Republic of Iraq
Ministry of Higher Education
and Scientific Research
Al-Nahrain University
College of Science
Department of Mathematics
and Computer Applications

Estimation of Reliability Function for Inverse Gaussian Distribution Model with Application by Using Monte Carlo Simulation

A Thesis

Submitted to the College of Science Al-Nahrain University as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Applied Mathematics

By
Omar Ismael AL-Timeemy
(B.Sc., Al-Nahrain University, 2005)

Supervised by

Asst. Prof. Dr. Akram M. Al-Abood

May *Jamady Al-awal*
2008 *1429*

بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

اقْرَأْ بِاسْمِ رَبِّكَ الَّذِي خَلَقَ ﴿١﴾ خَلَقَ الْإِنسَانَ مِنْ عَلَقٍ ﴿٢﴾ اقْرَأْ
وَرَبُّكَ الْأَكْرَمُ ﴿٣﴾ الَّذِي عَلَّمَ بِالْقَلَمِ ﴿٤﴾ عَلَّمَ الْإِنسَانَ مَا لَمْ يَعْلَمْ ﴿٥﴾

صدق الله العظيم

In the name of Allah, Most Gracious, Most Merciful.

Proclaim! (or read!) in the name of thy Lord and Cherisher, Who created, Created man, out of a (mere) clot of congealed blood, Proclaim! And thy Lord is Most Bountiful He Who taught (the use of) the pen, Taught man that which he knew not.

سورة العلق

DEDICATION

To...
All Persons who
Encouraged and
Supported Me in My Life

ACKNOWLEDGMENTS

I owe my gratitude to Allah and all the people who have made this work possible.

I wish to express my deep appreciation and sincere thanks to my supervisor Dr. Akram M. Al-Abood for his appreciable advices, important comments support and encouragement during the research.

I am indebted to the staff members of the department of mathematics and computer applications Dr. Ala'a , Dr. Ahlam , Dr. Radhi , Dr. Fadhil and Dr. Usama, for extremely high quality supervision on my undergraduate study ever since I joined the Department and support in this work. It has been a pleasure to learn from such extraordinary individuals.

Thanks are extended to the College of Science of Al-Nahrain University for giving me the chance to complete my postgraduate study.
Last but not least, I am deeply indebted to my family, father, mother and my brothers, Zaid, Abdulrahmaan, Ali, Mustafa, and my friends Salam, Akram, Mohammed, Suhaib, Bilal, Azad, Anas, Saddam, Noor, Sara, Semma and all my friends.

Omar Al-Timeemey

May 2008

Notes and Abbreviations

r.v	*random variable*
r.s	*random sample*
s.s	*sample space*
distn.	*distribution*
p.d.f	*probability density function*
c.d.f	*cumulative distribution function*
m.g.f	*moment generating function*
e.w.	*else were*
AR	*acceptance-rejection*
$\Phi(\,.\,)$	*cumulative standard normal distribution function*
M.M	*moment method*
m.ℓ.e	*Maximum Likelihood Estimate*
MVUE	*Minimum Variance Unbiased Estimator*
$\chi^2_{(n)}$	*Chi square Distribution with n degrees of freedom*
$G(\alpha,\beta)$	*Gamma Distribution with parameters* α *and* β
$Exp(\lambda)$	*Exponential Distribution With Parameters* λ
R(x)	*Reliability function of x*
h(x)	*hazard function of x*
MLM	*maximum likelihood method*
N(0,1)	*standard normal distribution*
MLE	*Maximum Likelihood Estimator*

Abstract

In this work we consider the Inverse Gaussian distribution model of two parameters, because it have many applications in the fields of statistics and reliability. Mathematical and statistical properties of the distribution are given together with illustration. Moments and higher moments of the distribution properties and of the reliability and hazard functions are discussed theoretically.

Two methods of estimation namely moments method and maximum likelihood method are used to estimate the distribution parameters. The obtained estimators are utilized together with Basu method to estimate the reliability and the hazard function.

These methods are discussed theoretically and applied practically by using three procedures of generating random sample from the distribution. Bias measure is used to compare between these procedures.

The computer programs are coding in appendices by the run is made by using “MathCAD 14”.

List of Contents

Introduction

The inverse Gaussian distribution was originally discovered by Schrödinger in 1915 as the probability distribution of the first passage time in Brownian motion [38]. Because of the inverse relationship between the cumulant generating function of the first passage time distribution and that of the normal distribution, Tweedie (1945) proposed the name inverse Gaussian for the first passage time distribution[41]. The distribution was next given by Wald (1947) who derived it as a limiting form for the distribution of sample size in a sequential probability ratio test [45]. Because of this derivation, the distribution is also known as Wald's distribution, particularly in the Russian literature. However, from the viewpoint of statistics, it might more appropriately be called Tweedie's distribution. It had remained virtually unnoticed until Tweedie (1957) investigated its basic characteristics, established some important statistical properties and depicted certain analogies between its statistical analysis and that of the normal distribution [42], [43].A characterization of the inverse Gaussian distribution by Khatri (1962) paralleled the usual characterization of the normal distribution by the independence of sample mean and variance, further reflecting this analogy [26]. Wasan and his associates (1968, 1969) investigated some analytical and characteristic properties of this class of distributions, particularly for the limiting forms [46], [47]. Chhikara (1975) and Chhikara and Folks (1974, 1975, 1976, 1977,1978) have developed further its statistical theory, provided statistical methods based upon the inverse Gaussian, particularly in the field of reliability[7], [8], [9], [10], [11], [15]. The interpretation of the inverse

Gaussian random variable as a first passage time suggests its potential useful applications in studying life time or number of event occurrences for a wide range of fields, For example, Sheppard (1962) proposed it for the distribution of the time spent by an injected labelled substance, called tracer, in a biological system [39]. Hasofer (1964) considered the inverse Gaussian model for the emptiness of dam[18]. Lancaster (1972) used it as a model for duration of strikes [29]. Banerjee and Bhattacharyya (1976) applied it in a study of purchase incidence models [2]. Bardsley (1980) applied the inverse Gaussian distribution for wind energy [3]. Dennis etal (1991) used the inverse Gaussian distribution to describe the time to extinction of endangered species [12]. Barndorff-Neielsen (1994) used the inverse Gaussian distribution as a model for the electrical networks when it has the structures of a rooted tree [4]. Koichi etal (1997) proposed the application of inverse Gaussian distribution to occupational exposure data [27]. Durham and Padgett (1997) found that for certain materials, such as carbon fiber composites, the inverse Gaussian distribution provides a better fit as a material strength model [13]. Hamsa (1997) used the inverse Gaussian distribution to estimate the return periods of floods and droughts of the Blue-Nile river [17]. Huberman et al (1998) showed that the number of links an internet user follows before the page value first reaches the stopping threshold has an asymptotic inverse Gaussian distribution [20].

The aim of this work is to find the estimators of the reliability and the hazard functions of the inverse Gaussian distribution by different methods theoretically, and then applied them practically to find our best estimator by using Monte-Carlo simulation.

This thesis includes three chapters. In chapter one, we present some important mathematical and statistical properties of inverse Gaussian distn. Moment properties of the distribution are illustrated and unified. Two methods of estimation for the distribution parameters are discussed theoretically.

In chapter two, we introduce some concepts of reliability and hazard functions, estimates the reliability function, illustration to the minimum variance unbiased estimator for the reliability function by three different cases.

In chapter three, we introduce the Monte Carlo simulation and its applications for parameters estimation given in chapter one and the reliability and the hazard functions given in chapter two practically by three procedures namely (IG-1), (IG-2) and (IG-3).

CHAPTER ONE

1

ON INVERSE GAUSSIAN DISTRIBUTION

1.1 Introduction

In this chapter, some mathematical and statistical properties of inverse Gaussian distn. have been presented.

This chapter involves five sections. In section (1.2) we give some basic concepts of inverse Gaussian distn., while in section (1.3) we illustrate moments and higher moments properties of the distn. In section (1.4) we considere two methods of parameters estimation namely moments method and maximum likelihood method, these methods discussed theoretically. In section (1.5) we prove some related theorems concerning the disn..

1.2 Some Basic Concepts of Inverse Gaussian Distribution

In this section we shall give some mathematical and statistical properties of the inverse Gaussian distribution.

Definition (1.1) [42]

A continuous r.v. X is said to have inverse Gaussian distn., denoted by $X \sim IG(\mu, \lambda)$ if X has p.d.f

$$f(x;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi}}\, x^{\frac{-3}{2}}\, e^{\left[\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right]}, \quad 0 < x < \infty \tag{1.1}$$

$$= 0, e.w.; \quad \mu > 0, \lambda > 0$$

Where μ and λ are respectively known as the scale and shape parameters.

To verify that $f(x;\mu,\lambda)$ of eq. (1.1) is valid p.d.f., we have to show that

(i) $f(x;\mu,\lambda)>0,\ \forall\ x,\mu,\lambda\in(0,\infty)$, obvious.

(ii) The integration of eq. (1.1) is unity.

We forward a new approach for satisfying condition (ii).

For the purpose of our approach, we need the following result:

From advanced calculus [1]

$$\int_0^\infty z^{\frac{-1}{2}} e^{\frac{-z}{2}}\, dz = \sqrt{2\pi} \tag{1.2}$$

Let $I = \int_0^\infty f(x)dx = \int_0^\infty \sqrt{\frac{\lambda}{2\pi}}\, x^{\frac{-3}{2}} e^{\left[\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right]} dx$

Set $w = \frac{x}{\mu}$ or equivalently $x=w\mu$ implies $dx = \mu\, dw$, then

$$I = \sqrt{\frac{\lambda}{2\pi\mu}} \int_0^\infty w^{\frac{-3}{2}} e^{\left[\frac{-\lambda(w-1)^2}{2\mu w}\right]} dw$$

For simplicity, we set $\theta = \frac{\lambda}{\mu}$, then I becomes

$$I = \sqrt{\frac{\theta}{2\pi}} \int_0^\infty w^{\frac{-3}{2}} e^{\left[\frac{-\theta(w-1)^2}{2w}\right]} dw$$

$$= \sqrt{\frac{\theta}{2\pi}} \int_0^1 w^{\frac{-3}{2}} e^{\left[\frac{-\theta(w-1)^2}{2w}\right]} dw + \sqrt{\frac{\theta}{2\pi}} \int_1^\infty w^{\frac{-3}{2}} e^{\left[\frac{-\theta(w-1)^2}{2w}\right]} dw$$

consider the transformation $Y = \min(W, \frac{1}{W})$

For $0<w<1$, $y=\min(w,\frac{1}{w})=w \Rightarrow dy=dw$

for $1<w<\infty$, $y=\min(w,\frac{1}{w})=\frac{1}{w} \Rightarrow dy=\frac{-1}{w^2}dw$

Therefore

$$I=\sqrt{\frac{\theta}{2\pi}}\int_0^1 y^{\frac{-3}{2}} e^{\left[\frac{-\theta(y-1)^2}{2y}\right]} dy + \sqrt{\frac{\theta}{2\pi}}\int_0^1 y^{-\frac{1}{2}} e^{\left[\frac{-\theta(y-1)^2}{2y}\right]} dy$$

$$=\sqrt{\frac{\theta}{2\pi}}\int_0^1 (y^{\frac{-3}{2}}+y^{\frac{-1}{2}}) e^{\left[\frac{-\theta(y-1)^2}{2y}\right]} dy$$

consider the transformation $z=\frac{\theta(y-1)^2}{y}$

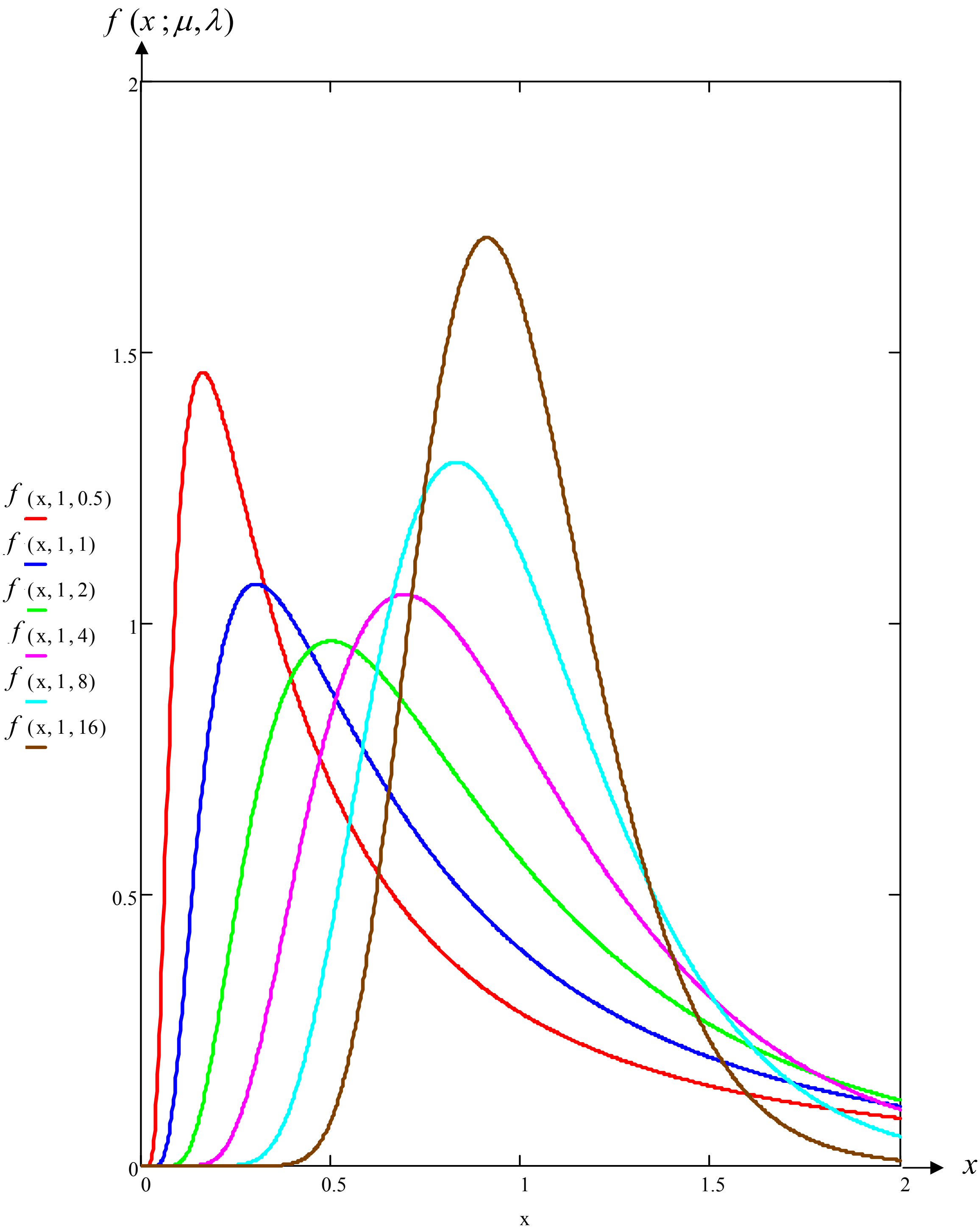
for $0<y<1$, $\sqrt{z}=\frac{-\sqrt{\theta}(y-1)}{\sqrt{y}}$ with $\frac{-dz}{\sqrt{\theta z}}=(y^{\frac{-1}{2}}+y^{\frac{-3}{2}})dy$

So $I=\frac{1}{\sqrt{2\pi}}\int_0^\infty z^{\frac{-1}{2}} e^{\frac{-z}{2}} dz$

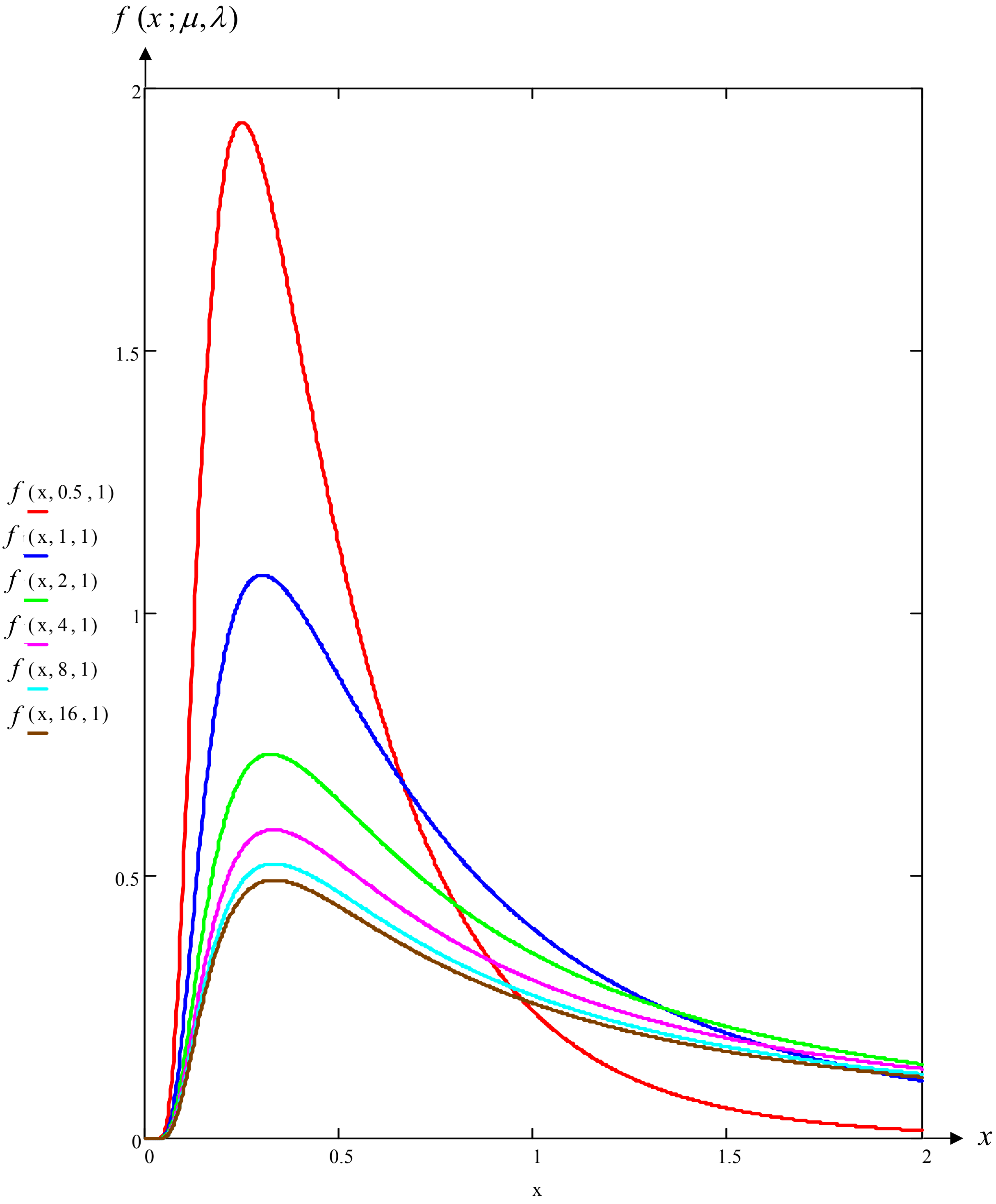
Using eq.(1.2), we have

$$I=\frac{1}{\sqrt{2\pi}}.\sqrt{2\pi}=1$$

The inverse Gaussian distn. depends on two parameters μ and λ and a wide variety of distribution shapes can be generated by suitable choice of μ and λ. Figures (1) and (2) show respectively a graphically representation of some p.d.f.'s for fixed μ and λ varying and for fixed λ and μ varying.



Fig(1): Inverse Gaussian p.d.f.'s with μ =1 and λ = 0.5,1,2,4,8,16.



Fig(2): Inverse Gaussian p.d.f.'s with μ = 0.5,1,2,4,8,16 and λ =1.

The graph of $IG(\mu,\lambda)$ as shown in figure (1) and figure (2):

1- Have the $x-axis$ as a horizontal asymptote.

2- Increasing for $0 < x < \mu\left[\sqrt{\frac{9\mu^2}{4\lambda^2}+1}-\frac{3\mu}{2\lambda}\right]$ and decreasing for $\mu\left[\sqrt{\frac{9\mu^2}{4\lambda^2}+1}-\frac{3\mu}{2\lambda}\right] < x < \infty$

3- Has maximum point at $x=\mu\left[\sqrt{\frac{9\mu^2}{4\lambda^2}+1}-\frac{3\mu}{2\lambda}\right]$

4- The total area under the curve and above the +ive x-axis is unity.

5- There is a single inflection point which can not be evaluated analytically from the solution by equating the 2nd derivative of eq. (1.1) to zero. An approximate solution can be made when some values of the parameters μ and λ are specified, for instant, when μ=λ=1 , we have $x=0.678$

1.2.1 The Cumulative Distribution Function

The c.d.f of inverse Gaussian distn. is known by the following integral:

$$F(x)=Pr(X\le x)=\int_0^x\sqrt{\frac{\lambda}{2\pi}}\,w^{\frac{-3}{2}}\,e^{\left[\frac{-\lambda(w-\mu)^2}{2\mu^2 w}\right]}dw \qquad (1.3)$$

It is possible to express the formulation of the c.d.f of inverse Gaussian distn. in terms of the c.d.f of standardized normal distn. as follows:

$$f(x;\mu,\lambda)=\sqrt{\frac{\lambda}{2\pi}}\quad x^{\frac{-3}{2}}\,e^{\left[\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right]},\quad 0<x<\infty$$

$= 0$, e.w.

$$F(x)=\Pr(X\le x)=\int_0^x\sqrt{\frac{\lambda}{2\pi}}\,t^{\frac{-3}{2}}\,e^{\left[\frac{-\lambda(t-\mu)^2}{2\mu^2 t}\right]}\,dt$$

Set $y=\dfrac{\sqrt{\lambda}(t-\mu)}{\mu\sqrt{t}}\Rightarrow dy=\dfrac{\sqrt{\lambda}}{\mu}\dfrac{\sqrt{t}-(t-\mu)\frac{1}{2\sqrt{t}}}{t}dt$

$$dy=\frac{\sqrt{\lambda}}{\mu}\frac{t+\mu}{2t^{\frac{3}{2}}}dt\Rightarrow\sqrt{\lambda}\,t^{\frac{-3}{2}}dt=\frac{2\mu}{t+\mu}dy$$

Since $\dfrac{2\mu}{t+\mu}=\dfrac{\mu+\mu}{t+\mu}=\dfrac{(t+\mu)-(t-\mu)}{t+\mu}$

$$=1-\frac{t-\mu}{t+\mu}=1-\frac{t-\mu}{\sqrt{(t+\mu)^2}}$$

$$=1-\frac{(t-\mu)}{\sqrt{(t-\mu)^2+4\mu t}}=1-\frac{\sqrt{\lambda}(t-\mu)}{\sqrt{\lambda(t-\mu)^2+4\mu\lambda t}}$$

$$=1-\frac{\frac{\sqrt{\lambda}(t-\mu)}{\mu\sqrt{t}}}{\sqrt{\frac{\lambda(t-\mu)^2}{\mu^2 t}+\frac{4\lambda}{\mu}}}$$

$$=1-\frac{y}{\sqrt{y^2+\frac{4\lambda}{\mu}}}$$

$$\sqrt{\lambda}\, t^{\frac{-3}{2}} dt = \left[1 - \frac{y}{\sqrt{y^2 + \frac{4\lambda}{\mu}}} \right] dy$$

$$F(x) = \int_{-\infty}^{\frac{\sqrt{\lambda}(x-\mu)}{\mu\sqrt{x}}} \frac{1}{\sqrt{2\pi}} \left[1 - \frac{y}{\sqrt{y^2 + \frac{4\lambda}{\mu}}} \right] e^{\frac{-1}{2} y^2} dy$$

$$= \int_{-\infty}^{\frac{\sqrt{\lambda}(x-\mu)}{\mu\sqrt{x}}} \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2} y^2} dy - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\sqrt{\lambda}(x-\mu)}{\mu\sqrt{x}}} \frac{y}{\sqrt{y^2 + \frac{4\lambda}{\mu}}} e^{\frac{-1}{2} y^2} dy$$

$$F(x) = \Phi\left[\frac{\sqrt{\lambda}(x-\mu)}{\mu\sqrt{x}} \right] - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\sqrt{\lambda}(x-\mu)}{\mu\sqrt{x}}} \frac{y}{\sqrt{y^2 + \frac{4\lambda}{\mu}}} e^{\frac{-1}{2} y^2} dy$$

Consider the integral $I = \displaystyle\int_{-\infty}^{\frac{\sqrt{\lambda}(x-\mu)}{\mu\sqrt{x}}} \frac{-1}{\sqrt{2\pi}} \frac{y}{\sqrt{y^2 + \frac{4\lambda}{\mu}}} e^{\frac{-1}{2} y^2} dy$

set $w = -\sqrt{y^2 + \frac{4\lambda}{\mu}} \Rightarrow y = \sqrt{w^2 - \frac{4\lambda}{\mu}}$

$$dy = \frac{w}{\sqrt{w^2 - \frac{4\lambda}{\mu}}} dw \Rightarrow y dy = w dw \;, w^2 \neq \frac{4\lambda}{\mu}$$

$$I = \int_{-\infty}^{\frac{-\sqrt{\lambda}(x+\mu)}{\mu\sqrt{x}}} \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}(w^2 - \frac{4\lambda}{\mu})} dw$$

$$= e^{\frac{2\lambda}{\mu}} \int_{-\infty}^{\frac{-\sqrt{\lambda}(x+\mu)}{\mu\sqrt{x}}} \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}w^2} dw = e^{\frac{2\lambda}{\mu}} \Phi\left[\frac{-\sqrt{\lambda}(x+\mu)}{\mu\sqrt{x}}\right]$$

$$F(x) = \Phi\left[\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} - 1\right)\right] + e^{\frac{2\lambda}{\mu}} \Phi\left[-\sqrt{\frac{\lambda}{x}}\left(1 + \frac{x}{\mu}\right)\right] \qquad (1.4)$$

Thus when both parameters μ and λ are known, the inverse Gaussian distn. can be evaluated using the normal distn. table.

There is another technique for finding $F(x) = Pr(X \le x)$ for some x by a statistical table suggested by Wasan and Roy [46], base on the following property:

Property (1.2.1.1) [46]

Let $X \sim IG(\mu, \lambda)$, then the r.v $Y = \frac{\lambda X}{\mu^2} \sim IG(\alpha, \alpha^2)$ where $\alpha = \frac{\lambda}{\mu}$.

To find the c.d.f of *Y* we have

$F(x) = \Pr(X \le x)$ and $Y = \frac{\lambda X}{\mu^2}$

Then the c.d.f of *Y*, say *G(y)* is:

$$G(y) = \Pr(Y \le y) = \Pr(\frac{\lambda X}{\mu^2} \le y)$$

$$= \Pr(X \le \frac{\mu^2 y}{\lambda}) = F(\frac{\mu^2 y}{\lambda})$$

$$G(y) = \Pr(Y \le y) = \Phi\left(\frac{y-\alpha}{\sqrt{y}}\right) + e^{2\alpha}\Phi\left(\frac{-(y+\alpha)}{\sqrt{y}}\right) \tag{1.5}$$

We note that the c.d.f of *Y* contain the parameter α only.

According to this property a table of the values of *G(y)* was accomplished by Wasan and Roy(1969) for some *y* and $\alpha = \frac{\lambda}{\mu}$.

