Republic of Iraq
Ministry of Higher Education
and Scientific Research
Al-Nahrain University
College of Science
Department of Mathematics
and Computer Applications

Taylor Expansion Method for Solving the Non-Linear Integral and Integro-Differential Equations

A Thesis
Submitted to the Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By
Huda Abdul-Razaq Mohammad
(B.Sc. Al-Nahrain University, 2005)

Supervised by
Assist.Prof.Dr. Ahlam Jameel Khaleel

Ramadan 1429
September 2008

بِســــمِ ٱللهِ ٱلرَحَمـــنْ ٱلرَحيــمْ

﴿ فَتَعَالَى ٱللهُ ٱلمَلِكُ ٱلحَقُ وَلَا تَعجَل بِٱلقُرءَانِ

مِن قَبلِ أَن يُقضَى إِلَيكَ وَحيُهُ وَ قُل رَبِي

زِدنِي عِلمًا ﴾

صَـــدَقَ ٱللهُ ٱلعَظيْــم

(سورة طــه ، الآية ١١٤)

ألإهِداء

إلى السَفينةِ التي عَبرت بيْ بَرَ الأمانْ...أبي

إلى الشَمعةِ التي أضاءت ليْ دَربيْ...أمي

إلى الأصدِقاء ألذينَ يُعينونَني على أهوالَ

ألزمانْ...إخوَتي

إلى الإنسان الذي يُغرقُني في بحَرِ العِلمَ

والآمالْ...أستاذتي

هُدى الجنابي

Acknowledgements

Thanks to Allah for his help and for giving me the ability to complete this work.

I would like to express my deep appreciation and gratitude to my supervisor Dr. Ahlam J. Khaleel for her continuous help and advice during the preparation of this work.

I'm deeply indebted to the Department of Mathematics and to the College of Science, Al-Nahrain University for giving me the chance to complete my study. Also, I would like to express my thanks to my family and all my friends who help me.

Huda Abdul-Razaq

September, 2008

Supervisor Certification

I certify that this thesis was prepared under my supervision at the Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

Signature:
Name: Dr. Ahlam J. Khaleel
Address: Assistant Professor
Date: / / 2008

In view of the available recommendations, I forward this thesis for debate by the examining committee.

Signature:
Name: Dr. Akram M. Al-Abood
Address: Assistant Professor
Head of the Department of Mathematics and Computer Applications
Date: / / 2008

Examining Committee Certification

We certify that we have read this thesis entitled "**Taylor Expansion Method for Solving the Non-Linear Integral and Integro-Differential Equations"** an as examining committee examined the student (**Huda Abdul-Razaq Al-Janaby)** in its contents and in what it connected with, and that is in our opinion it meet the standards of a thesis for the degree of Master of Science in Mathematics.

(Chairman)	***(Member)***
Signature:	***Signature:***
Name: Dr. Akram M. Al-Abood	***Name: Dr. Salowa Salman***
Date:	***Date:***
(Member)	***(Member and Supervisor)***
Signature:	***Signature:***
Name: Dr. Zeinab A. Salman	***Name: Dr. Ahlam J. Khaleel***
Date:	***Date:***

Approved by the College of Science

Signature:

Name: Dr. Laith Abdul Azizz Al-Ani

Dean of the College of Science,

Al-Nahrain University

Date: / / 2008

Abstract

The main purpose of this work is to study Volterra-Fredholm integral and integro-differential equations.

This study include the classification of Volterra-Fredholm integral and integro-differential equations.

Also, some theorems for the existence and uniqueness of the solution for linear Volterra-Freadholm integral and integro-differential equations are presented.

Moreover, Taylor expansion method for solving special types of non-linear Volterra-Freadholm integral and integro-differential equations with some illustrate examples are discussed.

CONTENTS

Introduction

The integral and integro-differential equations are encountered in various fields of science and numerous applications say in elasticity, plasticity, heat, mass transfer, oscillation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering, economics and medicine, [Jerri A., 1985].

Recall that the one-dimensional integral equation is an equation in which the integration is carried out with respect to one independent variable, [Delves L. and Walsh J., 1974].

Many researchers and authors studied the one-dimensional integral equations say, [Hochstadt H., 1973] discussed the existence of the unique solution for the one-dimensional non-linear integral equations, [Delves L. and Walsh J., 1974] gave some numerical solutions for the one-dimensional integral equations, [Jerri A., 1985] gave some approximated methods for solving the one-dimensional integral equations with some real life applications, [Al-Shather A., 1999] studied the one-dimensional singular integral equations, [Al-Shakry A., 2001] descried the one-dimensional delay integral equations with their solutions, [Najieb S., 2002] studied the one-dimensional fuzzy integral equations, [Al-Shather A., 2003] introduced some approximated solutions for solving the one-dimensional fractional integral equations with or without delays, [Mustafa M., 2004] devoted the numerical solutions for systems of the one-dimensional integral equations via spline functions, [Al-Jawary M., 2005] used some numerical methods for solving systems of the one-dimensional linear Volterra integral equations, [Ibrahim G., 2005] used some numerical methods for solving systems of the one-dimensional linear Fredholm integral equations, [Abdul-Jabbar R., 2005] presented the inverse problem for the one-dimensional fractional integral equations.

Also, many researchers and authors studied the integro-differential equations say, [Choi M., 1993] studied the collocation method to solve integro-differential equations in which memory kernels have a singularity at $t = 0$, [Al-Timeme A., 2003] studied the resolvent kernel method for solving linear Volterra integro-differential equations, [Yaslan H. and Dascioglu A., 2006] developed a Chebyshev collocation method to find the approximated solutions for non-linear Volterra-Fredholm integro-differential equations.

Recall that the multi-dimensional integral equation is an equation in which the integrations are carried out with respect to multiple independent variables, [Ladopoulas G., 1988]. Many researchers concerned with the multi-dimensional integral equations say, [Ladopoulas G., 1988] gave some numerical solutions of the multi-dimensional singular integral equations namely, the quadrature methods, [Xu Y. and Zhou A., 2004] used the collocation method for solving the multi-dimensional integral equations, [Cardone A., Mession E. and Russa E., 2006] used some iterative methods for solving the two-dimensional Volterra-Fredholm integral equations namely, Numman series method and successive method, [Al-Niamey L., 2006] used the degenerate kernel method for solving the multi-dimensional integral equations.

The aim of this work is to devote Volterra-Fredholm integral and integro-differential equations with their classification. Also, some theorems that grantee the existence of a unique solution for the linear Volterra-Fredholm integral and integro-differential equations are presented. Moreover, special types of non-linear Volterra-Fredholm integral and integro-differential equations via Taylor expansion method is described.

This thesis consists of three chapters.

In chapter one, we classify the integral and integro-differential equations into linear/nonlinear, Fredholm/Volterra/Volterra-Fredholm, first kind/second kind, homogeneous/nonhomogeneous and singular/nonsingular. Also, some existence and uniqueness theorems of the solution for the non-singular linear Volterra-Fredholm integral and integro-differential equations are given. Moreover, some real life applications of integral and integro-differential equations are illustrated.

In chapter two, we use Taylor expansion method to solve the linear Fredholm, Volterra and Volterra-Fredholm integral and integro-differential equations.

In chapter three, we use Taylor expansion method to solve special types of the nonlinear integral and integro-differential equations. These types are in which their kernel are product of two functions, the first function depends on two variables (the independent variable and its dummy variable) and the second function is a power function for the unknown function of these integral and integro-differential equations.

Introduction:-

Recall that the integral equations are important not only to mathematicians but also may be used to model many real life applications in physics, biology and engineering, [Hochstadt H., 1973].

This chapter concerns with the integral and integro-differential equations with their classification and some real life applications. Also, the existence and uniqueness theorems of the solution for special types of the non-singular integral and integro-differential equations.

This chapter consist of three sections.

In section one, we gave a classification of the integral and integro-differential equation. Also, we gave some types of the singular integral equations.

In section two, we give two theorems for the existence of the unique solution for the linear Volterra-Fredholm integral and integro-differential equations.

In section three, we gave some real life applications for the integral and integro-differential equations.

1.1 Classification of Integral and Integro-Differential Equations:-

An integral equation is an equation in which the unknown function appears under one or more integral signs. The integral equations occur naturally in many fields of mechanics and mathematical physics, [Jerri A., 1985]. They also arise as representation formulas for the solution of differential equations. Indeed, a differential equation can be replaced by an integral equation which incorporates it's boundary conditions, [Tricami F., 1985].

An integral equation is called linear in $u(x)$ if no nonlinear functions of the unknown function $u(x)$ are involved otherwise it is nonlinear, [Chambers L., 1976].

The general form of the linear integral equation that contains n integral operators is

$$h(x)u(x) = g(x) + \sum_{i=1}^{n} \lambda_i \int_a^{b_i(x)} k_i(x,t)u(t)\,dt \qquad (1.1)$$

where g, h, b_i and k_i are known functions. The function g is said to be the driving term and k_i is said to be the i-th kernel function that depends on x, t, λ_i is a scalar parameter, a is a known constant and u is the unknown function that must be determined.

On the other hand, the general form of the nonlinear integral equation is

$$h(x)u(x) = g(x) + \sum_{i=1}^{n} \lambda_i \int_a^{b_i(x)} k_i(x,t,u(t))\,dt \qquad (1.2)$$

where g, h, b_i, a, λ_i are defined similar to the previous and k_i is the i-th kernel function that depends on x, t and $u(t)$.

An integro-differential equation is an equation in which the unknown function appears under integral and derivative signs. The integro-differential equations are the natural mathematical model for representing a physically interesting situation, [Linz P., 1985].

The general form of the m-th order linear integro-differential equation that contains n integral equations is

$$\sum_{i=0}^{m} h_i(x)u^{(i)}(x) = g(x) + \sum_{i=1}^{n} \lambda_i \int_a^{b_i(x)} k_i(x,t)u(t)\,dt \qquad (1.3)$$

where g, h_i, b_i and k_i are known functions such that $h_m \neq 0$. The function g is said to be the driving term and k_i is said to be the i-th kernel function, λ_i is a scalar parameter, a is a known constant and u is the unknown function that must be determined.

On the other hand, the general form of the m-th order nonlinear integro-differential equation that contains n integral operators is

$$\sum_{i=0}^{m} h_i(x)u^{(i)}(x) = g(x) + \sum_{i=1}^{n} \lambda_i \int_a^{b_i(x)} k_i(x,t,u(t))\,dt \qquad (1.4)$$

where g, h_i, b_i, a, λ_i, u are defined similar to the previous and k_i is the i-th kernel function that depends on x, t and $u(t)$.

The following equations are special types of equations (1.1)-(1.4) that are of main interest:

(I) The Linear Integral and Integro-Differential Equations:-

The linear integral and integro-differential equations that contain n linear integral operators can be divided into three kinds:-

(1) The Linear Fredholm Integral and Integro-Differential Equations:-

If $b_i(x) = b_i$, where b_i is a known constant for each $i = 1, 2, \ldots, n$ then equation (1.1) and equation (1.3) are said to be the general form of the linear Fredholm integral equation and the general form of the m-th order linear Fredholm integro-differential equation respectively and take the following forms:-

$$h(x)u(x) = g(x) + \sum_{i=1}^{n} \lambda_i \int_a^{b_i} k_i(x,t)u(t)\,dt\,,\ a \le x \le \max_{1\le i\le n}\{b_i\} \qquad (1.5)$$

and

$$\sum_{i=0}^{m} h_i(x)u^{(i)}(x) = g(x) + \sum_{i=1}^{n} \lambda_i \int_a^{b_i} k_i(x,t)u(t)\,dt\,,\ a \le x \le \max_{1\le i\le n}\{b_i\} \qquad (1.6)$$

respectively.

These integral and integro-differential equations can be divided into the following types:-

(a) If $h(x) = 0$ for each $a \le x \le \max_{1\le i\le n}\{b_i\}$ then equation (1.5) becomes

$$g(x) = -\sum_{i=1}^{n} \lambda_i \int_a^{b_i} k_i(x,t)u(t)dt\,,\ a \le x \le \max_{1\le i\le n}\{b_i\}$$

and it is said to be the linear Fredholm integral equation of the first kind.

(b) If $h(x) = 1$ for each $a \le x \le \max_{1\le i\le n}\{b_i\}$ then equation (1.5) becomes

$$u(x) = g(x) + \sum_{i=1}^{n} \lambda_i \int_a^{b_i} k_i(x,t)u(t)dt\,,\ a \le x \le \max_{1\le i\le n}\{b_i\}$$

and it is said to be the linear Fredholm integral equation of the second kind. Also, if equation (1.6) takes the form:

$$\sum_{i=0}^{m} h_i(x)u^{(i)}(x) = g(x) + \sum_{i=1}^{n} \lambda_i \int_a^{b_i} k_i(x,t)u(t)dt\,,\ a \le x \le \max_{1\le i\le n}\{b_i\}$$

then it is said to be the m-th order linear Fredholm integro-differential equation of the second kind.

(c) If $g(x) = 0$ for each $a \le x \le \max_{1\le i\le n}\{b_i\}$ then equations (1.5) and (1.6) becomes

$$h(x)u(x) = \sum_{i=1}^{n} \lambda_i \int_a^{b_i} k_i(x,t)u(t)dt\,,\ a \le x \le \max_{1\le i\le n}\{b_i\}$$

and

$$\sum_{i=0}^{m} h_i(x)u^{(i)}(x) = \sum_{i=1}^{n} \lambda_i \int_a^{b_i} k_i(x,t)u(t)dt\,,\ a \le x \le \max_{1\le i\le n}\{b_i\}$$

are said to be the homogeneous linear Fredholm integral and integro-differential equations respectively.

(2) The Linear Volterra Integral and Integro-Differential Equations:-

If $b_i(x) = x$ for some $i \in \{1, 2, \ldots, n\}$ then equation (1.1) and equation (1.3) are said to be the general form of the linear Volterra integral equation and the general form of the m-th order linear Volterra integro-differential equation respectively. Therefore

$$g(x) = -\lambda \int_a^{x} k(x,t)u(t)dt,\ \ x \ge a$$

and

$$u(x) = g(x) + \lambda \int_a^{x} k(x,t)u(t)dt\,,\ x \ge a$$

are said to be the linear Volterra integral equations of the first and second kinds respectively. Morever,

$$h(x)u(x) = \lambda \int_a^x k(x,t)u(t)dt\ ,\ x \ge a$$

is said to be the homogeneous linear Volterra integral equation.

On the other hand,

$$\sum_{i=0}^{m} h_i(x)u^{(i)}(x) = g(x) + \lambda \int_a^x k(x,t)u(t)dt\ ,\ x \ge a$$

is said to be the m-th order linear Volterra integro-differential equation of the second kind. Also,

$$\sum_{i=0}^{m} h_i(x)u^{(i)}(x) = \lambda \int_a^x k(x,t)u(t)dt\ ,\ x \ge a$$

is said to be the m-th order homogeneous linear Volterra integro-differential equation.

(3) The Linear Volterra-Fredholm Integral and Integro-Differential Equations:-

If $n = 2$, $b_1(x) = x$ and $b_2(x) = b$ where b is a known constant then equation (1.1) and equation (1.3) are said to be the general form of the linear Volterra-Fredholm integral equation and the general form of the m-th order linear Volterra-Fredholm integro-differential equation respectively. Therefore

$$g(x) = -\lambda_1 \int_a^x k_1(x,t)u(t)dt - \lambda_2 \int_a^b k_2(x,t)u(t)dt\ ,\ a \le x \le b$$

and

$$u(x) = g(x) + \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt\,,\ a \le x \le b$$

are said to be the linear Volterra-Fredholm integral equations of the first and second kinds respectively. Moreover,

$$h(x)u(x) = \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt\,,\ a \le x \le b$$

is said to be the homogeneous linear Volterra-Fredholm integral equation.

On the other hand,

$$\sum_{i=0}^{m} h_i(x)u^{(i)}(x) = g(x) + \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt\,,\ a \le x \le b$$

is said to be the m-th order linear Volterra-Fredholm integro-differential equation of the second kind. Also,

$$\sum_{i=0}^{m} h_i(x)u^{(i)}(x) = \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt\,,\ a \le x \le b$$

is said to be the m-th order homogeneous linear Volterra-Fredholm integro-differential equation.

(II) The Nonlinear Integral and Integro-Differential Equations:-

The nonlinear integral and integro-differential equations that contain n nonlinear integral operators can be divided into three types:-

(1) The Nonlinear Fredholm Integral and Integro-Differential Equations:-

If $b_i(x) = b_i$, where b_i is a known constant for each $i = 1, 2, \ldots, n$ then equation (1.2) and equation (1.4) are said to be the general form of the nonlinear Fredholm integral equation and the general form of the m-th order nonlinear

Fredholm integro-differential equation respectively and take the following forms:-

$$h(x)u(x) = g(x) + \sum_{i=1}^{n} \lambda_i \int_a^{b_i} k_i(x,t,u(t))\ dt\,,\ a \le x \le \max_{1\le i\le n}\{b_i\} \qquad (1.7)$$

and

$$\sum_{i=0}^{m} h_i(x)u^{(i)}(x) = g(x) + \sum_{i=1}^{n} \lambda_i \int_a^{b_i} k_i(x,t,u(t))\ dt\,,\ a \le x \le \max_{1\le i\le n}\{b_i\} \qquad (1.8)$$

respectively.

These integral and integro-differential equations can be divided into the following types:-

(a) If $h(x) = 0$ for each $a \le x \le \max_{1\le i\le n}\{b_i\}$ then equation (1.7) becomes

$$g(x) = -\sum_{i=1}^{n} \lambda_i \int_a^{b_i} k_i(x,t,u(t))dt,\ \ a \le x \le \max_{1\le i\le n}\{b_i\}$$

and it is said to be the nonlinear Fredholm integral equation of the first kind.

(b) If $h(x) = 1$ for each $a \le x \le \max_{1\le i\le n}\{b_i\}$ then equation (1.7) becomes

$$u(x) = g(x) + \sum_{i=1}^{n} \lambda_i \int_a^{b_i} k_i(x,t,u(t))\,dt\,,\ a \le x \le \max_{1\le i\le n}\{b_i\}$$

and it is said to be the nonlinear Fredholm integral equation of the second kind. Also, if equation (1.8) takes the form:

$$\sum_{i=0}^{m} h_i(x)\,u^{(i)}(x) = g(x) + \sum_{i=1}^{n} \lambda_i \int_a^{b_i} k_i(x,t,u(t))dt\,,\ a \le x \le \max_{1\le i\le n}\{b_i\}$$

then it is said to be the m-th order nonlinear Fredholm integro-differential equation of the second kind.

(c) If $g(x) = 0$ for each $a \le x \le \max_{1\le i\le n}\{b_i\}$ then equations (1.7) and (1.8) becomes

$$h(x)u(x)=\sum_{i=1}^{n}\lambda_i\int_a^{b_i}k_i(x,t,u(t))dt\,,\ a\le x\le \max_{1\le i\le n}\{b_i\}$$

and

$$\sum_{i=0}^{m}h_i(x)u^{(i)}(x)=\sum_{i=1}^{n}\lambda_i\int_a^{b_i}k_i(x,t,u(t))dt\,,\ a\le x\le \max_{1\le i\le n}\{b_i\}$$

are said to be the homogeneous nonlinear Fredholm integral and integro-differential equations respectively.

(2) The Nonlinear Volterra Integral and Integro-Differential Equations:-

If $b_j(x)=x$ for some $j\in\{1,2,\ldots,n\}$ then equation (1.2) and equation (1.4) are said to be the general form of the nonlinear Volterra integral equation and the general form of the m-th order nonlinear Volterra integro-differential equation respectively. Therefore

$$g(x)=-\lambda\int_a^{x}k(x,t,u(t))dt,\ \ x\ge a$$

and

$$u(x)=g(x)+\lambda\int_a^{x}k(x,t,u(t))\,dt\,,\ x\ge a$$

are said to be the nonlinear Volterra integral equations of the first and second kinds respectively. Morever,

$$h(x)u(x)=\lambda\int_a^{x}k(x,t,u(t))dt\,,\ x\ge a$$

is said to be the homogeneous nonlinear Volterra integral equation.

On the other hand,

$$\sum_{i=0}^{m} h_i(x)u^{(i)}(x) = g(x) + \lambda\int_a^x k(x,t,u(t))dt\,,\ x \ge a$$

is said to be the m-th order nonlinear Volterra integro-differential equation of the second kind. Also,

$$\sum_{i=0}^{m} h_i(x)u^{(i)}(x) = \lambda\int_a^x k(x,t,u(t))dt\,,\ x \ge a$$

is said to be the m-th order homogeneous nonlinear Volterra integro-differential equation.

