition the solution $\mathrm{P}(t)$ of the Riccati equation (2.24), $\mathrm{P}_{11}(t)$, $\mathrm{P}_{12}(t)$, and $\mathrm{P}_{22}(t)$ according to partitioning $\tilde{x}(t)=[x(t),x_r(t)]$, we get:

$$\begin{bmatrix} -\dot{P}_{11}(t) & -\dot{P}_{12}(t) \\ -\dot{P}^{T}_{12}(t) & -\dot{P}_{22}(t) \end{bmatrix} = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P^{T}{}_{12}(t) & P_{22}(t) \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} +$$

$$\begin{bmatrix} A^{T} & 0 \\ 0 & A_r^{T} \end{bmatrix} \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P^{T}{}_{12}(t) & P_{22}(t) \end{bmatrix} +$$

$$\begin{bmatrix} D^{T}(t)Q^{*}D(t) & \vdots & -D^{T}(t)Q^{*}D_r(t) \\ \cdots & \cdots & \cdots \\ -D_r^{T}(t)Q^{*}D(t) & \vdots & D_r^{T}(t)Q^{*}D_r(t) \end{bmatrix} -$$

$$\begin{bmatrix} P_{11}B_u R^{-1}B_u^{T}P_{11} & \vdots & P_{11}B_u R^{-1}B_u^{T}P_{12} \\ \ldots & \cdots & \cdots \\ P_{12}^{T}B_u R^{-1}B_u^{T}P_{11} & \vdots & P_{12}^{T}B_u R^{-1}B_u^{T}P_{12} \end{bmatrix}$$

$$\begin{bmatrix} -\dot{P}_{11}(t) & -\dot{P}_{12}(t) \\ -\dot{P}^{T}_{12}(t) & -\dot{P}_{22}(t) \end{bmatrix} = \begin{bmatrix} P_{11}(t)A & P_{12}(t)A_r \\ P_{12}^{T}(t)A & P_{22}(t)A_r \end{bmatrix} +$$

$$\begin{bmatrix} A^{T}P_{11}(t) & A^{T}P_{12}(t) \\ A_r^{T}P_{12}^{T}(t) & A_r^{T}P_{22}(t) \end{bmatrix} +$$

$$\begin{bmatrix} D^{T}(t)Q^{*}D(t) & \vdots & -D^{T}(t)Q^{*}D_r(t) \\ \cdots & \cdots & \cdots \\ -D_r^{T}(t)Q^{*}D(t) & \vdots & D_r^{T}(t)Q^{*}D_r(t) \end{bmatrix} -$$

$$\begin{bmatrix} P_{11}B_u R^{-1}B_u^{T}P_{11} & \vdots & P_{11}B_u R^{-1}B_u^{T}P_{12} \\ \ldots & \cdots & \cdots \\ P_{12}^{T}B_u R^{-1}B_u^{T}P_{11} & \vdots & P_{12}^{T}B_u R^{-1}B_u^{T}P_{12} \end{bmatrix}$$

Here the matrices $P_{11}(t)$, $P_{12}(t)$ and $P_{22}(t)$ are obtained by partitioning the matrix $P(t)$ according to the portioning $\tilde{x}(t) = col[x(t), x_r(t)]$; they satisfy the matrix differential equations:

$$-\dot{P}_{11}(t) = P_{11}(t)A + A^T P_{11}(t) + D^T(t)Q^* D(t) - P_{11}^T(t)B_u R^{-1} B_u^T P_{11}(t);$$

$$P_{11}(t_f) = H_{11} \tag{2.51}$$

$$-\dot{P}_{12}(t) = P_{12}(t)A_r + A^T P_{12}(t) - D^T(t)Q^* D_r(t) - P_{11}(t)B_u R^{-1} B_u^T P_{12}(t)$$

$$P_{12}(t_f) = H_{12} \tag{2.52}$$

$$-\dot{P}_{22}(t) = P_{22}(t)A_r + A_r^T P_{22}(t) + D_r^T(t)Q^* D_r(t) - P_{12}^T(t)B_u R^{-1} B_u^T P_{12}(t);$$

$$P_{22}(t_f) = H_{22} \tag{2.53}$$

And

$K = [K_1, -K_2]$, where

$$K_1 = R^{-1} B_u^T P_{11}(t) \tag{2.54}$$

$$K_2 = -R^{-1} B_u^T P_{12}(t) \tag{2.55}$$

Since assumptions (I ,II ,III ,IV ,V , VI, VII, $VIII$) are satisfy then the mean, the correlation function and the covariance matrix can be found:

(1) From (2.34), we have:

$$\begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t;K) & \Phi_{12}(t;K) \\ \Phi_{21}(t;K) & \Phi_{22}(t;K) \end{bmatrix} \begin{bmatrix} x(0) \\ x_r(0) \end{bmatrix} + \int_0^t \begin{bmatrix} \Phi_{11}(t,t;K) & \Phi_{12}(t,t;K) \\ \Phi_{21}(t,t;K) & \Phi_{22}(t,t;K) \end{bmatrix} \begin{bmatrix} 0 \\ B_{x_r} x_r(t) \end{bmatrix} dt$$

$$x(t) = \Phi_{11}(t;K)x(0) + \Phi_{12}(t;K)x_r(0) + \int_0^t \Phi_{12}(t,t;K) B_{x_r} x_r(t) dt \tag{2.56}$$

$$x_r(t) = \Phi_{21}(t;K)x(0) + \Phi_{22}(t;K)x_r(0) + \int_0^t \Phi_{22}(t,t\,;K)\mathrm{B}_{x_r}x_r(t\,)dt \qquad (2.57)$$

Using (2.56), (2.57), to find the mean, one get:

$$\begin{bmatrix} E[x(t)] \\ \\ E[x_r(t)] \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t;K)E[x(0)] + \Phi_{12}(t;K)E[x_r(0)] + \int_0^t \Phi_{12}(t,t\,;K)\mathrm{B}_{x_r}E[x_r(t\,)]dt \\ \\ \Phi_{21}(t;K)E[x(0)] + \Phi_{22}(t;K)E[x_r(0)] + \int_0^t \Phi_{22}(t,t\,;K)\mathrm{B}_{x_r}E[x_r(t\,)]dt \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}$$

Where $\Phi(t\,;K\,)$ is the partition into $\Phi(t\,;K\,) = \begin{bmatrix} \Phi_{11}(t\,;K\,) & \Phi_{12}(t\,;K\,) \\ \Phi_{21}(t\,;K\,) & \Phi_{22}(t\,;K\,) \end{bmatrix}$

Since the mean of the initial condition and the mean of the white noise is zero.

(2) Using (2.56) and (2.57) to find the correlation function:

$$R_{\tilde{x}}(t_1,t_2) = E\,[\tilde{x}(t_1)\tilde{x}^T(t_2)]$$

$$\stackrel{\Delta}{\equiv} E\left\{ \begin{bmatrix} x\,(t_1) \\ x_r\,(t_1) \end{bmatrix} \begin{bmatrix} x^T\,(t_2) & x_r^T\,(t_2) \end{bmatrix} \right\}$$

$$= \begin{bmatrix} E\,[x\,(t_1)x^T\,(t_2)] & E\,[x\,(t_1)x_r^T\,(t_2)] \\ E\,[x_r^T\,(t_1)x\,(t_2)] & E\,[x_r\,(t_1)x_r^T\,(t_2)] \end{bmatrix}$$

$$\stackrel{\Delta}{\equiv} \begin{bmatrix} R_{xx}\,(t_1,t_2) & R_{xx_r}\,(t_1,t_2) \\ R_{x_rx}\,(t_1,t_2) & R_{x_rx_r}\,(t_1,t_2) \end{bmatrix}$$

where,

$$R_{xx}(t_1,t_2) = E\left\{ \left[\Phi_{11}(t_1;K)x(0) + \Phi_{12}(t_1;K)x_r(0) + \int_0^{t_1} \Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}x_r(t)dt \right] \cdot \right.$$

$$\left. \left[\Phi_{11}(t_2;K)x(0) + \Phi_{12}(t_2;K)x_r(0) + \int_0^{t_2} \Phi_{12}(t_2,s;K)\mathrm{B}_{x_r}x_r(s)ds \right]^T \right\}$$

$$R_{xx}(t_1,t_2) = E\left[\Phi_{11}(t_1;K)x(0)x^T(0)\Phi_{11}^T(t_2;K) + \Phi_{11}(t_1;K)x(0)x_r^T(0)\Phi_{12}^T(t_2;K) + \right.$$

$$\int_0^{t_2} \Phi_{11}(t_1;K)x(0)x_r^T(s)\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)ds +$$

$$\Phi_{12}(t_1;K)x_r(0)x^T(0)\Phi_{11}^T(t_2;K) + \Phi_{12}(t_1;K)x_r(0)x_r^T(0)\Phi_{12}^T(t_2;K) +$$

$$\int_0^{t_2} \Phi_{12}(t_1;K)x_r(0)x_r^T(s)\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)ds +$$

$$\int_0^{t_1} \Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}x_r(t)x^T(0)\Phi_{11}^T(t_2;K)dt +$$

$$\int_0^{t_1} \Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}x_r(t)x_r^T(0)\Phi_{12}^T(t_2;K)dt +$$

$$\left. \int_0^{t_1}\int_0^{t_2} \Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}x_r(t)x_r^T(s)\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)\,dt\,ds \right]$$

$$R_{xx}(t_1,t_2)=\Phi_{11}(t_1;K)E[x(0)x^T(0)]\Phi_{11}^T(t_2;K)+\Phi_{11}(t_1;K)E[x(0)x_r^T(0)]\Phi_{12}^T(t_2;K)+$$
$$\int_0^{t_2}\Phi_{11}(t_1;K)E[x(0)x_r^T(s)]\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)ds+$$
$$\Phi_{12}(t_1;K)E[x_r(0)x^T(0)]\Phi_{11}^T(t_2;K)+\Phi_{12}(t_1;K)E[x_r(0)x_r^T(0)]\Phi_{12}^T(t_2;K)+$$
$$\int_0^{t_2}\Phi_{12}(t_1;K)E[x_r(0)x_r^T(s)]\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)ds+$$
$$\int_0^{t_1}\Phi_{12}(t_1,t;K)E[x_r(t)x^T(0)]\mathrm{B}_{x_r}\Phi_{11}^T(t_2;K)dt+$$
$$\int_0^{t_1}\Phi_{12}(t_1,t;K)E[x_r(t)x_r^T(0)]\mathrm{B}_{x_r}\Phi_{12}^T(t_2;K)dt+$$
$$\int_0^{t_1}\int_0^{t_2}\Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}E[x_r(t)x_r^T(s)]\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)dt\,ds$$

$$R_{xx}(t_1,t_2)=\Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{11}^T(t_2;K)+\Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{12}^T(t_2;K)+$$
$$\int_0^{t_1}\int_0^{t_2}\Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}S_{x_r}d(t-s)\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)dt\,ds$$
$$=\Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{11}^T(t_2;K)+\Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{12}^T(t_2;K)+$$
$$\int_0^{\min(t_1,t_2)}\Phi_{12}(t_1,s;K)\mathrm{B}_{x_r}S_{x_r}\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)ds \qquad (2.58)$$

The last equation can be found (see Appendix [A.3]).

From (2.56), (2.57), we get:

$$R_{xx_r}(t_1,t_2)=E\left\{\left[\Phi_{11}(t_1;K)x(0)+\Phi_{12}(t_1;K)x_r(0)+\int_0^{t_1}\Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}x_r(t)dt\right]\cdot\right.$$
$$\left.\left[\Phi_{21}(t_2;K)x(0)+\Phi_{22}(t_2;K)x_r(0)+\int_0^{t_2}\Phi_{22}(t_2,s;K)\mathrm{B}_{x_r}x_r(s)ds\right]^T\right\}$$

$$R_{xx_r}(t_1,t_2) = E\Big[\ \Phi_{11}(t_1;K)x(0)x^T(0)\Phi_{21}^T(t_2;K) + \Phi_{11}(t_1;K)x(0)x_r^T(0)\Phi_{22}^T(t_2;K) +$$

$$\int_0^{t_2} \Phi_{11}(t_1;K)x(0)x_r^T(s)\mathrm{B}_{x_r}^T\Phi_{22}^T(t_2,s;K)ds +$$

$$\Phi_{12}(t_1;K)x_r(0)x^T(0)\Phi_{21}^T(t_2;K) + \Phi_{12}(t_1;K)x_r(0)x_r^T(0)\Phi_{22}^T(t_2;K) +$$

$$\int_0^{t_2} \Phi_{12}(t_1;K)x_r(0)x_r^T(s)\mathrm{B}_{x_r}^T\Phi_{22}^T(t_2,s;K)ds +$$

$$\int_0^{t_1} \Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}x_r(t)x^T(0)\Phi_{21}^T(t_2;K)dt +$$

$$\int_0^{t_1} \Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}x_r(t)x_r^T(0)\Phi_{22}^T(t_2;K)dt +$$

$$\int_0^{t_1}\int_0^{t_2} \Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}x_r(t)x_r^T(s)\mathrm{B}_{x_r}^T\Phi_{22}^T(t_2,s;K)dt\,ds\ \Big]$$

$$R_{xx_r}(t_1,t_2)=\Phi_{11}(t_1;K)E[x(0)x^T(0)]\Phi_{21}^T(t_2;K)+\Phi_{11}(t_1;K)E[x(0)x_r^T(0)]\Phi_{22}^T(t_2;K)+$$
$$\int_0^{t_2}\Phi_{11}(t_1;K)E[x(0)x_r^T(s)]\mathrm{B}_{x_r}^T\Phi_{22}^T(t_2,s;K)\,ds+$$
$$\Phi_{12}(t_1;K)E[x_r(0)x^T(0)]\Phi_{21}^T(t_2;K)+\Phi_{12}(t_1;K)E[x_r(0)x_r^T(0)]\Phi_{22}^T(t_2;K)+$$
$$\int_0^{t_2}\Phi_{12}(t_1;K)E[x_r(0)x_r^T(s)]\mathrm{B}_{x_r}^T\Phi_{22}^T(t_2,s;K)ds+$$
$$\int_0^{t_1}\Phi_{12}(t_1,t;K)E[x_r(t)x^T(0)]\mathrm{B}_{x_r}\Phi_{21}^T(t_2;K)dt+$$
$$\int_0^{t_1}\Phi_{12}(t_1,t;K)E[x_r(t)x_r^T(0)]\mathrm{B}_{x_r}\Phi_{22}^T(t_2;K)dt+$$
$$\int_0^{t_1}\int_0^{t_2}\Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}E[x_r(t)x_r^T(s)]\mathrm{B}_{x_r}^T\Phi_{22}^T(t_2,s;K)dt\,ds$$

$$R_{xx_r}(t_1,t_2)=\Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K)+\Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K)+$$
$$\int_0^{t_1}\int_0^{t_2}\Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}S_{x_r}d(t-s)\mathrm{B}_{x_r}^T\Phi_{22}^T(t_2,s;K)dt\,ds$$
$$=\Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K)+\Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K)+$$
$$\int_0^{\min(t_1,t_2)}\Phi_{12}(t_1,s;K)\mathrm{B}_{x_r}S_{x_r}\mathrm{B}_{x_r}^T\Phi_{22}^T(t_2,s;K)ds \tag{2.59}$$

And similarly to $R_{x_rx}(t_1,t_2)$,

$R_{x_r}(t_1,t_2)=E\left[x_r(t_1)x_r^T(t_2)\right]$, by using (2.57), we obtain:

$$R_{x_rx_r}(t_1,t_2)=\Phi_{21}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K)+\Phi_{22}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K)+$$
$$\int_0^{\min(t_1,t_2)}\Phi_{22}(t_1,s;K)\mathrm{B}_{x_r}S_{x_r}\mathrm{B}_{x_r}^T\Phi_{22}^T(t_2,s;K)ds \tag{2.60}$$

The correlation matrix can be obtain from correlation function (see remark (1.11):

$$\Sigma_{\tilde{x}}(t,t) = R_{\tilde{x}}(t,t) = E[\tilde{x}(t)\tilde{x}^T(t)]$$

$$\overset{\Delta}{\equiv} E\left\{ \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} \begin{bmatrix} x^T(t) & x_r^T(t) \end{bmatrix} \right\}$$

$$= \begin{bmatrix} E[x(t)x^T(t)] & E[x(t)x_r^T(t)] \\ E[x_r^T(t)x(t)] & E[x_r(t)x_r^T(t)] \end{bmatrix} \overset{\Delta}{\equiv} \begin{bmatrix} \Sigma_{xx}(t) & \Sigma_{xx_r}(t) \\ \Sigma_{x_r x}(t) & \Sigma_{x_r x_r}(t) \end{bmatrix}$$

$$\Sigma_{xx}(t) = \Phi_{11}(t;K)\Sigma_x(0)\Phi_{11}^T(t;K) + \Phi_{12}(t;K)\Sigma_{x_r}(0)\Phi_{12}^T(t;K) + \int_0^t \Phi_{12}(t,s;K)\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T \Phi_{12}^T(t,s;K)ds \quad (2.61)$$

$$\Sigma_{xx_r}(t) = \Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K) + \Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K) + \int_0^{\min(t_1,t_2)} \Phi_{12}(t_1,s;K)\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T \Phi_{22}^T(t_2,s;K)ds \quad (2.62)$$

And similarly

$$\Sigma_{x_r x_r}(t) = \Phi_{21}(t;K)\Sigma_x(0)\Phi_{21}^T(t;K) + \Phi_{22}(t;K)\Sigma_{x_r}(0)\Phi_{22}^T(t;K) + \int_0^t \Phi_{22}(t,s;K)\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T \Phi_{22}^T(t,s;K)ds \quad (2.63)$$

Concluding remarks and that completes the proof

1) In passing, we note that this system, (2.29) is not completely controllable from u.

2) We observed that P_{11} of (2.51), and therefore also $K_1 = R^{-1}\mathrm{B}_u^T \mathrm{P}_{11}(t)$, is completely independent of the properties of the disturbances, and is in fact obtained by solving the deterministic regulator problem with the

disturbances omitted. Once P_{11} and K_1 have been found, (2.52) can be solved to determine P_{12} and from this $K_2 = -R^{-1}\mathrm{B}_u^T \mathrm{P}_{12}(t)$.

3) The state correlation matrix can also be computed as the solution of a differential equation. This differential equation is obtained by differentiating equation (2.61) to get:

Since $\Phi(t;K) = e^{(\tilde{A}-\tilde{B}K)t}$

Expanding from (2.61), and using the remarks (1.11) point (2) to obtain:

$$\dot{\Sigma}_{\tilde{x}}(t) = (\tilde{A} - \tilde{B}_u K)\Sigma_{\tilde{x}}(t) + \Sigma_{\tilde{x}}(t)(\tilde{A} - \tilde{B}_u K)^T + \tilde{B}_x \tilde{S}_x \tilde{B}_x^T \tag{2.64}$$

4) The steady-state correlation matrix of the state can also be found from the matrix differential equation (2.64) provided that the system is time-invariant and stable. In steady-state, the derivative of the correlation matrix equal to zero then:

$$0 = (\tilde{A} - \tilde{B}_u K)\Sigma_{\tilde{x}}(\infty) + \Sigma_{\tilde{x}}(\infty)(\tilde{A} - \tilde{B}_u K)^T + \tilde{B}_x \tilde{S}_x \tilde{B}_x^T \tag{2.65}$$

It is quite helpful to realize that this algebraic matrix equation in $\Sigma_x(\infty)$ has a unique solution (see remark (1.3)). Matrix equation in (1.54) is also known ***Lyapunov equation*** (see theorem (1.7)).

The above result is very interesting in computing $\Sigma_{\tilde{x}}(t)$ directly by solving some ordinary differential equation, similar to differential Riccati equation .

The following problem formulation 2 and its corresponding lemma (1.1), are some generalized result of theorem (2.1), where the stochastic process disturbance are defined to be lied in both the dynamical system and its target dynamical one.

