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Analytical and Approximate Solutions for Volterra Integro-Differential Equations and It's Applications

A Thesis

Submitted to the College of Science, Al-Nahrain University as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

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بسم الله الرحمن الرحيم

يَٰٓأَيُّهَا ٱلَّذِينَ
ءَامَنُوٓاْ إِذَا قِيلَ لَكُمۡ تَفَسَّحُواْ فِي ٱلۡمَجَٰلِسِ فَٱفۡسَحُواْ يَفۡسَحِ
ٱللَّهُ لَكُمۡۖ وَإِذَا قِيلَ ٱنشُزُواْ فَٱنشُزُواْ يَرۡفَعِ ٱللَّهُ ٱلَّذِينَ ءَامَنُواْ
مِنكُمۡ وَٱلَّذِينَ أُوتُواْ ٱلۡعِلۡمَ دَرَجَٰتٖۚ وَٱللَّهُ بِمَا تَعۡمَلُونَ خَبِيرٞ ١١

صدق الله العظيم

سورة المجادلة

الآية (11)

Deduction

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Firas Shakir Ahmed
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Supervisors Certification

We, certify that this thesis entitled " **Analytical and Approximate Solutions for Volterra Intgro – Differential Equations and It's Applications** " was prepared by " Firas Shakir Ahmed " under our supervision at the College of Science / Al-Nahrain University as a partial fulfillment of the requirements for the degree of Master of Science in Applied Mathematics.

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We, the examining committee certify that we have read this thesis entitled "**Analytical and Approximate Solutions for Volterra Integro-Differential Equations and It's Applications**" and examined the student **''Firas Shakir Ahmed''** in its contents and that is in our opinion, it is accepted for the degree of Master of Science in Mathematics.

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Summary

The main objective of this thesis is to study and solve the Volterra integral and integro-differential equation and some scientific models for real life problems.

This objective may be divided into three sub objectives, as follows:

The first one is to classify and study the subject of the integral equations and to supply the basic definitions related to the Volterra integral and integro-differential equations.

In the second sub objective we used the " dafterdar-jafari method " to solve Volterra integral and integro-differential equations.

The third sub objective is to introduce another iterative method which is called the power series method to provide a solution of the Volterra integral and integro-differential equations.

Finally, the application of the dafterdar-jafari method and the power series method are presented for finding the analytic and approximate solution for some real life scientific problems, namely, Volterra's population model, the hybrid selection model, Riccati equation and the logistic model. Also, it is important to remark that the computer programs are coded using the computer software Mathematica.8.

List of Symbols and Abbreviations

ADM	Adomian Decomposition Method.
a	The birth rate coefficient.
b	The crowding coefficient.
c	The toxicity coefficient.
$a(x)$, $b(x)$, $c(x)$	Continuous functions.
DJM	Dafterdar-Jafari Method
HPM	Homotopy Perturbation Method
IVP	Initial Value Problem.
κ	Non-dimensional parameter.
PSM	Power Series Method
$P(T)$	The population size at time T.
P_0	The initial population size.
u(t)	The scaled population of identical individuals at a time t.
λ	The Lagrange multiplier.
μ	Positive constant.
VIE	Volterra Integral Equation.
VIDE	Volterra Integro-Differential Equation.
VPM	Volterra's Population Model.
VIM	Variational Iteration Method.

Contents

Introduction

The study of integral equations might be regarded of more fundamental importance than the differential equations because an integral equation often provides less restriction and more useful model than differential equations. Moreover, in numerical analysis, differentiation increases error, but integration will tend to smooth errors out [14].

The advantage of integral equations is witnessed by the increasing frequency of integral equations in the literature and in many fields of applied mathematics, since some problems have their mathematical representation appear directly, and in a very natural way, in terms of integral equations. Other problems, whose direct representation is in term of differential equations, have their auxiliary conditions replaced by integral equations more elegantly and compactly than the differential equations.

Historically, Volterra started the working on integral equations in 1884, but his serious study began in 1896. The name integral equations was given by du Bois-Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908. During the last years, many researches had been studies concerned with the solution of integral equations and integro-differential equations. Burner in 1974 [8], studied the approximate solution of first kind integral equations of Volterra type. In 1988 Burner [9], study the application of certain spline collection methods to Volterra integro-differential equations of a certain order, and so on, Hosseini in 2003 [22], extend the Tau method to integro-differential equations to produce a numerical solution. In 2005 Al-Jawary [6], investigate the numerical solution of a system of linear second kind Volterra integral equations. In 2006 Hashim [21] used the Adomian Decomposition Method (ADM) for solving boundary value problems for the fourth-order integro-differential equations. Abbasbandy in 2008 [3] used the Variational Iteration Method (VIM) to solve nonlinear Volterra integro-differential equations. Ali, in 2010 [4], introduce the fractional integro-differential equations using a modified type of operators, which consists of the same order fractional differentiation and fractional integration which have been solved using the modified Adomian Decomposition Method. Erfanian in 2011 [18], introduce an approach by an optimization method to find approximate solution for a class of nonlinear Volterra integral equations of the first and second kinds. In 2012 Mirazaee [30], had used the repeated Simpson's quadrature rule to solve the linear Volterra integral equations of the first kind by approximating the integral. In 2012 Venkatesh [40], studied the Legendre wavelets for the solution of boundary value problems for a class of higher order Volterra integro-differential equations. Wadeá in 2012 [41], studied the approximate solution of fractional integro-differential equations using Variational Iteration Method (VIM). Rashidinia in 2013 [33], developed and modified Taylor expansion method for approximating the solution of linear Fredholm and Volterra integro-differential equations. Alao et al. in 2014 [7], using both ADM and VIM on various types of integro-differential equations, which are the Fredholm, Volterra and Fredholm- Volterra equations.

In this thesis, an analytic and approximation solutions for Volterra integral and integro-differential equations is presented.

The Dafterdar-Jafari method (DJM) and the power series method (PSM) are implemented independently to the integral equations. The DJM has been extensively used by many researchers for the treatment of linear and nonlinear ordinary and partial differential equations of integer and fractional order . In [10] Bhalekar applied the DJM to solve partial differential equations. In [11] Bahlekar solved the evolution equation using DJM. In [16] Dafterdar solved the fractional boundary value problems with dirichlet boundary conditions using the DJM. It is important to notice that the DJM will converges to the exact solution, if it exists, through successive approximations. The PSM and DJM results demonstrate that the methods has many merits such as being derivative-free, overcome the difficulty arising in calculating Adomian polynomials to handle the nonlinear terms in ADM, it does not require to calculate Lagrange multiplier as in the VIM and no needs to construct a homotopy and solve the corresponding algebraic equations as in Homotopy Perturbation Method (HPM).

Moreover, in this thesis the proposed DJM and PSM will be used for the first time to solve real life applications, such as,Volterra's population model, the Hybrid selection model, the Riccati equation and the Logistic differential equation.

Recently many attempts have been made to develop analytic and approximate methods to solve the Volterra's population model. Although, such methods have been successfully applied, but some difficulties have appeared. Wazwaz [43] used ADM to solve the governing problem. Moreover, Mohyud-Din et al.[29] employed the combining of the HPM and Pade technique to obtain the numerical solutions to Volterra's population model. Parand et al. [31] established a method based on collocation approach using Sinc functions and Rational Legendre functions. Also, [28] a numerical method based on hybrid function consist of block-pulse and Lagrange-interpolating polynomials approximations was proposed to solve Volterra's population model and in [27] Khana used New Homotopy Perturbation Method (NHPM) which is an improvement of the classical HPM. Al-Khaled [5] implemented the ADM and Sinc-Galerkin method for the solution of some mathematical population growth models. In addition, Ramezani et al. [32] applied the spectral method to solve Volterra's population on a semi infinite interval.

The work in this thesis is divided into five chapters; the first chapter is an introductory chapter presents the basic and main aspects of the subject of integral equations in order to give a wide range of background to the readers concerned with the subject of integral equations. The second chapter consists the DJM for solving Volterra integral and integro-differential equation. Chapter three the PSM used and implemented to solve the Volterra integral and integro-differential equation. In Chapter four, some scientific models are presented to evaluate the analytic and approximate solutions of such models using DJM and PSM. Finally, in chapter five, conclusion and recommendations are presented.

The analytical and approximate results are presented for some selected illustrative examples which are programmed and coded using the computer program mathematica 8.

Chapter One

Introductory Concepts of Integral Equations

1.1 Introduction

Theory of integral equations has close contacts with many different areas of mathematics. Foremost among these are differential equations and operator theory. Any problems in the field of ordinary and partial differential equations can be recast as integral equations. Many existence and uniqueness results can then be derived from the corresponding results in integral equations. Also, many problems of mathematical physics can be stated in the form of integral equations, and therefor it is suffice to say that there is almost no area of applied mathematics and mathematical physics where integral equations do not play a role. In another meaning, one can view the subject of integral equations as an extension of linear algebra. Especially in dealing with linear integral equations the fundamental concepts of linear vector spaces, eigen values and eigen functions will play a significant role, [20]. This chapter consist of six sections. In section 1.2 some preliminaries related to the concept of integral equations are given. In section 1.3 the classification of integral equations is given for completeness purpose, wile in section 1.4 the relationship between IVP and volterra integral equation have been discussed. In section 1.5 the powerful of Padé approximate have been studied for function and polynomials, as well as some illustrative examples are given. Finely sections 1.6 and 1.7 present the basic ideas of the adomian decomposition method and varitional iteration method for solving approximatly nonlinear operator equations.

1.2 Preliminaries

An integral equation is an equation in which the unknown function u appears under an integral sign [42]. A standard integral equation in u is of the form:

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t,u(t))dt, x \in [0,t] \qquad (1.1)$$

where g and h are two functions represent the limits of integration, λ is a constant parameter plays the role of an eigen value, and k is a function of two variables x and t called the kernel or the nucleus of the integral equation. The function u that will be determined appears under the integral sign, and it may be appears inside the integral sign and outside the integral sign as well. The functions f and k are given in advance. It is to be noted that the limits of integration g and h may be both variables, constants, or mixed. An integro-differential equation is an integral equation in which the unknown function u appears under an integral sign and contains an ordinary derivative of u up to certain order as well. A standard integro-differential equation is of the form:

$$u^{(n)}(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t,u(t))\, dt, n \in N \qquad (1.2)$$

where $u^{(n)}(x) = \dfrac{d^n u}{dx^n}$ and $u(0), u'(0), ..., u^{(n-1)}(0)$ are the initial conditions.

The representation of some problems has its mathematical appearance, directly and in a very natural way, in term of integral equations. Other problems, whose direct representation is in terms of differential equations and their auxiliary conditions, may also be reduced to integral equation. Integral and integro-differential equations arise in many scientific and engineering applications (for more details see [25]). Volterra integral equations and Volterra integro-differential equations can be obtained from converting initial value problems with prescribed initial values. However, Fredholm integral equations and Fredholm integro-differential equations can be derived from boundary value problems with given boundary conditions.

One of the simplest integral equations arises from the shop stocking problem [14]. It is found that a proportion $k(t)$ remains unsold at time t after the shop has purcased the goods. It is required to find the rate at which the shop should purchase the goods, so that the stock of the goods in the shop remains constant. Suppose that the shop commences business in the goods by purchasing an amount A of the goods at zero time, and buys at a rate $\phi(t)$ subsequently. Over the time interval $\tau \le t \le \tau + \delta\tau$, an amount $\phi(t)\delta\tau$ is bought by the shop, and at time t the portion of this remaining unsold is $K(t-\tau)\phi(t)\delta\tau$. Thus the amount of goods remaining unsold at time t and which was bought to that time, is given by

$$AK(t) + \int_0^t K(t-\tau)\phi(\tau)\,d\tau \tag{1.3}$$

This is the total stock of the shop and is to remain constant at its initial value and so

$$A = AK(t) + \int_0^t K(t-\tau)\phi(\tau)\,d\tau \tag{1.4}$$

The required restocking rate $\phi(t)$ is the solution of this integral equation.

Another problem is the smoke absorption or filtration a cigarette, [25] which can also be represented as an integral equation. The deposited weight per until length $w(x,t)$ of certain tobacco component at distance x (measured from the original $t=0$ position of the burning tip) at time t after lighting a cigarette is represented by the integral equation:

$$w(x,t) = w(x,0) + a\,b\,v\,e^{-bx}\int_0^1 w(v,\tau)\,e^{bw}\,d\tau,\ \ x = vt \tag{1.5}$$

where $w(x,0)$ is the initial value of $w(x,t)$ at x before burning, a the constant function of the weight that is drawn through the cigarette ($1-a$ in the air), b the absorption coefficient of the cigarette acting as a filter, and v the assumed constant velocity of the moving burning tip where $x=vt$, [25].

1.3 Classification of the Integral Equations

Integral equations appears in many types. Such types depend mainly on the limits of integration and the kernel of the equation [42]. Two distinct ways that depend on the limits of integration are used to characterize integral equations, namely:

1. If the limits of integration are fixed, the integral equation is called a Fredholm integral equation which given in the form [42]:

$$u(x)=f(x)+\lambda\int_a^b k(x,t,u(t))dt \tag{1.6}$$

where a and b are constants.

2. If at least one limit is a variable, then the equation is called a Volterra integral equation given in the form [42]:

$$u(x)=f(x)+\lambda\int_a^x k(x,t,u(t))dt \tag{1.7}$$

Moreover, two other distinct kinds, that depend on the appearance of the unknown function u, are defined as follows:

1) If the unknown function u appears only under the integral sign of Fredholm or Volterra equation, the integral equation is called a first kind Fredholm or Volterra integral equation, respectively.

2) If the unknown function u appears both inside and outside the integral sign of Fredholm or Volterra equation, the integral equation is called a second kind Fredholm or Volterra integral equation respectively. In all

Fredholm or Volterra integral equations presented above, if the function f is identically zero, the resulting equation:

$$u(x) = \lambda \int_a^b k(x,t,u(t))dt \tag{1.8}$$

or

$$u(x) = \lambda \int_a^x k(x,t,u(t))dt \tag{1.9}$$

is called homogeneous Fredholm or homogeneous Volterra integral equation respectively.