(3) The Nonlinear Volterra-Fredholm Integral and Integro-Differential Equations:-

If $n = 2, b_1(x) = x$ and $b_2(x) = b$ where b is a known constant then equation (1.2) and equation (1.4) are said to be the general form of the nonlinear Volterra-Fredholm integral equation and the general form of the m-th order nonlinear Volterra-Fredholm integro-differential equation respectively. Therefore

$$g(x) = -\lambda_1\int_a^x k_1(x,t,u(t))dt - \lambda_2\int_a^b k_2(x,t,u(t))dt\,,\ a \le x \le b$$

and

$$u(x) = g(x) + \lambda_1\int_a^x k_1(x,t,u(t))\,dt + \lambda_2\int_a^b k_2(x,t,u(t))\,dt\,,\ a \le x \le b$$

are said to be the nonlinear Volterra-Fredholm integral equations of the first and second kinds respectively. Moreover,

$$h(x)u(x) = \lambda_1\int_a^x k_1(x,t,u(t))dt + \lambda_2\int_a^b k_2(x,t,u(t))dt\,,\ a \le x \le b$$

is said to be the homogeneous nonlinear Volterra-Fredholm integral equation.

On the other hand,

$$\sum_{i=0}^{m} h_i(x)u^{(i)}(x) = g(x) + \lambda_1 \int_a^x k_1(x,t,u(t))dt + \lambda_2 \int_a^b k_2(x,t,u(t))dt\,,\ a \le x \le b$$

is said to be the m-th order nonlinear Volterra-Fredholm integro-differential equation of the second kind. Also,

$$\sum_{i=0}^{m} h_i(x)u^{(i)}(x) = \lambda_1 \int_a^x k_1(x,t,u(t))dt + \lambda_2 \int_a^b k_2(x,t,u(t))dt\,,\ a \le x \le b$$

is said to be the m-th order homogeneous nonlinear Volterra-Fredholm integro-differential equation.

Remark (1.1):-

The integral and integro-differential equations in which the range of integration is infinite, or in which the kernel $k_i(x,t)$ is discontinuous for some $i \in \{1, 2, \cdots, n\}$ are called singular integral and integro-differential equations otherwise are called nonsingular. For examples, the integral and integro-differential equations

$$g(x) = \int_0^\infty \sin(xt)u(t)dt\,,$$

$$u'(x) = 3x + \int_0^\infty e^{-xt}\, u(t)\, dt,$$

$$g(x) = \int_0^x \frac{u(t)}{\sqrt{x-t}} dt$$

and

$$u'(x) = 5x^2 + \int_0^x \frac{u(t)}{x-t}\,dt$$

are singular integral and integro-differential equations.

If $k_i(x,t) = \cot\left(\frac{t-x}{2}\right), i \in \{1,2,\cdots,n\}$

in equation (1.1) and equation (1.3) then these equations are said to be linear Fredholm singular integral and integro-differential equations with Hilbert kernel.

Also, if $k_i(x,t) = \dfrac{g(x,t)}{(x-t)^\alpha}$, $\alpha > 0$, $i \in \{1,2,\cdots,n\}$ (1.9)

where g is a known function of x and t, then equation (1.1) and equation (1.3) are said to be linear Fredholm singular integral and integro-differential equations with weakly singular kernel.

Notice that, if $\alpha = 1$ in equation (1.9) then equation (1.1) and equation (1.3) are said to be linear Fredholm singular integral and integro-differential equations with Cauchy kernel.

1.2 Existence and Uniqueness Theorems of the Solution for Linear Volterra-Fredholm Integral and Integro-Differential Equations:-

In this section, we give two theorems for the existence of a unique solution of linear Volterra-Freadholm integral and integro-differential equations. These theorems are generalization of the theorems that appeared in [Chambes L., 1976].

Theorem (1.1):-

Consider the linear Volterra-Fredholm integral equation of the second kind:

$$u(x) = g(x) + \lambda_1 \int_a^x k_1(x,t)u(t)\,dt + \sum_{i=2}^{n} \lambda_i \int_a^{b_i} k_i(x,t)u(t)\,dt,\; a \le x \le b = \max_{2 \le i \le n} \{b_i\} \tag{1.10}$$

If $|k_1(x,t)| \le L_1$ for each $a \le t \le x \le b$, $|k_i(x,t)| \le L_i$ for each $a \le t, x \le b$, $i = 2,3,\ldots,n$ and $\sum_{i=1}^{n} |\lambda_i| L_i (b-a)$ then equation (1.10) has a unique solution for each continuous function g.

Proof:-

By rewriting the above equation in an operator equation one can have:

$u = Tu$

where $Tu = g(x) + \lambda_1 \int_a^x k_1(x,t)u(t)\,dt + \sum_{i=2}^{n} \lambda_i \int_a^{b_i} k_i(x,t)u(t)\,dt$

It is known that the set of all continuous functions defined on the interval $[a,b]$ is a complete metric space with the following distance

$$d(u_1,u_2) = \sup_{a \le x \le b} |u_1(x) - u_2(x)|$$

Next, we show that T is a contraction mapping. To do this, consider

$d(Tu_1,Tu_2) =$

$$\sup_{a \le x \le b} \left| \lambda_1 \int_a^x k_1(x,t)u_1(t)dt + \sum_{i=2}^{n} \lambda_i \int_a^{b_i} k_i(x,t)u_1(t)dt - \lambda_1 \int_a^x k_1(x,t)u_2(t)dt - \sum_{i=2}^{n} \lambda_i \int_a^{b_i} k_i(x,t)u_2(t)dt \right|$$

$$= \sup_{a \le x \le b} \left| \lambda_1 \int_a^x k_1(x,t)[u_1(t) - u_2(t)]dt + \sum_{i=2}^{n} \lambda_i \int_a^{b_i} k_i(x,t)[u_1(t) - u_2(t)]dt \right|$$

Therefore

$$d(Tu_1,Tu_2) \le \sup_{a\le x\le b}|u_1(x)-u_2(x)|\left[|\lambda_1|L_1\int_a^x dt + \sum_{I=2}^{n}|\lambda_i|L_i\int_a^{b_i} dt\right]$$

$$= \sup_{a\le x\le b}|u_1(x)-u_2(x)|\left[|\lambda_1|L_1(x-a) + \sum_{i=2}^{n}|\lambda_i|L_i(b_i-a)\right]$$

$$\le d(u_1,u_2)\sum_{i=1}^{n}|\lambda_i|L_i(b-a).$$

But $\sum_{i=1}^{n}|\lambda_i|L_i(b-a)<1.$ Thus T is a contraction mapping. By using Banach fixed point theorem one can have T has a unique fixed point and hence equation (1.10) has a unique solution.

To illustrate this theorem, consider the following example.

Example (1.1):-

Consider the linear Volterra-Fredholm integral equation of the second kind:

$$u(x) = g(x) + \int_0^x 3xt\,u(t)dt + \int_0^{\frac{1}{2}} x^2 \sin t\, u(t)dt$$

$|k_1(x,t)| = |3xt| = 3|x||t| \le \frac{3}{4}$, $\quad |k_2(x,t)| = |x^2\sin t| = |x^2||\sin t| \le \frac{1}{4}$. Therefore $L_1 = \frac{3}{4}$ and $L_2 = \frac{1}{4}$. But $\lambda_1 = \lambda_2 = 1.$ Hence

$$\sum_{i=1}^{2}\lambda_i L_i(b-a) = \left(\frac{3}{4}+\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{1}{2} < 1.$$

Thus by using theorem (1.1), the above equation has a unique solution for each continuous function g.

Theorem (1.2):-

Consider the first order linear Volterra-Fredholm integro-differential equation:

$$u'(x) = g(x) + \lambda_1 \int_a^x k_1(x,t)u(t)dt + \sum_{i=2}^{n} \lambda_i \int_a^{b_i} k_i(x,t)u(t)dt,\ a \le x \le b = \max_{2 \le i \le n} \{b_i\} \tag{1.11.a}$$

together with the initial condition

$$u(a) = \alpha \tag{1.11.b}$$

If $|k_1(x,t)| \le L_1$ for each $a \le t \le x \le b$, $|k_i(x,t)| \le L_i$ for each $a \le t, x \le b$ and $i = 2,3,\ldots,n$ and $\left[\dfrac{|\lambda_1|L_1 + 2\sum_{i=2}^{n}|\lambda_i|L_i}{2}(b-a)^2\right] < 1$, then equations (1.11) has a unique solution.

Proof:-

By integrating both sides of equation (1.11.a) with respect to x, one can get:

$$u(x) = \alpha + \int_a^x g(z)dz + \lambda_1 \int_a^x \int_a^z k_1(z,t)u(t)dtdz + \sum_{i=2}^{n} \lambda_i \int_a^x \int_a^{b_i} k_i(z,t)u(t)dtdz \tag{1.12}$$

We rewrite equation (1.12) as $Tu = u$, where

$$Tu = \alpha + \int_a^x g(z)dz + \lambda_1 \int_a^x \int_a^z k_1(z,t)u(t)dtdz + \sum_{i=2}^{n} \lambda_i \int_a^x \int_a^{b_i} k_i(z,t)u(t)dtdz$$

Next, we show that T is a contractive mapping. To do this, consider

$$d(Tu_1, Tu_2) =$$

$$\sup_{a \le x \le b} \left| \lambda_1 \int_a^x \int_a^z k_1(z,t)[u_1(t) - u_2(t)]dtdz + \sum_{i=2}^{n} \lambda_i \int_a^x \int_a^{b_i} k_i(z,t)[u_1(t) - u_2(t)]dtdz \right|$$

where

$$Tu_1 = \int_a^x g(z)dz + \lambda_1 \int_a^x \int_a^z k_1(z,t)u_1(t)dtdz + \lambda_2 \int_a^x \int_a^b k_2(z,t)u_1(t)dtdz$$

and

$$Tu_2 = \int_a^x g(z)dz + \lambda_1 \int_a^x \int_a^z k_1(z,t)u_2(t)dtdz + \lambda_2 \int_a^x \int_a^b k_2(z,t)u_2(t)dtdz$$

$$d(Tu_1, Tu_2)$$

$$\le \sup_{a \le x \le b} |u_1(x) - u_2(x)| \left[|\lambda_1| L_1 + \sum_{i=2}^{n} |\lambda_i| L_i \int_a^x \int_a^{b_i} dtdz \right] \int_a^x \int_a^z dtdz$$

$$= d(u_1, u_2) \left[|\lambda_1| L_1 \left(\frac{(x-a)^2}{2} \right) + (x-a) \sum_{i=2}^{n} |\lambda_i| L_i (b_i - a) \right]$$

$$\le d(u_1, u_2) \left[|\lambda_1| L_1 \frac{(b-a)^2}{2} + \sum_{i=2}^{n} |\lambda_i| L_i (b-a)^2 \right]$$

$$= \left[\frac{|\lambda_1| L_1 + 2\sum_{i=2}^{n} |\lambda_i| L_i}{2} (b-a)^2 \right] d(u_1, u_2). \text{But} \left[\frac{|\lambda_1| L_1 + 2\sum_{i=2}^{n} |\lambda_i| L_i}{2} (b-a)^2 \right] < 1,$$

therefore T is a contractive mapping. By using Banach fixed point theorem, T has exactly one fixed point and hence equations (1.11) has a unique solution.

1.3 Some Real Life Applications for Integral and Integro-Differential Equations:-

In this section, we give some real life applications for the integral and integro-differential equations namely, Human population and the Contact problem for applications of the integral equations and nuclear reactor dynamics and Electricity for applications of the integro-differential equations.

(a) Human population, [Jerri A., 1985]:-

Let the number of people presented at time $x=0$ be n_0. If we look at the survival or insurance tables, we find that there is some sort of a survival function $f(x)$ which gives the fraction of people surviving to age x. It is assumed that these people are either male or female. The surviving population $n_s(x)$ at time x is

$$n_s(x)=n_0\,f(x) \tag{1.13}$$

where $n_s(0)=n_0\,f(0)=n_0$.

Under normal circumstances there is a continuous addition to the population through new births. If children are born at an average rate $r(x)$, then in a particular time interval $\Delta_i\tau$ about the time τ_i, there are $r(\tau_i)\Delta_i\tau$ children added who, if they survive, will be of age $x-\tau_i$ at time x. A fraction $f(x-\tau_i)$ of these children will survive to age $x-\tau_i$, so the final addition to the population at time x, from the children born in the interval $\Delta_i\tau$ about time τ_i, is $f(x-\tau_i)r(\tau_i)\Delta_i\tau$

Now if this process is repeated for all the m subintervals of the time interval $(0,x)$, we obtain the partial sum

$$b_m(x) = \sum_{i=1}^{m} f(x-\tau_i)\, r(\tau_i)\Delta_i\tau \tag{1.14}$$

as the number of people added through new births which, if passed to the limit, becomes the integral

$$b(x) = \int_0^x f(x-\tau)\, r(\tau)\, d\tau \tag{1.15}$$

If this is added to $n_s(x)$ in equation (1.13) (the survivors of the initial population), we obtain the total population at time x as

$$n(x) = n_s(x) + b(x) = n_0 f(x) + \int_0^x f(x-\tau)\, r(\tau)\, d\tau \tag{1.16}$$

It is reasonable now to assume that the rate of birth $r(x)$ is proportional to $n(x)$, the number of the population present at time x,

$$r(x) = k\, n(x) \tag{1.17}$$

From equations (1.16) and (1.17) it follows that

$$n(x) = n_0 f(x) + k\int_0^x f(x-\tau)\, n(\tau)\, d\tau \tag{1.18}$$

which is a Volterra integral equation of the second kind in $n(x)$ with a difference kernel $k\, f(x-\tau)$.

(b) The Contact problem, [Badr A., 2000]:-

Consider the semi-symmetric problem, when the tangent force $q(x)$ is related with the normal pressure $p(x)$ in the contact region of the two surfaces, by the relation:

$$q(x) = k_1\, p(x)$$

where k_1 is the friction coefficient. Also, the normal stress τ_{xt} with the tangent stress σ_t satisfying the relation:

$$\tau_{xt} = k_1 \sigma_t$$

For the displacement components v_1^* and v_2^* in the t-direction we have the relation:

$$\frac{dv_1^*}{dx} = \frac{q(x)}{G_1}, \qquad \frac{dv_2^*}{dx} = \frac{q(x)}{G_2} \tag{1.19}$$

where G_1 and G_2 are the displacement compressible materials of two surfaces $g_1(x)$ and $g_2(x)$ respectively. Such problem reduces to the following integral equation:

$$k_2 \frac{G_1 + G_2}{G_1 G_2} \int_0^x u(t)\,dt + (v_1 + v_2) \int_{-1}^{1} k\left(\frac{x-t}{\lambda}\right) u(t)dt = \delta - g_1(x) - g_2(x),$$

$$\text{for} \quad \lambda \in [0,\infty), \quad k(t) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\tanh u}{\pi} e^{iut} du$$

under the condition:

$$\int_{-1}^{1} u(t)\,dt = p < \infty, \quad u(-1) = u(1) = 0, \quad (p \text{ is constant})$$

where the contact domain between the two surfaces $g_1(x)$ and $g_2(x)$,δ is the rigid displacement under the action of a force p, k_2 is a physical constant, k is the discontinuous kernel of the problem with singularity at the point $x = t$, and $v_i = \dfrac{1 - \mu_i^2}{\pi E_i}$ $(i = 1,2)$, μ_1, μ_2 are Poisson's coefficients and E_1, E_1 are Young coefficients and u is the unknown potential function which is continuous through the interval of integration [-1,1]. The kernel can be written in the following form:

$$k(t)=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\tanh u}{\pi}e^{iut}du=-\ln\left|\tanh\frac{\pi t}{4}\right|$$

If $\lambda\to\infty$ and the term $\left(\frac{x-t}{\lambda}\right)$ is very small, such that it satisfies the condition $\tanh z\approx z$, then we have

$$\ln\left|\tanh\frac{\pi t}{4}\right|=\ln t-d,\qquad \left(d=\ln\frac{4\lambda}{\pi}\right) \tag{1.20}$$

Hence, equation (1.19) with the aid of equation (1.20) can be adapted in the form

$$\int_0^x u(t)\,dt+v\int_{-1}^{1}[-\ln|t-x|+d\,]u(t)\,dt=g^*(x) \tag{1.21}$$

where

$$v=\frac{(v_1+v_2)G_1G_2}{k_2(G_1+G_2)},\qquad g^*(x)=\frac{[\delta-g_1(x)-g_2(x)]G_1G_2}{k_2(G_1+G_2)}$$

Differentiating equation (1.21) with respect to x, one can get

$$u(x)+v\int_{-1}^{1}\frac{u(t)}{t-x}dt=g(x),\quad \left(g(x)=\frac{d\,g^*(x)}{dx}\right) \tag{1.22}$$

Equation (1.27) represents a Fredholm integral equation of the second kind with Cauchy singular kernel.

(c) Nuclear reactor dynamics, [Linz P., 1985]:-

The relation between the temperature of the reactor $\beta(x,t)$ and the power produced $u(t)$, can be described by the rather complication set of equations:

$$\frac{du(x)}{dx}=\int_{-\infty}^{\infty}\alpha(x)\,\beta(x,t)dt,\ -\infty<x<\infty$$

$$\frac{\partial\beta(x,t)}{\partial x} = \frac{\partial^2\beta(x,t)}{\partial t^2} + e(x)u(t), \ -\infty < x < \infty, \ t > 0$$

Additional conditions will be taken as:

$u(0) = 0, \ \beta(x,0) = f(x)$

and

$$\lim_{x\to\pm\infty} \beta(x,t) = \lim_{x\to\pm\infty} \frac{\partial}{\partial x}\beta(x,t) = 0 .$$

The first equation expresses the power production as a function of the temperature and the second equation is simply a diffusion equation with an added source term due to the power generated by the reactor.

By some more manipulation, and applying the full Fourier transform to second equation and using integration by parts with condition at infinity, the Fourier transform $B(w,t)$ satisfies:

$$\frac{\partial B(w,t)}{\partial x} = -w^2 B(w,t) + u(t)E(w)$$

together with the initial condition:

$B(w,0) = F(w)$

After using inverse Fourier transform in the solutions of the above differential equation and by substituting the result $\beta(x,t)$ into first equation and exchanging order of integration, one can obtain:

$$\frac{du(x)}{dx} = \frac{1}{\sqrt{2\pi}} \int_0^x u(t) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha(x) e^{-iwx} e^{-w^2(x-t)} E(w) dx dw dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha(x) e^{-iwx} e^{-w^2 x} F(w) dx dw$$

This equation can be written in the explicit form:

$$\frac{du(x)}{dx} = \int_0^x k(x,t)u(t)dt + g(x) \tag{1.23}$$

where $k(x,t) = -\int_{-\infty}^{\infty} A(-w)E(w)e^{-w^2(x-t)}dw,$

and $g(x) = -\int_{-\infty}^{\infty} A(-w)F(w)e^{-w^2 x}dw$

where A, E and F are Fourier transforms to $\alpha,\ e$ and f respectively. Equation (1.23) represents a first order linear Volterra integro-differential equation of the second kind.

(d) Electricity, [Burton T., 1983]:-

If a single-loop circuit contains resistance R, capacitance C and L with impressed voltage E, then the basic series circuit equation is

$$L\frac{dI}{dx} + RI(x) + \frac{1}{c}q(x) = E(x),\ x \geq 0$$

where the current is defined by $I = \frac{dq}{dx}$. Therefore

$$q(x) = q_0 + \int_0^x I(\xi)\,d\xi,\ x \geq 0$$

where q_0 is the initial charge on capacitor. Thus

$$L\frac{dI(x)}{dx} + RI(x) + \frac{1}{c}\left[q_0 + \int_0^x I(\xi)\,d\xi\right] = E(x),\ x \geq 0$$

which is a first order linear Volterra integro-differential equation of the second kind.

Introduction:-

It is known that, when a Taylor series is truncated to a finite number of terms the result is a Taylor polynomial. This Taylor polynomial is used to approximate functions numerically, [Taylor B., 1985].

Taylor expansion method can be used to solve the linear Fredholm integral equations of the second kind, [Kanwal R. and Liv K., 1989].

The aim of this chapter is to use Taylor expansion method to solve the homogenous linear Fredholm integral equations. Also we use this method to solve the linear Volterra-Fredholm integral and integro-differential equations.

This chapter consist of two sections.

In section one, we use Taylor expansion method to solve the linear homogenous and nonhomogenous Fredholm integral equations. Also this method can be also used to solve the linear Volterra integral equations of the second kind and the linear Volterra-Fredholm integral equations of the second kind that contains two integral operators.

In section two, we will solve the same types by using the same method followed in section one but in the linear integro-differential equations and we gave some examples of this types.

2.1 Taylor Expansion Method for Solving the Linear Integral Equations:-

In this section, we use Taylor expansion method to solve the linear Fredholm, Volterra and Volterra-Fredholm integral equations with some illustrative examples.

2.1.1 Taylor Expansion Method for Solving the Linear Fredholm Integral Equations:-

As seen before, [Kanwal R. and Liv K., 1989] used Taylor expansion method for solving the non-homogeneous linear Fredholm integral equations.