2.3.2 Problem Formulation 2:

Consider the non-linear dynamical control system:

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}_u u(t) + \mathrm{B}_x x(t)$$

$$\dot{x}_r(t) = \mathrm{A}_r x_r(t) + \mathrm{B}_{x_r} x_r(t) \qquad \text{(The target dynamic space)}$$

(2.66)

Where $x \in R^n, x_r \in R^n$. $x(0), x_r(0)$ are random variables, $\mathrm{A} \in R^{n\times n}, \mathrm{A}_r \in R^{n\times n}$ constant matrices, $\mathrm{B}_u \in R^{n\times m}$ constant matrix, $u(t) \in R^m$ the control input, $\mathrm{B}_{x_r} \in R^{n\times r}, \mathrm{B}_x \in R^{n\times r}$ constant matrices, $x_r(t) \in R^r, x(t) \in R^r$ (white noises stochastic process with intensity $S_x(t) > 0, S_{x_r}(t) > 0$). The controlled variable, the output of target dynamic system, and the performance index are defined in the problem formulation 1.

Lemma (2.1)

Consider the non-linear stochastic dynamical optimal control defined in problem formulation 2:

Assuming that:

I. The initial conditions are zero-mean random vectors with the following correlation matrices (The second-order moment matrix):

$E[x(0)x^T(0)] = \Sigma_x(0)$, and $E\{x(0)\} = \mathbf{0}$

$E[x_r(0)x_r^T(0)] = \Sigma_{x_r}(0)$, and $E\{x_r(0)\} = \mathbf{0}$

The pair $(\mathrm{A}, \mathrm{B}_u)$ of a non-linear dynamical control system is controllable

II. A_r is stable matrix.

III. $E[x(0)x_r^T(0)] = \mathbf{0}$

IV. $E[x_r(0)x^T(0)] = \mathbf{0}$

V. $E[x(0)x^T(t)] = \mathbf{0}$

VI. $E[x(0)x_r^T(t)] = \mathbf{0}$

VII. $E[x_r(0)x^T(t)] = \mathbf{0}$

VIII. $E[x_r(0)x_r^T(t)] = \mathbf{0}$

IX. $E[x(t)x^T(0)] = \mathbf{0}$

X. $E[x(t)x_r^T(0)] = \mathbf{0}$

XI. $E[x_r(t)x^T(0)] = \mathbf{0}$

XII. $E[x_r(t)x_r^T(0)] = \mathbf{0}$

XIII. $E[x_r(t)x^T(s)] = \mathbf{0}$

XIV. $E[x(t)x_r^T(s)] = \mathbf{0}$

XV. R, Q, and H are positive-definite, symmetric, matrices.

Then:

a) The optimal control law is obtained as:

$$u(t) = -K\tilde{x}(t)$$

Furthermore, it can be found by partitioning the Riccati equation $\mathrm{P}_{11}(t)$, $\mathrm{P}_{12}(t)$, and $\mathrm{P}_{22}(t)$ are the solution of the matrix differential equations if possible. (see the equations (2.51), (2.52), (2.53) in theorem (2.1)).

b) The optimal cost value is obtained to be:

$$J_{SR} = \frac{1}{2} tr \left\{ \mathrm{P}(0)\Sigma_{\tilde{x}}(0) + \int_0^{t_f} \mathrm{P}(t)\tilde{B}_x \tilde{S}_x \tilde{B}_x^T dt \right\}$$

Where $\mathrm{P}(0) = \begin{bmatrix} \mathrm{P}_{11}(0) & \mathrm{P}_{12}(0) \\ \mathrm{P}_{12}^T(0) & \mathrm{P}_{22}(0) \end{bmatrix}$, $\Sigma_{\tilde{x}}(0) = \begin{bmatrix} E[x(0)x^T(0)] & E[x(0)x_r^T(0)] \\ E[x_r(0)x^T(0)] & E[x_r(0)x_r^T(0)] \end{bmatrix}$,

$$\tilde{B}_x \tilde{B}_x^T = \begin{bmatrix} B_x \\ B_{x_r} \end{bmatrix} \begin{bmatrix} B_x^T & B_{x_r}^T \end{bmatrix} = \begin{bmatrix} B_x B_x^T & B_x B_{x_r}^T \\ B_{x_r} B_x^T & B_{x_r} B_{x_r}^T \end{bmatrix},$$

$$P(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}^T(t) & P_{22}(t) \end{bmatrix}$$

$$\tilde{S}_x = E\left\{ \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} \begin{bmatrix} x^T(s) & x_r^T(s) \end{bmatrix} \right\} = \begin{bmatrix} E[x(t)x^T(s)] & E[x(t)x_r^T(s)] \\ E[x_r(t)x^T(s)] & E[x_r(t)x_r^T(s)] \end{bmatrix}$$

c) The correlation function, correlation matrix as well as the mean, vector of $x(t)$, are found to be:

$$R_{xx}(t_1,t_2) = \Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{11}^T(t_2;K) + \Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{12}^T(t_2;K) +$$

$$\int_0^{\min(t_1,t_2)} \Phi_{11}(t_1,s;K) B_x S_x B_x^T \Phi_{11}^T(t_2,s;K)\,ds +$$

$$\int_0^{\min(t_1,t_2)} \Phi_{12}(t_1,s;K) B_{x_r} S_{x_r} B_{x_r}^T \Phi_{12}^T(t_2,s;K)\,ds$$

$$R_{xx_r}(t_1,t_2) = \Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K) + \Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K) +$$

$$\int_0^{\min(t_1,t_2)} \Phi_{11}(t_1,s;K) B_x S_x B_x^T \Phi_{21}^T(t_2,s;K)\,ds +$$

$$\int_0^{\min(t_1,t_2)} \Phi_{12}(t_1,s;K) B_{x_r} S_{x_r} B_{x_r}^T \Phi_{22}^T(t_2,s;K)\,ds$$

$$R_{x_r x_r}(t_1,t_2) = \Phi_{21}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K) + \Phi_{22}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K) +$$

$$\int_0^{\min(t_1,t_2)} \Phi_{21}(t_1,s;K) B_x S_x B_x^T \Phi_{21}^T(t_2,s;K)\,ds +$$

$$\int_0^{\min(t_1,t_2)} \Phi_{22}(t_1,s;K) B_{x_r} S_{x_r} B_{x_r}^T \Phi_{22}^T(t_2,s;K)\,ds$$

$$\Sigma_{xx}(t) = \Phi_{11}(t;K)\Sigma_x(0)\Phi_{11}^T(t;K) + \Phi_{12}(t;K)\Sigma_{x_r}(0)\Phi_{12}^T(t;K) +$$

$$\int_0^t \Phi_{11}(t,s;K)B_x S_x B_x^T \Phi_{11}^T(t,s;K)ds + \int_0^t \Phi_{12}(t,s;K)B_{x_r} S_{x_r} B_{x_r}^T \Phi_{12}^T(t,s;K)ds$$

$$\Sigma_{xx_r}(t_1,t_2) = \Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K) + \Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K) +$$

$$\int_0^t \Phi_{11}(t_1,s;K)B_x S_x B_x^T \Phi_{21}^T(t_2,s;K)ds +$$

$$\int_0^t \Phi_{12}(t_1,s;K)B_{x_r} S_{x_r} B_{x_r}^T \Phi_{22}^T(t_2,s;K)ds$$

$$\Sigma_{x_r x_r}(t) = \Phi_{21}(t;K)\Sigma_x(0)\Phi_{21}^T(t;K) + \Phi_{22}(t;K)\Sigma_{x_r}(0)\Phi_{22}^T(t;K) +$$

$$\int_0^t \Phi_{21}(t,s;K)B_x S_x B_x^T \Phi_{21}^T(t,s;K)ds +$$

$$\int_0^t \Phi_{22}(t,s;K)B_{x_r} S_{x_r} B_{x_r}^T \Phi_{22}^T(t,s;K)ds$$

And

$$E[x(t)] = \Phi_{11}(t;K)E[x(0)] + \Phi_{12}(t;K)E[x_r(0)] +$$

$$\int_0^t \left\{ \Phi_{11}(t,t;K)B_x E[x(t)] + \Phi_{12}(t,t;K)B_{x_r} E[x_r(t)] \right\} dt$$

$$= 0$$

$$E[x_r(t)] = \Phi_{21}(t;K)E[x(0)] + \Phi_{22}(t;K)E[x_r(0)] +$$

$$\int_0^t \left\{ \Phi_{21}(t,t;K)B_x E[x(t)] + \Phi_{22}(t,t;K)B_{x_r} E[x_r(t)] \right\} dt$$

$$= 0$$

Proof

Define the augmented state between state space $x(t)$ and target state space $x_r(t)$:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_r(t) \end{bmatrix} = \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{A}_r \end{bmatrix} \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} + \begin{bmatrix} \mathrm{B}_u \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} \mathrm{B}_x(t)x(t) \\ \mathrm{B}_{x_r}(t)x_r(t) \end{bmatrix} \tag{2.67}$$

To complete this lemma using the equations (2.31) to (2.48), from theorem (2.1), then we obtain:

$$J_{SR} = \frac{1}{2} tr \left\{ \mathrm{P}(0)\Sigma_{\tilde{x}}(0) + \int_0^{t_f} \mathrm{P}(t)\tilde{\mathrm{B}}_x \tilde{S}_x \tilde{\mathrm{B}}_x^T dt \right\}$$

$$= \frac{1}{2} tr \left\{ \begin{bmatrix} \mathrm{P}_{11}(0) & \mathrm{P}_{12}(0) \\ \mathrm{P}_{12}^T(0) & \mathrm{P}_{22}(0) \end{bmatrix} \begin{bmatrix} \Sigma_x(0) & \Sigma_{xx_r}(0) \\ \Sigma_{x_r x}(0) & \Sigma_{x_r}(0) \end{bmatrix} + \right.$$

$$\left. \int_0^{t_f} \begin{bmatrix} \mathrm{P}_{11}(t) & \mathrm{P}_{12}(t) \\ \mathrm{P}_{12}^T(t) & \mathrm{P}_{22}(t) \end{bmatrix} \begin{bmatrix} \mathrm{B}_x S_x \mathrm{B}_x^T & 0 \\ 0 & \mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T \end{bmatrix} dt \right\} \tag{2.68}$$

where all the weights are non-negative. From R is positive-definite and $(\tilde{\mathrm{A}}, \tilde{\mathrm{B}}_u)$ are controllable, the Hamiltonian approach (see subsection (2.2.1)) can be applied to compute $K = (R^{-1}\mathrm{B}_u^T \mathrm{P}_{11}(t), -R^{-1}\mathrm{B}_u^T \mathrm{P}_{12}(t))$ Furthermore, it can be found by partitioning the Riccati equation $\mathrm{P}_{11}(t)$, $\mathrm{P}_{12}(t)$, and $\mathrm{P}_{22}(t)$ are the solution of the matrix differential equations. See the details in theorem (2.1).

Since assumptions $(I, II, III, IV, V, VI, VII, VIII, IX, X, XI, XII, XIII, XIV, XV)$, are satisfy then the mean, the correlation function and the covariance matrix can be found:

From (2.34), we have:

1) $$\begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t;K) & \Phi_{12}(t;K) \\ \Phi_{21}(t;K) & \Phi_{22}(t;K) \end{bmatrix} \begin{bmatrix} x(0) \\ x_r(0) \end{bmatrix} + \int_0^t \begin{bmatrix} \Phi_{11}(t,t;K) & \Phi_{12}(t,t;K) \\ \Phi_{21}(t,t;K) & \Phi_{22}(t,t;K) \end{bmatrix} \begin{bmatrix} \mathrm{B}_x x(t) \\ \mathrm{B}_{x_r} x_r(t) \end{bmatrix} dt$$

$$x(t) = \Phi_{11}(t;K)x(0) + \Phi_{12}(t;K)x_r(0) + \int_0^t \{ \Phi_{11}(t,t;K)\mathrm{B}_x x(t) + \Phi_{12}(t,t;K)\mathrm{B}_{x_r} x_r(t) \} dt \tag{2.69}$$

$$x_r(t) = \Phi_{21}(t;K)x(0) + \Phi_{22}(t;K)x_r(0) + \int_0^t \{\Phi_{21}(t,t;K)\mathrm{B}_x x(t) + \Phi_{22}(t,t;K)\mathrm{B}_{x_r} x_r(t)\} dt \tag{2.70}$$

From (2.69), (2.70), we compute the mean:

$$E[x(t)] = \Phi_{11}(t;K)E[x(0)] + \Phi_{12}(t;K)E[x_r(0)] + \int_0^t \left\{ \Phi_{11}(t,t;K)\mathrm{B}_x E[x(t)] + \Phi_{12}(t,t;K)\mathrm{B}_{x_r} E[x_r(t)] \right\} dt$$

$$= 0$$

$$E[x_r(t)] = \Phi_{21}(t;K)E[x(0)] + \Phi_{22}(t;K)E[x_r(0)] + \int_0^t \{\Phi_{21}(t,t;K)\mathrm{B}_x E[x(t)] + \Phi_{22}(t,t;K)\mathrm{B}_{x_r} E[x_r(t)]\} dt$$

$$= 0$$

Since the mean of the initial condition and the mean of the white noise is zero.

2) The correlation function can be found from (2.69), (2.70), we obtain:

$$R_{\tilde{x}}(t_1,t_2) = E[\tilde{x}(t_1)\tilde{x}^T(t_2)]$$

$$\overset{\Delta}{\equiv} E\left\{ \begin{bmatrix} x(t_1) \\ x_r(t_1) \end{bmatrix} \begin{bmatrix} x^T(t_2) & x_r^T(t_2) \end{bmatrix} \right\}$$

$$= \begin{bmatrix} E[x(t_1)x^T(t_2)] & E[x(t_1)x_r^T(t_2)] \\ E[x_r^T(t_1)x(t_2)] & E[x_r(t_1)x_r^T(t_2)] \end{bmatrix} \quad \text{where,}$$

$$\overset{\Delta}{\equiv} \begin{bmatrix} R_{xx}(t_1,t_2) & R_{xx_r}(t_1,t_2) \\ R_{x_r x}(t_1,t_2) & R_{x_r x_r}(t_1,t_2) \end{bmatrix}$$

$$R_{xx}(t_1,t_2) = E\Bigg\{ \Bigg[\Phi_{11}(t_1;K)x(0) + \Phi_{12}(t_1;K)x_r(0) +$$

$$\int_0^t \left\{ \Phi_{11}(t_1,t;K)\mathrm{B}_x x(t) + \Phi_{12}(t_1,t;K)\mathrm{B}_{x_r} x_r(t) \right\} dt \Bigg] \cdot$$

$$\Bigg[\Phi_{11}(t_2;K)x(0) + \Phi_{12}(t_2;K)x_r(0) +$$

$$\int_0^{t_2} \left\{ \Phi_{11}(t_2,s;K)\mathrm{B}_x x(s) + \Phi_{12}(t_2,s;K)\mathrm{B}_{x_r} x_r(s) \right\} ds \Bigg]^T$$

$$R_{xx}(t_1,t_2) = E\Big[\ \Phi_{11}(t_1;K)x(0)x^T(0)\Phi_{11}^T(t_2;K) + \Phi_{11}(t_1;K)x(0)x_r^T(0)\Phi_{12}^T(t_2;K) +$$

$$\int_0^{t_2}\{\ \Phi_{11}(t_1;K)x(0)x^T(s)\mathrm{B}_{x_r}^T\Phi_{11}^T(t_2,s;K) +$$

$$\Phi_{11}(t_1;K)x(0)x_r^T(s)\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)\ \}ds +$$

$$\Phi_{12}(t_1;K)x_r(0)x^T(0)\Phi_{11}^T(t_2;K) +$$

$$\Phi_{12}(t_1;K)x_r(0)x_r^T(0)\Phi_{12}^T(t_2;K) +$$

$$\int_0^{t_2}\{\ \Phi_{12}(t_1;K)x_r(0)x^T(s)\mathrm{B}_x^T\Phi_{11}^T(t_2,s;K) +$$

$$\Phi_{12}(t_1;K)x_r(0)x_r^T(s)\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)\ \}ds +$$

$$\int_0^{t_1}\{\ \Phi_{11}(t_1,t;K)\mathrm{B}_x x(t)x^T(0)\Phi_{11}^T(t_2;K) +$$

$$\Phi_{11}(t_1,t;K)\mathrm{B}_x x(t)x_r^T(0)\Phi_{12}^T(t_2;K)\ \}dt +$$

$$\int_0^{t_1}\int_0^{t_2}\Phi_{11}(t_1,t;K)\mathrm{B}_x x(t)x_r^T(s)\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)dt\,ds +$$

$$\int_0^{t_1}\int_0^{t_2}\Phi_{11}(t_1,t;K)\mathrm{B}_x x(t)x_r^T(s)\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)dt\,ds +$$

$$\int_0^{t_1}\ \{\ \Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}x_r(t)x^T(0)\Phi_{11}^T(t_2;K) +$$

$$\Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}x_r(t)x_r^T(0)\Phi_{12}^T(t_2;K)\ \}dt +$$

$$\int_0^{t_1}\int_0^{t_2}\Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}x_r(t)x^T(s)\mathrm{B}_x^T\Phi_{11}^T(t_2,s;K)dt\,ds +$$

$$\int_0^{t_1}\int_0^{t_2}\Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}x_r(t)x_r^T(s)\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)dt\,ds$$

$$
\begin{aligned}
R_{xx}(t_1,t_2) = {} & \Phi_{11}(t_1;K)E[x(0)x^T(0)]\Phi_{11}^T(t_2;K) + \\
& \Phi_{11}(t_1;K)E[x(0)x_r^T(0)]\Phi_{12}^T(t_2;K) + \\
& \int_0^{t_2}\{\ \Phi_{11}(t_1;K)E[x(0)x^T(s)]\mathrm{B}_x\Phi_{11}^T(t_2,s;K) + \\
& \Phi_{11}(t_1;K)E[x(0)x_r^T(s)]\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)\ \}ds + \\
& \Phi_{12}(t_1;K)E[x_r(0)x^T(0)]\Phi_{11}^T(t_2;K) + \Phi_{12}(t_1;K)E[x_r(0)x_r^T(0)]\Phi_{12}^T(t_2;K) + \\
& \int_0^{t_2}\{\ \Phi_{12}(t_1;K)E[x_r(0)x^T(s)]\mathrm{B}_x^T\Phi_{11}^T(t_2,s;K) + \\
& \Phi_{12}(t_1;K)E[x_r(0)x_r^T(s)]\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)\ \}ds + \\
& \int_0^{t_1}\{\ \Phi_{11}(t_1,t;K)\mathrm{B}_x E[x(t)x^T(0)]\Phi_{11}^T(t_2;K) + \\
& \Phi_{11}(t_1,t;K)\mathrm{B}_x E[x(t)x_r^T(0)]\Phi_{12}^T(t_2;K)\ \}dt + \\
& \int_0^{t_1}\int_0^{t_2}\Phi_{11}(t_1,t;K)\mathrm{B}_x E[x(t)x^T(s)]\mathrm{B}_x^T\Phi_{11}^T(t_2,s;K)dt\,ds + \\
& \int_0^{t_1}\int_0^{t_2}\Phi_{11}(t_1,t;K)\mathrm{B}_x E[x(t)x_r^T(s)]\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)dt\,ds + \\
& \int_0^{t_1}\ \{\ \Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}E[x_r(t)x^T(0)]\Phi_{11}^T(t_2;K) + \\
& \Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}E[x_r(t)x_r^T(0)]\Phi_{12}^T(t_2;K)\ \}dt + \\
& \int_0^{t_1}\int_0^{t_2}\Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}E[x_r(t)x^T(s)]\mathrm{B}_x^T\Phi_{11}^T(t_2,s;K)dt\,ds + \\
& \int_0^{t_1}\int_0^{t_2}\Phi_{12}(t_1,t;K)\mathrm{B}_{x_r}E[x_r(t)x_r^T(s)]\mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)dt\,ds
\end{aligned}
\tag{2.71}
$$

The 10th and 15-th terms of equation (2.71) can be found (see in Appendix [A.3]).

$$R_{xx}(t_1,t_2)=\Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{11}^T(t_2;K)+\Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{12}^T(t_2;K)+$$
$$\int_0^{\min(t_1,t_2)}\Phi_{11}(t_1,s;K)\mathrm{B}_x S_x \mathrm{B}_x^T\Phi_{11}^T(t_2,s;K)ds+$$
$$\int_0^{\min(t_1,t_2)}\Phi_{12}(t_1,s;K)\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T\Phi_{12}^T(t_2,s;K)ds \qquad (2.72)$$

