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The Numerical Solution of Linear Variable Order Fractional Differential Equations Using Bernstein Polynomials

A Thesis
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بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ وَاللَّهُ بِمَا تَعْمَلُونَ خَبِيرٌ

سورة المجادلة الاية **(11)**

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Contents

List of Symbols

Symbol	Meaning
$\Gamma(x)$	Gamma Function
$\beta(p,q)$	Beta function
$\beta_x(p,q)$	Incomplete beta function of argument *x*
$\gamma^*(c,x)$	Incomplete gamma function of argument *x*
I^{α}	Riemann-Liouville fractional integral of order α
I^{0}	Identity operator
I_{AR}^{α}	Abel-Riemann fractional integral of order α
${}_xD_{a+}^{\alpha}$, ${}_xD_{b-}^{\alpha}$	Abel-Riemann fractional derivatives of order α from The left and right respectively
D_{AR}^{α}	Abel-Riemann fractional derivative of order α
${}^{c}D_{x}^{\alpha}$	Caputo fractional derivative of order α
$\lceil\alpha\rceil$	Smallest integer greater than or equal to α
D^{α}	Grünwald fractional derivatives
E_v	Mittag-Leffler function
${}_aD_t^{\alpha(t)}$	left Riemann-Liouville fractional derivative of order $\alpha(t)$
${}_tD_b^{\alpha(t)}$	Right Riemann-Liouville fractional derivative of order $\alpha(t)$
${}_a^cD_t^{\alpha(t)}$	Type *I* left Caputo derivative of order $\alpha(t)$
${}_t^cD_b^{\alpha(t)}$	Type *I* right Caputo derivative of order $\alpha(t)$
$B_{i,n}(x)$	Bernstein basis polynomials of degree *n*
$B_n(x)$	Bernstein polynomial

List of Symbols

$\Phi(t)$	Represent $(n+1) \times 1$ vector of Bernstein basis polynomials
H	Dual matrix of $\Phi(t)$
Q	Hilbert matrix
$D^{(1)}$	Derivative of the vector $\Phi(t)$ of order 1
$D^{q}\Phi(t)$	Fractional derivative of $\Phi(t)$ of order q
$D^{q(t)}\Phi(t)$	Fractional derivative of $\Phi(t)$ of order *q(t)*
$D^{\beta_i(t)}\Phi(t)$	Fractional derivative of $\Phi(t)$ of β_i in the caputo sense

Abstract

The main theme of this thesis is oriented about three objects:

The first objective is to study the basic concepts of fractional calculus and variable-order fractional differential equations.

The second objective is about solving numerically the variable-order fractional differential equations using operational matrices of Bernstein polynomials.

The proposed approach will transform the variable-order fractional differential equations into the product of some matrices which can be considered as a linear system of algebraic equations, after solving the resulting system the numerical solution can be obtained.

The third objective is to find the numerical solution of multiterm variable-order fractional differential equations using operational matrices of Bernstein polynomials, also the proposed method will transform the multiterm variable-order fractional differential equations into the product of matrices in other words into a system of linear algebraic equations, and the numerical solution will be reached after solving the resulting system.

Introduction

The expression of fractional calculus is more than 300 years old. It is a generalization of the ordinary differentiation and integration to non-integer (arbitrary) order. The subject is as old as the calculus of differentiation and return to times when Leibniz, Gauss, and Newton invented this kind of calculation.

In a letter to L'Hospital in 1695 Leibniz raised the following question ''Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders??" The story goes that L'Hospital was somewhat curious about that question and answered by another question to Leibniz. "What if the order will be 1/2?" Leibniz in a letter dated September 30, 1695 answered: "It will lead to a paradox, from which one day useful consequences will be drawn", [Miller,1993].

The question raised by Leibniz for a fractional derivative was an ongoing topic in the last 300 years.

Several mathematicians contributed to this subject through the years, people like Liouville, Riemann, and Weyl made major contributions to the theory of fractional calculus. The story of the fractional calculus persistent with contributions from Fourier, Abel, Leibniz, Grünwald, and Letnikov [Gorenflo,2008].

Nowadays, the fractional calculus attracts many scientists and engineers, there are several applications of this mathematical phenomenon in mechanics, physics, chemistry, control theory and so on [Caponetto,2010], [Magin, 2006], [Oldham,1974] and [Podlubny,1999].

Fractional differential equations are generalized of classical integer order ones which are obtained by replacing integer order derivatives by fractional ones.

Their advantages comparing which integer order differential equations are the capability of simulating natural physical process and dynamic system more accurately [Chen,2007].

Most fractional order differential equations do not have exact solutions, so approximate and numerical techniques must to used [Mohammed,2016].

Several numerical and approximate methods to solve fractional order differential equations have been given such as Adomian decomposition method [Momani,2006], variational iteration method [Sweilam,2007], homotopy analysis method [Tan,2008], homotopy perturbation method [Khader,2012], collocation method [Bhrawy,2013] and [Bhrawy,2014], wavelet method [Heydari,2014]], finite element method [Ma,2014] and spectral tau method [Bhrawy,2015].

However, several numbers of algorithms for solving fractional-order differential equations have been investigated. Suarez [Suarez, 1997] used the eigenvector expansion method to find the solution of motion containing fractional derivative.

Podlubny [Podlubny,1999] used the Laplace transform method to solve fractional differential equations numerically with Riemann-Lioville derivatives definition as well as the fractional partial differential equations with constant coefficients, Meerscharet and Tadjeran [Meerscharet,2006] proposed the finite difference method to find the numerical solution of two-sided space fractional partial differential equations. Momani [Momani,2007] used a numerical algorithm to solve the fractional convection-diffusion equation nonlinear source term. Odibat and Momani [Odibat,2009] used the variation iteration method to exhaust fractional partial differential equations in fluid mechanics.

Jafari and Seifi [Jafari,2009] solved a system of nonlinear fractional differential equations using homotopy analysis method. Wu [Wu,2009] derived a wavelet operational method to solve fractional differential equations numerically. Chen and Wu [Chen Y,2010] used wavelet method to find the numerical solution for the class of fractional convection-diffusion equation with variable coefficients. Geng [Geng,2011] suggested wavelet method for solving nonlinear partial differential equations of fractional order. Guo [Guo,2013] used the fractional variation homotopy perturbation iteration method to solve a fractional diffusion equation.

Recently, more and more researchers are finding that numerous important dynamical problems exhibit fractional order behavior which may vary with space and time. This fact illustrates that variable order calculus provides an effective mathematical scope for the description of complex dynamical problems. The concept of a variable order operator is a much more recent expansion, which is a new orientation in science.

Different authors have proposed several definitions of variable order differential operators, each of these with a specific meaning to suit desired goals. The variable order operator definitions recently proposed in the science include Riemann-Liouville definition, Caputo definition, Marchaud definition, Coimbra definition and Grünwald definition [Lerenzo,2007].

Since the kernel of variable order operators is very complex for having a variable-exponent, the numerical solutions of variable order fractional differential equations are highly difficult to obtain, and have not attracted much attention. Thus, the development of numerical techniques to solve variable order fractional differential equations has not taken off [Liu,2016].

There are slight references arisen on discussion of variable order fractional differential equation. In several emerging, most authors adopt different methods to deduce an approximate scheme. For example, Coimbra [Coimbra,2003], employed a consistent approximation with first-order accurate for the solution of variable order differential equations.

Soon et al. [Soon,2005] proposed a second-order Runge-Kutta method which is consisting at an explicit Euler predictor step followed by an implicit Euler corrector step to numerically integrate the variable order differential equation.

Lin et al. [Lin,2009] studied the stability and the convergence of an explicit finite-difference approximation for the variable-order fractional diffusion equation with a nonlinear source term. Zhuang et al. [Zhuang,2009] obtained explicit and implicit Euler approximations for the fractional advection–diffusion nonlinear equation of variable-order. Aiming a variable-order anomalous sub diffusion equation, Chen et al. [Chen,2010] employed two numerical schemes one fourth order spatial accuracy and with first order temporal precision, the other with fourth order spatial accuracy and second order temporal accuracy.

Operational matrices recently were adapted for solving several kinds of fractional differential equations. The use of numerical techniques in conjunction with operational matrices in some orthogonal polynomials, for the solution of fractional differential equations on finite and infinite intervals, produced highly accurate solutions for such equations [Bhrawy,2015].

Bhrawy [Bhrawy,2015] discusses spectral techniques based on operational matrices of fractional derivatives and integrals for solving several kinds of linear and nonlinear of fractional differential equations.

The operational matrices of fractional derivatives and integrals, for several polynomials on bounded domains, such as the Legendre, Chebyshev, Jacobi and Bernstein polynomials uses them with different spectral techniques for solving the aforementioned equations on bounded domains. The operational matrices of fractional derivatives and integrals are as well presented for orthogonal Laguerre and modified generalized Laguerre polynomials [Bhrawy,2015].

In this thesis the operational matrices of Bernstein polynomials will be used to solve variable-order and multiterm variable-order fractional differential equations.

Bernstein polynomials play a prominent role in various areas of mathematics, these polynomials have extremely been used in the solution of integral equations, differential equations and approximation theory [Doha,2011], [Maleknejad,2011].

The operational matrices for Bernstein polynomials are introduced in order to solve different types of differential equations among them [Maleknejad,2012] used the operational matrices for Bernstein polynomials for solve nonlinear Volterra-Fredholm-Hammerstein integral equations, [Hashemizadeh,2013] have been used operational matrices of Bernstein polynomials for solving physiology problems, [Bataineh,2016] have been used operational matrices of Bernstein polynomials for solving high order delay differential equations.

This thesis consists of three chapters.

In chapter one which is entitled basic concepts of fractional calculus we discuss the following concepts:

Gamma and Beta functions, fractional order integrations, fractional order derivatives finally variable order fractional derivatives and integrations.

Chapter two handles the numerical solution of variable-order linear fractional differential equations using Bernstein polynomials operational matrices

Chapter three is about the numerical solution of the multiterm variable-order linear fractional differential equations using Bernstein polynomials operational matrices.

It is remarkable to notice that all the computer programs or calculations have been made by using Mathcad 14.