In this section, we use the same method to solve the homogenous linear Fredholm integral equations. To do this, first consider the linear Fredholm integral equation of the second kind:

$$u(x) = g(x) + \lambda \int_a^b k(x,t) u(t)\, dt, \ a \le x \le b \tag{2.1}$$

Assume that the solution u can be approximated in terms of Taylor polynomials of the form:

$$u(x) \approx u^*(x) = \sum_{i=0}^{n} \frac{1}{i!} u^{(i)}(c) (x-c)^i, \ a \le c \le b \tag{2.2}$$

which is a Taylor polynomial of degree n at $x = c$ and $u^{(i)}(c), i = 0, 1, \ldots, n$ are the unknown coefficients that must be determined.

Therefore

$$u(t) \approx u^*(t) = \sum_{i=0}^{n} w(i)\, z_i(t) \tag{2.3}$$

where $w(i) = \frac{1}{i!} u^{(i)}(c)$ and $z_i(t) = (t-c)^i$, $i = 0, 1, \ldots, n.$

By substituting equation (2.3) into equation (2.1) one can get:

$$u(x) = g(x) + \lambda \sum_{i=0}^{n} w(i) \int_a^b k(x,t)\, z_i(t)\, dt$$

To find the approximated solution of equation (2.1), we differentiate the above equation j-times with respect to x to get:

$$u^{(j)}(x) = g^{(j)}(x) + \lambda \sum_{i=0}^{n} w(i) \int_a^b \frac{\partial^j}{\partial x^j} k(x,t)\, z_i(t)\, dt,\ j = 0,1,\ldots,n$$

and hence

$$u^{(j)}(c) = g^{(j)}(c) + \lambda \sum_{i=0}^{n} w(i) \int_a^b \left(\frac{\partial^j}{\partial x^j} k(x,t)\bigg|_{x=c} z_i(t) \right) dt$$

$$= g^{(j)}(c) + \lambda \sum_{i=0}^{n} \frac{1}{i!} u^{(i)}(c) \int_a^b \left(\frac{\partial^j}{\partial x^j} k(x,t)\bigg|_{x=c} (t-c)^i \right) dt$$

$$= g^{(j)}(c) + \lambda \sum_{i=0}^{n} \frac{1}{i!} u^{i}(c) k_{i,j},\ j = 0,1,\ldots,n$$

where

$$k_{i,j} = \int_a^b \left(\frac{\partial^j}{\partial x^j} k(x,t) \right)\bigg|_{x=c} (t-c)^i\, dt,\ i,j = 0,1,\ldots,n \tag{2.4}$$

Therefore from equation (2.4) one can obtain the following linear system:

$$U - \lambda K Y = G$$

where

$$U = \begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ u^{(n)}(c) \end{bmatrix},\ K = \begin{bmatrix} k_{0,0} & k_{1,0} & \cdots & k_{n,0} \\ k_{0,1} & k_{1,1} & \cdots & k_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ k_{0,n} & k_{1,n} & \cdots & k_{n,n} \end{bmatrix},\ Y = \begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ \frac{1}{n!} u^{(n)}(c) \end{bmatrix}$$

and $G = \begin{bmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(n)}(c) \end{bmatrix}$.

The above system can be rewritten as

$$K^{*}U = G \tag{2.5}$$

where

$$K^{*} = \begin{bmatrix} 1-\lambda k_{0,0} & -\lambda k_{1,0} & -\lambda \frac{1}{2!}k_{2,0} & \cdots & -\lambda \frac{1}{(n-1)!}k_{n-1,0} & -\lambda \frac{1}{n!}k_{n,0} \\ -\lambda k_{0,1} & 1-\lambda k_{1,1} & -\lambda \frac{1}{2!}k_{2,1} & \cdots & -\lambda \frac{1}{(n-1)!}k_{n-1,1} & -\lambda \frac{1}{n!}k_{n,1} \\ -\lambda k_{0,2} & -\lambda k_{1,2} & 1-\lambda \frac{1}{2!}k_{2,2} & \cdots & -\lambda \frac{1}{(n-1)!}k_{n-1,2} & -\lambda \frac{1}{n!}k_{n,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\lambda k_{0,n-1} & -\lambda k_{1,n-1} & -\lambda \frac{1}{2!}k_{2,n-1} & \cdots & 1-\lambda \frac{1}{(n-1)!}k_{n-1,n-1} & -\lambda \frac{1}{n!}k_{n,n-1} \\ -\lambda k_{0,n} & -\lambda k_{1,n} & -\lambda \frac{1}{2!}k_{2,n} & \cdots & -\lambda \frac{1}{(n-1)!}k_{n-1,n} & 1-\lambda \frac{1}{n!}k_{n,n} \end{bmatrix} \tag{2.6}$$

This linear system can be solved by any suitable method to find the values of $u(c), u'(c), \ldots, u^{(n)}(c)$. These values are substituted in equation (2.2) to get the approximated solution of equation (2.1) .

Second, consider the homogenous linear Fredholm integral equation of the second kind:

$$u(x) = \lambda \int_{a}^{b} k(x,t)u(t)dt, \; a \le x \le b \tag{2.7}$$

Assume that the solution u can be approximated as in equation (2.2). In this case, equation (2.7) can be written as

$$u(x)=\lambda\sum_{i=0}^{n}w(i)\int_{a}^{b}k(x,t)\,z_i(t)\,dt$$

where w, z_i are defined previously. Therefore

$$u^{(j)}(x)=\lambda\sum_{i=0}^{n}w(i)\int_{a}^{b}\frac{\partial^j}{\partial x^j}k(x,t)\,z_i(t)\,dt,\ j=0,1,\ldots,n$$

and this implies that

$$u^{(j)}(c)=\lambda\sum_{i=0}^{n}u^{(i)}(c)\frac{1}{i!}k_{i,j},\ j=0,1,\ldots,n$$

where

$$k_{i,j}=\int_{a}^{b}\left(\frac{\partial^j}{\partial x^j}k(x,t)\right)\Bigg|_{x=c}(t-c)^i\,dt,\ i,j=0,1,\ldots,n \tag{2.8}$$

Therefore from the above equation one can obtain the following homogenous linear system:

$$(\mathrm{I}-\lambda K)U=0 \tag{2.9}$$

where I is the $(n+1)\times(n+1)$ identity matrix,

$$K=\begin{bmatrix} k_{0,0} & k_{1,0} & \cdots & \frac{1}{n!}k_{n,0} \\ k_{0,1} & k_{1,1} & \cdots & \frac{1}{n!}k_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ k_{0,n} & k_{1,n} & \cdots & \frac{1}{n!}k_{n,n} \end{bmatrix} \text{ and } U=\begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ u^{(n)}(c) \end{bmatrix}.$$

It is clear that $U=0$ for any values of λ. But if

$$|\mathrm{I}-\lambda K|=0 \tag{2.10}$$

then λ is said to be the generalized eigenvalue of the pair of matrices (I,K). So by solving the above characteristic equation one can get the values of λ which can be substituted into equation (2.9) to find the corresponding eigenvectors U.

To illustrate this method, consider the following examples.

Example (2.1):-

Consider the linear Fredholm integral equation of the second kind:

$$u(x)=\frac{4}{5}x^2+\int_0^1 x^2t^2\,u(t)\,dt,\ 0\le x\le 1 \qquad (2.11)$$

We solve this example by using Taylor expansion method. To do this, assume that the solution u can be approximated in terms of Taylor polynomials of the form:

$$u(x)\approx\sum_{i=0}^{2}\frac{1}{i!}u^{(i)}\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)^i=u\left(\frac{1}{2}\right)+u'\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)+\frac{u''\left(\frac{1}{2}\right)}{2!}\left(x-\frac{1}{2}\right)^2.$$

which is a Taylor polynomial of degree 2 at $x=\frac{1}{2}$ and $u^{(i)}\left(\frac{1}{2}\right), i=0,1,2$ are the unknown coefficients that must be determined .

Moreover since $g(x)=\frac{4}{5}x^2$ then $g'(x)=\frac{8}{5}x$ and $g''(x)=\frac{8}{5}$.

Thus $G=\begin{bmatrix} g\left(\frac{1}{2}\right) \\ g'\left(\frac{1}{2}\right) \\ g''\left(\frac{1}{2}\right)\end{bmatrix}=\begin{bmatrix}\frac{1}{5}\\ \frac{4}{5}\\ \frac{8}{5}\end{bmatrix}$.

Next, we substitute $i, j=0,1,2$ in equation (2.4) to get:

$$k_{0,0}=\int_0^1 k\left(\frac{1}{2},t\right)dt=\frac{1}{12},\ k_{1,0}=\int_0^1 k\left(\frac{1}{2},t\right)\left(t-\frac{1}{2}\right)dt=\frac{1}{48},$$

$$k_{2,0}=\int_0^1 k\left(\frac{1}{2},t\right)\left(t-\frac{1}{2}\right)^2 dt=\frac{1}{120},\ k_{0,1}=\int_0^1 \frac{\partial}{\partial x}k(x,t)\bigg|_{x=\frac{1}{2}} dt=\int_0^1 \frac{\partial}{\partial x}x^2t^2\bigg|_{x=\frac{1}{2}} dt=\frac{1}{3},$$

$$k_{1,1}=\int_0^1 \frac{\partial}{\partial x}k(x,t)\bigg|_{x=\frac{1}{2}}\left(t-\frac{1}{2}\right)dt=\int_0^1 \frac{\partial}{\partial x}x^2t^2\bigg|_{x=\frac{1}{2}}\left(t-\frac{1}{2}\right)dt=\frac{1}{12},$$

$$k_{2,1}=\int_0^1 \frac{\partial}{\partial x}k(x,t)\bigg|_{x=\frac{1}{2}}\left(t-\frac{1}{2}\right)^2 dt=\int_0^1 \frac{\partial}{\partial x}x^2t^2\bigg|_{x=\frac{1}{2}}\left(t-\frac{1}{2}\right)^2 dt=\frac{1}{30},$$

$$k_{0,2}=\int_0^1 \frac{\partial^2}{\partial x^2}k(x,t)\bigg|_{x=\frac{1}{2}} dt=\int_0^1 \frac{\partial^2}{\partial x^2}x^2t^2\bigg|_{x=\frac{1}{2}} dt=\frac{2}{3},$$

$$k_{1,2}=\int_0^1 \frac{\partial^2}{\partial x^2}k(x,t)\bigg|_{x=\frac{1}{2}}\left(t-\frac{1}{2}\right)dt=\int_0^1 \frac{\partial^2}{\partial x^2}x^2t^2\bigg|_{x=\frac{1}{2}}\left(t-\frac{1}{2}\right)dt=\frac{1}{6}$$

and

$$k_{2,2}=\int_0^1 \frac{\partial^2}{\partial x^2}k(x,t)\bigg|_{x=\frac{1}{2}}\left(t-\frac{1}{2}\right)^2 dt=\int_0^1 \frac{\partial^2}{\partial x^2}x^2t^2\bigg|_{x=\frac{1}{2}}\left(t-\frac{1}{2}\right)^2 dt=\frac{1}{15}.$$

Thus $K=\begin{bmatrix}\frac{1}{12} & \frac{1}{48} & \frac{1}{120}\\ \frac{1}{3} & \frac{1}{12} & \frac{1}{30}\\ \frac{2}{3} & \frac{1}{6} & \frac{1}{15}\end{bmatrix}$ and $K^*=\begin{bmatrix}\frac{11}{12} & -\frac{1}{48} & -\frac{1}{240}\\ -\frac{1}{3} & \frac{11}{12} & -\frac{1}{60}\\ -\frac{2}{3} & -\frac{11}{12} & \frac{29}{30}\end{bmatrix}$.

Hence, equation (2.5) becomes

$$\begin{bmatrix}\frac{11}{12} & -\frac{1}{48} & -\frac{1}{240}\\ -\frac{1}{3} & \frac{11}{12} & -\frac{1}{60}\\ -\frac{2}{3} & -\frac{1}{6} & \frac{29}{30}\end{bmatrix}\begin{bmatrix}u\left(\frac{1}{2}\right)\\ u'\left(\frac{1}{2}\right)\\ u''\left(\frac{1}{2}\right)\end{bmatrix}=\begin{bmatrix}\frac{1}{5}\\ \frac{4}{5}\\ \frac{8}{5}\end{bmatrix}.$$

Then the solution of this linear system is $u\left(\frac{1}{2}\right)=\frac{1}{4}, u'\left(\frac{1}{2}\right)=1$ and $u''\left(\frac{1}{2}\right)=2$.

Thus $u(x)\approx x^2$ is the approximated solution of equation (2.11). Notice that, this solution is the exact solution of equation (2.11).

Example (2.2):-

Consider the linear Fredholm integral equation of the second kind:

$$u(x)=\sin x+x^3(\cos(1)-\sin(1))+\int_0^1 x^3 t\,u(t)\,dt,\ 0\le x\le 1 \qquad (2.12)$$

This example is constructed such that the exact solution of this it is $u(x)=\sin x$. Assume that the solution u can be approximated in terms of Taylor polynomials of the form:

$$u(x)\approx\sum_{i=0}^{1}\frac{1}{i!}u^{(i)}(0)\,x^i=u(0)+u'(0)\,x\,.$$

which is a Taylor polynomial of degree 1 at $x=0$ and $u^{(i)}(0), i=0,1$ are the unknown coefficients that must be determined. Moreover since $g(x)=\sin x+x^3(\cos(1)-\sin(1))$ then $g'(x)=\cos x+3x^2(\cos(1)-\sin(1))$.

Thus $G=\begin{bmatrix} g(0)\\ g'(0)\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$.

Next, we substitute $i,j=0,1$ in equation (2.4) to get:

$$k_{0,0}=\int_0^1 k(0,t)\,dt=0,\ k_{1,0}=\int_0^1 k(0,t)\,t\,dt=0,$$

$$k_{0,1}=\int_0^1 \frac{\partial}{\partial x}k(x,t)\bigg|_{x=0}dt=\int_0^1 \frac{\partial}{\partial x}x^3 t\bigg|_{x=0}dt=0$$

and

$$k_{1,1} = \int_0^1 \frac{\partial}{\partial x} k(x,t)\Big|_{x=0} t\,dt = \int_0^1 \frac{\partial}{\partial x} x^3 t\Big|_{x=0} t\,dt = 0.$$

Thus $K = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $K^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Hence, equation (2.5) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u(0) \\ u'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then the solution of this linear system is $u(0) = 0$ and $u'(0) = 1$. Thus $u(x) \approx x$ is the approximated solution of equation (2.12). By substituting this approximated solution into the right hand side of equation (2.12) one can get:

$$\sin x + x^3(\cos(1) - \sin(1)) + \int_0^1 x^3 t\, u(t)\,dt \approx \sin x + x^3\left(\frac{1}{3} + \cos(1) - \sin(1)\right)$$
$$\neq u(x) \approx x.$$

So, we must increase the value of n. Therefore let $n = 2$, then the approximated solution of equation (2.12) takes the form:

$$u(x) \approx u(0) + u'(0)\,x + u''(0)\frac{x^2}{2!}.$$

Moreover since $g'(x) = \cos x + 3x^2(\cos(1) - \sin(1))$

then $g''(x) = -\sin x + 6x(\cos(1) - \sin(1))$.

Thus $G = \begin{bmatrix} g(0) \\ g'(0) \\ g''(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Next, we substitute $i, j = 0,1,2$ in equation (2.4) to get:

$$k_{0,0} = k_{1,0} = 0,\ k_{2,0} = \int_0^1 k(0,t)\,t^2\,dt = 0,$$

$$k_{0,1}=k_{1,1}=0,\ k_{2,1}=\int_0^1 \frac{\partial}{\partial x}\,k(x,t)\Big|_{x=0} t^2\,dt=\int_0^1 \frac{\partial}{\partial x}\,x^3 t\Big|_{x=0} t^2\,dt=0,$$

$$k_{0,2}=\int_0^1 \frac{\partial^2}{\partial x^2}\,k(x,t)\Big|_{x=0} dt=\int_0^1 \frac{\partial^2}{\partial x^2}\,x^3 t\Big|_{x=0} dt=0,$$

$$k_{1,2}=\int_0^1 \frac{\partial^2}{\partial x^2}\,k(x,t)\Big|_{x=0} t\,dt=\int_0^1 \frac{\partial^2}{\partial x^2}\,x^3 t\Big|_{x=0} t\,dt=0$$

and

$$k_{2,2}=\int_0^1 \frac{\partial^2}{\partial x^2}\,k(x,t)\Big|_{x=0} t^2\,dt=\int_0^1 \frac{\partial^2}{\partial x^2}\,x^3 t\Big|_{x=0} t^2\,dt=0\,.$$

Thus $K=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ and $K^*=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$.

Hence, equation (2.5) becomes

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}u(0)\\u'(0)\\u''(0)\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}.$$

Then the solution of this linear system is $u(0)=u''(0)=0$ and $u'(0)=1$. Thus $u(x)\approx x$ is the approximated solution of equation (2.12). But this solution is not sufficient satisfactory. So we must increase the value of n. Therefore let $n=3$, then the approximated solution of equation (2.12) takes the form:

$$u(x)\approx u(0)+u'(0)\,x+u''(0)\frac{x^2}{2!}+u'''(0)\frac{x^3}{3!}.$$

By following the same previous steps, the system given by equation (2.5) becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & -2 & -\frac{3}{4} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} u(0) \\ u'(0) \\ u''(0) \\ u'''(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1+6(\cos(1)-\sin(1)) \end{bmatrix}.$$

Then the solution of this linear system is $u(0) = u''(0) = 0$, $u'(0) = 1$ and $u'''(0) = \frac{5}{4} + \frac{15}{2}(\cos(1) - \sin(1)) \approx -1.009$. Thus $u(x) \approx x - 1.009\frac{x^3}{3!}$ is the approximated solution of equation (2.12). This solution is also not sufficient satisfactory. So we must increase the value of n. The following table shows that the approximated solutions of equation (2.12) for $n = 4,5,\ldots,9$.

Table (2.1) represents the approximated solutions of equation (2.12) for different values of n

n	$u^*(x)$
4	$x - 1.009\frac{x^3}{3!}$
5	$x - \frac{x^3}{3!} + \frac{x^5}{5!}$
6	$x - \frac{x^3}{3!} + \frac{x^5}{5!}$
7	$x - 0.987\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$
8	$x - 0.987\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$
9	$x - 0.987\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$

From this table, for $n=9$, the approximated solution of equation (2.12) takes the form:

$$u(x)\approx x-0.987\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}.$$

By substituting this approximated solution into the right hand side of equation (2.12) one can get:

$$\sin x+x^3(\cos(1)-\sin(1))+\int_0^1 x^3 t\,u(t)\,dt\approx\sin x+4.333\times10^{-4}x^3$$

$$\approx u(x)\approx x-0.987\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}.$$

Example (2.3):-

Consider the homogeneous linear Fredholm integral equation:

$$u(x)=\lambda\int_0^1\left(x^2+\frac{16}{9}t\right)u(t)\,dt,\ 0\le x\le 1 \tag{2.13}$$

We solve this example by using Taylor expansion method. To do this, assume that the solution u can be approximated in terms of Taylor polynomials of the form:

$$u(x)\approx u(0)+u'(0)x.$$

Next, we substitute $i,j=0,1$ in equation (2.8) to get:

$$k_{0,0}=\int_0^1 k(0,t)\,dt=\int_0^1\frac{16}{9}t\,dt=\frac{8}{9},\ k_{0,1}=\int_0^1 k(0,t)\,t\,dt=\int_0^1\frac{16}{9}t^2\,dt=\frac{16}{27},$$

$$k_{1,0}=\int_0^1\frac{\partial}{\partial x}k(x,t)\Big|_{x=0}dt=0\ \text{ and }\ k_{1,1}=\int_0^1\frac{\partial}{\partial x}k(x,t)\Big|_{x=0}t\,dt=0.$$

Thus $K = \begin{bmatrix} \frac{9}{8} & \frac{16}{27} \\ 0 & 1 \end{bmatrix}$. In this case, equation (2.10) becomes

$$|\mathrm{I} - \lambda K| = \begin{vmatrix} 1 - \frac{9}{8}\lambda & -\lambda\frac{16}{27} \\ 0 & 1 \end{vmatrix} = 0$$

and this implies that $\lambda = \frac{9}{8}$. By substituting $\lambda = \frac{9}{8}$ into equation (2.9) one can obtain the following linear system:

$$\begin{bmatrix} 0 & -\frac{2}{3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u(0) \\ u'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

which has the solution $u'(0) = 0$. Therefore $u(x) \approx u(0), u(0) \neq 0$ is the approximated eigenfunction of the pair of operators $\left(\mathrm{I}, \int_0^1 \left(x^2 + \frac{16}{9}t \right).dt \right)$ corresponding to the generalized eigenvalue $\lambda = \frac{9}{8}$. By substituting $(\lambda, u(t))$ into the right hand side of equation (2.13) one can get:

$$\begin{aligned} \lambda\int_0^1 \left(x^2 + \frac{16}{9}t \right) u(t)\, dt &\approx \frac{9}{8}u(0)\int_0^1 \left(x^2 + \frac{16}{9}t \right) dt \\ &= \frac{9}{8}\left(x^2 + \frac{9}{8} \right) u(0) \\ &\neq u(x) \approx u(0). \end{aligned}$$

So, we must increase the value of n. Therefore let $n = 2$, then the approximated solution of equation (2.12) takes the form:

$$u(x) \approx u(0) + u'(0)\,x + u''(0)\frac{x^2}{2!}.$$

By substituting $i, j = 0,1,2$ in equation (2.8) one can get:

$$K=\begin{bmatrix}\frac{8}{9} & \frac{16}{27} & \frac{2}{9}\\ 0 & 0 & 0\\ 2 & 1 & \frac{1}{3}\end{bmatrix}.$$

In this case, equation (2.10) becomes

$$\left|\mathrm{I}-\lambda K\right|=\begin{vmatrix}1-\frac{8}{9}\lambda & -\frac{16}{27}\lambda & -\frac{2}{9}\lambda\\ 0 & 1 & 0\\ -2\lambda & -\lambda & 1-\frac{1}{3}\lambda\end{vmatrix}=0$$

and this implies that $\lambda_1=-9$ and $\lambda_2=\frac{3}{4}$.