<u>Definition (1.1), [20]</u>

An integral equation is termed linear if the integral operator:

$$L\bullet = \int_{a(x)}^{b(x)} k(x,t)\bullet dt$$

which satisfies the linearity condition

$$L[c_1u_1(x)+c_2u_2(x)] = c_1L[u_1(x)] + c_2L[u_2(x)]$$

where c_1, c_2 are two constants and u_1, u_2 are two continous functions.

The most general linear integral equations is of the form [35]

$$u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt \tag{1.10}$$

where the general form of nonlinear integral equations may be written as follows [35]:

$$u(x) = f(x) + \lambda \int_a^x k(x,t,u(t))dt \tag{1.11}$$

where $k(x,t,u(t))$ is nonlinear function in u .

Definition (1.2), [20]:

If the kernel k of the linear integral equation depends only on the difference $x-t$, i.e., if the kernel is of the form $k(x,t)=k(x-t)$, then k is called difference kernel, and the integral equation is called integral equation of convolution type.

Definition (1.3), [20]:

The kernel k is called symmetric, if it has the property $k(x,t)=k(t,x)$, and kernel k is called antisymmetric, if it has the property $k(x,t)=-k(t,x)$.

Definition (1.4), [20]:

The integral equation is said to be singular if either, the domain of definition tends to ∞, or if the kernel has a singularity within its region of definition.

1.4 Converting the Initial Value Problem into Volterra Integral Equation

In this section, the technique that is followed to convert an IVP to an equivalent VIE will be studied [42]. For simplicity reasons, we will apply this process to a second order initial value problem given by

$$y''(x)+p(x)y'(x)+q(x)y(x)=g(x), x\geq 0 \tag{1.12}$$

subject to the initial conditions $y(0)=\alpha$, $y'(0)=\beta$, where α and β are constants. The functions p, q are analytic functions, and g is continuous through the interval of discussion. To achieve our goal, we first set:

$$y''(x)=u(x) \tag{1.13}$$

where u is a continuous function. Integrating both sides of Eq.(1.13) from 0 to x, yields to:

$$y'(x)-y'(0)=\int_0^x u(t)dt, \tag{1.14}$$

or equivalently:

$$y'(x) = \beta + \int_0^x u(t)dt. \tag{1.15}$$

Againe, integrating both sides of Eq.(1.15) from 0 to x give :

$$y(x) - y(0) = \beta x + \int_0^x \int_0^x u(t)dtdx, \tag{1.16}$$

It is normal to outline the formula that will reduce the multiple integrals to single integrals. We will first show that the double integrals can be reduce to a single integral by using the formula :

$$\int_0^x \int_0^{x_1} F(t)dtdx_1 = \int_0^x (x-t)F(t)dt. \tag{1.17}$$

This can be easily proved by two ways. The first way is to set

$$G(x) = \int_0^x (x-t)F(t)dt, \tag{1.18}$$

Where G(0)=0. Differentianting both sides of Eq. (1.18) gives

$$G'(x) = \int_0^x F(t)dt, \tag{1.19}$$

Obtained by using Leibnitz rule. Now by integrating both sides of the last equation from 0 to x yeilds :

$$G(x) = \int_0^x \int_0^{x_1} F(t)dtdx \tag{1.20}$$

Consequently, the right side of the two equations (1.18) and (1.20) are equivelent. This complete the proof.

For the second method, the concept of integration by part will be use. Recall that

$$\int u\,dv = uv - \int v\,du,$$
$$u(x_1) = \int_0^x F(t)\,dt, \tag{1.21}$$

Then we find

$$\int_0^x \int_0^{x_1} F(t)dtdx_1 = x_1 \int_0^{x_1} F(t)dt \Big|_0^x - \int_0^x x_1 F(x_1)dx_1$$

$$= x\int_0^x F(t)dt - \int_0^x x_1 F(x_1)dx_1 \tag{1.22}$$

$$= \int_0^x (x-t)F(t)dt$$

Obtained by setting $x_1 = t$.

The general formula that convert multiple integrals to a single integral is given by

$$\int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} u(t)dt\,dx_{n-1} \ldots dx_1 = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} u(t)\,dt \tag{1.23}$$

where n is positive integer, and a is a constant, and hence Eq.(1.16) may be rewritten as:

$$y(x) = \alpha + \beta x + \int_0^x (x-t)u(t)dt \tag{1.24}$$

Substituting Eq.(1.13), Eq.(1.15), and Eq.(1.24) back into the initial value problem Eq.(1.12) yields the following VIE:

$$\begin{aligned} &u(x) + p(x)\,[\beta + \int_0^x u(t)\,dt] + q(x)\,[\alpha + \beta x \\ &+ \int_0^x (x-t)\,u(t)\,dt] = g(x) \end{aligned} \tag{1.25}$$

The last equation can be written in the standard VIE form as:

$$u(x) = f(x) + \int_0^x k(x,t)\,u(t)\,dt, \tag{1.26}$$

Where $k(x,t) = -p(x) - q(x)(x-t)$ and

$f(x) = g(x) - [\beta p(x) + \alpha q(x) + \beta x q(x)]$

The reader can see that Eq.(1.26) is a VIE of the second kind. The advantage here is that the auxiliary conditions are automatically satisfied in the process of formulating the resulting integral equation. The other advantage is in the case when both differential, as well as, integral forms do not have exact, closed-form solutions in term of elementary functions. In this case, we must detour the approach to numerical or approximate computations, where the integral representation is more suitable [25]. In other words, sometimes the problem can be expressed together by an integral equation and a differential equation. In some cases where numerical values are required it may be advisable to transform a problem from differential equation to integral equation because in the numerical analysis differentiation increases the error, but integration will tend out to smooth errors [14].

1.5 The Padé Approximants

In this section, the powerful Padé approximants [45] will be investigated. First of all, the construction of Padé approximants for functions and polynomials will be discussed. Then, some examples to see the implementation of Padé approximants will be given.

Polynomials are frequently used to approximate power series. However, polynomials tend to exhibit oscillations that may produce an approximation error bounds. In addition, polynomials can never blow up in a finite plane; and this makes the singularities not apparent. To overcome these difficulties, the Taylor series is best manipulated by Padé Approximants for numerical approximations. Padé approximants represents a function by the ratio of two polynomials. The coefficients of the polynomials in the numerator and in the denominator are determined by using the coefficients in the Taylor expansion of the function. Padé rational approximations are widely used in numerical analysis and fluid mechanics, because they are more efficient than polynomials. In the following, we will introduce the simple and the straightforward method to construct Padé Approximants. Suppose that f is a function has the Taylor series expansion given by:

$$f(x) = \sum_{k=0}^{\infty} c_k x^k$$

The procedure is to seek a rational function for the series. We can use the coefficients of the series to represent the function by a ratio of two polynomials. Padé approximants, symbolized by $[m/n]$, is a rational function defined by:

$$[m/n] = \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m}{b_0 + b_1 x + b_2 x^n + \ldots + b_n x^n}$$

The basic idea is to match the series coefficients as far as possible. Even though the series has a finite region of convergence, the limit of the function as $x \to \infty$ may be obtained, if $m = n$, then the approximants $[m/n]$ are called diagonal approximants. We note that there are $m+1$ independent coefficients in the numerator and $n+1$ coefficients in the denominator. To make the system determinable, let $b_0 = 1$. We then have n independent coefficients in the denominator and $m+n+1$ independent coefficients in all. Now the $[m/n]$ approximant can fit the power series through orders 1, x, $x^2, \ldots,$ x^{m+n}. In addition, the Padé approximants will converge on the entire real axis if the function f has no singularities. It was discussed by many authors that the diagonal Padé approximants, where $m = n$, are more accurate and efficient. Assuming that f can be manipulated by the diagonal Padé approximants, where $m = n$. This admits the use of :

$$\frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n}{1 + b_1 x + b_2 x^n + \ldots + b_n x^n} = c_0 + c_1 x + c_2 x^2 + \ldots + c_{2n} x^{2n}$$

By using cross multiplication, we find that

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = c_0 + (c_1 + b_1 c_0)x + (c_2 + b_1 c_1 + b_2 c_0)x^2$$
$$+ (c_3 + b_1 c_2 + b_2 c_1 + b_3 c_0)x^3 + \ldots$$

Equating the related powers of x leads to :

coefficient of x^0: $a_0 = c_0$,

coefficient of x^1: $a_1 = c_1 + b_1 c_0$,

coefficient of x^2: $a_2 = c_2 + b_1 c_1 + b_2 c_0$,

coefficient of x^3: $a_3 = c_3 + b_1 c_2 + b_2 c_1 + b_3 c_0, \cdots$

coefficient of x^n : $a_n = c_n + \sum_{k=1}^{n} b_k c_{n-k}$.

This completes the determination of the constants of the polynomials in the numerator and in the denominator. Thus we have

$$[1/1] = \frac{a_0 + a_1 x}{b_0 + b_1 x}, \lim_{x\to\infty}[1/1] = \frac{a_1}{b_1}$$

$$[2/2] = \frac{a_0 + a_1 x + a_2 x^2}{b_0 + b_1 x + b_2 x^2}, \lim_{x\to\infty}[2/2] = \frac{a_2}{b_2}$$

$$[3/3] = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{b_0 + b_1 x + b_2 x^2 + b_3 x^3}, \lim_{x\to\infty}[3/3] = \frac{a_3}{b_3}$$

$$\vdots$$

$$[m/m] = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_m x^m}{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots + b_m x^m}, \lim_{x\to\infty}[m/m] = \frac{a_m}{b_m}$$

Now, we give some examples as an allustration:

Example (1.1)[45] :

Consider the function $f(x) = \sqrt{\dfrac{1+3x}{1+x}}, x \in (-\infty,-1) \cup (-\dfrac{1}{3},\infty)$

The Taylor series for f is given by :

$$f(x) = 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 - \frac{37}{8}x^4 + \frac{75}{8}x^5 + \frac{327}{16}x^6 + \frac{753}{16}x^7 + O(x^8)$$

Thus, [2/2] approximants is defined by ;

$$[2/2] = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2}$$

To determine the five coefficients of the two polynomials, the $[2/2]$ approximants must fit the Taylor series of f through the order of $1,\ x,...,\ x^4$, hence we set :

$$\frac{a_0+a_1x+a_2x^2}{b_0+b_1x+b_2x^2}=1+x+\frac{3}{2}x^2+\frac{5}{2}x^3-\frac{37}{8}x^4$$

Cross multiplying yields

$$a_0+a_1x+a_2x^2=1+(b_1+1)x+(b_1+b_2-\frac{3}{2})x^2+(-\frac{3}{2}b_1+b_2+\frac{5}{2})x^3$$
$$+(\frac{5}{2}b_1-\frac{3}{2}b_2-\frac{37}{8})x^4$$

Equating the powers of x leads to:

coefficient of x^0: $a_0=1$,

coefficient of x^1: $a_1=b_1+1$,

coefficient of x^2: $a_2=b_1+b_2-\frac{3}{2}$,

coefficient of x^3: $0=-\frac{3}{2}b_1+b_2+\frac{5}{2}$

coefficient of x^4: $0=\frac{5}{2}b_1-\frac{3}{2}b_2-\frac{37}{8}$

Hence, the solution of this system of equations is;

$$a_0=1,\ a_1=\frac{9}{2},\ a_2=\frac{19}{4},$$

$$b_1=\frac{7}{2},\ b_2=\frac{11}{4}.$$

Consequently, the $[2/2]$ Padé approximants is

$$[2/2]=\frac{1+\frac{9}{2}x+\frac{19}{4}x^2}{1+\frac{7}{2}x+\frac{11}{4}x^2}$$

To determine the Padé approximants $[3/3]$, we first set :

$$\frac{a_0+a_1x+a_2x^2+a_3x^3}{b_0+b_1x+b_2x^2+b_3x^3}=1+x+\frac{3}{2}x^2+\frac{5}{2}x^3-\frac{37}{8}x^4+\frac{75}{8}x^5+\frac{327}{16}x^6$$

Cross multiplying, and equating the coefficient of like powers of x and solving the resulting system of equations leads to :

$$a_0=1,\ a_1=\frac{13}{2},\ a_2=\frac{27}{2},\ a_3=\frac{71}{8}$$

$$b_1=\frac{11}{2},\ b_2=\frac{19}{2},\ b_3=\frac{41}{8}.$$

This gives :

$$[3/3]=\frac{1+\frac{13}{2}x+\frac{27}{2}x^2+\frac{71}{8}x^3}{1+\frac{11}{2}x+\frac{19}{2}x^2+\frac{41}{8}x^3}$$

<u>Example (1.2)[45]</u>:

Consider the function $f(x)=\dfrac{\ln(1+x)}{x},\ x\geq -1, x\neq 0$

The Taylor series for f is given by

$$f(x)=1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\frac{x^5}{6}+\frac{x^6}{7}-\frac{x^7}{8}+O(x^8)$$

To establish [3/3] Approximants, we set

$$[3/3]=\frac{a_0+a_1x+a_2x^2+a_3x^3}{b_0+b_1x+b_2x^2+b_3x^3}$$

To determine the unknowns, we proceed as before and therefore we set

$$\frac{a_0+a_1x+a_2x^2+a_3x^3}{b_0+b_1x+b_2x^2+b_3x^3}=1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\frac{x^5}{6}+\frac{x^6}{7}$$

Cross multiplying and proceeding as before we find

$$a_0=1,\ a_1=\frac{17}{14},\ a_2=\frac{1}{3},\ a_3=\frac{1}{140}$$

$$b_1 = \frac{12}{7},\ b_2 = \frac{6}{7},\ b_3 = \frac{4}{35}.$$

So that the Padé approximants is

$$[3/3] = \frac{1 + \frac{17}{14}x + \frac{1}{3}x^2 + \frac{1}{140}x^3}{1 + \frac{12}{7}x + \frac{6}{7}x^2 + \frac{4}{35}x^3}$$

To determine the Padé approximants $[4/4]$, we set:

$$[4/4] = \frac{a_0 + a_1x + a_2x^2 + a_3x^3 + +a_4x^4}{b_0 + b_1x + b_2x^2 + b_3x^3 + +b_4x^4}$$

Proceeding as before we obtain:

$$[4/4] = \frac{1 + \frac{17}{8}x + \frac{11}{3}x^2 + \frac{1}{140}x^3 + +\frac{13}{140}x^4}{1 + \frac{12}{17}x + \frac{16}{7}x^2 + \frac{14}{35}x^3 + +\frac{12}{90}x^4}.$$