Chapter One

Basic Concepts of Fractional Calculus

1.1 Introduction:

This chapter consists of seven sections, in section 1.2 Gamma and Beta function were given, in section 1.3 some definitions of fractional order integration are presented, in section 1.4 some definitions of fractional order derivatives are given, in section 1.5 the definition of Mittag-Leffler function will be given, while in section 1.6 two analytical methods for solving differential equations of fractional order are introduced, finally in section 1.7 we present some definitions of variable order fractional derivatives.

1.2 The Gamma and Beta Functions, [Oldham,1974]:

The complete gamma function $\Gamma(x)$ plays an important role in the theory of fractional calculus. A comprehensive definition of $\Gamma(x)$ is that provided by Euler limit:

$$\Gamma(x) = \lim_{N\to\infty}\left(\frac{N!\,N^{x}}{x(x+1)(x+2)\dots(x+N)}\right), x > 0 \qquad \dots(1.1)$$

But the integral transform definition given by:

$$\Gamma(x) = \int_{0}^{\infty} y^{x-1}e^{-y}dy\,, x > 0 \qquad \dots(1.2)$$

is often more useful, although it is restricted to positive value of x. An integration by parts applied to eq.(1.2) leads to the recurrence relationship:

$$\Gamma(x + 1) = x\Gamma(x) \qquad \dots(1.3)$$

This is the most important property of gamma function. The same result is a simple consequence of eqation (1.1), since $\Gamma(1)=1$, this recurrence shows that for positive integer n:

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n)\\ &= n! \end{aligned} \qquad \ldots(1.4)$$

The following are the most important properties of the gamma function:

1. $\Gamma\left(\frac{1}{2}-n\right)=\frac{(-4)^n n!\sqrt{\pi}}{(2n)!}$

2. $\Gamma\left(\frac{1}{2}+n\right)=\frac{(2n)!\sqrt{\pi}}{4^n n!}$

3. $\Gamma(-x)=\frac{-\pi\csc(\pi x)}{\Gamma(x+1)}$

4. $\Gamma(nx)=\sqrt{\frac{2\pi}{n}}\left[\frac{nx}{\sqrt{2\pi}}\right]^n\prod_{k=0}^{n-1}\Gamma\left(x+\frac{k}{n}\right), n\in\mathbb{N}^+$

Note Gamma function is also defined for $\mathbb{R}\backslash\{0,-1,-2,-3,\ldots\}$

A function that is closely related to the gamma function is the complete beta function $\beta(p,q)$. For positive value of the two parameters p and q; the function is defined by the beta integral:

$$\beta(p,q)=\int_0^1 y^{p-1}(1-y)^{q-1}\,dy\,,\ p,\ q>0 \qquad \ldots(1.5)$$

which is also known as the Euler's integral of the first kind. If either p or q is nonpositive, the integral diverges otherwise $\beta(p,q)$ is defined by the relationship:

$$\beta(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad \ldots(1.6)$$

where p and $q > 0$.

Both beta and gamma functions have "incomplete" analogues. The incomplete beta function of argument x is defined by the integral:

$$\beta_x(p,q) = \int_0^x y^{p-1}(1-y)^{q-1}\,dy \qquad \ldots(1.7)$$

and the incomplete gamma function of argument x is defined by:

$$\gamma^*(c,x) = \frac{c^{-x}}{\Gamma(x)}\int_0^c y^{x-1}e^{-y}\,dy$$

$$= e^{-x}\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+c+1)} \qquad \ldots(1.8)$$

$\gamma^*(c,x)$ is a finite single-valued analytic function of x and c.

1.3 Fractional Integration:

There are many literatures introduce different definitions of fractional integrations, among them:

1. Riemann-Liouville integral, [Oldham,1974]:

The generalization to non-integer α of Riemann-Liouville integral can be written for suitable function $f(x)$, $x \in \mathbb{R}$; as:

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s)\,ds\,,\ \alpha > 0 \qquad \ldots(1.9)$$

and $I^0 f(x) = f(x)$ is the identity operator.

The properties of the operator I^α can be founded in [Podlbuny,1999] for $\beta \geq 0$, $\alpha > 0$, as follows:

$$\left.\begin{aligned} &1.\ I^\alpha I^\beta f(x) = I^{\alpha+\beta} f(x) \\ &2.\ I^\alpha I^\beta f(x) = I^\beta I^\alpha f(x) \end{aligned}\right\} \qquad \ldots(1.10)$$

2. Weyl fractional integral, [Oldham,1974]:

The left hand fractional order integral of order $\alpha > 0$ of a given function f is defined as:

$$ {}_{-\infty}I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{1-\alpha}} dy \qquad \ldots(1.11)$$

and the right fractional order integral of order $\alpha > 0$ of a given function f is given by:

$$ {}_{\infty}I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(y)}{(y-x)^{1-\alpha}} dy $$

3. Abel-Riemann fractional integral, [Mittal,2008]:

The Abel-Riemann fractional integral of any order $\alpha > 0$, for a function $f(x)$ with $x \in \mathbb{R}^+$ is defined as:

$$ I_{AR}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-\tau)^{\alpha-1} f(\tau) d\tau, \; x > 0, \alpha > 0 \qquad \ldots(1.12)$$

$I_{AR}^{0} = I$ (Identity operator)

The Abel-Riemann integral possess the semigroup property:

$$ I_{AR}^{\alpha} I_{AR}^{\beta} = I_{AR}^{\alpha+\beta} \text{ For all } \alpha, \beta \geq 0 \qquad \ldots(1.13)$$

1.4 Fractional Derivatives:

Many literatures discussed and presented fractional derivatives of certain functions, therefore in this section, some definitions of fractional derivatives are presented:

1. Riemann-Liouville fractional derivatives, [Oldham,1974], [Nishimoto,1983]:

Among the most important formulae used in fractional calculus is the Riemann-Liouville formula. For a given function *f(x)*, $\forall\ x \in [a,b]$; the left and right hand Riemann-Liouville fractional derivatives of order $\alpha > 0$ where *m* is a natural number, are given by:

$$ {}_xD^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x \frac{f(t)}{(x-t)^{\alpha-m+1}} dt \qquad \ldots(1.14)$$

$$ {}_xD^{\alpha}_{b-} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b \frac{f(t)}{(x-t)^{\alpha-m+1}} dt \qquad \ldots(1.15)$$

where $m - 1 < \alpha \leq m,\ m \in \mathbb{N}$.

2. The Abel-Riemann fractional derivative, [Mittal,2008]:

The Abel-Riemann fractional derivative of order $\alpha > 0$ is defined as the inverse of the corresponding A-R fractional integral, i.e.,

$$D^{\alpha}_{AR} I^{\alpha}_{AR} = I \qquad \ldots(1.16)$$

For positive integer *m*, such that $m-1 < \alpha \leq m$, where *I* is the identity operator.

$$(D^{m}_{AR} I^{m-\alpha}_{AR}) I^{\alpha}_{AR} = D^{m}_{AR} (I^{m-\alpha}_{AR} I^{\alpha}_{AR}) = D^{m}_{AR} I^{m}_{AR} = I$$

i.e.,

$$D^{\alpha}_{AR} f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dx^m} \displaystyle\int_a^x \frac{f(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m \\ \dfrac{d^m}{dx^m} f(x), & \alpha = m \end{cases} \qquad \ldots(1.17)$$

3. Caputo fractional derivative, [Caputo,1967], [Minadri,1997]:

In the late sixties of the last century, an alternative definition of fractional derivatives was introduced by Caputo. Caputo and Minadri used this definition in their work on the theory of viscoelasticity. According to Caputo's definition:

$$ {}^{c}D_{x}^{\alpha} = I^{m-\alpha}D^{m}, \text{ for } m-1<\alpha\le m $$

which means that:

$$ {}^{c}D_{x}^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_{0}^{x}\dfrac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}}d\tau, & m-1<\alpha<m \\ \dfrac{d^{m}}{dx^{m}} f(x), & \alpha=m \end{cases} $$

For the Caputo derivative, we have:

i. ${}^{c}D_{x}^{\alpha}C = 0$, ($c$ is a constant)

ii. $${}^{c}D_{x}^{\alpha}x^{j} = \begin{cases} 0, & for\ j\in\mathbb{N}\cup\{0\}\ and\ j<\lceil\alpha\rceil \\ \dfrac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}x^{j-\alpha}, & for\ j\in\mathbb{N}\cup\{0\}\ and\ j\ge\lceil\alpha\rceil \end{cases} \qquad \ldots(1.18)$$

We utilize the ceiling function $\lceil\alpha\rceil$ to denote the smallest integer greater than or equal to α.

The basic properties of the Caputo fractional derivative are:

1. Caputo introduced an alternative definition, which has the preference of defining integer order initial conditions for fractional order differential equations.

2. $I^{\alpha}\ {}^{c}D_{x}^{\alpha}f(x) = f(x) - \sum\limits_{k=0}^{m-1} f^{(k)}(0^{+})\dfrac{x^{k}}{k!}$.