And similarly, by using (2.69), (2.70), to get:

$$R_{xx_r}(t_1,t_2)=\Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K)+\Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K)+$$
$$\int_0^{\min(t_1,t_2)}\Phi_{11}(t_1,s;K)\mathrm{B}_x S_x \mathrm{B}_x^T\Phi_{21}^T(t_2,s;K)ds+$$
$$\int_0^{\min(t_1,t_2)}\Phi_{12}(t_1,s;K)\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T\Phi_{22}^T(t_2,s;K)ds \qquad (2.73)$$

Using (2.70), to obtain $R_{x_r x_r}(t_1,t_2)=E\left[x_r(t_1)x_r^T(t_2)\right]$

$$R_{x_r x_r}(t_1,t_2)=\Phi_{21}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K)+\Phi_{22}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K)+$$
$$\int_0^{\min(t_1,t_2)}\Phi_{21}(t_1,s;K)\mathrm{B}_x S_x \mathrm{B}_x^T\Phi_{21}^T(t_2,s;K)ds+$$
$$\int_0^{\min(t_1,t_2)}\Phi_{22}(t_1,s;K)\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T\Phi_{22}^T(t_2,s;K)ds \qquad (2.74)$$

The correlation matrix can be obtain from correlation function (2.72), (see remark (1.11):

$$\Sigma_{\tilde{x}}(t,t) = R_{\tilde{x}}(t,t) = E[\tilde{x}(t)\tilde{x}^T(t)]$$

$$\overset{\Delta}{\equiv} E\left\{ \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} \begin{bmatrix} x^T(t) & x_r^T(t) \end{bmatrix} \right\}$$

$$= \begin{bmatrix} E[x(t)x^T(t)] & E[x(t)x_r^T(t)] \\ E[x_r^T(t)x(t)] & E[x_r(t)x_r^T(t)] \end{bmatrix} \overset{\Delta}{\equiv} \begin{bmatrix} \Sigma_{xx}(t) & \Sigma_{xx_r}(t) \\ \Sigma_{x_r x}(t) & \Sigma_{x_r x_r}(t) \end{bmatrix}$$

$$\Sigma_{xx}(t) = \Phi_{11}(t;K)\Sigma_x(0)\Phi_{11}^T(t;K) + \Phi_{12}(t;K)\Sigma_{x_r}(0)\Phi_{12}^T(t;K) +$$
$$\int_0^t \Phi_{11}(t,s;K)\mathrm{B}_x S_x \mathrm{B}_x^T \Phi_{11}^T(t,s;K)ds +$$
$$\int_0^t \Phi_{12}(t,s;K)\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T \Phi_{12}^T(t,s;K)ds \qquad (2.75)$$

And similarly use (2.73), we gets:

$$\Sigma_{xx_r}(t,t) = \Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K) + \Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K) +$$
$$\int_0^t \Phi_{11}(t_1,s;K)\mathrm{B}_x S_x \mathrm{B}_x^T \Phi_{21}^T(t_2,s;K)ds +$$
$$\int_0^t \Phi_{12}(t_1,s;K)\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T \Phi_{22}^T(t_2,s;K)ds \qquad (2.76)$$

And from (2.74), we have:

$$\Sigma_{x_r x_r}(t) = \Phi_{21}(t;K)\Sigma_x(0)\Phi_{21}^T(t;K) + \Phi_{22}(t;K)\Sigma_{x_r}(0)\Phi_{22}^T(t;K) +$$
$$\int_0^t \Phi_{21}(t,s;K)\mathrm{B}_x S_x \mathrm{B}_x^T \Phi_{21}^T(t,s;K)ds +$$
$$\int_0^t \Phi_{22}(t,s;K)\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T \Phi_{22}^T(t,s;K)ds \qquad (2.77)$$

To extend result and the develop are making step-by–step, the following problem formulation is reached in process.

2.3.3 Problem Formulation 3:

Consider the non-linear dynamical control system:

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}_u u(t) + \mathrm{B}_x x(t)$$

$$\dot{x}_r(t) = \mathrm{A}_r x_r(t) + \mathrm{B}_{u_r} u_r(t) + \mathrm{B}_{x_r} x_r(t) \qquad \text{(The target dynamic space)}$$

(2.78)

Where $x \in R^n, x_r \in R^n$. $x(0), x_r(0)$ are random variables, $\mathrm{A} \in R^{n\times n}, \mathrm{A}_r \in R^{n\times n}$ constant matrices, $\mathrm{B}_{u_r} \in R^{n\times m}, \mathrm{B}_u \in R^{n\times m}$ constant matrix, $u_r(t) \in R^m, u(t) \in R^m$ the control input, $\mathrm{B}_{x_r} \in R^{n\times r}, \mathrm{B}_x \in R^{n\times r}$ constant matrices, $x_r(t) \in R^r, x(t) \in R^r$ (white noises stochastic process with intensity $S_x(t) > 0, S_{x_r}(t) > 0$). The controlled variable, the output of target dynamic system, and the performance index are defined by as the problem formulation 1.

Lemma (2.2)

Consider the non-linear stochastic dynamical optimal control defined in problem formulation 3.

Assuming that:

I. The pair $(\mathrm{A}, \mathrm{B}_u)$, $(\mathrm{A}_r, \mathrm{B}_{u_r})$ of a non-linear dynamical control system are controllable . And define

$u_r = K_r x_r$ such that K_r selected to make $(\mathrm{A}_r - \mathrm{B}_{u_r} K_r)$ is a asymptotically stable matrix.

II. The initial conditions are zero-mean random vectors with the following correlation matrices (The second-order moment matrix):

$E[x(0)x^T(0)] = \Sigma_x(0)$, and $E\{x(0)\} = \mathbf{0}$

$E[x_r(0)x_r^T(0)] = \Sigma_{x_r}(0)$, and $E\{x_r(0)\} = \mathbf{0}$

Assuming that the conditions of lemma (2.1) are satisfied:

a) $E[x(0)x_r^T(0)] = \mathbf{0}$

b) $E[x_r(0)x^T(0)] = \mathbf{0}$

c) $E[x(0)x^T(t)] = \mathbf{0}$

d) $E[x(0)x_r^T(t)] = \mathbf{0}$

e) $E[x_r(0)x^T(t)] = \mathbf{0}$

f) $E[x_r(0)x_r^T(t)] = \mathbf{0}$

g) $E[x(t)x^T(0)] = \mathbf{0}$

h) $E[x(t)x_r^T(0)] = \mathbf{0}$

i) $E[x_r(t)x^T(0)] = \mathbf{0}$

j) $E[x_r(t)x_r^T(0)] = \mathbf{0}$

k) $E[x_r(t)x^T(s)] = \mathbf{0}$

l) $E[x(t)x_r^T(s)] = \mathbf{0}$

m) R, Q, and H are positive-definite, symmetric, matrices.

Then :

1) The optimal control law is obtained as:

$u_r(t) = -K_r x_r(t)$, and

$u(t) = -K\tilde{x}(t)$, where

$K = [K_1, -K_2]$, and can be define in (2.54), (2.55), where

$\mathrm{P}_{11}(t)$, $\mathrm{P}_{12}(t)$, and $\mathrm{P}_{22}(t)$ are the solution of the matrix differential equations:

$$-\dot{\mathrm{P}}_{11}(t) = \mathrm{P}_{11}(t)\mathrm{A} + \mathrm{A}^T \mathrm{P}_{11}(t) + D^T(t)Q^*D(t) - \mathrm{P}_{11}(t)\mathrm{B}_u R^{-1}\mathrm{B}_u^T \mathrm{P}_{11}(t);$$

$$\mathrm{P}_{11}(t_f) = \mathrm{H}_{11}$$

$$-\dot{\mathrm{P}}_{12}(t) = \mathrm{P}_{12}(t)(\mathrm{A}_r - \mathrm{B}_r K_r) + \mathrm{A}^T \mathrm{P}_{12}(t) - D^T(t)Q^*D_r(t) - \mathrm{P}_{11}(t)\mathrm{B}_u R^{-1}\mathrm{B}_u^T \mathrm{P}_{12}(t)$$

$$\mathrm{P}_{12}(t_f) = \mathrm{H}_{12}$$

$$-\dot{\mathrm{P}}_{22}(t) = \mathrm{P}_{22}(t)(\mathrm{A}_r - \mathrm{B}_r K_r) + (\mathrm{A}_r - \mathrm{B}_r K_r)^T \mathrm{P}_{22}(t) + D_r^T(t)Q^*D_r(t) - \mathrm{P}_{12}^T(t)\mathrm{B}_u R^{-1}\mathrm{B}_u^T \mathrm{P}_{12}(t);$$

$$\mathrm{P}_{22}(t_f) = \mathrm{H}_{22}$$

And

$$K = [K_1, K_2] = [R^{-1}\mathrm{B}_u^T \mathrm{P}_{11}(t), -R^{-1}\mathrm{B}_u^T \mathrm{P}_{12}(t)]$$

2) The optimal cost value is obtained to be:

$$J_{SR} = \frac{1}{2} tr \left\{ \mathrm{P}(0)\Sigma_{\tilde{x}}(0) + \int_0^{t_f} \mathrm{P}(t)\tilde{\mathrm{B}}_x \tilde{S}_x \tilde{\mathrm{B}}_x^T dt \right\}$$

Where $\mathrm{P}(0) = \begin{bmatrix} \mathrm{P}_{11}(0) & \mathrm{P}_{12}(0) \\ \mathrm{P}_{12}^T(0) & \mathrm{P}_{22}(0) \end{bmatrix}$, $\Sigma_{\tilde{x}}(0) = \begin{bmatrix} E[x(0)x^T(0)] & E[x(0)x_r^T(0)] \\ E[x_r(0)x^T(0)] & E[x_r(0)x_r^T(0)] \end{bmatrix}$,

$$\tilde{\mathrm{B}}_x \tilde{\mathrm{B}}_x^T = \begin{bmatrix} \mathrm{B}_x \\ \mathrm{B}_{x_r} \end{bmatrix} \begin{bmatrix} \mathrm{B}_x^T & \mathrm{B}_{x_r}^T \end{bmatrix} = \begin{bmatrix} \mathrm{B}_x \mathrm{B}_x^T & \mathrm{B}_x \mathrm{B}_{x_r}^T \\ \mathrm{B}_{x_r} \mathrm{B}_x^T & \mathrm{B}_{x_r} \mathrm{B}_{x_r}^T \end{bmatrix},$$

$$\mathrm{P}(t) = \begin{bmatrix} \mathrm{P}_{11}(t) & \mathrm{P}_{12}(t) \\ \mathrm{P}_{12}^T(t) & \mathrm{P}_{22}(t) \end{bmatrix}$$

$$\tilde{S}_x = E\left\{ \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} \begin{bmatrix} x^T(s) & x_r^T(s) \end{bmatrix} \right\} = \begin{bmatrix} E[x(t)x^T(s)] & E[x(t)x_r^T(s)] \\ E[x_r(t)x^T(s)] & E[x_r(t)x_r^T(s)] \end{bmatrix}$$

3) The correlation function, correlation matrix as well as the mean, vector of $x(t)$, are found to be:

$$R_{\tilde{x}\tilde{x}}(t_1,t_2) = E[\tilde{x}(t_1)\tilde{x}^T(t_2)]$$

$$\overset{\Delta}{\equiv} E\left\{ \begin{bmatrix} x(t_1) \\ x_r(t_1) \end{bmatrix} \begin{bmatrix} x^T(t_2) & x_r^T(t_2) \end{bmatrix} \right\}$$

$$= \begin{bmatrix} E[x(t_1)x^T(t_2)] & E[x(t_1)x_r^T(t_2)] \\ E[x_r^T(t_1)x(t_2)] & E[x_r(t_1)x_r^T(t_2)] \end{bmatrix}$$

$$\overset{\Delta}{\equiv} \begin{bmatrix} R_{xx}(t_1,t_2) & R_{xx_r}(t_1,t_2) \\ R_{x_r x}(t_1,t_2) & R_{x_r x_r}(t_1,t_2) \end{bmatrix}$$

$$R_{xx}(t_1,t_2) = \Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{11}^T(t_2;K) + \Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{12}^T(t_2;K) +$$

$$\int_0^{\min(t_1,t_2)} \Phi_{11}(t_1,s;K)\mathrm{B}_x S_x \mathrm{B}_x^T \Phi_{11}^T(t_2,s;K)ds +$$

$$\int_0^{\min(t_1,t_2)} \Phi_{12}(t_1,s;K)\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T \Phi_{12}^T(t_2,s;K)ds$$

$$R_{xx_r}(t_1,t_2) = \Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K) + \Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K) +$$

$$\int_0^{\min(t_1,t_2)} \Phi_{11}(t_1,s;K)\mathrm{B}_x S_x \mathrm{B}_x^T \Phi_{21}^T(t_2,s;K)ds +$$

$$\int_0^{\min(t_1,t_2)} \Phi_{12}(t_1,s;K)\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T \Phi_{22}^T(t_2,s;K)ds$$

$$R_{x_r x_r}(t_1,t_2) = \Phi_{21}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K) + \Phi_{22}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K) +$$

$$\int_0^{\min(t_1,t_2)} \Phi_{21}(t_1,s;K)\mathrm{B}_x S_x \mathrm{B}_x^T \Phi_{21}^T(t_2,s;K)ds +$$

$$\int_0^{\min(t_1,t_2)} \Phi_{22}(t_1,s;K)\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T \Phi_{22}^T(t_2,s;K)ds$$

$$\Sigma_{\tilde{x}}(t,t) = R_{\tilde{x}}(t,t) = E[\tilde{x}(t)\tilde{x}^T(t)]$$

$$\stackrel{\Delta}{\equiv} E\left\{ \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} \begin{bmatrix} x^T(t) & x_r^T(t) \end{bmatrix} \right\}$$

$$= \begin{bmatrix} E[x(t)x^T(t)] & E[x(t)x_r^T(t)] \\ E[x_r^T(t)x(t)] & E[x_r(t)x_r^T(t)] \end{bmatrix} \stackrel{\Delta}{\equiv} \begin{bmatrix} \Sigma_{xx}(t) & \Sigma_{xx_r}(t) \\ \Sigma_{x_r x}(t) & \Sigma_{x_r x_r}(t) \end{bmatrix}$$

$$\Sigma_{xx}(t) = \Phi_{11}(t;K)\Sigma_x(0)\Phi_{11}^T(t;K) + \Phi_{12}(t;K)\Sigma_{x_r}(0)\Phi_{12}^T(t;K) +$$

$$\int_0^t \Phi_{11}(t,s;K)\mathrm{B}_x S_x \mathrm{B}_x^T \Phi_{11}^T(t,s;K)ds +$$

$$\int_0^t \Phi_{12}(t,s;K)\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T \Phi_{12}^T(t,s;K)ds$$

$$\Sigma_{xx_r}(t_1,t_2) = \Phi_{11}(t_1;K)\Sigma_x(0)\Phi_{21}^T(t_2;K) + \Phi_{12}(t_1;K)\Sigma_{x_r}(0)\Phi_{22}^T(t_2;K) +$$

$$\int_0^t \Phi_{11}(t_1,s;K)\mathrm{B}_x S_x \mathrm{B}_x^T \Phi_{21}^T(t_2,s;K)ds +$$

$$\int_0^t \Phi_{12}(t_1,s;K)\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T \Phi_{22}^T(t_2,s;K)ds$$

$$\Sigma_{x_r x_r}(t) = \Phi_{21}(t;K)\Sigma_x(0)\Phi_{21}^T(t;K) + \Phi_{22}(t;K)\Sigma_{x_r}(0)\Phi_{22}^T(t;K) +$$

$$\int_0^t \Phi_{21}(t,s;K)\mathrm{B}_x S_x \mathrm{B}_x^T \Phi_{21}^T(t,s;K)ds +$$

$$\int_0^t \Phi_{22}(t,s;K)\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r}^T \Phi_{22}^T(t,s;K)ds$$

And

$$E[x(t)] = \Phi_{11}(t;K)E[x(0)] + \Phi_{12}(t;K)E[x_r(0)] +$$

$$\int_0^t \Phi_{11}(t,t;K)\mathrm{B}_x E[x(t)] + \Phi_{12}(t,t;K)\mathrm{B}_{x_r} E[x_r(t)]dt$$

$$= 0$$

$$E[x_r(t)] = \Phi_{21}(t;K)E[x(0)] + \Phi_{22}(t;K)E[x_r(0)] +$$
$$\int_0^t \left\{ \Phi_{21}(t,t;K)\mathrm{B}_x E[x(t)] + \Phi_{22}(t,t;K)\mathrm{B}_{x_r} E[x_r(t)] \right\} dt$$
$$= 0$$

Proof

Since $(\mathrm{A}_r, \mathrm{B}_{u_r})$ is controllable matrix , so it is always possible to find K_r such that $(\mathrm{A}_r - \mathrm{B}_{u_r} K_r)$ is asymptotically stable matrix using pole placement method.

The feedback gain K_r is obtained as follows:

Consider the single variable time invariant equation:

$$\dot{x}_r(t) = \mathrm{A}_r x_r(t) + \mathrm{B}_{u_r} u_r(t)$$

where $\mathrm{A} \in R^{n \times n}$ and $\mathrm{B} \in R^{n \times 1}$ and linear state feedback $u(t) = -\mathrm{K}_r x_r(t)$ where $\mathrm{K}_r \in R^{1 \times n}$

Step (1)

Check the controllability condition for the system. If the system is completely state controllable, i.e.,$(\mathrm{A}_r, \mathrm{B}_{u_r})$ is controllable, and then use the following steps.

Step (2)

From the characteristic polynomial for matrix A,

$$|\lambda \mathrm{I} - \mathrm{A}| = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n$$

and then determine the values of $a_1, a_2, \ldots, a_n$

Step (3)

Determine the transformation matrix T that transforms the system state equation into the controllable canonical form (If the given system equation is already in the controllable canonical form, then $\mathrm{T}=\mathrm{I}$). It is not necessary to write the state equation in the controllable canonical form. All we need here is to find the transformation matrix T which is given by

$$\mathrm{T}=\mathrm{MW} \qquad (2.79)$$

where M is the controllability matrix

$$\mathrm{M}=[\mathrm{B}_r \vdots \mathrm{A}_r\mathrm{B}_r \vdots \mathrm{A}_r^2\mathrm{B}_r \vdots \ \ldots \ \vdots \mathrm{A}_r^{n-1}\mathrm{B}_r] \qquad (2.80)$$

where W is defined by

$$\mathrm{W}=\begin{pmatrix} a_{n-1} & a_{n-2} & \ldots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \ldots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix} \qquad (2.81)$$

where the a_i's are coefficients of the characteristic polynomial of step (2).

Step (4)

Using the desired eigenvalues (desired closed-loop poles), write the desired characteristic polynomial as:

$$(\lambda-\mu_1)(\lambda-\mu_2)\ldots(\lambda-\mu_n) \equiv \lambda^n+\alpha_1\lambda^{n-1}+\ldots+\alpha_{n-1}\lambda+\alpha_n$$

where the values of $\alpha_1,\alpha_2,\ldots,\alpha_n$ can be determined.