1.6 Adomian Decomposition Method (ADM)

The basic idea of Adomian Decomposition Method (ADM) will be introduced and for more details see [2]. To introduce the basic idea of the ADM, consider the operator equation $Fu = G$, where F represents a general nonlinear ordinary differential operator and G is a given function. Suppose the nonlinear operator F can be decomposed as:

$$Lu + Ru + Nu = G \tag{1.27}$$

where, N is a nonlinear operator, L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of order less than L and G is the nonhomogeneous term. The method is based on applying the operator L^{-1} formally to the expression :

$$Lu = G - Ru - Nu \tag{1.28}$$

so by using the given conditions, we obtain:

$$u = h + L^{-1}G - L^{-1}Ru - L^{-1}Nu \tag{1.29}$$

where, h is the solution of the homogeneous equation $Lu=0$, with the initial boundary conditions. The problem now is the decomposition of the nonlinear term Nu. To do this, Adomian developed a very elegant technique as follows:

The Adomian technique consists of approximating the solution of Eq.(1.27) as an infinite series:

$$u=\sum_{n=0}^{\infty}u_n \qquad (1.30)$$

and decomposing the nonlinear term Nu as:

$$f(u)=Nu=\sum_{n=0}^{\infty}A_n$$

where A_n are the so-called Adomian polynomials of u_0, u_1,..., u_n that are the terms of the analytical expansion of Nu, where

$$u=\sum_{i=0}^{\infty}\lambda^i u_i$$

around $\lambda=0$. That is:

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}f\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}$$

The Adomian polynomials are not unique and can be generated from the Taylor expansion of f about the first component u_0, i.e.,

$$f(u)=\sum_{n=0}^{\infty}\frac{f^{(n)}u_0}{n!}(u-u_0)^{(n)}$$

In [2], Adomians polynomials are arranged to have the form:

$$A_0=f(u_0)$$
$$A_1=u_1f'(u_0)$$
$$A_2=u_2f'(u_0)+\frac{u_1^2}{2!}f''(u_0)$$
$$A_3=u_3f'(u_0)+u_1u_2f''(u_0)+\frac{u_1^3}{3!}f'''(u_0)$$

Now, we parameterize Eq.(1.29) in the form:

$$u = h + L^{-1}G - \lambda L^{-1}Ru - \lambda L^{-1}Nu$$

where, λ is just an identifier for collecting terms in a suitable way such that A_n depends on u_0, u_1,..., u_n and we will later set $\lambda = 1$.

$$\sum_{n=0}^{\infty}\lambda^n u_n = h + L^{-1}G - \lambda L^{-1}R\sum_{n=0}^{\infty}\lambda^n u_n - \lambda L^{-1}\sum_{n=0}^{\infty}\lambda^n A_n$$

Equating the coefficients of equal powers of λ, we obtain:

$$\begin{aligned} u_0 &= h + L^{-1}G \\ u_1 &= -L^{-1}R(u_0) - L^{-1}(A_0) \\ u_2 &= -L^{-1}R(u_1) - L^{-1}(A_1) \\ &\vdots \end{aligned} \tag{1.31}$$

and in general

$$u_n = -L^{-1}R(u_{n-1}) - L^{-1}(A_{n-1})$$

Finally, an n-term that approximates the solution is given by:

$$\varphi_N(x) = \sum_{n=0}^{N-1}u_n(x), N = 1,2,3,....$$

Example(1.3),[42]:

Consider the linear VIDE :

$$u'(x) = 1 - \int_0^x u(t)dt, u(0) = 0 \tag{1.32}$$

Apllying the integral operator L^{-1} defined by

$$L^{-1}(\bullet) = \int_0^x (\bullet)dt,$$

to both sides of Eq.(1.32), and using the initial condition, yields to :

$$u(x) = x - L^{-1}\left(\int_0^x u(t)dt\right).$$

Using the decomposition series Eq.(1.30) and recurence relation Eq.(1.31) will be give :

$$u_0(x) = x,$$

$$u_1(x) = -L^{-1}\left(\int_0^x u_0(t)dt\right) = \frac{-1}{3!}x^3,$$

$$u_2(x) = -L^{-1}\left(\int_0^x u_1(t)dt\right) = \frac{1}{5!}x^5,$$

$$u_3(x) = -L^{-1}\left(\int_0^x u_2(t)dt\right) = \frac{-1}{7}x^7,$$

and so on. This gives the solution in a series form

$$u(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

and hence the exact solution is given by

$$u(x) = \sin(x).$$

1.7 The Variational Iteration Method

The basic idea of The Variational Iteration Method (VIM) will be introduced and for more details see [23], [24]. Consider the differential equation:

$$Lu + Nu = g(x) \tag{1.33}$$

where L and N are linear and nonlinear operators, respectively, and $g(x)$ is the source in homogeneous term. The VIM introduces a correction functional for Eq.(1.33) fo the form :

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)(Lu_n(t) + N\tilde{u}_n(t) - g(t))dt \tag{1.34}$$

where λ is a general Lagrange's multiplier that can be identified optimally via the variational theory, and $\tilde{u}_n$ as a restricted variation which means $\delta\tilde{u}_n = 0$. It is to be noted that the Lagrange multiplier λ may be a constant or a function.

First, it is required to determine the Lagrange multiplier $\lambda(t)$ that can be identified optimally via integration by parts and by using a restricted variation. It was found in [17], [37] that a general formula for $\lambda(t)$ for the $n-th$ order differential equation

$$u^{(n)} + f\,(u(x), u'(x), u''(x), \ldots\ldots, u^{(n)}(x)) = 0 \qquad (1.35)$$

can be proved to be of the form :

$$\lambda(t) = (-1)^n \frac{1}{(n-1)!}(t-x)^{(n-1)} \qquad (1.36)$$

This means that for first-order ordinary differential equations (ODEs), $\lambda(t) = -1$, and for the second-order ODEs, $\lambda(t) = (t-x)$, and so on. Although the proof of formula given in Eq.(1.36) is given in [24], [45] in details. Having determined $\lambda(t)$, an iteration formula or a recurrence relation should be used for the determination of the successive approximations $u_{n+1}(x), n = 0, 1, \ldots$ of the solution $u(x)$. It is well known that in recursion, an initial value is needed to perform the iteration process. Based on this, we can use any selective function for the zeroth approximation u_0. However, using the initial values $u(0)$, $u'(0)$, and $u''(0)$ are normally used for the zeroth approximation u_0 as will be seen later. Consequently, the solution is given by:

$$u(x) = \lim_{n\to\infty} u_n(x). \qquad (1.37)$$

Example (1.4),[13]:

Consider the nonlinear VIDE :

$$u'(x) = -1 + \int_0^x u^2(t)\,dt,\ u(0) = 0 \qquad (1.38)$$

Using VIM, the correction functional for Eq.(1.38) is :

$$u_{n+1}(x) = u_n(x) + \int_0^x \left[u_n'(s) + 1 - \int_0^s u^2(t)\,dt \right] ds \qquad (1.39)$$

Now, take the initial approximation $u_0(x) = -x$.

The first three iterates are obtained from Eq.(1.39) and given by :

$$u_1(x) = -x + \frac{1}{12}x^4,$$

$$u_2(x) = -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{12960}x^{10},$$

$$u_3(x) = -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{12960}x^{10} - \frac{37}{7076160}x^{13}$$
$$+ \frac{109}{914457600}x^{16} - \frac{1}{558472320}x^{19} + \frac{1}{77598259200}x^{22}.$$

Chapter Two

Solution of Volterra Integral and Integro-differential Equations Using Dafterdar-Jafari Method

2.1 Introduction

Daftardar-Gejji and Jafari in 2006 [15] have proposed a new method for solving linear and nonlinear functional equations namely Dafterdar-Jafari Method (DJM). The method converges to the exact solution, if it exists, through successive approximations. For concrete problems, a few number of approximations can be used for numerical purposes with high degree of accuracy. The DJM is simple to understand and easy to implement using computer packages and yields better results and does not require any restrictive assumptions for nonlinear terms as required by some existing techniques [10]. This chapter consist three sections, in section 2.2 the analysis of DJM have been discused for general nonlinear operator equations. In section 2.3 the DJM have been applied for volterra integral and integro-differential equations using illustrative examples.

2.2 Analysis of the DJM

Consider the general functional equation [15]:

$$u = N(u) + f \tag{2.1}$$

where N is a nonlinear operator and f is a known function. We are looking for a solution u of Eq.(2.1) having the series form:

$$u = \sum_{i=0}^{\infty} u_i \tag{2.2}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right] \tag{2.3}$$

From Eq.(2.2) and Eq.(2.3) which implies that Eq.(2.1) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}[N(\sum_{j=0}^{i} u_j) - N(\sum_{j=0}^{i-1} u_j)] \qquad (2.4)$$

We define the recurrence relation:

$$\left.\begin{cases} u_0 = f \,, \\ u_1 = N(u_0), \\ u_{m+1} = N(u_0 + ...+ u_m) - N(u_0 + ...+ u_{m-1}), \; m = 1, 2, ... \end{cases}\right\} \qquad (2.5)$$

Then

$$u = \sum_{i=1}^{\infty} u_i + f \qquad (2.6)$$

The m-term approximate solution of Eq.(2.1) is given by $u = u_0 + u_1 + \cdots + u_{m-1}$.

Now, the condition to ensure convergence of the DJM will be presented:

Theorem(2.1),[12]:

If $N \in C^{(\infty)}$ in a neighborhood of u_0 and

$\left\| N^{(n)}(u_0) \right\| = Sup \left\{ N^{(n)}(u_0)(h_1,...,h_n) : \left\| h_i \right\| \le 1, 1 \le i \le n \right\} \le L$

for any n and for some real $L > 0$ and $\left\| u_i \right\| \le M < 1/e, i = 1, 2, ...$ then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent, and moreover,

$\left\| G_n \right\| \le LM^n e^{n-1}(e-1), n = 1, 2, \cdots$

Theorem(2.2),[12]:

If $N \in C^{(\infty)}$ and $N^{(n)}(u_0) \le M \le e^{-1}$, for all n, then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent.

2.3 Test Examples

In this section, the DJM will be implemented to some examples, linear and nonlinear Volterra integral and integro-differential equations. Moreover, we compare the results with ADM and VIM, [13], [42].

Example (2.1):

Consider the following linear VIE [42]:

$$u(x) = 1 + \int_0^x (t - x)u(t)dt \tag{2.7}$$

To solve Eq.(2.7) by DJM, we follow the recurence relation given in Eq.(2.5) , then we have:

$$u_0 = 1,$$

$$u_1 = N(u_0) = \int_0^x [(t - x)\, u_0(t)]\, dt = \frac{-1}{2!}x^2,$$

$$u_2 = N(u_0 + u_1) - N(u_0) = \int_0^x [(t - x)(u_0(t) + u_1(t))]\, dt - u_1 = \frac{1}{4!}x^4,$$

$$\begin{aligned} u_3 &= N(u_0 + u_1 + u_2) - N(u_0 + u_1) \\ &= \int_0^x [(t - x)(u_0 + u_1 + u_2)]\, dt - \int_0^x [(t - x)(u_0 + u_1)]\, dt \\ &= \frac{-1}{6!}x^6, \end{aligned}$$

and so on. The solution in a series form is given by

$$u(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \tag{2.8}$$

which is the same results obtained by ADM in [42] that converges to the exact solution $u(x) = \cos(x)$.

Example (2.2):

Consider the following nonlinear VIE [42]:

$$u(x) = x + \int_0^x u^2(t)dt \tag{2.9}$$

To solve Eq.(2.9) by DJM, we follow the recurence relation given in Eq.(2.5) , then we have:

$$u_0 = x,$$

$$u_1 = N(u_0) = \int_0^x [u_0^2(t)]\,dt = \frac{1}{3}x^3,$$

$$u_2 = N(u_0+u_1) - N(u_0) = \int_0^x [(u_0+u_1)^2]dt - u_1 = \frac{2}{15}x^5 + \frac{1}{63}x^7,$$

$$u_3 = N(u_0+u_1+u_2) - N(u_0+u_1) = \int_0^x [(u_0+u_1+u_2)^2]dt - \int_0^x [(u_0+u_1)^2]dt$$
$$= \frac{4}{105}x^7 + \frac{38}{2835}x^9 + \ldots up\ to\ x^{15}$$

and so on. The solution in a series form is obtained to be:

$$u(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots \tag{2.10}$$

which is the same results obtained by ADM in [42] that converges to the exact solution $u(x) = \tan(x)$.

Example (2.3):

Consider the linear VIDE [42]:

$$u'(x) = 1 - \int_0^x u(t)dt,\ u(0) = 0 \tag{2.11}$$

To solve Eq.(2.11) we have to convert it to integral equation, therefore, integrate Eq.(2.11) from 0 to x and applying the initial conditions, then yields to:

$$u(x) = x - \int_0^x \int_0^x u(t)dtdx = x - \int_0^x (x-t)\,u(t)\,dt \qquad (2.12)$$

Now, by applying DJM to Eq.(2.12), we find:

$$u_0 = x,$$

$$u_1 = N(u_0) = \int_0^x [(x-t)u_0(t)]dt = -\frac{1}{3!}x^3,$$

$$u_2 = N(u_0+u_1) - N(u_0) = \int_0^x [(x-t)(u_0+u_1)]dt - u_1 = \frac{1}{5!}x^5,$$

$$u_3 = N(u_0+u_1+u_2) - N(u_0+u_1)$$
$$= \int_0^x [(x-t)(u_0+u_1+u_2)]dt - \int_0^x [(x-t)(u_0+u_1)]dt = -\frac{1}{7!}x^7,$$

and so on. The solution in a series form is given by

$$u(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \qquad (2.13)$$

which is the same results obtained by ADM in [42] and hence the exact solution is given by $u(x) = \sin(x)$.