3. Caputo's fractional differentiation is linear operator, similar to integer order differentiation:

$$ {}^{c}D_{x}^{\alpha}\,[\lambda f(x) + \mu\, g(x)] = \lambda\, {}^{c}D_{x}^{\alpha}\, f(x) + \mu\, {}^{c}D_{x}^{\alpha}\, g(x) $$

4. Grünwald fractional derivatives, [Oldham,1974]:

The Grünwald derivatives of any integer order to any function, can take the form:

$$ \mathrm{D}^{\alpha} f(x) = \lim_{N\to\infty} \left\{ \frac{\left(\frac{x}{N}\right)^{-\alpha}}{\Gamma(-\alpha)} \sum_{j=0}^{N-1} \frac{\Gamma(j-\alpha)}{\Gamma(j+1)} f\left(x - j\frac{x}{N}\right) \right\} \qquad \ldots(1.19) $$

1.5 Mittag-Leffler functions [Oldham,1974]:

In this section, the definition and some properties of two classical Mittag-Leffler functions are presented. We start with the function:

$$ E_{v} = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(vk+1)}, \quad z \in \mathbb{C};\ \mathcal{R}e(v) > 0 $$

Which is known as the Mittag-Leffler function, and as an example when $v = 1$ and $v = 2$:

$$ E_{1}(z) = e^{z} \text{ and } E_{2}(z) = \cosh(z) $$

While when $v = n \in \mathbb{N}$, the following differentiation formula hold for the function $E_{n}(\lambda z^{n})$:

$$ \left(\frac{d}{dz}\right)^{n} E_{n}(\lambda z^{n}) = \lambda E_{n}(\lambda z^{n}), n \in \mathbb{N};\ \lambda \in \mathbb{C} $$

and

$$ \left(\frac{d}{dz}\right)^{n} \left[z^{n-1} E_{n}\left(\frac{\lambda}{z^{n}}\right)\right] = \frac{(-1)^{n}}{z^{n+1}} E_{n}\left(\frac{\lambda}{z^{n}}\right), z \neq 0;, n \in \mathbb{N};\ \lambda \in \mathbb{C} $$

Also, when $v = \frac{1}{n}$ $(n \in \mathbb{N}\{1\})$, the function $E_{\frac{1}{n}}(z)$ has the following representation:

$$E_{\frac{1}{n}}(z) = e^{z^n}\left[1 + n\int_0^z e^{-t^n}\left(\sum_{k=1}^{n-1}\frac{t^{k-1}}{\Gamma\left(\frac{k}{n}\right)}\right)dt\right],\ \ n \in \mathbb{N}\{1\}$$

A two-parameter function of the Mittag-Leffler type is defined by the series expansion:

$$E_{v,y}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(vk+y)},\ \ v > 0,\ \ y > 0 \qquad \ldots(1.20)$$

It follows from the definition (1.20) that:

$$E_{1,1}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty}\frac{z^k}{\mathrm{k}!} = e^z$$

$$E_{1,2}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty}\frac{z^k}{(\mathrm{k}+1)!} = \frac{e^z-1}{z}$$

$$E_{1,3}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty}\frac{z^k}{(\mathrm{k}+2)!} = \frac{e^z-z-1}{z^2}$$

and in general:

$$E_{1,m}(z) = \frac{1}{z^{m-1}}\left\{e^z - \sum_{k=0}^{m-2}\frac{z^k}{k!}\right\}$$

1.6 Analytic Methods for Solving Fractional Order Differential Equations, [Oldham,1974]:

In the present section, some analytical methods are presented for solving fractional order differential equations, and among such method:

1.6.1 The Inverse Operator Method:

Consider the fractional order differential equation:

$$\frac{d^\alpha f}{dx^\alpha} = F \qquad \ldots(1.21)$$

where f is an unknown function and $\frac{d^{\alpha}}{dx^{\alpha}}$ is a fractional order derivative of Riemann-Liouville sense, hence upon taking the inverse operator $\frac{d^{-\alpha}}{dx^{-\alpha}}$ to the both sides of eq.(1.21) gives:

$$f = \frac{d^{-\alpha}F}{dx^{-\alpha}} \qquad \ldots(1.22)$$

additional terms must be added to eq.(1.22), which are:

$$c_1 x^{\alpha-1}, c_2 x^{\alpha-2}, \ldots, c_m x^{\alpha-m}$$

and hence:

$$f - \frac{d^{-\alpha}}{dx^{-\alpha}} \frac{d^{\alpha}}{dx^{\alpha}} f = c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + \ldots + c_m x^{\alpha-m}$$

where $c_1, c_2, \ldots, c_m$ are an arbitrary constants to be determined from the initial conditions and $m-1 < \alpha \leq m$.

Thus:

$$f - c_1 x^{\alpha-1} - c_2 x^{\alpha-2} - \ldots - c_m x^{\alpha-m} = \frac{d^{-\alpha}}{dx^{-\alpha}} \frac{d^{\alpha}}{dx^{\alpha}} f = \frac{d^{-\alpha}}{dx^{-\alpha}} F$$

Hence, the most general solution of eq.(1.21) is given by:

$$f = \frac{d^{-\alpha}}{dx^{-\alpha}} F + c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + \ldots + c_m x^{\alpha-m}$$

where $m-1 < \alpha \leq m$.

As an illustration, we shall consider the following example:

Example (1.1):

Consider the fractional order differential equation:

$$\frac{d^{3/2}}{dx^{3/2}} f(x) = x^5 \qquad \dots(1.23)$$

Applying $\frac{d^{-3/2}}{dx^{-3/2}}$ to the both sides of eq.(1.23), we get:

$$f(x) = \frac{d^{-3/2} x^5}{dx^{-3/2}} + c_1 x^{1/2} + c_2 x^{-1/2}$$

<u>1.6.2 Laplace Transform Method:</u>

In this section, we shall seek a transform of $d^m f \big/ dx^m$ for all m and differintegrable *f*, i.e., we wish to relate:

$$\mathcal{L}\left\{\frac{d^m f}{dx^m}\right\} = \int_0^\infty \exp(-sx) \frac{d^m f}{dx^m} dx$$

to the Laplace transform $\mathcal{L}$ *{f}* of the differintegrable function. Let us first recall the well-known transforms of integer-order derivatives:

$$\mathcal{L}\left\{\frac{d^m f}{dx^m}\right\} = s^m \mathcal{L}\{f\} - \sum_{k=0}^{m-1} s^{m-k-1} \frac{d^k f}{dx^k}(0), \quad m = 1,2,3, \dots$$

and multiple integrals:

$$\mathcal{L}\left\{\frac{d^m f}{dx^m}\right\} = s^m \mathcal{L}\{f\}, \ \mathrm{m} = 0, -1, -2, \dots \qquad \dots(1.24)$$

and note that both formulae are embraced by:

$$\mathcal{L}\left\{\frac{d^m f}{dx^m}\right\} = s^m \mathcal{L}\{f\} - \sum_{k=0}^{m-1} s^k \frac{d^{m-k-1} f(0)}{dx^{m-k-1}}, \quad m = 0, \pm 1, \pm 2, \dots \qquad \dots(1.25)$$

Also, formula (1.25), can be generalized to include noninteger *m* by the simple extension:

$$\mathcal{L}\left\{\frac{d^m f}{dx^m}\right\} = s^m \mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^k \frac{d^{m-k-1} f(0)}{dx^{m-k-1}} \qquad \dots(1.26)$$

where *n* is the integer such that $n-1 < m \le n$. The sum is empty vanishes when $m \le 0$.

In proving (1.26), we first consider $m < 0$, so that the Riemann-Liouville definition:

$$\frac{d^m f}{dx^m} = \frac{1}{\Gamma(-m)} \int_0^x \frac{f(y)}{[x-y]^{m+1}} dy, \ \mathrm{m} < 0$$

May be adopted and upon direct application of the convolution theorem:

$$\mathcal{L}\left\{\int_0^x f_1(x-y) f_2(y) dy\right\} = \mathcal{L}\{f_1\}\, \mathcal{L}\{f_2\}$$

Then gives:

$$\mathcal{L}\left\{\frac{d^m f}{dx^m}\right\} = \frac{1}{\Gamma(-m)} \mathcal{L}\{x^{-1-m}\} \mathcal{L}\{f\} = s^m \mathcal{L}\{f\}, \ m < 0 \ \ldots\ldots\ldots\ldots\ldots\ldots \ldots(1.27)$$

So that equation (1.24) generalized unchanged for negative *m*.

For noninteger positive *m*, we use the result, [Oldham,1974]:

$$\left[\frac{d^m f}{dx^m}\right] = \frac{d^n}{dx^n}\left[\frac{d^{m-n} f}{dx^{m-n}}\right]$$

where *n* is the integer such that $n-1 < m \leq n$.

Now, on application of the formula (1.25), we find that:

$$\mathcal{L}\left\{\frac{d^m f}{dx^m}\right\} = \mathcal{L}\left\{\frac{d^n}{dx^n}\left[\frac{d^{m-n} f}{dx^{m-n}}\right]\right\}$$

$$= s^n \mathcal{L}\left\{\frac{d^{m-n} f}{dx^{m-n}}\right\} - \sum_{k=0}^{n-1} s^k \frac{d^{n-k-1}}{dx^{n-k-1}}\left[\frac{d^{m-n} f}{dx^{m-n}}\right](0) \qquad \ldots(1.28)$$

The difference $m-n$ being negative, the first right-hand term may be evaluated by use of (1.28).since $m-n < 0$,the composition rule may be applied to the terms within the summation. The result:

$$\mathcal{L}\left\{\frac{d^m f}{dx^m}\right\} = s^m \mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^k \frac{d^{m-k-1} f(0)}{dx^{m-k-1}}, \qquad 0 < m \neq 1,2,\ldots$$

Follows from these two operations and is seen to be incorporated in (1.26).

The transformation (1.26) is a very simple generalization of the classical formula for the Laplace transform of the derivative or integral of f. No similar generalization exists, however, for the classical formulae, [Oldham,1974]:

$$\mathcal{L}\left\{\frac{-f}{x}\right\} = \frac{d^{-1}L\{f\}}{ds^{-1}}(s) - \frac{d^{-1}\mathcal{L}\{f\}}{ds^{-1}}(\infty)$$

$$\mathcal{L}\{-xf\} = \frac{d\mathcal{L}\{f\}}{ds}$$

$$\mathcal{L}\{[-x]^n f\} = \frac{d^n\mathcal{L}\{f\}}{ds^n}, \qquad n = 1,2,\ldots$$

As a final result of this section we shall establish the useful formula:

$$\mathcal{L}\left\{\exp(-kx)\frac{d^m}{dx^m}[fe^{kx}]\right\} = [s+k]^m\mathcal{L}\{f\}$$

As an illustration, we consider the following example:

Example (1.2), [Abdulkhalik,2008]:

Consider the semi differential equation:

$$\frac{df}{dx} + \frac{d^{1/2}f}{dx^{1/2}} - 2f = 0 \qquad \ldots\ldots(1.29)$$

By using Laplace transformation

$$sF(s) - f(0) + \sqrt{s}F(s) - \frac{d^{1/2}}{dx^{1/2}}f(0) - 2F(s) = 0$$

And after making use of the equations (1.25) and (1.27), we get:

$$F(s) = \frac{f(0)+\frac{d^{1/2}f}{dx^{1/2}}(0)}{s+\sqrt{s}-2} = \frac{c}{[\sqrt{s}-1][\sqrt{s}+2]}$$

where c is a constant. A partial fraction decomposition gives:

$$F(s) = \frac{A}{3[\sqrt{s}-1]} - \frac{B}{3[\sqrt{s}+2]}$$

which upon Laplace inversion produces

$$f(x) = \frac{c}{3}\left[\frac{1}{\sqrt{\pi x}} + \exp(x)\,erfc(-\sqrt{x})\right] - \frac{c}{3}\left[\frac{1}{\sqrt{\pi x}} - 2\exp(4x)\,erfc(2\sqrt{x})\right]$$

$$= \frac{c}{3}\left[2\exp(4x)\, erfc\left(2\sqrt{x}\right) + \exp(x)\, erfc\left(-\sqrt{x}\right)\right]$$

As the solution of the semi differential equation.