Step (5)

The required state feedback gain matrix K_r can be determined from this equation

$$\mathrm{K}_r=[\alpha_n-a_n \vdots \alpha_{n-1}-a_{n-1} \vdots \ \ldots \ \vdots \alpha_2-a_2 \vdots \alpha_1-a_1]\mathrm{T}^{-1} \qquad (1.82)$$

Set $(A_r - B_{u_r} K_r) = A_0$ the system (2.78), is then become as:

$$\dot{x}(t) = Ax(t) + B_u u(t) + B_x x(t)$$

$$\dot{x}_r(t) = A_0 x_r(t) + B_{x_r} x_r(t)$$

Then we define the augmented state between state space $x(t)$ and target state space $x_r(t)$:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_r(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} + \begin{bmatrix} B_u \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} B_x(t)x(t) \\ B_{x_r}(t)x_r(t) \end{bmatrix} \tag{2.83}$$

Since

$(A_r - B_{u_r} K_r) = A_0$ is found to be a stabile matrix and the vector $\begin{bmatrix} B_u \\ B_{u_r} \end{bmatrix}$ becomes $\begin{bmatrix} B_u \\ 0 \end{bmatrix}$ and from comparison point of view between the augmented system (2.78), and the augmented system (2.83), of the lemma (2.2), we have that both are identical and the same proof and results may be adapted. And it can be found by partitioning the Riccati equation $P_{11}(t)$, $P_{12}(t)$, and $P_{22}(t)$ are the solution of the matrix differential equations:

$$-\dot{P}_{11}(t) = P_{11}(t)A + A^T P_{11}(t) + D^T(t)Q^* D(t) -$$
$$P_{11}(t)B_u R^{-1} B_u^T P_{11}(t);$$

$$P_{11}(t_f) = H_{11}$$

$$-\dot{P}_{12}(t) = P_{12}(t)(A_r - B_{u_r} K_r) + (A_r - B_{u_r} K_r)^T P_{12}(t) - D^T(t)Q^* D_r(t) -$$
$$P_{11}(t)B_u R^{-1} B_u^T P_{12}(t)$$

$$P_{12}(t_f) = H_{12}$$

$$-\dot{P}_{22}(t) = P_{22}(t)(A_r - B_{u_r} K_r) + (A_r - B_{u_r} K_r)^T P_{22}(t) + D_r^T(t)Q^* D_r(t) -$$
$$P_{12}^T(t)B_u R^{-1} B_u^T P_{12}(t);$$

$$P_{22}(t_f) = H_{22}$$

And

$$K = [K_1, K_2] = [R^{-1} B_u^T P_{11}(t), -R^{-1} B_u^T P_{12}(t)]$$

Can be obtained easily from the lemma (2.1), by using that $A_r = A_0 = (A_r - B_{u_r} K_r)$, and that completes the proof.

Chapter Three

Some Mathematical Illustrations

The mathematical applications using result of chapter two have been adapted in this chapter. Some numerical examples ranking from simple to harder, have been presented and discussed. The solvability of the presented examples are based on the theoretical result of theorems (2.1), and lemmas (2.2), (2.3), of chapter two. Step-by-step computational have been used to make our illustrations easily and understandable.

Illustration (3.1) ***(Angular Velocity Tracking System)[20]***

Suppose we wish that the angular velocity, which is controlled variable $z(t)$, follows as accurately as possible a reference variable $z_r(t)$, which may be described as exponentially correlated noise with time constant q and value s .

The system state differential equation is:

$$\dot{x}(t) = -a\,x(t) + b\,u(t) \tag{3.1}$$

where a, b are constants, and

Suppose that the controlled variable:

$$z(t) = x(t) \tag{3.2}$$

follows as accurately as possible a reference variable:

$$z_r(t) = x_r(t) \tag{3.3}$$

where $z_r(t)$ is the solution of

$$\dot{x}_r(t) = -\frac{1}{q}x_r(t) + x_r(t) \tag{3.4}$$

The white noise $x_r(t)$ has intensity $S_{x_r} = 2s^2/q$, where q is a time constant, and s is rms value.

The augmented state differential equation is given by:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_r(t) \end{bmatrix} = \begin{bmatrix} -a & 0 \\ 0 & \frac{-1}{q} \end{bmatrix} \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_r(t) \tag{3.5}$$

With $\tilde{x}(t) = col[x(t), x_r(t)]$. For the optimization cost function:

$$J = \frac{1}{2} E \left\{ \int_0^{\infty} \tilde{z}^T(t) Q^* \tilde{z}(t) + u^T(t)\, r\, u(t) dt \right\} \tag{3.6}$$

Where $\tilde{z}(t) = [z(t), -z_r(t)]$, (3.6) become:

$$J = \frac{1}{2} E \left\{ \int_0^{\infty} [z(t) - z_r(t)]^T Q^* [z(t) - z_r(t)] + r u^2(t) dt \right\} \tag{3.7}$$

where r is a suitable weighting factor.

Remarks (3.1)

The following assumptions are assumed for this illustration:

I. The initial conditions are zero-mean random vectors with the following correlation matrices (The second-order moment matrix):

$E[x(0)x^T(0)] = \Sigma_x(0)$, and $E\{x(0)\} = \mathbf{0}$

$E[x_r(0)x_r^T(0)] = \Sigma_{x_r}(0)$, and $E\{x_r(0)\} = \mathbf{0}$

The pair $(\mathrm{A}, \mathrm{B}_u)$ of a non-linear dynamical control system is controllable.

II. A_r is a stable matrix.

III. $E[x(0)x_r^T(0)] = \mathbf{0}$

IV. $E[x_r(0)x^T(0)] = \mathbf{0}$

V. $E[x(0)x_r^T(t)] = \mathbf{0}$

VI. $E[x_r(0)x_r^T(t)] = \mathbf{0}$

VII. $E[x_r(t)x^T(0)] = \mathbf{0}$

VIII. $E[x_r(t)x_r^T(0)] = \mathbf{0}$

To follow our suggested procedure, see the steps of the main theorem (2.1), one can do as follows:

Step (1)

Substituting (3.2), (3.3) into (3.7), one can get:

$$J = \frac{1}{2} E \left\{ \int_0^{\infty} [x(t) - x_r(t)]^T Q^* [x(t) - x_r(t)] + r\, u^2(t) dt \right\}$$
$$= \frac{1}{2} E \left\{ \int_0^{\infty} \left[\begin{bmatrix} x^T(t) & x_r^T(t) \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} Q^* \begin{bmatrix} 1, & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} + r u^2(t) dt \right] \right\} \tag{3.8}$$

Let

$$Q^* = I \,,$$
$$Q(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1, & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.9}$$

From (3.9), the cost function (3.8), can be written as:

$$J = \frac{1}{2} E \left\{ \int_0^{\infty} \left\{ \tilde{x}^T(t) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tilde{x}(t) + r\, u^2(t) dt \right\} \right\}$$

It follows from theorem (2.1), that the optimal tracking law is given by:

Step (2)

Set $u(t) = -K\tilde{x}(t)$

$$= -(K_1, -K_2) \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} \tag{3.10}$$

$\Rightarrow$

$$u(t) = -K_1 x(t) + K_2 x_r(t)$$

Step (3)

Find the solution of the following algebraic Riccati equation:

To find the steady-state value of the feedback gain, we must find the steady-state value of P_{11} that we use (2.51), one can get:

$$0 = P_{11}(t)A + A^T P_{11}(t) + D^T(t)Q^* D(t) -$$
$$P_{11}^T(t)B_u R^{-1} B_u^T P_{11}(t)$$

$\Rightarrow$

$$-\frac{b^2}{r} P_{11}^2 - 2aP_{11} + 1 = 0$$

$\Rightarrow$

$$P_{11} = \frac{r}{b^2}\left(-a + \sqrt{a^2 + \frac{b^2}{r}}\right) \tag{3.11}$$

By using (2.52), to solve P_{12}:

$$0 = P_{12}(t)A_r + A^T P_{12}(t) - D^T(t)Q^* D_r(t) - P_{11}(t)B_u R^{-1} B_u^T P_{12}(t)$$

$$\frac{-1}{q} P_{12} - aP_{12} - \frac{b^2}{r}\left(\frac{r}{b^2}\left(-a + \sqrt{a^2 + \frac{b^2}{r}}\right)\right)P_{12} - 1 = 0$$

$\Rightarrow$

$$-1 - \sqrt{a^2 + \frac{b^2}{r}}\, P_{12} - \frac{1}{q} P_{12} = 0$$

$$P_{12} = \frac{-1}{\frac{1}{q} + \sqrt{a^2 + \frac{b^2}{r}}} \tag{3.12}$$

Finally, solution of (2.53), for P_{22} gives:

$$0 = P_{22}(t)A_r + A_r^T P_{22}(t) + D_r^T(t)Q^* D_r(t) - P_{12}^T(t)B_u R^{-1} B_u^T P_{12}(t)$$

$$\frac{-1}{q}\mathrm{P}_{22}-\frac{1}{q}\mathrm{P}_{22}+1-\frac{b^2}{r}\mathrm{P}_{12}^2=0$$

$$-\frac{2}{q}\mathrm{P}_{22}+1-\frac{b^2}{r}\left(\frac{-1}{\frac{1}{q}+\sqrt{a^2+\frac{b^2}{r}}}\right)^2=0$$

$\Rightarrow$

$$\mathrm{P}_{22}=\frac{q}{2}\frac{a^2+\frac{1}{q^2}+\frac{2}{q}\sqrt{a^2+\frac{b^2}{r}}}{\left(\frac{1}{q}+\sqrt{a^2+\frac{b^2}{r}}\right)^2} \qquad (3.13)$$

Step (4)

To find the steady-state value of the feedback gain $[K_1,K_2]$, substituting (3.11) into (2.54), we obtained:

$$K_1=\frac{1}{b}(-a+\sqrt{a^2+\frac{b^2}{r}}) \qquad (3.14)$$

And substituting (3.12) into (2.55), we obtained:

$$K_2=\frac{\frac{b}{r}}{\frac{1}{q}+\sqrt{a^2+\frac{b^2}{r}}} \qquad (3.15)$$

Step (5)

Design the feedback system as follows:

Substituting (3.10), into (3.5), one can get:

$$\begin{bmatrix}\dot{x}(t)\\ \dot{x}_r(t)\end{bmatrix}=\begin{bmatrix}-a & 0\\ 0 & -\frac{1}{q}\end{bmatrix}\begin{bmatrix}x(t)\\ x_r(t)\end{bmatrix}+\begin{bmatrix}b\\ 0\end{bmatrix}(-K_1,K_2)\begin{bmatrix}x(t)\\ x_r(t)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}x_r(t)$$

$\Rightarrow$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_r(t) \end{bmatrix} = \begin{bmatrix} -a - bK_1 & bK_2 \\ 0 & -\dfrac{1}{q} \end{bmatrix} \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_r(t) \tag{3.16}$$

Step (6)

Let the following numerical values have been adapted and as given in, (see [20]):

$a = 0.5 s^{-1}$

$b = 150\, rad/(V s^2)$

$q = 1\ s$

$s = 30\, rad/s$

$r = 1000 (rad)^2/(V^2 s^2)$

Substituting these values into (3.11), (3.12), (3.13), (3.14), and (3.15), we have:

$\overline{P} = \begin{bmatrix} 0.1897 & -0.1733 \\ -0.1733 & 0.1621 \end{bmatrix}$, where $\overline{P}_{ij}$, $i, j = 1,2$, denoted the elements of $\overline{P}$.

$K_1 = 0.02846$

$K_2 = 0.02600$

Then (3.16), can be determined as follows:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_r(t) \end{bmatrix} = \begin{bmatrix} -4.769 & 3.9 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_r(t) \tag{3.17}$$

Step (7)

In order to find the analytical solution to following system, we have done as follows:

The solution of (3.17), can be found:

The eigenvalues of matrix, $A = \begin{bmatrix} -4.769 & 3.9 \\ 0 & -1 \end{bmatrix}$ are:

$$|l I - A| = \begin{vmatrix} (l + 4.769) & -3.9 \\ 0 & (l + 1) \end{vmatrix} = 0$$

$$\overset{\Delta}{\equiv} l^2 + 5.769 l + 4.769 = 0$$

$\Rightarrow$

$$l_1 = -1$$
$$l_2 = -4.769$$

The eigenvectors of matrix A are:

$$V = [u_1 \quad u_2 \quad \mathbf{L} \quad u_n]$$
$$Au_i = l_i u_i \,, i = 1, 2, ..., n$$
$$\begin{bmatrix} -4.769 & 3.9 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} -4.769 u_{11} \\ -4.769 u_{12} \end{bmatrix}$$

$$Au_2 = l_2 u_2$$
$$\begin{bmatrix} -4.769 & 3.9 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} -u_{21} \\ -u_{22} \end{bmatrix}$$

$\Rightarrow$

$$V = \begin{bmatrix} 1 & 0.7191 \\ 0 & 0.6949 \end{bmatrix} \tag{3.18}$$

And

$$V^{-1} = \begin{bmatrix} 1 & -1.0348 \\ 0 & 1.4390 \end{bmatrix} \tag{3.19}$$

Since the eigenvalues $l_1, l_2, ..., l_n$ of the matrix A are distinct, then , we have:

$$e^{At} = V e^{(V^{-1}AV)t} V^{-1} = V e^{Zt} V^{-1} \tag{3.20}$$

where

$$Z = V^{-1}AV$$

$$= \begin{bmatrix} 1 & -1.0348 \\ 0 & 1.4390 \end{bmatrix} \begin{bmatrix} -4.769 & 3.9 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0.7191 \\ 0 & 0.6949 \end{bmatrix}$$

$$Z = \begin{bmatrix} -4.769 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.21}$$

From (3.21), equation (3.20), become:

$$e^{At} = Ve^{Zt}V^{-1} \rightarrow$$

$$= \begin{bmatrix} 1 & 0.7191 \\ 0 & 0.6949 \end{bmatrix} \begin{bmatrix} e^{-4.769t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & -1.0348 \\ 0 & 1.4390 \end{bmatrix}$$

$\Rightarrow$

$$e^{At} = \begin{bmatrix} e^{-4.769t} & -1.0348e^{-4.769t} + 1.0347e^{-t} \\ 0 & e^{-t} \end{bmatrix} \tag{3.22}$$

Then (3.17), become:

$$\tilde{x}(t) = e^{At}\tilde{x}(0) + e^{At}\int_0^t e^{-At}\tilde{B}_x\tilde{x}(t)\,dt \tag{3.23}$$

Substituting (3.22), into (3.23), we have:

$$\tilde{x}(t) = \begin{bmatrix} e^{-4.769t} & -1.0348e^{-4.769t} + 1.0347e^{-t} \\ 0 & e^{-t} \end{bmatrix}\tilde{x}(0) +$$

$$\int_0^t \begin{bmatrix} e^{-4.769(t-t)} & -1.0348e^{-4.769(t-t)} + 1.0347e^{-(t-t)} \\ 0 & e^{-(t-t)} \end{bmatrix}\tilde{B}_x\tilde{x}(t)\,dt$$

$\Rightarrow$

$$\begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} = \begin{bmatrix} e^{-4.769t} & -1.0348e^{-4.769t}+1.0347e^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} x(0) \\ x_r(0) \end{bmatrix} +$$

$$\int_0^t \begin{bmatrix} e^{-4.769(t-t)} & -1.0348e^{-4.769(t-t)}+1.0347e^{-(t-t)} \\ 0 & e^{-(t-t)} \end{bmatrix} \begin{bmatrix} 0 \\ x_r(t) \end{bmatrix} dt$$

$$x(t) = e^{-4.769t}x(0) - 1.0348\ e^{-4.769t}x_r(0) +$$

$$1.0347\,e^{-t}x_r(0) + \int_0^t \left(-1.0348\ e^{-4.769(t-t)}x_r(t) + 1.0347e^{-(t-t)}x_r(t)\right)dt \tag{3.24}$$

$$x_r(t) = e^{-t}x_r(0) + \int_0^t e^{-(t-t)}x_r(t)\,dt \tag{3.25}$$

Step (8)

Find the statistical properties of this solution and as follows:

From (3.24), (3.25), the mean can be compute :

Step (8.1)

On using the assumption (I, II) of remarks (3.1), we have:

$$E[x(t)] = e^{-4.769t}E[x(0)] - 1.0348\ \ e^{-4.769t}E[x_r(0)] + 1.0347e^{-t}E[x_r(0)] +$$

$$\int_0^t \left(-1.0348\ e^{-4.769(t-t)}E[x_r(t)] + 1.0347e^{-(t-t)}E[x_r(t)]\right)dt$$

$$= 0$$

$$E[x_r(t)] = e^{-t}E[x_r(0)] + \int_0^t e^{-(t-t)}E[x_r(t)]dt$$

$$= 0$$

Step (8.2)

To compute the correlation function as discussed in subsection (1.6.3), using the assumptions of remark (3.1), we have then:

$$R_{\tilde{x}\tilde{x}}(t_1,t_2)=E[\tilde{x}(t_1)\tilde{x}^T(t_2)]$$

$$\overset{\Delta}{\equiv} E\left\{ \begin{bmatrix} x(t_1) \\ x_r(t_1) \end{bmatrix} \begin{bmatrix} x^T(t_2) & x_r^T(t_2) \end{bmatrix} \right\}$$

$$=\begin{bmatrix} E[x(t_1)x^T(t_2)] & E[x(t_1)x_r^T(t_2)] \\ E[x_r^T(t_1)x(t_2)] & E[x_r(t_1)x_r^T(t_2)] \end{bmatrix}$$

$$\overset{\Delta}{\equiv} \begin{bmatrix} R_{xx}(t_1,t_2) & R_{xx_r}(t_1,t_2) \\ R_{x_r x}(t_1,t_2) & R_{x_r x_r}(t_1,t_2) \end{bmatrix}$$

Use (3.24), we get:

$$R_{xx}(t_1,t_2)=E[x(t_1)x^T(t_2)]$$

$$\begin{aligned}
&= e^{-4.769t_1}\Sigma_x(0)e^{-4.769t_2} + \\
&\quad 1.0708\ e^{-4.769t_1}\Sigma_{x_r}(0)\ e^{-4.769t_2} - 1.0707\ e^{-4.769t_1}\Sigma_{x_r}(0)\ e^{-t_2} - \\
&\quad 1.0707\ e^{-t_1}\Sigma_{x_r}(0)\ e^{-4.769t_2} + 1.0706\ e^{-t_1}\ \Sigma_{x_r}(0)\ e^{-t_2} + \\
&\quad \int_0^{\min(t_1,t_2)} \Big\{ 1.0708\ e^{-4.769(t_1+t_2)}(1800)\ e^{9.538s} - \\
&\qquad 1.0707\ e^{-4.769t_1}\ e^{-t_2}(1800)\ e^{5.769s} - \\
&\qquad 1.0707\ e^{-t_1}(1800)\ e^{-4.769t_2}\ e^{5.796s} + \\
&\qquad 1.0706\ e^{-(t_1+t_2)}(1800)\ e^{2s} \Big\} ds
\end{aligned}$$

$$R_{xx}(t_1,t_2) = \Sigma_x(0)\ e^{-4.769(t_1+t_2)} +$$
$$1.0708\,\Sigma_{x_r}(0)\ e^{-4.769(t_1+t_2)} -$$
$$1.0707\ e^{-4.769t_1}\Sigma_{x_r}(0)\ e^{-t_2} -$$
$$1.0707\ e^{-t_1}\Sigma_{x_r}(0)\ e^{-4.769t_2} +$$
$$1.0706\ e^{-4.769t_1}\ e^{-t_1}\Sigma_{x_r}(0)\ e^{-t_2} +$$
$$202.09\,\Sigma_{x_r}(0)\ e^{-4.769(t_1+t_2)}\ e^{9.538\min(t_1,t_2)} -$$
$$334.07\ e^{-4.769t_1}\ e^{-t_2}\ e^{5.769\min(t_1,t_2)} -$$
$$334.07\ e^{-t_1}\,\Sigma_{x_r}(0)\ e^{-4.769t_2}\ e^{5.769\min(t_1,t_2)} +$$
$$963\ e^{-(t_1+t_2)}\ e^{2\min(t_1,t_2)}$$

For arbitrary given $\Sigma_x(0)$, $\Sigma_{x_r}(0)$.

From (3.24), and (3.25), we have:

$$R_{xx_r}(t_1,t_2) = E[x(t_1)x_r^T(t_2)] = R_{x_r x}(t_1,t_2) = E[x_r(t_1)x^T(t_2)]$$
$$= -1.0348\,e^{-4.769t_1}\Sigma_{x_r}(0)\,e^{-t_2} + 1.0347\,e^{-t_1}\Sigma_{x_r}(0)\,e^{-t_2} +$$
$$\int_0^{\min(t_1,t_2)} \left\{ -1.0348(1800)\,e^{-4.769(t_1-s)}\ e^{-(t_2-s)} + 1.0347\,e^{-(t_1-s)}(1800)\,e^{-(t_2-s)} \right\} ds$$

$$R_{xx_r}(t_1,t_2) = E[x(t_1)x_r^T(t_2)] = R_{x_r x}(t_1,t_2) = E[x_r(t_1)x^T(t_2)]$$
$$= -1.0348\,e^{-4.769t_1}\Sigma_{x_r}(0)\,e^{-t_2} + 1.0347\,e^{-(t_1+t_2)}\Sigma_{x_r} -$$
$$322.87\,e^{-4.769t_1}e^{-t_2}\ e^{5.769\min(t_1,t_2)} + 931.23\,e^{-(t_1+t_2)}\ e^{2\min(t_1,t_2)}$$

For arbitrary given $\Sigma_{x_r}(0)$.