1.3 Moments and Higher Moments Properties of Inverse Gaussian Distribution [24]

Moments are set of constants used for measuring distn. properties and under certain circumstances they specify the distn. The moments of r.v. *X* (or distn.) are defined in terms of the mathematical expectation of certain power of *X* when they exist. For instance,

$\mu'_r = E(X^r)$ is called the r^{th} moment of *X* about the origin and $\mu_r = E[(X - \mu)^r]$ is called the r^{th} central moment of *X*. That is

$$\mu'_r = E(X^r) = \begin{cases} \sum_x x^r f(x), & X \text{ is discrete r.v.} \\ \int_x x^r f(x)dx, & X \text{ is continuous r.v.} \end{cases}$$

and

$$\mu_r = E\left[(X-\mu)^r\right] = \begin{cases} \sum_x (x-\mu)^r f(x), & X \text{ is discrete r.v.} \\ \int_x (x-\mu)^r f(x)dx, & X \text{ is continuous r.v.} \end{cases}$$

Provided the sum or integral converges absolutely.

The generating functions reflect certain properties of the distn., they could be used to generate moments. Sometimes they are defining some specific distn.'s, and also have a particular usefulness in connection with sums of independent, r.v.'s.

First, we shall consider a function of a real t called the moment generating function, denoted by $M(t)$, which can be used to generate moments of r.v X.

For continuous r.v X, the m.g.f is defined by

$M(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$, provided the integral converge absolutely.

To find the m.g.f of inverse Gaussian distn.:

Set $\mu = \frac{\lambda}{\alpha}$ in eq. (1.1) then we have $X \sim IG(\frac{\lambda}{\alpha}, \lambda)$, with p.d.f

$$f(x) = \sqrt{\frac{\lambda}{2\pi}}\, x^{\frac{-3}{2}}\, e^{\left[\alpha - \frac{\alpha^2 x}{2\lambda} - \frac{\lambda}{2x}\right]} \qquad (1.6)$$

So,

$$M(t) = \int_0^{\infty} e^{tx} \sqrt{\frac{\lambda}{2\pi}}\, x^{\frac{-3}{2}}\, e^{\left[\alpha - \frac{\alpha^2 x}{2\lambda} - \frac{\lambda}{2x}\right]} dx$$

$$= e^{\alpha} \int_0^{\infty} \sqrt{\frac{\lambda}{2\pi}}\, x^{\frac{-3}{2}}\, e^{\left[-\left(\frac{\alpha^2 - 2\lambda t}{2\lambda}\right)x - \frac{\lambda}{2x}\right]} dx$$

$$= e^{\alpha - (\alpha^2 - 2\lambda t)^{\frac{1}{2}}} \int_0^{\infty} \sqrt{\frac{\lambda}{2\pi}}\, x^{\frac{-3}{2}}\, e^{\left[(\alpha^2 - 2\lambda t)^{\frac{1}{2}} - \left(\frac{\alpha^2 - 2\lambda t}{2\lambda}\right)x - \frac{\lambda}{2x}\right]} dx \qquad (1.7)$$

Since the integrand of eq.(1.7)is the p.d.f of r.v $X \sim IG\left((\alpha^2 - 2\lambda t)^{1/2}, \lambda\right)$.

It follows, the integral side of eq.(1.7) is unity

Hence,

$$M(t) = e^{\alpha - (\alpha^2 - 2t)^{\frac{1}{2}}}$$

Substitutes $\alpha = \dfrac{\lambda}{\mu}$, then we have

$$M(t) = e^{\frac{\lambda}{\mu}\left[1 - \left(1 - \frac{2\mu^2 t}{\lambda}\right)^{\frac{1}{2}}\right]}, \quad t < \frac{\lambda}{2\mu^2} \tag{1.8}$$

The theory of mathematical analysis show that the existence of $M(t)$ for $t < \dfrac{\lambda}{2\mu^2}$ implies that the derivatives of $M(t)$ of all orders exist at $t = 0$.

Thus the r^{th} moment of X about the origin is

$$\mu'_r = E(X^r) = \frac{d^r M(t)}{dt^r}\Big|_{t=0}, \; r = 1, 2, 3, \ldots..$$

The following mathematical representation of the r^{th} moments about origin is given by Tweedie (1957) [43].

$$\mu'_r = \mu^r \sum_{i=0}^{r-1} \frac{(r-1+i)!}{i!(r-i-1)!\left(\frac{2\lambda}{\mu}\right)^i} \tag{1.9}$$

Now, to find the r^{th} moments about the mean we have

$$\mu_r = E\left[(x-\mu)^r\right]$$

By the binomial theorem [1] we have:

$$(x-\mu)^r = \sum_{i=0}^{r}(-1)^i \binom{r}{i} \mu^i x^{r-i}$$

So,

$$\mu_r = E\left[\sum_{i=0}^{r}(-1)^i \binom{r}{i} \mu^i x^{r-i}\right] = \sum_{i=0}^{r}(-1)^i \binom{r}{i} \mu^i E(x^{r-i})$$

$$\mu_r = \sum_{i=0}^{r}(-1)^i \binom{r}{i} \mu^i \mu'_{r-i} \qquad (1.10)$$

(i) Mean

E(X) = μ = μ′$_1$ is called the mean of r.v *X*. It is a measure of central tendency. Use of eq. (1.9) with $r = 1$, we have

$$E(X) = \mu \qquad (1.11)$$

(ii) Variance

$Var(X) = \sigma^2 = E[(X - \mu)^2]$ is called the variance of r.v *X*. It is a measure of dispersion. Use of eq.'s (1.9) and (1.10) with $r = 2$, we have

$$\sigma^2 = \frac{\mu^3}{\lambda} \qquad (1.12)$$

(iii) Coefficient of Variation

$c.v = \dfrac{\sigma}{\mu}$ is called the variational coefficient of r.v *X*. It is a measure of dispersion. For inverse Gaussian case, we have

$$c.v = \frac{\sigma}{\mu} = \sqrt{\frac{\mu}{\lambda}} \qquad (1.13)$$

(iv) Coefficient of Skewness

$\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}}$ is called the coefficient of Skewness. It is a measure of the departure of the frequency curve from symmetry. If $\gamma_1 = 0$, the curve is not skewed, $\gamma_1 > 0$, the curve is positively skewed, and $\gamma_1 < 0$, the curve is negatively skewed [24]. Use of eq.'s (1.9) and (1.10) with $r = 3$, we have

$$\mu_3 = E[(X - \mu)^3] = \frac{3\mu^5}{\lambda^2}$$

Thus,

$$\gamma_1 = \frac{\left(\frac{3\mu^5}{\lambda^2}\right)}{\left(\frac{\mu^3}{\lambda}\right)^{\frac{3}{2}}} = 3\sqrt{\frac{\mu}{\lambda}} \tag{1.14}$$

(v) Coefficient of Kurtosis

$\gamma_2 = \frac{\mu_4}{\mu_2^2} - 3$ is called the coefficient of kurtosis. It is a measure of the degree of flattening of the frequency curve. If $\gamma_2 = 0$, the curve is called mesokurtic, if $\gamma_2 > 0$, the curve is called leptokurtic, and if $\gamma_2 < 0$, the curve is called platykurtic [24].

Use of eq.'s (1.10) and (1.9) with $r = 4$, we have

$$\mu_4 = E[(X - \mu)^4] = 15\frac{\mu^7}{\lambda^3} + 3\frac{\mu^6}{\lambda^2}$$

Thus,

$$\gamma_2 = \frac{\left[15\frac{\mu^7}{\lambda^3} + 3\frac{\mu^6}{\lambda^2}\right]}{\left(\frac{\mu^3}{\lambda}\right)^2} = 3 + 15\frac{\mu}{\lambda} \tag{1.15}$$

(vi) Mode

A mode of a disn. is the value x of r.v X that maximize the p.d.f $f(x)$. For continuous distn.'s, the mode x is a solution of

$$\frac{df(x)}{dx} = 0 \quad and \quad \frac{d^2 f(x)}{dx^2} < 0.$$

A mode is a measure of location. Also we note that the mode may not exist or we may have more than one mode.

For inverse Gaussian case with p.d.f of (1.1), the natural logarithm of $f(x)$ is

$$\ln f(x) = \frac{1}{2}\ln\left(\frac{\lambda}{2\pi}\right) - \frac{3}{2}\ln(x) - \frac{\lambda(x-\mu)^2}{2\mu^2 x}$$

$$\frac{d\ln f(x)}{dx} = \frac{\lambda}{2x^2} - \frac{3}{2x} - \frac{\lambda}{2\mu^2}$$

For maximum

Set $\frac{d\ln f(x)}{dx} = 0$ implies $\lambda x^2 + 3\mu^2 x - \lambda\mu^2 = 0$

Implies $x = \mu\left[-\frac{3\mu}{2\lambda} \pm \sqrt{\frac{9\mu^2}{4\lambda^2} + 1}\right]$

We know that $x > 0$ then a mode of inverse Gaussian distn. is:

$$x = \mu\left[\sqrt{\frac{9\mu^2}{4\lambda^2} + 1} - \frac{3\mu}{2\lambda}\right] \tag{1.16}$$

(vii) Median

A median of a disn. is defined to be the value x of r.v X such that $F(x) = \Pr(X \le x) = \frac{1}{2}$. The median is measure of location.

.... For inverse Gaussian case, the c.d.f given by equation (1.4), we have

$$\frac{1}{2} = \Phi\left[\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} - 1\right)\right] + e^{\frac{2\lambda}{\mu}}\Phi\left[-\sqrt{\frac{\lambda}{x}}\left(1 + \frac{x}{\mu}\right)\right]$$

1.4 Point Estimation[31]

The point estimation concerned with inference about the unknown parameters of a distn. from a sample. It provides a single value for each unknown parameter.

The following definitions are needed for the interest of this work.

Definition (1.2) (statistic) [31]

A statistic is a function of one or more r.v.'s which does not depends on any unknown parameters.

Definition (1.3) (Point Estimator) [31]

Any statistic whose value used to estimate the unknown parameter θ for some function of θ say $\tau(\theta)$ is called point estimator.

Point estimation admits two problems:

First, developing methods of obtaining a statistic, to represent or estimate the unknown parameters in the p.d.f such statistic is called point estimator.

Second, selecting criteria and technique to define and find best estimator among many possible estimators.

1.4.1 Methods of Finding Estimators [31]

Assume that $X_1, X_2, ..., X_n$ be a r.s. of size n from a distn. whose $p.d.f$ $f(x, \underset{\sim}{\theta})$, $\underset{\sim}{\theta} = (\theta_1, \theta_2, ..., \theta_k)$ is a vector of unknown parameters. On the basis of the observed values $x_1, x_2, ..., x_n$ of r.v.'s $X_1, X_2, ..., X_n$ the object is to find statistics, say $U_i = u_i(X_1, X_2, ..., X_n)$, $i = 1, 2, ..., k$, whose values to be used as estimators for θ_i, $i = 1, 2, ...k$.

Several methods can be found in the literature such as:

Moments method, Maximum likelihood method, Bayesian method, Least square method, Minimum chi-square method, Minimum distance method and Modified moment method.

For inverse Gaussian case we shall discuss two methods theoretically namely the method of moments and the maximum likelihood method.

1.4.1.1 Estimation of parameters by Moments Method [35]

Let $X_1, X_2, ..., X_n$ be a r.s of size n from a distn. whose $p.d.f$ $f(x, \underset{\sim}{\theta})$, $\underset{\sim}{\theta} = (\theta_1, \theta_2, ..., \theta_k)$ is a vector of k unknown parameters, let $\mu'_r = E\left(X^r\right)$ be the r^{th} distn. moment about origin and $M_r = \frac{1}{n}\sum_{i=1}^{n} X_i^r$ be the r^{th} sample moment about origin. The $M.M$ can be described as follows:

Since, we have k unknown parameters, equate

μ'_r to M_r at $\underset{\sim}{\theta} = \underset{\sim}{\hat{\theta}}$ .That is $\mu'_r = M_r$ at $\underset{\sim}{\theta} = \underset{\sim}{\hat{\theta}}$, $r = 1, 2, ..., k$

For these k eq.^{,s}, we find a unique solution for $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k$ and we say that $\hat{\theta}_r (r = 1, 2, \ldots, k)$ is an estimate of θ_r obtained by $M.M$ and the corresponding statistic $\hat{\Theta}_r$ is the $M.M$ estimator of θ_r.

For inverse Gaussian distn. case, we have two unknown parameters μ and λ and if a r.s of size n is taken, then we set

$$\mu_r' = M_r \quad at \qquad \mu = \hat{\mu} \ , \quad \lambda = \hat{\lambda} \ , \quad r = 1,2$$

r=1, we have $\mu_1' = E(X) = \mu$ and $M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$, then

$$\hat{\mu} = \bar{X} \tag{1.17}$$

where $\hat{\mu}$ is the $M.M$ estimator for μ.

r=2,we have $\mu_2' = E(X^2) = \mu^2 + \frac{\mu^3}{\lambda}$ and $M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{n-1}{n} S^2 + \bar{X}^2$

Implies $\bar{X}^2 + \frac{\bar{X}^3}{\hat{\lambda}} = \frac{n-1}{n} S^2 + \bar{X}^2$

$$\hat{\lambda} = \frac{n\bar{X}^3}{(n-1)S^2} \tag{1.18}$$

where $S^2 = \frac{1}{n-1}\sum_{i=0}^{n}\left(X_i - \bar{X}\right)^2$ and $\hat{\lambda}$ is the $M.M$ estimator for λ.

Definition (1.4) (Likelihood function) [35]

The likelihood function of a r.s $X_1, X_2, \ldots, X_n$ of size n from a distn. having p.d.f $f(x, \underset{\sim}{\theta})$ (where $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ is a vector of unknown parameters) defined to be the joint p.d.f of the n r.v^{,s} $X_1, X_2, \ldots, X_n$ which is considered as a function of $\underset{\sim}{\theta}$ and denoted by $L(\underset{\sim}{\theta}, \underset{\sim}{x})$, that is

$$L(\underset{\sim}{\theta},\underset{\sim}{x}) = f(\underset{\sim}{x},\underset{\sim}{\theta}) = \prod_{i=1}^{n} f(x_i,\underset{\sim}{\theta})$$

1.4.1.2 Estimation of Parameters by Maximum Likelihood Method [35]

Let $L(\underset{\sim}{\theta},\underset{\sim}{x})$ be the likelihood function of a r.s $X_1, X_2,\ldots, X_n$ of size n from a distn. whose p.d.f $f(x,\underset{\sim}{\theta})$, $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ is a vector of unknown parameters.

Let

$$\hat{\underset{\sim}{\theta}} = \underset{\sim}{u}(\underset{\sim}{x}) = \left(u_1(\underset{\sim}{x}), u_2(\underset{\sim}{x}), \ldots, u_k(\underset{\sim}{x})\right)$$

be a vector function of the observations $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$

If $\hat{\underset{\sim}{\theta}}$ have the value of $\underset{\sim}{\theta}$ which maximizes $L(\hat{\underset{\sim}{\theta}},\underset{\sim}{x})$ then $\hat{\underset{\sim}{\theta}}$ is the m.ℓ.e of $\underset{\sim}{\theta}$ and the corresponding statistic $\hat{\underset{\sim}{\Theta}}$ is the $M.L.E$ of $\underset{\sim}{\theta}$.

We note that

(i) Many likelihood functions satisfy the condition that the m.ℓ.e is a solution of the likelihood eq.'s

$$\frac{\partial L(\underset{\sim}{\theta},\underset{\sim}{x})}{\partial \theta_r} = 0, \text{ at } \underset{\sim}{\theta} = \hat{\underset{\sim}{\theta}} \quad r=1,2,\ldots,k.$$

(ii) Since $L(\underset{\sim}{\theta},\underset{\sim}{x})$ and $\ln L(\underset{\sim}{\theta},\underset{\sim}{x})$ have their maximum at the same value of $\underset{\sim}{\theta}$ so sometimes it is easier to find the maximum of the logarithm of the likelihood.

In such case, the m.ℓ.e $\hat{\underset{\sim}{\theta}}$ of $\underset{\sim}{\theta}$ which maximizes $L(\underset{\sim}{\theta},\underset{\sim}{x})$ may be given the solution of the likelihood eq.'s

$$\frac{\partial \ln L(\underset{\sim}{\theta},\underset{\sim}{x})}{\partial \theta_r} = 0 \quad \text{at} \quad \underset{\sim}{\theta} = \hat{\underset{\sim}{\theta}} \quad ,r=1,2,\ldots,k$$

For inverse Gaussian distn. case

Let $X_1, X_2, \ldots, X_n$ be a r.s. of size n from $IG(\mu, \lambda)$ where the distn. p.d.f is given by (1.1). The likelihood function is

$$L(\mu,\lambda,\underset{\sim}{x}) = f(\underset{\sim}{x},\mu,\lambda)$$

$$= \prod_{i=1}^{n} f(x_i,\mu,\lambda)$$

$$= \prod_{i=1}^{n} \sqrt{\frac{\lambda}{2\pi}}\, x_i^{\frac{-3}{2}}\, e^{\left[\frac{-\lambda(x_i-\mu)^2}{2\mu^2 x_i}\right]}$$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \prod_{i=1}^{n} x_i^{\frac{-3}{2}}\, e^{\left[-\frac{\lambda}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}\right]}$$

$$\ln L(\mu,\lambda;\underset{\sim}{x}) = \frac{n}{2}\ln(\lambda) - \frac{n}{2}\ln(2\pi) - \frac{3}{2}\sum_{i=1}^{n}\ln(x_i) - \frac{\lambda}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}$$

$$\ln L(\mu,\lambda;\underset{\sim}{x}) = \frac{n}{2}\ln(\lambda) - \frac{n}{2}\ln(2\pi) - \frac{3}{2}\sum_{i=1}^{n}\ln(x_i) - \frac{\lambda}{2\mu^2}\sum_{i=1}^{n}x_i + \frac{n\lambda}{\mu} - \frac{\lambda}{2}\sum_{i=1}^{n}\frac{1}{x_i}$$

(1.19)

$$\frac{\partial \ln L(\mu,\lambda;\underset{\sim}{x})}{\partial \mu} = \frac{\lambda}{\mu^3}\sum_{i=1}^{n}x_i - \frac{n\lambda}{\mu^2} \qquad (1.20)$$

$$\frac{\partial \ln L(\mu,\lambda;\underset{\sim}{x})}{\partial \lambda} = \frac{n}{2\lambda} - \frac{1}{2\mu^2}\sum_{i=1}^{n}x_i + \frac{n}{\mu} - \frac{1}{2}\sum_{i=1}^{n}\frac{1}{x_i} \qquad (1.21)$$

Set $\dfrac{\partial \ln L(\mu,\lambda;\underset{\sim}{x})}{\partial \mu} = \dfrac{\partial \ln L(\mu,\lambda;\underset{\sim}{x})}{\partial \lambda} = 0$ *at* $\mu = \hat{\mu}, \lambda = \hat{\lambda}$ then

From eq. (1.20) we have

$\dfrac{\hat{\lambda}}{\hat{\mu}^3}\sum_{i=1}^{n} x_i - \dfrac{n\hat{\lambda}}{\hat{\mu}^2} = 0$ implies $n\hat{\mu} = \sum_{i=1}^{n} x_i$, then

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x} \tag{1.22}$$

From eq. (1.21) we have

$$\frac{n}{2\hat{\lambda}} - \frac{1}{2\hat{\mu}^2}\sum_{i=1}^{n} x_i + \frac{n}{\hat{\mu}} - \frac{1}{2}\sum_{i=1}^{n}\frac{1}{x_i} = 0$$

implies $\hat{\lambda} = \dfrac{n}{\left[\dfrac{\sum_{i=1}^{n} x_i}{\hat{\mu}^2} - \dfrac{2n}{\hat{\mu}} + \sum_{i=1}^{n}\dfrac{1}{x_i}\right]}$ but $\hat{\mu} = \bar{x}$

Hence

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n}\left(\frac{1}{X_i} - \frac{1}{\bar{X}}\right)} \tag{1.23}$$

$\bar{X}$ and $\dfrac{n}{\sum_{i=1}^{n}\left(\frac{1}{X_i} - \frac{1}{\bar{X}}\right)}$ are the *M.L.E* for μ and λ respectively.

1.5 Quality of Estimation

In this section, we shall introduce some definitions and theorems concern the quality of estimators which reach to the best estimators for the unknown parameters.

Definition (1.5) [35]

Let the statistic $\hat{\theta} = u(X_1, X_2, \ldots, X_n)$ be an estimator of the unknown parameter θ, then $\hat{\theta}$ is said to be

(i) Unbiased estimator if and only if $E(\hat{\theta}) = \theta$, otherwise $\hat{\theta}$ is called biased estimator for θ. The term $E(\hat{\theta}) - \theta$ is called the bias term.

(ii) Consistent estimator if $\underset{n \to \infty}{Lim}\ pr\left(\left|\hat{\theta} - \theta\right| < \varepsilon\right) = 0$.

(iii) Asymptotically unbiased if $\underset{n \to \infty}{Lim}\ E(\hat{\theta}) = \theta$.

Definition (1.6) [35]

Let the statistic $\hat{\theta} = u(X_1, X_2, \ldots, X_n)$ be an estimator of the unknown parameter θ, then $\hat{\theta}$ is said to be a minimum variance unbiased estimator (*MVUE*) for θ if:

1. $\hat{\theta}$ is an unbiased estimator for θ.
2. The variance of $\hat{\theta}$ is less than or equal to the variance of every other unbiased estimators of θ.

Definition (1.7) (Sufficient statistic) [35]

Let $X_1, X_2, \ldots, X_n$ be a r.s. of size n from a distn. whose p.d.f. $f(\underset{\sim}{x}; \underset{\sim}{\theta})$, where $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_m)$ is a vector of unknown parameters and $Y_i = u_i(X_1, X_2, \ldots, X_n)$, $i = 1, 2, \ldots, m$ be m statistics whose joint p.d.f. $g(\underset{\sim}{y}, \underset{\sim}{\theta})$. Then the m statistics are called jointly sufficient statistics for $\underset{\sim}{\theta}$ iff:

$$\frac{f(\underset{\sim}{x}; \underset{\sim}{\theta})}{g(\underset{\sim}{y}; \underset{\sim}{\theta})} = H(\underset{\sim}{x})$$

where $H(\underset{\sim}{x})$ does not depend on $\underset{\sim}{\theta}$ for all fixed values of $y_i = u_i(\underset{\sim}{x})$, $i = 1, 2, \ldots, m$.

Theorem (1.2) [34]

If $Y_i = u_i(X_1, X_2, \ldots, X_n)$, $i = 1, 2, \ldots, m$ is a set of jointly sufficient statistics, then any set of one to one functions or transformation of $Y_1, Y_2, \ldots, Y_m$ is also jointly sufficient statistics.

Theorem (1.3) (Neymann Factorization Theorem) [35]

Let $X_1, X_2, \ldots, X_n$ be a r.s. of size n from a distn. whose p.d.f. $f(x; \underset{\sim}{\theta})$, where $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_m)$ is a vector of unknown parameters. A set of statistics $y_i = u_i(\underset{\sim}{x})$, $i = 1, 2, \ldots, m$ are jointly sufficient statistics for $\underset{\sim}{\theta}$ iff, we can find two non-negative functions k_1 and k_2 such that

$$\begin{aligned} f(\underset{\sim}{x}; \underset{\sim}{\theta}) &= f(x_1, x_2, \ldots, x_n; \theta_1, \theta_2, \ldots, \theta_m) \\ &= k_1[u_1(\underset{\sim}{x}), u_2(\underset{\sim}{x}), \ldots, u_m(\underset{\sim}{x}); \theta_1, \theta_2, \ldots, \theta_m] \cdot k_2(\underset{\sim}{x}) \end{aligned}$$

where $k_2(\underset{\sim}{x})$ is free of $\underset{\sim}{\theta}$ for every values of $y_1, y_2, \ldots, y_m$ of $Y_1, Y_2, \ldots, Y_m$.

Theorem (1.4) [19]

If a sufficient statistic $Y = u(X_1, X_2, \ldots, X_n)$ for θ exist and if the M.L.E $\hat{\theta}$ of θ also exist uniquely then $\hat{\theta}$ is a function of Y .

For $IG(\mu,\lambda)$ case, we have two unknown parameters μ and λ, where we assume a r.s. $X_1, X_2, \ldots, X_n$ is a available, then the joint p.d.f. can be written as

$$f(\underset{\sim}{x}, \mu, \lambda) = \prod_{i=1}^{n} f(x, \mu, \lambda)$$

$$= \prod_{i=1}^{n} \sqrt{\frac{\lambda}{2\pi}}\, x_i^{\frac{-3}{2}}\, e^{\left[\frac{-\lambda(x_i-\mu)^2}{2\mu^2 x_i}\right]}$$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \prod_{i=1}^{n} x_i^{\frac{-3}{2}}\, e^{\left[-\frac{\lambda}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}\right]}$$

$$= \lambda^{\frac{n}{2}} \exp\left[\frac{\lambda n}{\mu} - \frac{\lambda}{2\mu^2}\sum_{i=1}^{n} x_i - \frac{\lambda}{2}\sum_{i=1}^{n} x_i^{-1}\right] . (2\pi)^{\frac{2}{n}} \prod_{i=1}^{n} x_i^{\frac{3}{2}} \tag{1.24}$$

$$= k_1\left[\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i^{-1}, \mu, \lambda\right] \cdot k_2(\underset{\sim}{x})$$

where $k_2(\underset{\sim}{x}) = (2\pi)^{\frac{2}{n}} \prod_{i=1}^{n} x_i^{\frac{3}{2}}$

Thus according to factorization theorem (1.3), the statistics $Y_1 = \sum_{i=1}^{n} X_i$ and $Y_2 = \sum_{i=1}^{n} X_i^{-1}$ are jointly sufficient statistics for μ and λ.

Definition (1.8) (Complete family) [35]

Let $X_1, X_2, X_3, \ldots, X_n$ be a r.s of size n from a distn. whose p.d.f belong to the family of p.d.f^s $\{f(x;\underset{\sim}{\theta}), \underset{\sim}{\theta} = (\theta_1, \theta_2, \theta_3, \ldots, \theta_m), \underset{\sim}{\theta} \in \Omega^m\}$, Ω^m is a parameter space, and let $u(x_1, x_2, x_3, \ldots, x_n)$ be a continuous function of $x_1, x_2, x_3, \ldots, x_n$. If $E[u(\underset{\sim}{x})] = 0$, implies $u(\underset{\sim}{x}) = 0$, then the family $\{f(x;\underset{\sim}{\theta}), \underset{\sim}{\theta} \in \Omega^m\}$ is called a complete family of p.d.f^{,s} .

Theorem (1.5) (Lehman-scheffe'-1st Theorem) [35]

Let $X_1, X_2, \ldots, X_n$ be a r.s. of size n from a distn. whose p.d.f. $f(\underset{\sim}{x};\underset{\sim}{\theta}), \theta \in \Omega$. Let $Y = u(\underset{\sim}{x})$ be a sufficient statistic for θ whose p.d.f. belong to the complete family $\{g(y;\theta), \theta \in \Omega\}$.

If $\Phi(Y)$ is a function of Y which is an unbiased estimator for θ, then $\Phi(Y)$ is a unique *MVUE* for θ.

Definition (1.9) (The Exponential Family of p.d.f.'s) [35]

Consider the family $\{f(x;\underset{\sim}{\theta}), \underset{\sim}{\theta} \in \Omega^m\}$ of p.d.f.'^{s} which can be expressed as:

$$f(x;\theta) = q(\underset{\sim}{\theta}).s(x).\exp\left[\sum_{j=1}^{m} p_j(\theta) k_j(x)\right], \ a < x < b$$

$=0$, e.w.