1.7 Variable Order Fractional Derivatives for Functions of One Variable [Dina,2015]:

Our goal in this section is to consider fractional derivatives of variable order, with α depending on time. In fact, some phenomena at physics are better described when the order of the fractional operator is not constant, for example, in the discussion process in an inhomogeneous or heterogeneous medium. Or processes where changes at the environment modify the dynamic of the particle [Chechkin,2005], [Santamaria,2006], [Sun,2009].

1.7.1 Variable order Caputo fractional derivatives:

Motivated by the above considerations, we introduce three types for Caputo fractional derivatives. The order of the derivative is considered as a function *α(t)* by taking values on the open interval (0,1). To start, we define two different kinds of Riemann-Liouville fractional derivatives.

Definition 1.1, [Dina,2015] (Riemann–Liouville fractional derivatives of order α(t)-types I and II):

Given a function $u : [a, b] \longrightarrow R$,

1. The type *I* left Riemann-Liouville fractional derivative of order *α(t)* is defined by:

$$ {}_aD_t^{\alpha(t)}u(t) = \frac{1}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_a^t (t-\tau)^{-\alpha(t)}u(\tau)d\tau$$

2. The type *I* right Riemann-Liouville fractional derivative of order *α(t)* is defined by

$${}_tD_b^{\alpha(t)}u(t) = \frac{-1}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_t^b (\tau - t)^{-\alpha(t)}u(\tau)d\tau$$

3. The type *II* left Riemann-Liouville fractional derivative of order *α(t)* is defined by

$${}_aD_t^{\alpha(t)}u(t) = \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha(t))}\int_a^t (t-\tau)^{-\alpha(t)}u(\tau)d\tau\right)$$

4. The type *II* right Riemann-Liouville fractional derivative of order α(t) is defined by

$${}_tD_b^{\alpha(t)}u(t) = \frac{d}{dt}\left(\frac{-1}{\Gamma(1-\alpha(t))}\int_t^b (\tau - t)^{-\alpha(t)}u(\tau)d\tau\right)$$

The Caputo derivatives are given using the previous Riemann–Liouville fractional derivatives as follows:

Definition 1.2, [Dina,2015] (Caputo fractional derivatives of order α(t)-types I, II and III):

Given a function *u : [a, b]* ⟶ *R*

1. The type *I* left Caputo derivative of order *α(t)* is defined by:

$${}_a^cD_t^{\alpha(t)}u(t) = {}_aD_t^{\alpha(t)}\big(u(t) - u(a)\big)$$
$$= \frac{1}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_a^t (t-\tau)^{-\alpha(t)}[u(\tau) - u(a)]d\tau$$

2. The type *I* right Caputo derivative of order *α(t)* is defined by:

$${}_t^cD_b^{\alpha(t)}u(t) = {}_tD_b^{\alpha(t)}\big(u(t) - u(b)\big)$$
$$= \frac{-1}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_t^b (\tau - t)^{-\alpha(t)}[u(\tau) - u(b)]d\tau$$

3. The type *II* left Caputo derivative of order *α(t)* is defined by:

$${}_a^cD_t^{\alpha(t)}u(t) = {}_aD_t^{\alpha(t)}\big(u(t) - u(a)\big)$$

$$= \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha(t))}\int_a^t (t-\tau)^{-\alpha(t)}[u(\tau)-u(a)]d\tau\right)$$

4. The type *II* right Caputo derivative of order *α(t)* is defined by:

$${}_t^cD_b^{\alpha(t)}u(t) = {}_tD_b^{\alpha(t)}\big(u(t)-u(b)\big)$$

$$= \frac{d}{dt}\left(\frac{-1}{\Gamma(1-\alpha(t))}\int_t^b (\tau-t)^{-\alpha(t)}[u(\tau)-u(b)]d\tau\right)$$

5. The type *III* left Caputo derivative of order *α(t)* is defined by

$${}_a^cD_t^{\alpha(t)}u(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_a^t (t-\tau)^{-\alpha(t)}u'(\tau)d\tau$$

6. The type *III* right Caputo derivative of order *α(t)* is defined by:

$${}_t^cD_b^{\alpha(t)}u(t) = \frac{-1}{\Gamma(1-\alpha(t))}\int_t^b (\tau-t)^{-\alpha(t)}u'(\tau)d\tau$$

<u>Remark 1.1, [Jinsheng,2014]:</u>

From the definition (1.2)(5), we can get the following formula ($0 < \alpha(t) \le 1$):

$$\mathrm{D}^{\alpha(t)}\mathrm{x}^{\beta} = \begin{cases} 0 & \beta = 0 \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha(t))}\mathrm{x}^{\beta-\alpha(t)} & \beta = 1,2,3\ ... \end{cases} \qquad \ldots(1.30)$$

<u>Remark 1.2:</u>

In contrast with the case when α is a constant, definitions of different types do not coincide.

Chapter Two

Numerical Solution for Variable Order Fractional Differential Equations Using Bernstein Polynomials

2.1 Introduction:

This chapter consists of six sections, section 2.2 is oriented about the definition of Bernstein polynomials and their properties, in section 2.3 a matrix representation for the Bernstein polynomials is given, in section 2.4 function approximation using Bernstein polynomials and its operational matrices are presented, section 2.5 focused on the numerical solution of variable-order fractional differential equation using Bernstein operational matrices, finally in section 2.6 some illustrative examples are given.

2.2 Bernstein polynomials [Korovkin,2001]:

In the mathematical field for numerical analysis 'a Bernstein polynomial' named after Sergei Natanovich Bernstein, is a polynomial in the Bernstein form, that is a linear combination of Bernstein basis polynomials.

Polynomials in Bernstein form were first used by Bernstein in a constructive proof for the Stone-Weierstrass approximation theorem.

Definition2.1, [Korovkin,2001]:

The $n + 1$ Bernstein basis polynomials of degree n are defined as:

$$B_{i,n}(x) = \binom{n}{i} x^i (1 - x)^{n-i} \quad , i = 0,1,2, \dots, n$$

Where $\binom{n}{i}$ is a binomial coefficient.

The Bernstein basis polynomials of degree n form a basis at the vector space Π_n of polynomials of degree at most n.

A linear combination of Bernstein basis polynomials:

$$B_n(x) = \sum_{i=0}^{n} c_i B_{i,n}(x)$$

is called a Bernstein polynomial or polynomial in Bernstein form of degree n.

The coefficients c_i are called Bernstein coefficients or Bézier coefficients.

2.2.1 Properties of Bernstein Polynomials:

The Bernstein basis polynomials have the following properties:

1- $B_{i,n}(x) = 0$, if $i < 0$ or $i > n$.

2- $B_{i,n}(0) = \delta_{i,0}$ and $B_{i,n}(1) = \delta_{i,n}$, where δ is the Kronecker delta function given by:

$$\delta_{i,n} = \begin{cases} 1, & if\ i = n \\ 0, & if\ i \neq n \end{cases}$$

3- $B_{i,n}(x)$ has a root with multiplicity i at a point $x = 0$.

4- $B_{i,n}(x)$ has a root with multiplicity $(n - i)$ at a point $x = 1$.

5- $B_{i,n}(x) \geq 0$ for $x \in [0,1]$.

6- $B_{i,n}(1 - x) = B_{n-i,n}(x)$.

7- The derivative can be written as a combination of two polynomials of lower degree:

$$B'_{i,n}(x) = n\left(B_{i-1,n-1}(x) - B_{i,n-1}(x)\right)$$

8- The integral is constant for a given n

$\int_0^1 B_{i,n}(x)dx = \frac{1}{n+1}, \forall i = 0,1,\dots,n$

9- If $n \neq 0$, then $B_{i,n}(x)$ has a unique local maximum of the interval [0,1] at $x = \frac{i}{n}$. This maximum takes the value:

$i^i n^{-n}(n-i)^{n-i}\binom{n}{i}$

10- The Bernstein basis polynomials of degree *n* form a partition of unity:

$\sum_{i=0}^{n} B_{i,n}(x) = \sum_{i=0}^{n}\binom{n}{i} x^i(1-x)^{n-i} = (x+(1-x))^n = 1$

11- By taking the first derivative of $(x+y)^n$, where $y = 1-x$, it can show that:

$\sum_{i=0}^{n} iB_{i,n}(x) = nx$

12- The second derivative of $(x+y)^n$ where $y = 1-x$ can be used to show

$\sum_{i=1}^{n} i(i-1)B_{i,n}(x) = n(n-1)x^2$

13- A Bernstein polynomial can always be written as a linear combination for polynomials of higher degree:

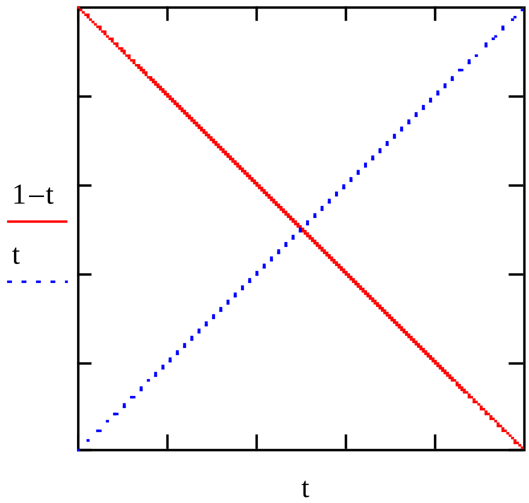
$B_{i,n-1}(x) = \frac{n-i}{n}B_{i,n}(x) + \frac{i+1}{n}B_{i+1,n}(x)$