By substituting these values into equation (2.9) one can get the following linear systems:

$$\begin{bmatrix}9 & \frac{16}{3} & 2\\ 0 & 1 & 0\\ 18 & 9 & 4\end{bmatrix}\begin{bmatrix}u(0)\\ u'(0)\\ u''(0)\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

and

$$\begin{bmatrix}\frac{1}{3} & -\frac{4}{9} & -\frac{1}{6}\\ 0 & 1 & 0\\ -\frac{3}{2} & -\frac{3}{4} & \frac{3}{4}\end{bmatrix}\begin{bmatrix}u(0)\\ u'(0)\\ u''(0)\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

respectively. The solution of these systems are

$$U_1=\begin{bmatrix}u(0)\\ 0\\ -\frac{9}{2}u(0)\end{bmatrix}=u(0)\begin{bmatrix}1\\ 0\\ -\frac{9}{2}\end{bmatrix}, u(0)\neq 0$$

and

$$U_2 = \begin{bmatrix} u(0) \\ 0 \\ 2u(0) \end{bmatrix} = u(0)\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, u(0) \neq 0$$

respectively. Therefore $\lambda_1 = -9$ and $\lambda_2 = \frac{3}{4}$ are the generalized eigenvalues corresponding to the eigenfunctions

$$u_1(x) = u(0) - \frac{9}{4}u(0)x^2, u(0) \neq 0$$

and

$$u_2(x) = u(0) + u(0)x^2, u(0) \neq 0$$

respectively.

By substituting the eigenvalues $\lambda_1 = -9$ and $\lambda_2 = \frac{3}{4}$ and the corresponding eigenfunctions u_1 and u_2 into the right hand side of equation (2.13) one can have:

$$\begin{aligned} \lambda_1 \int_0^1 \left(x^2 + \frac{16}{9}t \right) u_1(t)\,dt &\approx -9u(0)\int_0^1 \left(x^2 + \frac{16}{9}t \right)\left(1 - \frac{9}{4}t^2 \right) \\ &= u(0)\left(1 - \frac{9}{4}x^2 \right) \\ &\approx u_1(x). \end{aligned}$$

and

$$\begin{aligned} \lambda_2 \int_0^1 \left(x^2 + \frac{16}{9}t \right) u_2(t)\,dt &\approx \frac{3}{4}u(0)\int_0^1 \left(x^2 + \frac{16}{9}t \right)\left(1 + t^2\right)dt \\ &= u(0)\left(1 + x^2\right) \\ &\approx u_2(x). \end{aligned}$$

Therefore $\left(-9, u(0)\left(1-\frac{9}{4}x^2\right)\right)$ and $\left(\frac{3}{4}, u(0)\left(1+x^2\right)\right)$ are the exact eigenpair of the pair of operators $\left(\mathrm{I}, \int_0^1 \left(x^2+\frac{16}{9}t\right).dt\right)$.

2.1.2 Taylor Expansion Method for Solving the Linear Volterra Integral Equations:-

Consider the linear Volterra integral equation of the second kind:

$$u(x) = g(x) + \lambda \int_a^x k(x,t)u(t)\,dt, \;\; x \geq a \tag{2.14}$$

Assume that the solution u can be approximated in terms of Taylor polynomials of the form:

$$u(x) \approx u^*(x) = \sum_{i=0}^{n} \frac{1}{i!} u^{(i)}(c)(x-c)^i, \;\; c \geq a \tag{2.15}$$

which is a Taylor polynomial of degree n at $x = c$, and $u^{(i)}(c), i = 0, 1, \ldots, n$ are the unknown coefficients that must be determined. By substituting equation (2.15) in equation (2.14) one can get:

$$u(x) = g(x) + \lambda \sum_{i=0}^{n} w(i) \int_a^x k(x,t)\, z_i(t)\,dt$$

where w and z_i are defined previously.

Then by differentiating the above equation j-times with respect to x one can have:

$$u^{(j)}(x) = g^{(j)}(x) + \lambda \sum_{i=0}^{n} w(i) \frac{d^j}{dx^j} \int_a^x k(x,t)\, z_i(t)\,dt$$

Hence

$$u^{(j)}(c)= g^{(j)}(c) + \lambda\sum_{i=0}^{n}\frac{1}{i!}u^{i}(c)k_{i,j},\ j=0,1,\ldots,n \tag{2.16}$$

where

$$k_{i,j}=\left[\frac{d^{j}}{dx^{j}}\int_{a}^{x}k(x,t)(t-c)^{i}\,dt\right]\Bigg|_{x=c},\ i,j=0,1,\ldots,n \tag{2.17}$$

Therefore from equation (2.17) one can get the following linear system:

$$U-\lambda KY=G$$

where

$$U=\begin{bmatrix}u(c)\\ u'(c)\\ \vdots\\ u^{(n)}(c)\end{bmatrix},\ K=\begin{bmatrix}k_{0,0} & k_{1,0} & \cdots & k_{n,0}\\ k_{0,1} & k_{1,1} & \cdots & k_{n,1}\\ \vdots & \vdots & \ddots & \vdots\\ k_{0,n} & k_{1,n} & \cdots & k_{n,n}\end{bmatrix},\ Y=\begin{bmatrix}u(c)\\ u'(c)\\ \vdots\\ \frac{1}{n!}u^{(n)}(c)\end{bmatrix}$$

and $G=\begin{bmatrix}g(c)\\ g'(c)\\ \vdots\\ g^{(n)}(c)\end{bmatrix}$.

The above system can be rewritten as

$$K^{*}U=G \tag{2.18}$$

where K^{*} is defined by equation (2.6).

This linear system can be solved by any suitable method to find the values of $u(c), u'(c),\ldots, u^{(n)}(c)$. These values are substituted in equation (2.15) to get the approximated solution of equation (2.14) .

Remark (2.1):-

In equation (2.15), if $c = a$, then by substituting $x = a$ in equation (2.14) one can get $u(a) = g(a)$. In this case, the values of $u'(a), u''(a), \ldots, u^{(n)}(a)$ can be obtained by solving the linear system

$$K_1 U_1 = G_1 \tag{2.19}$$

where

$$K_1^* = \begin{bmatrix} 1-\lambda k_{1,1} & -\lambda \frac{1}{2!} k_{2,1} & \cdots & -\lambda \frac{1}{n!} k_{n,1} \\ -\lambda k_{1,2} & 1-\lambda \frac{1}{2!} k_{2,2} & \cdots & -\lambda \frac{1}{n!} k_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda k_{1,n} & -\lambda \frac{1}{2!} k_{2,n} & \cdots & 1-\lambda \frac{1}{n!} k_{n,n} \end{bmatrix},$$

$$U_1 = \begin{bmatrix} u'(a) \\ u''(a) \\ \vdots \\ u^{(n)}(a) \end{bmatrix} \text{ and } G_1 = \begin{bmatrix} g'(a) + \lambda k_{0,1} u(a) \\ g''(a) + \lambda k_{0,2} u(a) \\ \vdots \\ g^{(n)}(a) + \lambda k_{0,n} u(a) \end{bmatrix}.$$

To illustrate this method, consider the following example.

Example (2.4):-

Consider the linear Volterra integral equation of the second kind:

$$u(x) = \frac{-1}{20} x^5 + \frac{1}{3} x^4 + x^3 - \frac{9}{2} x^2 - \frac{1}{12} x + \frac{13}{10} + \int_1^x (x-t) u(t)\, dt \tag{2.20}$$

We solve this example by using Taylor expansion method. To do this, first, assume that the solution u can be approximated in terms of Taylor polynomials of the form:

$$u(x) \approx u(1) + u'(1)(x-1) + \frac{u''(1)}{2!}(x-1)^2 + \frac{u'''(1)}{3!}(x-1)^3 .$$

which is a Taylor polynomial of degree 3 at $x=1$ and $u^{(i)}(1), i=0,1,2,3$ are the unknown coefficients that must be determined .

Moreover since $g(x) = \frac{-1}{20}x^5 + \frac{1}{3}x^4 + x^3 - \frac{9}{2}x^2 - \frac{1}{12}x + \frac{13}{10}$ then $g(1) = -2.$

Thus the above approximated solution of equation (2.19) becomes

$$u(x) \approx -2 + u'(1)(x-1) + \frac{u''(1)}{2!}(x-1)^2 + \frac{u'''(1)}{3!}(x-1)^3.$$

Since $g'(x) = \frac{-1}{4}x^4 + \frac{4}{3}x^3 + 3x^2 - 9x - \frac{1}{12}$, $g''(x) = -x^3 + 4x^2 + 6x - 9$ and $g'''(x) = -3x^2 + 8x + 6$, thus $g'(1) = -5$, $g''(1) = 0$ and $g'''(1) = 11$.

Next, we substitute $i = 0,1,2,3$ and $j = 1,2,3$ into equation (2.17) one can have:

$$k_{0,1} = \left[\frac{d}{dx}\int_1^x (x-t)\,dt \right]\Bigg|_{x=1} = 0, \; k_{1,1} = \left[\frac{d}{dx}\int_1^x (x-t)(t-1)dt \right]\Bigg|_{x=1} = 0,$$

$$k_{2,1} = \left[\frac{d}{dx}\int_1^x (x-t)(t-1)^2\,dt \right]\Bigg|_{x=1} = 0, \; k_{3,1} = \left[\frac{d}{dx}\int_1^x (x-t)(t-1)^3\,dt \right]\Bigg|_{x=1} = 0,$$

$$k_{0,2} = \left[\frac{d^2}{dx^2}\int_1^x (x-t)\,dt \right]\Bigg|_{x=1} = 1, \; k_{1,2} = \left[\frac{d^2}{dx^2}\int_1^x (x-t)(t-1)dt \right]\Bigg|_{x=1} = 0,$$

$$k_{2,2} = \left[\frac{d^2}{dx^2}\int_1^x (x-t)(t-1)^2\,dt \right]\Bigg|_{x=1} = 0, \; k_{3,2} = \left[\frac{d^2}{dx^2}\int_1^x (x-t)(t-1)^3\,dt \right]\Bigg|_{x=1} = 0,$$

$$k_{0,3} = \left[\frac{d^3}{dx^3}\int_1^x (x-t)\,dt \right]\Bigg|_{x=1} = 0, \; k_{1,3} = \left[\frac{d^3}{dx^3}\int_1^x (x-t)(t-1)dt \right]\Bigg|_{x=1} = 1,$$

$$k_{2,3} = \left[\frac{d^3}{dx^3} \int_1^x (x-t)(t-1)^2 \, dt \right] \Bigg|_{x=1} = 0$$

and $k_{3,3} = \left[\dfrac{d^3}{dx^3} \displaystyle\int_1^x (x-t)(t-1)^3 \, dt \right] \Bigg|_{x=1} = 0.$

Thus $K_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $G_1 = \begin{bmatrix} -5 \\ -2 \\ 11 \end{bmatrix}$.

Therefore, equation (2.19) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u'(1) \\ u''(1) \\ u'''(1) \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 11 \end{bmatrix}.$$

Then the solution of this linear system is $u'(1) = -5, u''(1) = -2$ and $u'''(0) = 6$.

Hence $u(x) \approx -2 - 5(x-1) - (x-1)^2 + (x-1)^3 = x^3 - 4x^2 + 1$ is the approximated solution of equation (2.20). Notice that, this solution is the exact solution of equation (2.20).

Second, assume that the solution u can be approximated in terms of Taylor polynomials of the form:

$$u(x) \approx u\left(\frac{3}{2}\right) + u'\left(\frac{3}{2}\right)\left(x - \frac{3}{2}\right) + \frac{u''\left(\frac{3}{2}\right)}{2!}\left(x - \frac{3}{2}\right)^2 + \frac{u'''\left(\frac{3}{2}\right)}{3!}\left(x - \frac{3}{2}\right)^3.$$

which is a Taylor polynomial of degree 3 at $x = \dfrac{3}{2}$ and $u^{(i)}(\dfrac{3}{2}), i = 0,1,2,3$ are the unknown coefficients that must be determined .

In this case,

$$G=\begin{bmatrix} g\left(\frac{3}{2}\right) \\ g'\left(\frac{3}{2}\right) \\ g''\left(\frac{3}{2}\right) \\ g'''\left(\frac{3}{2}\right) \end{bmatrix},\ U=\begin{bmatrix} u\left(\frac{3}{2}\right) \\ u'\left(\frac{3}{2}\right) \\ u''\left(\frac{3}{2}\right) \\ u'''\left(\frac{3}{2}\right) \end{bmatrix} \text{ and } K^*=\begin{bmatrix} \frac{7}{8} & \frac{1}{24} & -\frac{1}{128} & \frac{1}{960} \\ -\frac{1}{2} & \frac{9}{8} & -\frac{1}{48} & \frac{1}{384} \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

Therefore, equation (2.18) becomes

$$\begin{bmatrix} \frac{7}{8} & \frac{1}{24} & -\frac{1}{128} & \frac{1}{960} \\ -\frac{1}{2} & \frac{9}{8} & -\frac{1}{48} & \frac{1}{384} \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u\left(\frac{3}{2}\right) \\ u'\left(\frac{3}{2}\right) \\ u''\left(\frac{3}{2}\right) \\ u'''\left(\frac{3}{2}\right) \end{bmatrix}=\begin{bmatrix} -\frac{37}{8} \\ -\frac{21}{4} \\ 1 \\ 6 \end{bmatrix}.$$

Then the solution of this linear system is $u\left(\frac{3}{2}\right)=-\frac{37}{8},\ u'\left(\frac{3}{2}\right)=-\frac{21}{4},\ u''\left(\frac{3}{2}\right)=1$ and $u'''(0)=6$.

In this case $u(x)\approx-\frac{37}{8}-\frac{21}{4}\left(x-\frac{3}{2}\right)+\frac{1}{2!}\left(x-\frac{3}{2}\right)^2+\left(x-\frac{3}{2}\right)^3=x^3-4x^2+1$ is the approximated solution of equation (2.20).

2.1.3 Taylor Expansion Method for Solving the Linear Volterra-Fredholm Integral Equations:-

Consider the linear Volterra-Fredholm integral equation of the second kind:

$$u(x) = g(x) + \lambda_1 \int_a^x k_1(x,t)u(t)\,dt + \lambda_2 \int_a^b k_2(x,t)u(t)\,dt\,,\ a \le x \le b \tag{2.21}$$

Assume that the solution u can be approximated as in equation (2.2). By substituting equation (2.3) into equation (2.21) one can get:

$$u(x) = g(x) + \lambda_1 \sum_{i=0}^{n} w(i) \int_a^x k_1(x,t)\, z_i(t)\,dt + \lambda_2 \sum_{i=0}^{n} w(i) \int_a^b k_2(x,t)\, z_i(t)\,dt$$

$$= g(x) + \sum_{i=0}^{n} w(i) \left[\lambda_1 \int_a^x k_1(x,t)\, z_i(t)\,dt + \lambda_2 \int_a^b k_2(x,t)\, z_i(t)\,dt \right]$$

where w and z_i are defined previously.

To find the approximated solution of equation (2.21), we must differentiate the above equation j-times with respect to x to get:

$$u^{(j)}(x) = g^{(j)}(x) + \sum_{i=0}^{n} w(i) \left[\lambda_1 \frac{d^j}{dx^j} \int_a^x k_1(x,t)\, z_i(t)\,dt + \lambda_2 \int_a^b \frac{\partial^j}{\partial x^j} k_2(x,t)\, z_i(t)\,dt \right]$$

and hence

$$u^{(j)}(c) = g^{(j)}(c) + \sum_{i=0}^{n} w(i) \left[\lambda_1 \left\{ \frac{d^j}{dx^j} \int_a^x k_1(x,t)\, z_i(t)\,dt \right\} \Bigg|_{x=c} + \lambda_2 \int_a^b \left(\frac{\partial^j}{\partial x^j} k_2(x,t) \Bigg|_{x=c} z_i(t) \right) dt \right]$$

$$= g^{(j)}(c) + \sum_{i=0}^{n} \frac{1}{i!} u^{(i)}(c) \left[\lambda_1 \left\{ \frac{d^j}{dx^j} \int_a^x k_1(x,t)(t-c)^i\,dt \right\} \Bigg|_{x=c} + \lambda_2 \int_a^b \left(\frac{\partial^j}{\partial x^j} k_2(x,t) \Bigg|_{x=c} (t-c)^i \right) dt \right]$$

$$= g^{(j)}(c) + \sum_{i=0}^{n} \frac{1}{i!} u^{(i)}(c) \left[\lambda_1 k_{i,j} + \lambda_2 k_{i,j}^{*} \right],\ j = 0,1,\ldots,n$$

where

$$k_{i,j} = \left[\frac{d^j}{dx^j} \int_a^x k_1(x,t)(t-c)^i\,dt \right] \Bigg|_{x=c},\ i, j = 0,1,\ldots,n \tag{2.22}$$

and

$$k^*_{i,j} = \int_a^b \frac{\partial^j}{\partial x^j} k_2(x,t)\bigg|_{x=c} (t-c)^i \, dt, \; i, j = 0,1,\ldots,n \tag{2.23}$$

Therefore from the above equations one can obtain the following linear system:

$U - KY = G$

where

$$U = \begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ u^{(n)}(c) \end{bmatrix}, \; K = \begin{bmatrix} \lambda_1 k_{0,0} + \lambda_2 k^*_{0,0} & \lambda_1 k_{1,0} + \lambda_2 k^*_{1,0} & \cdots & \lambda_1 k_{n,0} + \lambda_2 k^*_{n,0} \\ \lambda_1 k_{0,1} + \lambda_2 k^*_{0,1} & \lambda_1 k_{1,1} + \lambda_2 k^*_{1,1} & \cdots & \lambda_1 k_{n,1} + \lambda_2 k^*_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 k_{0,n} + \lambda_2 k^*_{0,n} & \lambda_1 k_{1,n} + \lambda_2 k^*_{1,n} & \cdots & \lambda_1 k_{n,n} + \lambda_2 k^*_{n,n} \end{bmatrix},$$

$$Y = \begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ \frac{1}{n!} u^{(n)}(c) \end{bmatrix} \text{ and } G = \begin{bmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(n)}(c) \end{bmatrix}.$$

The above system can be rewritten as

$$K^* U = G \tag{2.24}$$

where

$$K^* = \begin{bmatrix} 1 - \lambda_1 k_{0,0} - \lambda_2 k_{0,0} & -\lambda_1 k^*_{1,0} - \lambda_2 k^*_{0,0} & \vdots & -\lambda_1 \frac{1}{n!} k_{n,0} - \lambda_2 \frac{1}{n!} k^*_{0,0} \\ -\lambda_1 k_{0,1} - \lambda_2 k^*_{0,0} & 1 - \lambda_1 k_{1,1} - \lambda_2 k^*_{1,1} & \vdots & -\lambda_1 \frac{1}{n!} k_{n,1} - \lambda_2 \frac{1}{n!} k^*_{0,1} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda_1 k_{0,n} - \lambda_2 k^*_{0,n} & -\lambda_1 k_{1,n} - \lambda_2 k^*_{1,n} & \vdots & 1 - \lambda_1 \frac{1}{n!} k_{n,n} - \lambda_2 \frac{1}{n!} k^*_{n,n} \end{bmatrix}.$$

The solution of this linear system $u(c), u'(c), \ldots, u^{(n)}(c)$ can be substituted into equation (2.2) to get the approximated solution of equation (2.21) .

To illustrate this method, consider the following example.

Example (2.5):-

Consider the linear Volterra-Fredholm integral equation of the second kind:

$$u(x) = 3x - \frac{3}{2}x^2 - 3x^3 + \int_0^x 2xu(t)\,dt + \int_0^1 x^2 u(t)\,dt, \quad 0 \le x \le 1 \qquad (2.25)$$

We solve this example by using Taylor expansion method. To do this, assume that the solution u can be approximated in terms of Taylor polynomials of the form:

$$u(x) \approx u(0) + u'(0)\,x + u''(0)\frac{x^2}{2!}.$$

Moreover since $g(x) = 3x - \frac{3}{2}x^2 - 3x^3$ then $g'(x) = 3 - 3x - 9x^2$ and $g''(x) = -3 - 18x$.