Example (2.4):

Consider the nonlinear VIDE [13]:

$$u'(x) = -1 + \int_0^x u^2(t)dt,\; u(0) = 0 \qquad (2.14)$$

To solve Eq.(2.14) we have to convert it to integral equation, therefore, integrate Eq.(2.14) from 0 to x and applying the initial conditions, then yields to:

$$u(x) = -x + \int_0^x \int_0^x u^2(t)dtdx = -x + \int_0^x (x-t)u^2(t)dt \qquad (2.15)$$

Now, by apply the DJM to Eq.(2.15), we find:

$$u_0 = -x,$$

$$u_1 = N(u_0) = \int_0^x [(x-t)u_0^2(t)]dtdt = \frac{1}{12}x^4,$$

$$u_2 = N(u_0+u_1) - N(u_0) = \int_0^x [(x-t)(u_0+u_1)^2]dt - u_1$$

$$= -\frac{1}{252}x^7 + \frac{1}{12960}x^{10},$$

$$u_3 = N(u_0+u_1+u_2) - N(u_0+u_1)$$

$$= \int_0^x (x-t)(u_0+u_1+u_2)^2 dt - \int_0^x (x-t)(u_0+u_1)^2 dt$$

$$= \frac{1}{11340}x^{10} - \frac{37}{7076160}x^{13} + \frac{109}{914457600}x^{16}$$

$$-\frac{1}{558472320}x^{19} + \frac{1}{77598259200}x^{22}$$

and so on. The solution in a series form is given by

$$u(x) = -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{6048}x^{10} - \frac{37}{7076160}x^{13}$$
$$+\frac{109}{914457600}x^{16} - \frac{1}{558472320}x^{19} + \frac{1}{77598259200}x^{22} \qquad (2.16)$$

which is the same results obtained by VIM in [13].

Chapter Three
Solution of Volterra Integral and Integro-differential Equations Using Power Series Method

3.1 Introduction

Tahmasbi A. and Fard O. S. in 2008 [38], [39] proposed the power series method (PSM) and then implemented to solve Volterra integral and integro-differential equations. The obtained analytic approximate solutions of applying the PSM is in perfect agreement with the results obtained with those methods available in the literature. This chapter consist three sections, in section 3.2 the statement of PSM is illustrative. In section 3.3 the PSM have been applied for volterra integral and inegro-differential equations using illustrative examples.

3.2 The Power Series Method

Consider the equation [38], [39]:

$$u(x) = f(x) + \int_0^x k(x,t)[u(t)]^p dt \qquad (3.1)$$

In Eq.(3.1), the functions $f(x)$ and $k(x,t)$ are known, and $u(x)$ is the unknown function to be determined, also $p \geq 1$ is a positive integer. Suppose the solution of Eq.(3.1) with $e_0 = u(0) = f(0)$ as the initial condition to be as follows:

$$u(x) = e_0 + e_1 x, \qquad (3.2)$$

where, e_1 is a unknown parameter.

If we substitute Eq.(3.2) into Eq.(3.1) the following linear algebraic equation will be obtained:

$$(a_1 e_1 - b_1)x + Q(x^2) = 0, \qquad (3.3)$$

where, a and b are known constant and $Q(x^2)$ is a polynomial with the order greater than unity.

By neglecting $Q(x^2)$ in Eq.(3.3) and solving the algebraic equation $a_1e_1 = b_1$, the unknown parameter e_1 and therefore is the coefficient of x in Eq.(3.2) are obtaind.

In the next step, we assume that the solution of Eq.(3.1) to be,

$$u(x) = e_0 + e_1x + e_2x^2, \tag{3.4}$$

here, e_0 and e_1 both are known and e_2 is unknown parameter.

By substituting Eq.(3.4) into Eq.(3.1), we have the following system,

$$(a_2e_2 - b_2)x^2 + Q(x^3) = 0. \tag{3.5}$$

By neglecting $Q(x^3)$ in Eq.(3.5) and solving the algebraic equation $a_2e_2 = b_2$, the unknown parameter e_2 and therefore the coefficient of x^2 in Eq.(3.4) is obtains. By repeating the above procedure for m iteration, a power series of the following form will be derived:

$$u(x) = e_0 + e_1x + e_2x^2 + ... + e_mx^m, \tag{3.6}$$

Eq.(3.6) is an approximation for the exact solution $u(x)$ of the integral equation Eq.(3.1).

The following theorems shows the convrgence of PSM for nonlinear volterra integral equation and integro-differential equation.

<u>Theorem (3.1),[39]:</u>

Let $u = u(x)$ be the exact solution of the following VIE :

$$u(x) = f(x) + \int_0^x k(x,t)[u(t)]^p dt \tag{3.7}$$

Then, the power series method obtains the Taylor expansion of $u(x)$.

Corollary(3.2)[39]:

If the exact solution to Eq.(3.7) be a polynomial, then the power series method will obtain the real solution.

Theorem (3.3)[19]:

Let $u = u(x)$ be the exact solution of the following Volterra integro-differential equation

$$u'(x) = g(x, u(x)) + \int_0^x k(x, t, u(t), u'(t))dt, \; u(0) = a \tag{3.8}$$

Furthermore assume that $u(x)$ has a power series representation. Then, the power series method obtains it (the Taylor expansion of $u(x)$).

3.3 Test Examples

In this section, the PSM will be implemented to some examples, linear and nonlinear VIE, linear and nonlinear VIDE. Moreover, we compare the results with the results obtained by ADM and VIM.

Example (3.1):

Consider the following linear VIE [42]

$$u(x) = 1 + \int_0^x (t - x)\, u(t)\, dt \tag{3.9}$$

Now, applying the PSM to Eq.(3.9), we suppose that the solution of Eq.(3.9) with

$u_0(x) = u(0) = e_0 = 1$ is $u_1(x) = e_0 + e_1 x$, and hence

$$u_1(x) = 1 + e_1 x \tag{3.10}$$

Substitute Eq.(3.10) in Eq.(3.9), then,

$$1 + e_1 x = 1 + \int_0^x (t - x)(1 + e_1 t)dt, \tag{3.11}$$

After simplifying, we get, $e_1 = -\frac{x}{2}$ and hence,

$$u_1(x) = 1 - \frac{1}{2}x^2, \tag{3.12}$$

which is the first approximation for the solution of Eq.(3.9). The solution of Eq.(3.9) can be supposed as:

$$u(x) = 1 - \frac{1}{2}x^2 + e_2 x^2, \tag{3.13}$$

Substitute Eq.(3.13) in Eq.(3.9), we have,

$$1 - \frac{1}{2}x^2 + e_2 x^2 = 1 + \int_0^x (t - x) 1 - \frac{1}{2}t^2 + e_2 t^2 dt, \tag{3.14}$$

After simplifying, we get, $e_2 = \frac{x^2}{24}$ and hence,

$$u_2(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4, \tag{3.15}$$

Proceeding in this way, we get, the solution in a series form which is given by

$$u(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots \tag{3.16}$$

which is the same results found by DJM and obtained by ADM in [42] that converges to the exact solution $u(x) = \cos(x)$.

Example (3.2):

Consider the following nonlinear VIE [42]

$$u(x) = x + \int_0^x u^2(t) dt \tag{3.17}$$

By applying the PSM to Eq.(3.17), we suppose the solution of Eq.(3.17) with

$u_0(x) = u(0) = e_0 = 0$ is $u_1(x) = e_0 + e_1 x$, and hence

$$u_1(x) = 0 + e_1 x = e_1 x \tag{3.18}$$

Substitute Eq.(3.18) in Eq.(3.17), we have,

$$e_1 x = x + \int_0^x (e_1 t)^2 dt, \tag{3.19}$$

After simplifying, we get, $e_1 = 1$ and hence,

$$u_1(x) = x, \tag{3.20}$$

which is the first approximation for the solution of Eq.(3.17). The solution of Eq.(3.17) can be supposed as:

$$u(x) = x + e_2 x^2, \tag{3.21}$$

Substitute Eq.(3.21) in Eq.(3.17), we have,

$$x + e_2 x^2 = x + \int_0^x (t + e_2 t^2)^2 dt, \tag{3.22}$$

After simplifying, we get, $e_2 = \frac{1}{3}x$ and hence,

$$u_2(x) = x - \frac{1}{3}x^3, \tag{3.23}$$

Proceeding in this way, we get that, the solution in a series form given by

$$u(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + ... \tag{3.24}$$

which is the same results found by DJM and obtained by ADM in [42] that converges to the exact solution $u(x) = \tan(x)$.

Example (3.3):

Consider the linear VIDE [42]:

$$u'(x) = 1 - \int_0^x u(t)dt, \; u(0) = 0 \tag{3.25}$$

To solve Eq.(3.25), integrate Eq.(3.25) from 0 to x and applying the initial condition, then we have:

$$u(x) = x - \int_0^x \int_0^x u(t)dtdx = x - \int_0^x (x - t)u(t)dt \tag{3.26}$$

By using the PSM to solve Eq.(3.26), we suppose the solution of Eq.(3.26) with $u_0(x) = u(0) = e_0 = 0$ is $u_1(x) = e_0 + e_1 x$, and hence

$$u_1(x) = 0 + e_1 x = e_1 x \tag{3.27}$$

Substitute Eq.(3.27) in Eq.(3.26), we have,

$$e_1 x = x + \int_0^x (t - x) e_1 t \, dt, \tag{3.28}$$

By integrating and solving, we get,

$(e_1 - 1)\text{x} - \frac{e_1 x^3}{6} = 0$, by neglecting $(-\frac{e_1 x^3}{6})$, therefore $e_1 = 1$ and hence,

$$u_1(x) = x, \tag{3.29}$$

which is the first approximation for the solution of Eq.(3.26). The solution of Eq.(3.26) can be supposed as:

$$u(x) = x + e_2 x^2, \tag{3.30}$$

Substitute Eq.(3.30) in Eq.(3.26), we have,

$$x + e_2 x^2 = x + \int_0^x (x - t)(t + e_2 t^2) dt \tag{3.31}$$

By integrating and solving, we get,

$(e_2 + \frac{x}{6})\text{x}^2 - \frac{e_2 x^4}{12} = 0$, by neglecting $(-\frac{e_2 x^4}{12})$, therefore $e_2 = -\frac{1}{6} x$ and hence,

$$u_2(x) = x - \frac{1}{6} x^3, \tag{3.32}$$

Proceeding in this way, we get, The solution in a series form is given by

$$u(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots \tag{3.33}$$

which is the same results found by DJM and obtained by ADM in [42] and hence the exact solution is given by $u(x) = \sin(x)$.

Example (3.4):

Consider the nonlinear VIDE [13]:

$$u'(x) = -1 + \int_0^x u^2(t)dt, \; u(0) = 0 \qquad (3.34)$$

To solve Eq.(3.34), integrate Eq.(3.34) from 0 to x and apply the initial condition, then we have:

$$u(x) = -x + \int_0^x \int_0^x u^2(t)dtdx = -x + \int_0^x (x-t)u^2(t)dt \qquad (3.35)$$

Now, applying PSM to Eq.(3.35), we suppose the solution of Eq.(3.35) with $u_0(x) = u(0) = e_0 = 0$ is $u_1(x) = e_0 + e_1 x$, and hence

$$u_1(x) = 0 + e_1 x = e_1 x \qquad (3.36)$$

Substitute Eq.(3.36) in Eq.(3.35), we have,

$$e_1 x = -x + \int_0^x (x-t)(e_1 t)^2 dt, \qquad (3.37)$$

By integrating and solving, we get,

$(e_1 + 1)\text{x} - \dfrac{e_1^2 x^4}{12} = 0$, by neglecting $(-\dfrac{e_1 x^4}{12})$, therefore, $e_1 = -1$ and hence,

$$u_1(x) = -x, \qquad (3.38)$$

which is the first approximation for the solution of Eq.(3.35). The solution of Eq.(3.35) can be supposed as:

$$u(x) = -x + e_2 x^2, \qquad (3.39)$$

Substitute (3.39) in (3.35), we have,

$$x + e_2 x^2 = -x + \int_0^x (x-t)(t + e_2 t^2)^2 dt,$$

By integrating and solving, we get,

$(e_2 - \dfrac{x^2}{12})\text{x}^2 - (\dfrac{e_2 x^5}{10} - \dfrac{e_2^2 x^6}{30}) = 0$, by neglecting $(\dfrac{e_2 x^5}{10} - \dfrac{e_2^2 x^6}{30})$, therefore, $e_2 = \dfrac{x^2}{12}$, and hence,

$$u_2(x) = -x + \frac{1}{12}x^4, \tag{3.40}$$

Proceeding in this way, we get, the solution in a series form is given by

$$\begin{aligned} u(x) &= -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{6048}x^{10} - \frac{37}{7076160}x^{13} \\ &+ \frac{109}{914457600}x^{16} - \frac{1}{558472320}x^{19} + \frac{1}{77598259200}x^{22} \end{aligned} \tag{3.41}$$

which is the same results found by DJM and obtained by VIM in [13].

Chapter Four

Analytic and Approximate Solutions of Some Real Life Models

4.1 Introduction

The VIDE's arises from the mathematical modeling of various physical, engineering and biological models, for example the population growth of a species in a closed system. These models help us to understand different factors like the behavior of the population evolution over a period of time.

In this chapter, the applications of the DJM and PSM for Volterra's population model will be presented to find the approximate solutions. The main attractive features of the current methods are being derivative-free, overcome the difficulty in some existing techniques, simple to understand. It is economical in terms of computer power/memory and does not involve tedious calculations such as Adomian polynomials in ADM, construct a homotopy as in HPM and solve the corresponding algebraic equations, and calculate Lagrange multiplier as in VIM. The numerical solution are obtained by combining the DJM and PSM with Padé technique. Moreover, the comparison of the achieved results are compared with existing results by ADM and HPM also presented [29], [43]. Also, the DJM and PSM will be implemented to evaluate a solution for some scientific models. We will focus our work on three well-known nonlinear equations, namely the Hybrid selection model, the Riccati equation, and the Logistic equation.

4.2 Approximate Solutions for Volterra's Population Model

In this section, an approximate methods have been implemented to obtain a solutions for Volterra's population model of population growth of a species in a closed system, besides the DJM and the PSM, the ADM and the VIM are implemented independently to the model in the literature [42], [43]. The Padé approximants, that often show superior performance over series approximations, are effectively used in the analysis to capture the essential behavior of the population $u(t)$ of identical individuals.