And similarly from (3.25), to find

$$R_{x_r x_r}(t_1,t_2) = [x_r(t_1)x_r^T(t_2)]$$
$$= e^{-t_1}\Sigma_{x_r}(0)\, e^{-t_2} + \int_0^{\min(t_1,t_2)} 1800\ e^{-(t_1-s)}e^{-(t_2-s)}ds$$
$$= e^{-(t_1+t_2)}\Sigma_{x_r}(0) + 900\ e^{-(t_1+t_2)}\ e^{2\min(t_1,t_2)}$$

For arbitrary given $\Sigma_{x_r}(0)$.

Step (8.3)

The correlation matrix can be obtain from correlation function (see remark (1.11):

$$R_{xx}(t,t) = \Sigma_{xx}(t)$$
$$= \Sigma_x(0)\, e^{-9.538t} + 1.0708\Sigma_{x_r}(0)\, e^{-9.538t} - 1.0707\, e^{-5.769t}\Sigma_{x_r}(0) -$$
$$1.0707\, e^{-5.769t}\Sigma_{x_r}(0) + 1.0706\, e^{-2t}\Sigma_{x_r}(0) + 202.09 - 334.07 - 334.07 + 963$$

For arbitrary given $\Sigma_x(0),\ \Sigma_{x_r}(0)$.

$$R_{xx_r}(t_1,t_2) = R_{x_r x}(t_1,t_2) = \Sigma_{xx_r}(t) = \Sigma_{x_r x}(t)$$
$$= -1.0348\ e^{-5.769t}\Sigma_{x_r}(0) + 1.0347\ e^{-2t}\Sigma_{x_r} - 322.87 + 931.23$$

For arbitrary given $\Sigma_{x_r}(0)$.

$$\Sigma_{x_r x_r}(t) = e^{-2t}\Sigma_{x_r}(0) + 900\, e^{-2t}\ e^{2t}$$
$$= 900 + e^{-2t}\Sigma_{x_r}(0)$$

For arbitrary given $\Sigma_{x_r}(0)$.

Step (8.4)

The correlation matrix are depending on time-varying and the steady-state can be obtained by let $t \to \infty$,

$$\Sigma_{xx}(\infty) = \lim_{t\to\infty}[\ \Sigma_x(0)\ e^{-9.538t} + 1.0708\Sigma_{x_r}(0)\, e^{-9.538t} - 1.0707\ e^{-5.769t}\Sigma_{x_r}(0) - 1.0707\ e^{-5.769t}\Sigma_{x_r}(0) + 1.0706\ e^{-2t}\Sigma_{x_r}(0) + 202.09 - 334.07 - 334.07 + 963\ ]$$

$$= 202.09 - 334.07 - 334.07 + 963$$
$$= 496.95$$

$$\Sigma_{xx_r}(\infty) = \Sigma_{x_r x}(\infty)$$
$$= \lim_{t\to\infty}[\ -1.0348\ e^{-5.769t}\Sigma_{x_r}(0) + 1.0347\ e^{-2t}\Sigma_{x_r} - 322.87 + 931.23\ ]$$
$$= 608.36$$

$$\Sigma_{x_r x_r}(\infty) = \lim_{t\to\infty} e^{-2t}\Sigma_{x_r}(0) + 900$$
$$= 900$$

Hence

$$\begin{bmatrix} \Sigma_{xx}(\infty) & \Sigma_{xx_r}(\infty) \\ \Sigma_{x_r x}(\infty) & \Sigma_{x_r x_r}(\infty) \end{bmatrix} = \begin{bmatrix} 496.95 & 608.36 \\ 608.36 & 900 \end{bmatrix} \tag{3.26a}$$

For comparison point of view we have, consider the steady-state correlation matrix of the state $\tilde{x}(t)$ of the closed-loop augmented system (3.17), can be found by using ***Lyapunov equation***, from (2.61), we obtain:

$$\begin{bmatrix} -4.769 & 3.9 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \Sigma_{x_{11}}(\infty) & \Sigma_{x_{12}}(\infty) \\ \Sigma_{x_{21}}(\infty) & \Sigma_{x_{22}}(\infty) \end{bmatrix} +$$

$$\begin{bmatrix} \Sigma_{x_{11}}(\infty) & \Sigma_{x_{12}}(\infty) \\ \Sigma_{x_{21}}(\infty) & \Sigma_{x_{22}}(\infty) \end{bmatrix}\begin{bmatrix} -4.769 & 0 \\ 3.9 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1800 \end{bmatrix} = 0$$

On using simple calculating, one can found:

$$\begin{bmatrix} \Sigma_{x_{11}}(\infty) & \Sigma_{x_{12}}(\infty) \\ \Sigma_{x_{21}}(\infty) & \Sigma_{x_{22}}(\infty) \end{bmatrix} = \begin{bmatrix} 497.5 & 608.4 \\ 608.4 & 900. \end{bmatrix} \tag{3.26b}$$

As one can see from both results of steady-state (3.36a), and (3.36b), they are identical and this will give a good justification for our work, even our results are more general since it represents the transient and steady-state behavior.

Step (9)

Computing the performance index of the problem and using concluding remarks (2.4) points (4) and (5), it follows that:

$$J_{SR} = tr\left\{ \mathrm{P}(\infty)\tilde{\mathrm{B}}_x \tilde{S}_x \tilde{\mathrm{B}}_x^T \right\} \tag{3.27}$$

With intensity $S_{x_r} = \begin{bmatrix} 0 & 0 \\ 0 & 1800 \end{bmatrix}$, $\mathrm{P}(\infty)$ is found in (see step 3 to step 6) , we get:

$$J_{SR} = tr\left(\begin{bmatrix} 0.1897 & -0.1733 \\ -0.1733 & 0.1621 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1800 \end{bmatrix} \right) = 291.8\,rad^2/s^2 \tag{3.28}$$

Illustration (3.2) ***(Optimal Stochastic Control)***

Consider the system differential equation is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} x_1(t) \\ 0 \end{bmatrix} \tag{3.29}$$

The white noise $\tilde{x}(t)$ and its intensity matrix, has the following:

$$\mathrm{B}_x S_x \mathrm{B}_x = \begin{bmatrix} 0.025 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.30}$$

Suppose that the controlled variable:

$$\begin{gathered} \tilde{z}(t) = \tilde{D}(t)\tilde{x}(t) \\ \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 1, & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{gathered} \tag{3.31}$$

And let the system:

$$\begin{bmatrix}\dot{x}_{r_1}(t)\\ \dot{x}_{r_2}(t)\end{bmatrix}=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}\begin{bmatrix}\dot{x}_{r_1}(t)\\ \dot{x}_{r_2}(t)\end{bmatrix}+\begin{bmatrix}2\\ 0\end{bmatrix}u_r(t)+\begin{bmatrix}x_{r_1}(t)\\ 0\end{bmatrix} \quad (3.32)$$

The white noise $\tilde{x}(t)$ and its intensity matrix, has the following:

$$\mathrm{B}_{x_r} S_{x_r} \mathrm{B}_{x_r} = \begin{bmatrix}0.005 & 0\\ 0 & 0\end{bmatrix} \quad (3.33)$$

follows as accurately as possible a reference variable:

$$\tilde{z}_r(t)=\tilde{D}_r(t)\tilde{x}_r(t)$$

$$\begin{bmatrix}z_{r_1}(t)\\ z_{r_2}(t)\end{bmatrix}=\begin{bmatrix}4, & -2\end{bmatrix}\begin{bmatrix}x_{r_1}(t)\\ x_{r_2}(t)\end{bmatrix} \quad (3.34)$$

The aim is to make system (3.29) follows the system (3.32), by using a suitable controller, and one can do this aim as follows:

The augmented state differential equation is given by

$$\tilde{x}(t)=col[x_1(t),x_2(t),x_{r_1}(t),x_{r_2}(t)].$$

For the optimization cost function:

$$J=\frac{1}{2}E\left\{\int_0^{\infty}\tilde{\tilde{z}}(t)^T Q^{*}\tilde{\tilde{z}}(t)+u^T(t)\,r\,u(t)dt\right\} \quad (3.35)$$

Where $\tilde{\tilde{z}}(t)=[\tilde{z}(t),\tilde{z}_r(t)]$, r is a suitable weighting factor.

Substituting (3.31), (3.34), into (3.35), we get:

$$J=\frac{1}{2}E\left\{\int_0^{\infty}[\tilde{z}(t)-\tilde{z}_r(t)]^T Q^{*}[\tilde{z}(t)-\tilde{z}_r(t)]+r\,u^2(t)dt\right\}$$

$$J = \frac{1}{2} E \left\{ \int_0^{\infty} [\tilde{x}(t) - \tilde{x}_r(t)]^T Q^* [\tilde{x}(t) - \tilde{x}_r(t)] + r u^2(t) dt \right\}$$

$$= \frac{1}{2} E \left\{ \int_0^{\infty} \begin{bmatrix} x_1^T(t) & x_2^T(t) & x_{r_1}^T(t) & x_{r_2}^T(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \\ -2 \end{bmatrix} Q^* \begin{bmatrix} 1 & 0 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_{r_1}(t) \\ x_{r_2}(t) \end{bmatrix} + \right.$$

$$\left. r u^2(t) dt \right\}$$

(3.36)

Remarks (3.2)

The following assumptions are assumed for this illustration:

1) The initial conditions are zero-mean random vectors with the following correlation matrices (The second-order moment matrix):

$$E[x(0)x^T(0)] = \Sigma_x(0), \text{ and } E\{x(0)\} = \mathbf{0}$$

$$E[x_r(0)x_r^T(0)] = \Sigma_{x_r}(0), \text{ and } E\{x_r(0)\} = \mathbf{0}$$

The pair $(\mathrm{A}, \mathrm{B}_u)$ of a non-linear dynamical control system is controllable [see theorem (1.1)].

2) A_r is stable matrix.

3) $E[x(0)x_r^T(0)] = \mathbf{0}$

4) $E[x_r(0)x^T(0)] = \mathbf{0}$

5) $E[x(0)x_r^T(t)] = \mathbf{0}$

6) $E[x_r(0)x_r^T(t)] = \mathbf{0}$

7) $E[x_r(t)x^T(0)] = \mathbf{0}$

8) $E[x_r(t)x_r^T(0)] = \mathbf{0}$

Step (1)

$$\begin{bmatrix} \dot{x}_{r_1}(t) \\ \dot{x}_{r_2}(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \dot{x}_{r_1}(t) \\ \dot{x}_{r_2}(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u_r(t) + \begin{bmatrix} x_{r_1}(t) \\ 0 \end{bmatrix} \qquad (3.37)$$

The feedback gain K_r is obtained by using pole placement method (see lemma (2.2)), such that (3.37), is stable.

as follows:

Step (1.1)

Check the controllability condition for the system (2.37). If the system is completely state controllable, i.e.,(A_r, B_r) is controllable :

$$M = [B_r \;\vdots\; A_r B_r]$$

$$= \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}, \quad |M| \neq 0$$

Hence rank $(M = 2)$. Therefore (A_r, B_r) is completely state controllable.

Step (1.2)

From the characteristic polynomial (eigenvalues), for matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} (\lambda - 3) & -1 \\ -1 & (\lambda - 3) \end{vmatrix}$$

$$\equiv \lambda^2 - 6\lambda + 8,$$

then $a_1 = -6, a_2 = 8$

Step (2)

Step (2.1)

Determine the transformation matrix T, for details one can see [26] , for transformation

$$\mathrm{T} = \mathrm{MW}$$

where M is the controllability matrix of substep (1.1), and using the result of substep (1.2),

W is defined by

$$\mathrm{W} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 1 \\ 1 & 0 \end{bmatrix}$$

And hence

$$\mathrm{T} = \mathrm{MW}$$

$$= \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -6 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 2 \\ 2 & 0 \end{bmatrix}$$

And

$$\mathrm{T}^{-1} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$$

Step (2.2)

Let the desired eigenvalues (desired closed-loop poles), are selected as:

$$s_1 = -2, s_2 = -5$$

$$(l - m_1)(l - m_2) = l^2 + 5l + 2l + 10$$

$$\equiv l^2 + 7l + 10$$

Then

$a_1 = 7,\ a_2 = 10$

Step (2.3)

The state feedback gain matrix

K_r can be determined using the result of substeps (2.1), (2.2), and (2.3) as follows:

$$K_r = [a_2 - a_2 \vdots a_1 - a_1]\mathrm{T}^{-1}$$

$$= [10 - 8 \vdots 7 + 6]\begin{bmatrix} 0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$$

$$K_r = [6.5 \quad 20.5]$$

Step (3)

Define the robust controller $u_r = -K_r x_r(t)$

Using the results of step 2, we have that:

$$\begin{bmatrix} \dot{x}_{r_1}(t) \\ \dot{x}_{r_2}(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} \dot{x}_{r_1}(t) \\ \dot{x}_{r_2}(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u_r(t) + \begin{bmatrix} x_{r_1}(t) \\ 0 \end{bmatrix}$$

The feedback of the following system, where $u(t) = -K_r x_r(t)$, (K_r is calculating in step 2), is given by

$$\begin{bmatrix} \dot{x}_{r_1}(t) \\ \dot{x}_{r_2}(t) \end{bmatrix} = \begin{bmatrix} -10 & -40 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} \dot{x}_{r_1}(t) \\ \dot{x}_{r_2}(t) \end{bmatrix} + \begin{bmatrix} x_{r_1}(t) \\ 0 \end{bmatrix} \quad (3.38)$$

$\Rightarrow$

Eigenvalues of matrix $\begin{bmatrix} -10 & -40 \\ 1 & 3 \end{bmatrix}$ are $l_1 = -2, \quad l_2 = -5$

Step (4)

Define the augmented matrix for system (3.29), and (2.38), to have:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{r_1}(t) \\ \dot{x}_{r_2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \vdots & 0 & 0 \\ -10 & -1 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & -10 & -40 \\ 0 & 0 & \vdots & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{r_1}(t) \\ x_{r_2}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} x_1(t) \\ 0 \\ \vdots \\ x_{r_1}(t) \\ 0 \end{bmatrix} \tag{3.39}$$

From the equation (3.36):

$$J = \frac{1}{2} E \left\{ \int_0^\infty [\dot{x}(t) - \dot{x}_r(t)]^T Q^* [\dot{x}(t) - \dot{x}_r(t)] + r\,u^2(t)dt \right\}$$

$$= \frac{1}{2} E \left\{ \int_0^\infty \left[x_1^T(t) \quad x_2^T(t) \quad x_{r_1}^T(t) \quad x_{r_2}^T(t) \right] \begin{bmatrix} 1 \\ 0 \\ 4 \\ -2 \end{bmatrix} Q^* \left[1 \quad 0 \quad 4 \quad -2\right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_{r_1}(t) \\ x_{r_2}(t) \end{bmatrix} + r\,u^2(t)\,dt \right\}$$

Let

$$Q^* = I\,,$$

$$Q(t) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[1 \quad 0 \quad 4 \quad -2\right] \tag{3.40}$$

$$= \begin{bmatrix} 1 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 16 & -8 \\ -2 & 0 & -8 & 4 \end{bmatrix}$$

Since the eigenvalues of $Q(t)$ are positive, so its clear that $Q(t)$ is positive-semidefinite

Step (5)

The general performance index is defined as:

From (3.40), the cost function (3.36), can be written:

$$J=\frac{1}{2}E\left\{\int_0^{\infty}\left[x_1^T(t)\quad x_2^T(t)\quad x_{r_1}^T(t)\quad x_{r_2}^T(t)\right]\begin{bmatrix}1 & 0 & 4 & -2\\ 0 & 0 & 0 & 0\\ 4 & 0 & 16 & -8\\ -2 & 0 & -8 & 4\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_{r_1}(t)\\ x_{r_2}(t)\end{bmatrix}+ ru^2(t)dt\right\} \tag{3.41}$$

Step (6)

Find the optimal control $u=-Kx(t)$, for all system as follows:

Find the solution of Riccati equation:

It follows from theorem (2.1), that the optimal tracking law is given (3.10):

$$\begin{aligned}u(t)&=-K\tilde{x}(t)\\ &=-(K_1,-K_2)\tilde{x}(t)\end{aligned}$$

To find the steady-state value of the feedback gain, we must find the steady-state value of $P_{11}, P_{12}, P_{21}, P_{22}$, are obtained by partition the matrix P according to the partitioning $\tilde{x}(t)=col[x_1(t), x_2(t), x_{r_1}(t), x_{r_2}(t)]$; from Riccati equation (2.24):

$$\begin{bmatrix} 0 & -10 & \vdots & 0 & 0 \\ 1 & -1 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & -10 & 1 \\ 0 & 0 & \vdots & -40 & 3 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & \vdots & P_{13} & P_{14} \\ P_{21} & P_{22} & \vdots & P_{23} & P_{24} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ P_{31} & P_{32} & \vdots & P_{33} & P_{34} \\ P_{41} & P_{42} & \vdots & P_{43} & P_{44} \end{bmatrix} +$$

$$\begin{bmatrix} P_{11} & P_{12} & \vdots & P_{13} & P_{14} \\ P_{21} & P_{22} & \vdots & P_{23} & P_{24} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ P_{31} & P_{32} & \vdots & P_{33} & P_{34} \\ P_{41} & P_{42} & \vdots & P_{43} & P_{44} \end{bmatrix} \begin{bmatrix} 0 & 1 & \vdots & 0 & 0 \\ -10 & -1 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & -10 & -40 \\ 0 & 0 & \vdots & 1 & 3 \end{bmatrix} +$$

$$\begin{bmatrix} 1 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 16 & -8 \\ -2 & 0 & -8 & 4 \end{bmatrix} -$$

$$\begin{bmatrix} P_{11} & P_{12} & \vdots & P_{13} & P_{14} \\ P_{21} & P_{22} & \vdots & P_{23} & P_{24} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ P_{31} & P_{32} & \vdots & P_{33} & P_{34} \\ P_{41} & P_{42} & \vdots & P_{43} & P_{44} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1000 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & \vdots & P_{13} & P_{14} \\ P_{21} & P_{22} & \vdots & P_{23} & P_{24} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ P_{31} & P_{32} & \vdots & P_{33} & P_{34} \\ P_{41} & P_{42} & \vdots & P_{43} & P_{44} \end{bmatrix} = 0 \qquad (3.42)$$

On using the definition of Riccati equation and its partions (see theorem (2.1)), and the symmetry of $\overline{P}_{ij}$, $i, j = 1, 2,$ can be exploited by reducing (3.42), to set of $\frac{n(n+1)}{2} = 10$ equations, we obtain:

$\Rightarrow$

$$-10P_{21} - 10P_{12} + 1 - 1000P_{21}^2 = 0$$

$$-10P_{22}+P_{11}-P_{12}-1000P_{12}P_{22}=0$$

$$-10P_{23}-10P_{13}+P_{14}+4-1000P_{12}P_{23}=0$$

$$-10P_{24}-40P_{13}+3P_{14}-2-1000P_{12}P_{24}=0$$

$$P_{12}-P_{22}+P_{21}-P_{22}-1000P_{22}^{2}=0$$

$$P_{13}-P_{23}-10P_{23}+P_{24}-1000P_{22}P_{23}=0$$

$$P_{14}-P_{24}-40P_{23}+3P_{24}-1000P_{22}P_{24}=0$$

$$-10P_{33}+P_{43}-10P_{33}+P_{34}+16-1000P_{32}^{2}=0$$

$$-10P_{34}+P_{44}-40P_{33}+3P_{34}-8-1000P_{32}P_{24}=0$$

$$-40P_{34}+3P_{44}-40P_{43}+3P_{44}+4-1000P_{42}^{2}=0$$

(3.43)

Then solving the system (3.43), on using A MATLAB program, to have that:

$$\overline{P}=\begin{bmatrix} 0.5392 & 0.0499 & \vdots & 0.3167 & -2.5700 \\ 0.0499 & 0.0487 & \vdots & -0.1729 & -2.2269 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0.3167 & -0.1729 & \vdots & 2.5199 & 17.2137 \\ -2.5700 & -2.2269 & \vdots & 17.2137 & 229.6763 \end{bmatrix}$$

Where

$$P_{11}=\begin{bmatrix} 0.5392 & 0.0499 \\ 0.0499 & 0.0487 \end{bmatrix} \qquad (3.44)$$

And

$$P_{12}=\begin{bmatrix} 0.3167 & -0.1729 \\ -0.1729 & -2.2269 \end{bmatrix} \qquad (3.45)$$

$$P_{21}=\begin{bmatrix} 0.3167 & -0.1729 \\ -2.5700 & -2.2269 \end{bmatrix}, \quad P_{22}=\begin{bmatrix} 2.5199 & 17.2137 \\ 17.2137 & 229.6763 \end{bmatrix}$$

One can noted that the eigenvalues of $\overline{P}$ are $(0.0243, 0.02489, 1.4880, 231.0229)$ positive then $\overline{P}$ which are unique positive definite solution.