Such p.d.f. is said to be a member of exponential class of p.d.f.'^{s} and satisfying the following conditions:

(i) Neither a nor b depends on $\underset{\sim}{\theta}=(\theta_1,\theta_2,\ldots,\theta_m)$.

(ii) $p_j(\underset{\sim}{\theta})$ is nontrivial, functionally independent, continuous functions of θ_j , $j=1,2,\ldots,m$.

(iii) $k_j^{'}(x)\neq 0$ and $s(x)$ is continuous function of x for $a<x<b$.

Now, if a r.s. $X_1,X_2,\ldots,X_n$ is taken from a distn. whose p.d.f. $f(x;\underset{\sim}{\theta})$. Then the joint p.d.f. of the sample set $\{X_i\}$ is

$$f(\underset{\sim}{x},\underset{\sim}{\theta})=\prod_{i=1}^{n} f(x_i,\underset{\sim}{\theta})=\prod_{i=1}^{n}\left\{q(\underset{\sim}{\theta})\,.s(x_i).\exp\left[\sum_{j=1}^{m} p_j(\underset{\sim}{\theta})k_j(x_i)\right]\right\}$$

$$=\left[q(\underset{\sim}{\theta})\right]^n Exp\left[\sum_{j=1}^{m} p_j(\underset{\sim}{\theta})\sum_{i=1}^{n} k_j(x_i)\right].\prod_{i=1}^{n} s(x_i)$$

Then according to the Factorization theorem (1.3),

The statistics $Y_1=\sum_{i=1}^{n} k_1(X_i)$, $Y_2=\sum_{i=1}^{n} k_2(X_i),\ldots,Y_m=\sum_{i=1}^{n} k_m(X_i)$ are jointly sufficient statistics for the m parameters $\theta_1,\theta_2,\ldots,\theta_m$.

Theorem (1.6) (Lehman-scheffe'-2nd Theorem) [35]

Let $X_1,X_2,\ldots,X_n$ be a r.s. of size n from a distn. whose p.d.f. $f(x;\underset{\sim}{\theta})$, $\underset{\sim}{\theta}=(\theta_1,\theta_2,\ldots,\theta_m)$ belong to the exponential family and let $Y_1,Y_2,\ldots,Y_m$ be jointly sufficient statistics for $\theta_1,\theta_2,\ldots,\theta_m$, then the family of p.d.f.'s $\{g(y;\underset{\sim}{\theta}),\underset{\sim}{\theta}\in\Omega^m\}$ is complete and the statistics $Y_1,Y_2,\ldots,Y_m$ are jointly complete sufficient statistics for $\theta_1,\theta_2,\ldots,\theta_m$.

For $X \sim IG(\mu, \lambda)$ with p.d.f.

$$f(x;\mu,\lambda)=\sqrt{\frac{\lambda}{2\pi}}\; x^{\frac{-3}{2}}\, e^{\left[\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right]},\quad 0<x<\infty$$

which can be written as a member of the exponential family as

$$f(x;\mu,\lambda)=\left[\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\lambda}{\mu}}\right] x^{\frac{-3}{2}}\, e^{\frac{\lambda x}{2\mu^2}+\frac{\lambda}{2x}}$$

where $p_1(\mu,\lambda)=\dfrac{\lambda}{2\mu^2}$, $p_2(\mu,\lambda)=\dfrac{\lambda}{2}$, $k_1(x)=x$, $k_2(x)=\dfrac{1}{x}$,

$$q(\mu,\lambda)=\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\lambda}{\mu}},\; s(x)=x^{\frac{-3}{2}}$$

Now, if a sample set $\{X_i\}$ is available, then the statistic $Y_1=\sum_{i=1}^{n} k_1(x_i)=\sum_{i=1}^{n} X_i$ and $Y_2=\sum_{i=1}^{n} k_2(x_i)=\sum_{i=1}^{n} X_i^{-1}$ are jointly complete sufficient statistics for (μ,λ).

We note that according to the theorems (1.2) and (1.4)

The statistics $\bar{X}$ and $\sum_{i=1}^{n}\left(\frac{1}{X_i}-\frac{1}{\bar{X}}\right)$ are also jointly sufficient statistics for μ and λ.

Now

$$E(Y_1)=E\left(\sum_{i=1}^{n} X_i\right)=\sum_{i=1}^{n} E(X_i)=\sum_{i=1}^{n}\mu=n\mu$$

Therefore $\dfrac{Y_1}{n}=\bar{X}$ is the *MVUE* for μ.

To find an unbiased estimator for λ, we need the following theorem:

Theorem (1.7) *[42]*

Let $X_1, X_2, \ldots, X_n$ be a r.s of size n from IG(μ,λ), then the statistics $\bar{X}$ and $V = \lambda \sum_{i=1}^{n} \left(\frac{1}{X_i} - \frac{1}{\bar{X}} \right)$ are stochastically independent, and

$V \sim \chi^2(n-1)$.

Now according to theorem (1.7) the statistic V has p.d.f

$$g(v) = \frac{v^{\frac{n-1}{2}-1} e^{\frac{-v}{2}}}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)}, \; 0 < v < \infty$$

$$= 0, \; e.w$$

also the *MLE* for λ as given in eq.(1.23)

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} \left(\frac{1}{X_i} - \frac{1}{\bar{X}} \right)}$$

then

$$E\left[\hat{\lambda}\right] = E\left[\frac{n}{\sum_{i=1}^{n} \left(\frac{1}{X_i} - \frac{1}{\bar{X}} \right)} \right] = E\left(\frac{n\lambda}{V} \right) = n\lambda E\left(\frac{1}{V}\right)$$

$$E\left[\hat{\lambda}\right]=n\lambda .E\left(V^{-1}\right)=n\lambda\int_{0}^{\infty}v^{-1}\frac{v^{\frac{n-1}{2}-1}e^{\frac{-v}{2}}}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)}dv \qquad (1.25)$$

From advance calculus we have

$$\Gamma(\eta+1)=\eta\Gamma(\eta)$$

Then, $\Gamma\left(\frac{n-3}{2}+1\right)=\frac{n-3}{2}\Gamma\left(\frac{n-3}{2}\right)$

Now, eq.(1.25) becomes

$$E\left[\hat{\lambda}\right]=\frac{n\lambda}{n-3}\int_{0}^{\infty}\frac{v^{\frac{n-3}{2}-1}e^{\frac{-v}{2}}}{2^{\frac{n-3}{2}}\Gamma\left(\frac{n-3}{2}\right)}dv \qquad (1.26)$$

The integral of the last eq. is unity, then

$$E\left[\hat{\lambda}\right]=\frac{n}{(n-3)}\lambda$$

Therefore, the statistic

$$\hat{\hat{\lambda}}=\frac{n-3}{n}\hat{\lambda}=\frac{n-3}{n}\frac{n}{\sum_{i=1}^{n}\left(\frac{1}{X_i}-\frac{1}{\bar{X}}\right)}$$

$$\hat{\hat{\lambda}}=\frac{n-3}{\sum_{i=1}^{n}\left(\frac{1}{X_i}-\frac{1}{\bar{X}}\right)} \qquad (1.27)$$

is unbiased estimator for λ.

1.6 Some Related Theorems:

.. In this section we shall give three theorems explain the relationship between the inverse Gaussian distribution and the other distributions.

Theorem (1.8)[40]

Let the r.v $X \sim IG(\mu,\lambda)$ then, $Y = \dfrac{\lambda(X-\mu)^2}{2\mu^2 X} \sim \chi^2_{(1)}$

Proof:

By using the m.g.f technique, let $M_Y(t)$ be the m.g.f of the r.v Y then

$$M_Y(t) = E\left(e^{tY}\right) = E\left(e^{t\frac{\lambda(X-\mu)^2}{\mu^2 X}}\right)$$

$$= \int_0^\infty \sqrt{\frac{\lambda}{2\pi}}\, x^{\frac{-3}{2}} e^{t\frac{\lambda(x-\mu)^2}{\mu^2 x}} e^{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}}\, dx$$

$$= \int_0^\infty \sqrt{\frac{\lambda}{2\pi}}\, x^{\frac{-3}{2}} e^{\left[(2t-1)\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right]}\, dx$$

$$M_Y(t) = \frac{1}{(1-2t)^{\frac{1}{2}}} \int_0^\infty \sqrt{\frac{\lambda(1-2t)}{2\pi}}\, x^{\frac{-3}{2}} e^{\left[\frac{-(1-2t)\lambda(x-\mu)^2}{2\mu^2 x}\right]}\, dx$$

where the integral $\int_0^\infty \sqrt{\frac{\lambda(1-2t)}{2\pi}}\, x^{\frac{-3}{2}} e^{\left[\frac{-(1-2t)\lambda(x-\mu)^2}{2\mu^2 x}\right]}\, dx$ is unity

Hence

$M_Y(t) = \dfrac{1}{(1-2t)^{\frac{1}{2}}}$, which is the m.g.f of the r.v Y~ $\chi^2_{(1)}$

Theorem (1.9) [46]

If the r.v $X \sim IG(\mu,\lambda)$, then for fixed λ and μ approaches infinity the r.v $Y = \dfrac{1}{X} \sim G\left(\dfrac{1}{2},\dfrac{\lambda}{2}\right)$

Proof:

Let

$$g(x) = \lim_{\mu\to\infty} f(x) = \sqrt{\frac{\lambda}{2\pi}}\, x^{\frac{-3}{2}} e^{\lim\limits_{\mu\to\infty}\left[\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right]}$$

$$= \sqrt{\frac{\lambda}{2\pi}}\, x^{\frac{-3}{2}} e^{\lim\limits_{\mu\to\infty}\left[\frac{2\lambda(x-\mu)}{4\mu x}\right]}$$

$$= \sqrt{\frac{\lambda}{2\pi}}\, x^{\frac{-3}{2}} e^{\frac{-\lambda}{2x}}$$

Now to find the p.d.f of the r.v Y. The function $y = \dfrac{1}{x}$ define one to one transformation that maps the space $A = \{x : 0 < x < \infty\}$ onto the space $B = \{y : 0 < y < \infty\}$, with inverse $x = \dfrac{1}{y}$ and $\dfrac{dx}{dy} = \dfrac{-1}{y^2}$ then:

$$h(y) = g\left(\frac{1}{x}\right)\left|\frac{dx}{dy}\right|$$

$$= \sqrt{\frac{\lambda}{2\pi}}\, y^{\frac{-1}{2}} e^{-\frac{\lambda y}{2}}$$

$$h(y)=\frac{1}{\Gamma\left(\frac{1}{2}\right)\left(\frac{2}{\lambda}\right)^{\frac{1}{2}}}\,y^{\frac{1}{2}-1}\,e^{-\frac{y}{\left(\frac{2}{\lambda}\right)}}$$ Which is the p.d.f of the r.v $Y \sim G\left(\frac{1}{2},\frac{\lambda}{2}\right)$.

To best of our knowledge, the following theorem seems to be new:

Theorem (1.10)

If X and Y are two independent r.v's with $X \sim \exp\left(\frac{\mu}{2}\right)$ and $Y \sim G\left(\frac{3}{2},\frac{2}{\mu}\right)$. Define the transformation

$Z=\sqrt{XY}$ and $W=\sqrt{\frac{X}{Y}}$ then the conditional distn. of W given $Z=z$ is inverse Gaussian distn. $IG(\frac{\mu}{2},\mu z)$, where Z has gamma distn.

Proof:

The joint p.d.f of X and Y are

$f(x,y)=f_1(x)f_2(y)$

$$=\frac{\mu}{\Gamma\left(\frac{3}{2}\right)\sqrt{2}}\,y^{\frac{1}{2}}\,e^{-\left(\frac{2x}{\mu}+\frac{\mu y}{2}\right)}, \qquad \begin{array}{l}0<x<\infty\\ 0<y<\infty\end{array}$$

$=0,\ e.w.$

The functions $z=\sqrt{xy}$ and $w=\sqrt{\frac{x}{y}}$ define one to one transformation that maps the space $A=\{(x,y):0<x<\infty,0<y<\infty\}$ onto the space $B=\{(z,w):0<z<\infty,0<w<\infty\}$, with inverse $x=zw$ and $y=\frac{z}{w}$ with

$$J=\begin{vmatrix}\frac{dx}{dz} & \frac{dx}{dw}\\ \frac{dy}{dz} & \frac{dy}{dw}\end{vmatrix}=\begin{vmatrix}w & z\\ \frac{1}{w} & \frac{-z}{w^2}\end{vmatrix}=\frac{-2z}{w}$$

Then the joint p.d.f of Z and W are

$$g(z,w)=f\left(zw,\frac{z}{w}\right)|J|$$

$$g(z,w)=\frac{\sqrt{2\mu}}{\Gamma\left(\frac{3}{2}\right)}z^{\frac{3}{2}}w^{\frac{-3}{2}}e^{-\left(\frac{2zw}{\mu}+\frac{\mu z}{2w}\right)},\quad 0<z<\infty,\ 0<w<\infty$$

$$=0,\ e.w.$$

Now, the marginal p.d.f of Z is:

$$g_1(z)=\int_w f(z,w)\ dw$$

$$=\frac{\sqrt{2\mu}\,z^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)}\int_0^\infty w^{\frac{-3}{2}}e^{-\left(\frac{2zw}{\mu}+\frac{\mu z}{2w}\right)}\ dw$$

Set $V=\frac{W}{Z}\ \Rightarrow W=VZ\ \Rightarrow\ dw=zdv$

$$g_1(z)=\frac{\sqrt{2\mu}\; z}{\Gamma\left(\frac{3}{2}\right)}\int_0^\infty v^{\frac{-3}{2}}\, e^{-\left(\frac{2z^2 v}{\mu}+\frac{\mu}{2v}\right)}\,dv$$

$$g_1(z)=\frac{2\sqrt{\pi}\; z\; e^{-2z}}{\Gamma\left(\frac{3}{2}\right)}\int_0^\infty \sqrt{\frac{\mu}{2\pi}}\, v^{\frac{-3}{2}}\, e^{2z-\left(\frac{4z^2 v}{2\mu}+\frac{\mu}{2v}\right)}\,dv$$

where the integral $\int_0^\infty \sqrt{\frac{\mu}{2\pi}}\, v^{\frac{-3}{2}}\, e^{2z-\left(\frac{4z^2 v}{2\mu}+\frac{\mu}{2v}\right)}\,dv$ is unity, then

$$g_1(z)=\frac{2\sqrt{\pi}\; z\; e^{-2z}}{\Gamma\left(\frac{3}{2}\right)}$$

From advance calculus [1]: $\sqrt{\pi}=\Gamma\left(\frac{1}{2}\right)$ and $\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

So

$g_1(z)=4z\; e^{-2z}=\dfrac{1}{\Gamma(2)\left(\frac{1}{2}\right)^2}\, z\; e^{-2z}$ which is the p.d.f of the r.v

$Z \sim G\left(2,\frac{1}{2}\right)$.

The conditional p.d.f of W given $Z=z$ is

$$h(W\,|Z=z)=\frac{g(z,w)}{g_1(z)}$$

$$= \frac{\dfrac{\sqrt{2\mu}}{\Gamma\left(\dfrac{3}{2}\right)} z^{\frac{3}{2}} w^{\frac{-3}{2}} e^{-\left(\frac{2z\,w}{\mu} + \frac{\mu\,z}{2w}\right)}}{\dfrac{2\sqrt{\pi}\; z\; e^{-2z}}{\Gamma\left(\dfrac{3}{2}\right)}}$$

$$= \sqrt{\frac{\mu z}{2\pi}}\; w^{\frac{-3}{2}} e^{2z - \left(\frac{2z\,w}{\mu} + \frac{\mu\,z}{2w}\right)}$$

$$h(W \,|Z = z) = \sqrt{\frac{\mu z}{2\pi}}\; w^{\frac{-3}{2}} e^{\frac{-\mu z\left(w - \frac{\mu}{2}\right)^2}{\frac{\mu^2}{2} w}}$$

which is the p.d.f of the r.v $W \sim IG\left(\dfrac{\mu}{2}, \mu z\right)$

CHAPTER TWO

2

PROPERTIES AND ESTIMATION OF THE RELIABILITY AND HAZARD FUNCTIONS OF INVERSE GAUSSIAN DISTRIBUTION

2.1 Introduction

The world has witnessed since the beginning of the fifties of the last century on the appearance of a variety of electronic equipments. Those equipments and gadgets were contributing and still contributing in an increase rate to facilitating daily life. Today they are indispensable items in our day to day life. It was only natural that those developments be accompanied by the problem of determining dependability of those devices and its life time. Failure of a device causes despair on the side of users in addition of course to the incurred costs and time losses. This is of course if it was possible to restore the original functionality of the device.

Thus determining the so called "life time" of a device seem to be an urgent and sometimes the most important requirement when studying it's economic feasibility.

Most of the times determining the life-time of a device can not be determined because of the variety of causes that leads to it's failure. Some of those causes are controllable and some are not in addition due to human operation error. Thus the life time of a device becomes a random variable.

Since the forties of the last century, statisticians began to construct the theory of reliability as a sub-disspline of statistics to deal with life-time problems.

At the beginning reliability theory took the direction of using observational data to derive the specific distribution of the random variable of the life time of a specified device. The approach was to fit observational data to a known distribution. It was observed that special categories of distributions are especially suitable for reliability. Those require the random distribution to have positive values and its probability density function flexible enough to be right skewed.

The probability of survival of aging devices should be decreasingly small, thus Weibull distribution, and log-normal distribution are special cases were used in reliability theory. The inverse Gaussian distribution used in this work was not less important. Thus the study of this distribution in reliability as enunciated by Chhikara and Folks [11] is involved in this work.

This chapter involves four sections. In section (2.2) we gave some basic concepts of reliability, while in section (2.3) we gave some important properties of the reliability and hazard functions of the inverse Gaussian distn.. In section (2.4) we use the obtained estimators of parameters from chapter one together with Basu method to estimate the reliability and hazard functions of the inverse Gaussian distn.

2.2 Reliability Concept

In this section we shall give some concepts and properties of the reliability function.

Definition (2.1) (Reliability) [28]

Reliability is known as the probability that a device will perform its intended functions satisfactorily for a specified period of time under specified operating conditions.

Probability theory has been used to analyze the reliability of components as well as the reliability of systems consisting of these components.

Since the performance of a system usually depends on the performance of its components, the reliability of a system is a function of the reliability of its components. The intended function of the device is supposedly understood and the degree of success of the device's performance of the intended function can be measured so that we can easily conclude if the performance is satisfactory or not. Time is an important factor in defining reliability, for instance if a newly purchased device can perform its intended functions satisfactorily, the question arise what is the probability that it will continue to perform satisfactorily for a specified period of time,in other words, what will be the life of this device? The lifetime of the device can be treated as a r.v with a statistical probability distn.

Further, the operating conditions, such as stress, load, temperature, pressure, and/or other environmental factors, under which the device is expected to operate, must be specified.

Definition (2.2) (Reliability Function) [32]

Let X be a r.v representing the lifetime of a device. The units of measurement for the lifetime may be a time unit such as seconds, hours, days, and years or a usage unit such as miles driven and cycles of operation. If the random variable X is continuous and can take only nonnegative values. Its statistical distn. can be described by its p.d.f $f(x)$, its c.d.f $F(x)$, and/or its characteristics such as mean and variance. Given that we understand the intended functions, the operating conditions, and the satisfactory performance of the device when it is new, we need only to deal with the probability that the device can last beyond a specified period x. Thus, the reliability function of the device, denoted by $R(x)$, is given by

$$R(x) = Pr(X > x) = \int_x^{\infty} f(w)\, dw = 1 - F(x) \qquad (2.1)$$

2.2.1 Some Properties of Reliability Function [32]

Based on the definition (2.2), we can illustrate some properties of the reliability function as follows:

1- $0 \le R(x) \le 1$

2- $R(0) = 1$ and $R(\infty) = 0$.

3- The function $R(x)$ is a non-increasing function of x.

4- The function $R(x)$ is continuous from the left at each x.

Definition (2.3) (Hazard Function) [32]

The failure rate function, or the hazard function, denoted by $h(x)$, is defined to be the probability that a device will fail in the next time unit given that it has been working properly up to time x, that is,

$$h(x) = \lim_{\Delta x \to 0} \Pr(X \le x + \Delta x \mid X > x) = \frac{f(x)}{R(x)} \qquad (2.2)$$

The cumulative failure rate function, or the cumulative hazard function, denoted by $H(x)$, is defined to be

$$H(x) = \int_0^x h(w)\,dw \qquad (2.3)$$

The failure rate function is often used to indicate the health condition of a working device. A high failure rate indicates a bad health condition because the probability for the device to fail in the next instant of time is high.

2.2.2 Relationships Among h(x), R(x) and f(x) [28]

It is obvious that one of the functions $f(x)$, $F(x)$, $R(x)$, $h(x)$ is adequate to specify completely the lifetime distribution of a device. These functions are satisfied the well-known relations

1- $h(x) = \dfrac{\frac{-d}{dx}R(x)}{R(x)} = \dfrac{f(x)}{[1 - F(x)]}$

2- $R(x) = e^{-\int_0^x h(w)\,dw}$

3- $f(x) = h(x).R(x) = h(x).e^{-\int_0^x h(w)dw}$

2.2.3 Reasons for Collecting Reliability Data [28]

There are many possible reasons for collecting reliability data. Examples include the following:

- Assessing characteristics of materials over a warranty period or over the product's design life.
- Predicting product reliability.
- Predicting product warranty costs.
- Providing needed inputs for system-failure risk assessment.
- Assessing the effect of a proposed design change.
- Assessing whether customer requirements and government regulations have been met.
- Tracking the product in the field to provide information on causes of failure and methods of improving product reliability.
- Supporting programs to improve reliability through the use of laboratory experiments, including accelerated life tests.
- Comparing components from two or more different manufacturers, materials, production periods, operating environments, and so on.
- Checking the veracity of an advertising claim.

2.3 Properties of Reliability and Hazard Functions of the Inverse Gaussian Distribution

In this section we shall give some mathematical properties of the reliability and the hazard functions of the inverse Gaussian distn.

2.3.1 Reliability Function [28]

The reliability function of the inverse Gaussian distn. can be obtained in terms of the c.d.f of eq.(1.4) as follows:

$$F(x) = \Phi\left[\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}-1\right)\right] + e^{\frac{2\lambda}{\mu}}\,\Phi\left[-\sqrt{\frac{\lambda}{x}}\left(1+\frac{x}{\mu}\right)\right]$$

Then the reliability function of the inverse Gaussian distn. is

$$R(x) = 1 - F(x) = 1-\Phi\left[\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}-1\right)\right] - e^{\frac{2\lambda}{\mu}}\,\Phi\left[-\sqrt{\frac{\lambda}{x}}\left(1+\frac{x}{\mu}\right)\right]$$

$$R(x) = \Phi\left[\sqrt{\frac{\lambda}{x}}\left(1-\frac{x}{\mu}\right)\right] - e^{\frac{2\lambda}{\mu}}\,\Phi\left[-\sqrt{\frac{\lambda}{x}}\left(1+\frac{x}{\mu}\right)\right] \qquad (2.4)$$

2.3.2 Hazard Function [28]

The hazard function of the inverse Gaussian distn. can be obtained in terms of the p.d.f of eq.(1.1) and the reliability function of eq. (2.4) as follows:

$$h(x) = \frac{f(x)}{R(x)} = \frac{\sqrt{\frac{\lambda}{2\pi}}\, x^{\frac{-3}{2}}\, e^{\left[\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right]}}{\Phi\left[\sqrt{\frac{\lambda}{x}}\left(1-\frac{x}{\mu}\right)\right] - e^{\frac{2\lambda}{\mu}}\,\Phi\left[-\sqrt{\frac{\lambda}{x}}\left(1+\frac{x}{\mu}\right)\right]} \qquad (2.5)$$

The expression for *h(x)* is rather complicated but it is not difficult to compute its value for any given μ and λ. Several typical failure rate curves are given in Figure (2.1). Inspection of these curves makes it obvious that the failure rate is not monotonic for all μ and λ. However, one might be led to ask whether it is monotonic for some parameter values. We shall show that *h(x),* in general, is non-monotonic.

2.3.2.1 Properties of Hazard Function [11]

In this sub-section we shall give some properties of the hazard function as follows:

1- *h(x)* is an increasing function for $x < x_m$, where x_m is the mode of the inverse Gaussian distn. given by eq.(1.16) [11].

Because the p.d.f of the inverse Gaussian distn. is an increasing function and the reliability function is decreasing function from 0 to x_m where x_m is the mode of the inverse Gaussian function, it follow that the hazard function is an increasing function for $x < x_m$.

2 – *h(x)* is a decreasing function for $x > x_0$ where $x_0 = \dfrac{2\lambda}{3}$.

To prove this property we shall test the derivative of the natural log of *h(x)* as follows:

$$\frac{d}{dx}\left[\ln h(x)\right] = \frac{h'(x)}{h(x)} = \frac{P(x)}{R(x)}\left[\int_x^{\infty} \frac{f'(t)}{P(t)}dt + \frac{f(x)}{P(x)}\right]$$

where

$$P(x) = \frac{d}{dx}\ln\left[f(x)\right] = \frac{3}{2x} + \frac{\lambda}{2\mu^2} + \frac{\lambda}{2x^2}$$

Then, we find:

$$P'(x) = \frac{-3}{2x^2} + \frac{\lambda}{x^3}$$

Therefore;

$$P'(x) < 0, \quad x > \frac{2\lambda}{3} \qquad \text{and} \qquad P'(x) > 0, \quad x < \frac{2\lambda}{3} \tag{2.6}$$

Then, we can verify that $x_m < x_0 = \frac{2\lambda}{3}$.

3- Has a maximum value at the point x^* when x^* in the interval $[x_m, x_0]$ [11].

2.3.2.2 Asymptotic of Hazard Function [11]

Examination of the graph in figures (2.1.a) and (2.1.b) indicates there exists a nonzero asymptotic value of *h(x)* unlike the failure rate of the log normal which approaches zero asymptotically.