Remark 2.1 [Kenneth,2000]:

1- Bernstein polynomials of degree 1 are:

$B_{0,1}(t) = 1-t$, $B_{1,1}(t) = t$

And can be plotted for $0 \leq t \leq 1$ as:



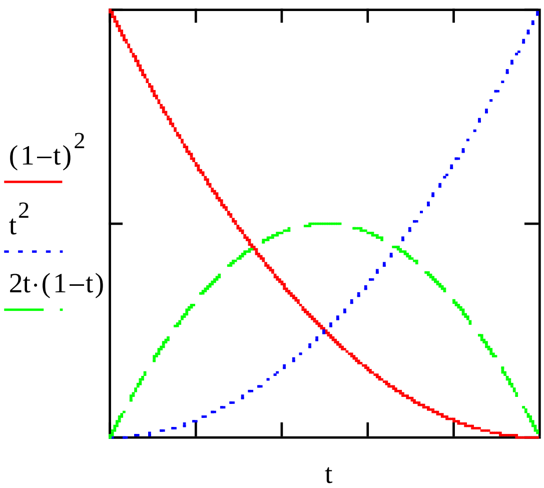
2- Bernstein polynomials of degree 2 are:

$B_{0,2}(t) = (1 - t)^2$

$B_{1,2}(t) = 2t(1 - t)$

$B_{2,2}(t) = t^2$

And can be plotted for $0 \le t \le 1$, as:



3- Bernstein polynomials of degree 3 are:

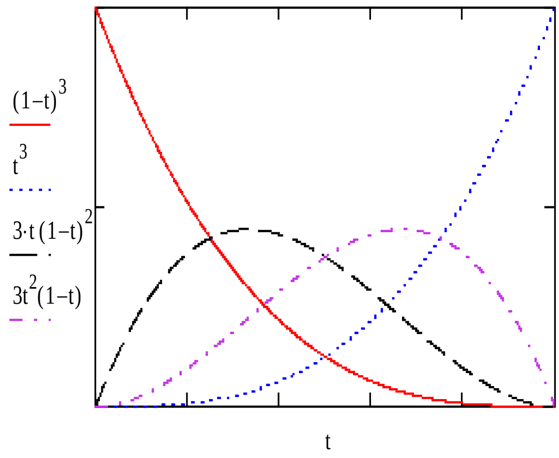
$B_{0,3}(t) = (1 - t)^3$

$B_{1,3}(t) = 3t(1 - t)^2$

$B_{2,3}(t) = 3t^2(1 - t)$

$B_{3,3}(t) = t^3$

And can be plotted for $0 \le t \le 1$, as:



2.2.2 A Recursive Definition of the Bernstein Polynomials, [Kenneth,2000]:

The Bernstein polynomials of degree n can be defined by blending together two Bernstein polynomials of degree $n - 1$. That is, the i^{th} n^{th}-degree Bernstein polynomial can be written as:

$$B_{i,n}(t) = (1 - t)B_{i,n-1}(t) + tB_{i-1,n-1}(t)$$

To show this, we need only use the definition of the Bernstein polynomials and some simple algebra:

$$(1 - t)B_{i,n-1}(t) + tB_{i-1,n-1}(t) = (1-t)\binom{n-1}{i}t^i(1-t)^{n-1-i} + t\binom{n-1}{i-1}t^{i-1}(1-t)^{n-1-(i-1)}$$

$$= \binom{n-1}{i}t^i(1-t)^{n-i} + \binom{n-1}{i-1}t^i(1-t)^{n-i}$$

$$= \left[\binom{n-1}{i} + \binom{n-1}{i-1}\right]t^i(1-t)^{n-i}$$

$$= \binom{n}{i}t^i(1-t)^{n-i}$$

$$= B_{i,n}(t)$$

2.3 A Matrix Representation for the Bernstein Polynomials, [Kenneth,2000]:

In many applications, a matrix formulation for the Bernstein polynomials is useful. These are straightforward to develop if one only looks on a linear combination in terms of dot products. Given a polynomial written as a linear combination of the Bernstein basis functions:

$$B(t) = c_0 B_{0,n}(t) + c_1 B_{1,n}(t) + \cdots + c_n B_{n,n}(t) \qquad \ldots(2.1)$$

It is easy to write this as a dot product of two vectors:

$$B(t) = [B_{0,n}(t) \quad B_{1,n}(t) \quad \cdots \quad B_{n,n}(t)] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} \qquad \ldots(2.2)$$

$$B(t) = [1 \quad t \quad t^2 \ldots \quad t^n] \begin{bmatrix} b_{0,0} & 0 & 0 & \cdots & 0 \\ b_{1,0} & b_{1,1} & 0 & \cdots & 0 \\ b_{2,0} & b_{2,1} & b_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n,0} & b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} \qquad \ldots(2.3)$$

where the $b_{i,j}$ are the coefficients of the power basis that are used to determine the respective Bernstein polynomials. We note that the matrix in this case is lower triangular.

In the quadratic case ($n = 2$), the Bernstein polynomial is:

$$B(t) = [1 \quad t \quad t^2] \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

and in the cubic case ($n = 3$), the Bernstein polynomial is:

$$B(t) = [1 \quad t \quad t^2 \quad t^3] \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ 1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Now, we define:

$$\Phi(t) = [B_{0,n}(t), B_{1,n}(t), \dots, B_{n,n}(t)]^T \qquad \dots(2.4)$$

or in matrix form:

$$\Phi(t) = AT_n(t) \qquad \dots(2.5)$$

where

$$A = \begin{bmatrix} (-1)^0\binom{n}{0} & (-1)^1\binom{n}{0}\binom{n-0}{1} & \dots & (-1)^{n-0}\binom{n}{0}\binom{n-0}{n-0} \\ 0 & (-1)^0\binom{n}{1}\binom{n-1}{0} & \dots & (-1)^{n-1}\binom{n}{1}\binom{n-1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (-1)^0\binom{n}{n} \end{bmatrix}$$

and

$$T_n(t) = [1, t, t^2, \dots, t^n]^T$$

Clearly:

$$T_n(t) = A^{-1}\Phi(t) \qquad \dots(2.6)$$

2.4 Function Approximation using Bernstein Polynomials [Saadatmandi,2013]:

A function $u(t) \in L^2(0,1)$ can be expressed in terms of the Bernstein polynomials basis. In practice only the first $(n + 1)$ terms of Bernstein polynomials are considered. Hence:

$$u(t) = \sum_{i=0}^{n} c_i B_{i,n}(t) = c^T \Phi(t) \qquad \dots(2.7)$$

where $c = [c_{0,} c_{1,\dots,} c_n]^T$ and $\Phi(t) = [B_{0,n}(t), B_{1,n}(t), \dots, B_{n,n}(t)]^T$. Then we have:

$$c = H^{-1}(u, \Phi(t)) \qquad \dots(2.8)$$

where H is an $(n + 1) \times (n + 1)$ matrix, which is called the dual matrix of $\Phi(t)$.

$$H = \int_0^1 \Phi(t)\Phi^T(t)dt \ = \int_0^1 \left(AT_n(t)\right)\left(AT_n(t)\right)^T dt$$

$$= A\left(\int_0^1 T_n(t){T_n}^T(t)dt\right)A^T$$

$$= AQA^T \qquad \dots (2.9)$$

where Q is a Hilbert matrix given by:

$$Q=\begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n+1} \end{bmatrix} \qquad \dots(2.10)$$

Then:

$$c_i = \int_0^1 u(t)d_{i,n}(t)dt, \quad i = 0,1,\dots,n$$

$d_{i,n}$ has derived explicit representations:

$$d_{j,n}(t) = \sum_{k=0}^{n} \lambda_{j,k}B_{k,n}(t),\ j = 0,1,\dots,n$$

For the dual basis functions, defined by the coefficients:

$$\lambda_{j,k} = \frac{(-1)^{j+k}}{\binom{n}{j}\binom{n}{k}}\sum_{i=0}^{\min(j,k)}(2i+1)\binom{n+i+1}{n-j}\binom{n-i}{n-j}\binom{n+i+1}{n-k}\binom{n-i}{n-k}$$

$$\dots(2.11)$$

for $j,k = 0,1,\dots,n$.

2.4.1 Operational Matrix of $D^q\Phi(t)$ Based on Bernstein Polynomials, [Saadatmandi,2013]:

The derivative of the vector $\Phi(t)$ can be expressed by:

$$\frac{d}{dt}\Phi(t) = D^{(1)}\Phi(t)$$

where $D^{(1)}$ is the $(n+1)\times(n+1)$ operational matrix of derivative and

$$\frac{d}{dt}\Phi(t) = D^{(1)}\Phi(t) = \frac{d}{dt}[AT_n(t)] = A\frac{d}{dt}[1,t,t^2,\dots,t^n]^T = AD^{(1)}A^{-1}\Phi(t)$$

... (2.12)

where:

$$D^{(1)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n & 0 \end{bmatrix} \qquad \ldots(2.13)$$