Step (7)

To find the steady-state value of the feedback gain $[K_1, K_2]$, substituting (3.44) into (2.54), (see the result of theorem (2.1)) we obtained:

$$K_1 = R^{-1} \mathrm{B}_u^T \mathrm{P}_{11}$$

$$= 1000 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5392 & 0.0499 \\ 0.0499 & 0.0487 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} 49.9 & 48.7 \end{bmatrix} \tag{3.46}$$

And substituting (3.45) into (2.55), we obtain:

$$K_2 = R^{-1} \mathrm{B}_u^T \mathrm{P}_{12}$$

$$= 1000 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0.3167 & -0.1729 \\ -0.1729 & -2.2269 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} -0.1729 & -2.2269 \end{bmatrix} \tag{3.47}$$

Step (8)

substituting (3.10), into (3.39), then substituting (3.46), (3.47), into (3.39), one can get, the following feedback dynamic system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \cdots \\ \dot{x}_{r1}(t) \\ \dot{x}_{r2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \vdots & 0 & 0 \\ -59.9 & -49.7 & \vdots & 0.1729 & 2.2269 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & -10 & -40 \\ 0 & 0 & \vdots & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_{r1}(t) \\ x_{r2}(t) \end{bmatrix} + \begin{bmatrix} x_1(t) \\ 0 \\ \cdots \\ x_{r1}(t) \\ 0 \end{bmatrix} \tag{3.48}$$

Step (9)

On solving analytically the system (3.48), one can do as follows:

Step (9.1)

The eigenvalues of matrix:

$$A = \begin{bmatrix} 0 & 1 & \vdots & 0 & 0 \\ -59.9 & -49.7 & \vdots & 0.1729 & 2.2269 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & -10 & -40 \\ 0 & 0 & \vdots & 1 & 3 \end{bmatrix}$$

$$\left| \lambda I - A \right| = 0$$

$\Rightarrow$

$$\lambda_1 = -1.2360$$
$$\lambda_2 = -48.4640$$
$$\lambda_3 = -5$$
$$\lambda_4 = -2$$

Step (9.2)

The eigenvectors of matrix A are:

$$V = [u_1 \quad u_2 \quad \cdots \quad u_n]$$
$$A u_i = \lambda_i u_i \, , \; i = 1, 2, \ldots, n$$

$\Rightarrow$

$$V = \begin{bmatrix} 0.6290 & -0.0206 & -0.0006 & 0.0075 \\ -0.7774 & 0.9998 & 0.0032 & -0.0151 \\ 0 & 0 & -0.9923 & 0.9804 \\ 0 & 0 & 0.1240 & -0.1961 \end{bmatrix} \qquad (3.49)$$

And

$$V^{-1} = \begin{bmatrix} 1.6315 & 0.0337 & 0.0175 & 0.1475 \\ 1.2686 & 1.0264 & -0.0034 & -0.0470 \\ 0 & 0 & -2.6874 & -13.4372 \\ 0 & 0 & -1.6999 & -13.5993 \end{bmatrix} \qquad (3.50)$$

Step (9.3)

Since the eigenvalues $l_1, l_2, ..., l_n$ of the matrix A are distinct, we have:

$$e^{At} = Ve^{(V^{-1}AV)t}V^{-1} = Ve^{Zt}V^{-1} \tag{3.51}$$

where

$Z = V^{-1}AV$

$$Z = \begin{bmatrix} 1.6315 & 0.0337 & 0.0175 & 0.1475 \\ 1.2686 & 1.0264 & -0.0034 & -0.0470 \\ 0 & 0 & -2.6874 & -13.4372 \\ 0 & 0 & -1.6999 & -13.5993 \end{bmatrix} \begin{bmatrix} 0 & 1 & \vdots & 0 & 0 \\ -59.9 & -49.7 & \vdots & 0.1729 & 2.2269 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & -10 & -40 \\ 0 & 0 & \vdots & 1 & 3 \end{bmatrix} .$$

$$\begin{bmatrix} 0.6290 & -0.0206 & -0.0006 & 0.0075 \\ -0.0206 & 0.9998 & 0.0032 & -0.0151 \\ 0 & 0 & -0.9923 & 0.9804 \\ 0 & 0 & 0.1240 & -0.1961 \end{bmatrix}$$

$$Z = \begin{bmatrix} -1.236 & 0 & 0 & 0 \\ 0 & -48.464 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \tag{3.52}$$

From (3.51), and (3.52),we obtain:

$$e^{At} = Ve^{Zt}V^{-1}$$

$$\begin{bmatrix} 0.6290 & -0.0206 & -0.0006 & 0.0075 \\ -0.7774 & 0.9998 & 0.0032 & -0.0151 \\ 0 & 0 & -0.9923 & 0.9804 \\ 0 & 0 & 0.1240 & -0.1961 \end{bmatrix} .$$

$$\begin{bmatrix} e^{-1.236t} & 0 & 0 & 0 \\ 0 & e^{-48.464t} & 0 & 0 \\ 0 & 0 & e^{-5t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{bmatrix} .$$

$$\begin{bmatrix} 1.6315 & 0.0337 & 0.0175 & 0.1475 \\ 1.2686 & 1.0264 & -0.0034 & -0.0470 \\ 0 & 0 & -2.6874 & -13.4372 \\ 0 & 0 & -1.6999 & -13.5993 \end{bmatrix} \quad (3.53)$$

$\Rightarrow$

Let $e^{At} = \begin{bmatrix} g_{11}(t) & g_{12}(t) & g_{13}(t) & g_{14}(t) \\ g_{21}(t) & g_{22}(t) & g_{23}(t) & g_{24}(t) \\ g_{31}(t) & g_{32}(t) & g_{33}(t) & g_{34}(t) \\ g_{41}(t) & g_{42}(t) & g_{43}(t) & g_{44}(t) \end{bmatrix}$, where

$$g_{11}(t) = 1.0262\, e^{-1.236t} - 0.0261\, e^{-48.464t}$$

$$g_{12}(t) = 0.0211\, e^{-1.236t} - 0.0211\, e^{-48.464t}$$

$$g_{13}(t) = 0.011\, e^{-1.236t} + 0.00007\, e^{-48.464t} + 0.0016\, e^{-5t} - 0.0127\, e^{-2t}$$

$$g_{14}(t) = 0.0927\, e^{-1.236t} + 0.0009\, e^{-48.464t} + 0.0008\, e^{-5t} - 0.1019\, e^{-2t}$$

$$g_{21}(t) = 1.2683\, e^{-1.236t} + 1.2683\, e^{-48.464t}$$

$$g_{22}(t) = -0.0261\, e^{-1.236t} + 0.0261\, e^{-48.464t}$$

$$g_{23}(t) = 0.0136\, e^{-1.236t} - 0.0033\, e^{-48.464t} - 0.0085\, e^{-5t} + 0.0256\, e^{-2t}$$

$$g_{24}(t) = -0.1142\, e^{-1.236t} - 0.0469\, e^{-48.464t} - 0.0429\, e^{-5t} + 0.2053\, e^{-2t}$$
$$g_{31}(t) = 0$$
$$g_{32}(t) = 0$$
$$g_{33}(t) = 2.6667\, e^{-5t} - 1.6665\, e^{-2t}$$
$$g_{34}(t) = 13.3337\, e^{-5t} - 13.3327\, e^{-2t}$$
$$g_{41}(t) = 0$$
$$g_{42}(t) = 0$$
$$g_{43}(t) = -0.33325\, e^{-5t} + 0.3333\, e^{-2t}$$
$$g_{44}(t) = -1.6662\, e^{-5t} + 2.6668\, e^{-2t}$$
$$g_{21}(t) = 1.2683\, e^{-1.236t} + 1.2683\, e^{-48.464t}$$
$$g_{22}(t) = -0.0261\, e^{-1.236t} + 0.0261\, e^{-48.464t}$$
$$g_{23}(t) = 0.0136\, e^{-1.236t} - 0.0033\, e^{-48.464t} - 0.0085\, e^{-5t} + 0.0256\, e^{-2t}$$
$$g_{24}(t) = -0.1142\, e^{-1.236t} - 0.0469\, e^{-48.464t} - 0.0429\, e^{-5t} + 0.2053\, e^{-2t}$$
$$g_{31}(t) = 0$$
$$g_{32}(t) = 0$$
$$g_{33}(t) = 2.6667\, e^{-5t} - 1.6665\, e^{-2t}$$
$$g_{34}(t) = 13.3337\, e^{-5t} - 13.3327\, e^{-2t}$$
$$g_{41}(t) = 0$$
$$g_{42}(t) = 0$$
$$g_{43}(t) = -0.33325\, e^{-5t} + 0.3333\, e^{-2t}$$
$$g_{44}(t) = -1.6662\, e^{-5t} + 2.6668\, e^{-2t}$$

(3.55)

And hence

$$\tilde{x}(t) = e^{At}\tilde{x}(0) + e^{At}\int_0^t e^{-At}\tilde{B}_x \tilde{x}(t)\,dt \qquad (3.56)$$

Substituting (3.55), into (3.56), we get:

$$x_1(t) = 1.0262\ e^{-1.236t} x_1(0) - 0.0261\ e^{-48.464t} x_1(0) + 0.0211\ e^{-1.236t} x_2(0) - 0.0211 e^{-48.464t} x_2(0) + 0.011 e^{-1.236t} x_{r_1}(0) + 0.00007 e^{-48.464t} x_{r_1}(0) + 0.0016 e^{-5t} x_{r_1}(0) - 0.0127 e^{-2t} x_{r_1}(0) + 0.0927 e^{-1.236t} x_{r_2}(0) + 0.0009 e^{-48.464t} x_{r_2}(0) + 0.0008\ e^{-5t} x_{r_2}(0) - 0.1019\ e^{-2t} x_{r_2}(0) +$$

$$\int_0^t \Big\{ 1.0262\ e^{-1.236(t-t)} x_1(t) - 0.0261\ e^{-48.464(t-t)} x_1(t) + 0.011 e^{-1.236(t-t)} x_{r_1}(t) + 0.00007\ e^{-48.464(t-t)} x_{r_1}(t) + 0.0016\ e^{-5(t-t)} x_{r_1}(t) - 0.0127\ e^{-2(t-t)} x_{r_1}(t) \Big\} dt$$

(3.57)

$$x_2(t) = 1.2683 e^{-1.236t} x_1(0) + 1.2683 e^{-48.464t} x_1(0) - 0.0261 e^{-1.236t} x_2(0) + 0.0261 e^{-48.464t} x_2(0) + 0.0136 e^{-1.236t} x_{r_1}(0) - 0.0033 e^{-48.464t} x_{r_1}(0) - 0.0085 e^{-5t} x_{r_1}(0) + 0.0256 e^{-2t} x_{r_1}(0) - 0.1142 e^{-1.236t} x_{r_2}(0) - 0.0469 e^{-48.464t} x_{r_2}(0) - 0.0429 e^{-5t} x_{r_2}(0) + 0.2053 e^{-2t} x_{r_2}(0) + \int_0^t \Big\{ 1.2683 e^{-1.236(t-t)} x_1(t) + 1.2683 e^{-48.464(t-t)} x_1(t) + 0.0136 e^{-1.236(t-t)} x_{r_1}(t) - 0.0033 e^{-48.464(t-t)} x_{r_1}(t) - 0.0085\ e^{-5(t-t)} x_{r_1}(t) + 0.0256\ e^{-2(t-t)} x_{r_1}(t) \Big\} dt$$

(3.58)

$$x_{r_1}(t) = 2.6667 e^{-5t} x_{r_1}(0) - 1.6665 e^{-2t} x_{r_1}(0) + 13.3337 e^{-5t} x_{r_2}(0) - 13.3327 e^{-2t} x_{r_2}(0) + \int_0^t \Big\{ 2.6667 e^{-5(t-t)} x_{r_1}(t) - 1.6665 e^{-2(t-t)} x_{r_1}(t) \Big\} dt$$

(3.59)

$$x_{r_2}(t) = -0.3332e^{-5t}x_{r_1}(0) + 0.3333e^{-2t}x_{r_1}(0)$$
$$-1.6662e^{-5t}x_{r_2}(0) + 2.6668e^{-2t}x_{r_2}(0) +$$
$$\int_0^t \left\{ -0.3332e^{-5(t-t)}x_{r_1}(t) + 0.3333e^{-2(t-t)}x_{r_1}(t) \right\} dt \qquad (3.60)$$

Step (10)

Find the statistical properties of this solution and as follows:

From (3.57), (3.58), (3.59), and (3.60), we get:

Step (10.1)

On using the assumption (1, 2) of remarks (3.2), we have:

$$E[x_1(t)] = 1.0262\,e^{-1.236t}E[x_1(0)] - 0.0261\,e^{-48.464t}E[x_1(0)] +$$
$$0.0211\,e^{-1.236t}E[x_2(0)] - 0.0211\,e^{-48.464t}E[x_2(0)] +$$
$$0.011\,e^{-1.236t}E[x_{r_1}(0)] + 0.00007e^{-48.464t}E[x_{r_1}(0)] +$$
$$0.0016\,e^{-5t}E[x_{r_1}(0)] - 0.0127e^{-2t}E[x_{r_1}(0)] +$$
$$0.0927\,e^{-1.236t}E[x_{r_2}(0)] + 0.0009\,e^{-48.464t}E[x_{r_2}(0)] +$$
$$0.0008\,e^{-5t}E[x_{r_2}(0)] - 0.1019\,e^{-2t}E[x_{r_2}(0)] +$$
$$\int_0^t \Big\{ 1.0262e^{-1.236(t-t)}E[x_1(t)] - 0.0261e^{-48.464(t-t)}E[x_1(t)] +$$
$$0.011e^{-1.236(t-t)}E[x_{r_1}(t)] + 0.00007e^{-48.464(t-t)}E[x_{r_1}(t)] +$$
$$0.0016e^{-5(t-t)}E[x_{r_1}(t)] - 0.0127e^{-2(t-t)}E[x_{r_1}(t)] \Big\} dt = 0$$

$$E[x_2(t)] = 1.2683\, e^{-1.236t} E[x_1(0)] + 1.2683\, e^{-48.464t} E[x_1(0)] -$$
$$0.0261\, e^{-1.236t} E[x_2(0)] + 0.0261\, e^{-48.464t} E[x_2(0)] +$$
$$0.0136\, e^{-1.236t} E[x_{r_1}(0)] - 0.0033\, e^{-48.464t} E[x_{r_1}(0)] -$$
$$0.0085\, e^{-5t} E[x_{r_1}(0)] + 0.0256\, e^{-2t} E[x_{r_1}(0)] -$$
$$0.1142\, e^{-1.236t} E[x_{r_2}(0)] - 0.0469\, e^{-48.464t} E[x_{r_2}(0)] -$$
$$0.0429\, e^{-5t} E[x_{r_2}(0)] + 0.2053\, e^{-2t} E[x_{r_2}(0)] +$$
$$\int_0^t \Big\{ 1.2683\, e^{-1.236(t-t)} E[x_1(t)] + 1.2683\, e^{-48.464(t-t)} E[x_1(t)] +$$
$$0.0136\, e^{-1.236(t-t)} E[x_{r_1}(t)] - 0.0033\, e^{-48.464(t-t)} E[x_{r_1}(t)] -$$
$$0.0085\, e^{-5(t-t)} E[x_{r_1}(t)] + 0.0256\, e^{-2(t-t)} E[x_{r_1}(t)] \Big\} dt = 0$$

$$E[x_{r_1}(t)] = 2.6667\, e^{-5t} E[x_{r_1}(0)] - 1.6665\, e^{-2t} E[x_{r_1}(0)] + 13.3337 e^{-5t} E[x_{r_2}(0)]$$
$$-13.3327\, e^{-2t} E[x_{r_2}(0)] + \int_0^t \Big\{ 2.6667\, e^{-5(t-t)} E[x_{r_1}(t)] -$$
$$1.6665\, e^{-2(t-t)} E[x_{r_1}(t)] \Big\} dt = 0$$

$$E[x_{r_2}(t)] = -0.3332\, e^{-5t} E[x_{r_1}(0)] + 0.3333\, e^{-2t} E[x_{r_1}(0)] - 1.6662\, e^{-5t} E[x_{r_2}(0)] +$$
$$2.6668\, e^{-2t} E[x_{r_2}(0)] + \int_0^t \Big\{ -0.3332\, e^{-5(t-t)} E[x_{r_1}(t)] +$$
$$0.3333\, e^{-2(t-t)} E[x_{r_1}(t)] \Big\} dt = 0$$

Step (10.2)

The correlation function can be found see subsection (1.6.3), and using the assumption of remark (3.2). From (3.56), we get:

$$R_{x_1 x_1}(t_1,t_2)] = E[x_1(t_1) x_1^T (t_1)]$$

$$= 1.053\, e^{-1.236 t_1} \Sigma_{x_1}(0) e^{-1.236 t_2} - 0.0267\, e^{-1.236 t_1} \Sigma_{x_1}(0)\, e^{-48.464 t_2} -$$
$$0.0267\, e^{-48.464 t_1} \Sigma_{x_1}(0)\, e^{-1.236 t_2} + 0.0006\, e^{-48.464 t_1} \Sigma_{x_1}(0)\, e^{-48.464 t_2} +$$
$$0.0004\, e^{-1.236 t_1} \Sigma_{x_2}(0)\, e^{-1.236 t_2} - 0.0004\, e^{-1.236 t_1} \Sigma_{x_2}(0)\, e^{-48.464 t_2} -$$
$$0.0004\, e^{-48.464 t_1} \Sigma_{x_2}(0)\, e^{-1.236 t 2} + 0.0004\, e^{-48.464 t_1} \Sigma_{x_2}(0)\, e^{-48.464 t_2} +$$
$$0.0001\, e^{-1.236 t_1} \Sigma_{x_{r_1}}(0)\, e^{-1.236 t_2} - 0.0001\, e^{-1.236 t_1} \Sigma_{x_{r_1}}(0) e^{-2 t_2} -$$
$$0.0001\, e^{-2 t_1} \Sigma_{x_{r_1}}(0)\, e^{-1.236 t_2} - 0.00016\, e^{-2 t_1} \Sigma_{x_{r_1}}(0)\, e^{-2 t_2} +$$
$$0.0085\, e^{-1.236 t_1} \Sigma_{x_{r_2}}(0) e^{-1.236 t_2} - 0.0074\, e^{-1.236 t_1} \Sigma_{x_{r_2}}(0)\, e^{-2 t_2} -$$
$$0.0074\, e^{-2 t_1} \Sigma_{x_{r_2}}(0)\, e^{-1.236 t_2} + 0.103\, e^{-2 t_1} \Sigma_{x_{r_2}}(0)\, e^{-2 t_2} +$$

$$\int_0^{\min(t_1,t_2)} \Big\{ 1.053(0.025)\, e^{-1.236(t_1 - s)}\, e^{-1.236(t_2 - s)} -$$
$$0.0267(0.025)\, e^{-1.236(t_1 - s)}\, e^{-48.464(t_2 - s)} -$$
$$0.0267(0.025)\, e^{-48.464(t_1 - s)}\, e^{-1.236(t_2 - s)} +$$
$$0.0006(0.025)\, e^{-48.464(t_1 - s)}\, e^{-48.464(t_2 - s)} +$$
$$0.0001(0.005)\, e^{-1.236(t_1 - s)}\, e^{-1.236(t_2 - s)} -$$
$$0.0001(0.005)\, e^{-1.236(t_1 - s)}\, e^{-2(t_2 - s)} -$$
$$0.0001(0.005)\, e^{-2(t_1 - s)}\, e^{-1.236(t_2 - s)} +$$
$$0.0001(0.005)\, e^{-2(t_1 - s)}\, e^{-2(t_2 - s)} \Big\} ds$$

$$R_{x_1x_1}(t_1,t_2)] = E[x_1(t_1)x_1^T(t_1)]$$
$$= 1.053\ e^{-1.236(t_1+t_2)}\Sigma_{x_1}(0) - 0.0267\, e^{-1.236t_1}\Sigma_{x_1}(0)\, e^{-48.464t_2} -$$
$$0.0267\, e^{-48.464t_1}\Sigma_{x_1}(0)\ e^{-1.236t_2} + 0.0006\ e^{-48.464(t_1+t_2)}\Sigma_{x_1}(0) +$$
$$0.0004\ e^{-1.236t_1}\Sigma_{x_2}(0)e^{-1.236t_2} - 0.0004\ e^{-1.236t_1}\Sigma_{x_2}(0)e^{-48.464t_2} -$$
$$0.0004\ e^{-48.464t_1}\Sigma_{x_2}(0)\, e^{-1.236t\,2} + 0.0004\ e^{-48.464(t_1+t_2)}\Sigma_{x_2}(0) +$$
$$0.0001\, e^{-1.236(t_1+t_2)}\Sigma_{x_{r_1}}(0) - 0.0001\ e^{-1.236t_1}\Sigma_{x_{r_1}}(0)\ e^{-2t_2} -$$
$$0.0001\ e^{-2t_1}\Sigma_{x_{r_1}}(0)\ e^{-1.236t_2} - 0.00016\ e^{-2(t_1+t_2)}\Sigma_{x_{r_1}}(0) +$$
$$0.0085\, e^{-1.236(t_1+t_2)}\,\Sigma_{x_{r_2}}(0) - 0.0074\, e^{-1.236t_1}\Sigma_{x_{r_2}}(0)\, e^{-2t_2} -$$
$$0.0074\, e^{-2t_1}\Sigma_{x_{r_2}}(0)\, e^{-1.236t_2} + 0.103\, e^{-2(t_1+t_2)}\Sigma_{x_{r_2}}(0) +$$
$$0.01064\ e^{-1.236(t_1+t_2)}\ e^{2.472\min(t_1,t_2)}$$