To find the asymptotic value we first find the upper and lower bounds for *h(x)* as follows:

The failure rate *h(x)* can be written as:

$$\frac{1}{h(x)} = \int_x^{\infty} \left(\frac{x}{t}\right)^{\frac{3}{2}} e^{\left[\frac{-\lambda}{2\mu^2}\left\{(t-x) - \mu^2 \frac{(t-x)}{tx}\right\}\right]} dt$$

Letting $z = (t-x)$ we obtain

$$\frac{1}{h(x)} = e^{\frac{\lambda}{2x}} \int_0^{\infty} \left(1 + \frac{z}{x}\right)^{-\frac{3}{2}} e^{\left[\frac{-\lambda}{2x}\left(1+\frac{z}{x}\right)^{-1} - \frac{\lambda z}{2\mu^2}\right]} dz \tag{2.7}$$

For any $x > 0$ and $z > 0$ we have:

$$e^{\frac{-\lambda}{2x}} \leq e^{\left[\frac{-\lambda\left(1+\frac{z}{x}\right)^{-1}}{2x}\right]} \leq 1 \qquad (2.8)$$

Due to the first inequality in (2.8) it follows from (2.7) that

$$\frac{1}{h(x)} > \int_{0}^{\infty}\left(1+\frac{z}{x}\right)^{\frac{-3}{2}} e^{\left(\frac{-\lambda z}{2\mu^2}\right)} dz$$

Set $y = 1+\frac{z}{x} \Rightarrow z = yx - 1 \Rightarrow dz = xdy$, then

$$\frac{1}{h(x)} > x e^{\frac{\lambda x}{2\mu^2}} \int_{1}^{\infty} y^{\frac{-3}{2}} e^{\left(\frac{-\lambda xy}{2\mu^2}\right)} dy$$

Integrated the right side by parts, on gets for an upper bound of the failure rate,

$$h(x) \leq \frac{\lambda}{2\mu^2} A^{-1}(x)$$

where

$$A(x) = 1 - \left(\frac{3}{2}\right)\int_{x}^{\infty} y^{\frac{-5}{2}} e^{\left[-\lambda x \frac{(y-1)}{2\mu^2}\right]} dy$$

On the other hand, due to the second inequality in (2.8), it follows from (2.7) that a lower bound for *h(x)* given by

$$h(x) \geq \frac{\lambda}{2\mu^2} A^{-1}(x) e^{\frac{-\lambda}{2x}}$$

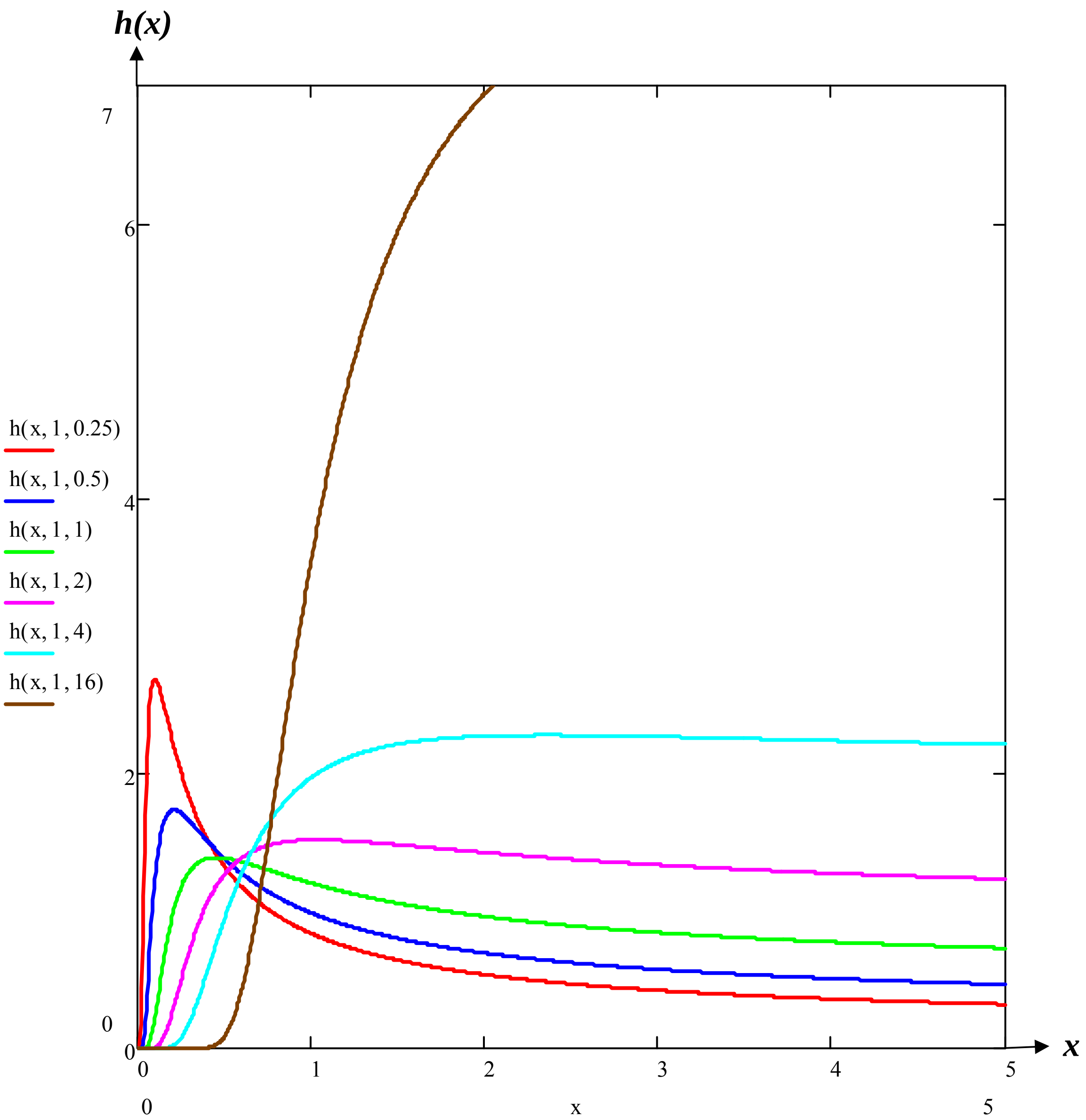
or

$$h(x) \geq \frac{\lambda}{2\mu^2} e^{\frac{-\lambda}{2x}}$$

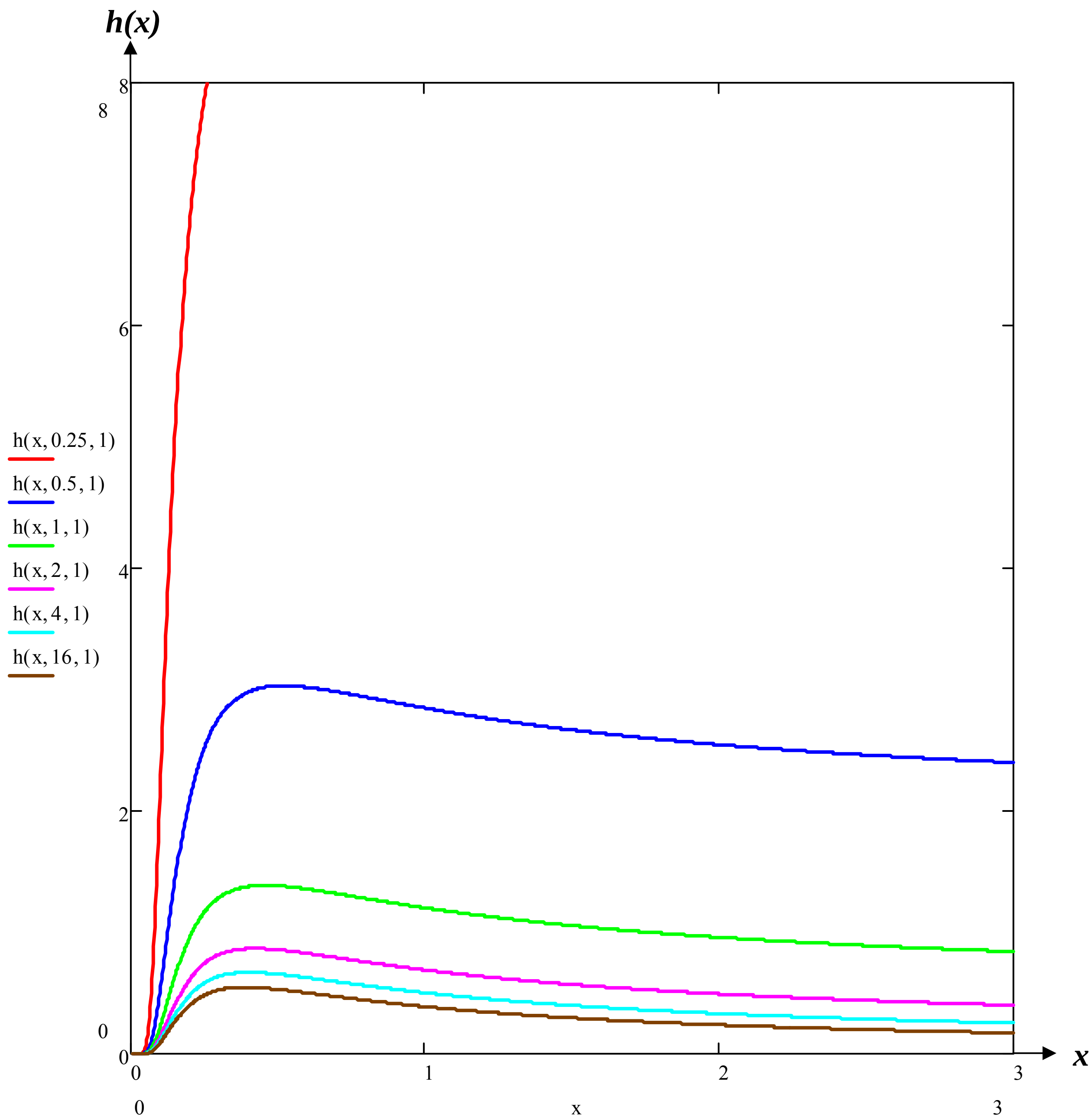
The hazard function is asymptotically equal to $\frac{\lambda}{2\mu^2}$ when x approaches to infinity.

We now have a good idea of the general behavior of *h(x),* it increases from zero to a modal value, then decreases, approaching its asymptotic value $\frac{\lambda}{2\mu^2}$.

From figures (2.1.a) and (2.1.b) one can observe that *h(x)* is virtually non-decreasing for all x when λ is large relative to μ.



Fig(2.1.a): failure rate of Inverse Gaussian distribution with μ =1 and λ = 0.25,0.5,1,2,4,16



Fig(2.1.b): failure rate of Inverse Gaussian distribution with μ =0.25,0.5,1,2,4,16 and λ = 1

2.4 Estimation of the Reliability and Hazard Functions of the Inverse Gaussian Distribution

In chapter one, we observe estimators by two different methods namely *MM* and *MLM* for the parameters of inverse Gaussian distribution.
These estimators could be used to estimate the reliability and hazard functions.

2.4.1 Estimation by Using MLM Estimators

From chapter one, the maximum likelihood estimators of μ and λ as given by (1.22) and (1.27) are

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{X}$$

and

$$\hat{\hat{\lambda}} = \frac{(n-3)}{\sum_{i=1}^{n}\left[\frac{1}{x_i} - \frac{1}{\bar{x}}\right]}$$

accordingly the *M.L.E* of *R(x)* and *h(x)* is now obtained by replacing μ and λ in (2.4) and (2.5) by their estimates $\hat{\mu}$ *and* $\hat{\hat{\lambda}}$ given in (1.22) and (1.27).

Accordingly the *M.L.E* of *R(x)* is:

$$\hat{R}_{mlm}(x) = \Phi\left[\sqrt{\frac{\hat{\hat{\lambda}}}{x}}\left(1 - \frac{x}{\hat{\mu}}\right)\right] - e^{\frac{2\hat{\hat{\lambda}}}{\hat{\mu}}}\,\Phi\left[-\sqrt{\frac{\hat{\hat{\lambda}}}{x}}\left(1 + \frac{x}{\hat{\mu}}\right)\right] \qquad (2.9)$$

and the M.L.E. of *h(x)* is:

$$\hat{h}_{mlm}(x) = \frac{\sqrt{\frac{\hat{\hat{\lambda}}}{2\pi}}\, x^{\frac{-3}{2}}\, e^{\left[\frac{-\hat{\hat{\lambda}}(x-\hat{\mu})^2}{2\hat{\mu}^2 x}\right]}}{\Phi\left[\sqrt{\frac{\hat{\hat{\lambda}}}{x}}\left(1-\frac{x}{\hat{\mu}}\right)\right] - e^{\frac{2\hat{\hat{\lambda}}}{\hat{\mu}}}\,\Phi\left[-\sqrt{\frac{\hat{\hat{\lambda}}}{x}}\left(1+\frac{x}{\hat{\mu}}\right)\right]} \tag{2.10}$$

2.4.2 Estimation by Using M.M Estimators

From chapter one, the *M.M* estimators of μ and λ as given by (1.17) and (1.18) are

$$\hat{\mu} = \overline{X}$$

and

$$\hat{\lambda} = \frac{n\overline{X}^{3}}{(n-1)S^{2}}$$

accordingly the estimators of *R(x)* and *h(x)* is now obtained by replacing μ and λ in (2.4) and (2.5) by their estimates $\hat{\mu}$ *and* $\hat{\lambda}$ given in (1.17) and (1.18).

Accordingly the estimator of *R(x)* is:

$$\hat{R}_{mm}(x) = \Phi\left[\sqrt{\frac{\hat{\lambda}}{x}}\left(1-\frac{x}{\hat{\mu}}\right)\right] - e^{\frac{2\hat{\lambda}}{\hat{\mu}}}\,\Phi\left[-\sqrt{\frac{\hat{\lambda}}{x}}\left(1+\frac{x}{\hat{\mu}}\right)\right] \tag{2.11}$$

and the estimator of *h(x)* is:

$$\hat{h}_{mm}(x) = \frac{\sqrt{\dfrac{\hat{\lambda}}{2\pi}}\, x^{\frac{-3}{2}}\, e^{\left[\dfrac{-\hat{\lambda}(x-\hat{\mu})^2}{2\hat{\mu}^2 x}\right]}}{\Phi\left[\sqrt{\dfrac{\hat{\lambda}}{x}}\left(1-\dfrac{x}{\hat{\mu}}\right)\right] - e^{\frac{2\hat{\lambda}}{\hat{\mu}}}\, \Phi\left[-\sqrt{\dfrac{\hat{\lambda}}{x}}\left(1+\dfrac{x}{\hat{\mu}}\right)\right]} \tag{2.12}$$

We note that the estimators of *R(x)* and *h(x)* given in eq.'s (2.9) , (2.10) ,(2.11) and (2.12) are biased estimators for *R(x)* and *h(x)*.
So we need to find new estimator forms to be unbiased estimators for *R(x)* and *h(x)*.

To derive these estimators we shall use Basu method [5], but before describing this method we need the following theorem.

Theorem (2.1) (Rao-Blackwell theorem) [35]

Let X and Y are two independent r.v's such that $E(Y) = \mu_y$ and $Var(Y) = \sigma_y^2$, let $E(y \mid X = x) = \phi(X)$, then

1- $E\left(\phi(X)\right) = \mu_Y$

2- $Var\left(\phi(X)\right) \le Var(Y)$

2.4.3 Basu Method [5]

Let $\underset{\sim}{X} = (X_1, X_2, X_3, ..., X_n)$ be a r.s of size *n* from a distn. with density $f(x; \underset{\sim}{\theta})$, where $\underset{\sim}{\theta} = (\theta_1, \theta_2, \theta_3, ..., \theta_d)$ is a vector of unknown parameters, $d \ge 1$. Let $\hat{\underset{\sim}{\theta}} = \hat{\underset{\sim}{\theta}}(\underset{\sim}{X})$ be a complete sufficient statistic for θ and let its density be given by $f_{\hat{\theta}}(\hat{\theta})$.

The r.s $\underset{\sim}{X} = (X_1, X_2, X_3, ..., X_n)$ may possibly be thought of as made of two independent components (ξ) and $\underset{\sim}{V} = (V_1, V_2, V_3, ..., V_{n-1})$

of sizes 1 and *(n - 1)* respectively where ξ may be any one of the $X_i^{;s}$ and $\underset{\sim}{V}$ comprises the remaining $(n-1)x_i^{;s}$, $i=(1,2,3,...,n)$. If $\underset{\sim}{\theta}^* = \underset{\sim}{\theta}^*(\underset{\sim}{V})$ is the *M.V.U.E.* of $\underset{\sim}{\theta}$ from $\underset{\sim}{V}$, we may find the joint density of ξ and $\underset{\sim}{\theta}^*$, and hence that of ξ and $\hat{\theta}$ (which is always possible for the cases under consideration) from which the conditional disn. $f\left(\xi/\underset{\sim}{\hat{\theta}}\right)$ of ξ given $\hat{\theta}$ is obtained.

Now, let $I_X(\xi)$ be a function defined as the following:

$$I_X(\xi) = \begin{cases} 0 & ,\xi \le x \\ 1 & ,\xi > x \end{cases} \tag{2.13}$$

Then,

$$E\left[I_X(\xi)\right] = \Pr(X \ge x) = R(x).$$

It means that this function is an unbiased estimator of *R(x),* hence by theorem (2.1) and theorem (1.5), the unique *M.V.U.E*, of *R(x)* is:

$$R^*(x) = E\left[I_x(\xi)/\hat{\theta}\right] = \int_x^\infty f\left(\xi/\hat{\theta}(\underset{\sim}{X})\right) d\xi$$

Now, we want to apply this method to estimate *R(x)* for the inverse Gaussian distn., by taking three different cases for the parameters of the distn. as follows:

(i) When μ is unknown and λ is known.

(ii) When μ is known and λ is unknown.

(iii) When μ is unknown and λ is unknown.

Now, we shall explain every cases individually:

Case(i) The M.V.U.E of R(x) When μ is Unknown and λ is Known.

To find the *M.V.U.E* of *R(x)* when μ is unknown and λ is known we shall use Basu method as follows:

Let $X_1, X_2, X_3, \ldots, X_n$ be a r.s of IG(μ,λ) then $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ ~IG(μ,nλ) is a complete sufficient statistic for the parameter μ, so $\bar{Y} = \frac{\sum_{i=2}^{n} X_i}{(n-1)}$ ~IG(μ,(n-1)λ) is a *M.V.U.E* for μ.

Then, the joint p.d.f of the two independent r.v's X_1 *and* $\bar{Y}$ is:

$$f(x_1, \bar{y}) = f_1(x_1).f_2(\bar{y})$$

$$f(x_1, \bar{y}) = \frac{\sqrt{n-1}\lambda}{2\pi (x_1 \bar{y})^{\frac{3}{2}}} \exp\left[\frac{-\lambda}{2\mu^2}\left\{\frac{(x_1-\mu)^2}{x_1} + \frac{(n-1)(\bar{y}-\mu)^2}{\bar{y}}\right\}\right], \quad \begin{array}{c} 0 < x_1 < \infty \\ 0 < \bar{y} < \infty \end{array}$$
$$= 0, \quad e.w$$

Now, by using the transformation $\bar{X} = \frac{(n-1)\bar{Y} + X_1}{n}$ we obtain the joint p.d.f of $X_1, \bar{X}$ as:

$$f(x_1, \bar{x}) = \frac{n(n-1)\lambda}{2\pi\left[x_1(n\bar{x} - x_1)\right]^{\frac{3}{2}}} \exp\left[\frac{-\lambda}{2\mu^2}\frac{(x_1-\mu)^2}{x_1} + \frac{\left[n(\bar{x}-\mu) - (x_1-\mu)\right]^2}{n\bar{x} - x_1}\right], 0 < x_1 < n\bar{x}$$
$$= 0, \quad e.w$$

We already know that the p.d.f of the r.v $\bar{X}$ is:

$$g(\bar{x}) = \sqrt{\frac{n\lambda}{2\pi}}\, \bar{x}^{\frac{-3}{2}} \exp\left[\frac{-n\lambda(\bar{x}-\mu)^2}{2\mu^2 \bar{x}}\right], \quad 0 < \bar{x} < \infty$$
$$= 0, \quad e.w$$

So, the conditional distn. of X_1 given $\bar{X} = \bar{x}$ is:

$$h(x_1|\bar{x}) = \frac{f(x_1,\bar{x})}{g(\bar{x})}$$

$$h(x_1|\bar{x}) = \frac{\sqrt{n\lambda}(n-1)\bar{x}^{\frac{3}{2}}}{\sqrt{2\pi}\left[x_1(n\bar{x}-x_1)\right]^{\frac{3}{2}}}\exp\left[\frac{-n\lambda(\bar{x}-x_1)^2}{2x_1\bar{x}\left(n\bar{x}-x_1\right)}\right], \quad 0<x_1<n\bar{x}$$

$$= 0, \quad e.w$$

Hence, we derive the *M.V.U.E* of *R(x)* as:

$$R^*(x) = \int_x^\infty h(x_1|\bar{x})dx_1$$

Now, with some simplification steps, we found that:

$$R^*(x) = \begin{cases} 1, & x<0 \\ 0, & x>n\bar{x} \\ \Phi(-w) - \left(\frac{n-2}{n}\right)e^{2\left[\frac{(n-1)\lambda}{n\bar{x}}\right]}\Phi(-w'), & e.w \end{cases} \tag{2.14}$$

where

$$w = \frac{\sqrt{n\lambda}(x-\bar{x})}{\left[x\bar{x}(n\bar{x}-x)\right]^{1/2}}, \quad w' = \frac{\sqrt{\lambda}\left[n\bar{x}+(n-2)x\right]}{\left[nx\bar{x}(n\bar{x}-x)\right]^{1/2}}$$

Case(ii) The M.V.U.E of R(x) When μ is Known and λ is Unknown.

To find the *M.V.U.E* of *R(x)* when μ is known and λ is unknown we shall use Basu method as follows:

Let $X_1, X_2, X_3, \ldots, X_n$ be a r.s of IG(μ,λ) then the statistic T= $\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{x_i}$ is a complete sufficient statistic for the parameter λ, and from theorem (1.8) we have $\frac{\lambda T}{\mu^2} \sim \chi^2_{(n)}$ then if $Y = \sum_{i=2}^{n} \frac{(x_i - \mu)^2}{x_i}$ implies that $\frac{\lambda Y}{\mu^2} \sim \chi^2_{(n-1)}$

Then the joint p.d.f of the two independent r.v's X_1 *and* Y is:

$f(x_1, y) = f_1(x_1) f_2(y)$

$$= \frac{\lambda^{\frac{n}{2}}}{\sqrt{\pi}\, 2^{\frac{n}{2}} \Gamma\left(\frac{n-1}{2}\right) \mu^{n-1}} x_1^{\frac{-3}{2}} y^{\frac{n-3}{2}} \exp\left[\frac{-\lambda}{2\mu^2}\left(y + \frac{(x_1 - \mu)^2}{x_1}\right)\right], \quad \begin{array}{l} 0 < x_1 < \infty \\ 0 < y < \infty \end{array}$$

$$= 0, \; e.w$$

Now, by using the transformation $T = Y + \frac{(X_1 - \mu)^2}{X_1}$ we obtain the joint p.d.f of X_1, T as:

$$f(x_1, t) = \frac{\lambda^{\frac{n}{2}}}{\sqrt{\pi}\, 2^{\frac{n}{2}} \Gamma\left(\frac{n-1}{2}\right) \mu^{n-1}} x_1^{\frac{-3}{2}} \left[t - \frac{(x_1 - \mu)^2}{x_1}\right]^{\frac{n-3}{2}} \exp\left[\frac{-\lambda t}{2\mu^2}\right], \quad 0 < \frac{(x_1 - \mu)^2}{x_1} < t$$

$$= 0, \; e.w$$

We already know that the p.d.f of the r.v T is:

$$g(t) = \frac{\lambda^{\frac{n}{2}}}{2^{\frac{n}{2}} \mu^n \Gamma\left(\frac{n}{2}\right)} t^{\frac{n-2}{2}} e^{\frac{-\lambda t}{2\mu^2}}, \quad 0 < t < \infty$$
$$= 0, \ e.w$$

So, the conditional distn. of X_1 given $T = t$ is:

$$h(x_1|t) = \frac{f(x_1,t)}{g(t)}$$

$$h(x_1|\bar{x}) = \frac{\mu}{\beta\left(\frac{1}{2},\frac{n-1}{2}\right)\left(tx_1{}^3\right)^{\frac{1}{2}}}\left[1 - \frac{(x_1-\mu)^2}{tx_1}\right]^{\frac{n-3}{2}}, \quad 0 < \frac{(x_1-\mu)^2}{x_1} < t$$
$$= 0, \ e.w$$

Consider the inequality $\frac{(x_1-\mu)^2}{x_1} < t$

Then, $\frac{1}{2}\left[2\mu + t - \left(4\mu t + t^2\right)^{\frac{1}{2}}\right] < x_1 < \frac{1}{2}\left[2\mu + t + \left(4\mu t + t^2\right)^{\frac{1}{2}}\right]$

Set,

L= $\frac{1}{2}\left[2\mu + t - \left(4\mu t + t^2\right)^{\frac{1}{2}}\right]$, U= $\frac{1}{2}\left[2\mu + t + \left(4\mu t + t^2\right)^{\frac{1}{2}}\right]$

Hence, we derive the *M.V.U.E* of *R(x)* as:

$$R^*(x) = \int_L^U h(x_1|\bar{x})\,dx_1$$

Now, with some simplification steps, we found that:

$$R^*(x)=\begin{cases}0, & x>U\\ 1, & x<L\\ F_{t,(n-1)}(-w_1)-\left(\dfrac{t+4\mu}{t}\right)^{\frac{n-2}{2}}F_{t,(n-1)}(-w_1'), & e.w\end{cases} \qquad (2.15)$$

where

$$w_1=\frac{\sqrt{n-1}(x-\mu)}{\left[tx-(x-\mu)^2\right]^{1/2}}\ ,\quad w_1'=\frac{\sqrt{n-1}\left[x+\mu\right]}{\left[tx-(x-\mu)^2\right]^{1/2}}$$

and $F_{t,(n-1)}$ denotes the c.d.f of student's *t* distribution with *(n-1)* degree of freedom.