4.2.1 Volterra's Population Model (VPM)[43]

In this section, the Volterra's model for population growth of a species within a closed system will studied. The Volterras population model is characterized by the nonlinear Volterra integro-differential equation:

$$\frac{dP}{dT} = aP - bP^2 - cP\int_0^T P(x)dx, \; P(0) = P_0 \tag{4.1}$$

where $P = P(T)$ denotes the population size at time T; a, b, and c are constant and positive parameters, in which a is the birth rate coefficient, b is the crowding coefficient and c is the toxicity coefficient. Also P_0 is the initial population size at time T. The coefficient c indicates the essential behavior of the population evolution before its level falls to zero in the long run. When $b = 0$ and $c = 0$, Eq.(4.1) becomes the Malthus differential equation:

$$\frac{dP}{dT} = aP, \; P(0) = P_0 \tag{4.2}$$

The Malthus Eq.(4.2) assumes that the population growth is proportional to the current population. Eq.(4.2) is separable with a solution given by:

$$P(T) = P_0 e^{aT} \tag{4.3}$$

It is obvious that Eq.(4.1) represents a population growth for $a > 0$, and a population decay for $a < 0$. When $c = 0$, Eq.(4.1) becomes the logistic growth model that reads:

$$\frac{dP}{dT} = aP - bP^2, \; P(0) = P_0 \tag{4.4}$$

Verhulst instituted the logistic growth model Eq.(4.4) that eliminates the undesirable effect of unlimited growth by introducing the growth-limiting term $-bP^2$. The solution to logistic growth model Eq.(4.4) is

$$P(T) = \frac{a\,P_0\,e^{aT}}{a + b\,P_0\,(e^{aT} - 1)}, \tag{4.5}$$

where

$$\lim_{T \to \infty} P(T) = \frac{a}{b}$$

Volterra introduced an integral term $-cP\int_0^T P(x)\,dx$ to the logistic growth model Eq.(4.4) to get the Volterras population growth model Eq.(4.1). The additional integral term characterizes the accumulated toxicity produced since time zero. Many time scales and population scales may be applied. However, the scale time population by introducing the non-dimensional variables will be applied:

$$t = \frac{cT}{b},\; u = \frac{b\,P}{a} \tag{4.6}$$

to obtain the non-dimensional Volterra's population growth model:

$$\kappa \frac{du}{dt} = u - u^2 - u\int_0^t u(x)\,dx,\; u(0) = u_0 \tag{4.7}$$

where $u = u(t)$ is the scaled population of identical individuals at a time t, and the non-dimensional parameter $\kappa = c/(ab)$ is a prescribed parameter. Volterra introduced this model for a population u of identical individuals which exhibits crowding and sensitivity to the amount of toxins produced. A considerable amount of research work has been investiged to determine numerical and analytic solutions of the population growth model Eq.(4.4) [34], [35], [36]. The analytical solution:

$$u(t)=u_0 e^{\left(\frac{1}{\kappa}\int_0^t [1-u(\tau)-\int_0^\tau u(x)dx\,]\,d\tau\right)} \tag{4.8}$$

which shows that $u(t)>0$ for all t if the initial population $u_0>0$. However, this closed form solution cannot lead to an insight into the behavior of the population evolution. As a result, researchs were directed towards the analysis of the population rapid rise along the logistic curve followed by its decay to zero in the long run. The non-dimensional parameter κ plays a great role in the behavior of u concerning the rapid rise to a certain amplitude followed by an exponential decay to extinction. For κ small, the population is not sensitive to toxins, whereas the population is strongly sensitive to toxins for large κ [43].

4.2.2 Solution of Volterra's Population Model by ADM

In this section, the population growth model characterized by nonlinear VIDE:

$$\kappa\frac{du}{dt}=u-u^2-u\int_0^t u(x)dx\,,\; u(0)=u_0$$

which have been considered, and is also studied in [29], [44]:

$$\frac{du}{dt}=10u(t)-10u^2(t)-10u(t)\int_0^t u(x)dx\,,\; u(0)=0.1 \tag{4.9}$$

where the initial condition $u(0)=0.1$ and the nondimensional parameter $\kappa=0.1$ were used by Wazwaz A. M. in [43] for simplicity reasons.

To begin, it is convenient to rewrite Eq.(4.9) in an operator form as:

$$Lu(t)=10u(t)-10u^2(t)-10\int_0^t u(t)u(x)dx\,,\; u(0)=0.1 \tag{4.10}$$

where the differential operator L is defined by $L\bullet=\frac{d}{dt}\bullet$.

It is clear that L is invertible so that the integral operator is defined by:

$$L^{-1}(.) = \int_0^t (.)dt$$

Applying L^{-1} to both sides of Eq.(4.10) and using the initial condition leads to:

$$u(t) = 0.1 + L^{-1}(10u(t) - 10u^2(t) - 10\int_0^t u(t)u(x)dx) \tag{4.11}$$

With this method, we usually represent u in Eq.(4.11) by the decomposition series:

$$u(t) = \sum_{n=0}^{\infty} u_n(t) \tag{4.12}$$

Accordingly, the main concern here is to formally determine the components $u_n(t)$, u_0. To do so, we substitute Eq.(4.12) into both sides of Eq.(4.11) to obtain

$$\sum_{n=0}^{\infty} u_n(t) = 0.1 + L^{-1}\left(10\sum_{n=0}^{\infty} u_n(t) - 10\sum_{n=0}^{\infty} A_n(t) - 10\int_0^t B_n(x,t)dx \right)$$

where the nonlinear terms $u^2(t)$ and $u(x)u(t)$ are represented by the so-called Adomian polynomials $A_n(t)$ and $B_n(x,t)$, respectively. In other words, we set

$$u^2(t) = \sum_{n=0}^{\infty} A_n(t),$$

$$u(x)u(t) = \sum_{n=0}^{\infty} B_n(x,t).$$

The evaluation of the Adomian polynomials $A_n(t)$, $B_n(x,t)$ has been discussed for various classes of nonlinearities in [1]. For convenience, we list below, by using the algorithms introduced in [1], few of the Adomian polynomials for $A_n(t)$ and $B_n(x,t)$ that are used in the literature:

For $A_n(t)$, we find:

$$A_0(t)=u_0^2(t),\ A_1(t)=2u_0(t)u_1(t),\ A_2(t)=u_1^2(t)+2u_0(t)u_2(t),$$
$$A_3(t)=2u_1(t)u_2(t)+2u_0(t)u_3(t),$$
$$A_4(t)=u_2^2(t)+2u_1(t)u_3(t)+2u_0(t)u_4(t),...$$

For $B_n(x,t)$, we find:

$$B_0(x,t)=u_0(x)u_0(t),\ B_1(x,t)=u_0(x)u_1(t)+u_1(x)u_0(t),$$
$$B_2(x,t)=u_0(x)u_2(t)+u_1(x)u_1(t)+u_2(x)u_0(t),$$
$$B_3(x,t)=u_0(x)u_3(t)+u_1(x)u_2(t)+u_2(x)u_1(t)+u_3(x)u_0(t),$$
$$B_4(x,t)=u_0(x)u_4(t)+u_1(x)u_3(t)+u_2(x)u_2(t)+u_3(x)u_1(t)+u_4(x)u_0(t),...$$

To determine the components $u_0,\ u_1,\ u_2,...$ of $u(t)$, the following recursive relationship will be applied:

$$u_0(t)=0.1,$$

$$u_{k+1}(t)=L^{-1}(10u_k(t)-10A_k(t)-10\int_0^t B_k(x,t)dx),\ k\geq 0$$

With $u_0(t)$ defined as shown above, this can be very valuable practically in evaluating the other components. It then follows

$$u_0(t)=0.1,$$

$$u_1(t)=L^{-1}(10u_0(t)-10A_0(t)-10\int_0^t B_0(x,t)dx)=0.9t+0.05t^2,$$

$$u_2(t)=L^{-1}(10u_1(t)-10A_1(t)-10\int_0^t B_1(x,t)dx)=3.6t^2-\frac{7}{12}t^3+\frac{1}{60}t^4,$$

$$u_3(t)=L^{-1}(10u_2(t)-10A_2(t)-10\int_0^t B_2(x,t)dx)=$$
$$6.9t^3-3.1541667t^4+0.2424t^5-0.0047222t^6,$$

$$u_4(t)=L^{-1}(10u_3(t)-10A_3(t)-10\int_0^t B_3(x,t)dx)=$$
$$-2.4t^4-9.3516667t^5+1.6631944t^6-0.08273809t^7$$
$$+0.00123016t^8,$$

The components u_5, u_6, u_7 and u_8 were also determined and will be used, but for brevity not listed. This completes the formal determination of the approximation of u given by [43]

$$u(t) = 0.1 + 0.9t + 3.55t^2 + 6.31666667t^3 - 5.5375t^4 - 63.70916667t^5 - 156.0804167t^6 - 18.47323411t^7 + 1056.288569t^8 + O(t^9). \quad (4.13)$$

4.2.3 Solution of Volterra's population model by VIM

In this subsection the Volterra's population model is solved by using the VIM [42], which is characterized by the nonlinear VIDE Eq.(4.9).

To solve Eq.(4.9) by VIM, we procesed as following:

The correction functional for Eq.(4.9) is:

$$u_{n+1} = u_n + \int_0^t \lambda(t)[u_n'(t) - 10u_n(t) + 10u_n^2(t) + 10u_n(t)\int_0^t u_n(x)dx\,]dt \quad (4.14)$$

where we used $\lambda(t) = -1$ for the first-order integro-differential equation Eq.(4.9). The zeroth approximation u_0 can be selected by using the initial value $u(0) = 0.1$. Selecting $u_0(t) = 0.1$, and using Eq.(4.14) the following successive approximations have been obtained:

$$u_0(t) = 0.1,$$

$$u_1(t) = u_0 - \int_0^t [u_0'(t) - 10u_0(t) - 10u_0^2(t) - 10u_0(t)\int_0^t u_0(x)dx\,]dt$$

$$= 0.1 + 0.9t - 0.05t^2,$$

$$u_2(t) = u_1 - \int_0^t [u_1'(t) - 10u_1(t) - 10u_1^2(t) - 10u_1(t)\int_0^t u_1(x)dx\,]dt$$

$$= 0.1 + 0.9t + 3.55t^2 - 3.2833333t^3 - 0.7708333t^4 + 0.0699999t^5 - 0.0013888t^6,$$

$$u_3(t) = u_2 - \int_0^t [u_2'(t) - 10u_2(t) - 10u_2^2(t) - 10u_2(t)\int_0^t u_2(x)dx\,]dt$$

$$= 0.1 + 0.9t + 3.55t^2 + 6.31667t^3 - 24.7375t^4 - 19.1225t^5 + \cdots$$

$$u_4(t) = u_3 - \int_0^t [u_3'(t) - 10u_3(t) - 10u_3^2(t) - 10u_3(t)\int_0^t u_3(x)dx]dt$$

$$= 0.1 + 0.9t + 3.55t^2 + 6.31667t^3 - 5.5375t^4 - 94.4292t^5 + \cdots$$

The components u_5, u_6, u_7 and u_8 were also evaluted and will be used, but for brevity not listed. This completes the formal determination of the approximation of $u(t)$ given by :

$$\begin{aligned} u(t) &= 0.1 + 0.9t + 3.55t^2 + 6.31666667t^3 - 5.5375t^4 \\ &-63.7091667t^5 - 156.08041667t^6 - 18.47323413t^7 \\ &+1056.288569t^8 + O(t^9) \end{aligned} \tag{4.15}$$

The approximation in Eq.(4.15) is in a very good agreement with the results obtained by using ADM .

4.2.3.1 Analysis and Numerical Results

The results obtained here can be discussed to get more details about the mathematical structure of $u(t)$. In particular, we seek to study the rapid growth along the logistic curve that will reach a peak, then followed by the slow exponential decay where $u(t) \to 0$ as $t \to \infty$, [43]. In order to study the mathematical structure of $u(t)$ the Padé approximants is used. Using the approximation obtained for $u(t)$ in Eq.(4.15), we find

$$[4/4] = \frac{0.1 + 0.4687931832t + 0.924957398t^2 + 0.9231294892t^3 + 0.4004234788t^4}{1 - 4.312068168t + 12.5581875t^2 - 13.88063927t^3 + 10.8683047t^4}$$

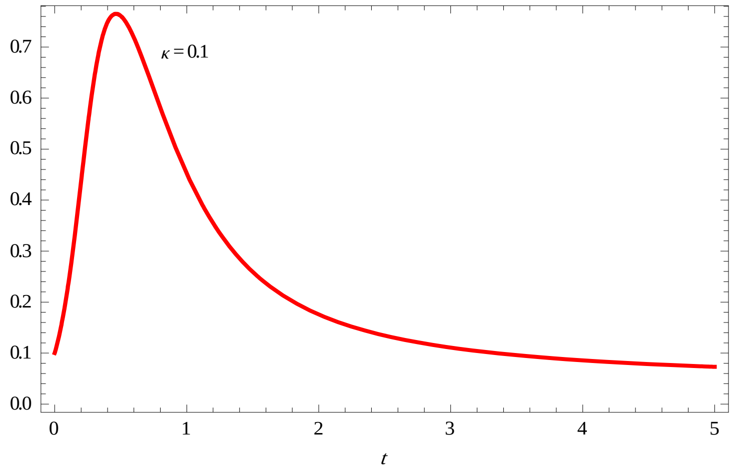
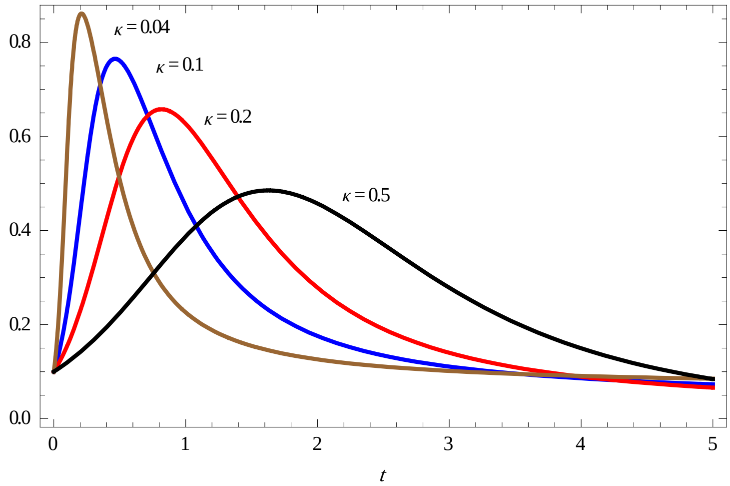


***Figure (4.1): Relation between Padé approximants* [4/4] *of* $u(t)$ *and* t *for* $u(0) = 0.1, \kappa = 0.1.$**



Figure(4.2): Relation between Padé approximants [4/4] of $u(t)$ and t for $u(0) = 0.1, \kappa = 0.04, 0.1, 0.2$ *and* 0.5.

which is the same obtained by ADM. From Fig.(4.1), we can easily observe that for $u(0) = 0.1$ and $\kappa = 0.1$, we obtain $u_{max} = 0.7651130891$ occurs at $t_{critical} = 0.4644767409$. However, Fig.(4.2) shows the Padé approximants [4/4] of $u(t)$ for $u(0) = 0.1$ and for $\kappa = 0.04, 0.1, 0.2$ *and* 0.5, which is the same obtained by ADM. The key finding of this graph is that when κ increases, the amplitude of $u(t)$ decreases, whereas the exponential decay increases. Table (4.1) below summarizes the relation between κ, and $t_{critical}$. The exact values of u_{max} were evaluated by using

$$u_{max} = 1 + \kappa \ln\left(\frac{\kappa}{1+\kappa-u_0}\right) \tag{4.16}$$

obtained by TeeBest [36].