By using eq.(2.12), it is clear that:

$$\frac{d^n}{dt^n}\,\Phi(t) = (D^{(1)})^n\Phi(t)$$

Where $n \in \mathbb{N}$ and the superscript, in $D^{(1)}$, denotes matrix powers. Thus:

$$D^{(n)} = (D^{(1)})^n, n = 1,2,\ldots \qquad \ldots(2.14)$$

<u>**Theorem 2.1, [Saadatmandi,2013]**</u>**:**

Let $\Phi(t)$ be Bernstein vector defined in (2.4) and also suppose $q > 0$, then:

$$D^q\Phi(t) \simeq D^{(q)}\Phi(t)$$

where $D^{(q)}$ is the $(n+1)\times(n+1)$ operational matrix of fractional derivative of order q in caputo sense and is defined as follows:

$$D^{(q)} = \begin{bmatrix} \sum_{j=\lceil q\rceil}^{n}\omega_{0,j,0} & \sum_{j=\lceil q\rceil}^{n}\omega_{0,j,1} & \cdots & \sum_{j=\lceil q\rceil}^{n}\omega_{0,j,n} \\ \vdots & \vdots & & \vdots \\ \sum_{j=\lceil q\rceil}^{n}\omega_{i,j,0} & \sum_{j=\lceil q\rceil}^{n}\omega_{i,j,1} & \cdots & \sum_{j=\lceil q\rceil}^{n}\omega_{i,j,n} \\ \vdots & \vdots & & \vdots \\ \sum_{j=\lceil q\rceil}^{n}\omega_{n,j,0} & \sum_{j=\lceil q\rceil}^{n}\omega_{n,j,1} & \cdots & \sum_{j=\lceil q\rceil}^{n}\omega_{n,j,n} \end{bmatrix}$$

Here $\omega_{i,j,l}$ is given by:

$$\omega_{i,j,l} = (-1)^{j-i}\binom{n}{i}\binom{n-i}{j-i}\frac{\Gamma(j+1)}{\Gamma(j+1-q)}\sum_{k=0}^{n}\lambda_{l,k}\mu_{k,j} \qquad \ldots(2.15)$$

where $\lambda_{l,k}$ is given in eq.(2.11) and:

$\mu_{kj} = \sum_{s=k}^{n}(-1)^{s-k}\binom{n}{k}\binom{n-k}{s-k}\frac{1}{\mathrm{j-q+s+1}}$

Proof:

Using the linearity property of D^q

$D^q\big(c_1 f_1(t) + c_2 f_2(t)\big) = c_1 D^q f_1(t) + c_2 D^q f_2(t)$

and the equation:

$B_{i,n}(t) = \sum_{j=i}^{\mathrm{n}}(-1)^{j-i}\binom{n}{i}\binom{n-i}{j-i}t^j, \quad i = 0,1,\dots,n$

we have:

$$D^q B_{i,n}(t) = \sum_{j=i}^{\mathrm{n}}(-1)^{j-i}\binom{n}{i}\binom{n-i}{j-i}D^q(t^j)$$

$$= \sum_{j=\lceil q\rceil}^{\mathrm{n}}(-1)^{j-i}\binom{n}{i}\binom{n-i}{j-i}\frac{\Gamma(\mathrm{j}+1)}{\Gamma(\mathrm{j}+1-\mathrm{q})}t^{j-q}, \quad i = 0,1,\dots,n$$

…(2.16)

Now, approximate t^{j-q} by Bernstein polynomials, we have:

$$t^{j-q} \simeq \sum_{l=0}^{\mathrm{n}} u_{l,j}B_{l,n}(t) \quad \text{…(2.17)}$$

where :

$$u_{l,j} = \int_0^1 t^{j-q}\, d_{l,n}(t)dt = \sum_{k=0}^{\mathrm{n}}\lambda_{l,k}\int_0^1 t^{j-q}\, B_{k,n}(t)dt$$

$$= \sum_{k=0}^{\mathrm{n}}\lambda_{l,k}\sum_{s=k}^{\mathrm{n}}(-1)^{s-k}\binom{n}{k}\binom{n-k}{s-k}\int_0^1 t^{j-q+s}\, dt$$

$$= \sum_{k=0}^{\mathrm{n}}\lambda_{l,k}\sum_{s=k}^{\mathrm{n}}(-1)^{s-k}\binom{n}{k}\binom{n-k}{s-k}\frac{1}{\mathrm{j-q+s+1}}$$

$$= \sum_{k=0}^{n}\lambda_{l,k}\mu_{k,j}$$

Employing equations (2.16) and (2.17), we get:

$$D^q B_{i,n}(t) \simeq \sum_{j=\lceil q\rceil}^{\mathrm{n}}\sum_{l=0}^{\mathrm{n}}(-1)^{j-i}\binom{n}{i}\binom{n-i}{j-i}\frac{\Gamma(\mathrm{j}+1)}{\Gamma(\mathrm{j}+1-\mathrm{q})}u_{l,j}B_{l,n}(t)$$

$$= \sum_{l=0}^{n} \left(\sum_{j=\lceil q \rceil}^{n} \omega_{i,j,l} \right) B_{l,n}(t) \qquad \ldots(2.18)$$

where $\omega_{i,j,l}$ is given in eq. (2.15). Rewrite eq. (2.18) as a vector form, we have for all $i = 0,1,\ldots,n$:

$$D^{q} B_{i,n}(t) \simeq \left[\sum_{j=\lceil q \rceil}^{n} \omega_{i,j,0} \quad \sum_{j=\lceil q \rceil}^{n} \omega_{i,j,1} \quad \cdots \quad \sum_{j=\lceil q \rceil}^{n} \omega_{i,j,n} \right] \Phi(t)$$

2.4.2 Operational Matrix of $D^{q(t)}\Phi(t)$ Based on Bernstein Polynomials:

In order to transform both integer and fractional order differential operators into matrix forms.

Firstly, the following equation can be easily obtained for the first order differential operator:

By equations (2.5) and (2.12):

$$\frac{d}{dt} \Phi(t) = A D^{(1)} A^{-1} \Phi(t)$$

By combining (2.7) with (2.12), the following result is obtained:

$$\frac{d}{dt} u(t) = \frac{d}{dt} [c^{T} \Phi(t)] = c^{T} \frac{d}{dt} \Phi(t) = c^{T} A D^{(1)} A^{-1} \Phi(t) \qquad \ldots(2.19)$$

Secondly, using (1.30) the following equation can be obtained for the variable order fractional differential operator:

$$D_t^{q(t)} \Phi(t) = D_t^{q(t)} [A T_n(t)]$$

$$= A D_t^{q(t)} [1, t, t^2, \ldots, t^n]^{T}$$

$$= A G A^{-1} \Phi(t) \qquad \ldots(2.20)$$

where:

$$G(t) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-q(t))} t^{-q(t)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-q(t))} t^{-q(t)} & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-q(t))} t^{-q(t)} \end{bmatrix}$$

…(2.21)

By combining (2.7) with (2.20), the following result is obtained:

$$D_t^{q(t)}u(t) = D_t^{q(t)}[c^T\Phi(t)]$$

$$= c^T D_t^{q(t)}\Phi(t)$$

$$= c^T AGA^{-1}\Phi(t) \qquad \ldots(2.22)$$

2.5 Bernstein Operational Matrix of Variable Order Fractional Derivative for Solving variable Order Fractional Differential Equations:

In this section the Bernstein polynomials and it is operational matrices are used to solve the variable order fractional differential equation:

$$D_t^{q(t)}u(t) + \lambda_1 u'(t) + \lambda_2 u(t) = f(t), \quad u(0) = u_0 \qquad \ldots(2.23)$$

where $f(t) \in L^2[0,1]$ is known, $u(t) \in L^2[0,1]$ is the unknown function which we want to approximate, λ_1, λ_2 and u_0 are all constants.

In order to solve eq. (2.23), we set the approximate solution of equation (2.23) to be:

$$u(t) \approx \sum_{i=0}^{n} c_i B_{i,n}(t) = c^T\Phi(t) \qquad \ldots(2.24)$$

Where $\mathrm{c} = [\mathrm{c}_0, \mathrm{c}_1, \ldots, \mathrm{c}_n]^T$ and $\Phi(t) = [B_{0,n}(t), B_{1,n}(t), \ldots, B_{n,n}(t)]^T$.

Operating $D_t^{q(t)}$ to the both sides of eq. (2.24) and by using equation (2.22), thus we have:

$$D_t^{q(t)}u(t) = c^T D_t^{q(t)}\Phi(t) = c^T AGA^{-1}\Phi(t) \qquad \ldots(2.25)$$

Substituting eqs. (2.24) and (2.25) into eq. (2.23), then it can be written in the following form:

$$c^T AGA^{-1}\Phi(t) + \lambda_1 c^T AD^{(1)}A^{-1}\Phi(t) + \lambda_2 c^T \Phi(t) = f(t) \qquad \ldots(2.26)$$

$$c^T \Phi(0) = \mathrm{u}_0$$

Consequently, by calculating the values of Φ and f on [0,1], using $t_i = \frac{2i+1}{2(n+1)}$, for $i = 0,1,\ldots,n$, therefore we get the following system of algebraic equations:

$$c^T AGA^{-1}\Phi(t_i) + \lambda_1 c^T AD^{(1)}A^{-1}\Phi(t_i) + \lambda_2 c^T \Phi(t_i) = f(t_i) \qquad \ldots(2.27)$$

$$c^T \Phi(0) = \mathrm{u}_0$$

One can obtain the unknown c by solving a system of algebraic equations given by (2.27) and by substituting into eq. (2.24), the desired solution is obtained.

2.6 Illustrative Examples:

In this section, two illustrative examples are presented and we compare the numerical solution for variable order fractional differential equations that we have been obtained by using Bernstein polynomials with the analytical solution and with the existing methods such as [Chen,2015] in order to illustrate the efficiency and simplicity of the proposed method.

Example 2.1:

Consider the following linear variable order fractional differential equation:

$$D_t^{q(t)} u(t) + u(t) = f(t),\ t \in [0,1] \qquad \ldots(2.28)$$

$$u(0) = 5$$

Where:

$$q(t) = 2e^t,\ f(t) = \frac{3t^{(3-q(t))}}{\Gamma(4-q(t))} - \frac{t^{(1-q(t))}}{\Gamma(2-q(t))} + t^3 - t + 5$$

The exact solution of this equation is given by $\mathrm{u(t)} = t^3 - t + 5$.