And from (3.59), we get:

$$R_{x_{r_1}x_{r_1}}(t_1,t_2) = 7.1112\ e^{-5t_1}\Sigma_{x_{r_1}}(0)\ e^{-5t_2} - 4.444\ e^{-5t_1}\Sigma_{x_{r_1}}(0)\ e^{-2t_2} -$$
$$4.444\ e^{-2t_1}\Sigma_{x_{r_1}}(0)\ e^{-5t_2} + 2.7772\ e^{-2t_1}\Sigma_{x_{r_1}}(0)\ e^{-2t_2} +$$
$$177.7875\ e^{-5t_1}\,\Sigma_{x_{r_2}}(0)\, e^{-5t_2} - 177.7742\ e^{-5t_1}\,\Sigma_{x_{r_2}}(0)\ e^{-2t_2} -$$
$$177.7742\, e^{-2t_1}\,\Sigma_{x_{r_2}}(0)\ e^{-5t_2} + 177.7608\ e^{-2t_1}\,\Sigma_{x_{r_2}}(0)\ e^{-2t_2} +$$
$$\int_0^{\min(t_1,t_2)} \left\{ \begin{array}{l} 7.1112(0.005)\ e^{-5(t_1-s)}\ e^{-5(t_2-s)} - \\ 4.444(0.005)\, e^{-5(t_1-s)}\ e^{-2(t_2-s)} - \\ 4.444(0.005)\ e^{-2(t_1-s)}\ e^{-5(t_2-s)} + \\ 2.7772(0.005)\ e^{-2(t_1-s)}\ e^{-2(t_2-s)} \end{array} \right\} ds$$

$$R_{x_{r_1}x_{r_1}}(t_1,t_2)=7.1112\ e^{-5(t_1+t_2)}\Sigma_{x_{r_1}}(0)+0.0035\ e^{-5(t_1+t_2)}\ e^{10\min(t_1,t_2)}-$$
$$4.444\ e^{-5t_1}\Sigma_{x_{r_1}}(0)\ e^{-2t_2}-0.0031\ e^{-5t_1}\ e^{-2t_2}\ e^{7\min(t_1,t_2)}-$$
$$4.444\ e^{-2t_1}\Sigma_{x_{r_1}}(0)\,e^{-5t_2}-0.0031\,e^{-2t_1}e^{-5t_2}\,e^{7\min(t_1,t_2)}+$$
$$2.7772\ e^{-2(t_1+t_2)}\Sigma_{x_{r_1}}(0)+0.0034\ e^{-2(t_1+t_2)}\ e^{4\min(t_1,t_2)}$$

And similarly one can compute $(R_{x_1x_2}, R_{x_1x_{r_1}}, R_{x_1x_{r_2}}, R_{x_2x_1}, R_{x_2x_2},$

$R_{x_2x_{r_1}}, R_{x_2x_{r_2}}, R_{x_{r_1}x_1}, R_{x_{r_1}x_2}, R_{x_{r_1}x_{r_2}}, R_{x_{r_2}x_1}, R_{x_{r_2}x_2}, R_{x_{r_2}x_{r_1}}, R_{x_{r_2}x_{r_2}})$,

using the same procedure.

Step (10.3)

The correlation matrix can be obtain from correlation function (see remark (1.11):

$$R_{x_1x_1}(t,t)=\Sigma_{x_1x_1}(t)$$
$$=1.053\,e^{-2.472t}\Sigma_{x_1}(0)-0.0534\,\Sigma_{x_1}(0)\ e^{-49.7t}+$$
$$0.0006\ e^{-96.928t}\Sigma_{x_1}(0)+0.0004\ e^{-2.472t}\Sigma_{x_2}(0)-$$
$$0.0008\Sigma_{x_2}(0)\,e^{-49.7t}+0.0004\ e^{-96.928t}\Sigma_{x_2}(0)+$$
$$0.0001\,e^{-2.472t}\ \Sigma_{x_{r_1}}(0)-0.0002\ e^{-3.236t}\Sigma_{x_{r_1}}(0)-$$
$$0.00016\ e^{-4t}\Sigma_{x_{r_1}}(0)+0.0085\,e^{-2.472t}\ \Sigma_{x_{r_2}}(0)-$$
$$0.0014\ e^{-3.236t}\ \Sigma_{x_{r_2}}(0)+0.103\,e^{-4t}\Sigma_{x_{r_2}}(0)+0.0106$$

$$R_{x_{r_1}x_{r_1}}(t,t)=\Sigma_{x_{r_1}x_{r_1}}(t)$$
$$=7.1112\ e^{-5(t_1+t_2)}\Sigma_{x_{r_1}}(0)+0.0035-0.0062-8.888\,e^{-7t}+$$
$$2.7772\ e^{-2(t_1+t_2)}\Sigma_{x_{r_1}}(0)+0.0034$$

Step (10.4)

The correlation matrix are depending on time-varying and the steady-state can be obtained by let $t \to \infty$, (see remarks (1.11)):

$$\Sigma_{x_1x_1}(\infty)=\lim_{t\to\infty}[1.053\,e^{-2.472t}\Sigma_{x_1}(0)-0.0534\,\Sigma_{x_1}(0)\,e^{-49.7t}+$$
$$0.0006\,e^{-96.928t}\Sigma_{x_1}(0)+0.0004\,e^{-2.472t}\Sigma_{x_2}(0)-$$
$$0.0008\Sigma_{x_2}(0)\,e^{-49.7t}+0.0004\,e^{-96.928t}\Sigma_{x_2}(0)+$$
$$0.0001\,e^{-2.472t}\Sigma_{x_{r_1}}(0)-0.0002\,e^{-3.236t}\Sigma_{x_{r_1}}(0)-$$
$$0.00016\,e^{-4t}\Sigma_{x_{r_1}}(0)+0.0085\,e^{-2.472t}\Sigma_{x_{r_2}}(0)-$$
$$0.0014\,e^{-3.236t}\Sigma_{x_{r_2}}(0)+0.103\,e^{-4t}\Sigma_{x_{r_2}}(0)+0.0106$$

$$\Sigma_{x_1x_1}(\infty)=0.0106$$

$$\Sigma_{x_{r_1}x_{r_1}}(\infty)=\lim_{t\to\infty}[7.1112\,e^{-5(t_1+t_2)}\Sigma_{x_{r_1}}(0)+0.0035-0.0062-8.888\,e^{-7t}+$$
$$2.7772\,e^{-2(t_1+t_2)}\Sigma_{x_{r_1}}(0)+0.0034]$$
$$=0.0007$$

Step (11)

For comparison point of view we have, consider the steady-state correlation matrix of the state $\tilde{x}(t)$ of the closed-loop augmented system (3.48), can be found by using ***Lyapunov equation***, from (2.61), we obtain:

$$\begin{bmatrix} 0 & 1 & M & 0 & 0 \\ -59.9 & -49.7 & M & 0.1729 & 2.2269 \\ \mathbf{L} & \mathbf{L} & \mathbf{L} & \mathbf{L} & \mathbf{L} \\ 0 & 0 & M & -10 & -40 \\ 0 & 0 & M & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} \Sigma_{x_{11}}(\infty) & \Sigma_{x_{12}}(\infty) & M & \Sigma_{x_{13}}(\infty) & \Sigma_{x_{14}}(\infty) \\ \Sigma_{x_{21}}(\infty) & \Sigma_{x_{22}}(\infty) & M & \Sigma_{x_{23}}(\infty) & \Sigma_{x_{24}}(\infty) \\ \mathbf{L} & \mathbf{L} & \mathbf{L} & \mathbf{L} & \mathbf{L} \\ \Sigma_{x_{31}}(\infty) & \Sigma_{x_{32}}(\infty) & M & \Sigma_{x_{33}}(\infty) & \Sigma_{x_{34}}(\infty) \\ \Sigma_{x_{41}}(\infty) & \Sigma_{x_{42}}(\infty) & M & \Sigma_{x_{43}}(\infty) & \Sigma_{x_{44}}(\infty) \end{bmatrix} +$$

$$\begin{bmatrix} \Sigma_{x_{11}}(\infty) & \Sigma_{x_{12}}(\infty) & M & \Sigma_{x_{13}}(\infty) & \Sigma_{x_{14}}(\infty) \\ \Sigma_{x_{21}}(\infty) & \Sigma_{x_{22}}(\infty) & M & \Sigma_{x_{23}}(\infty) & \Sigma_{x_{24}}(\infty) \\ \mathbf{L} & \mathbf{L} & \mathbf{L} & \mathbf{L} & \mathbf{L} \\ \Sigma_{x_{31}}(\infty) & \Sigma_{x_{32}}(\infty) & M & \Sigma_{x_{33}}(\infty) & \Sigma_{x_{34}}(\infty) \\ \Sigma_{x_{41}}(\infty) & \Sigma_{x_{42}}(\infty) & M & \Sigma_{x_{43}}(\infty) & \Sigma_{x_{44}}(\infty) \end{bmatrix} +$$

$$\begin{bmatrix} 0 & -59.9 & M & 0 & 0 \\ 1 & -49.7 & M & 0 & 0 \\ \mathbf{L} & \mathbf{L} & \mathbf{L} & \mathbf{L} & \mathbf{L} \\ 0 & 0.1729 & M & -10 & 1 \\ 0 & 2.2269 & M & -40 & 3 \end{bmatrix} + \begin{bmatrix} 0.025 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.005 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0$$

$\Rightarrow$

$$\begin{bmatrix} \Sigma_{x_{11}}(\infty) & \Sigma_{x_{12}}(\infty) & \vdots & \Sigma_{x_{13}}(\infty) & \Sigma_{x_{14}}(\infty) \\ \Sigma_{x_{21}}(\infty) & \Sigma_{x_{22}}(\infty) & \vdots & \Sigma_{x_{23}}(\infty) & \Sigma_{x_{24}}(\infty) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Sigma_{x_{31}}(\infty) & \Sigma_{x_{32}}(\infty) & \vdots & \Sigma_{x_{33}}(\infty) & \Sigma_{x_{34}}(\infty) \\ \Sigma_{x_{41}}(\infty) & \Sigma_{x_{42}}(\infty) & \vdots & \Sigma_{x_{43}}(\infty) & \Sigma_{x_{44}}(\infty) \end{bmatrix} =$$

$$\begin{bmatrix} 0.0106 & 0.0002 & 0 & 0.0002 \\ 0.0002 & 0 & 0 & 0 \\ 0 & 0 & 0.0007 & 0.0043 \\ 0.0002 & 0 & 0.0043 & 0.0571 \end{bmatrix} \qquad (3.61)$$

As one can see the identical agreements between the two result

Step (12)

Computing the performance index of the problem and using concluding remarks (2.4) points (4) and (5), it follows that:

see concluding remarks points (4), (5), we get:

$$J_{SR} = tr\left\{P(\infty)\tilde{B}_x \tilde{S}_x \tilde{B}_x\right\},$$

$$\Rightarrow$$

$$J_{SR} = tr\left(\begin{bmatrix} 0.5392 & 0.0499 & \vdots & 0.3167 & -2.5700 \\ 0.0499 & 0.0487 & \vdots & -0.1729 & -2.2269 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0.3167 & -0.1729 & \vdots & 2.5199 & 17.2137 \\ -2.5700 & -2.2269 & \vdots & 17.2137 & 229.6763 \end{bmatrix}\begin{bmatrix} 0.025 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.005 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$= tr\begin{bmatrix} 0.0135 & 0 & 0.0016 & 0 \\ 0.0012 & 0 & -0.0009 & 0 \\ 0.0079 & 0 & 0.0126 & 0 \\ -0.0643 & 0 & 0.0861 & 0 \end{bmatrix}$$

$$J_{SR} = 0.0261 \qquad (3.62)$$

Conclusions

1) The tracking approach between the some stochastic dynamic system and desired one, seems a good task to find a suitable controller that guarantee the asymptotic stability of the error dynamic between the state-space and desired state-space.
2) As one can see from the illustration, if one can follow the theoretical result of the presented work, the illustration and then the solution of tracking problem may be obtained relatively easily The present approach as one can see from the illustration, and for us, it seems a practical one and good enough.
3) The complexity of finding the solution of tracking problem is ranking from easier to harder depending on the nature of dynamic system (time-variant system, time-invariant system etc., and its order (first order, second order, etc.).
4) The solution of differential Riccati equation that corresponding its optimal tracking problem becomes very complicated when final time of performance index is varying, and on easier solution may be found for steady-state case where the differential Riccati equation converting into algebraic Riccati one.
5) Some numerical solution may be needed for illustration of some tracking problem when the transient behavior is required but not in the steady-state one.

Future Work

In the future work, one can precede and developed his work based on the present study to include a large class;

1) The studying the state variables are not directly available (measurable), but only measurable in a noisy environment, the relevant methodological result, on which the optimal control (LQG) theory is based, may be considered. Since the most interesting version it states that, assuming for the control a quite general dependence on the past output and an equally general structure for the cost function, in case of linear systems and white Gaussian noises, an optimal solution will be existed for which the control may be a function of the optimal state estimate; and dynamic estimator.
2) Studying the tracking problem with additional equality and inequality constraints and some kind of nonlinearity as well as some kind of uncertainty may be considered in future.
3) Studying the case when the problem of tracking using Ito-stochastic dynamic system derived by Brownian motion may also be developed.

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APPNDIX [A]

A.1 NON-LINEAR MODEL AND LINEARIZATION [20]

It would seem apparent that the linear model is more amenable to analytic consideration than the corresponding non-linear system model.

A linear model can of course arise in two different ways (1) when the system it self is linear, and (2) as an approximation to non-linear system. The procedure for carrying out the second arrangement is referred to as linearization.

The basic idea of linearization is to expand the variables in a Taylor series expansion about an equilibrium point x_e and then truncate this series to retain only the corresponding linear terms.

Consider the system:

$$\dot{x}(t) = f\,[x(t), u(t), t] \qquad \text{(A.1)}$$

if $u_0(t)$ is a given input to a system described by the state differential equation (A.1), and $x_0(t)$ is a known solution of (A.1), we can find approximation to neighboring solutions, for small deviations in the initial state and the input, from a linear state differential equation. Suppose that $x_0(t)$ satisfies

$$\dot{x}_0(t) = f[x_0(t), u_0(t), t] \qquad t_0 \le t \le t_1$$

we refer to u_0 as a ***nominal input*** to x_0 as a ***nominal trajectory.*** Often we can assume that the system is operated close to nominal conditions, which means that u and x deviate only slightly from u_0 and x_0.

Let $$u(t) = u_0(t) + \tilde{u}(t) \qquad t_0 \le t \le t_1$$

$$x(t_0) = x_0(t_0) + \tilde{x}(t_0) \qquad \text{(A.2)}$$

where $\delta u(t)$ and $\delta x(t_0)$ are small perturbations.

Correspondingly , let us introduce $\delta x(t)$ by:

$$x(t) = x_0(t) + \delta x(t) \qquad t_0 \le t \le t_1$$

Now, substitute x and u into the state differential equation and make a Taylor expansion. It follows that:

$$\dot{x}_0(t) + \delta\dot{x}(t) = f[x_0(t), u_0(t), t] + J_x[x_0(t), u_0(t), t]\delta x(t)$$
$$+ J_u[x_0(t), u_0(t), t]\delta u(t) + h(t) \qquad t_0 \le t \le t_1 \qquad \text{(A.3)}$$

J_x and J_u are the Jacobian matrices of f with respect to x and u, respectively, that is, J_x is matrix the (i, j)-*th* element of which is

$$(J_x)_{i,j} = \frac{\partial f_i}{\partial y_j}$$

where f_i is the *i-th* component of f and y_j the *j-th* component of x. J_u is similarly defined. The term *h(t)* is an expression that is supposed to be "small" with respect to δx and δu. Neglecting h, we see that δx and δu approximately satisfy the ***linear*** equation

$$\delta\dot{x}(t) = \mathrm{A}(t)\delta x(t) + \mathrm{B}(t)\delta u(t) \qquad t_0 \le t \le t_1 \qquad \text{(A.4)}$$

where $\mathrm{A}(t) = J_x[x_0(t), u_0(t), t]$ and $\mathrm{B}(t) = J_u[x\ (t), u_0(t), t]$. (A.4) is called the ***linearized state differential equation for time-invariant*** the linearization leads to the equation:

$$\delta\dot{x}(t) = \mathrm{A}(t)\delta x(t) + \mathrm{B}(t)\delta u(t)$$

A.2 Homogeneous State Equations and it Solution [27]

Consider

$$\dot{x}(t) = \mathrm{A}x(t) \ , \ x(t_0) = x_0 \qquad \text{(A.5)}$$

Where $x(t)$ is n-vector, A is $n \times n$ constant matrix, and $x(t_0) \in R^n$. The solution of equation (A.5) can be written as

$$x(t) = e^{\mathrm{A}t} x_0 \qquad \text{(A.6)}$$

We can expand e^{At} into a power series in t ,we have

$$x(t)=\left(I+\frac{At}{1!}+\frac{(At)^2}{2!}+\frac{(At)^3}{3!}+\ldots+\frac{(At)^n}{n!}+\ldots\right)x_0$$

in order to write explicit analytical solution we may use the following:

If the eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$ of the matrix A are distinct, then

$$e^{At}=Ve^{(V^{-1}AV)t}V^{-1}=Ve^{Zt}V^{-1}$$

Where V is $n\times n$ nonsingular matrix, whose columns are the eigenvectors corresponding to the eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$, and $Z=V^{-1}AV$. Once A is given, the eigenvalues are determined, then a necessary transformation matrix V can be obtained by this method and

$$e^{At}=Ve^{Zt}V^{-1}$$

$$=V\begin{pmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{pmatrix}V^{-1} \qquad \text{(A.7)}$$

Remark

Consider $\dot{x}(t)=Ax(t)$, where A is $n\times n$ constant matrix, if $V=[u_1 \quad u_2 \quad \ldots \quad u_n]$, where $Au_i=\lambda_i u_i \quad ,i=1,2,\ldots,n$, on setting $w=Vx$, then

$\dot{w}=V\dot{x}\Rightarrow \dot{w}=V\,Ax \Rightarrow \dot{w}=V\,AV^{-1}w$, and hence $\dot{w}=V\,AV^{-1}w$, where $V\,AV^{-1}$ is a diagonal (block) similar matrix to A and its diagonalization architecture depends on the nature of eigenvalues whether they are real and distinct, real and have multiplicity of some order, complex and distinct, complex and have some multiplicity or even mixed of real and complex, distinct or not. In most cases the solution can be found explicitly and hence its diagonalization can be formed easily. This is sometime called a Jordan canonical forms or a general diagonalization. The selection of transformation matrix V can be easily obtained depending on the eigenvectors corresponding to distinct eigenvalues or generalized eigenvectors when some multiplicity of eigenvalues are available, to ensure the nonsingularty of a matrix V .