Table(4.1):Approximation of u_{max} and exact value of u_{max} for κ = 0.04,0.1,0.2 and 0.5

κ	Critical t	Approx. u_{max}	Exact u_{max}
0.04	0.2102464442	0.8612401810	0.8737199832
0.1	0.4644767409	0.7651130891	0.7697414491
0.2	0.8168581213	0.6579123099	0.6590503816
0.5	1.6266622270	0.4852823490	0.4851902914

4.2.4 Solution of the Volterra's Population Model by DJM

In this section the Volterra's population model is solved by DJM. Recall the population growth model characterized by the nonlinear VIDE Eq.(4.9) and to solve this model by DJM, we integrate Eq.(4.9) with respect to t from 0 to t and applying the initial condition, then we have:

$$u(t) = 0.1 + \int_0^t [10u(t) - 10u^2(t) - 10u(t)\int_0^t u(x)dx\,]dt \tag{4.17}$$

Now, by applying NIM to Eq.(4.17), we get:

$$u_0 = 0.1,$$

$$u_1 = N(u_0) = \int_0^t [10u_0(t) - 10u_0^2(t) - 10u_0(t)\int_0^t u_0(x)dx\,]dt$$

$$= \int_0^t [10(0.1) - 10(0.1)^2 - 10(0.1)\int_0^t 0.1dx\,]dt = 0.9t - 0.05t^2,$$

$$u_2 = N(u_0+u_1) - N(u_0) = \int_0^t [10(u_0(t)+u_1(t)) - 10(u_0(t)+u_1(t)^2) -$$

$$10(u_0(t)+u_1(t))\int_0^t (u_0(x)+u_1(x))dx\,]dt - u_1,$$

$$= \int_0^t [10(0.1+(0.9t-0.05t^2)) - 10(0.1+(0.9t-0.05t^2))^2 -$$

$$10(0.1+(0.9t-0.05t^2))\int_0^t (0.1+(0.9x-0.05x^2))dx\,]dt - (0.9t-0.05t^2)$$

$$= 3.6\,\mathrm{t}^2 - 3.28333t^3 - 0.770833t^4,$$

$$u_3 = N(u_0+u_1+u_2) - N(u_0+u_1) = \int_0^t [10(u_0+u_1+u_2) - 10(u_0+u_1+u_2)^2$$

$$-10(u_0+u_1+u_2)\int_0^t (u_0(x)+u_1(x)+u_2(x))dx\,] - \int_0^t [10(u_0+u_1)$$

$$-10(u_0+u_1)^2 - 10(u_0+u_1)\int_0^t (u_0(x)+u_1(x))dx\,]$$

$$= 9.6t^3 - 23.9667t^4 - 19.1925t^5 + 38.1065t^6$$

$$u_4 = N(u_0+u_1+u_2+u_3) - N(u_0+u_1+u_2) = \int_0^t [10(u_0+u_1+u_2+u_3) -$$

$$10(u_0+u_1+u_2+u_3)^2 - 10(u_0+u_1+u_2+u_3)\int_0^t (u_0(x)+u_1(x)+u_2(x)+$$

$$u_3(x))dx\,] - \int_0^t [10(u_0+u_1+u_2) - 10(u_0+u_1+u_2)^2 - 10(u_0+u_1+u_2)$$

$$\int_0^t (u_0(x)+u_1(x)+u_2(x))dx\,],$$

$$= 19.2t^4 - 75.3067t^5 - 73.2967t^6 + 290.762t^7 + 540.771t^8$$

The components u_5, u_6, u_7 and u_8 were also evaluted and will be used, but for brevity not listed. This completes the formal determination of the approximation of $u(t)$ given by

$$\begin{aligned} u(x) &= 0.1 + 0.9t + 3.55t^2 + 6.31666667t^3 - 5.5375t^4 \\ &\quad - 63.7091667t^5 - 156.08041667t^6 - 18.47323413t^7 \\ &\quad + 1056.288569t^8 + O(t^9) \end{aligned} \tag{4.18}$$

The approximation given in Eq.(4.18) is in a very good agreement with the results obtained by using ADM, HPM and VIM in [29], [43], [42]. It is worth to mention here the DJM is straightforward without required to calculating Adomian polynomials to handle the nonlinear terms as in ADM and construct a homotopy as in HPM and calculate Lagrange multiplier as in VIM.

4.2.4.1 Analysis and Numerical Results

To examine more closely the mathematical structure of u as shown above, we seek to study the rapid growth along the logistic curve that will reach a peak, then followed by the slow exponential decay where $u(t) \to 0$ as $t \to \infty$, [43]. Polynomials are frequently used to approximate power series.Which tends to exhibit oscillations that may produce an approximation error bounds. In addition, polynomials can never blow up in a finite plane; and this makes the singularities not apparent. To overcome these difficulties, the Taylor series is best manipulated by Padé approximants for numerical approximations. Padé Approximants [45] have the advantage of manipulating the polynomial approximation into a rational function to gain more information about $u(t)$. The coefficients of the polynomials in the numerator and in the denominator are determined by using the coefficients in the Taylor expansion of the function. Using the approximation obtained for $u(t)$ in Eq.(4.18), we find :

$$[4/4] = \frac{0.1 + 0.4687931832t + 0.924957398t^2 + 0.9231294892t^3 + 0.4004234788t^4}{1 - 4.312068168t + 12.5581875t^2 - 13.88063927t^3 + 10.8683047t^4}$$

which is in a very good agreement with results obtained by using ADM and VIM.

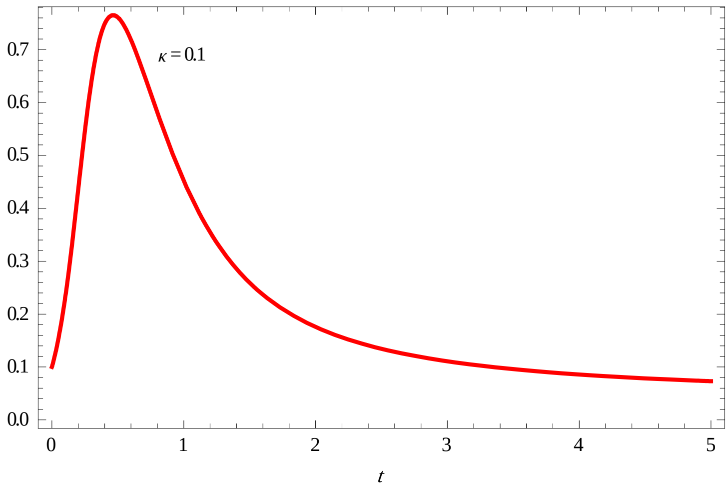
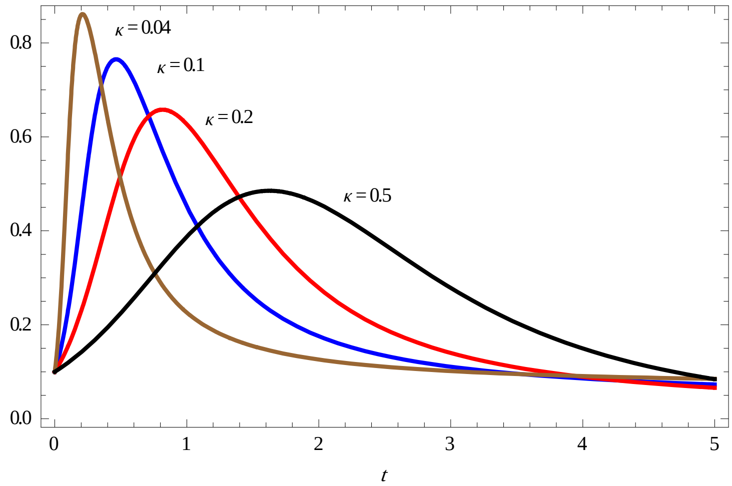


Figure (4.3): Padé approximant $[4/4]$ ***shows a rapid growth followed by a slow exponential decay***.



Figure(4.4): Relation between Padé approximants $[4/4]$ ***of*** $u(t)$ ***and*** t ***for*** $u(0)=0.1, \kappa = 0.04, 0.1, 0.2$ ***and*** 0.5.

Figure(4.3) shows the behavior of $u(t)$ and explores the rapid growth that will reach a peak followed by a slow exponential decay. Which is the same results obtained by ADM and VIM. Also Figure(4.4), is the same results that is obtained by ADM and VIM. From Fig.(4.3), we can easily observe that for $u(0)=0.1$ and $\kappa = 0.1$, we obtain $u_{max} = 0.7651130891$ which occurs at $t_{critical} = 0.4644767409$. Table (4.2) summarizes the relation between κ and $t_{critical}$. The exact values of u_{max} were evaluated by using Eq.(4.16) obtained by [36].

Table(4.2): Approximation of u_{max} and exact value of u_{max} for $\kappa = 0.04, 0.1, 0.2$ and 0.5

κ	Critical t	Approx. u_{max}	Exact u_{max}
0.04	0.2102464442	0.8612401810	0.8737199832
0.1	0.4644767409	0.7651130891	0.7697414491
0.2	0.8168581213	0.6579123099	0.6590503816
0.5	1.6266622270	0.4852823490	0.4851902914

4.2.5 Solution of Volterra's Population Model by PSM

In this section, the Volterra's population model is solved by using PSM. Recall that the population growth model is characterized by the nonlinear VIE Eq.(4.17) , and by applying the PSM to Eq.(4.17).

Suppose the solution of Eq.(4.17) with $u_0(t) = u(0) = e_0 = 0.1$ is $u_1(t) = e_0 + e_1 t$ and hence

$$u_1(t) = 0.1 + e_1 t \tag{4.19}$$

Substituting Eq.(4.19) into Eq.(4.17), yields to:

$$\begin{aligned} 0.1 + e_1 t &= 0.1 + \int_0^t [10(0.1 + e_1 t) - 10(0.1 + e_1 t)^2 \\ &\quad - 10(0.1 + e_1 t)\int_0^t (0.1 + e_1 x)dx\,]dt \end{aligned} \tag{4.20}$$

After simplifying, we get $e_1 = 0.9$ and hence,

$$u_1(t) = 0.1 + 0.9t \tag{4.21}$$

Which is the first approximtion for the solution of Eq.(4.17).

The second approximate solution of Eq.(4.17) can be supposed as:

$$u(t) = 0.1 + 0.9t + e_2 t^2 \tag{4.22}$$

and substitute Eq.(4.22) in Eq.(4.17), will give:

$$0.1+e_1 t+e_2 t^2 = 0.1+\int_0^t [10(0.1+e_1 t+e_2 t^2)-10(0.1+e_1 t+e_2 t^2)^2 \\ -10(0.1+e_1 t+e_2 t^2)\int_0^t (0.1+e_1 x+e_2 x^2)dx\,]dt \tag{4.23}$$

After simplifying, the second constant e_2 is found to be $e_2 = 3.55$ and hence,

$$u_2(t) = 0.1+0.9t+3.55t^2 \tag{4.24}$$

Similarly, proceeding in this way, the following approximate solution have been obtained,

$$u(t) = 0.1+0.9t+3.55t^2+6.316667t^3-5.5375t^4- \\ 63.7091667t^5-156.0804167t^6-18.4732341t^7 \\ +1056.288569t^8+O(t^9). \tag{4.25}$$

4.2.5.1 Analysis and Numerical Results

In order to study the mathematical structure of $u(t)$, the Padé pproximants that presented and implemented in chapter one is used. Using the approximation obtained for $u(t)$ in Eq.(4.25), we find

$$[4/4] = \frac{0.1+0.4687931832t+0.924957398t^2+0.9231294892t^3+0.4004234788t^4}{1-4.312068168t+12.5581875t^2-13.88063927t^3+10.8683047t^4}$$

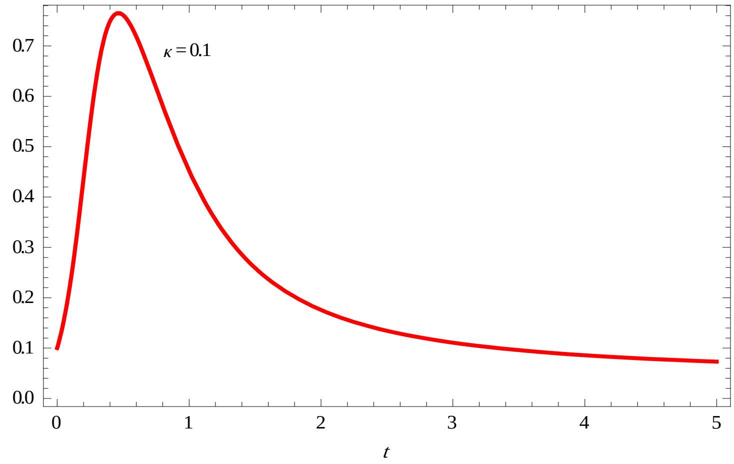
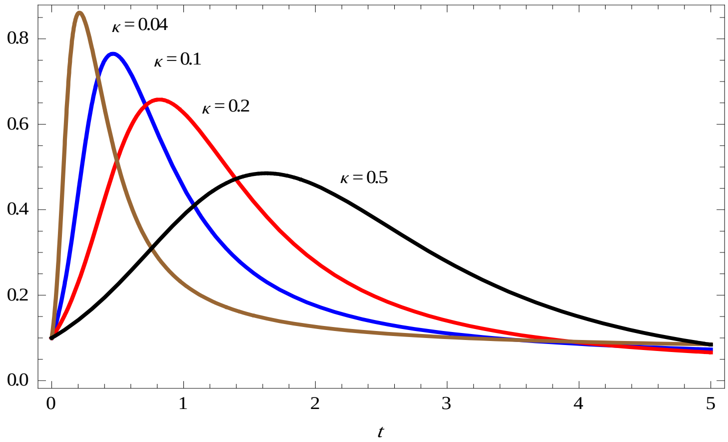


Figure (4.5): Relation between Padé approximants $[4/4]$ of $u(t)$ and t for $u(0)=0.1, \kappa=0.1$.