We suppose the approximate solution of equation (2.28), for *n=3* to be:

$$u(t) = \sum_{i=0}^{3} c_i B_{i,3}(t) = c^T \Phi(t) \qquad \ldots(2.29)$$

where:

$$\mathrm{c} = [\mathrm{c}_0, \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3]^T$$

and

$$\Phi(t) = [B_{0,3}(t), B_{1,3}(t), B_{2,3}(t), B_{3,3}(t)]^T$$

and

$$B_{0,3}(t) = 1 - 3t + 3t^2 - t^3$$

$$B_{1,3}(t) = 3t - 6t^2 + 3t^3$$

$$B_{2,3}(t) = 3t^2 - 3t^3$$

$$B_{3,3}(t) = t^3$$

By applying the proposed method given in section 2.5 eq. (2.28) can be transformed into the following equation:

$$c^T A G A^{-1} \Phi(t) + c^T \Phi(t) = f(t) \qquad \ldots(2.30)$$

$$c^T \Phi(0) = 5$$

Where:

$$G(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-q(t))} t^{-q(t)} & 0 & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-q(t))} t^{-q(t)} & 0 \\ 0 & 0 & 0 & \frac{\Gamma(4)}{\Gamma(4-q(t))} t^{-q(t)} \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and by taking $t_i = \frac{2i+1}{2^n}$, for $i = 0,1,\dots,n$ we get a system of algebraic equations in terms of c as follows:

$$c^T AGA^{-1}\Phi(t_i) + c^T\Phi(t_i) = f(t_i),\ i = 0,1,2,3 \qquad \dots(2.31)$$

$$c^T\Phi(0) = 5$$

Solving the obtained system, one can get the unknown $c = [5,4.667,4.333,4.999]^T$ and by substituting into eq. (2.29) hence the approximate solution of eq. (2.28) is reached.

Table 2.1 represent the approximate solution of example (2.1) using the proposed method compared with the exact solution.

Table 2.1

Comparison between the exact solution with the proposed method.

t	***Exact solution***	***The proposed method***
0.2	4.808	4.808
0.4	4.664	4.664
0.6	4.616	4.616
0.8	4.712	4.711
1	5	5

Figure 2.1 represent a comparison between the analytical and the numerical solution of example (2.1)

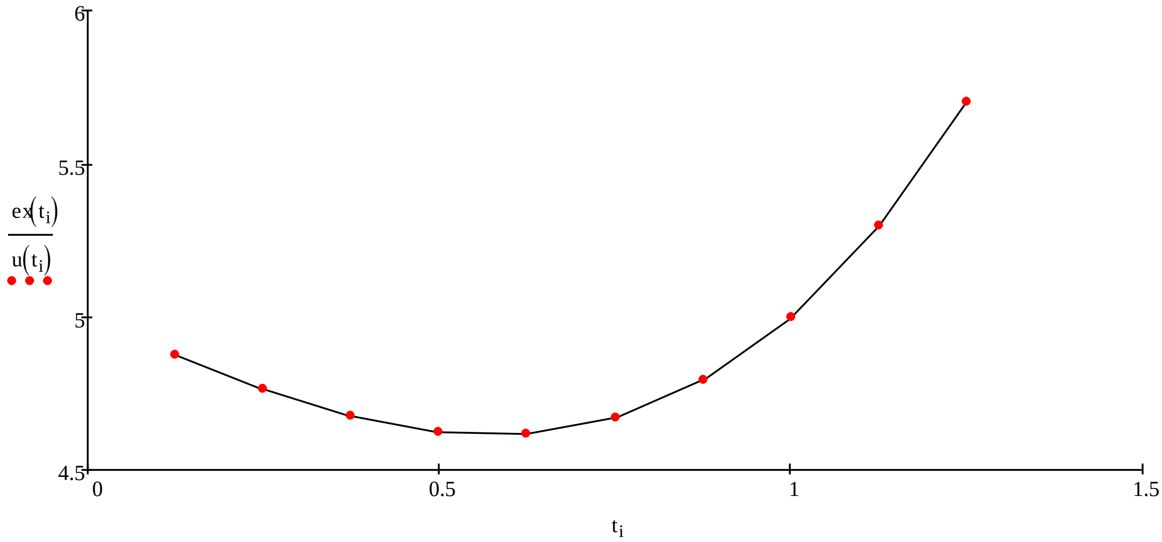


Figure 2.1: The analytical and the numerical solution of example 2.1.

Example 2.2:

Consider the following linear variable order fractional differential equation:

$$D_t^{q(t)}u(t) - 10u'(t) + u(t) = f(t),\ t \in [0,1] \qquad \dots(2.32)$$

$$u(0) = 5$$

where $q(t) = \frac{t+2e^t}{7}$, $f(t) = 10\left(\frac{t^{(2-q(t))}}{\Gamma(3-\mathrm{q}(t))} + \frac{t^{(1-q(t))}}{\Gamma(2-q(t))}\right) + 5t^2 - 90t - 95.$

The exact solution of this equation is given by $u(t) = 5(1+t)^2$.

We suppose the approximate solution of equation (2.32), for $n = 3$ to be:

$$u(t) = \sum_{i=0}^{3} c_i B_{i,3}(t) = c^T \Phi(t) \qquad \dots(2.33)$$

where:

$$\Phi(t) = [B_{0,3}(t), B_{1,3}(t), B_{2,3}(t), B_{3,3}(t)]^T$$

$$\mathrm{c} = [\mathrm{c}_0, \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3]^T$$

and

$$B_{0,3}(t) = 1 - 3t + 3t^2 - t^3$$

$$B_{1,3}(t) = 3t - 6t^2 + 3t^3$$

$$B_{2,3}(t) = 3t^2 - 3t^3$$

$$B_{3,3}(t) = t^3$$

By applying the proposed method given in section 2.5 eq. (2.32) can be transformed into the following equation:

$$c^T AGA^{-1}\Phi(t) - 10c^T AD^{(1)}A^{-1}\Phi(t) + c^T\Phi(t) = f(t) \qquad \dots(2.34)$$

$$c^T\Phi(0) = 5$$

where:

$$G(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-q(t))}t^{-q(t)} & 0 & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-q(t))}t^{-q(t)} & 0 \\ 0 & 0 & 0 & \frac{\Gamma(4)}{\Gamma(4-q(t))}t^{-q(t)} \end{bmatrix}$$

$$D^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and by taking $t_i = \frac{2i+1}{2^n}$, for $i = 0,1,\dots,n$; we get a system of algebraic equations in terms of c as follows:

$$c^T AGA^{-1}\Phi(t_i) - 10c^T AD^{(1)}A^{-1}\Phi(t_i) + c^T\Phi(t_i) = f(t_i), i = 0,1,2,3 \qquad \dots(2.35)$$

$c^T\Phi(0) = 5$

Solving the obtained system, one can get the unknown $c = [5,8.333,13.334,20]^T$ and by substituting c into eq. (2.33) hence the approximate solution of eq. (2.32) is reached.

Table 2.2 represent the approximate solution of example (2.2) using the proposed method compared with the method of [Chen,2015] and the exact solution.

Table 2.2

Comparison between the exact solution with the proposed method and the method [Chen,2015].

t	***Exact solution***	***The proposed method***	***Method [Chen,2015]***
0.2	7.2	7.2	7.2
0.4	9.8	9.8	9.8
0.6	12.8	12.8	12.8
0.8	16.2	16.201	16.2
1	20	20	20

Figure 2.2 represent a comparison between the analytical and the numerical solution of example (2.2).

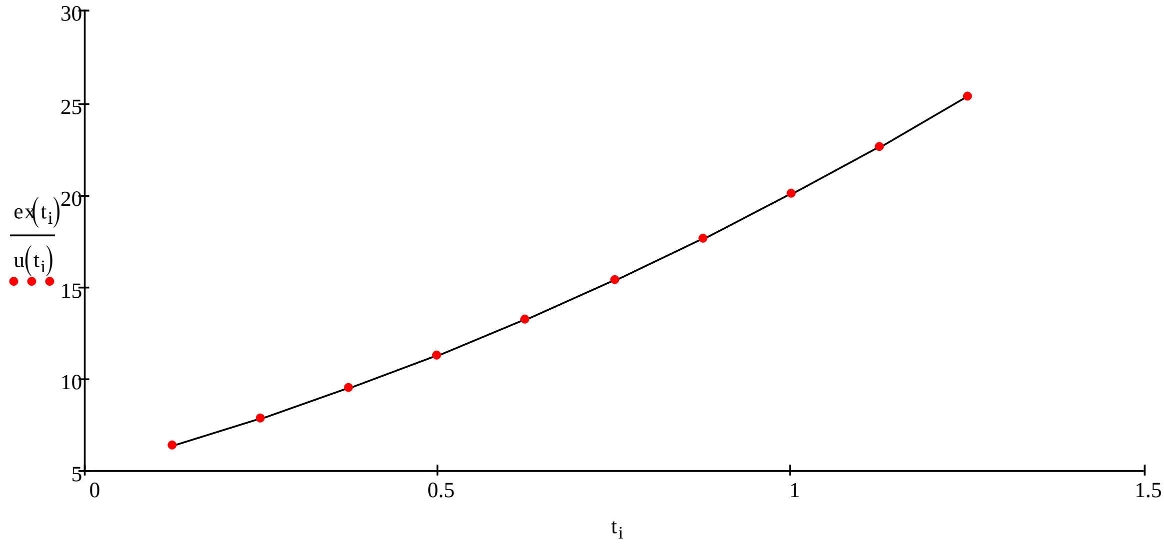


Figure 2.2: Analytical and numerical solution of example 2.2.

Chapter Three

Bernstein Operational Matrices for Solving Multiterm Variable-Order Fractional Differential Equations

3.1 Introduction:

This chapter consists of four sections, in section 3.2 operational matrices of multiterm variable-order fractional derivative based on Bernstein polynomials are given, section 3.3 focused on the numerical solution of multiterm variable order fractional differential equations using Bernstein operational matrices, finally in section 3.4 some illustrative examples are given.

3.2 Operational Matrices of $D^{\alpha(t)}\Phi(t)$ and $D^{\beta_i(t)}\Phi(t)$, $i = 1, 2, \dots, k$; Based on Bernstein Polynomials:

Define:

$$\Phi(t) = [B_{0,n}(t), B_{1,n}(t), \dots, B_{n,n}(t)]^T$$

and

$$T_n(t) = [1, t, t^2, \dots, t^n]^T$$

Then:

$$\Phi(t) = AT_n(t)$$

where A is given by equation (2.5).