Thus $G = \begin{bmatrix} g(0) \\ g'(0) \\ g''(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$.

Next, we substitute $i,\ j = 0,1,2$ into equations (2.22)-(2.23) to get:

$$k_{0,0} = k_{1,0} = k_{2,0} = 0,\ \ k_{0,1} = \left[\frac{d}{dx}\int_0^x 2x\,dt\right]\Bigg|_{x=0} = 0,$$

$$k_{1,1} = \left[\frac{d}{dx}\int_0^x 2xt\,dt\right]\Bigg|_{x=0} = 0,\ \ k_{2,1} = \left[\frac{d}{dx}\int_0^x 2xt^2\,dt\right]\Bigg|_{x=0} = 0,$$

$$k_{0,2} = \left[\frac{d^2}{dx^2}\int_0^x 2x\,dt\right]\Bigg|_{x=0} = 4,\ \ k_{1,2} = \left[\frac{d^2}{dx^2}\int_0^x 2xt\,dt\right]\Bigg|_{x=0} = 0,$$

$$k_{2,2} = \left[\frac{d^2}{dx^2}\int_0^x 2xt^2\,dt\right]\Bigg|_{x=0} = 0,$$

$$k^*_{0,0} = \int_0^1 x^2\Big|_{x=0} dt = 0, \quad k^*_{1,0} = \int_0^1 x^2\Big|_{x=0} t\,dt = 0, \; k^*_{2,0} = \int_0^1 x^2\Big|_{x=0} t^2\,dt = 0,$$

$$k^*_{0,1} = \int_0^1 \frac{d}{dx}x^2\Big|_{x=0} dt = 0, \; k^*_{1,1} = \int_0^1 \frac{d}{dx}x^2\Big|_{x=0} t\,dt = 0, \; k^*_{2,1} = \int_0^1 \frac{d}{dx}x^2\Big|_{x=0} t^2\,dt = 0,$$

$$k^*_{0,2} = \int_0^1 \frac{d^2}{dx^2}x^2\Big|_{x=0} dt = 2, \; k^*_{1,2} = \int_0^1 \frac{d^2}{dx^2}x^2\Big|_{x=0} t\,dt = 1$$

and

$$k^*_{2,2} = \int_0^1 \frac{d^2}{dx^2}x^2\Big|_{x=0} t^2\,dt = \frac{2}{3}.$$

Thus $K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 6 & 1 & \frac{2}{3} \end{bmatrix}$ and $K^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & -1 & \frac{2}{3} \end{bmatrix}$

Hence, equation (2.24) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & -1 & \frac{2}{3} \end{bmatrix}\begin{bmatrix} u(0) \\ u'(0) \\ u''(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}.$$

Then the solution of this linear system is $u(0) = u''(0) = 0$ and $u'(0) = 3$. Thus $u(x) \approx 3x$ is the approximated solution of equation (2.25). Notice that, this solution is the exact solutions of equation (2.25).

2.2 Taylor Expansion Method for Solving the Linear Integro-Differential Equations:-

In this section, we use Taylor expansion method for solving the linear Fredholm, Volterra and Volterra-Fredholm integro-differential equations.

2.2.1 Taylor Expansion Method for Solving the Linear Fredholm Integro-Differential Equations:-

Consider the first order linear Fredholm integro-differential equation of the second kind:

$$u'(x) = g(x) + \lambda \int_a^b k(x,t) u(t)\, dt,\ a \le x \le b \qquad (2.26.a)$$

together with the boundary condition:

$$\alpha_1 u(a) + \alpha_2 u(c) + \alpha_3 u(b) = \beta,\ a \le c \le b \qquad (2.26.b)$$

where $\alpha_1, \alpha_{2,}, \alpha_3$ and β are known constants.

Assume that the solution u can be approximated as in equation (2.2). By substituting equation (2.3) into equation (2.26.a) one can get:

$$u'(x) = g(x) + \lambda \sum_{i=0}^{n} w(i) \int_a^b k(x,t) z_i(t)\, dt$$

To find the approximated solution of equation (2.26), we differentiate the above equation j-times with respect to x to get:

$$u^{(j+1)}(x) = g^{(j)}(x) + \lambda \sum_{i=0}^{n} w(i) \int_a^b \frac{\partial^j}{\partial x^j} k(x,t) z_i(t)\, dt,\ j = 0,1,\ldots,n-1$$

and hence

$$u^{(j+1)}(c) = g^{(j)}(c) + \lambda \sum_{i=0}^{n} w(i) \int_a^b \left(\frac{\partial^j}{\partial x^j} k(x,t) \bigg|_{x=c} z_i(t) \right) dt$$

$$= g^{(j)}(c) + \lambda \sum_{i=0}^{n} \frac{1}{i!} u^{(i)}(c) \int_a^b \left(\frac{\partial^j}{\partial x^j} k(x,t) \bigg|_{x=c} (t-c)^i \right) dt$$

$$= g^{(j)}(c) + \lambda \sum_{i=0}^{n} \frac{1}{i!} u^{(i)}(c) k_{i,j}, \; j = 0,1,\ldots,n-1$$

where

$$k_{i,j} = \int_a^b \frac{\partial^j}{\partial x^j} k(x,t) \bigg|_{x=c} (t-c)^i \, dt, \; i = 0,1,\ldots,n, \; j = 0,1,\ldots,n-1 \tag{2.27}$$

Therefore from the above equation one can obtain the following linear system:

$$U - \lambda K Y = G \tag{2.28}$$

where

$$U = \begin{bmatrix} u'(c) \\ u''(c) \\ \vdots \\ u^{(n)}(c) \end{bmatrix}, K = \begin{bmatrix} k_{0,0} & k_{1,0} & \cdots & k_{n,0} \\ k_{0,1} & k_{1,1} & \cdots & k_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ k_{0,n-1} & k_{1,n-1} & \cdots & k_{n,n-1} \end{bmatrix}, Y = \begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ \frac{1}{n!} u^{(n)}(c) \end{bmatrix}$$

and $G = \begin{bmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(n-1)}(c) \end{bmatrix}$.

Now, we substitute $x = a$ and $x = b$ into equation (2.2) to get:

$$u(a) \approx \sum_{i=0}^{n} \frac{1}{i!} u^{(i)}(c)(a-c)^i \text{ and } u(b) \approx \sum_{i=0}^{n} \frac{1}{i!} u^{(i)}(c)(b-c)^i$$

Therefore, equation (2.26.b) becomes

$$\sum_{i=0}^{n} \frac{1}{i!} u^{(i)}(c) \left[\alpha_1 (a-c)^i + \alpha_3 (b-c)^i \right] + \alpha_2 u(c) = \beta$$

Next, we consider the following two cases:

Case (1):-

If $\alpha_2 \neq 0$ then

$$u(c)=\frac{1}{\alpha_2}\left\{\beta-\sum_{i=0}^{n}\frac{1}{i!}u^{(i)}(c)\left[\alpha_1(a-c)^i+\alpha_3(b-c)^i\right]\right\}$$

This equation together with the linear system given by equation (2.28) can be written as in the following linear system:

$$U^*-\lambda K^* Y=G^* \qquad (2.29)$$

where

$$U^*=\begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ u^{(n)}(c) \end{bmatrix},\ G^*=\begin{bmatrix} \dfrac{\beta}{\alpha_2} \\ g(c) \\ \vdots \\ g^{(n-1)}(c) \end{bmatrix}$$

and

$$K^*=\begin{bmatrix} -\dfrac{1}{\alpha_2\lambda}[\alpha_1+\alpha_3] & -\dfrac{1}{\alpha_2\lambda}[\alpha_1(a-c)+\alpha_3(b-c)] & \cdots & -\dfrac{1}{\alpha_2\lambda}\left[\alpha_1(a-c)^n+\alpha_3(b-c)^n\right] \\ k_{0,0} & k_{1,0} & \cdots & k_{n,0} \\ \vdots & \vdots & \ddots & \vdots \\ k_{0,n-1} & k_{1,n-1} & \cdots & k_{n,n-1} \end{bmatrix}$$

The above system can be rewritten as

$$K_1U^*=G^* \qquad (2.30)$$

where

$$K_1=\begin{bmatrix} 1+\dfrac{1}{\alpha_2}[\alpha_1+\alpha_3] & \dfrac{1}{\alpha_2}[\alpha_1(a-c)+\alpha_3(b-c)] & \cdots & \dfrac{1}{\alpha_2}\left[\alpha_1(a-c)^n+\alpha_3(b-c)^n\right] \\ -\lambda k_{0,0} & 1-\lambda k_{1,0} & \cdots & -\lambda\dfrac{1}{n!}k_{n,0} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda k_{0,n-1} & -\lambda k_{1,n-1} & \cdots & -\lambda\dfrac{1}{n!}k_{n,n-1} \end{bmatrix}$$

The solution of this linear system $u(c), u'(c), \ldots, u^{(n)}(c)$ can be substituted into equation (2.2) to get the approximated solution of the boundary value problem given by equations (2.26) .

Case (2):

If $\alpha_2 = 0$, then the boundary condition given by equation (2.26.b) becomes

$$\alpha_1 \sum_{i=0}^{n} \frac{1}{i!} u^{(i)}(c)(a-c)^i + \alpha_3 \sum_{i=0}^{n} \frac{1}{i!} u^{(i)}(c)(b-c)^i = \beta$$

Now, consider the following two cases:

Case (I):-

If $\alpha_1 + \alpha_3 \neq 0$ then the above boundary condition can be written as

$$u(c) = \frac{1}{\alpha_1 + \alpha_3} \left[\beta - \alpha_1 \sum_{i=1}^{n} \frac{1}{i!} u^{(i)}(c)(a-c)^i - \alpha_3 \sum_{i=1}^{n} \frac{1}{i!} u^{(i)}(c)(b-c)^i \right]$$

This equation together with the linear system given by equation (2.29) can be written as in the following linear system:

$$U^* - \lambda K^* Y = G^* \qquad (2.31)$$

where

$$U^* = \begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ u^{(n)}(c) \end{bmatrix}, \quad G^* = \begin{bmatrix} \dfrac{\beta}{\alpha_1 + \alpha_3} \\ g(c) \\ \vdots \\ g^{(n-1)}(c) \end{bmatrix}$$

$$K^* = \begin{bmatrix} 0 & -\frac{\alpha_1}{\lambda(\alpha_1+\alpha_3)}(a-c) - \frac{\alpha_3}{\lambda(\alpha_1+\alpha_3)}(b-c) & \cdots & -\frac{\alpha_1}{\lambda(\alpha_1+\alpha_3)}(a-c)^n - \frac{\alpha_3}{\lambda(\alpha_1+\alpha_3)}(b-c)^n \\ k_{0,0} & k_{1,0} & \cdots & k_{n,0} \\ \vdots & \vdots & \ddots & \vdots \\ k_{0,n-1} & k_{1,n-1} & \cdots & k_{n,n-1} \end{bmatrix}$$

and $Y = \begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ \frac{1}{n!}u^{(n)}(c) \end{bmatrix}$.

This system can be rewritten as

$$K_1 U^* = G^* \tag{2.32}$$

where

$$K_1 = \begin{bmatrix} 1 & \frac{\alpha_1}{\lambda(\alpha_1+\alpha_3)}(a-c) + \frac{\alpha_3}{\lambda(\alpha_1+\alpha_3)}(b-c) & \cdots & \frac{\alpha_1}{\lambda(\alpha_1+\alpha_3)n!}(a-c)^n + \frac{\alpha_3}{\lambda(\alpha_1+\alpha_3)n!}(b-c)^n \\ -\lambda k_{0,0} & 1-\lambda k_{1,0} & \cdots & -\lambda \frac{1}{n!} k_{n,0} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda k_{0,n-1} & -\lambda k_{1,n-1} & \cdots & 1-\lambda \frac{1}{n!} k_{n,n-1} \end{bmatrix}$$

By solving this system one can get the values of $u(c), u'(c), \ldots, u^{(n)}(c)$.

<u>Case (II):-</u>

If $\alpha_1 + \alpha_3 = 0$ then the boundary condition given by equation (2.26.b) can be written as

$$\sum_{i=1}^{n} \frac{1}{i!} u^{(i)}(c)(a-c)^i - \sum_{i=1}^{n} \frac{1}{i!} u^{(i)}(c)(b-c)^i = \frac{\beta}{\alpha_1}$$

This equation together with the linear system given by equation (2.29) can be written as in the following linear system:

$$U^* - \lambda K^* Y = G^* \tag{2.33}$$

where

$$U^* = \begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ u^{(n)}(c) \end{bmatrix}, \quad G^* = \begin{bmatrix} \dfrac{\beta}{\alpha_1} \\ g(c) \\ \vdots \\ g^{(n-1)}(c) \end{bmatrix}$$

$$K^* = \begin{bmatrix} \dfrac{1}{\lambda} & \dfrac{(a-c)+(b-c)}{\lambda} & \cdots & \dfrac{1}{\lambda}\left[(a-c)^n + (b-c)^n\right] \\ k_{0,0} & k_{1,0} & \cdots & k_{n,0} \\ \vdots & \vdots & \ddots & \vdots \\ k_{0,n-1} & k_{1,n-1} & \cdots & k_{n,n-1} \end{bmatrix} \text{ and } Y = \begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ \dfrac{1}{n!}u^{(n)}(c) \end{bmatrix}.$$

This system can be rewritten as

$$K_1 U^* = G^* \tag{2.34}$$

where

$$K_1 = \begin{bmatrix} 0 & -(a-c)-(b-c) & \cdots & -\dfrac{1}{n!}\left[(a-c)^n + (b-c)^n\right] \\ -\lambda k_{0,0} & 1-\lambda k_{1,0} & \cdots & -\lambda \dfrac{1}{n!} k_{n,0} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda k_{0,n-1} & -\lambda k_{1,n-1} & \cdots & 1-\lambda \dfrac{1}{n!} k_{n,n-1} \end{bmatrix}.$$

By solving this system one can get the values of $u(c), u'(c), \ldots, u^{(n)}(c)$.

To illustrate this method, consider the following examples.

Example (2.6):-

Consider the first order linear Fredholm integro-differential equation of the second kind:

$$u'(x) = 4x^3 - \frac{32}{3}x^2 + \int_0^2 x^2 t\, u(t)\, dt,\ \ 0 \le x \le 2 \qquad (2.35.a)$$

together with the boundary condition:

$$2u(0) + 4u(1) - \frac{1}{4}u(2) = 0 \qquad (2.35.b)$$

We solve this example by using Taylor expansion method. To do this, assume that the solution u can be approximated in terms of Taylor polynomials of the form:

$$u(x) \approx u(1) + u'(1)(x-1) + \frac{u''(1)}{2!}(x-1)^2.$$

Moreover since $g(x) = 4x^3 - \frac{32}{3}x^2$ then $g'(x) = 12x^2 - \frac{64}{3}x$.

Therefore $g(1) = -\frac{20}{3}$ and $g'(1) = -\frac{28}{3}$.

From the boundary condition given by equation (2.35.b) one can deduce that:

$\alpha_1 = 2, \alpha_2 = 4, \alpha_3 = \frac{-1}{4}$ and $\beta = 0$.

Thus $G^* = \begin{bmatrix} \frac{\beta}{\alpha_2} \\ g(1) \\ g'(1) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{20}{3} \\ -\frac{28}{3} \end{bmatrix}$.

Next, we substitute $i = 0,1,2,\ j = 0,1$ into equation (2.27) to get:

$k_{0,0} = \int_0^2 k(1,t)dt = 2,\ k_{1,0} = \int_0^2 k(1,t)(t-1)dt = \frac{2}{3},$

$$k_{2,0}=\int_0^2 k(1,t)(t-1)^2\,dt=\frac{2}{3},\quad k_{0,1}=\int_0^2 \frac{\partial}{\partial x}k(x,t)\bigg|_{x=1}dt=4,$$

$$k_{1,1}=\int_0^2 \frac{\partial}{\partial x}k(x,t)\bigg|_{x=1}(t-1)dt=\frac{4}{3}\ \text{ and }\ k_{2,1}=\int_0^2 \frac{\partial}{\partial x}k(x,t)\bigg|_{x=1}(t-1)^2\,dt=\frac{4}{3}.$$

Since $\lambda=1$, then

$$-\frac{1}{\alpha_2\lambda}\left[\alpha_1+\alpha_3\right]=-\frac{7}{16},\quad -\frac{1}{\alpha_2\lambda}\left[\alpha_1(a-c)+\alpha_3(b-c)\right]=\frac{9}{16}$$

and

$$-\frac{1}{\alpha_2\lambda}\left[\alpha_1(a-c)^2+\alpha_3(b-c)^2\right]=-\frac{7}{16}.$$

Thus $K^*=\begin{bmatrix} -\frac{7}{16} & \frac{9}{16} & -\frac{7}{16} \\ 2 & \frac{2}{3} & \frac{2}{3} \\ 4 & \frac{4}{3} & \frac{4}{3} \end{bmatrix}$.

Hence, equation (2.29) becomes

$$\begin{bmatrix} u(1) \\ u'(1) \\ u''(1) \end{bmatrix}-\begin{bmatrix} -\frac{7}{16} & \frac{9}{16} & -\frac{7}{16} \\ 2 & \frac{2}{3} & \frac{2}{3} \\ 4 & \frac{4}{3} & \frac{4}{3} \end{bmatrix}\begin{bmatrix} u(1) \\ u'(1) \\ \frac{1}{2}u''(1) \end{bmatrix}=\begin{bmatrix} 0 \\ -\frac{20}{3} \\ -\frac{28}{3} \end{bmatrix}.$$

Then the solution of this linear system is $u(1)=\frac{18}{35}$, $u'(1)=\frac{452}{35}$ and $u''(1)=\frac{1044}{35}$. Thus $u(x)\approx\frac{18}{35}+\frac{452}{35}(x-1)+\frac{1044}{35}(x-1)^2=\frac{88}{35}-\frac{592}{35}x+\frac{522}{35}x^2$ is the approximated solution of equations (2.35). By substituting this approximated solution into the right hand side of equation (2.35.a) one can get:

$$4x^3 - \frac{32}{3}x^2 + \int_0^2 x^2 t\,u(t)\,dt = 4x^3 + \frac{312}{35}x^2$$

$$\neq u'(x) \approx -\frac{592}{35} + \frac{1044}{35}x.$$

Therefore, we must increase the value of n. So, assume that the solution of equations (2.35) takes the form:

$$u(x) \approx u(1) + u'(1)(x-1) + \frac{u''(1)}{2!}(x-1)^2 + \frac{u'''(1)}{3!}(x-1)^3.$$

In this case

$$G^* = \begin{bmatrix} \frac{\beta}{\alpha_2} \\ g(1) \\ g'(1) \\ g''(1) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{20}{3} \\ -\frac{28}{3} \\ \frac{8}{3} \end{bmatrix} \text{ and } K^* = \begin{bmatrix} -\frac{7}{6} & \frac{9}{16} & -\frac{7}{16} & \frac{9}{16} \\ 2 & \frac{2}{3} & \frac{2}{3} & \frac{2}{5} \\ 4 & \frac{4}{3} & \frac{4}{3} & \frac{4}{5} \\ 4 & \frac{4}{3} & \frac{4}{3} & \frac{4}{5} \end{bmatrix}.$$

Hence, equation (2.29) becomes

$$\begin{bmatrix} u(1) \\ u'(1) \\ u''(1) \\ u'''(1) \end{bmatrix} - \begin{bmatrix} -\frac{7}{16} & \frac{9}{16} & -\frac{7}{16} & \frac{9}{16} \\ 2 & \frac{2}{3} & \frac{2}{3} & \frac{2}{5} \\ 4 & \frac{4}{3} & \frac{4}{3} & \frac{4}{5} \\ 4 & \frac{4}{3} & \frac{4}{3} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} u(1) \\ u'(1) \\ \frac{1}{2}u''(1) \\ \frac{1}{6}u'''(1) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{20}{3} \\ -\frac{28}{3} \\ \frac{8}{3} \end{bmatrix}.$$

Then the solution of this linear system is $u(1) = 7.712 \times 10^{-2}$, $u'(1) = -1.409$, $u''(1) = -16.226$ and $u'''(1) = -28.226$.