Case(iii) The M.V.U.E of R(x) When Both μ and λ are Unknown

To find the *M.V.U.E* of *R(x)* when μ is unknown and λ is unknown we shall use Basu method as follows:

Let $X_1, X_2, X_3, ..., X_n$ be a r.s of IG(μ,λ) then the statistic T=($\bar{X}$, V) where $\bar{X}\sim$ IG(μ,nλ) $V=\sum_{i=1}^{n}\left(\frac{1}{X_i}-\frac{1}{\bar{X}}\right)$, are complete sufficient statistic for (μ,λ), and from theorem (1.7) we have $\bar{X}$ and V are stochastically independent, and $V\sim\frac{1}{\lambda}\chi^2(n-1)$. Let $\bar{Y}=\frac{\sum_{i=2}^{n}X_i}{(n-1)}$~IG (μ, (n-1) λ) and $\lambda V_1=\lambda\sum_{i=2}^{n}\left(\frac{1}{X_i}-\frac{1}{\bar{X}}\right)\sim\chi^2_{(n-2)}$, Then the joint p.d.f of the three independent r.v's $X_1, \bar{Y}$ and V is:

$$f(x_1,\bar{y},v_1)=\frac{\sqrt{n-1}\,\lambda^{\frac{n}{2}}v_1^{\frac{n-4}{2}}}{\pi\Gamma\left(\frac{n-2}{2}\right)\sqrt{2^n x_1^3\bar{y}^3}}\mathrm{e}^{\left[-\frac{\lambda}{2\mu^2}\left\{\frac{(x_1-\mu)^2}{x_1}+\frac{(n-1)(\bar{y}-\mu)^2}{\bar{y}}\right\}-\frac{\lambda v_1}{2}\right]},\quad \begin{array}{l}0<x_1<\infty\\ 0<\bar{y}<\infty\\ 0<v_1<\infty\end{array}$$

$$=0,\quad e.w$$

Now, by using the transformations

$$X_1=X_1\ ,\quad \bar{X}=\frac{(n-1)\bar{Y}+X_1}{n}\ ,\quad V=V_1+\frac{(n-1)}{\bar{Y}}+\frac{1}{X_1}-\frac{n}{\bar{X}}$$

we obtain the joint p.d.f of $X_1,\bar{X}$ *and* V as:

$$f(x_1,\bar{x},v)=\frac{n(n-1)\lambda^{\frac{n}{2}}}{\pi\Gamma\left(\frac{n-2}{2}\right)\sqrt{2^n x_1^3(n\bar{x}-x_1)^3}}\left[v-\frac{n(x_1-\bar{x})}{x_1\bar{x}(n\bar{x}-x_1)}\right]^{\frac{n-4}{2}}$$

$$\mathrm{e}^{\left[-\frac{\lambda}{2\mu^2}\left\{\frac{(x_1-\mu)^2}{x_1}+\frac{[n(\bar{x}-\mu)-(x_1-\mu)]^2}{n\bar{x}-x_1}\right\}-\frac{\lambda}{2}\left(v-\frac{n(x_1-\bar{x})^2}{x_1\bar{x}(n\bar{x}-x_1)}\right)\right]},\quad \begin{array}{l}0<x_1<n\bar{x}\\ 0<\frac{n(x_1-\bar{x})^2}{x_1\bar{x}(n\bar{x}-x_1)}<v\end{array}$$

$$=0,\quad e.w$$

The joint p.d.f of $\bar{X}$ *and* V is:

$$g(\bar{x},v)=\frac{1}{\Gamma\left(\frac{n-1}{2}\right)}\left[\frac{n\lambda^n v^{n-3}}{\pi 2^n\bar{x}^3}\right]^{\frac{1}{2}}e^{\left(-\frac{n\lambda(\bar{x}-\mu)^2}{2\mu^2\bar{x}}-\frac{\lambda v}{2}\right)},\quad \begin{array}{l}\bar{x}>0\\ v>0\end{array}$$

$$=0,\ e.w$$

So, the conditional distn. of X_1 given $T=(\bar{x},v)$ is:

$$h(x_1|t)=\frac{f(x_1,t)}{g(t)}$$

$$h(x_1|\bar{x},v)=\frac{\sqrt{n}(n-1)}{\beta\left(\frac{1}{2},\frac{n-2}{2}\right)}\left[\frac{\bar{x}^{-3}}{vx_1^{\,3}(n\bar{x}-x_1)^3}\right]^{\frac{1}{2}}\left[1-\frac{n(x_1-\bar{x})^2}{vx_1\bar{x}(n\bar{x}-x_1)}\right]^{\frac{n-4}{2}},\quad L_1<x_1<U_1$$

$$=0,\ e.w$$

where

$$L_1=\frac{\bar{x}}{2[n+v\bar{x}]}\left\{n(2+v\bar{x})-\left[4n(n-1)v\bar{x}+\left(nv\bar{x}\right)^2\right]^{\frac{1}{2}}\right\}$$

$$U_1=\frac{\bar{x}}{2[n+v\bar{x}]}\left\{n(2+v\bar{x})+\left[4n(n-1)v\bar{x}+\left(nv\bar{x}\right)^2\right]^{\frac{1}{2}}\right\}$$

Hence, we derive the M.V.U.E of *R(x)* as:

$$R^*(x)=\int_x^{U_1}h(x_1|\bar{x},v)\,dx_1$$

Now, with some simplification steps, we found that:

$$R^*(x)=\begin{cases}0, & x>U\\ 1, & x<L\\ F_{t,(n-2)}(-w_2)-\frac{n-2}{2}\left(1+\frac{4(n-1)}{nv\bar{x}}\right)^{\frac{n-3}{2}}F_{t,(n-2)}(-w_2'), & e.w\end{cases}$$

(2.16)

where

$$w_2 = \frac{\left(n\left(n-2\right)\right)^{\frac{1}{2}}\left(x-\bar{x}\right)}{\left[vx\bar{x}\left(n\bar{x}-x\right)-n\left(x-\bar{x}\right)^2\right]^{\frac{1}{2}}} \quad , \quad w_2' = \frac{\left(n\left(n-2\right)\right)^{\frac{1}{2}}\left(\bar{x}+(n-2)\frac{x}{n}\right)}{\left[vx\bar{x}\left(n\bar{x}-x\right)-n\left(x-\bar{x}\right)^2\right]^{\frac{1}{2}}}$$

CHAPTER THREE

3

MONTE CARLO APPLICATIONS

3.1 Introduction

After constructing a mathematical model for the problem under consideration, the next step is to derive a solution. There are analytic and numerical solution methods. The analytic solution is usually obtained directly from its mathematical representation in the form of formula, while the numerical solution is generally an approximate solution obtained as a result of substitution of numerical values for the variables and parameters of the model. Many numerical methods are iterative, that is, each successive step in the solution uses the result from the previous step such as Newton's method for approximating the root of non-linear eq.. Two special types of numerical methods simulation and the Monte Carlo are designed for a solution of deterministic and stochastic problem [21].

Simulation in a wide sense is defined as a numerical technique for conducting experiments on a digital computer which involve certain types of mathematical and logical models that describe the behavior of system over extended periods of real time, for example, simulating a football game, supersonic jet flight, a telephone communication system, wind tunnel [16], a large scale military battle (to evaluate defensive or offensive weapon system), or a mainterinance operation (to determine the optimal size of repair crews) and a live applications of real equipment in mock combat scenarios or firing range, these allow pilots, tank derivers and others soldiers to practice the physical activates of a war with their real equipment, etc.

Whereas simulation in a narrow sense (also called stochastic simulation) is defined as experimenting with the model over time, it includes sampling stochastic variates from probability distn.. Often simulation is viewed as a "Method of Last Resort" to be used when every things else has failed. Software building and technical development have made simulation one of the most widely used and accepted tools for designers in the system analysis and operation research.

This chapter involves six sections. In section (3.2) we gave a historical review on Monte-Carlo simulation. In section (3.3) we discussed the random numbers generation. In section (3.4) introduced two methods to generate random variates from continuous probability distn., namely Acceptance-Rejection method and transformation with multiple root method, in section (3.5) these two methods applied together with Box and Muller method on three procedures for generating random variates from inverse Gaussian distn.. In section (3.6), the simulated inverse Gaussian samples are observed in section (3.5) used to estimate the distribution parameters, reliability function, and hazard function by three methods given in section (2.4) of chapter two.

3.2 Monte Carlo Simulation

The Monte Carlo method provides approximate solutions to a variety of mathematical problems by performing statistical sampling experiments on a digital computer. The method applies to problems with no probabilistic content as well as to those with inherent probabilistic structure. Among all numerical methods that rely on N-point evaluations in M-dimensional space to produce an approximate solution, the Monte Carlo method has absolute error of estimate that decreases as $N^{-1/2}$ whereas, in the absence of exploitable special structure all others have errors that decrease as $N^{-1/M}$ at best.[21]

The method is called after the city in the Monaco principality, because of a roulette, a simple random number generator. The name and the systematic development of Monte Carlo methods dates from about 1944.[30]

There are however a number of isolated and undeveloped instances on much earlier occasions. For example, in the second half of the nineteenth century a number of people performed experiments, in which they threw a needle in a haphazard manner onto a board ruled with parallel straight lines and inferred the value of $\pi = 3.14\ldots$ from observations of the number of intersections between needle and lines. An account of this playful diversion (indulged in by certain Captain Fox, amongst others, whilst recovering from wounds incurred in the American Civil War) occurs in a paper Hall (A. HALL 1873. " On an experimental determination of π").[16]

In 1899 Lord Rayleigh use simulation and show that a one-dimensional random walk without absorbing barriers could provide an approximate solution to a parabolic differential equation [37]. In early part of the twentieth century, British statistical schools indulged in a fair amount of unsophisticated Monte Carlo work. Most of this seems to have been of didactic character and rarely used for research or discovery. Only on a few rare occasions was the emphasis on original discovery rather than comforting verification. In 1908 Student (W.S. Gosset) used experimental sampling to help him towards his discovery of the t-distribution and the estimate of its correlation coefficient [16]. Kolmogorov (1931) showed the relationship between Markov stochastic processes and certain integro-differential equations [30]. The real use of Monte Carlo methods as a research tool stems from work on the atomic bomb during the second world war, this work involved a direct simulation of the probabilistic problems concerned with random neutron diffusion in fissile material; but even at an early stage of these investigations, Von Neumann and Ulam refined this particular " Russian roulette" and "splitting" methods. About 1948 Fermi, Metropolis, and Ulam obtained Monte Carlo estimates for the eigenvalues of Schrodinger equation[33]. Shortly thereafter Monte Carlo methods used to evaluate complex multidimensional integrals, stochastic problems, and deterministic problems if they have the same formal expressions as some stochastic process. Also Monte Carlo method is used for solution of certain integrals and differential equations, sampling of random variates from probability distn.'s, and for analyzing complex problem (such as radiation transport to rivers).

Useful references related to Monte Carlo simulation by James E. Gentle(2003) [21] and George S. Fishman (1996) [16].

3.3 Random Number Generation

Many techniques for generating random numbers on digital computer by Monte Carlo method and simulation have been suggested, and used in recent years. Some of these methods are based on random phenomena, others on deterministic recurrence procedures.

Initially manual methods were used to generate, a sequence of numbers such as coin flipping, dice rolling, card shuffling, and roulette wheels, but these methods were too slow for general use and moreover the generated sequence of such methods could not reproduced .

With the computer aid it become possible to obtained random numbers. In (1951) Von Neumann suggested the mid-square method using the arithmetic operations of a computer [44]. His idea was to take the square of the preceding random number and extract the middle digits. For instance, suppose we wish to generate 4-digits numbers.

1 – Choose any 4-digits to generate 4-digits numbers, say 3201.

2 – Square it, to have 10246401.

3 – The next 4-digits numbers is the middle 4-digit in step (2), that is 2464.

4 – Repeat the process.

This method proved slow and awkward for statistical analysis, furthermore the sequence tend to cyclicity, and once a zero is encountered the sequence terminates.

One method of generating random numbers on digital computer was published by RAND Corporation (1955), consist of preparing a table of million random digits stored in the computer memory [36]. The advantage of this method is reproducibility and its disadvantage, was it's slow and the risk of exhausting the table.

We say that, the random numbers generated by any method is a "good" one if the random numbers are uniformly distributed, statistically independent and reproducible, more over the method is necessarily fast and requires minimum capacity in the computer memory.

The Congruential methods for generating pseudorandom numbers are designed specifically to satisfy as many of these requirements as possible.

These methods produce a nonrandom sequence of numbers according to some recursive formula based on calculating the residues module of some integer m of a linear transformation. The Congruential methods are based on a fundamental congruence relationship, which may be formulated as:

$$X_{i+1} = (aX_i + c)(\mathrm{mod}\, m),\ i = 1,2,...,m\ . \tag{3.1}$$

where *a* is the multiplier, *c* is the increment , and *m* is the modulus (*a, c, m* are non-negative integers), (mod *m*) mean that eq.(3.1) can be written as:

$$X_{i+1} = aX_i + c - m[z] \tag{3.2}$$

where $[z] = \left[\dfrac{aX_i + c}{m}\right]$ is the greater integer in *z*.

Given an initial starting value X_1 with fixed values of *a*, *c* and *m*, then eq. (3.2) yields congruence relationship (modulo *m*) for any values *i* of the sequence $\{X_i\}$. The seq. $\{X_i\}$ will repeat itself in at most m steps and will be therefore periodic.

For example:

Let $a = c = X_1 = 4$, and $m = 9$,then the sequence obtained from the recursive formula

$X_{i+1} = (4X_i + 4)(\text{mod } 9)$ is $X_i = 4,2,3,7,5,6,10,4...,\ i = 1,2,3,...$.

The random number on the unit interval [0,1] can be obtained by:

$$U_i = \frac{X_i}{m},\ i = 1,2,...,m \tag{3.3}$$

It follows from eq.(3.3) that $X_i \le m$, $\forall i$, this inequality mean that the period of the generator cannot exceed m, that is, the sequence $\{X_i\}$ contains at most *m* distinct numbers. So we should choose *m* as large as possible to ensure, a sufficiently large sequence of distinct numbers in the cycle.

It is noted in the literature, [30] that good statistical result can be achieved from computers by choosing $a = 2^{7+1}$, $c = 1$, and $m = 2^{35}$.

3.4 Random Variates Generation From Continuous Distribution

Many methods and procedures are proposed in the literature for generating random variates from different distn. We shall utilize two methods namely, Acceptance-Rejection method and Transformation with multiple root method.

3.4.1 Acceptance-Rejection Method

This method is due to Von Numann [44]. This method can be applied to generate variable from an appropriate distn. and subjecting it to a test to determine whether or not it will be acceptable for use.

To carry out the method, we represent the p.d.f. $f(x)$ of the generated r.v X as $f(x)=c\,h(x)g(x)$ where $c\geq 1$, *h(x)* is also a p.d.f. and $0\leq g(x)\leq 1$. Then we generate two r.v's U and Y from $U(0,1)$ and *h(y),* respectively, and test to see whether or not the inequality $U\leq g(Y)$ holds.

1- If the inequality holds, then accept *Y=X* as a variate generated from $f(x)$.

2- If the inequality is violated, reject the pair *U, Y* and try again.

Theorem (3.1) [44]:

Let X be a random variable distributed with the p.d.f. $f(x)$, $x\in I$, which is represented as $f(x)=c\,h(x)g(x)$ where $c\geq 1$, *h(x)* is also a p.d.f. and $0\leq g(x)\leq 1$.

Let U and Y be a distributed $U(0,1)$ and *h(y),* respectively, then $\Pr\left[Y=x\,|U\leq g(Y)\right]=f(x)$.

Proof:

$$\Pr\left[Y=x\,|U\leq g(Y)\right]=\frac{\Pr\left[Y=x,U\leq g(Y)\right]}{\Pr\left[U\leq g(Y)\right]}$$

$$=\frac{\Pr\left[Y=x\,|U\leq g(Y)\right]}{\int\limits_{x}\Pr\left[Y=x,U\leq g(Y)\right]dx}$$

Using Bayes theorem, we have:

$$\Pr\left[Y=x\,|U\leq g(Y)\right]=\frac{\Pr\left(U\leq g(Y)|Y=x\right)\Pr\left(Y=x\right)}{\int\limits_{x}\Pr\left[U\leq g(Y),Y=x\right]\Pr\left(Y=x\right)dx}$$

Since,

$\Pr\left[U\leq g(Y)|Y=x\right]=\Pr\left[U\leq g(x)\right]=g(x)$ and $\Pr(Y=x)=h(x)$

Therefore ;

$$\Pr\left[Y = x \mid U \le g(Y)\right] = \frac{g(x)h(x)}{\int_x g(x)h(x)dx} = \frac{g(x)h(x)}{\int_x \frac{f(x)}{c}dx} = \frac{g(x)h(x)}{\frac{1}{c}}$$

$$= c\,h(x)g(x).$$

The efficiency of Acceptance-Rejection Method is to determined by the inequality $U \le g(Y)$ where efficiency= $\Pr\left[U \le g(Y)\right] = \frac{1}{c} = p$.

Because the trails are independent, the probability of success in each trials is $p = \frac{1}{c}$. The number of trials *N* before a successful pair *(U,Y)* has geometric distn. with p.d.f.

$$\Pr(N = n) = p(1-p)^{n-1},\ n = 1,2,3,...$$
$$= 0 \qquad ,e.w.$$

With the expected number of trails $E(N) = \frac{1}{p} = c$.

The AR-Algorithm describes the necessary steps of generating a random variable by Acceptance-Rejection Method.

AR-Algorithm:

1- Generate *U* from $U(0,1)$.

2- Generate *Y* from *h(y)*.

3- If $U \le g(Y)$, deliver (we accept) *Y=X* as a random variable generated from the p.d.f. $f(x)$. Go to step (5).

4- Else go to step (1).

5- Stop.

We note that, for the acceptance-rejection method to be practiced interest, the following criteria must be used.

a- It should be easy to generate from *h(x)*.

b- The efficiency (probability) of the procedure $\frac{1}{c}$ should be large, that is, c closed to one.

3.4.2 Transformation with multiple root method [23]

Consider the problem of generation of a random vector X where

$$g(X) = V \tag{3.4}$$

and a value of x is sought for each value of v that is generated.

When a single-valued inverse does not exist, more than one value of x satisfies (3.4).

For a specific roots of (3.4) denoted $x_1, x_2, x_3, \ldots, x_k$. The problem is how to determine the multinomial probabilities for choosing each of the k roots.

If X and V are discrete random variables then probability can be associated with each of k roots. The conditional probability with which the i^{th} root should be chosen, $p_i(v_0)$, is easily seen to be

$$\begin{aligned} p_i(v_0) &= p(X = x_i | V = v_0) \\ &= \frac{p(X = x_i, V = v_0)}{p(V = v_0)} = \frac{p(X = x_i)}{\sum_{j=i}^{k} p(X = x_j)} \end{aligned} \tag{3.5}$$

For the continuous case, a similar expression will be developed for an interval about v_0, Then the limit will be taken as the interval shrinks to the point v_0, the result is not generally a simple ratio of the likelihood of the i^{th} root to the sum of the likelihoods of the k roots.

Suppose X and V are absolutely continuous random variables. Let $f(x)$ and $F(x)$ denote the p.d.f and c.d.f of X, respectively. Let g be such that the first derivative of g, g', exists, are continuous, and nonzero, except on a closed set of values for X with probability zero. Consider the interval $(v_0 - h, v_0 + h)$, where $h > 0$. According to the inverse function theorem, for h sufficiently small, the inverse image of $(v_0 - h, v_0 + h)$ is comprised of k disjoint intervals about the k distinct roots. Let the interval containing the i^{th} root, x_i, be denoted (y_{i1}, y_{i2}). If $p(v_0)$ is the probability with which an observation should be chosen from the i^{th} interval given that V is in the interval $(v_0 - h, v_0 + h)$, then, similar to (3.5)

$$p_i^h(v_0) = \frac{P(y_{i1} < X < y_{i2})}{\sum_{j=1}^{k} P(y_{i1} < X < y_{i2})} = \frac{F(y_{i1}) - F(y_{i2})}{\sum_{j=1}^{k}\left[F(y_{i1}) - F(y_{i2})\right]}$$

Since selection is to be made among the k points $x_1, x_2, x_3, \ldots, x_k$ (having observed the point v_0), and these points are the limits $\lim_{h \to 0}\left[(y_{j1}, y_{j2})\right] = x_j$ and $\lim_{h \to 0}\left[(v_0 - h, v_0 + h)\right] = v_0$ then $p_i(v_0) = \lim_{h \to 0}\left[p_i^h(v_0)\right]$ will yield the conditional probability with which the i^{th} root should be selected. Hence

$$p_i(v_0) = \lim_{h \to 0}\left[p_i^h(v_0)\right]$$

$$= \left\{1 + \sum_{j=1, j \neq i}^{k} \lim_{h \to 0}\left[\frac{F(y_{j2}) - F(y_{j1})}{F(y_{i2}) - F(y_{i1})}\right]\right\}^{-1}$$

$$= \left\{1 + \sum_{j=1, j \neq i}^{k} \lim_{h \to 0}\left[\frac{(y_{j2} - y_{j1})/h}{(y_{i2} - y_{i1})/h} \cdot \frac{\left[F(y_{j2}) - F(y_{j1})/(y_{j2} - y_{j1})\right]}{\left[F(y_{i2}) - F(y_{i1})\right]/(y_{i2} - y_{i1})}\right]\right\}^{-1}$$

$$p_i(v_0) = \left\{ 1 + \sum_{j=1, j \neq i}^{k} \left| \frac{g'(x_i)}{g'(x_j)} \right| . \frac{f(x_j)}{f(x_i)} \right\}^{-1} \tag{3.6}$$

To generate random variates from inverse Gaussian distribution we shall use the transformation with multiple root method in follows:

Let the r.v $X \sim IG(\mu, \lambda)$ then from theorem (1.5.1) we have:

$$V = g(x) = \frac{\lambda (X - \mu)^2}{2\mu^2 X} \sim \chi^2_{(1)} \tag{3.7}$$

Observations from $\chi^2_{(1)}$ are easily generated as the squares of standard normal variates. For each chi-square variate, v_0, we must solve (3.7) for *x* to obtain a corresponding observation from the inverse Gaussian distribution. For any v_0>0 there are exactly two roots of the associated quadratic eq. which can always be expressed as

$$x_1 = \mu + \frac{\mu^2 v_0}{2\lambda} - \frac{\mu}{2\lambda}\sqrt{4\mu\lambda v_0 + \mu^2 v_0^{\,2}} \quad \text{and} \quad x_2 = \frac{\mu^2}{x_1}$$

Since the relationship which exists between the roots of any quadratic equation implies here that $x_1 x_2 = \mu^2$.

The difficulty in generating observations with the inverse Gaussian distribution now lies in choosing between the two roots. From (3.5) we have, x_1 should be chosen with the probability

$$p_1(v_0) = 1 - p_2(v_0) = \left\{ 1 + \left| \frac{g'(x_1)}{g'(x_2)} \right| . \frac{f(x_2)}{f(x_1)} \right\}^{-1}$$

By using eq.(1.1) we have

$$\frac{f(x_2)}{f(x_1)} = \left(\frac{x_1}{\mu} \right)^3 \quad \text{and} \quad \frac{g'(x_1)}{g'(x_2)} = -\left(\frac{\mu}{x_1} \right)^2 \Rightarrow \left| \frac{g'(x_1)}{g'(x_2)} \right| = \left(\frac{\mu}{x_1} \right)^2$$

Hence the smaller root, x_1, should be chosen with the probability

$$p_1(v_0) = \frac{\mu}{\mu + x_1} \tag{3.8}$$

So for each random observation from $\chi^2_{(1)}$, v_0 , the smaller root calculated. An auxiliary Bernoulli trail is then preformed with $p_1(v_0) = \frac{\mu}{\mu + x_1}$. If the trail results in a " success", x_1 is chosen; otherwise the larger root $x_2 = \frac{\mu^2}{x_1}$, is chosen. So the algorithm of generating random variates from inverse Gaussian distn. by using transformation with multiple roots can be illustrated as follows:

Multiple roots algorithm:

1- Read μ, λ.

2- Generate X from N(0,1).

3-Set $Z=X^2$.

4- Set $V_1 = \mu + \frac{\mu^2 Z}{2\lambda} - \frac{\mu}{2\lambda}\sqrt{4\mu\lambda Z + \mu^2 Z^2}$.

5- Generate U_3 from $U\left(0,1\right)$.

6- If $U_3 < \frac{\mu}{\mu + V_1}$ Deliver $V=V_1$ as a r.v. generated from IG(μ,λ).

7- Else deliver $V = \frac{\mu^2}{V_1}$ as a r.v. generated from IG(μ,λ).

8- Stop.

The problem now is how to generate a r.v. from N(0,1). So we shall use three different procedures for generating r.v.'s from N(0,1) as follows:

3.5 Procedures for Generating Random Variates for inverse Gaussian Distribution

In this section we shall consider three different procedures to generate random variates from the inverse Gaussian distn.

3.5.1 Procedure (IG-1):

The procedure is based on Acceptance-Rejection method to generate standard normal variates and the transformation with multiple root method to generate inverse Gaussian variate as follows:

The p.d.f. of r.v. $X \sim N\left(0,1\right)$ is

$f\left(x\right)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, -\infty < x < \infty$ where we make use the inequality

$e^{\frac{-x^2}{2}} \le \frac{2}{1+x^2}$.

To apply the Acceptance-Rejection method, we need to write the p.d.f. as $f\left(x\right)=c\,h\left(x\right)g\left(x\right)$ as shown in section (3.4.1).

Now, we consider the inequality

$e^{\frac{-x^2}{2}} \le \frac{2}{1+x^2} \Rightarrow \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}} \le \frac{1}{\sqrt{2\pi}}\frac{2}{1+x^2}$, then

$$f\left(x\right)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \le \frac{1}{\sqrt{2\pi}}\frac{2}{1+x^2}=\varphi\left(x\right) .$$

$$c\,h\left(x\right)=\varphi\left(x\right) \Rightarrow c=\int_{-\infty}^{\infty}\varphi\left(x\right)dx=\int_{-\infty}^{\infty}\frac{2}{\sqrt{2\pi}\left(1+x^2\right)}dx \Rightarrow c=\sqrt{2\pi} .$$

$$g(x)=\frac{f(x)}{\varphi(x)}=\frac{1}{2}\left(1+x^2\right)e^{-\frac{x^2}{2}}$$, where $0\le g(x)\le 1$.

Set $u_2=H(y)\Rightarrow u_2=\frac{1}{2}+\tan^{-1}(y)$ implies $y=\tan\left[\pi\left(u_2-\frac{1}{2}\right)\right]$.

The efficiency (probability) of the method is equal to $\frac{1}{c}=\frac{1}{\sqrt{2\pi}}\approx 0.40$

And the number of trials equal to $c=\sqrt{2\pi}\approx 2.51$

<u>Algorithm (IG-1)</u>

1- Read μ, λ.

2- Generate U_1 and U_2 from $U(0,1)$.

3- Set $Y=\tan\left[\pi\left(u_2-\frac{1}{2}\right)\right]$.

5- If $U_1>g(Y)$ go to step (2).

6- Else set $X=Y$ as a r.v. generated from $N(0,1)$.

7- Set $Z=X^2$.

8- Set $V_1=\mu+\frac{\mu^2 Z}{2\lambda}-\frac{\mu}{2\lambda}\sqrt{4\mu\lambda Z+\mu^2 Z^2}$.

9- Generate U_3 from $U(0,1)$.

10- If $U_3<\frac{\mu}{\mu+V_1}$ deliver $V=V_1$ as a r.v. generated from IG(μ,λ).

11- Else deliver $V=\frac{\mu^2}{V_1}$ as a r.v. generated from IG(μ,λ)

12- Stop.

3.5.2 Procedure (IG-2):

This Procedure due to Box and Muller (1958) [6], where the inverse Gaussian variates is generated by utilizing the standard normal distn.

If U_1 and U_2 is a r.s. of size 2 from $U\left(0,1\right)$, then the r.v.'s

$X_1=\left(-2\ln U_1\right)^{\frac{1}{2}}\cos\left(2\pi U_2\right), X_2=\left(-2\ln U_1\right)^{\frac{1}{2}}\sin\left(2\pi U_2\right)$ represent a r.s. of size 2 from $N\left(0,1\right)$.

Since, the joint distn. of U_1 and U_2 are:

$$g\left(u_1,u_2\right)=1, 0<u_i<1, i=1,2.$$
$$=0, e.w.$$

The function $X_1=\left(-2\ln U_1\right)^{\frac{1}{2}}\cos\left(2\pi U_2\right), X_2=\left(-2\ln U_1\right)^{\frac{1}{2}}\sin\left(2\pi U_2\right)$ is defined $\left(1-1\right)$ transformation that maps $A=\left\{\left(u_1,u_2\right):o<x<1, i=1,2\right\}$ on to the space $B=\left\{\left(x_1,x_2\right):-\infty<x_i<\infty, i=1,2\right\}$ with inverse transforms

$$x_1^2+x_2^2=\left(-2\ln u_1\right)\cos^2\left(2\pi u_2\right)+\left(-2\ln u_1\right)\sin^2\left(2\pi u_2\right)$$
$$=-2\ln u_1\left[\cos^2\left(2\pi u_2\right)+\sin^2\left(2\pi u_2\right)\right]$$
$$x_1^2+x_2^2=-2\ln u_1 \Rightarrow u_2=\frac{1}{2\pi}\tan^{-1}\left(\frac{x_1}{x_2}\right)$$

with Jacobin transformation

$$J=\frac{\partial\left(u_1,u_2\right)}{\partial\left(u_1,u_2\right)}=\begin{vmatrix} -x_1e^{-\frac{1}{2}\left(x_1^2+x_2^2\right)} & -x_2e^{-\frac{1}{2}\left(x_1^2+x_2^2\right)} \\ \frac{1}{2\pi}\frac{\left(\frac{-x_2}{x_1^2}\right)}{1+\left(\frac{x_1}{x_2}\right)^2} & \frac{1}{2\pi}\frac{\left(\frac{1}{x_1}\right)}{1+\left(\frac{x_2}{x_1}\right)^2} \end{vmatrix}$$
$$=\frac{-1}{2\pi}e^{-\frac{1}{2}\left(x_1^2+x_2^2\right)}$$

Then, the joint distn. of X_1 and X_2 is:

$$f\left(x_1,x_2\right)=g\left(e^{-\frac{1}{2}\left(x_1^2+x_2^2\right)},\frac{1}{2\pi}\tan^{-1}\left(\frac{x_2}{x_1}\right)\right)\left|J\right|$$

$$=\frac{1}{2\pi}e^{-\left(x_1^2+x_2^2\right)},-\infty<x<\infty,\ i=1,2\ .$$

$\underset{\sim}{X}=\left(X_1,X_2\right)$ distributed as a r.v. vector of size 2 from $N\left(0,1\right)$., and we shall use the Transformation with multiple root method to generate inverse Gaussian variates.