***Figure(4.6): Relation between Padé approximants* $[4/4]$ *of* $u(t)$ *and* t *for* $u(0) = 0.1, \kappa = 0.04, 0.1, 0.2$ *and* 0.5.**

Figure(4.5) shows the relation between the Padé approximants $[4/4]$ of $u(t)$ and t. Which is the same results obtained by ADM, VIM and DJM. Also Figure(4.6), is the same results is that obtained by ADM, VIM and NIM. From Fig.4.5, we can easily observe that for $u(0) = 0.1$ and $\kappa = 0.1$, we obtain $u_{max} = 0.7651130891$ which occurs at $t_{critical} = 0.4644767409$. Table (4.3) summarizes the relation between κ, u_{max} and $t_{critical}$. Moreover, the results of the corresponding absolute errors $\varepsilon = |exact\ solution - approximate\ solution|$ are also presented for the Padé approximants and $[4/4]$. The exact values of u_{max} were evaluated by using Eq.(4.16) obtained by TeeBest in [36]. It can seen clearly from Table (4.3) that the results obtained by PSM is in a very good agreement with the exact solution and the absolute errors are decreases when the values of κ are increases as $u(t)$ becomes smooth.

Table (4.3): Approximation of u_{max} and exact value of u_{max} and absolute errors for $\kappa = 0.04, 0.1, 0.2$ and 0.5 with the Padé approximants $[4/4]$

κ	Critical t	Approx. u_{max}	Exact u_{max}	ε for [4/4]
0.04	0.2102464442	0.8612401810	0.8737199832	4.62836×10^{-3}
0.1	0.4644767409	0.7651130891	0.7697414491	1.13807×10^{-3}
0.2	0.8168581213	0.657912310	0.6590503816	9.20576×10^{-5}
0.5	1.6266622270	0.4852823490	0.4851902914	1.24798×10^{-2}

4.3 Convergence of PSM for Volterra's population model

Volterra's population model given by Eq.(4.9) may be generalized to the following inetgro-differential equation :

$$u'(x) = g(x, u(x)) + u(x)\int_0^x k(x, t, u(t), u'(t))dt, \; u(0) = a \qquad (4.26)$$

In the following theorem we prove convergence of the PSM for Volterra's population model. The proof of the convergence theorem is a modification to the proof of theorem (3.3).

Theorem (4.1):

Let u be the exact solution of the generalized Volterra's population model (4.26) and assume that u has a power series representation. Then, the approximate solution obtaind by PSM converge to exact solution.

Proof : Assume the approximation solution to Eq.(4.26) be as follows

$$\tilde{u} = e_0 + e_1 x + e_2 x^2 + ...$$

Hence, it suffices to prove that $e_m = \dfrac{u^m(0)}{m!}, m = 1, 2, 3, ...$

If $m=0$ and with the cooperation of the initial condition, we get

$$a=\tilde{u}(0)=e_0+e_1 0+e_2 0+\ldots=e_0 \rightarrow e_0=a$$

If $m=1$, and since $u(x)$ is the exact solution of Eq.(4.26), then it satisfies this integro-differential equation. i.e.

$$u'(x)=g(x,u(x))+u(x)\int_0^x k(x,t,u(t),u'(t))dt \qquad (4.27)$$

and hence, substituting $x=0$ in Eq.(4.27) gives

$$u'(0)=g(0,u(0))+u(0)\int_0^0 k(x,t,u(t),u'(t))dt = g(0,u(0)) \qquad (4.28)$$

and since $u'(x)=\tilde{u}'(x)=e_1+2e_2x+3e_3x^2+\ldots$

and therefore, $u'(0)=\tilde{u}'(0)$, then $g(0,u(0))=u'(0)$

If $m=2$, then differentiate Eq.(4.26) with respect to x and substitute $u=u(x)$, we get

$$\begin{aligned} u''(x)&=\frac{\partial}{\partial x}g(x,u(x))+\frac{\partial}{\partial u}g(x,u(x))u'(x) \\ &+u'(x)\int_0^x k(x,t,u,u')dt+u(x)[k(x,t,u,u') \qquad (4.29) \\ &+\int_0^x \frac{\partial}{\partial x}k(x,t,u,u')]dt \end{aligned}$$

Evaluate Eq.(4.29) at $x=0$, gives

$$\begin{aligned} u''(0)&=\frac{\partial}{\partial x}g(0,u(0))+\frac{\partial}{\partial u}g(0,u(0))u'(0) \\ &+u'(0)\int_0^0 k(x,t,u,u')dt+u(0)[k(x,t,u,u') \qquad (4.30) \\ &+\int_0^0 \frac{\partial}{\partial x}k(x,t,u,u')]dt \end{aligned}$$

$$=\frac{\partial}{\partial x}g(0,u(0))+\frac{\partial}{\partial u}g(0,u(0))u'(0)+u(0)k(0,u(0),u'(0)) \qquad (4.31)$$

and since

$$u''(x) = \tilde{u}''(x), then\ u''(0) = \tilde{u}''(0)$$

Therefore,

$$2e_2 = \frac{\partial}{\partial x} g(0,u(0)) + \frac{\partial}{\partial u} g(0,u(0))u'(0) + u(0)k(0,u(0),u'(0))$$

and hence

$$2e_2 = \frac{\partial}{\partial x} g(0,e_0) + \frac{\partial}{\partial u} g(0,e_0)e_1 + e_0 k(0,e_0,e_1) \qquad (4.32)$$

Comparing Eq.(4.31) and Eq.(4.32), we get $2e_2 = u''(0) \to e_2 = \frac{u''(0)}{2!}$.

So by inducting, one can show that $e_3 = \frac{u''(0)}{3!}$, $e_4 = \frac{u''(0)}{4!}$

By mathematical induction and so in general :

$$e_m = \frac{u^m(0)}{m!},\ m = 1,2,3,...$$

This completes the proof of Theorem 4.1. □

4.4 The Hybrid Selection Model, [46]

The Hybrid selection model with constant coefficients has the following model:

$$u' = ku(1-u)(2-u),\ u(0) = 0.5. \qquad (4.18)$$

where k is a positive constant that depends on the genetic characteristic. In the hybrid model, u is the portion of population of a certain characteristic and t is the time measured in generations.

4.4.1 Solving the Hybrid Selection Model by DJM

To solve Eq.(4.18), we integrate the differential equation from 0 to t and applying the initial condition, then the following nonlinear VIE is obtained:

$$u(t) = 0.5 + \int_0^t [ku(1-u)(2-u)]dt \tag{4.19}$$

Now, applying the DJM to Eq.(4.19) it is found that:

$$u_0 = \frac{1}{2},$$

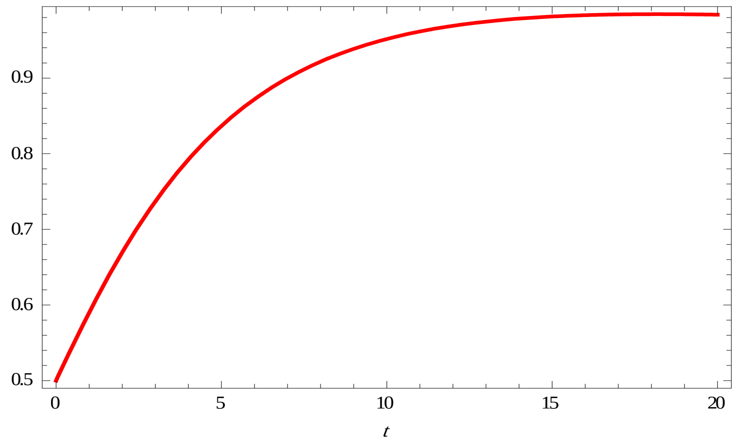
$$u_1 = N(u_0) = \int_0^t [ku_0(1-u_0)(2-u_0)]dt = \frac{3}{8}kt,$$

$$\begin{aligned} u_2 &= N(u_0+u_1) - N(u_0) \\ &= \int_0^t [k(u_0+u_1)(1-(u_0+u_1))(2-(u_0+u_1))]dt - u_1 \\ &= \frac{3}{64}(kt)^2 - \frac{9}{128}(kt)^3 - \frac{27}{2048}(kt)^4, \end{aligned}$$

$$\begin{aligned} u_3 &= N(u_0+u_1+u_2) - N(u_0+u_1) = \int_0^t [k(u_0+u_1+u_2) \\ &\quad (1-(u_0+u_1+u_2))(2-(u_0+u_1+u_2))]dt - \\ &\quad \int_0^t [k(u_0+u_1)(1-(u_0+u_1))(2-(u_0+u_1))]dt \\ &= \frac{17}{256}(kt)^3 - \frac{63}{2048}(kt)^4 + \frac{27}{2560}(kt)^5. \end{aligned}$$

and so on. This gives

$$\begin{aligned} u(t) &= 0.5 + \frac{3}{8}kt - \frac{3}{64}(kt)^2 - \frac{17}{256}(kt)^3 \\ &\quad - \frac{125}{4096}(kt)^4 + \frac{721}{81920}(kt)^5 + \ldots \end{aligned} \tag{4.20}$$



Figure(4.7): The solution $u(t)$ for $k = 0.25$, and $0 \le t \le 20$

This in turn gives

$$u(t) = \frac{\sqrt{1+3e^{3kt}} - 1}{\sqrt{1+3e^{3kt}}}$$

which is the exact solution of the problem and it is the same results obtained by VIM [46]. Figure(4.7) shows the solution u which is an increasing function bounded by $u = 1$.

4.4.2 Solving the Hybrid Selection Model by PSM

Now, applying PSM to solve Eq.(4.19), we suppose the solution of Eq.(4.19) with $u_0(t) = u(0) = e_0 = 0.5$ is, $u_1(t) = e_0 + e_1 t$, which implies

$$u_1(t) = 0.5 + e_1 t, \qquad (4.21)$$

Substitute Eq.(4.21) into Eq.(4.19), yields to:

$$0.5 + e_1 t = 0.5 + \int_0^t [k(0.5+e_1 t)(1-(0.5+e_1 t))(2-(0.5+e_1 t))]dt, \qquad (4.22)$$

After simplifying, we get, $e_1 = \frac{3}{8}k$ and hence:

$$u_1(t) = 0.5 + \frac{3}{8}kt, \qquad (4.23)$$

which is the first approximation for the solution of Eq.(4.19).

Similarly, the solution of Eq.(4.19) can be supposed as:

$$u(t) = 0.5 + \frac{3}{8}kt + e_2 t^2, \tag{4.24}$$

and substitute Eq.(4.24) into Eq.(4.19), give:

$$\begin{aligned} 0.5 + \frac{3}{8}kt + e_2 t^2 = 0.5 + \int_0^t [k\,(0.5 + \frac{3}{8}kt + e_2 t^2)\\ (1 - (0.5 + \frac{3}{8}kt + e_2 t^2))\\ (2 - (0.5 + \frac{3}{8}kt + e_2 t^2))]dt. \end{aligned} \tag{4.25}$$

After simplifying, we get, $e_2 = \frac{3}{64}k^2$ and hence,

$$u_2(t) = 0.5 + \frac{3}{8}kt + \frac{3}{64}(kt)^2, \tag{4.26}$$

Proceeding in this way, we get:

$$\begin{aligned} u(t) = 0.5 + \frac{3}{8}kt + \frac{3}{64}(kt)^2 - \frac{17}{256}(kt)^3\\ - \frac{125}{4096}(kt)^4 + \frac{721}{81920}(kt)^5 + \ldots \end{aligned} \tag{4.27}$$

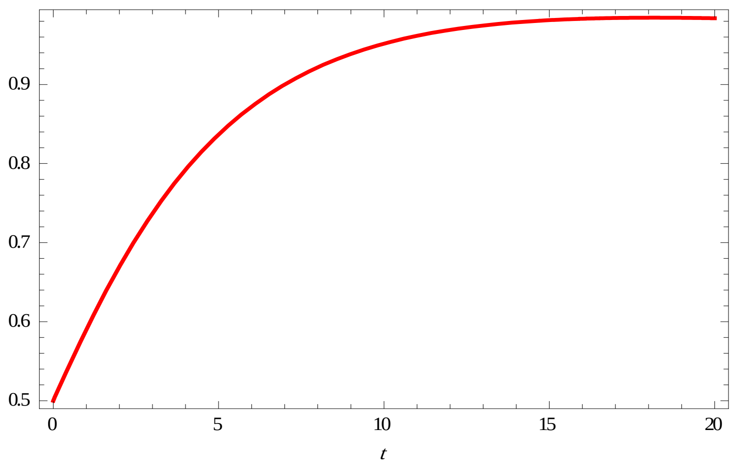


Figure (4.8): The solution $u(t)$ for $k = 0.25$, and $0 \le t \le 20$

This in turn gives

$$u(t)=\frac{\sqrt{1+3e^{3kt}}-1}{\sqrt{1+3e^{3kt}}}.$$

which is the exact solution of the problem and it is the same results obtained by DJM and VIM in [46].