Thus $u(x) \approx 1.486 - 1.409x - 8.113(x-1)^2 - 4.704(x-1)^3$

is the approximated solution of equations (2.35). By substituting this approximated solution into the right hand side of equation (2.35.a) one can get:

$$4x^3 - \frac{32}{3}x^2 + \int_0^2 x^2 t\, u(t)\, dt = 4x^3 - 18.742x^2$$

$$\neq u(x) \approx 1.486 - 1.409x - 8.113(x-1)^2 - 4.704(x-1)^3.$$

Therefore, we must increase the value of n. So, assume that the solution of equations (2.35) takes the form:

$$u(x) \approx \sum_{i=0}^{4} \frac{1}{i!} u^{(i)}(1)(x-1)^i.$$

By following the same previous steps, the system given by equation (2.29) becomes

$$\begin{bmatrix} u(1) \\ u'(1) \\ u''(1) \\ u'''(1) \\ u^{(4)}(1) \end{bmatrix} - \begin{bmatrix} -\frac{7}{16} & \frac{9}{16} & -\frac{7}{16} & \frac{9}{16} & -\frac{7}{16} \\ 2 & \frac{2}{3} & \frac{2}{3} & \frac{2}{5} & \frac{2}{5} \\ 4 & \frac{4}{3} & \frac{4}{3} & \frac{4}{5} & \frac{4}{5} \\ 4 & \frac{4}{3} & \frac{4}{3} & \frac{4}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\ u'(1) \\ \frac{1}{2}u''(1) \\ \frac{1}{6}u'''(1) \\ \frac{1}{24}u^{(4)}(1) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{20}{3} \\ -\frac{28}{3} \\ \frac{8}{3} \\ 24 \end{bmatrix}.$$

Then the solution of this linear system is $u(1) = 1$, $u'(1) = 4$, $u''(1) = 12$, $u'''(1) = 24$ and $u^{(4)}(1) = 24$. Thus $u(x) \approx 1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3 + (x-1)^4 = x^4$ is the approximated solution of equations (2.35). Notice that, this solution is the exact solutions of equations (2.35).

Example (2.7):-

Consider the first order linear Fredholm integro-differential equation of the second kind:

$$u'(x) = 1 - 8x + 3\int_0^2 xt\, u(t)\, dt, \; 0 \le x \le 2 \qquad (2.36.a)$$

together with the boundary condition:

$$u(0)+u(2)=2 \tag{2.36.b}$$

We solve this example by using Taylor expansion method. To do this, assume that the solution u can be approximated in terms of Taylor polynomials of the form:

$$u(x)\approx u(1)+u'(1)(x-1)+\frac{u''(1)}{2!}(x-1)^2.$$

Moreover since $g(x)=1-8x$ then $g'(x)=-8$.

Therefore $g(1)=-7$ and $g'(1)=-8$.

From the boundary condition given by equation (2.36.b) one can deduce that:

$\alpha_1=1, \alpha_2=0, \alpha_3=1, \alpha_1+\alpha_3=2\neq 0$ and $\beta=2$.

Thus $G^*=\begin{bmatrix}\dfrac{\beta}{\alpha_1+\alpha_3}\\ g(1)\\ g'(1)\end{bmatrix}=\begin{bmatrix}1\\-7\\-8\end{bmatrix}$.

Next, we substitute $i=0,1,2,\ j=0,1$ into equation (2.27) and since $\lambda=1$, then

$$\frac{\alpha_1}{\alpha_1+\alpha_3}(a-c)+\frac{\alpha_3}{\alpha_1+\alpha_3}(b-c)=0$$

and

$$\frac{\alpha_1}{2(\alpha_1+\alpha_3)}(a-c)^2+\frac{\alpha_3}{2(\alpha_1+\alpha_3)}(b-c)^2=\frac{1}{4}.$$

Thus $K_1=\begin{bmatrix}1 & 0 & \dfrac{1}{4}\\ -6 & -1 & -1\\ -6 & -2 & 0\end{bmatrix}$.

Hence, equation (2.32) becomes

$$\begin{bmatrix} 1 & 0 & \frac{1}{4} \\ -6 & -1 & -1 \\ -6 & -2 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\ u'(1) \\ u''(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -8 \end{bmatrix}.$$

Then the solution of this linear system is $u(1)=1, u'(1)=1$ and $u''(1)=0$. Thus $u(x)\approx x$ is the approximated solution of equation (2.36). Notice that this solution is the exact solution of equation (2.36).

Example (2.8):-

Consider the first order linear Fredholm integro-differential equation of the second kind:

$$u'(x) = 1 - 8x + 3\int_0^2 x\,t\,u(t)\,dt,\; 0 \le x \le 2 \qquad (2.37.a)$$

together with the boundary condition:

$$u(0) - u(2) = -2 \qquad (2.37.b)$$

We solve this example by using Taylor expansion method. To do this, assume that the solution u can be approximated in terms of Taylor polynomials of the form:

$$u(x) \approx u(1) + u'(1)(x-1) + \frac{u''(1)}{2!}(x-1)^2.$$

Moreover since $g(x) = 1 - 8x$ then $g'(x) = -8$.

Therefore $g(1) = -7$ and $g'(1) = -8$.

From the boundary condition given by equation (2.36.b) one can deduce that:

$\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = -1, \alpha_1 + \alpha_3 = 0$ and $\beta = -2$.

Thus $G^* = \begin{bmatrix} \dfrac{\beta}{\alpha_1} \\ g(1) \\ g'(1) \end{bmatrix} = \begin{bmatrix} -2 \\ -7 \\ -8 \end{bmatrix}$.

Next, we substitute $i = 0,1,2,\ j = 0,1$ into equation (2.27) and since $\lambda = 1$, then

$(a-c)+(b-c)=0$ and $\frac{1}{2}\left[(a-c)^2+(b-c)^2\right]=1$.

Thus $K_1 = \begin{bmatrix} 0 & -2 & 0 \\ -6 & -1 & -1 \\ -6 & -2 & 0 \end{bmatrix}$.

Hence, equation (2.34) becomes

$$\begin{bmatrix} 0 & -2 & 0 \\ -6 & -1 & -1 \\ -6 & -2 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\ u'(1) \\ u''(1) \end{bmatrix} = \begin{bmatrix} -2 \\ -7 \\ -8 \end{bmatrix}.$$

Then the solution of this linear system is $u(1)=1, u'(1)=1$ and $u''(1)=0$. Thus $u(x) \approx x$ is the approximated solution of equation (2.37). Notice that this solution is the exact solution of equation (2.37).

2.2.2 Taylor Expansion Method for Solving the Linear Volterra Integro-Differential Equations:-

Consider the first order linear Volterra integro-differential equation of the second kind:

$$u'(x) = g(x) + \lambda \int_a^x k(x,t)u(t)\,dt,\ \ x \ge a \tag{2.38.a}$$

together with the initial condition:

$$u(a)=\alpha \tag{2.38.b}$$

where α is a known constant.

Assume that the solution u can be approximated as in terms of Taylor polynomial of the form:

$$u(x)\approx u^{*}(x)=\sum_{i=0}^{n}\frac{1}{i!}u^{(i)}(a)(x-a)^{i}$$

then by using the above initial condition and the fact that $u'(a)=g(a)$, the above approximated solution reduces to

$$u(x)\approx \alpha+g(a)(x-a)+\sum_{i=2}^{n}\frac{1}{i!}u^{(i)}(a)(x-a)^{i}$$

where $u''(a), u'''(a), \ldots, u^{(n)}(a)$ are the unknowns coefficients that must be determined.

By substituting equation (2.15) into equation (2.38.a) one can get:

$$u'(x)=g(x)+\lambda\sum_{i=0}^{n}w(i)\int_{a}^{x}k(x,t)\,z_{i}(t)\,dt$$

where $w(i)=\frac{1}{i!}u^{(i)}(a)$ and $z_{i}(t)=(t-a)^{i}$, $i=0,1,\ldots,n.$

Then by differentiating the above equation j-times with respect to x and setting $x=a$ in the resulting equation one can have:

$$u^{(j+1)}(a)=g^{(j)}(a)+\lambda\sum_{i=0}^{n}\frac{1}{i!}u^{i}(a)k_{i,j},\ j=1,2,\ldots,n-1$$

where $k_{i,j}$ is defined as in equation (2.17).

Therefore from the above equation one can get the following system:

$$U-\lambda KY=G \tag{2.39}$$

where

$$U=\begin{bmatrix} u''(a) \\ u'''(a) \\ \vdots \\ u^{(n)}(a) \end{bmatrix},\ K=\begin{bmatrix} k_{2,1} & k_{3,1} & \cdots & k_{n,1} \\ k_{2,2} & k_{3,2} & \cdots & k_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ k_{2,n-1} & k_{3,n-1} & \cdots & k_{n,n-1} \end{bmatrix},\ Y=\begin{bmatrix} \frac{1}{2!}u''(a) \\ \frac{1}{3!}u'''(a) \\ \vdots \\ \frac{1}{n!}u^{(n)}(a) \end{bmatrix}$$

and $G=\begin{bmatrix} g'(a)+\lambda\alpha\ k_{0,1}+\lambda\, g(a)k_{1,1} \\ g''(a)+\lambda\alpha\, k_{0,2}+\lambda\, g(a)k_{1,2} \\ \vdots \\ g^{(n-1)}(a)+\lambda\alpha\ k_{0,n-1}+\lambda\, g(a)k_{1,n-1} \end{bmatrix}$.

The above system can be rewritten as

$K^{*}U=G$

where

$$K^{*}=\begin{bmatrix} 1-\frac{1}{2!}\lambda k_{2,1} & -\frac{1}{3!}\lambda k_{3,1} & \cdots & -\frac{1}{n!}\lambda k_{n,1} \\ -\frac{1}{2!}\lambda k_{2,2} & 1-\frac{1}{3!}\lambda k_{3,2} & \cdots & -\frac{1}{n!}\lambda k_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2!}\lambda k_{2,n-1} & -\frac{1}{3!}\lambda k_{3,n-1} & \cdots & -\frac{1}{n!}\lambda k_{n,n-1} \end{bmatrix}.$$

The values of $u''(a), u'''(a), \ldots, u^{(n)}(a)$ can be obtained from solving the above linear system.

To illustrate this method, consider the following example.

<u>Example (2.9):-</u>

Consider the first order linear Volterra integro-differential equation of the second kind:

$$u'(x) = 1 - \frac{5}{6}x^3 + \int_0^x (x+t)u(t)\,dt, \;\; x \geq 0 \tag{2.40.a}$$

together with the initial condition:

$$u(0) = 0 \tag{2.40.b}$$

We solve this example by using Taylor expansion method. To do this, let $n = 3,$ then the approximated solution of this example takes the form:

$$u(x) \approx u(0) + u'(0)\,x + u''(0)\frac{x^2}{2!} + u'''(0)\frac{x^3}{3!}.$$

But by using the above initial condition and the fact that $u'(0) = g(0) = 1,$ then the above approximated solution reduces to

$$u(x) \approx x + u''(0)\frac{x^2}{2!} + u'''(0)\frac{x^3}{3!}.$$

Moreover since $g(x) = 1 - \frac{5}{6}x^3$ then $g'(x) = -\frac{15}{6}x^2$ and $g''(x) = -5x$.

By substituting $i = 0,1,\; j = 1,2$ in equation (2.15) one can get:

$k_{0,1} = k_{1,1} = k_{0,2} = k_{1,2} = 0$.

But $u(0) = 0, u'(0) = 1$ and $\lambda = 1$, therefore

$$G = \begin{bmatrix} g'(0) + \lambda\alpha\, k_{0,1} + \lambda\, g(0)\, k_{1,1} \\ g''(0) + \lambda\alpha\, k_{0,2} + \lambda\, g(0)\, k_{1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By substituting $i = 2,3,\; j = 1,2$ in equation (2.15) one can have:

$k_{2,1} = k_{3,1} = k_{2,2} = k_{3,2} = 0.$

Hence, equation (2.31) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u''(0) \\ u'''(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

which has the solution $u''(0) = u'''(0) = 0$. Hence $u(x) \approx x$ is the approximated solution of equations (2.40). Notice that, this solution is the exact solutions of equations (2.40).

2.2.3 Taylor Expansion Method for Solving the Linear Volterra-Fredholm Integro-Differential Equations:-

Consider the first order linear Volterra-Fredholm integro-differential equation of the second kind:

$$u'(x) = g(x) + \lambda_1 \int_a^x k_1(x,t) u(t)\, dt + \lambda_2 \int_a^b k_2(x,t) u(t)\, dt \qquad (2.41.a)$$

together with the boundary condition:

$$\alpha_1 u(a) + \alpha_2 u(c) + \alpha_3 u(b) = \beta,\ a \le c \le b \qquad (2.41.b)$$

Here we assume that $\alpha_2 \neq 0$. The case $\alpha_2 = 0$ can be discussed similar to the previous section.

By following the same previous steps one can have:

$$u^{(j+1)}(c) = g^{(j)}(c) + \sum_{i=0}^{n} \frac{1}{i!} u^{(i)}(c) \left[\lambda_1 k_{i,j} + \lambda_2 k_{i,j}^{*}\right],\ j = 0,1,\ldots,n,$$

where $k_{i,j}$ and $k_{i,j}^{*}$ are defined by equations (2.21)-(2.22).

Therefore from the above equations one can obtain the following linear system:

$$U - KY = G \qquad (2.42)$$

where

$$U = \begin{bmatrix} u'(c) \\ u''(c) \\ \vdots \\ u^{(n)}(c) \end{bmatrix},$$

$$K = \begin{bmatrix} \lambda_1 k_{0,0} + \lambda_2 k^*_{0,0} & \lambda_1 k_{1,0} + \lambda_2 k^*_{1,0} & \cdots & \lambda_1 k_{n,0} + \lambda_2 k^*_{n,0} \\ \lambda_1 k_{0,1} + \lambda_2 k^*_{0,1} & \lambda_1 k_{1,1} + \lambda_2 k^*_{1,1} & \cdots & \lambda_1 k_{n,1} + \lambda_2 k^*_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 k_{0,n-1} + \lambda_2 k^*_{0,n-1} & \lambda_1 k_{1,n-1} + \lambda_2 k^*_{1,n-1} & \cdots & \lambda_1 k_{n,n-1} + \lambda_2 k^*_{n,n-1} \end{bmatrix},$$

$$Y = \begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ \frac{1}{n!} u^{(n)}(c) \end{bmatrix} \text{ and } G = \begin{bmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(n-1)}(c) \end{bmatrix}.$$

As seen before from the boundary condition:

$$u(c) = \frac{1}{\alpha_2}\left\{\beta - \sum_{i=0}^{n} \frac{1}{i!} u^{(i)}(c)\left[\alpha_1 (a-c)^i + \alpha_3 (b-c)^i\right]\right\}$$

This equation together with the linear system given by equation (2.42) can be written as in the following linear system:

$$K^* U = G^* \tag{2.43}$$

where

$$U^* = \begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ u^{(n)}(c) \end{bmatrix}, \; G^* = \begin{bmatrix} \frac{\beta}{\alpha_2} \\ g(c) \\ \vdots \\ g^{(n-1)}(c) \end{bmatrix}$$

and

$$K^* = \begin{bmatrix} 1 - \frac{1}{\alpha_2}[\alpha_1 + \alpha_3] & -\frac{1}{\alpha_2}[\alpha_1(a-c) + \alpha_3(b-c)] & \cdots & -\frac{1}{\alpha_2 n!}\left[\alpha_1 (a-c)^n + \alpha_3 (b-c)^n\right] \\ -\lambda_1 k_{0,0} - \lambda_2 k^*_{0,0} & 1 - \lambda_1 k_{1,0} + \lambda_2 k^*_{1,0} & \cdots & -\lambda_1 \frac{1}{n!} k_{n,0} - \lambda_2 \frac{1}{n!} k^*_{n,0} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda_1 k_{0,n-1} - \lambda_2 k^*_{0,n-1} & -\lambda_1 k_{1,n-1} - \lambda_2 k^*_{1,n-1} & \cdots & 1 - \lambda_1 \frac{1}{n!} k_{n,n-1} - \lambda_2 \frac{1}{n!} k^*_{n,n-1} \end{bmatrix}$$

The solution of this linear system $u(c), u'(c), \ldots, u^{(n)}(c)$ can be substituted into equation (2.2) to get the approximated solution of the boundary value problem given by equations (2.41) .

To illustrate this method, consider the following example.

Example (2.10):-

Consider the first order linear Volterra-Fredholm integro-differential equation of the second kind:

$$u'(x) = \frac{1}{2}\cos x\left(e^{-2} - e^{2}\right) - e^{-x} + (x+1)e^{-2x} + \int_{-1}^{x} e^{-x} t\, u(t)\, dt + \int_{-1}^{1} \cos x\, e^{-t} u(t)\, dt \tag{2.44.a}$$

where $-1 \le x \le 1$, together with the boundary condition:

$$2u(-1) + 5u(0) + u(1) = 2e + 5 + e^{-1} \tag{2.44.b}$$

We solve this example by using Taylor expansion method. To do this, assume that the solution u can be approximated in terms of Taylor polynomials of the form:

$$u(x) \approx u(0) + u'(0)x + \frac{u''(0)}{2!}x^{2}.$$

which is a Taylor polynomial of degree 2 at $x = 0$ and $u^{(i)}(0), i = 0,1,2$ are the unknown coefficients that must be determined .

Moreover since $g(x) = \frac{1}{2}\cos x\left(e^{-2} - e^{2}\right) - e^{-x} + (x+1)e^{-2x}$

then $g'(x) = \frac{1}{2}\sin x\left(e^{2} - e^{-2}\right) - 2(x+1)e^{-2x} + e^{-2x} + e^{-x}$.

Thus $G = \begin{bmatrix} g(0) \\ g'(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(e^{-2} - e^{2}\right) \\ 0 \end{bmatrix}$.

From the boundary condition given by equation (2.44.b) one can deduce that:

$\alpha_1 = 2, \alpha_2 = 5, \alpha_3 = 1$ and $\beta = 2e + 5 + e^{-1}$.

Thus $G^* = \begin{bmatrix} \frac{\beta}{\alpha_2} \\ g(0) \\ g'(0) \end{bmatrix} = \begin{bmatrix} \frac{2e+5+e^{-1}}{5} \\ \frac{1}{2}\left(e^{-2} - e^{2}\right) \\ 0 \end{bmatrix}$.

Next, by substituting $i = 0,1,2,\ j = 0,1$ in equations (2.21)-(2.22) and using $a = -1, b = 1, c = 0\ \alpha_1 = 2, \alpha_2 = 5, \alpha_3 = 1$ and $\beta = 2e + 5 + e^{-1}$ one can get:

$$K^* = \begin{bmatrix} -\frac{3}{5} & -\frac{1}{5} & -\frac{3}{5} \\ -\frac{1}{2} - e^{-1} + e^{1} & \frac{1}{3} - 2e^{-1} & -\frac{1}{4} - 5e^{-1} + e^{1} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} \end{bmatrix}.$$

Hence, equation (2.35) becomes

$$\begin{bmatrix} 1.6 & -0.2 & 0.3 \\ -1.8504 & 1.4024 & -0.3144 \\ -0.5 & 0.333 & 0.875 \end{bmatrix} \begin{bmatrix} u(0) \\ u'(0) \\ u''(0) \end{bmatrix} = \begin{bmatrix} 2.1608 \\ -3.627 \\ 0 \end{bmatrix}.$$

Which has the solution $u(0) = 1.0448,\ u'(0) = -0.9902$ and $u''(0) = 0.9735$.

Hence $u(x) \approx 1.0448 - 0.9902x + \frac{0.9735}{2!}x^2$ is the approximated solution of equations (2.44). By substituting this approximated solution into the right hand side of equation (2.44.a) one can get:

$$\frac{1}{2}\cos x\left(e^{-2}-e^{2}\right)-e^{-x}+(x+1)e^{-2x}+\int_{-1}^{x} e^{-x} t\,u(t)\,dt+\int_{-1}^{1}\cos x\, e^{-t} u(t)\,dt$$

$$\approx -0.016\cos x+e^{-x}\left(-1.974+0.522x^{2}-0.33x^{3}+0.121x^{4}\right)+(x+1)e^{-2x}$$

$$\neq u(x)\approx 1.0448-0.9902x+\frac{0.9735}{2!}x^{2}.$$

So we must increase the value of *n*. The following table shows that the approximated solutions of equations (2.44) for $n=3,4,5,6,7,8$.

Table (2.2) represents the approximated solutions of equation (2.36) for different values of n

n	$u^{*}(x)$
3	$1.017-0.988x+\frac{0.996}{2!}x^{2}-\frac{0.985}{3!}x^{3}$
4	$1.002-0.999x+\frac{0.999}{2}x^{2}-\frac{0.997}{3!}x^{3}+\frac{0.995}{4!}x^{4}$
5	$1.001-x-\frac{1}{2!}x^{2}-\frac{1}{3!}x^{3}+\frac{0.999}{4!}x^{4}-\frac{1}{5!}x^{5}$
6	$1-x+\frac{1}{2}x^{2}-\frac{1}{3!}x^{3}+\frac{1}{4!}x^{4}-\frac{1}{5!}x^{5}+\frac{1.004}{6!}x^{6}$
7	$1-x+\frac{1}{2!}x^{2}-\frac{1}{3!}x^{3}+\frac{1}{4!}x^{4}-\frac{1}{5!}x^{5}+\frac{1.004}{6!}x^{6}-\frac{1.017}{7!}x^{7}$
8	$1-x+\frac{1}{2!}x^{2}-\frac{1}{3!}x^{3}+\frac{1}{4!}x^{4}-\frac{1}{5!}x^{5}+\frac{1}{6!}x^{6}-\frac{1}{7!}x^{7}$

From this table, for $n=8$, the approximated solution of equations (2.44) takes the form:

$$u(x)\approx 1-x+\frac{1}{2!}x^{2}-\frac{1}{3!}x^{3}+\frac{1}{4!}x^{4}-\frac{1}{5!}x^{5}+\frac{1}{6!}x^{6}-\frac{1}{7!}x^{7}.$$

By substituting this approximated solution into the right hand side of equation (2.44.a) one can get:

$$\frac{1}{2}\cos x\left(e^{-2}-e^{2}\right)-e^{-x}+(x+1)e^{-2x}+\int_{-1}^{x}e^{-x}t\,u(t)dt$$

$$\approx u'(x)$$

$$\approx -1+x-\frac{1}{2!}x^{2}+\frac{1}{3!}x^{3}-\frac{1}{4!}x^{4}+\frac{1}{5!}x^{5}-\frac{1}{6!}x^{6}.$$

Introduction:-

The aim of this chapter is to use Taylor expansion method to solve special types of nonlinear Volterra-Fredholm integral and integro-differential equations with some illustrative examples.