Algorithm IG-2

1) Generate U_1 and U_2 from $U(0,1)$.
2) Set $X_1=\left(-2\ln U_1\right)^{1/2}\cos\left(2\pi U_2\right)$, $X_2=\left(-2\ln U_1\right)^{1/2}\sin\left(2\pi U_2\right)$.
3) Deliver $\underset{\sim}{X}=(X_1,X_2)$ as a random vector of size 2 generated from $N(0,1)$.
4) Set $Z_1=X_1^{\ 2}$, $Z_2=X_2^{\ 2}$
5) Set $V_{11}=\mu+\frac{\mu^2 Z_1}{2\lambda}-\frac{\mu}{2\lambda}\sqrt{4\mu\lambda Z_1+\mu^2 Z_1^{\ 2}}$.
6) Set $V_{21}=\mu+\frac{\mu^2 Z_2}{2\lambda}-\frac{\mu}{2\lambda}\sqrt{4\mu\lambda Z_2+\mu^2 Z_2^{\ 2}}$
7) Generate U_3 from $U\left(0,1\right)$.
8) If $U_3<\frac{\mu}{\mu+V_{11}}$ set $V_1=V_{11}$, else set $V_1=\frac{\mu^2}{V_{11}}$
9) If $U_3<\frac{\mu}{\mu+V_{21}}$ set $V_2=V_{21}$, else set $V_2=\frac{\mu^2}{V_{21}}$
10) Deliver $V=(V_1,V_2)$ as a random vector of size 2 generated from IG(μ,λ).
11) Stop.

3.5.3 Procedure (IG-3):

This procedure is based on Acceptance-Rejection method, where the Inverse Gaussian variate is generated by utilizing the standard normal distn. as follows:

Since the standard normal distribution is symmetric about origin, then the p.d.f. of r.v. $X \sim N^{+}(0,1)$ can be written as:

$$f(x \mid x>0)=\sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}}, 0<x<\infty$$
$$=0 \qquad , e.w.$$

where we use of inequality $(x-1)^2 \geq 0$.

To apply the Acceptance-Rejection method, we need to write the p.d.f. as $f(x)=c\,h(x)g(x)$ as shown in section (2.4.3).

Now, we consider the inequality $(x-1)^2 \geq 0 \Rightarrow x^2-2x+1 \geq 0$

$$\Rightarrow \frac{-x^2}{2} \leq \frac{1}{2}-x \Rightarrow e^{\frac{-x^2}{2}} \leq e^{\frac{1}{2}-x}$$

then $\sqrt{\frac{2}{\pi}}e^{\frac{-x^2}{2}} \leq \sqrt{\frac{2}{\pi}}e^{\frac{1}{2}-x}=\sqrt{\frac{2e}{\pi}}e^{-x}=\varphi(x)$.

$c\,h(x)=\varphi(x) \Rightarrow c=\int_0^\infty \varphi(x)\,dx=\sqrt{\frac{2e}{\pi}}\int_0^\infty e^{-x}dx$ implies $c=\sqrt{\frac{2e}{\pi}}$

$$h(x)=\frac{\varphi(x)}{c}=e^{-x}, 0<x<\infty$$
$$=0,\ e.w.$$

$$H(x)=\Pr(X \leq x)=\int_0^\infty h(t)dt=\begin{cases} 0 & , x \leq 0 \\ 1-e^{-x} & ,0<x<\infty \\ 1 & , x \to \infty \end{cases}$$

$$g(x)=\frac{f(x)}{\varphi(x)}=e^{-\frac{(x-1)^2}{2}} \text{ where } 0<g(x)<1.$$

Set $u_2=H(y) \Rightarrow u_2=1-e^{-y} \Rightarrow \mathrm{y}=-\ln(u_2)$.

The efficiency (probability) of the method is equal to $\frac{1}{c}=\sqrt{\frac{\pi}{2e}}\approx 0.76$ and the number of trails equal to $c=\sqrt{\frac{2e}{\pi}}\approx 1.32$

Algorithm (IG-3):

1- Read μ, λ.

2- Generate U_1 and U_2 from $U(0,1)$.

3- Set $Y=-\ln(u_2)$.

4- If $U_1>g(Y)$ go to step (2) .

5- Generate U_3 from $U(0,1)$.

6- If $U_3<\frac{1}{2}$ set $X=-Y$ as a r.v. generated from $N^-(0,1)$. Go to step(7).

7- Else set $X=Y$ as a r.v. generated from $N^+(0,1)$.

8- Set $Z=X^2$.

9- Set $V_1=\mu+\frac{\mu^2 Z}{2\lambda}-\frac{\mu}{2\lambda}\sqrt{4\mu\lambda Z+\mu^2 Z^2}$.

10- Generate U_4 from $U(0,1)$.

11- If $U_4<\frac{\mu}{\mu+V_1}$ deliver $V=V_1$ as a r.v. generated from IG(μ,λ).

12- Else deliver $V=\frac{\mu^2}{V_1}$ as a r.v. generated from IG(μ,λ).

13- Stop.

3.6 Monte Carlo Applications

In this section we shall utilize Monte Carlo method for estimate the parameters of inverse Gaussian distribution namely moment method and maximum likelihood method as given in sections (1.4.1.1) and (1.4.1.2) of chapter one, The simulated inverse Gaussian samples are observed by monte carlo method according to the three procedures given in sections (3.5.1), (3.5.2) and (3.5.3) seperatelly. These estimators are used to estimate the reliability and the hazard functions by three methods given by section (2.4) of chapter two.

3.6.1 Application of Estimation of Pararameter

In this section we shall use the three procedures given in sections (3.5.1), (3.5.2) and (3.5.3) to estimate the parameters of the inverse gaussian distn.

3.6.1.1 Application of Procedure (IG-1)

A computer program for procedure (IG-1) of section (3.5.1) was made to generate the IG(1,1) varaites is shown in program (1) of Appendix (A). Sample size n=5(1)10(2)20(5)30 are taken. For high accuracy the procedure repeats itself 500 times and the results of estimators together with the bias is display in table (3.1). we find practically that the efficiency of this procedure equal to 0.3998.

Table (3.1)

Parameters estimation using procedure (IG-1)

n	Moment method				Maximum likelihood method			
	$\hat{\mu}$	Bias $\hat{\mu}$	$\hat{\lambda}$	Bias $\hat{\lambda}$	$\hat{\mu}$	Bias $\hat{\mu}$	$\hat{\lambda}$	Bias $\hat{\lambda}$
5	1.031	0.013	1.996	0.996	1.031	0.013	1.02	0.02
6	0.99	0.01	1.834	0.834	0.99	0.01	1.072	0.072
7	0.99	0.01	1.454	0.454	0.99	0.01	1.023	0.023
8	0.991	0.009	1.442	0.442	0.991	0.009	0.989	0.011
9	0.991	0.009	1.44	0.44	0.991	0.009	0.996	0.004
10	1.005	0.005	1.381	0.381	1.005	0.005	0.965	0.035
12	0.991	0.009	1.34	0.34	0.991	0.009	0.968	0.032
14	0.991	0.009	1.28	0.82	0.991	0.009	1.033	0.033
16	0.993	0.007	1.317	0.317	0.993	0.007	0.982	0.018
18	1.001	0.001	1.206	0.206	1.001	0.001	0.991	0.009
20	0.999	0.001	1.261	0.261	0.999	0.001	0.976	0.024
25	0.998	0.002	1.237	0.237	0.998	0.002	1.003	0.003
30	1.002	0.002	1.192	0.192	1.002	0.002	1.009	0.009

3.6.1.2 Application of Procedure $(IG-2)$

A computer program for procedure (IG-2) of section (3.5.2) was made to generate the IG(1,1) varaites is shown in program (2) of Appendix (A). Sample size n=5(1)10(2)20(5)30 are taken. For high accuracy the procedure repeats itself 500 times and the results of estimators together with the bias is display in table (3.2).

Table (3.2)

Parameters estimation using procedure (IG-2)

n	Moment method				Maximum likelihood method			
	$\hat{\mu}$	Bias $\hat{\mu}$	$\hat{\lambda}$	Bias $\hat{\lambda}$	$\hat{\mu}$	Bias $\hat{\mu}$	$\hat{\lambda}$	Bias $\hat{\lambda}$
5	0.981	0.019	1.656	0.656	0.981	0.019	0.934	0.066
6	0.983	0.017	1.576	0.576	0.983	0.017	0.942	0.058
7	1.027	0.027	1.526	0.526	1.027	0.027	0.987	0.013
8	1.027	0.027	1.358	0.358	1.027	0.027	0.982	0.018
9	1.015	0.015	1.296	0.296	1.015	0.015	1.033	0.033
10	0.993	0.007	1.286	0.286	0.993	0.007	0.997	0.003
12	1.01	0.01	1.286	0.286	1.01	0.01	1.012	0.012
14	1.006	0.004	1.263	0.263	1.006	0.004	0.992	0.008
16	0.993	0.007	1.25	0.25	0.993	0.007	1.01	0.01
18	0.999	0.001	1.225	0.225	0.999	0.001	1.01	0.01
20	1.008	0.008	1.222	0.222	1.008	0.008	0.992	0.008
25	0.998	0.002	1.208	0.208	0.998	0.002	1	0
30	1.001	0.001	1.136	0.136	1.001	0.001	1.002	0.002

3.6.1.3 Application of Procedure $(IG-3)$

A computer program for procedure (IG-3) of section (3.5.3) was made to generate the IG(1,1) varaites is shown in program (3) of Appendix (A). Sample size n=5(1)10(2)20(5)30 are taken. For high accuracy the procedure repeats itself 500 times and the results of estimators together with the bias is display in table (3.3). we find practically that the efficiency of this procedure equal to 0.7599.

Table (3.3)

Parameters estimation using procedure (IG-3)

n	*Moment method*				*Maximum likelihood method*			
	$\hat{\mu}$	*Bias* $\hat{\mu}$	$\hat{\lambda}$	*Bias* $\hat{\lambda}$	$\hat{\mu}$	*Bias* $\hat{\mu}$	$\hat{\lambda}$	*Bias* $\hat{\lambda}$
5	**0.986**	**0.014**	**1.764**	**0.764**	**0.986**	**0.014**	**1.103**	**0.103**
6	**0.987**	**0.013**	**1.683**	**0.683**	**0.987**	**0.013**	**0.938**	**0.062**
7	**0.988**	**0.012**	**1.566**	**0.566**	**0.988**	**0.012**	**0.937**	**0.063**
8	**1.012**	**0.012**	**1.502**	**0.502**	**1.012**	**0.012**	**0.981**	**0.019**
9	**0.993**	**0.007**	**1.404**	**0.404**	**0.993**	**0.007**	**1.044**	**0.044**
10	**0.996**	**0.004**	**1.355**	**0.355**	**0.996**	**0.004**	**1.027**	**0.027**
12	**1.009**	**0.009**	**1.336**	**0.336**	**1.009**	**0.009**	**1.027**	**0.027**
14	**1.011**	**0.011**	**1.329**	**0.329**	**1.011**	**0.011**	**0.985**	**0.015**
16	**0.989**	**0.011**	**1.312**	**0.312**	**0.989**	**0.011**	**0.985**	**0.015**
18	**1.009**	**0.009**	**1.308**	**0.308**	**1.009**	**0.009**	**0.991**	**0.009**
20	**1.007**	**0.007**	**1.221**	**0.221**	**1.007**	**0.007**	**0.995**	**0.005**
25	**1.008**	**0.008**	**1.218**	**0.218**	**1.008**	**0.008**	**0.996**	**0.004**
30	**1.001**	**0.001**	**1.181**	**0.181**	**1.001**	**0.001**	**1.002**	**0.002**

3.6.2 Application of Estimation of reliability and hazard functions

In this section we shall use the estimators in section (3.6.1) to estimate the reliability and the hazard functions.

3.6.2.1 Application of Procedure (IG-1)

The estimators in table (3.1) are used to find the estimates of the reliability and the hazard functions by three methods given in section (2.4), the result is display in tables (3.4) and (3.5), the biased of the estimators shown in tables (3.6) and (3.7).

Table (3.4)

Estimation of *R(x)* using procedure (IG-1)

n	Estimation of $\hat{R}(x)$					
	True value	M.L.M	M.M	Basu (i)	Basu (ii)	Basu (iii)
5	0.4957	0.4969	0.5513	0.5317	0.501	0.4968
6	0.4959	0.4923	0.5458	0.5246	0.4772	0.4991
7	0.5038	0.4972	0.534	0.5322	0.4486	0.5024
8	0.499	0.5065	0.5545	0.5221	0.5295	0.5025
9	0.5021	0.4999	0.5315	0.519	0.5064	0.5016
10	0.4962	0.5002	0.5294	0.5195	0.4822	0.5001
12	0.5023	0.503	0.5219	0.5161	0.5042	0.5022
14	0.5018	0.4984	0.5288	0.5137	4886	0.4997
16	0.5007	0.5034	0.5204	0.5143	0.4972	0.5034
18	0.5008	0.5015	0.5225	0.5108	0.4978	0.5005
20	0.5013	0.499	0.5221	0.5083	0.4913	0.5003
25	0.4975	0.4989	0.5178	0.5059	0.5014	0.4987
30	0.5038	0.4991	0.5153	0.5073	0.5052	0.5003

Table (3.5)

Estimation of *h(x)* using procedure (IG-1)

n	Estimation of $\hat{h}(x)$					
	True value	M.L.M	M.M	Basu (i)	Basu (ii)	Basu (iii)
5	0.7793	0.7523	0.9213	0.7059	0.7575	0.7566
6	0.8186	0.829	0.9941	0.7832	0.861	0.823
7	0.7834	0.7626	0.8562	0.7124	0.8451	0.7648
8	0.8019	0.8224	0.9832	0.7979	0.7867	0.8198
9	0.8153	0.8263	0.9173	0.7959	0.8157	0.8236
10	0.7805	0.7728	0.8493	0.7441	0.8016	0.7767
12	0.8038	0.8129	0.8846	0.7923	0.8111	0.8105
14	0.8028	0.8005	0.8852	0.7767	0.8165	0.8015
16	0.7877	0.7827	0.8295	0.7631	0.7894	0.7859
18	0.7989	0.8021	0.8581	0.7874	0.8081	0.8007
20	0.8084	0.8126	0.8758	0.7977	0.8253	0.8104
25	0.7974	0.7953	0.8468	0.7861	0.7932	0.7962
30	0.8062	0.8158	0.8476	0.7927	0.7961	0.8038

Table (3.6)

Bias of Estimator $\left(\hat{R}(x)\right)$

n	Bias of Estimator $\left(\hat{R}(x)\right)$				
	M.L.M	M.M	Basu (i)	Basu (ii)	Basu (iii)
5	0.0011	0.0556	0.0058	0.0052	0.0011
6	0.0036	0.0499	0.0287	0.0187	0.0033
7	0.0066	0.0302	0.0284	0.0552	0.0014
8	0.0075	0.0555	0.0231	0.0305	0.0035
9	0.0022	0.0294	0.0169	0.0043	0.0005
10	0.004	0.0333	0.0233	0.014	0.0039
12	0.0007	0.0268	0.0138	0.0018	0.0001
14	0.0034	0.027	0.0118	0.0132	0.0022
16	0.0027	0.0197	0.0135	0.0036	0.0027
18	0.0006	0.0216	0.01	0.0031	0.0003
20	0.0023	0.0208	0.0069	0.0101	0.001
25	0.0014	0.0203	0.0084	0.0039	0.0012
30	0.0048	0.0114	0.0035	0.0013	0.0035

Table (3.7)

Bias of Estimator $\left(\hat{h}(x)\right)$

n	Bias of Estimation $\left(\hat{h}(x)\right)$				
	M.L.M	M.M	Basu (i)	Basu (ii)	Basu (iii)
5	0.027	0.1438	0.0734	0.0218	0.0227
6	0.0105	0.1755	0.0354	0.0424	0.0045
7	0.0208	0.0729	0.071	0.0617	0.0186
8	0.0205	0.1814	0.004	0.0152	0.0179
9	0.011	0.102	0.0194	0.0004	0.0083
10	0.0078	0.0687	0.0365	0.0211	0.0038
12	0.0091	0.0808	0.0116	0.0072	0.0067
14	0.0023	0.0824	0.0261	0.0137	0.0013
16	0.005	0.0418	0.0246	0.0016	0.0018
18	0.0033	0.0592	0.0114	0.0093	0.0033
20	0.0042	0.0674	0.0107	0.0169	0.002
25	0.0021	0.0494	0.0112	0.0041	0.0012
30	0.0096	0.0415	0.0135	0.0101	0.0024

3.6.2.2 Application of Procedure (IG-2)

The estimators given in table (3.2) are used to find the estimates of the reliability and the hazard functions by three methods given in section (2.4), the result is display in tables (3.8) and (3.9), the biased of the estimators shown in tables (3.10) and (3.11).

Table (3.8)

Estimation of $R(x)$ using procedure (IG-2)

n	Estimation of $\hat{R}(x)$					
	True value	M.L.M	M.M	Basu (i)	Basu (ii)	Basu (iii)
5	0.4981	0.5053	0.5416	0.5361	0.4781	0.5016
6	0.493	0.4996	0.5418	0.5296	0.4762	0.4981
7	0.4937	0.498	0.5407	0.5257	0.4778	0.4963
8	0.507	0.5001	0.5381	0.5251	0.4923	0.5018
9	0.4992	0.5031	0.5387	0.522	0.4703	0.5003
10	0.4961	0.5006	0.5341	0.5155	0.4764	0.5007
12	0.5049	0.5004	0.5301	0.5175	0.5016	0.5012
14	0.4939	0.5015	0.5317	0.5118	0.4942	0.4994
16	0.4962	0.501	0.5313	0.5092	0.4887	0.5011
18	0.5024	0.5002	0.5262	0.515	0.4895	0.5011
20	0.5	0.5015	0.5199	0.5155	0.4946	0.501
25	0.5031	0.5003	0.5202	0.5006	0.4978	0.5015
30	0.5002	0.5007	0.5152	0.5069	0.491	0.5004

Table (3.9)

Estimation of *h(x)* using procedure (IG-2)

n	Estimation of $\hat{h}(x)$					
	True value	M.L.M	M.M	Basu (i)	Basu (ii)	Basu (iii)
5	0.7967	0.8111	0.9224	0.7643	0.8572	0.8101
6	0.7814	0.7764	0.8999	0.7325	0.8145	0.7759
7	0.78	0.7641	0.8924	0.729	0.802	0.7656
8	0.8132	0.8172	0.9282	0.7784	0.8303	0.8144
9	0.7894	0.7963	0.8914	0.7627	0.8465	0.7957
10	0.8022	0.8115	0.9095	0.788	0.8527	0.8113
12	0.8056	0.8069	0.8892	0.7802	0.8049	0.8056
14	0.785	0.7898	0.8702	0.7702	0.7975	0.7892
16	0.7911	0.8065	0.8911	0.7935	0.8268	0.8063
18	0.7841	0.7622	0.8317	0.7448	0.784	0.7658
20	0.7872	0.7821	0.8287	0.7674	0.7936	0.7824
25	0.8138	0.8245	0.8779	0.8133	0.8278	0.8243
30	0.7953	0.7939	0.8312	0.7851	0.8106	0.7949

Table (3.10)

Bias of Estimator $\left(\hat{R}(x)\right)$

n	*Bias of Estimator* $\left(\hat{R}(x)\right)$				
	M.L.M	*M.M*	*Basu (i)*	*Basu (ii)*	*Basu (iii)*
5	**0.0072**	**0.0436**	**0.0381**	**0.02**	**0.0035**
6	**0.0066**	**0.0488**	**0.0365**	**0.0168**	**0.0051**
7	**0.0042**	**0.0469**	**0.0319**	**0.016**	**0.0026**
8	**0.0069**	**0.0311**	**0.018**	**0.0148**	**0.0052**
9	**0.0039**	**0.0395**	**0.0228**	**0.0289**	**0.0011**
10	**0.0044**	**0.038**	**0.0194**	**0.0198**	**0.0045**
12	**0.0044**	**0.0253**	**0.0127**	**0.0032**	**0.0036**
14	**0.0075**	**0.0378**	**0.0178**	**0.0003**	**0.0055**
16	**0.0064**	**0.0349**	**0.0128**	**0.0077**	**0.0047**
18	**0.0023**	**0.0238**	**0.0128**	**0.013**	**0.0013**
20	**0.0015**	**0.0199**	**0.0115**	**0.0054**	**0.001**
25	**0.0027**	**0.0171**	**0.0035**	**0.0053**	**0.0016**
30	**0.0005**	**0.015**	**0.0067**	**0.0092**	**0.0002**

Table (3.11)

Bias of Estimator $\left(\hat{h}(x)\right)$

n	*Bias of Estimation* $\left(\hat{h}(x)\right)$				
	M.L.M	*M.M*	*Basu (i)*	*Basu (ii)*	*Basu (iii)*
5	**0.0143**	**0.1256**	**0.0324**	**0.0604**	**0.0134**
6	**0.0049**	**0.1185**	**0.0489**	**0.0331**	**0.0054**
7	**0.0159**	**0.1124**	**0.051**	**0.0221**	**0.0144**
8	**0.004**	**0.1149**	**0.0348**	**0.017**	**0.0012**
9	**0.0069**	**0.102**	**0.0267**	**0.0572**	**0.0063**
10	**0.0093**	**0.1072**	**0.0142**	**0.0505**	**0.0091**
12	**0.0013**	**0.0836**	**0.0254**	**0.0007**	**0.0001**
14	**0.0048**	**0.0852**	**0.0148**	**0.0125**	**0.0042**
16	**0.0075**	**0.092**	**0.0055**	**0.0277**	**0.0073**
18	**0.0219**	**0.0504**	**0.0366**	**0.0027**	**0.0155**
20	**0.0051**	**0.0415**	**0.0198**	**0.0064**	**0.0048**
25	**0.0107**	**0.0641**	**0.0004**	**0.014**	**0.0105**
30	**0.0014**	**0.0359**	**0.0102**	**0.0153**	**0.0004**

3.6.2.3 Application of Procedure (IG-3)

The estimators given in table (3.3) are used to find the estimates of the reliability and the hazard functions by three methods given in section (2.4), the result is display in tables (3.12) and (3.13), the biased of the estimators shown in tables (3.14) and (3.15).

Table (3.12)

Estimation of *R(x)* using procedure (IG-3)

n	Estimation of $\hat{R}(x)$					
	True value	M.L.M	M.M	Basu (i)	Basu (ii)	Basu (iii)
5	0.4999	0.5076	0.5599	0.5385	0.5291	0.5019
6	0.5001	0.4924	0.5435	0.5327	0.5167	0.501
7	0.4996	0.4986	0.5348	0.5255	0.4859	0.5008
8	0.4976	0.5012	0.5404	0.515	0.4992	0.4981
9	0.5051	0.5008	0.5453	0.5212	0.525	0.5025
10	0.5009	0.5048	0.5448	0.5218	0.4949	0.5021
12	0.5055	0.5036	0.534	0.5211	0.4795	0.5038
14	0.4946	0.4984	0.5231	0.5094	0.4879	0.5004
16	0.5016	0.5003	0.5134	0.5118	0.4884	0.5015
18	0.4996	0.5005	0.5222	0.511	0.4848	0.5011
20	0.504	0.5019	0.5188	0.5116	0.5084	0.503
25	0.5043	0.5005	0.5199	0.5098	0.4966	0.502
30	0.498	0.5007	0.5182	0.5061	0.4964	0.5

Table (3.13)

Estimation of *h(x)* using procedure (IG-3)

n	*Estimation of* $\hat{h}(x)$					
	True value	*M.L.M*	*M.M*	*Basu (i)*	*Basu (ii)*	*Basu (iii)*
5	**0.7919**	**0.8057**	**0.9764**	**0.7594**	**0.773**	**0.8053**
6	**0.7902**	**0.7676**	**0.9082**	**0.7095**	**0.7315**	**0.7544**
7	**0.8044**	**0.8065**	**0.9086**	**0.7652**	**0.8275**	**0.8029**
8	**0.833**	**0.8697**	**1.009**	**0.8465**	**0.8732**	**0.8682**
9	**0.8144**	**0.8233**	**0.9618**	**0.791**	**0.7853**	**0.8204**
10	**0.7846**	**0.7949**	**0.8996**	**0.7593**	**0.8006**	**0.7891**
12	**0.7899**	**0.7975**	**0.8658**	**0.7669**	**0.8334**	**0.7933**
14	**0.8069**	**0.8143**	**0.8826**	**0.7968**	**0.8319**	**0.8112**
16	**0.8038**	**0.8072**	**0.8403**	**0.7891**	**0.8296**	**0.8052**
18	**0.7939**	**0.7916**	**0.8488**	**0.7764**	**0.8183**	**0.7918**
20	**0.8009**	**0.8031**	**0.8461**	**0.7879**	**0.7984**	**0.8013**
25	**0.8012**	**0.8**	**0.8479**	**0.7853**	**0.8063**	**0.8005**
30	**0.7922**	**0.7919**	**0.8363**	**0.7836**	**0.7988**	**0.7921**

Table (3.14)

Bias of Estimator $\left(\hat{R}(x)\right)$

n	Bias of Estimator $\left(\hat{R}(x)\right)$				
	M.L.M	M.M	Basu (i)	Basu (ii)	Basu (iii)
5	0.0077	0.06	0.0387	0.0292	0.002
6	0.0077	0.0433	0.0326	0.0166	0.0009
7	0.001	0.0352	0.0259	0.0137	0.001
8	0.0036	0.0428	0.0173	0.0016	0.0005
9	0.0043	0.0402	0.0161	0.0199	0.0026
10	0.0039	0.0439	0.0209	0.006	0.0012
12	0.0019	0.0285	0.0156	0.026	0.0018
14	0.002	0.0267	0.013	0.0086	0.0039
16	0.0013	0.0118	0.0102	0.0132	0
18	0.0009	0.0226	0.0114	0.0148	0.0015
20	0.0021	0.0148	0.0076	0.0009	0.0009
25	0.0038	0.0156	0.0056	0.0077	0.0023
30	0.0027	0.0201	0.008	0.0017	0.002

Table (3.15)

Bias of Estimator $\left(\hat{h}(x)\right)$

n	Bias of Estimation $\left(\hat{h}(x)\right)$				
	M.L.M	M.M	Basu (i)	Basu (ii)	Basu (iii)
5	0.0138	0.1845	0.0325	0.0189	0.0134
6	0.0226	0.118	0.0807	0.0587	0.0358
7	0.0021	0.1042	0.0392	0.0231	0.0015
8	0.0367	0.1761	0.0135	0.0402	0.0352
9	0.0089	0.1474	0.0234	0.0292	0.006
10	0.0103	0.1151	0.0252	0.0161	0.0046
12	0.0076	0.0759	0.023	0.0435	0.0034
14	0.0074	0.0757	0.0101	0.025	0.0043
16	0.0034	0.0365	0.0147	0.0231	0.0015
18	0.0023	0.0549	0.0175	0.0245	0.0021
20	0.0022	0.0452	0.013	0.0025	0.0004
25	0.0013	0.0485	0.0159	0.0051	0.0007
30	0.0003	0.0441	0.0087	0.0066	0.0001

CONCLUSIONS AND FUTURE WORK

Conclusions

1. *The best procedure for generating sample variates from inverse Gaussian distribution was procedure (IG-2), which depends on (Box and Muller) method, which consumes less time in comparison with the other two procedures of generation.*
2. *The theory and practice showed that the efficiency of procedure (IG-3) was better than procedures (IG-1).*
3. *For small and moderate sample sizes, the M.L.M gave estimates $\hat{\mu}$ and $\hat{\lambda}$ very close to the true values of μ and λ.*
4. *For large sample sizes, the M.M and M.L.M gave estimates close to the true values of μ and λ. But the M.L.M is superior than M.M.*
5. *For all sample sizes, the Basu(iii) and M.L.M methods gave estimates $\hat{R}(x)$ and $\hat{h}(x)$ very close to the true value of $R(x)$ and $h(x)$. But the Basu(iii) method was superior than M.L.M.*
6. *For all sample sizes, the Basu(i) and Basu(ii) methods were better than M.M for estimates $R(x)$ and $h(x)$.*
7. *The disadvantage of Monte-Carlo methods depends on generating pseudorandom variates and that might carry dirty data.*

Future Work

1. This work can be use for generalized inverse Gaussian distribution of three parameters and other life distribution.
2. Another methods can be used to estimate the distribution parameters μ, λ, $R(x)$ and $h(x)$ like least squares method, modified moment method, Minimum Chi-square method, Minimum Distance method, Bayesian Method, … etc.
3. It can be generate r.v.'s from inverse Gaussian distribution by other new procedures which can be compare their efficiency with our used procedures.
4. The bias of estimation is a r.v. of unknown distribution which can be investigated approximately by using well-known statistical tests such as Chi-Square Goodness-of-Fit Test, Kolmogorov-Smirnov Goodness-of-Fit Test, Serial Test, …etc.