4.5 The Riccati Equation

The Riccati equation is one of the most interesting nonlinear differential equations of the first order. It is written in the form:

$$u'=a(x)u+b(x)u^2+c(x), x\geq 0$$

where a , b , c are continuous functions of x and $c(x), b(x)\neq 0, \forall x\in[0,\infty)$

By integrating the equation from 0 to x, we get :

$$u(x)=u_0+\int_0^x [a(t)u(t)+b(t)u^2(t)+c(t)]dt$$

4.5.1 Solving the Riccati Equation by DJM

Consider the Riccati equation of the form [46]:

$$u'(t)=u^2(t)-2xu(t)+x^2+1,\ u(0)=0.5. \tag{4.28}$$

To solve Eq.(4.28), integrate from 0 to x, and apply the initial condition, we get:

$$u(t)=0.5+\int_0^x [u^2-2tu+x^2+1]dt. \tag{4.29}$$

Now, apply DJM to Eq.(4.29) we find:

$$u_0=\frac{1}{2},$$

$$u_1=N(u_0)=\int_0^x [u_0^2-2tu_0+x^2+1]dt$$

$$=u_1=\frac{5}{4}x+\frac{1}{2}x^2-\frac{1}{3}x^3,$$

$$u_2 = N(u_0 + u_1) - N(u_0) = \int_0^x [(u_0 + u_1)^2 - 2t(u_0 + u_1) + x^2 + 1]dt - u_1$$

$$= \frac{1}{8}x^2 + \frac{7}{48}x^3 - \frac{1}{28}x^4,$$

$$u_3 = N(u_0 + u_1 + u_2) - N(u_0 + u_1) = \int_0^x [(u_0 + u_1 + u_2)^2$$

$$- 2t(u_0 + u_1 + u_2) + x^2 + 1]dt - \int_0^x [(u_0 + u_1)^2 - 2t(u_0 + u_1) + x^2 + 1]dt$$

$$= \frac{1}{16}x^3 + \frac{1}{48}x^4 - \frac{7}{960}x^5.$$

and so on, this gives

$$u(x) = x + \frac{1}{2}(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 + \frac{1}{32}x^5 + ...) \qquad (4.30)$$

That converges to

$$u(x) = x + \frac{1}{2-x}, |x| < 2.$$

which is the exact solution of the problem and it is the same results obtained by VIM [46].

4.5.2 Solving the Riccati Equation by PSM

Now, applying the PSM to Eq.(4.29), and suppose the first approximate solution of Eq.(4.29) with $u_0(x) = u(0) = e_0 = 0.5$ is $u_1(x) = e_0 + e_1 x$, and hence

$$u_1(x) = 0.5 + e_1 x, \qquad (4.31)$$

Substitute Eq.(4.31) in Eq.(4.29), to get,

$$0.5 + e_1 x = 0.5 + \int_0^x [(0.5 + e_1 t)^2 - 2t(0.5 + e_1 t) + x^2 + 1]dt. \qquad (4.32)$$

After simplifying, we get, $e_1 = \frac{5}{4}$ and hence,

$$u_1(x) = 0.5 + \frac{5}{4}x, \qquad (4.33)$$

which is the first approximation for the solution of Eq.(4.29). The solution of Eq.(4.29) can be supposed as:

$$u(x) = 0.5 + \frac{5}{4}x + e_2 x^2, \tag{4.34}$$

Substitute Eq.(4.34) in Eq.(4.29), we have,

$$\begin{aligned} 0.5 + \frac{5}{4}x + e_2 x^2 = 0.5 + \int_0^x [(0.5 + \frac{5}{4}t + e_2 t^2)^2 - \\ 2t(0.5 + \frac{5}{4}t + e_2 t^2) + x^2 + 1]dt. \end{aligned} \tag{4.35}$$

After simplifying, we get, $e_2 = \frac{1}{8}$ and hence,

$$u_2(x) = 0.5 + \frac{5}{4}x + \frac{1}{8}x^2, \tag{4.36}$$

Proceeding in this way, we get,

$$u(x) = x + \frac{1}{2}(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 + \frac{1}{32}x^5 + ...) \tag{4.37}$$

That converges to

$$u(x) = x + \frac{1}{2-x}, |x| < 2.$$

which is the exact solution of the problem and it is the same results obtained by DJM and find VIM in [46].

4.6 The Logistic Differential Equation

The logistic function is the solution of the simple first-order nonlinear differential equation [47].

$$u' = \mu u(1-u), u(0) = \frac{1}{2}, \mu > 0, x \geq 0 \tag{4.38}$$

The logistic function finds many applications in a range of fields, including artificial neural networks, biology, biomathematics, demography, economics, chemistry, mathematical psychology, probability, sociology, political science, and statistics.

4.6.1 Solving the Logistic Differential Equation by DJM

To solve Eq.(4.38) using the DJM, integrate from 0 to x , and apply the initial condition, we get

$$u(x) = 0.5 + \int_0^x [\mu u(t)(1-u(t)] \, dt \tag{4.39}$$

Now applying the DJM to Eq.(4.39), one may find:

$$u_0 = \frac{1}{2},$$

$$u_1 = \int_0^x [\mu u_0(t)(1-u_0(t)]dt = \frac{\mu}{4}x,$$

$$u_2 = \int_0^x [\mu(u_0+u_1)(t)(1-(u_0+u_1)(t)]dt - u_1 = -\frac{\mu^3}{48}x^3,$$

$$\begin{aligned} u_3 &= \int_0^x [\mu(u_0+u_1+u_2)(t)(1-(u_0+u_1+u_2)(t)]dt \\ &\quad - \int_0^x [\mu(u_0+u_1)(t)(1-(u_0+u_1)(t)]dt \\ &= \frac{\mu^5}{480}x^5 - \frac{\mu^7}{16128}x^7. \end{aligned}$$

and so on, which gives

$$u(x) = \frac{1}{2} + \frac{\mu}{4}x - \frac{\mu^3}{48}x^3 + \frac{\mu^5}{480}x^5 - \frac{17\mu^7}{80640}x^7 - \frac{31\mu^9}{1451520}x^9 + \ldots \tag{4.40}$$

While the exact solutionis given by [40].

$$u(x) = \frac{e^{\mu x}}{1+e^{\mu x}}.$$

which is the same results obtained by VIM [47].

4.6.2 Solving the Logistic Differential Equation by PSM

Now, using the PSM to solve Eq.(4.39), we suppose the solution of Eq.(4.39) with $u_0(x) = u(0) = e_0 = 0.5$ is, $u_1(x) = e_0 + e_1 x$, and hence:

$$u_1(x) = 0.5 + e_1 x, \tag{4.41}$$

Substitute Eq.(4.41) into Eq.(4.39), then:

$$0.5 + e_1 x = 0.5 + \int_0^x [\mu(0.5 + e_1 t)(1 - (0.5 + e_1 t))]dt. \tag{4.42}$$

After simplifying, we get, $e_1 = \dfrac{\mu}{4}$ and hence:

$$u_1(x) = 0.5 + \frac{\mu}{4}x, \tag{4.43}$$

which is the first approximation for the solution of Eq.(4.39). The solution of Eq.(4.39) can be supposed as:

$$u(x) = 0.5 + \frac{\mu}{4}x + e_2 x^2, \tag{4.44}$$

Substitute Eq.(4.44) into Eq.(4.39), we have,

$$\begin{aligned} 0.5 + \frac{\mu}{4}x + e_2 x^2 = 0.5 + \int_0^x [\mu(0.5 + \frac{\mu}{4}t + e_2 t^2) \\ (1 - (0.5 + \frac{\mu}{4}t + e_2 t^2))]dt. \end{aligned} \tag{4.45}$$

After simplifying, weget, $e_2 = \dfrac{\mu^3}{48}$ and hence,

$$u(x) = 0.5 + \frac{\mu}{4}x + \frac{\mu^3}{48}x^2, \tag{4.46}$$

Proceeding in this way, we get,

$$\begin{aligned} u(x) = \frac{1}{2} + \frac{\mu}{4}x - \frac{\mu^3}{48}x^3 + \frac{\mu^5}{480}x^5 \\ - \frac{17\mu^7}{80640}x^7 - \frac{31\mu^9}{1451520}x^9 + \ldots \end{aligned} \tag{4.47}$$

While the exact solution is given by [47].

$$u(x) = \frac{e^{\mu x}}{1 + e^{\mu x}}.$$

which is the exact solution of the problem and it is the same results obtained by DJM and VIM in [47].

Chapter Five

Conclusions and Recommendations

The fundamental goals of this thesis are to construct an analytic and approximate solutions to VIDE and its applications to certain scientific models. The goals have been achieved by implementing the DJM [15] and the PSM [38], [39] in a straightforward manner to the nonlinear VIDE.

In this chapter we shall review the main results presented in the thesis and make some suggestions for future work.

5.1 Conclusions

There are three important points to be noted in this work. First, unlike the traditional grid points techniques used by TeeBest in [36], the solution is defined at grid points only, the solution here is given in a series form. Second, in [36], the scheme used 20 iteration steps, whereas less than 10 iteration steps were used in this thesis. Third, the high agreement of the approximations of the solution between the methods used at this work is clear and remarkable. This advantage over existing techniques demonstrates the reliability and the efficiency of these methods. Furthermore, the efficiency of these approaches can be dramatically enhanced by computing further terms or further components of the solution when the DJM or the PSM are used, respectively. In this thesis, a solution that is valid in the domain of definition is obtained and the mathematical structure of the solution was successfully enhanced by employing Padé approximants. The Padé approximants, that is often show superior performance over series approximations, provide a successful tool and promising scheme for identical applications. Among the obtained conclusions obtained from this thesis are as follows:

1. The DJM and the PSM have been successfully applied to Volterra's population model. The numerical results reveal that the proposed methods is very simple, straightforward and provided the same results obtained by ADM, HPM and VIM in [43], [29], [42].
2. The DJM and PSM unlike the mesh points schemes does not provide any linear or nonlinear system of equations.
3. The DJM and PSM does not require any discretization, linearization or small perturbations and therefore is capable of greatly reducing the size of calculations while still maintaining high accuracy of the numerical solution.
4. It is worth pointing out that the ADM requires to evaluate of the Adomian polynomials that mostly require tedious algebraic calculations. Also, HPM requires to construct a homotopy and solve the corresponding algebric equation and the VIM requires to calculate the Lagrange multiplier. It is interesting to point out that the DJM and the PSM reduces the volume of calculations in comparative with existing techniques, since the iteration is direct and straightforward.
5. Moreover, for nonlinear equations that arise frequently to express nonlinear phenomenon, the DJM and PSM are powerful and efficient, facilitate the computational work and give the solution rapidly in comparative with other numerical techniques.
6. Furthermore, the methods are used to solve some scientific models, namely, the hybrid selection model, the Riccati model and the logistic model to provide the analytic solution and the results showed the DJM and PSM provided the same results obtained by VIM, [46], [47].
7. The simple, easy-to-apply, economical in terms of computer power/memory and fast algorithm of the proposed methods is the main advantages over other existing methods.

Table (5.1): Comparison between the absolute errors of ADM, HPM, VIM, PSM and DJM

κ	ADM	HPM	VIM	PSM	DJM
0.1	$4.62836\text{x}10^{-3}$	$4.62836\text{x}10^{-3}$	$4.62836\text{x}10^{-3}$	$4.62836\text{x}10^{-3}$	$4.62836\text{x}10^{-3}$
0.2	$1.13807\text{x}10^{-3}$	$1.13807\text{x}10^{-3}$	$1.13807\text{x}10^{-3}$	$1.13807\text{x}10^{-3}$	$1.13807\text{x}10^{-3}$
0.5	$9.20568\text{x}10^{-5}$	$9.20568\text{x}10^{-5}$	$9.20576\text{x}10^{-5}$	$9.20576\text{x}10^{-5}$	$9.20568\text{x}10^{-5}$
0.04	$1.24798\text{x}10^{-2}$	$1.24798\text{x}10^{-2}$	$1.24798\text{x}10^{-2}$	$1.24798\text{x}10^{-2}$	$1.24798\text{x}10^{-2}$

It can be seen from the Table (5.1) that, the absolute errors of DJM and PSM are completely the same as the absolute errors of ADM, HPM and VIM as expected because the approximation in Eq.(4.18) and Eq.(4.25) is the same for all methods.

5.2 Recommendations

Our recommendations for future work are:

1. Solve the mixed Volterra-Fredholm integral equation by DJM or PSM.
2. Solving Burger's, coupled Burger's equation and system of two-dimensional Burger's equations by DJM or PSM.
3. Solving the Bratu-type equations by PSM.
4. Solving the Fokker-Planck equation by DJM or PSM.
5. Solving two forms of Blasiu's equation on a half-infinite domain by DJM or PSM.
6. Solving nonlinear systems of PDEs by DJM or PSM.
7. Solving the evolution equations by PSM.
8. Solving nonlinear Klein-Gordon Equations and nonlinear Schrodinger equations by PSM.

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المستخلص

الهدف الرئيسي لهذه الرسالة هو لدراسة وحل معادلة فولتيرا التكاملية (Volterra Integral Equations) والمعادلات التفاضلية التكاملية (Volterra Integro-Differential Equations) إضافة إلى دراسة بعض الأنظمة الحياتية الواقعية. حيث يمكن تمثيل هدف الرسالة بثلاث أهداف فرعية، وهي :

الهدف الأول تقديم بعض التعاريف والمفاهيم الأساسية في موضوع المعادلات التكاملية .

الهدف الثاني يتناول تنفيذ طريقة تكرارية جديدة وهي (Dafterdar-Jafari Method) لاستخلاص حل لمعادلات فولتيرا التكاملية والتفاضلية التكاملية الخطية وغير الخطية.

الهدف الثالث، استخدام الطريقة التكرارية المسماة طريقة متسلسلة القوى (Power Series Method) وتنفيذها لإيجاد الحل لمعادلات فولتيرا التكاملية والتفاضلية التكاملية الخطية وغير الخطية.

أخيرا، تم تطبيق Power Series Method و Dafterdar-Jafari Method لإيجاد الحل التقريبي التحليلي لبعض النماذج الحياتية العلمية.

جمهورية العراق

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حلول تحليلية وتقريبية لمعادلات فولتيرا التفاضلية - التكاملية وتطبيقاتها

رسالة

مقدمة إلى كلية العلوم/ جامعة النهرين
لكجزء من متطلبات نيل درجة ماجستير علومفي الرياضيات

من قبل

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