This chapter consist of two sections.

In section one, we use Taylor expansion method to solve special types of nonlinear Fredholm, Volterra and Volterra-Fredholm integral equations.

In section two, we use the same method to solve special types of first order nonlinear Volterra-Fredholm integro-differential equations.

3.1 Taylor Expansion Method for Solving the Nonlinear Integral Equations:-

In this section, we use Taylor expansion method to solve the nonlinear Fredholm, Volterra and Volterra-Fredholm integral equations with some illustrative examples.

3.1.1 Taylor Expansion Method for Solving the Nonlinear Fredholm Integral Equations:-

Consider the nonlinear Fredholm integral equation of the second kind:

$$u(x) = g(x) + \lambda \int_a^b k(x,t)\big(u(t)\big)^p \, dt, \quad a \le x \le b \tag{3.1}$$

where p is a nonnegative integer.

Assume that the solution can be approximated as in equation (2.2).

Let $q(t) = \big(u(t)\big)^p$. Assume that q can be approximated in terms of a Taylor polynomials of the form:

$$q(x) \approx q^*(x) = \sum_{i=0}^{n} \frac{1}{i!} q^{(i)}(c)\big(x-c\big)^i, \; a \le c \le b \tag{3.2}$$

which is a Taylor polynomial of degree n at $x = c$ and $u^{(i)}(c),\ i = 0, 1, \ldots, n$ are the unknown coefficients that must be determined. Therefore

$$q(t) \approx q^*(t) = \sum_{i=0}^{n} w(i)\, z_i(t) \tag{3.3}$$

where $w(i) = \frac{1}{i!} q^{(i)}(c)$ and $z_i(t) = \big(t-c\big)^i,\ i = 0,1,\ldots,n.$

By substituting equation (3.3) into equation (3.1) one can get:

$$u(x) = g(x) + \lambda \sum_{i=0}^{n} w(i) \int_a^b k(x,t)\, z_i(t)\, dt$$

To find the approximated solution of equation (3.1), we must differentiate the above equation j-times with respect to x and setting $x = c$ in the resulting equation one can get:

$$u^{(j)}(c) = g^{(j)}(c) + \lambda \sum_{i=0}^{n} w(i) \int_a^b \left(\frac{\partial^j}{\partial x^j} k(x,t) \bigg|_{x=c} z_i(t) \right) dt$$

$$= g^{(j)}(c) + \lambda \sum_{i=0}^{n} \frac{1}{i!} \left[(u(x))^p \right]^{(i)} \bigg|_{x=c} k_{i,j}, \; j = 0,1,\ldots,n \qquad (3.4)$$

where $k_{i,j}$ is given by equation (2.4).

Therefore from equation (3.4) one can obtain the following nonlinear system:

$$U - \lambda K Y = G \qquad (3.5)$$

where

$$U = \begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ u^{(n)}(c) \end{bmatrix}, \; K = \begin{bmatrix} k_{0,0} & k_{1,0} & \cdots & k_{n,0} \\ k_{0,1} & k_{1,1} & \cdots & k_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ k_{0,n} & k_{1,n} & \cdots & k_{n,n} \end{bmatrix}, \; Y = \begin{bmatrix} (u(c))^p \\ \left[(u(x))^p \right]' \Big|_{x=c} \\ \vdots \\ \frac{1}{n!} \left[(u(x))^p \right]^{(n)} \Big|_{x=c} \end{bmatrix}$$

and $G = \begin{bmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(n)}(c) \end{bmatrix}$.

This nonlinear system can be solved by any suitable method to find the values of $u(c), u'(c), \ldots, u^{(n)}(c)$ and by substituting these values into equation (2.2) one can get the approximated solution of equation (3.1) .

To illustrate this method, consider the following example.

Example (3.1):-

Consider the nonlinear Fredholm integral equation of the second kind:

$$u(x) = 1 - \frac{1}{4}x + \int_0^1 x t^3 (u(t))^2 \, dt, \quad 0 \le x \le 1 \tag{3.6}$$

We solve this example by using Taylor expansion method. To do this, let $n = 2$ and $c = \frac{1}{2}$ then the approximated solution of equation (3.6) takes the form:

$$u(x) \approx u\left(\frac{1}{2}\right) + u'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{u''\left(\frac{1}{2}\right)}{2!}\left(x - \frac{1}{2}\right)^2.$$

which is a Taylor polynomial of degree 2 at $x = \frac{1}{2}$ and $u^{(i)}\left(\frac{1}{2}\right), i = 0,1,2$ are the unknown coefficients that must be determined . Moreover since $g(x) = 1 - \frac{1}{4}x$ then $g'(x) = -\frac{1}{4}$ and $g''(x) = 0$.

Thus $G = \begin{bmatrix} g\left(\frac{1}{2}\right) \\ g'\left(\frac{1}{2}\right) \\ g''\left(\frac{1}{2}\right) \end{bmatrix} = \begin{bmatrix} \frac{7}{8} \\ -\frac{1}{4} \\ 0 \end{bmatrix}.$

Next, we substitute $i, j = 0,1,2$ in equation (2.4) to get

$$K = \begin{bmatrix} \frac{1}{8} & \frac{3}{80} & \frac{7}{480} \\ \frac{1}{4} & \frac{3}{40} & \frac{7}{240} \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $q(x) = (u(x))^2$, then $\frac{d}{dx}(u(x))^2 = 2u(x)u'(x)$ and

$$\frac{d^2}{dx^2}(u(x))^2 = 2\frac{d}{dx}[u(x)u'(x)] = 2(u'(x))^2 + 2u(x)u''(x).$$

Therefore $Y = \begin{bmatrix} \left(u\left(\frac{1}{2}\right)\right)^2 \\ 2u\left(\frac{1}{2}\right)u'\left(\frac{1}{2}\right) \\ \left(u'\left(\frac{1}{2}\right)\right)^2 + u\left(\frac{1}{2}\right)u''\left(\frac{1}{2}\right) \end{bmatrix}$.

Hence, equation (3.5) becomes

$$\begin{bmatrix} u\left(\frac{1}{2}\right) \\ u'\left(\frac{1}{2}\right) \\ u''\left(\frac{1}{2}\right) \end{bmatrix} - \begin{bmatrix} \frac{1}{8} & \frac{1}{10} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \left(u\left(\frac{1}{2}\right)\right)^2 \\ 2u\left(\frac{1}{2}\right)u'\left(\frac{1}{2}\right) \\ \left(u'\left(\frac{1}{2}\right)\right)^2 + u\left(\frac{1}{2}\right)u''\left(\frac{1}{2}\right) \end{bmatrix} = \begin{bmatrix} \frac{7}{8} \\ -\frac{1}{4} \\ 0 \end{bmatrix}$$

which has the two solutions

$$u\left(\frac{1}{2}\right) = 1,\ u'\left(\frac{1}{2}\right) = u''\left(\frac{1}{2}\right) = 0$$

and

$$u\left(\frac{1}{2}\right) = \frac{14}{5},\ u'\left(\frac{1}{2}\right) = \frac{18}{5} \text{ and } u''\left(\frac{1}{2}\right) = 0.$$

Thus $u_1(x)=1$ and $u_2(x)=\frac{14}{5}+\frac{18}{5}\left(x-\frac{1}{2}\right)=1+\frac{18}{5}x$.

But

$$1-\frac{1}{4}x+\int_0^1 xt^3\left(u_1(t)\right)^2 dt=1=u_1(x)$$

and

$$1-\frac{1}{4}x+\int_0^1 xt^3\left(u_2(t)\right)^2 dt=1+\frac{18}{5}x=u_2(x)\ .$$

Therefore $u_1(x)=1$ and $u_2(x)=1+\frac{18}{5}x$ are the exact solutions of equation (3.6).

3.1.2 Taylor Expansion Method for Solving the Nonlinear Volterra Integral Equations:-

Consider the nonlinear Volterra integral equation of the second kind:

$$u(x)=g(x)+\lambda\int_a^x k(x,t)\left(u(t)\right)^p\ dt,\ x\geq a \qquad (3.7)$$

where p is a nonnegative integer.

Let $q(t)=\left(u(t)\right)^p$ and assume that q can be approximated in terms of Taylor polynomial of the form:

$$q(x)\approx q^*(x)=\sum_{i=0}^{n}\frac{1}{i!}q^{(i)}(c)\left(x-c\right)^i,\ c\geq a$$

which is a Taylor polynomial of degree n at $x=c$ and $u^{(i)}(c),\ i=0,1,\ldots,n$ are the unknown coefficients that must be determined.

Therefore

$$q(t) \approx q^*(t) = \sum_{i=0}^{n} w(i)\, z_i(t) \tag{3.8}$$

where $w(i) = \frac{1}{i!} q^{(i)}(c)$ and $z_i(t) = (t-c)^i$, $i = 0,1,\ldots,n.$

By substituting equation (3.8) into equation (3.7) one can get:

$$u(x) = g(x) + \lambda \sum_{i=0}^{n} w(i) \int_a^x k(x,t)\, z_i(t)\, dt\,.$$

Hence

$$u^{(j)}(c) = g^{(j)}(c) + \lambda \sum_{i=0}^{n} \frac{1}{i!} \left[(u(x))^p \right]^{(i)} \Big|_{x=c} k_{i,j},\ j = 0,1,\ldots,n \tag{3.9}$$

where $k_{i,j}$ is given by equation (2.17).

Therefore from equation (3.9) one can obtain the following nonlinear system:

$$U - \lambda K Y = G \tag{3.10}$$

where $U = \begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ u^{(n)}(c) \end{bmatrix}$, $K = \begin{bmatrix} k_{0,0} & k_{1,0} & \cdots & k_{n,0} \\ k_{0,1} & k_{1,1} & \cdots & k_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ k_{0,n} & k_{1,n} & \cdots & k_{n,n} \end{bmatrix}$, $Y = \begin{bmatrix} (u(c))^p \\ \left[(u(x))^p\right]' \Big|_{x=c} \\ \vdots \\ \frac{1}{n!}\left[(u(x))^p\right]^{(n)} \Big|_{x=c} \end{bmatrix}$

and $G = \begin{bmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(n)}(c) \end{bmatrix}$.

This nonlinear system can be solved by any suitable method to find the values of $u(c), u'(c), \ldots, u^{(n)}(c)$ and by substituting these values into equation (2.15) one can get the approximated solution of equation (3.7) .

To illustrate this method, consider the following example.

Example (3.2):-

Consider the nonlinear Volterra integral equation of the second kind:

$$u(x) = 2x - \frac{4}{3}x^8 + \int_0^x x^2 t^2 (u(t))^3 \, dt, \quad x \geq 0 \tag{3.11}$$

We solve this example by using Taylor expansion method. To do this, let $n = 2$ and $c = 0$ then the approximated solution of equation (3.11) takes the form:

$$u(x) \approx u(0) + u'(0)\,x + \frac{u''(0)}{2!}x^2.$$

But $u(0) = 0,$ thus $u(x) \approx u'(0)\,x + \frac{u''(0)}{2!}x^2$

which is a Taylor polynomial of degree 2 at $x = 0$ and $u^{(i)}(0),\ i = 0,1,2$ are the unknown coefficients that must be determined . Moreover since $g(x) = 2x - \frac{4}{3}x^8$ then $g'(x) = 2 - \frac{32}{3}x^7$ and $g''(x) = -\frac{224}{3}x^6$.

Thus $G = \begin{bmatrix} g'(0) \\ g''(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$

Next, we substitute $i, j = 1,2$ in equation (2.17) to get:

$k_{1,1} = k_{2,1} = k_{1,2} = k_{2,2} = 0.$

Thus $K = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$

Since $q(x) = (u(x))^3,$ then $\frac{d}{dx}(u(x))^3 = 3(u(x))^2\, u'(x)$

and

$$\frac{d^2}{dx^2}(u(x))^3 = 3\frac{d}{dx}\left[(u(x))^2 u'(x)\right] = 6u(x)(u'(x))^2 + 3(u(x))^2 u''(x).$$

Therefore $Y = \begin{bmatrix} 3(u(0))^2 u'(0) \\ 3u(0)(u'(0))^2 + \frac{3}{2}(u(0))^2 u''(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

Hence, equation (3.10) becomes

$$\begin{bmatrix} u'(0) \\ u''(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Then the solution of this nonlinear system is $u'(0) = 2$ and $u''(0) = 0$. Thus $u(x) = 2x$ is the approximated solution of equation (3.11). Notice that this solution is the exact solution of equation (3.11).

Next, if we choose $c = 1$ and $n = 3$ then the approximated solution of equation (3.11) takes the form:

$$u(x) \approx u(1) + u'(1)(x-1) + \frac{u''(1)}{2!}(x-1)^2 + \frac{u'''(1)}{3!}(x-1)^3.$$

which is a Taylor polynomial of degree 3 at $x = 1$ and $u^{(i)}(1),\ i = 0,1,2,3$ are the unknown coefficients that must be determined.

Moreover since $g''(x) = -\frac{224}{3}x^6$ then $g'''(x) = -448x^5$.

Thus $G = \begin{bmatrix} g(1) \\ g'(1) \\ g''(1) \\ g'''(1) \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{26}{3} \\ -\frac{224}{3} \\ -448 \end{bmatrix}.$

Next, we substitute $i, j = 0,1,2,3$ in equation (2.17) to get:

$$K=\begin{bmatrix} \frac{1}{3} & -\frac{1}{12} & \frac{1}{30} & -\frac{1}{60} \\ \frac{5}{3} & -\frac{1}{6} & \frac{1}{15} & -\frac{1}{30} \\ \frac{20}{3} & \frac{5}{6} & \frac{1}{15} & -\frac{1}{30} \\ 20 & 10 & 2 & 0 \end{bmatrix}.$$

Since $\frac{d^2}{dx^2}(u(x))^3 = 3\frac{d}{dx}\left[(u(x))^2 u'(x)\right] = 6u(x)(u'(x))^2 + 3(u(x))^2 u''(x)$

then $\frac{d^3}{dx^3}(u(x))^3 = 6(u'(x))^3 + 18u(x)u'(x)u''(x) + 3(u(x))^2 u'''(x).$

Therefore $Y=\begin{bmatrix} (u(1))^3 \\ 3(u(1))^2 u'(1) \\ u(1)(u'(1))^2 + \frac{3}{2}(u(1))^2 u''(1) \\ (u'(1))^3 + 3u(1)u'(1)u''(1) + \frac{1}{2}(u(1))^2 u'''(1) \end{bmatrix}.$

Hence, equation (3.10) becomes

$$\begin{bmatrix} u(1) \\ u'(1) \\ u''(1) \\ u'''(1) \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & -\frac{1}{12} & \frac{1}{30} & -\frac{1}{60} \\ \frac{5}{3} & -\frac{1}{6} & \frac{1}{15} & -\frac{1}{30} \\ \frac{20}{3} & \frac{5}{6} & \frac{1}{15} & -\frac{1}{30} \\ 20 & 10 & 2 & 0 \end{bmatrix} \begin{bmatrix} (u(1))^3 \\ 3(u(1))^2 u'(1) \\ u(1)(u'(1))^2 + \frac{3}{2}(u(1))^2 u''(1) \\ (u'(1))^3 + 3u(1)u'(1)u''(1) + \frac{1}{2}(u(1))^2 u'''(1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} \\ -\frac{26}{3} \\ -\frac{224}{3} \\ -448 \end{bmatrix}.$$

Then the solution of this nonlinear system is $u(1)=2,\ u'(1)=2,$

$u''(1) = 2.783 \times 10^{-9}$ and $u'''(1) = 8.653 \times 10^{-8}$. Thus $u(x) \approx 2x$ is the approximated solution of equation (3.11). Notice that this solution is the exact solution of equation (3.11).

3.1.3 Taylor Expansion Method for Solving the Nonlinear Volterra-Fredholm Integral Equations:-

Consider the nonlinear Volterra-Fredholm integral equation of the second kind:

$$u(x) = g(x) + \lambda_1 \int_a^x k_1(x,t)(u(t))^{p_1}\, dt + \lambda_2 \int_a^b k_2(x,t)(u(t))^{p_2}\, dt,\ a \le x \le b \qquad (3.12)$$

where p_1 and p_2 are nonnegative integers.

Assume that the solution can be approximated as in equation (2.2). Let $q_1(t) = (u(t))^{p_1}$ and $q_2(t) = (u(t))^{p_2}$. Assume that q_1 and q_2 can be approximated in terms of Taylor polynomials of the forms:

$$q_1(x) \approx q_1^*(x) = \sum_{i=0}^{n} \frac{1}{i!} q_1^{(i)}(c)(x-c)^i,\ a \le c \le b$$

and

$$q_2(x) \approx q_2^*(x) = \sum_{i=0}^{n} \frac{1}{i!} q_2^{(i)}(c)(x-c)^i,\ a \le c \le b\,.$$

Therefore

$$q_1(t) \approx q_1^*(t) = \sum_{i=0}^{n} w_1(i)\, z_i(t) \qquad (3.13)$$

and

$$q_2(t) \approx q_2^*(t) = \sum_{i=0}^{n} w_2(i)\, z_i(t) \qquad (3.14)$$

where $w_1(i) = \frac{1}{i!} q_1^{(i)}(c)$, $w_2(i) = \frac{1}{i!} q_2^{(i)}(c)$ and $z_i(t) = (t-c)^i$, $i = 0,1,\ldots,n.$

By substituting equations (3.13)-(3.14) into equation (3.12) one can get:

$$u(x) = g(x) + \lambda_1 \sum_{i=0}^{n} w_1(i) \int_a^x k_1(x,t)\, z_i(t)\, dt + \lambda_2 \sum_{i=0}^{n} w_2(i) \int_a^b k_2(x,t)\, z_i(t)\, dt$$

Thus

$$u^{(j)}(c) = g^{(j)}(c) + \lambda_1 \sum_{i=0}^{n} \frac{1}{i!} \left[(u(x))^{p_1} \right]^{(i)} \Big|_{x=c} k_{i,j} + \lambda_2 \sum_{i=0}^{n} \frac{1}{i!} \left[(u(x))^{p_2} \right]^{(i)} \Big|_{x=c} k_{i,j}^{*} \tag{3.15}$$

where

$$k_{i,j} = \frac{d^j}{dx^j} \left[\int_a^x k_1(x,t)(t-c)^i\, dt \right] \Bigg|_{x=c}, \; i, j = 0,1,\ldots,n \tag{3.16}$$

and

$$k_{i,j}^{*} = \int_a^b \left(\frac{\partial^j}{\partial x^j} k_2(x,t) \right) \Bigg|_{x=c} (t-c)^i\, dt, \; i, j = 0,1,\ldots,n \tag{3.17}$$

Therefore from equation (3.15) one can obtain the following nonlinear system:

$$U - \lambda_1 K Y_1 - \lambda_2 K^{*} Y_2 = G$$

where

$$U = \begin{bmatrix} u(c) \\ u'(c) \\ \vdots \\ u^{(n)}(c) \end{bmatrix}, K = \begin{bmatrix} k_{0,0} & k_{1,0} & \cdots & k_{n,0} \\ k_{0,1} & k_{1,1} & \cdots & k_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ k_{0,n} & k_{1,n} & \cdots & k_{n,n} \end{bmatrix}, K^{*} = \begin{bmatrix} k_{0,0}^{*} & k_{1,0}^{*} & \cdots & k_{n,0}^{*} \\ k_{0,1}^{*} & k_{1,1}^{*} & \cdots & k_{n,1}^{*} \\ \vdots & \vdots & \ddots & \vdots \\ k_{0,n}^{*} & k_{1,n}^{*} & \cdots & k_{n,n}^{*} \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} (u(c))^{p_1} \\ \left[(u(x))^{p_1}\right]'\Big|_{x=c} \\ \vdots \\ \frac{1}{n!}\left[(u(x))^{p_1}\right]^{(n)}\Big|_{x=c} \end{bmatrix},\ Y_2 = \begin{bmatrix} (u(c))^{p_2} \\ \left[(u(x))^{p_2}\right]'\Big|_{x=c} \\ \vdots \\ \frac{1}{n!}\left[(u(x))^{p_2}\right]^{(n)}\Big|_{x=c} \end{bmatrix} \text{ and } G = \begin{bmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(n)}(c) \end{bmatrix}.$$

This nonlinear system can be solved by any suitable method to find the values of $u(c), u'(c), \ldots, u^{(n)}(c)$ and by substituting these values into equation (2.2) one can get the approximated solution of equation (3.12) .