REFERENCES

[1] Andre I. Khuri (2003), Advanced Calculus with Applications in Statistics. Second Edition, *John Wiley & Sons, Inc*., Hoboken, New Jersey.

[2] Banerjee, A. K. and Bhattacharyya, G. K. (1976), A purchase incidence model with inverse Gaussian interpurchase times. *J. Amer. Stat. Ass.* Vol. 71, pp. 823-829.

[3] Baradsley, W. E. (1980), Note on the use of the inverse Gaussian distribution for wind energy applications. *J. of App. Meteorology,* vol. 19, pp. 1126- 1130.

[4] Barndorff-Neielsen O. E. (1994), A Note on Electrical Networks and the Inverse Gaussian Distribution. *Adv. in App. Prob.*, vol. 26, pp. 63-67.

[5] Basu, A. P. (1964), Estimates of reliability for some distributions useful in life testing. *Technometrics,* vol. 6, pp. 215-219.

[6] Box, G. E. and Muller, M. E. (1958), A note on the generation of random normal deviates. *Ann. Math. Stat.*, vol. 29, pp.610-611.

[7] Chhikara, R. S. (1975), Optimum test for the comparison of two inverse Gaussian distribution means. *Austral. J. stat.* vol.17, pp. 77-83.

[8] Chhikara, R. S. and Folks, J. L. (1974), Estimation of the inverse Gaussian distribution function. *J. of Amaer. Stat. Assoc.* vol. 69, pp. 250-254.

[9] Chhikara, R. S. and Folks, J. L. (1975), Statistical distribution related to the inverse Gaussian. *Commun. In Stat.*, vol. 4, pp. 1081-1091.

[10] Chhikara, R. S. and Folks, J. L. (1976), Optimum test procedure for the mean of first passage time distribution in Brownian motion with positive drift (inverse Gaussian). *Technometrics,* vol. 18, pp. 189-193.

[11] Chhikara, R. S. and Folks, J. L. (1977), The Inverse Gaussian Distribution as a Lifetime Model. *Technometrics,* vol. 19, pp. 461-468.

[12] Dennis, B., Munholland, P. L. and Scott, M. J. (1991), Estimation of the Growth and Extinction Parameters for Endangered Species. *Ecological Monographs*, vol. 61, pp. 115-143.

[13] Durham, S. D. and Padgett, W. J. (1997), A Cumulative Damage Model for System Failure with Application to Carbon Fibers and Composites. *Technometrics*, vol. 39, pp. 34-44.

[14] Dyer, M., A., Frieze, and R. Kannan (1989), A random polynomial time algorithm for approximating the volume of convex bodies. *Proc. Twenty-First ACM Symposium on Theory of Computing*, pp. 375-381.

[15] Folks J. L. and Chhikara R. S. (1978), The inverse Gaussian distribution and its statistical application - a review, *J. Roy. Stat. Soc.* Series B, vol. 40, pp. 263-289.

[16] George S. Fishman (1996), Monte Carlo Concepts, Algorithms, and Applications. *Springer-Verlag New York, Inc.*

[17] Hamsa, F. (1997), On the inverse Gaussian distribution model in reliability with application. M.Sc. thesis college of Science of *Baghdad University*, Baghdad.

[18] Hastofer, A. M. (1964), A dam with inverse Gaussian input. *Proc. Comp. Phil. Soc.*, vol. 60, pp 931-933.

[19] Hogg, R.V. and Graig, A.T. (1978), Introdection to mathematical statistics. 4th edition, *Macmillan pub. Co.* New York.

[20] Huberman B. , Pirolli P. , Pitkow J. and Lukose R. (1998), Strong regularities in World Wide Web surfing. *Science*, vol. 280, pp. 95-97.

[21] James E. Gentle(2003), Random number generation and monte carlo method. second edition, *Springer Science+Business Media, Inc*.

[22] Jerrum, M., and A. Sinclair (1988), Conductance and the rapid mixing property for Markov chains: The approximation of the permanent resolved. *Proc. Twentieth ACM Symposium on Theory of Computing*, pp. 235-244.

[23] John R. Michael, William R. Schucany, Roy W. Haas (1976) , Generating Random Variates Using Transformations with Multiple Roots. *The American Statistician*, Vol. 30, No. 2. pp. 88-90.

[24] Jun Shao (1999), Mathematical Statistics. *Springer-Verlag New York, Inc.*

[25] Karp, R. M. (1985), Monte-Carlo algorithms for the planar multiterminal network reliability problem. *J. Complexity* vol. 1, pp. 45-64.

[26] Khatri, C. G. (1962), A characterization of the inverse Gaussian distribution. *ann. Math. Stat.*, vol. 33, pp.800-803

[27] Koichi T. , Shinji K. , Ichiro M. and Yukinori K.(1997), Application of Inverse Gaussian distribution to Occupational Exposure Data. *Ann. Occup. Hyg.*, vol. 41, pp. 505-514.

[28] Kuo, W. and Zuo, M. J. (2003), Optimal reliability modeling Principles and Applications. *John Wiley & Sons, Inc.*, Hoboken, New Jersey.

[29] Lancaster, A. (1972), A stochastic model for the duration of a strike. *J. R. stat. soc.* A, vol. 135, pp. 257-271.

[30] Law A. M. and Keliton W. D. (2000), Simulation Modeling and Analysis. *McGraw, Co., Raw – Hill International Series* Third Edition.

[31] Lehmann, E. L. and Casella, G. (1998), Theory of Point Estimation. Second Edition, *Springer-Verlag New York, Inc*.

[32] Meeker, W. Q. and Escobar L. U. (1998), Statistical Methods for Reliability Data. *John Wiley & Sons, Inc., Canada*.

[33] Metropolis, N. and Ulam, S.(1949), The Monte Carlo method, *J. Am. Stat. Assoc.*, vol. 44, pp. 335- 341.

[34] Mood, A. M. (1982), introduction to the theory of statistics. 3rd edition, *Mc Graw Hill international book Co. London.*

[35] Mukhopadhyay, N. (2000), Probability and Statistical Inference. *Marcel Dekker, Inc., New York.*

[36] Rand Corporation, A Million Random Digits With 1000,000 Normal Deviates. *Free Press, Clence, Illinois, (1955).*

[37] Rayleigh, Lord J. W. S. (1899). On James Bernoulli's theorem in probabilities. *Phil. Mag.*, vol. 47, pp. 246-251.

[38] Schrodinger, E. (1915), Zur Theorie der Fall - und Steigversuche an Teilchenn mit Brownsche Bewegung. *Physikalische Zeitschrift*, vol. 16 pp289-295.

[39] Sheppard, C. W. (1962), Basic Principles of the Tracer Method. *New York: Wiley.*

[40] Shuster, J. (1968), On the inverse Gaussian distribution function. *J. Amer. Stat. Assoc.* vol. 63, pp. 1514-1518

[41] Tweedie , M. C. (1945), Inverse statistical Variates. *Nature*, vol. 155, pp.453.

[42] Tweedie , M. C. (1957), statistical properties of inverse Gaussian distribution I. *ann. Math. Stat.*, vol. 28, pp.362-377

[43] Tweedie , M.C. (1957), statistical properties of inverse Gaussian distribution II. *ann. Math. Stat.*, vol. 28, pp.696-705

[44] Von Neumann J. (1951), Various Techniques Used in Connection With Random Digits. *U.S. Nat. Bur. Stand. Appl. Math. Ser.*, No.12, pp. $36-38$,.

[45] Wald, A. (1947), Sequential Analysis. *Wiley, New York.*

[46] Wasan, M. and Roy, L. K. (1969), Tables of inverse Gaussian percentage points. *Technometrics,* vol. 11, pp. 591-904.

[47] Wasan, M. T. (1968), First passage time distribution of Brownian motion with positive drift (inverse Gaussian distribution), Queens paper in pure and applied mathematics, 19. *Queen's University, Canada.*

APPENDIX

A

COMPUTER PROGRAMS FOR GENERATING RANDOM VARIATES FROM INVERSE GAUSSIAN DISTRIBUTION

Program(1) : procedure (IG-1)

Enter your values of μ, λ, n and m

$\lambda :=$ ▮ $\mu :=$ ▮ $n :=$ ▮ $m :=$ ▮

```
X := | for j ∈ 0.. m − 1
     |   for i ∈ 0.. (n − 1)
     |     u1 ← rnd(1)
     |     u2 ← rnd(1)
     |     b ← tan[(π·u2) − 5/2]
     |     while u1 > [1/2·(1 + b^2)·e^(−b^2/2)]
     |       u1 ← rnd(1)
     |       u2 ← rnd(1)
     |       b ← tan[(π·u2) − π/2]
     |     y ← b^2
     |     x1 ← μ + (μ^2·y)/(2·λ) − μ/(2·λ)·sqrt(4·μ·λ·y + μ^2·y^2)
     |     u ← rnd(1)
     |     x_(i,j) ← | x1      if |u| ≤ |μ/(μ + x1)|
     |               | μ^2/x1  otherwise
     |     x
     |   x
     | x
```

X = ▮

Program(2) : procedure (IG-2)

Enter your values of μ, λ, n and m

$\lambda :=$ ▮ $\mu :=$ ▮ $n :=$ ▮ $m :=$ ▮

```
X := | for j ∈ 0.. m − 1
     |   for i ∈ 0.. n − 1
     |     u1 ← rnd(1)
     |     u2 ← rnd(1)
     |     b1[i,j] ← √(−2·ln(u1))·cos(2π·u2)
     |     b2[i,j] ← √(−2·ln(u1))·sin(2π·u2)
     |     z1 ← (b1[i,j])^2
     |     z2 ← (b2[i,j])^2
     |     b11[i,j] ← μ + (μ^2·z1)/(2·λ) − μ/(2·λ)·√(4·μ·λ·z1 + μ^2·z1^2)
     |     b21[i,j] ← μ + (μ^2·z2)/(2·λ) − μ/(2·λ)·√(4·μ·λ·z2 + μ^2·z2^2)
     |     u3 ← rnd(1)
     |     y1[i,j] ← | b11[i,j]     if u3 ≤ μ/(μ + b11[i,j])
     |               | μ^2/b11[i,j] otherwise
     |     y2[i,j] ← | b21[i,j]     if u3 ≤ μ/(μ + b21[i,j])
     |               | μ^2/b21[i,j] otherwise
     |     y1
     |     y2
     |   y1
     |   y2
     | y1
     | y2
```

$X =$ ▮

Program(3) : procedure (IG-3)

Enter your values of μ, λ, n and m

$\lambda :=$ ▮ $\mu :=$ ▮ $n :=$ ▮ $m :=$ ▮

$i := 0..n-1 \qquad j := 0..m-1$

$$
b_{i,j} := \left|
\begin{array}{l}
u1 \leftarrow rnd(1) \\
u2 \leftarrow rnd(1) \\
y \leftarrow -\ln(u2) \\
\text{while } u1 > e^{\frac{-(y-1)^2}{2}} \\
\quad \left| \begin{array}{l} u1 \leftarrow rnd(1) \\ u2 \leftarrow rnd(1) \\ y \leftarrow -\ln(u2) \end{array} \right. \\
u3 \leftarrow rnd(1) \\
-y \quad \text{if } u3 < \frac{1}{2} \\
y \quad \text{if } u3 > \frac{1}{2}
\end{array}
\right.
$$

$$
X := \left|
\begin{array}{l}
\text{for } j \in 0..m-1 \\
\quad \text{for } t \in 0..n-1 \\
\quad \left| \begin{array}{l}
y \leftarrow \left(b_{t,j}\right)^2 \\
x1_{t,j} \leftarrow \mu + \frac{\mu^2 \cdot y}{2\cdot\lambda} - \frac{\mu}{2\cdot\lambda}\cdot\sqrt{4\cdot\mu\cdot\lambda\cdot y + \mu^2\cdot y^2} \\
u \leftarrow rnd(1) \\
x_{t,j} \leftarrow \left| \begin{array}{ll} x1_{t,j} & \text{if } u \le \frac{\mu}{\mu + x1_{t,j}} \\ \frac{\mu^2}{x1_{t,j}} & \text{otherwise} \end{array} \right. \\
x
\end{array} \right. \\
x
\end{array}
\right.
$$

$X =$ ▮

APPENDIX B

COMPUTER PROGRAMS OF ESTIMATION METHODS

Program(4) : procedure (IG-1)

Enter your values of μ, λ, n and m

$\lambda :=$ ▮ $\mu :=$ ▮ $n :=$ ▮ $m :=$ ▮

$$
\begin{array}{l}
X := \left| \begin{array}{l}
\text{for } j \in 0..m-1 \\
\quad \left| \begin{array}{l}
\text{for } i \in 0..(n-1) \\
\quad \left| \begin{array}{l}
u1 \leftarrow \text{rnd}(1) \\
u2 \leftarrow \text{rnd}(1) \\
b \leftarrow \tan\left[(\pi \cdot u2) - \frac{5}{2}\right] \\
\text{while } u1 > \left[\frac{1}{2}\cdot\left(1+b^2\right)\cdot e^{\frac{-b^2}{2}}\right] \\
\quad \left| \begin{array}{l}
u1 \leftarrow \text{rnd}(1) \\
u2 \leftarrow \text{rnd}(1) \\
b \leftarrow \tan\left[(\pi \cdot u2) - \frac{\pi}{2}\right]
\end{array} \right. \\
y \leftarrow b^2 \\
x1 \leftarrow \mu + \frac{\mu^2 \cdot y}{2\cdot\lambda} - \frac{\mu}{2\cdot\lambda}\cdot\sqrt{4\cdot\mu\cdot\lambda\cdot y + \mu^2\cdot y^2} \\
u \leftarrow \text{rnd}(1) \\
x_{i,j} \leftarrow \left| \begin{array}{ll}
x1 & \text{if } |u| \le \left|\frac{\mu}{\mu + x1}\right| \\
\frac{\mu^2}{x1} & \text{otherwise}
\end{array} \right. \\
x
\end{array} \right. \\
x
\end{array} \right. \\
x
\end{array} \right.
\end{array}
$$

$X =$ ▮

$i := 0..n-1 \quad j := 0..m-1$

$$XX_j := \frac{\sum_{i=0}^{n-1} X_{i,j}}{n} \qquad xb_j := \frac{1}{n}\cdot\sum_{i=0}^{n-1} X_{i,j} \qquad Xb := \frac{\sum_{i=0}^{m-1} xb_i}{m}$$

$$SA_j := \frac{1}{(n-1)}\cdot\left[\left[\sum_{i=0}^{n-1}\left(X_{i,j}-xb_j\right)^2\right]\right] \qquad \lambda1_j := \frac{n\cdot\left(xb_j\right)^3}{(n-1)\cdot SA_j} \qquad \lambda m := \frac{\sum_{i=0}^{m-1} \lambda1_i}{m}$$

$$\lambda3_j := \frac{(n-3)}{\sum_{i=0}^{n-1}\left(\frac{1}{X_{i,j}}-\frac{1}{xb_j}\right)} \qquad \lambda ml := \frac{\sum_{i=0}^{m-1} \lambda3_i}{m}$$

$$a1_{i,j} := \int_{-\infty}^{\sqrt{\frac{\lambda}{X_{i,j}}}\cdot\left(1-\frac{X_{i,j}}{\mu}\right)} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\,dw$$

$$b1_{i,j} := \left[e^{\frac{2\cdot\lambda}{\mu}}\cdot\int_{-\infty}^{\sqrt{\frac{\lambda}{X_{i,j}}}\cdot\left(1+\frac{X_{i,j}}{\mu}\right)} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\,dw\right] \qquad RR_{i,j} := a1_{i,j} - b1_{i,j}$$

$$RR_j := \frac{\sum_{i=0}^{n-1} R_{i,j}}{n} \qquad Rtrue := \frac{\sum_{i=0}^{m-1} RR_i}{m} \qquad htrue := \frac{\sqrt{\frac{\lambda}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}\cdot e^{\frac{-\lambda\cdot(Xb-\mu)^2}{2\cdot\mu^2\cdot Xb}}}{Rtrue}$$

$$a2_{i,j} := \int_{-\infty}^{\sqrt{\frac{\lambda ml}{X_{i,j}}}\cdot\left(1-\frac{X_{i,j}}{Xb}\right)} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\,dw$$

$$b2_{i,j} := \left[e^{\frac{2\cdot\lambda ml}{Xb}} \cdot \int_{-\infty}^{\left[\sqrt{\frac{\lambda ml}{X_{i,j}}}\cdot\left(1+\frac{X_{i,j}}{Xb}\right)\right]} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\, dw \right] \qquad Rcab_{i,j} := a2_{i,j} - b2_{i,j}$$

$$RCAB_j := \frac{\sum_{i=0}^{n-1} Rcab_{i,j}}{n} \qquad Rmlm := \frac{\sum_{i=0}^{m-1} RCAB_i}{m} \qquad hmlm := \frac{\sqrt{\frac{\lambda ml}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}}{Rmlm}$$

$$a3_{i,j} := \int_{-\infty}^{\sqrt{\frac{\lambda m}{X_{i,j}}}\cdot\left(1-\frac{X_{i,j}}{Xb}\right)} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\, dw$$

$$b3_{i,j} := \left[e^{\frac{2\cdot\lambda m}{Xb}} \cdot \int_{-\infty}^{\sqrt{\frac{\lambda m}{X_{i,j}}}\cdot\left(1+\frac{X_{i,j}}{Xb}\right)} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\, dw \right] \qquad Rcab2_{i,j} := a3_{i,j} - b3_{i,j}$$

$$RCAB2_j := \frac{\sum_{i=0}^{n-1} Rcab2_{i,j}}{n} \qquad Rmm := \frac{\sum_{i=0}^{m-1} RCAB2_i}{m} \qquad hmm := \frac{\sqrt{\frac{\lambda m}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}}{Rmm}$$

$$a4_{i,j} := \int_{-\infty}^{\frac{-\left[\sqrt{n\cdot\lambda}\cdot\left(X_{i,j}-Xb\right)\right]}{\sqrt{\left(X_{i,j}\right)\cdot Xb\cdot\left|n\cdot Xb-X_{i,j}\right|}}} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-w^2}{2}}\, dw$$

$$b4_{i,j} := \left[\int_{-\infty}^{\frac{-\sqrt{\lambda}\left[n\cdot Xb+(n-2)\cdot X_{i,j}\right]}{\left[n\cdot X_{i,j}\cdot Xb\cdot\left(\left|n\cdot Xb-X_{i,j}\right|\right)\right]^{\frac{1}{2}}}} \left[\frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-w^2}{2}} \cdot \left[\left(\frac{n-2}{n}\right)\cdot\left[e^{\frac{2\cdot(n-1)\cdot\lambda}{n\cdot Xb}}\right]\right]\right] dw \right]$$

$$Rstar1_{i,j} := a4_{i,j} - b4_{i,j} \qquad RSTAR1_j := \frac{\left[\sum_{i=0}^{n-1}\left(Rstar1_{i,j}\right)\right]}{n} \qquad RBasu1 := \frac{\left[\sum_{i=0}^{m-1}\left(RSTAR1_i\right)\right]}{m}$$

$$hBasu1 := \frac{\sqrt{\frac{\lambda ml}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}}{RBasu1}$$

$$y_j := \sum_{i=1}^{n-1} \frac{(X_{i,j}-\mu)^2}{X_{i,j}} \qquad y1 := \frac{\sum_{i=0}^{n-1} y_i}{n} \qquad xfirst := XX_1 \qquad t := xfirst + y1$$

$$a5_{i,j} := \int_{-\infty}^{\frac{-\sqrt{n-1}\cdot(X_{i,j}-\mu)}{\sqrt{\left|t\cdot X_{i,j}-(X_{i,j}-\mu)^2\right|}}} \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi(n-1)}\cdot\Gamma\left(\frac{n-1}{2}\right)\cdot\left(1+\frac{w^2}{n-1}\right)^{\frac{n}{2}}}\,dw$$

$$b5_{i,j} := \left[\left(\frac{t+4\mu}{t}\right)^{\frac{n-2}{2}} \cdot \int_{-\infty}^{\frac{-\sqrt{n-1}\cdot(X_{i,j}+\mu)}{\sqrt{\left|t\cdot X_{i,j}-(X_{i,j}-\mu)^2\right|}}} \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi(n-1)}\cdot\Gamma\left(\frac{n-1}{2}\right)\cdot\left(1+\frac{w^2}{n-1}\right)^{\frac{n}{2}}}\,dw\right]$$

$$Rstar2_{i,j} := a5_{i,j} - b5_{i,j} \qquad RSTAR2_j := \frac{\sum_{i=0}^{n-1} Rstar2_{i,j}}{n} \qquad RBasu2 := \frac{\sum_{i=0}^{m-1} RSTAR2_i}{m}$$

$$hBasu2 := \frac{\sqrt{\frac{\lambda ml}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}}{RBasu2} \qquad V1_j := \sum_{i=0}^{n-1}\left(\frac{1}{X_{i,j}} - \frac{1}{Xb}\right) \qquad V := \frac{\sum_{i=0}^{m-1} V1_i}{m}$$

$$a6_{i,j} := \int_{-\infty}^{\frac{-\sqrt{n(n-2)}\cdot(X_{i,j}-Xb)}{\sqrt{\left|V\cdot X_{i,j}\cdot Xb\cdot(n\cdot Xb-X_{i,j})-n\cdot(X_{i,j}-Xb)^2\right|}}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi(n-2)}\cdot\Gamma\left(\frac{n-2}{2}\right)\cdot\left(1+\frac{w^2}{n-2}\right)^{\frac{n-1}{2}}}\,dw$$

$$b6_{i,j} := \left[\left[\frac{4\cdot(n-1)}{n\cdot V\cdot Xb}+1\right]^{\frac{n-3}{2}} \cdot \left(\frac{n-2}{n}\right) \cdot \int_{-\infty}^{\frac{-\sqrt{n(n-2)}\cdot\left[Xb+\frac{(n-2)X_{i,j}}{n}\right]}{\sqrt{\left|V\cdot X_{i,j}\cdot Xb\cdot(n\cdot Xb-X_{i,j})-n\cdot(X_{i,j}-Xb)^2\right|}}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi(n-2)}\cdot\Gamma\left(\frac{n-2}{2}\right)\cdot\left(1+\frac{w^2}{n-2}\right)^{\frac{n-1}{2}}}\,dw\right]$$

$$Rstar3_{i,j} := a6_{i,j} - b6_{i,j}$$

$$RSTAR3_j := \frac{\sum_{i=0}^{n-1} Rstar3_{i,j}}{n} \qquad RBasu3 := \frac{\sum_{i=0}^{m-1} RSTAR3_i}{m} \qquad hBasu3 := \frac{\sqrt{\frac{\lambda ml}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}}{RBasu3}$$

$$ER1 := \int_0^{\infty} \left[Xb\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasXb := ER1 - \mu$$

$$ER2 := \int_0^{\infty} \left[\lambda ml\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad Bias\lambda ml := ER2 - \lambda$$

$$ER3 := \int_0^{\infty} \left[\lambda m\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad Bias\lambda m := ER3 - \lambda$$

$$ER4 := \int_0^{\infty} \left[Rmlm\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasRmlm := ER4 - Rtrue$$

$$ER5 := \int_0^{\infty} \left[Rmm\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasRmm := ER5 - Rtrue$$

$$ER6 := \int_0^{\infty} \left[RBasu1\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasRBasu1 := ER6 - Rtrue$$

$$ER7 := \int_0^{\infty} \left[RBasu2\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasRBasu2 := ER7 - Rtrue$$

$$ER8 := \int_0^{\infty} \left[RBasu3\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasRBasu3 := ER8 - Rtrue$$

$$ER9 := \int_0^{\infty} \left[hmlm\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad Biashmlm := ER9 - htrue$$

$$ER10 := \int_0^{\infty} \left[hmm\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad Biashmm := ER10 - htrue$$

$$ER11 := \int_{0}^{\infty} \left[hBasu1 \cdot \left[\left(\frac{\lambda}{2 \cdot \pi \cdot t^{3}} \right)^{\frac{1}{2}} \cdot e^{\left[\frac{-\lambda \cdot (t-\mu)^{2}}{2 \cdot \mu^{2} \cdot t} \right]} \right] \right] dt$$

BiashBasu1 := ER11 − htrue

$$ER12 := \int_{0}^{\infty} \left[hBasu2 \cdot \left[\left(\frac{\lambda}{2 \cdot \pi \cdot t^{3}} \right)^{\frac{1}{2}} \cdot e^{\left[\frac{-\lambda \cdot (t-\mu)^{2}}{2 \cdot \mu^{2} \cdot t} \right]} \right] \right] dt$$

BiashBasu2 := ER12 − htrue

$$ER13 := \int_{0}^{\infty} \left[hBasu3 \cdot \left[\left(\frac{\lambda}{2 \cdot \pi \cdot t^{3}} \right)^{\frac{1}{2}} \cdot e^{\left[\frac{-\lambda \cdot (t-\mu)^{2}}{2 \cdot \mu^{2} \cdot t} \right]} \right] \right] dt$$

BiashBasu3 := ER13 − htrue

Xb = ■ λm = ■ λml = ■

Rtrue = ■

Rmlm = ■ BiasRmlm = ■ RBasu1 = ■ BiasRBasu1 = ■

Rmm = ■ BiasRmm = ■ RBasu2 = ■ BiasRBasu2 = ■

RBasu3 = ■ BiasRBasu3 = ■

htrue = ■

hmlm = ■ Biashmm = ■ hmm = ■ Biashmlm = ■

hBasu1 = ■ BiashBasu1 = ■ hBasu2 = ■ BiashBasu2 = ■

hBasu3 = ■ BiashBasu3 = ■

Program(4) : procedure (IG-1)