A.2.1 Non –homogeneous State Equation [27]

Consider the non-homogeneous state equation:

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}y(t), \quad x(t_0) = x_0 \tag{A.8}$$

Where $x(t) = n - vector$, $x(t) = r - vector$, A is $n \times n$ constant matrix and B is $n \times r$ constant matrix is given by (A.8) as

$$\dot{x}(t) - \mathrm{A}x(t) = \mathrm{B}y(t)$$

and pre-multiplying both sides of this equation by $e^{-\mathrm{A}t}$, we obtain:

$$e^{-\mathrm{A}t}[\dot{x}(t) - \mathrm{A}x(t)] \equiv \frac{d}{dt}[e^{-\mathrm{A}t}x(t)] = e^{-\mathrm{A}t}\mathrm{B}y(t)$$

Integrating the preceding equation between t_0 and t gives,

$$e^{-\mathrm{A}(t-t_0)}x(t) = x(t_0) + \int_{t_0}^{t} e^{-\mathrm{A}(t-t_0)}\mathrm{B}y(t)dt$$

$$x(t) = e^{\mathrm{A}(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{\mathrm{A}(t-t)}\mathrm{B}y(t)dt \tag{A.9}$$

And

If $t_0 = 0$, then (A.9) become

$$x(t) = e^{\mathrm{A}t}x(0) + \int_{0}^{t} e^{\mathrm{A}(t-t)}\mathrm{B}y(t)dt \tag{A.10}$$

A.3 IMPULES AND CORRELATION FUNCTION [29]

The impulse function whose area is equal to unity is called the unit-impulse function or the Dirac delta function.

1) The unit-impulse function occurring at $t = t_0$ is usually denoted by $d(t - t_0)$.

2) The Dirac delta function $d(t - t_0)$ satisfies the following properties :

a) $d(t - t_0) = \begin{cases} 0 & for \quad t = t_0 \\ \infty & for \quad t \neq t_0 \end{cases}$

b) $\int_{-\infty}^{\infty} d(t - t_0)dt = 1$

A.3.1The Output Correlation Function [8]

Assuming $t_2 > t_1$,

$$R_x(t_1,t_2) = e^{\mathrm{A}t_1}\Sigma_x(0)e^{\mathrm{A}^T t_2} + \int_0^{t_1}\int_0^{t_2} e^{\mathrm{A}(t_1 - t)}\mathrm{B}S_x d(t - s)\mathrm{B}^T e^{\mathrm{A}^T (t_2 - s)} ds dt$$

(A.11)

$$R_x(t_1,t_2) = e^{\mathrm{A}t_1}\Sigma_x(0)e^{\mathrm{A}^T t_2} + \int_0^{t_1} e^{\mathrm{A}(t_1 - t)}\mathrm{B}S_x \mathrm{B}^T e^{\mathrm{A}^T (t_2 - t)} dt \qquad (A.12)$$

For $t_1 > t_2$, by integrating with respect to t we obtain:

$$R_x(t_1,t_2) = e^{\mathrm{A}t_1}\Sigma_x(0)e^{\mathrm{A}^T t_2} + \int_0^{t_2} e^{\mathrm{A}(t_1 - s)}\mathrm{B}S_x \mathrm{B}^T e^{\mathrm{A}^T (t_2 - s)} ds \qquad (A.13)$$

From (A.12) and (A.13) we obtain:

$$R_x(t_1,t_2) = e^{\mathrm{A}t_1}\Sigma_x(0)\, e^{\mathrm{A}^T t_2} + \int_0^{\min(t_1,t_2)} e^{\mathrm{A}(t_1 - s)}\mathrm{B}S_x \mathrm{B}^T e^{\mathrm{A}^T (t_2 - s)} ds$$

(A.14)

APPNDIX [B]

B.1 QUADRATIC OPTIMAL REGULATOR PROBLEMS [27]

Consider the optimal control problem that, given the system equation

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t) \qquad \text{(B.1)}$$

where

$$u(t) = -\mathrm{K}x(t) \qquad \text{(B.2)}$$

And the cost function is defined as follows:

$$J = \int_0^{\infty} x^T(t)Qx(t) + u^T(t)Ru(t)dt \qquad \text{(B.3)}$$

Where Q and R are positive definite, where the second term on the right-hand side of cost function (B.3) accounts for the expenditure of the energy of the control signals, and assume that the control vector *u(t)* is unconstrained.

The linear control law given by equation (B.2) is the feedback control law. Therefore, if the unknown elements of the matrix K are determined so as to minimize the cost function, then $u(t) = -\mathrm{K}x(t)$ is optimal for any initial $x(0)$.

Substituting equation (B.2) into (B.1), we obtain

$$\dot{x}(t) = \mathrm{A}x(t) - K\mathrm{B}x(t) = (\mathrm{A} - \mathrm{B}K)x(t)$$

We assume that the matrix $(\mathrm{A} - \mathrm{B}K)$ is stable on using controllability assumption, or that the eigenvalues of $(\mathrm{A} - \mathrm{B}K)$ have negative real parts. Substituting (B.2) into (B.3), yields

$$J = \int_0^{\infty} x^T(t)Qx(t) + x^T(t)K^T RKx(t)dt$$

$$= \int_0^{\infty} x^T(t)[Q + K^T RK]x(t)dt$$

Let

$$x^T(t)(Q+K^TRK)x(t)=-\frac{d}{dt}(x^T(t)\mathrm{P}x(t))$$

Where P is a positive-definite matrix. Then we obtain

$$x^T(t)(Q+K^TRK)x(t)=-(\dot{x}^T(t)\mathrm{P}x(t)+x^T(t)\mathrm{P}\dot{x})$$
$$=-x^T(t)[(\mathrm{A}-\mathrm{B}K)^T\mathrm{P}+\mathrm{P}(\mathrm{A}-\mathrm{B}K)]x(t)$$

Comparing both sides of this last equation and noting that this equation must hold true for any *x*, we require that

$$(\mathrm{A}-\mathrm{B}K)^T\mathrm{P}+\mathrm{P}(\mathrm{A}-\mathrm{B}K)=-(Q+K^TRK) \tag{B.4}$$

If $(\mathrm{A}-\mathrm{B}K)$ is stable matrix, there exist a positive –definite matrix P that satisfies (B.4).

Hence to determine the elements of P from equation (B.4)and if is positive definite.

The cost function *J* can be evaluated as

$$J=\int_0^\infty x^T(t)[Q+K^TRK]x(t)dt=-x^T\mathrm{P}x\Big|_0^\infty$$
$$=-x^T(\infty)\mathrm{P}x(\infty)+x^T(0)\mathrm{P}x(0)$$

Since eigenvalues of $(\mathrm{A}-\mathrm{B}K)$ are assumed to have negative real part, we have $x(\infty)\rightarrow 0$. Therefore, we obtain :

$$J=x^T(0)\mathrm{P}x(0) \tag{B.5}$$

Thus, the cost function *J* can be obtained in terms of the initial condition $x(0)$ and P.

To obtain the solution to the quadratic optimal control problem, we proceed as since *R* has been assumed to be positive-definite matrix or real symmetric matrix, we can write

$$R=H^TH$$

Where *H* is nonsingular matrix. Then (B.4), can be written as

$$(A^T - B^T K^T)P + P(A - BK) + Q + K^T H^T HK = 0$$

Which can be rewritten as

$$A^T P + PA + [HK - (H^T)^{-1} B^T P][HK - (H^T)^{-1} B^T P] - PBR^{-1} B^T P + Q = 0$$

The minimization of J with respect to K requires the minimization of

$x^T [HK - (H^T)^{-1} B^T P][HK - (H^T)^{-1} B^T P]$ with respect to *K.*

Since the last expression is non negative, the minimum occurs when it is zero, or when

$$HK = (H^T)^{-1} B^T P$$

Hence

$$K = H^{-1}(H^T)^{-1} B^T P = R^{-1} B^T P \qquad \text{(B.6)}$$

The optimal control law to the quadratic optimal control problem when the performance index is given by (B.2), is linear and is given by

$$u(t) = -Kx(t) = -R^{-1} B^T Px(t)$$

The matrix P in (B.6), must satisfy (B.4) or the following reduced equation:

$$A^T P + PA - PBR^{-1} B^T P + Q = 0 \qquad \text{(B.7)}$$

Equation (B.7), is called the reduced –matrix algebraic Riccati equation.

Finally, note that if the cost function is given in terms of the outputs vector rather than the state vector, that is

$J = \int_0^\infty Y^T(t)QY(t) + u^T(t)Ru(t)dt$ then the index can be modified by using the output equation

$Y = Cx(t)$, to

$$J = \int_0^\infty x^T(t)C^T QCx(t) + u^T(t)Ru(t)dt \qquad \text{(B.8)}$$

B.2 White noise can be expressed as the derivative of process with uncorrelated increments [20]

$$\Xi_x(t_1,t_2)=\begin{cases}\bar{\Sigma}_x(t_1) & for \quad t_2 \ge t_1 \ge t_0 \\ \bar{\Sigma}_x(t_2) & for \quad t_1 \ge t_2 \ge t_0\end{cases} \tag{B.9}$$

Proceeding completely formally, let us show that the covariance matrix of the derivative process

$\dot{x}(t)=\dfrac{dx(t)}{dt}, t \ge t_0$ consists of a delta function. For the mean of the derivative process, we have

$$E[\dot{x}(t)]=\frac{d}{dt}E[x(t)]=0 \qquad t \ge t_0 \tag{B.10}$$

For the covariance function of the derivative process we write, completely formally,

$$\Xi_x(t_1,t_2)=E[\dot{x}(t_1)\dot{x}(t_2)]$$

$$= \frac{\partial^2}{\partial t_1 \partial t_2} E[x(t_1)x^T(t_2)]$$

$$= \frac{\partial^2}{\partial t_1 \partial t_2} \Xi_x(t_1,t_2) \qquad t_1,t_2 \ge t_0 \tag{B.11}$$

Now , successively carrying out the partial differentiations, we obtain

$$\Xi_x(t,t)=\dot{\Sigma}_x(t_1)d(t_1-t_2) \qquad t_1 \ge t_2 \ge t_0 \tag{B.12}$$

Where

$$\dot{\Sigma}(t)=\frac{d\bar{\Sigma}(t)}{dt}$$

This shows that the derivative of a process with uncorrelated increments is a white noise. When each increment $x(t_2)-x(t_1)$ of the process has a variance matrix that may be written in the form $\bar{\Sigma}(t)=\int_{t_0}^{t_1} S(t)dt$ the intensity of the white noise process that derives from the process with uncorrelated increments is $S(t)$.

Theorem (B.1) [20]

The transition matrix $\Phi(t,t_0)$ of a linear differential system has the following properties:

1- $\Phi(t_2,t_1)\Phi(t_1,t_0)=\Phi(t_2,t_0)$, for all t_0, t_1, and $t_2 \in R$

2- $\Phi(t,t_0)$ is nonsingular, for all t and $t_0 \in R$.

3- $\Phi^{-1}(t,t_0)=\Phi(t_0,t)$, for all t and $t_0 \in R$.

4- $\frac{d}{dt}\Phi^{\mathrm{T}}(t_0,t)=-\mathrm{A}^{\mathrm{T}}(t)\Phi^{\mathrm{T}}(t,t_0)$, for all t and $t_0 \in R$.

(Where T denotes the transpose operator).

B.3 HOMOGENEOUS LINEAR TIME VARYING SYSTEM [26]

Consider the homogeneous linear vector matrix differential equation

$$\dot{x}(t)=\mathrm{A}(t)x(t), \quad x(t_0)=x_0 \tag{B.13}$$

Where $x(t)$ is state vector (n-vector), $\mathrm{A}(t)$ is $n\times n$ matrix whose elements are at least piecewise continuous functions of t in the interval $t_0 \le t \le t_1$, (or integrable). The solution of equation (B.13), is given by:

$$x(t)=\Phi(t,t_0)x(t_0) \tag{B.14}$$

Where $\Phi(t,t_0)$ is $n\times n$ non singular (transition or fundamental) matrix satisfying the following matrix differential equation

$$\dot{\Phi}(t,t_0)=\mathrm{A}(t)\Phi(t,t_0), \text{ and } \Phi(t_0,t_0)=\mathrm{I} \tag{B.15}$$

The fact, $\Phi(t,t_0)$ is the unique solution of equation (B.15), we can easily verify this, where the state transition matrix $\Phi(t,t_0)$ have the following properties, see theorem (B.1)

B.3.1 Non-homogeneous Linear Time-Varying System and it Solution

Consider the following state equation which is defined as:

$$\dot{x}(t) = \mathrm{A}(t)x(t) + \mathrm{B}(t)u(t), \quad x(t_0) = x_0 \tag{B.16}$$

Where $x(t)$ is state vector (n-vector), $u(t)$ is control vector (m-vector), $\mathrm{A}(t)$ is $n \times n$ matrix and $\mathrm{B}(t)$ is $n \times r$ matrix. We assume that the elements of $\mathrm{A}(t)$ and those of $\mathrm{B}(t)$ are absolutely integrable as functions of t in the interval $t_0 \le t \le t_1$. Let $\Phi(t, t_0)$ be the unique matrix which satisfies equation (B.15), rewritten as follows:

$$\frac{d}{dt}\Phi(t,t_0) = \mathrm{A}(t)\Phi(t,t_0), \quad \Phi(t_0,t_0) = \mathrm{I}$$

Assume that

$$x(t) = \Phi(t,t_0)y(t)$$

Then

$$\dot{x}(t) = \frac{d}{dt}\left[\Phi(\mathrm{t},\mathrm{t}_0)y(t)\right]$$

$$= \dot{\Phi}(t,t_0)y(t) + \Phi(\mathrm{t},\mathrm{t}_0)\dot{y}(t), \text{ (using (B.15))}$$

$$= \mathrm{A}(t)\Phi(t,t_0)y(t) + \Phi(t,t_0)\dot{y}(t)$$

$$\equiv \mathrm{A}(t)\Phi(t,t_0)y(t) + \mathrm{B}(t)u(t)$$

Thus

$$\Phi(t,t_0)\dot{y}(t) = \mathrm{B}(t)u(t) \Rightarrow \dot{y}(t) = \Phi^{-1}(t,t_0)\mathrm{B}(t)u(t)$$

Integrating both sides of the last equation from t_0 to t, we have

$$y(t) = y(t_0) + \int_{t_0}^{t} \Phi^{-1}(t,t_0)\mathrm{B}(t)u(t)dt \tag{B.17}$$

Since

$$y(t_0) = \Phi^{-1}(t_0,t_0)x(t_0) \Rightarrow y(t_0) = x(t_0) \Rightarrow y(t_0) \equiv x_0$$

The solution of equation (B.16), then can be given by

$$x(t) = \Phi(t, t_0) y(t), \qquad \text{(using (B.17))}$$

$$= \Phi(t, t_0) x_0 + \Phi(t, t_0) \int_{t_0}^{t} \Phi^{-1}(t, t_0) B(t) u(t) dt$$

From the properties of $\Phi(t, t_0)$, we have

$$x(t) = \Phi(t, t_0) x_0 + \int_{t_0}^{t} \Phi(t, t) B(t) u(t) dt \qquad \text{(B.18)}$$

B.4 RELATION INVOLVING THE TRACE [8]

Theorem (B.2)

The trace is invariant under cyclic perturbations:

$$tr(AB) = tr(BA)$$

Where AB is square, and therefore the trace is defined.

Proof

Let $A \in C^{n \times l}$ and $B \in C^{l \times n}$; then $AB \in C^{n \times n}$ and $BA \in C^{l \times l}$.

Expanding the matrix multiplication and noting that the trace is the sum of the diagonal elements of the matrix, we have:

$$tr(AB) = \sum_{k=1}^{n} \sum_{i=1}^{l} a_{ki} b_{ik} = \sum_{i=1}^{l} \sum_{k=1}^{n} b_{ik} a_{ki} = tr(BA)$$

Successive applications of the theorem (B.1), we have:

Lemma (B.1)

$$tr(ABC) = tr(CAB) = tr(BCA)$$

Provided ABC is square.

B.5 SINGULAR VALUE DECOMPOSITION [8]

The singular value decomposition (SVD) is a matrix factorization that has found a number of applications to engineering problems. The (SVD) of matrix $M \in \Im^{r \times m}$ is

$$M = USV^T = \sum_{i=1}^{p} s_i U_i V_i^T \tag{B.19}$$

Where

$V \in \Im^{n_v \times n_v}$, and $U \in \Im^{n_u \times n_u}$ are unitary matrices ($U^T U = I, V^T V = I$); $S \in \Re^{n_v \times n_u}$ is diagonal (but not necessarily square); and p equals the minimum of n_v and n_u. The singular values $\{s_1, \; s_2, \; \mathbf{L}, \; s_{nu}\}$ of M are defined as the positive square roots of the diagonal elements of $S^T S$, and are ordered from largest to smallest.

المستخلص

لقد ركزنا في هذه الرسالة على دراسة مسالة المتابعة **(Tracking Problem)** لمعادلات تفاضلية عشوائية مع الزمن بوجود المؤثر المتغير العشوائي من نوع ضوضاء ابيض **(White Noise)** و مسيطر مدخل **(Input Control)**.

لقد تم في هذا العمل التركيز على الانظمة الدينامية المتمثلة بالنظام الاصلي ونظام الهدف العشوائي المطلوب **(Desired Stochastic Process)** المشتقة من فوضوية عشوائية متغيرة مع الزمن من النوع الابيض **(White Noise)**.

اعطي اهتمام خاص على دراسة هذه الحالة لجعل سلوك النظام الدينامي الاصلي يتبع سلوك النظام المرغوب فيه بغض النظر عن تعددية المسيطر وظهور المتغير العشوائي الفوضوي، وباستعمال اسلوبية المتابعة المسيطرة للنظام.

لقد تم كذلك اشتقاق وتطوير المسيطر المتابع والذي يجعل النظام مستقرا و امثل للنظام الخطي الناتج من الفرق بين النظام الاصلي والنظام المرغوب فيه. لقد نوقشت وعرضت النظريات والفرضيات الرياضية مدعومة بالبراهين الضرورية اللازمة، وقُسم المسيطر فيه الى رصين (متين) ومسيطر أمثلي.

تم الحصول على المسيطر الامثل من خلال حل بعض المعادلات التفاضلية الخطية المعروفة باسم معادلات ريكاتي **(Riccati Equations)** بينما يحل المسيطر المتين (الرصين) بطرق اخرى مع ضمان شرط قابلية السيطرة للنظام الدينامي قيد الدراسة.

لقد نوقشت واشتقت المعادلات التفاضلية من نوع ريكاتي **(Riccati Equations)** والمرافقة لحلولية المسيطر العشوائي الامثل للنظام الدينامي.

اخيرا ، لقد تم عرض بعض التطبيقات الرياضية والمرتبطة بالسلوك الضوضائي **(White Noise)** متدرجة من الاسهل الى الاصعب ، واعتمدت على اسلوبية مشتقة من البرهان الرياضي في كيفية ايجاد الحلول.

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جــامعة النهـــريـن
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حول أمثلية أنظمة السيطرة المتابعة التصادفية اللاخطية

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