To illustrate this method, consider the following example.

Example (3.3):-

Consider the nonlinear Volterra-Fredholm integral equation of the second kind:

$$u(x) = -\frac{5}{2}x^2 - \frac{6295}{28}x + 3 - \frac{17}{3}x^6 - \frac{4}{5}x^7 - \frac{49}{3}x^5 - \frac{97}{4}x^4 - 19x^3 + \int_0^x \left(x^2 + t\right)(u(t))^2\, dt + \int_0^1 xt\left(u(t)\right)^3 dt \qquad (3.18)$$

We solve this example by using Taylor expansion method. To do this, let $n = 2$ and $c = \frac{1}{3}$ then the approximated solution of equation (3.18) takes the form:

$$u(x) \approx u\left(\frac{1}{3}\right) + u'\left(\frac{1}{3}\right)\left(x - \frac{1}{3}\right) + \frac{1}{2!}u''\left(\frac{1}{3}\right)\left(x - \frac{1}{3}\right)^2.$$

which is a Taylor polynomial of degree 2 at $x = \frac{1}{3}$ and $u^{(i)}\left(\frac{1}{3}\right), i = 0,1,2$ are the unknown coefficients that must be determined .

Morever since $g(x) = -\frac{5}{2}x^2 - \frac{6295}{28}x + 3 - \frac{17}{3}x^6 - \frac{4}{5}x^7 - \frac{49}{3}x^5 - \frac{97}{4}x^4 - 19x^3$

then $g'(x) = -5x - \frac{6295}{28} - \frac{102}{3}x^5 - \frac{28}{5}x^6 - \frac{245}{3}x^4 - \frac{388}{4}x^3 - 57x^2$

and $g''(x) = -5 - \frac{510}{3}x^4 - \frac{168}{5}x^5 - \frac{980}{3}x^3 - \frac{1164}{4}x^2 - 114x.$

Thus $G = \begin{bmatrix} g\left(\frac{1}{3}\right) \\ g'\left(\frac{1}{3}\right) \\ g''\left(\frac{1}{3}\right) \end{bmatrix} = \begin{bmatrix} -73.297 \\ -237.570 \\ -89.669 \end{bmatrix}.$

Next, we substitute $i, j = 0,1,2$ in equation (3.16) to get:

$$K = \begin{bmatrix} \frac{5}{54} & -\frac{1}{81} & \frac{7}{2916} \\ \frac{2}{3} & -\frac{1}{27} & \frac{2}{243} \\ 3 & \frac{1}{3} & \frac{2}{81} \end{bmatrix}.$$

Now, we substitute $i, j = 0,1,2$ in equation (3.17) to get:

$$K^* = \begin{bmatrix} \frac{1}{6} & \frac{1}{18} & \frac{1}{36} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{12} \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $q_1(x) = (u(x))^2$, then $\frac{d}{dx}(u(x))^2 = 2u(x)u'(x)$

and

$$\frac{d^2}{dx^2}(u(x))^2 = 2u(x)u''(x) + 2(u'(x))^2.$$

Thus $Y_1 = \begin{bmatrix} \left(u\left(\frac{1}{3}\right)\right)^2 \\ 2u\left(\frac{1}{3}\right)u'\left(\frac{1}{3}\right) \\ u\left(\frac{1}{3}\right)u''\left(\frac{1}{3}\right) + \left(u'\left(\frac{1}{3}\right)\right)^2 \end{bmatrix}.$

Since $q_2(x) = (u(x))^3$, then $\frac{d}{dx}(u(x))^3 = 3(u(x))^2 u'(x)$ and

$$\frac{d^2}{dx^2}(u(x))^3 = 3\frac{d}{dx}\left[(u(x))^2 u'(x)\right] = 6u(x)(u'(x))^2 + 3(u(x))^2 u''(x).$$

Therefore $Y_2 = \begin{bmatrix} \left(u\left(\frac{1}{3}\right)\right)^3 \\ 3\left(u\left(\frac{1}{3}\right)\right)^2 u'\left(\frac{1}{3}\right) \\ 3u\left(\frac{1}{3}\right)\left(u'\left(\frac{1}{3}\right)\right)^2 + \frac{3}{2}\left(u\left(\frac{1}{3}\right)\right)^2 u''\left(\frac{1}{3}\right) \end{bmatrix}.$

Hence

$$\begin{bmatrix} u\left(\frac{1}{3}\right) \\ u'\left(\frac{1}{3}\right) \\ u''\left(\frac{1}{3}\right) \end{bmatrix} - \begin{bmatrix} \frac{5}{54} & -\frac{1}{81} & \frac{7}{2916} \\ \frac{2}{3} & -\frac{1}{27} & \frac{2}{243} \\ 3 & \frac{1}{3} & \frac{2}{81} \end{bmatrix} \begin{bmatrix} \left(u\left(\frac{1}{3}\right)\right)^2 \\ 2u\left(\frac{1}{3}\right)u'\left(\frac{1}{3}\right) \\ u\left(\frac{1}{3}\right)u''\left(\frac{1}{3}\right) + \left(u'\left(\frac{1}{3}\right)\right)^2 \end{bmatrix} - \begin{bmatrix} \frac{1}{6} & \frac{1}{18} & \frac{1}{36} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{12} \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} \left(u\left(\frac{1}{3}\right)\right)^3 \\ 3\left(u\left(\frac{1}{3}\right)\right)^2 u'\left(\frac{1}{3}\right) \\ 3u\left(\frac{1}{3}\right)\left(u'\left(\frac{1}{3}\right)\right)^2 + \frac{3}{2}\left(u\left(\frac{1}{3}\right)\right)^2 u''\left(\frac{1}{3}\right) \end{bmatrix} = \begin{bmatrix} -73.297 \\ -237.570 \\ -89.669 \end{bmatrix}.$$

Then the solution of this nonlinear system is

$u\left(\frac{1}{3}\right) = 4.956,\ u'\left(\frac{1}{3}\right) = 6.834$ and $u''\left(\frac{1}{3}\right) = 8.833.$

Thus $u(x) = 3.168 + 3.890x + 4.417x^2$ is the approximated solution of equation (3.18). By substituting this approximated solution into the right hand side of equation (3.18) one can get:

$$-\frac{5}{2}x^2 - \frac{6295}{28}x + 3 - \frac{17}{3}x^6 - \frac{4}{5}x^7 - \frac{49}{3}x^5 - \frac{97}{4}x^4 - 19x^3 + \int_0^x \left(x^2 + t\right)(u(t))^2\,dt + \int_0^1 xt\left(u(t)\right)^3\,dt$$

$$\approx 3 + 84.046x + 2.520x^2 - 0.742x^3 - 1.145x^4 + 4.911x^5 + 6.174x^6 + 3.101x^7$$
$$\neq u(x) \approx 3.168 + 3.890x + 4.417x^2.$$

So, we must increase the value of n. By following the same previous steps for $n = 3, 4, 5$ one can get the same results. But for $n = 6$, the approximated solution of equation (3.18) takes the form:

$$u(x) \approx 2x^2 + 5x + 3.$$

By substituting this approximated solution into the right hand side of equation (3.18) one can get:

$$-\frac{5}{2}x^2-\frac{6295}{28}x+3-\frac{17}{3}x^6-\frac{4}{5}x^7-\frac{49}{3}x^5-\frac{97}{4}x^4-19x^3+$$

$$\int_0^x\left(x^2+t\right)(u(t))^2\,dt+\int_0^1 xt\left(u(t)\right)^3\,dt$$

$$=2x^2+5x+3\approx u(x).$$

In this case the above solution is the exact solution of equation (3.18).

3.2 Taylor Expansion Method for Solving the Nonlinear Volterra-Fredholm Integro-Differential Equations:-

Consider the first order nonlinear Volterra-Fredholm integro-differential equation of the second kind:

$$u'(x)=g(x)+\lambda_1\int_a^x k_1(x,t)\left(u(t)\right)^{p_1}\,dt+\lambda_2\int_a^b k_2(x,t)\left(u(t)\right)^{p_2}\,dt,\ a\le x\le b \qquad (3.19.a)$$

together with the initial condition:

$$u(a)=\alpha \qquad (3.19.b)$$

where p_1 and p_2 are nonnegative integers.

Assume that the solution can be approximated as in equation (2.15).

By substituting equation (3.13)-(3.14) into equation (3.19.a) one can get:

$$u'(x)=g(x)+\lambda_1\sum_{i=0}^{n}w_1(i)\int_a^x k_1(x,t)\,z_i(t)\,dt+\lambda_2\sum_{i=0}^{n}w_2(i)\int_a^b k_2(x,t)\,z_i(t)\,dt$$

Thus

$$u^{(j+1)}(a)=g^{(j)}(a)+\lambda_1\sum_{i=0}^{n}\frac{1}{i!}\left[\left(u(x)\right)^{p_1}\right]^{(i)}\Big|_{x=a}k_{i,j}+\lambda_2\sum_{i=0}^{n}\frac{1}{i!}\left[\left(u(x)\right)^{p_2}\right]^{(i)}\Big|_{x=a}k_{i,j}^{*} \qquad (3.20)$$

where $k_{i,j}$ and $k_{i,j}^{*}$ are defined as in equations (3.16) and (3.17).

Therefore from equation (3.20) one can obtain the following nonlinear system:

$$U - \lambda_1 K Y_1 - \lambda_2 K^* Y_2 = G$$

where

$$U = \begin{bmatrix} u'(a) \\ u''(a) \\ \vdots \\ u^{(n)}(a) \end{bmatrix}, K = \begin{bmatrix} k_{0,0} & k_{1,0} & \cdots & k_{n,0} \\ k_{0,1} & k_{1,1} & \cdots & k_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ k_{0,n-1} & k_{1,n-1} & \cdots & k_{n,n-1} \end{bmatrix},$$

$$K^* = \begin{bmatrix} k^*_{0,0} & k^*_{1,0} & \cdots & k^*_{n,0} \\ k^*_{0,1} & k^*_{1,1} & \cdots & k^*_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ k^*_{0,n-1} & k^*_{1,n-1} & \cdots & k^*_{n,n-1} \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} (u(a))^{p_1} \\ \left[(u(x))^{p_1}\right]'\Big|_{x=a} \\ \vdots \\ \frac{1}{n!}\left[(u(x))^{p_1}\right]^{(n)}\Big|_{x=a} \end{bmatrix}, \quad Y_2 = \begin{bmatrix} (u(a))^{p_2} \\ \left[(u(x))^{p_2}\right]'\Big|_{x=a} \\ \vdots \\ \frac{1}{n!}\left[(u(x))^{p_2}\right]^{(n)}\Big|_{x=a} \end{bmatrix} \text{ and } G = \begin{bmatrix} g(a) \\ g'(a) \\ \vdots \\ g^{(n-1)}(a) \end{bmatrix}.$$

To illustrate this method, consider the following example.

<u>Example (3.4):-</u>

Consider the first order nonlinear Volterra-Fredholm integro-differential equation of the second kind:

$$u'(x) = \frac{4}{5} - \frac{7}{12}x^4 - \frac{9}{2}x^4 + \int_0^x (x+t)(u(t))^2\,dt + \int_0^1 t(u(t))^3\,dt \tag{3.21}$$

together with the initial condition:

$u(0) = 0.$

We solve this example by using Taylor expansion method. To do this, assume that the approximated solution of equation (3.21) takes the form:

$$u(x) \approx u(0) + u'(0)\,x + \frac{u''(0)}{2!}x^2 + \frac{u'''(0)}{3!}x^3$$

$$= u'(0)\,x + \frac{u''(0)}{2!}x^2 + \frac{u'''(0)}{3!}x^3.$$

which is a Taylor polynomial of degree 3 at $x = 0$ and $u^{(i)}(0),\ i = 0,1,2,3$ are the unknown coefficients that must be determined . Moreover, Since

$g(x) = \frac{4}{5} - \frac{7}{12}x^4 - \frac{9}{2}x^4$ then $g'(x) = -\frac{7}{3}x^3$ and $g''(x) = {}^{5} -7x^2$.

Thus $G = \begin{bmatrix} g(0) \\ g'(0) \\ g''(0) \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ 0 \\ 0 \end{bmatrix}$.

Next, we substitute $i = 0,1,2,3$ and $j = 0,1,2$ in equation (3.16) to get:

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}.$$

Now, we substitute $i = 0,1,2,3$ and $j = 0,1,2$ in equation (3.17) to get:

$$K^* = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $q_1(x) = (u(x))^2$, then $\frac{d}{dx}(u(x))^2 = 2u(x)u'(x)$,

$$\frac{d^2}{dx^2}(u(x))^2 = 2u(x)u''(x) + 2(u'(x))^2$$

and $\dfrac{d^3}{dx^3}(u(x))^2 = 6u'(x)u''(x) + 2u(x)u'''(x)$.

Thus $Y_1 = \begin{bmatrix} (u(0))^2 \\ 2u(0)u'(0) \\ u(0)u''(0) + (u'(0))^2 \\ u'(0)u''(0) + \frac{1}{3}u(0)u'''(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (u'(0))^2 \\ u'(0)u''(0) \end{bmatrix}$.

Since $q_2(x) = (u(x))^3$, then $\dfrac{d}{dx}(u(x))^3 = 3(u(x))^2 u'(x)$,

$$\frac{d^2}{dx^2}(u(x))^3 = 3\frac{d}{dx}\left[(u(x))^2 u'(x)\right] = 6u(x)(u'(x))^2 + 3(u(x))^2 u''(x)$$

and $\dfrac{d^3}{dx^3}(u(x))^3 = 6(u'(x))^3 + 18u(x)u'(x)u''(x) + 3(u(x))^2 u'''(x)$.

Therefore $Y_2 = \begin{bmatrix} u((0))^3 \\ 3(u(0))^2 u'(0) \\ 3u(0)(u'(0))^2 + \frac{3}{2}(u(0))^2 u''(0) \\ (u'(0))^3 + 3u(0)u'(0)u''(0) + \frac{1}{2}(u(0))^2 u'''(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ (u'(0))^3 \end{bmatrix}$.

Hence

$$\begin{bmatrix} u'(0) \\ u''(0) \\ u'''(0) \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ (u'(0))^2 \\ u'(0)u''(0) \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ (u'(0))^3 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ 0 \\ 0 \end{bmatrix}.$$

Then the solution of this nonlinear system is

$u'(0) = 1, u''(0) = 0$ and $u'''(0) = 0$. Thus $u(x) \approx x$ is the approximated solution of equation (3.21). Notice that this approximated solution is the exact solution of equation (3.21).

Conclusions and Recommendations

From the present study, we can conclude the following:

1. Volterra-Fredholm integral and integro-differential equations are so difficult to solve analytically.
2. Fredholm and Volterra integral and integro-differential equations are special types of Volterra-Fredholm integral and integro-differential equations.
3. By using Taylor expansion method, the homogenous linear Fredholm integral equation may have more than one nontrivial solution.
4. Taylor expansion method gave more accurate results as the degree of Taylor polynomial increases.
5. Taylor expansion method can be used also to solve Volterra-Fredholm integral and integro-differential equations that contain integral operators. In this case, the approximated solution can be expressed as in equation (2.2), where $a \le c \le \max\limits_{1 \le i \le n} \{b_i\}$.

For future work, the following problems may be recommended:

1. Modify Taylor expansion method to solve the multi-dimensional integral and integro-differential equations.
2. Use other methods to solve the non-linear integral and integro-differential equations say, Chebyshev method.
3. Solve some real life applications in which its mathematical modeling can be represented as Volterra-Fredholm integral and integro-differential equations by using Taylor expansion method.
4. Solve another types of the Generalize nonlinear Volterra-Fredholm integral and integro-differential equations via Taylor expansion method and to solving the ordinary differential equations.

References

[1] Abdul-Jabbar R., "On Inverse Problems of Fractional Order Integro-Differential Equations", Ph.D. Thesis, College of Science, Al-Nahrain University, 2005.

[2] Al-Jawary M., "Numerical Methods for System of Integral Equations", M.Sc. Thesis, College of Education, Ibn-Al-Haitham, University of Baghdad, 2005.

[3] Al-Niamey L., "On the Multi-Dimensional Integral Equation", M.Sc. Thesis, College of Science, Al-Nahrain University, 2006.

[4] Al-Shakry A., "About Delay Integral Equations", M.Sc. Thesis, College of Science, University of Baghdad, 2001.

[5] Al-Shather A., "Approximate and Variational Approches for Solving Fractional Order Multiple Delay Integro-Differential Equation", Ph.D. Thesis, College of Science, Al-Nahrain University, 2003.

[6] Al-Shather A., " About the Singular Operator in Integral Equations", M.Sc. Thesis, College of Science, Al-Nahrain University, 1999.

[7] Al-Timeme A., "Approximated Methods for First Order Volterra Integro-Differential Equations", M.Sc. Thesis, College of Applied Science, University of Technology, 2003.

[8] Bader A., "On the Numerical Treatment of the Contact Problem", Integral J. Math. and Math. Sci., Vol.23, No.12, pp.865-871, 2000.

[9] Burton T., "Volterra Integral and Differential Equations", Academic Press, Inc., 1983.

[10] Cardone A., Messina E. and Russa E., "A fast Iterative Method for Discretized Volterra-Fredholm Integral Equations", Computational and Applied Mathematics, Vol.189, pp.568-579, 2006.

[11] Chambers L., "Integral Equations: A short Course", International Textbook Company Ltd., 1976.

[12] Choi M., "Collocation Approximations for Integro-Differential Equations", Bull. Korean Math. Soc., Vol.30, No.1, pp.35-51, 1993.

[13] Delves L. and Walsh J., "Numerical Solution of Integral Equations", Clarendon. Press, Oxford, 1974.

[14] Hochstadt H., "Integral Equations", A Wiley-Interscience Publication, New York, 1973.

[15] Ibrahim G., "Numerical Treatment of System of Fredholm Integral Equations", M.Sc. Thesis, College of Education, Ibn-Al-Haitham, University of Baghdad, 2005.

[16] Jerri A., "Introduction to Integral Equation with Application", Marcel Dekker, 1985.

[17] Kanwal R. and Liv K., "A type Expansion Approach for Solving Integral Equations", International J. of Mathematical Education in Science and Technology, Vol.20, No.3, pp.411-414, 1989.

[18] Ladopoulos E., "On the Numerical Solution of the Multi-Dimensional Singular Integral and Integral Equations, Used in the Theory of Linear Viscoelasticity", International J. Math. and Math. Sci., Vol.11, pp.561-574, 1988.

[19] Linz P., "Analytical and Numerical Methods for Volterra Equations", Siam Philadelphia, 1985.

[20] Mustafa M., "Numerical Solution for System of Volterra Integral Equations Using Spline Functions", Ph.D. Thesis, College of Science, Al-Mustansiriya University, 2004.

[21] Najieb S., "Method of Solving Linear Fuzzy Integral Equations", Ph.D. Thesis, College of Science, Al-Nahrain University, 2002.

[22] Taylor B., "Introduction to Functional Analysis", Angus Taylor, 1985.

[23] Tricomi F., "Integral Equations", Dover Publication, Inc. New York, 1985.

[24] Xu Y. and Zhau A., "Fast Boolean Approximation Methods for Solving Integral Equations in High Dimensions", J. Integral Equations and Applications, Vol.16, pp.83-92, 2004.

[25] Yaslan H. and Dasciaglu A., "Chebyshev Polynomial Solution of Nonlinear Fredholm-Volterra Integro-Differential Equations", J. Arts and Sciences, Vol.5, pp.89-101, 2006.

المستخلص

الهدف الرئيسي من هذا العمل هو دراسة معادلات فولتيرا-فريدهولم التكاملية والتكاملية التفاضلية.

هذه الدراسة شملت تصنيف معادلات فولتيرا-فريدهولم التكاملية والتكاملية التفاضلية.

كذلك تم تقديم بعض النظريات لوجود ووحدانية الحل لمعادلات فولتيرا-فريدهولم التكاملية والتكاملية التفاضلية الخطية.

أضافة الى ذلك تم مناقشة طريقة توسيع تيلر لحل معادلات فولتيرا-فريدهولم التكاملية والتكاملية التفاضلية الخطية و اللاخطية، مع بعض الامثلة التوضيحية.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة النهرين

كلية العلوم

قسم الرياضيات وتطبيقات الحاسوب

طريقة توسيع تيلر لحل المعادلات التكاملية والتكاملية التفاضلية اللاخطية

رسالة

كلية العلوم, جامعة النهرين مقدمة الى قسم الرياضيات وتطبيقات الحاسوب,

وهي جزء من متطلبات نيل درجة ماجستير علوم في الرياضيات

من قبل

هُدى عبد الرزاق محمد

(بكالوريوس علوم، جامعة النهرين، ٢٠٠٥)

بإشراف

أ.م.د.أحلام جميل خليل

رَمضان ١٤٢٩م

أيلول ٢٠٠٨هـ