Enter your values of μ, λ, n and m

$\lambda :=$ ▮ $\mu :=$ ▮ $n :=$ ▮ $m :=$ ▮

$$
\begin{aligned}
X := \; & \text{for } j \in 0..m-1 \\
& \quad \text{for } i \in 0..n-1 \\
& \quad\quad u1 \leftarrow \text{rnd}(1) \\
& \quad\quad u2 \leftarrow \text{rnd}(1) \\
& \quad\quad b1_{i,j} \leftarrow \sqrt{-2\cdot\ln(u1)}\cdot\cos(2\pi\cdot u2) \\
& \quad\quad b2_{i,j} \leftarrow \sqrt{-2\cdot\ln(u1)}\cdot\sin(2\pi\cdot u2) \\
& \quad\quad z1 \leftarrow \left(b1_{i,j}\right)^2 \\
& \quad\quad z2 \leftarrow \left(b2_{i,j}\right)^2 \\
& \quad\quad b11_{i,j} \leftarrow \mu + \frac{\mu^2\cdot z1}{2\cdot\lambda} - \frac{\mu}{2\cdot\lambda}\cdot\sqrt{4\cdot\mu\cdot\lambda\cdot z1 + \mu^2\cdot z1^2} \\
& \quad\quad b21_{i,j} \leftarrow \mu + \frac{\mu^2\cdot z2}{2\cdot\lambda} - \frac{\mu}{2\cdot\lambda}\cdot\sqrt{4\cdot\mu\cdot\lambda\cdot z2 + \mu^2\cdot z2^2} \\
& \quad\quad u3 \leftarrow \text{rnd}(1) \\
& \quad\quad y1_{i,j} \leftarrow \begin{cases} b11_{i,j} & \text{if } u3 \le \dfrac{\mu}{\mu + b11_{i,j}} \\[2ex] \dfrac{\mu^2}{b11_{i,j}} & \text{otherwise} \end{cases} \\
& \quad\quad y2_{i,j} \leftarrow \begin{cases} b21_{i,j} & \text{if } u3 \le \dfrac{\mu}{\mu + b21_{i,j}} \\[2ex] \dfrac{\mu^2}{b21_{i,j}} & \text{otherwise} \end{cases} \\
& \quad\quad y1 \\
& \quad\quad y2 \\
& \quad y1 \\
& \quad y2 \\
& y1 \\
& y2
\end{aligned}
$$

$X =$ ▮

$i := 0..n-1 \qquad j := 0..m-1$

$$XX_j := \frac{\sum_{i=0}^{n-1} X_{i,j}}{n} \qquad xb_j := \frac{1}{n}\cdot\sum_{i=0}^{n-1} X_{i,j} \qquad Xb := \frac{\sum_{i=0}^{m-1} xb_i}{m}$$

$$SA_j := \frac{1}{(n-1)}\cdot\left[\left[\sum_{i=0}^{n-1}\left(X_{i,j}-xb_j\right)^2\right]\right] \qquad \lambda1_j := \frac{n\cdot\left(xb_j\right)^3}{(n-1)\cdot SA_j} \qquad \lambda m := \frac{\sum_{i=0}^{m-1}\lambda1_i}{m}$$

$$\lambda3_j := \frac{(n-3)}{\sum_{i=0}^{n-1}\left(\frac{1}{X_{i,j}}-\frac{1}{xb_j}\right)} \qquad \lambda ml := \frac{\sum_{i=0}^{m-1}\lambda3_i}{m}$$

$$a1_{i,j} := \int_{-\infty}^{\sqrt{\frac{\lambda}{X_{i,j}}}\cdot\left(1-\frac{X_{i,j}}{\mu}\right)} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\,dw$$

$$b1_{i,j} := \left[e^{\frac{2\cdot\lambda}{\mu}}\cdot\int_{-\infty}^{\sqrt{\frac{\lambda}{X_{i,j}}}\cdot\left(1+\frac{X_{i,j}}{\mu}\right)} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\,dw\right] \qquad RR_{i,j} := a1_{i,j}-b1_{i,j}$$

$$RR_j := \frac{\sum_{i=0}^{n-1} R_{i,j}}{n} \qquad Rtrue := \frac{\sum_{i=0}^{m-1} RR_i}{m} \qquad htrue := \frac{\sqrt{\frac{\lambda}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}\cdot e^{\frac{-\lambda\cdot(Xb-\mu)^2}{2\cdot\mu^2\cdot Xb}}}{Rtrue}$$

$$a2_{i,j} := \int_{-\infty}^{\sqrt{\frac{\lambda ml}{X_{i,j}}}\cdot\left(1-\frac{X_{i,j}}{Xb}\right)} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\,dw$$

$$b2_{i,j} := \left[e^{\frac{2\cdot\lambda ml}{Xb}} \cdot \int_{-\infty}^{\left[\sqrt{\frac{\lambda ml}{X_{i,j}}}\cdot\left(1+\frac{X_{i,j}}{Xb}\right)\right]} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\, dw \right] \qquad Rcab_{i,j} := a2_{i,j} - b2_{i,j}$$

$$RCAB_j := \frac{\sum_{i=0}^{n-1} Rcab_{i,j}}{n} \qquad Rmlm := \frac{\sum_{i=0}^{m-1} RCAB_i}{m} \qquad hmlm := \frac{\sqrt{\frac{\lambda ml}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}}{Rmlm}$$

$$a3_{i,j} := \int_{-\infty}^{\sqrt{\frac{\lambda m}{X_{i,j}}}\cdot\left(1-\frac{X_{i,j}}{Xb}\right)} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\, dw$$

$$b3_{i,j} := \left[e^{\frac{2\cdot\lambda m}{Xb}} \cdot \int_{-\infty}^{\sqrt{\frac{\lambda m}{X_{i,j}}}\cdot\left(1+\frac{X_{i,j}}{Xb}\right)} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\, dw \right] \qquad Rcab2_{i,j} := a3_{i,j} - b3_{i,j}$$

$$RCAB2_j := \frac{\sum_{i=0}^{n-1} Rcab2_{i,j}}{n} \qquad Rmm := \frac{\sum_{i=0}^{m-1} RCAB2_i}{m} \qquad hmm := \frac{\sqrt{\frac{\lambda m}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}}{Rmm}$$

$$a4_{i,j} := \int_{-\infty}^{\frac{-\left[\sqrt{n\cdot\lambda}\cdot\left(X_{i,j}-Xb\right)\right]}{\sqrt{\left(X_{i,j}\right)\cdot Xb\cdot\left|n\cdot Xb-X_{i,j}\right|}}} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-w^2}{2}}\, dw$$

$$b4_{i,j} := \left[\int_{-\infty}^{\frac{-\sqrt{\lambda}\left[n\cdot Xb+(n-2)\cdot X_{i,j}\right]}{\left[n\cdot X_{i,j}\cdot Xb\cdot\left(\left|n\cdot Xb-X_{i,j}\right|\right)\right]^{\frac{1}{2}}}} \left[\frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-w^2}{2}} \cdot \left[\left(\frac{n-2}{n}\right)\cdot\left[e^{\frac{2\cdot(n-1)\cdot\lambda}{n\cdot Xb}} \right] \right] \right] dw \right]$$

$$Rstar1_{i,j} := a4_{i,j} - b4_{i,j} \qquad RSTAR1_j := \frac{\left[\sum_{i=0}^{n-1}\left(Rstar1_{i,j}\right)\right]}{n} \qquad RBasu1 := \frac{\left[\sum_{i=0}^{m-1}\left(RSTAR1_i\right)\right]}{m}$$

$$hBasu1 := \frac{\sqrt{\frac{\lambda ml}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}}{RBasu1}$$

$$y_j := \sum_{i=1}^{n-1} \frac{(X_{i,j} - \mu)^2}{X_{i,j}} \qquad y1 := \frac{\sum_{i=0}^{n-1} y_i}{n} \qquad xfirst := XX_1 \qquad t := xfirst + y1$$

$$a5_{i,j} := \int_{-\infty}^{\frac{-\sqrt{n-1}\cdot(X_{i,j}-\mu)}{\sqrt{\left|t\cdot X_{i,j}-(X_{i,j}-\mu)^2\right|}}} \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi\,(n-1)}\cdot\Gamma\left(\frac{n-1}{2}\right)\cdot\left(1+\frac{w^2}{n-1}\right)^{\frac{n}{2}}}\,dw$$

$$b5_{i,j} := \left[\left(\frac{t+4\mu}{t}\right)^{\frac{n-2}{2}} \cdot \int_{-\infty}^{\frac{-\sqrt{n-1}\cdot(X_{i,j}+\mu)}{\sqrt{\left|t\cdot X_{i,j}-(X_{i,j}-\mu)^2\right|}}} \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi\,(n-1)}\cdot\Gamma\left(\frac{n-1}{2}\right)\cdot\left(1+\frac{w^2}{n-1}\right)^{\frac{n}{2}}}\,dw\right]$$

$$Rstar2_{i,j} := a5_{i,j} - b5_{i,j} \qquad RSTAR2_j := \frac{\sum_{i=0}^{n-1} Rstar2_{i,j}}{n} \qquad RBasu2 := \frac{\sum_{i=0}^{m-1} RSTAR2_i}{m}$$

$$hBasu2 := \frac{\sqrt{\frac{\lambda ml}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}}{RBasu2} \qquad V1_j := \sum_{i=0}^{n-1}\left(\frac{1}{X_{i,j}} - \frac{1}{Xb}\right) \qquad V := \frac{\sum_{i=0}^{m-1} V1_i}{m}$$

$$a6_{i,j} := \int_{-\infty}^{\frac{-\sqrt{n(n-2)}\cdot(X_{i,j}-Xb)}{\sqrt{\left|V\cdot X_{i,j}\cdot Xb\cdot(n\cdot Xb - X_{i,j}) - n\cdot(X_{i,j}-Xb)^2\right|}}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi\,(n-2)}\cdot\Gamma\left(\frac{n-2}{2}\right)\cdot\left(1+\frac{w^2}{n-2}\right)^{\frac{n-1}{2}}}\,dw$$

$$b6_{i,j} := \left[\left[\frac{4\cdot(n-1)}{n\cdot V\cdot Xb}+1\right]^{\frac{n-3}{2}} \cdot \left(\frac{n-2}{n}\right) \cdot \int_{-\infty}^{\frac{-\sqrt{n(n-2)}\cdot\left[Xb+\frac{(n-2)X_{i,j}}{n}\right]}{\sqrt{\left|V\cdot X_{i,j}\cdot Xb\cdot(n\cdot Xb - X_{i,j}) - n\cdot(X_{i,j}-Xb)^2\right|}}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi\,(n-2)}\cdot\Gamma\left(\frac{n-2}{2}\right)\cdot\left(1+\frac{w^2}{n-2}\right)^{\frac{n-1}{2}}}\,dw\right]$$

$$Rstar3_{i,j} := a6_{i,j} - b6_{i,j}$$

$$RSTAR3_j := \frac{\sum_{i=0}^{n-1} Rstar3_{i,j}}{n} \qquad RBasu3 := \frac{\sum_{i=0}^{m-1} RSTAR3_i}{m} \qquad hBasu3 := \frac{\sqrt{\frac{\lambda ml}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}}{RBasu3}$$

$$ER1 := \int_0^{\infty} \left[Xb\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasXb := ER1 - \mu$$

$$ER2 := \int_0^{\infty} \left[\lambda ml\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad Bias\lambda ml := ER2 - \lambda$$

$$ER3 := \int_0^{\infty} \left[\lambda m\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad Bias\lambda m := ER3 - \lambda$$

$$ER4 := \int_0^{\infty} \left[Rmlm\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasRmlm := ER4 - Rtrue$$

$$ER5 := \int_0^{\infty} \left[Rmm\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasRmm := ER5 - Rtrue$$

$$ER6 := \int_0^{\infty} \left[RBasu1\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasRBasu1 := ER6 - Rtrue$$

$$ER7 := \int_0^{\infty} \left[RBasu2\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasRBasu2 := ER7 - Rtrue$$

$$ER8 := \int_0^{\infty} \left[RBasu3\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasRBasu3 := ER8 - Rtrue$$

$$ER9 := \int_0^{\infty} \left[hmlm\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad Biashmlm := ER9 - htrue$$

$$ER10 := \int_0^{\infty} \left[hmm\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad Biashmm := ER10 - htrue$$

$$ER11 := \int_0^{\infty} \left[hBasu1 \cdot \left[\left(\frac{\lambda}{2 \cdot \pi \cdot t^3} \right)^{\frac{1}{2}} \cdot e^{\left[\frac{-\lambda \cdot (t-\mu)^2}{2 \cdot \mu^2 \cdot t} \right]} \right] \right] dt \qquad BiashBasu1 := ER11 - htrue$$

$$ER12 := \int_0^{\infty} \left[hBasu2 \cdot \left[\left(\frac{\lambda}{2 \cdot \pi \cdot t^3} \right)^{\frac{1}{2}} \cdot e^{\left[\frac{-\lambda \cdot (t-\mu)^2}{2 \cdot \mu^2 \cdot t} \right]} \right] \right] dt \qquad BiashBasu2 := ER12 - htrue$$

$$ER13 := \int_0^{\infty} \left[hBasu3 \cdot \left[\left(\frac{\lambda}{2 \cdot \pi \cdot t^3} \right)^{\frac{1}{2}} \cdot e^{\left[\frac{-\lambda \cdot (t-\mu)^2}{2 \cdot \mu^2 \cdot t} \right]} \right] \right] dt \qquad BiashBasu3 := ER13 - htrue$$

Xb = ▮ λm = ▮ λml = ▮
Rtrue = ▮
Rmlm = ▮ BiasRmlm = ▮ RBasu1 = ▮ BiasRBasu1 = ▮
Rmm = ▮ BiasRmm = ▮ RBasu2 = ▮ BiasRBasu2 = ▮
RBasu3 = ▮ BiasRBasu3 = ▮
htrue = ▮
hmlm = ▮ Biashmm = ▮ hmm = ▮ Biashmlm = ▮
hBasu1 = ▮ BiashBasu1 = ▮ hBasu2 = ▮ BiashBasu2 = ▮
hBasu3 = ▮ BiashBasu3 = ▮

Program(4): procedure (IG-1)

Enter your values of μ, λ, n and m

$\lambda :=$ ▮ $\mu :=$ ▮ $n :=$ ▮ $m :=$ ▮

$i := 0..n-1 \quad j := 0..m-1$

```
b[i,j] := | u1 ← rnd(1)
          | u2 ← rnd(1)
          | y ← −ln(u2)
          | while u1 > e^(−(y−1)^2 / 2)
          |    | u1 ← rnd(1)
          |    | u2 ← rnd(1)
          |    | y ← −ln(u2)
          | u3 ← rnd(1)
          | −y  if u3 < 1/2
          | y   if u3 > 1/2
```

```
X := | for j ∈ 0..m − 1
     |   for t ∈ 0..n − 1
     |     | y ← (b[t,j])^2
     |     | x1[t,j] ← μ + μ^2·y/(2·λ) − μ/(2·λ)·√(4·μ·λ·y + μ^2·y^2)
     |     | u ← rnd(1)
     |     | x[t,j] ← | x1[t,j]       if u ≤ μ/(μ + x1[t,j])
     |     |          | μ^2/x1[t,j]   otherwise
     |     | x
     | x
```

$X =$ ▮

$$XX_j := \frac{\sum_{i=0}^{n-1} X_{i,j}}{n} \qquad xb_j := \frac{1}{n}\cdot\sum_{i=0}^{n-1} X_{i,j} \qquad Xb := \frac{\sum_{i=0}^{m-1} xb_i}{m}$$

$$SA_j := \frac{1}{(n-1)}\cdot\left[\left[\sum_{i=0}^{n-1}\left(X_{i,j}-xb_j\right)^2\right]\right] \qquad \lambda 1_j := \frac{n\cdot\left(xb_j\right)^3}{(n-1)\cdot SA_j} \qquad \lambda m := \frac{\sum_{i=0}^{m-1}\lambda 1_i}{m}$$

$$\lambda 3_j := \frac{(n-3)}{\sum_{i=0}^{n-1}\left(\frac{1}{X_{i,j}}-\frac{1}{xb_j}\right)} \qquad \lambda ml := \frac{\sum_{i=0}^{m-1}\lambda 3_i}{m}$$

$$a1_{i,j} := \int_{-\infty}^{\sqrt{\frac{\lambda}{X_{i,j}}}\cdot\left(1-\frac{X_{i,j}}{\mu}\right)} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\,dw$$

$$b1_{i,j} := \left[e^{\frac{2\cdot\lambda}{\mu}}\cdot\int_{-\infty}^{\sqrt{\frac{\lambda}{X_{i,j}}}\cdot\left(1+\frac{X_{i,j}}{\mu}\right)} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\,dw\right] \qquad RR_{i,j} := a1_{i,j} - b1_{i,j}$$

$$RR_j := \frac{\sum_{i=0}^{n-1} R_{i,j}}{n} \qquad Rtrue := \frac{\sum_{i=0}^{m-1} RR_i}{m} \qquad htrue := \frac{\sqrt{\frac{\lambda}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}\cdot e^{\frac{-\lambda\cdot(Xb-\mu)^2}{2\cdot\mu^2\cdot Xb}}}{Rtrue}$$

$$a2_{i,j} := \int_{-\infty}^{\sqrt{\frac{\lambda ml}{X_{i,j}}}\cdot\left(1-\frac{X_{i,j}}{Xb}\right)} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\,dw$$

$$b2_{i,j} := \left[e^{\frac{2\cdot\lambda ml}{Xb}} \cdot \int_{-\infty}^{\left[\sqrt{\frac{\lambda ml}{X_{i,j}}}\cdot\left(1+\frac{X_{i,j}}{Xb}\right)\right]} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\,dw \right] \qquad Rcab_{i,j} := a2_{i,j} - b2_{i,j}$$

$$RCAB_j := \frac{\sum_{i=0}^{n-1} Rcab_{i,j}}{n} \qquad Rmlm := \frac{\sum_{i=0}^{m-1} RCAB_i}{m} \qquad hmlm := \frac{\sqrt{\frac{\lambda ml}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}}{Rmlm}$$

$$a3_{i,j} := \int_{-\infty}^{\sqrt{\frac{\lambda m}{X_{i,j}}}\cdot\left(1-\frac{X_{i,j}}{Xb}\right)} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\,dw$$

$$b3_{i,j} := \left[e^{\frac{2\cdot\lambda m}{Xb}} \cdot \int_{-\infty}^{\sqrt{\frac{\lambda m}{X_{i,j}}}\cdot\left(1+\frac{X_{i,j}}{Xb}\right)} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-1}{2}\cdot w^2}\,dw \right] \qquad Rcab2_{i,j} := a3_{i,j} - b3_{i,j}$$

$$RCAB2_j := \frac{\sum_{i=0}^{n-1} Rcab2_{i,j}}{n} \qquad Rmm := \frac{\sum_{i=0}^{m-1} RCAB2_i}{m} \qquad hmm := \frac{\sqrt{\frac{\lambda m}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}}{Rmm}$$

$$a4_{i,j} := \int_{-\infty}^{\frac{-\left[\sqrt{n\cdot\lambda}\cdot\left(X_{i,j}-Xb\right)\right]}{\sqrt{\left(X_{i,j}\right)\cdot Xb\cdot\left|n\cdot Xb-X_{i,j}\right|}}} \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-w^2}{2}}\,dw$$

$$b4_{i,j} := \left[\int_{-\infty}^{\frac{-\sqrt{\lambda}\left[n\cdot Xb+(n-2)\cdot X_{i,j}\right]}{\left[n\cdot X_{i,j}\cdot Xb\cdot\left(\left|n\cdot Xb-X_{i,j}\right|\right)\right]^{\frac{1}{2}}}} \left[\frac{1}{\sqrt{2\cdot\pi}}\cdot e^{\frac{-w^2}{2}}\cdot\left[\left(\frac{n-2}{n}\right)\cdot\left[e^{\frac{2\cdot(n-1)\cdot\lambda}{n\cdot Xb}}\right]\right]\right] dw \right]$$

$$Rstar1_{i,j} := a4_{i,j} - b4_{i,j} \qquad RSTAR1_j := \frac{\left[\sum_{i=0}^{n-1} \left(Rstar1_{i,j}\right)\right]}{n} \qquad RBasu1 := \frac{\left[\sum_{i=0}^{m-1} \left(RSTAR1_i\right)\right]}{m}$$

$$hBasu1 := \frac{\sqrt{\frac{\lambda ml}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}}{RBasu1}$$

$$y_j := \sum_{i=1}^{n-1} \frac{(X_{i,j}-\mu)^2}{X_{i,j}} \qquad y1 := \frac{\sum_{i=0}^{n-1} y_i}{n} \qquad xfirst := XX_1 \qquad t := xfirst + y1$$

$$a5_{i,j} := \int_{-\infty}^{\frac{-\sqrt{n-1}\cdot(X_{i,j}-\mu)}{\sqrt{\left|t\cdot X_{i,j}-(X_{i,j}-\mu)^2\right|}}} \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi\,(n-1)}\cdot\Gamma\left(\frac{n-1}{2}\right)\cdot\left(1+\frac{w^2}{n-1}\right)^{\frac{n}{2}}}\,dw$$

$$b5_{i,j} := \left[\left(\frac{t+4\mu}{t}\right)^{\frac{n-2}{2}}\cdot\int_{-\infty}^{\frac{-\sqrt{n-1}\cdot(X_{i,j}+\mu)}{\sqrt{\left|t\cdot X_{i,j}-(X_{i,j}-\mu)^2\right|}}} \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi\,(n-1)}\cdot\Gamma\left(\frac{n-1}{2}\right)\cdot\left(1+\frac{w^2}{n-1}\right)^{\frac{n}{2}}}\,dw\right]$$

$$Rstar2_{i,j} := a5_{i,j} - b5_{i,j} \qquad RSTAR2_j := \frac{\sum_{i=0}^{n-1} Rstar2_{i,j}}{n} \qquad RBasu2 := \frac{\sum_{i=0}^{m-1} RSTAR2_i}{m}$$

$$hBasu2 := \frac{\sqrt{\frac{\lambda ml}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}}{RBasu2} \qquad V1_j := \sum_{i=0}^{n-1}\left(\frac{1}{X_{i,j}}-\frac{1}{Xb}\right) \qquad V := \frac{\sum_{i=0}^{m-1} V1_i}{m}$$

$$a6_{i,j} := \int_{-\infty}^{\frac{-\sqrt{n(n-2)}\cdot(X_{i,j}-Xb)}{\sqrt{\left|V\cdot X_{i,j}\cdot Xb\cdot(n\cdot Xb-X_{i,j})-n\cdot(X_{i,j}-Xb)^2\right|}}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi\,(n-2)}\cdot\Gamma\left(\frac{n-2}{2}\right)\cdot\left(1+\frac{w^2}{n-2}\right)^{\frac{n-1}{2}}}\,dw$$

$$b6_{i,j} := \left[\left[\frac{4\cdot(n-1)}{n\cdot V\cdot Xb}+1\right]^{\frac{n-3}{2}}\cdot\left(\frac{n-2}{n}\right)\cdot\int_{-\infty}^{\frac{-\sqrt{n(n-2)}\cdot\left[Xb+\frac{(n-2)X_{i,j}}{n}\right]}{\sqrt{\left|V\cdot X_{i,j}\cdot Xb\cdot(n\cdot Xb-X_{i,j})-n\cdot(X_{i,j}-Xb)^2\right|}}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi\,(n-2)}\cdot\Gamma\left(\frac{n-2}{2}\right)\cdot\left(1+\frac{w^2}{n-2}\right)^{\frac{n-1}{2}}}\,dw\right]$$

$$Rstar3_{i,j} := a6_{i,j} - b6_{i,j}$$

$$RSTAR3_j := \frac{\sum_{i=0}^{n-1} Rstar3_{i,j}}{n} \quad RBasu3 := \frac{\sum_{i=0}^{m-1} RSTAR3_i}{m} \quad hBasu3 := \frac{\sqrt{\frac{\lambda ml}{2\cdot\pi}}\cdot Xb^{\frac{-3}{2}}}{RBasu3}$$

$$ER1 := \int_0^{\infty} \left[Xb\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasXb := ER1 - \mu$$

$$ER2 := \int_0^{\infty} \left[\lambda ml\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad Bias\lambda ml := ER2 - \lambda$$

$$ER3 := \int_0^{\infty} \left[\lambda m\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad Bias\lambda m := ER3 - \lambda$$

$$ER4 := \int_0^{\infty} \left[Rmlm\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasRmlm := ER4 - Rtrue$$

$$ER5 := \int_0^{\infty} \left[Rmm\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasRmm := ER5 - Rtrue$$

$$ER6 := \int_0^{\infty} \left[RBasu1\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasRBasu1 := ER6 - Rtrue$$

$$ER7 := \int_0^{\infty} \left[RBasu2\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasRBasu2 := ER7 - Rtrue$$

$$ER8 := \int_0^{\infty} \left[RBasu3\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad BiasRBasu3 := ER8 - Rtrue$$

$$ER9 := \int_0^{\infty} \left[hmlm\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad Biashmlm := ER9 - htrue$$

$$ER10 := \int_0^{\infty} \left[hmm\cdot\left[\left(\frac{\lambda}{2\cdot\pi\cdot t^3}\right)^{\frac{1}{2}}\cdot e^{\left[\frac{-\lambda\cdot(t-\mu)^2}{2\cdot\mu^2\cdot t}\right]}\right]\right] dt \qquad Biashmm := ER10 - htrue$$

$$ER11 := \int_0^{\infty} \left[hBasu1 \cdot \left[\left(\frac{\lambda}{2 \cdot \pi \cdot t^3} \right)^{\frac{1}{2}} \cdot e^{\left[\frac{-\lambda \cdot (t-\mu)^2}{2 \cdot \mu^2 \cdot t} \right]} \right] \right] dt$$

$$BiashBasu1 := ER11 - htrue$$

$$ER12 := \int_0^{\infty} \left[hBasu2 \cdot \left[\left(\frac{\lambda}{2 \cdot \pi \cdot t^3} \right)^{\frac{1}{2}} \cdot e^{\left[\frac{-\lambda \cdot (t-\mu)^2}{2 \cdot \mu^2 \cdot t} \right]} \right] \right] dt$$

$$BiashBasu2 := ER12 - htrue$$

$$ER13 := \int_0^{\infty} \left[hBasu3 \cdot \left[\left(\frac{\lambda}{2 \cdot \pi \cdot t^3} \right)^{\frac{1}{2}} \cdot e^{\left[\frac{-\lambda \cdot (t-\mu)^2}{2 \cdot \mu^2 \cdot t} \right]} \right] \right] dt$$

$$BiashBasu3 := ER13 - htrue$$

Xb = ∎ λm = ∎ λml = ∎
Rtrue = ∎
Rmlm = ∎ BiasRmlm = ∎ RBasu1 = ∎ BiasRBasu1 = ∎
Rmm = ∎ BiasRmm = ∎ RBasu2 = ∎ BiasRBasu2 = ∎
RBasu3 = ∎ BiasRBasu3 = ∎
htrue = ∎
hmlm = ∎ Biashmm = ∎ hmm = ∎ Biashmlm = ∎
hBasu1 = ∎ BiashBasu1 = ∎ hBasu2 = ∎ BiashBasu2 = ∎
hBasu3 = ∎ BiashBasu3 = ∎

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم
قسم الرياضيات و تطبيقات الحاسوب

تخمين دالة المعولية لأنموذج التوزيع الكاوسي العكسي مع تطبيق بإستخدام محاكاة مونت كارلو العشوائية

رسالة

مقدمة إلى كلية العلوم - جامعة النهرين وهي جزء من متطلبات نيل درجة ماجستير في علوم الرياضيات التطبيقية

مِنْ قِبَل

عمر إسماعيل خليل التميمي

(بكالوريوس علوم، جامعة النهرين، 2005)

بأشراف

د. أكرم محمد العبود

جمادى الاولى 1429
آيار 2008