Consider:

$$D^{\alpha(t)}\Phi(t) = D^{\alpha(t)}[AT_n(t)] = AD^{\alpha(t)}[1\ \ t \ldots\ t^n]^T \qquad \ldots(3.1)$$

According to (1.30), we can get:

$$D^{\alpha(t)}\Phi(t) = A[0 \quad \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{1-\alpha(t)} \quad \cdots \quad \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{n-\alpha(t)}]^T$$

$$= A\begin{bmatrix} 0 & 0 & & \ldots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{-\alpha(t)} & & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{-\alpha(t)} \end{bmatrix}\begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix} \qquad \ldots(3.2)$$

$$= AMA^{-1}\Phi(t)$$

where:

$$M = \begin{bmatrix} 0 & 0 & & \ldots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{-\alpha(t)} & & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{-\alpha(t)} \end{bmatrix} \qquad \ldots(3.3)$$

AMA^{-1} is called the operational matrix of $D^{\alpha(t)}\Phi(t)$. Therefore, if we set:

$$u(t) = \sum_{i=0}^{n} c_i B_{i,n}(t) = c^T\Phi(t)$$

$$D^{\alpha(t)}u(t) \simeq D^{\alpha(t)}\left(c^T\Phi(t)\right) = c^T D^{\alpha(t)}\Phi(t) = c^T AMA^{-1}\Phi(t) \qquad \ldots(3.4)$$

Similarly:

$$D^{\beta_i(t)}\Phi(t) = D^{\beta_i(t)}[AT_n(t)] = AD^{\beta_i(t)}[1\ \ t \ldots\ t^n]^T \qquad \ldots(3.5)$$

According to (1.30), we can get:

$$D^{\beta_i(t)}\Phi(t) = A[0 \quad \frac{\Gamma(2)}{\Gamma(2-\beta_i(t))}t^{1-\beta_i(t)} \quad \cdots \quad \frac{\Gamma(n+1)}{\Gamma(n+1-\beta_i(t))}t^{n-\beta_i(t)}]^T$$

$$= A\begin{bmatrix} 0 & 0 & & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\beta_i(t))}t^{-\beta_i(t)} & & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & & \frac{\Gamma(n+1)}{\Gamma(n+1-\beta_i(t))}t^{-\beta_i(t)} \end{bmatrix}\begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix} \quad \ldots(3.6)$$

$$= AN_iA^{-1}\Phi(t)$$

where:

$$N_i = \begin{bmatrix} 0 & 0 & & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\beta_i(t))}t^{-\beta_i(t)} & & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & & \frac{\Gamma(n+1)}{\Gamma(n+1-\beta_i(t))}t^{-\beta_i(t)} \end{bmatrix} \quad \ldots(3.7)$$

AN_iA^{-1} is called the operational matrix of $D^{\beta_i(t)}\Phi(t)$. Thus:

$$D^{\beta_i(t)}u(t) \approx D^{\beta_i(t)}\left(c^T\Phi(t)\right) = c^T D^{\beta_i(t)}\Phi(t) = c^T AN_iA^{-1}\Phi(t) \quad (3.8)$$

3.3 Bernstein Operational Matrices of Variable Order Fractional Derivative for Solving Multiterm Variable-Order Linear Fractional Differential Equations:

The multiterm variable-order linear fractional differential equation is given as follows:

$$D^{\alpha(t)}u(t) + \sum_{j=0}^{k} a_j(t)D^{B_j(t)}u(t) = f(t), 0 < t < 1,\ m-1 < \alpha(t) \le m,$$

$$0 \le B_j(t) \le 1 \ \ldots\ldots\ldots\ldots\ldots(3.9)$$

$$u^{(i)}(0) = u_{i,}\ i = 0,1,\ldots,m-1$$

where $D^{\alpha(t)}u(t)$ and $D^{\beta_j(t)}u(t)$ are fractional derivative in the Caputo sense, $a_j(t)$ are continuous functions and $u_{i,}$ are all constants. When $\alpha(t)$ and $\beta_j(t), j = 1,2,\ldots,k$; are all constants, (3.9) becomes (3.10), namely:

$$D^{\alpha}u(t) + \sum_{j=0}^{k} a_j(t)D^{B_j}u(t) = f(t), 0 < t < 1,\ m-1 < \alpha(t) \le m,$$

$$0 \le B_j \le 1 \ \ldots\ldots\ldots\ldots\ldots(3.10)$$

$u^{(i)}(0) = u_i,\ i = 0,1,\dots,m-1$

In order to solve equation (3.9) we define the approximate solution of equation (3.9) as:

$$u(t) = \sum_{i=0}^{n} c_i B_{i,n}(t)$$

$$= c^T \Phi(t) \qquad \dots(3.11)$$

where $c = [c_0, c_1, \dots, c_n]^T$ and $\Phi(t) = [B_{0,n}(t), B_{1,n}(t), \dots, B_{n,n}(t)]^T$

Also, recall that:

$$D^{\alpha(t)} u(t) = D^{\alpha(t)}(c^T \Phi(t)\ ) = c^T A M A^{-1} \Phi(t) \qquad \dots(3.12)$$

Hence:

$$D^{B_j(t)} u(t) = D^{B_j(t)}\left(c^T \Phi(t)\right) = c^T A N_j A^{-1} \Phi(t) \qquad \dots(3.13)$$

Substituting equations (3.11), (3.12) and (3.13) into equation (3.9), therefore equation (3.9) will be transformed into the following form and as follows:

$$c^T A M A^{-1} \Phi(t) + \sum_{j=1}^{k} a_j(t)\, c^T A N_j A^{-1} \Phi(t) = f(t) \qquad \dots(3.14)$$

$$c^T A D^{(i)} A^{-1} \Phi(0) = u_i\ , i = 0,1,\dots,m-1$$

where $D^{(i)}$ is given by (2.14).

By taking the collection points $t_i = \frac{2i+1}{2(n+1)}$, for $i = 0,1,\dots,n$; equation (3.14) become an algebraic system of equations in terms of the unknown vector *c* as follows:

$$c^T A M A^{-1} \Phi(t_i) + \sum_{j=1}^{k} a_j(t)\, c^T A N_j A^{-1} \Phi(t_i) = f(t_i), i = 0,1,\dots,n$$

$$\dots(3.15)$$

$$c^T A D^{(i)} A^{-1} \Phi(0) = u_{i,}\ i = 0,1,\dots,m-1$$

The vector $c = [c_0, c_1, \dots, c_n]^T$ can be found by solving the resulting algebraic system of equations. Finally, the numerical solution $u(t)$ is obtained by equation (3.11).

3.4 Illustrative Examples:

In this section, to test the accuracy of the proposed method, we present some illustrative examples and we will compare the numerical solution for multiterm variable order linear fractional differential equation using Bernstein polynomials with the analytical solution and the method exist in [Liu,2016] in order to show the efficiency and simplicity of the proposed method.

Example 3.1:

Consider the following linear multiterm variable order linear fractional differential equation:

$$D^{2t}u(t) + \sqrt{t}D^{\frac{t}{3}}u(t) + t^{\frac{1}{3}}D^{\frac{t}{4}}u(t) + t^{\frac{1}{4}}D^{\frac{t}{5}}u(t) + t^{\frac{1}{5}}u(t) = g(t), t \in [0,1] \quad \dots(3.16)$$

$$u(0) = 2$$

$$u'(0) = 0$$

Where:

$$g(t) = -\frac{t^{2-2t}}{\Gamma(3-2t)} - \sqrt{t}\frac{t^{2-\frac{t}{3}}}{\Gamma\left(3-\frac{t}{3}\right)} - t^{\frac{1}{3}}\frac{t^{2-\frac{t}{4}}}{\Gamma\left(3-\frac{t}{4}\right)} - t^{\frac{1}{4}}\frac{t^{2-\frac{t}{5}}}{\Gamma\left(3-\frac{t}{5}\right)} + t^{\frac{1}{5}}(2-\frac{t^2}{2}) \quad \dots(3.17)$$

The analytic solution is given in [Liu,2016] by $u(t) = 2 - \frac{t^2}{2}$.

Consider:

$$u(t) = \sum_{i=0}^{3} c_i B_{i,3}(t) = c^T\Phi(t) \quad \dots(3.18)$$

In this case $n = 3$, where:

$$\mathrm{c} = [\mathrm{c}_0, \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3]^T \quad \text{and} \quad \Phi(t) = [B_{0,3}(t), B_{1,3}(t), B_{2,3}(t), B_{3,3}(t)]^T$$

and

$$B_{0,3}(t) = 1 - 3t + 3t^2 - t^3$$

$$B_{1,3}(t) = 3t - 6t^2 + 3t^3$$

$$B_{2,3}(t) = 3t^2 - 3t^3$$

$$B_{3,3}(t) = t^3$$

According to the above, thus we have:

$$D^{2t}u(t) = D^{2t}(c^T\Phi(t)) = c^T D^{2t}\Phi(t)$$

$$= c^T AMA^{-1}\Phi(t) \qquad \ldots(3.19)$$

where:

$$A = \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-2t)}t^{-2t} & 0 & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-2t)}t^{-q2t} & 0 \\ 0 & 0 & 0 & \frac{\Gamma(4)}{\Gamma(4-2t)}t^{-2t} \end{bmatrix}$$

Similarly:

$$D^{\frac{t}{3}}u(t) = D^{\frac{t}{3}}\left(c^T\Phi(t)\right) = c^T D^{\frac{t}{3}}\Phi(t)$$

$$= c^T AN_1A^{-1}\Phi(t) \qquad \ldots(3.20)$$

$$D^{\frac{t}{4}}u(t) = D^{\frac{t}{4}}\left(c^T\Phi(t)\right) = c^T D^{\frac{t}{4}}\Phi(t)$$

$$= c^T AN_2A^{-1}\Phi(t) \qquad \ldots(3.21)$$

$$D^{\frac{t}{5}}u(t) = D^{\frac{t}{5}}\left(c^T\Phi(t)\right) = c^T D^{\frac{t}{5}}\Phi(t)$$

$$= c^T A N_3 A^{-1} \Phi(t) \qquad \ldots(3.22)$$

where:

$$N_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\frac{t}{3})} t^{-\frac{t}{3}} & 0 & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\frac{t}{3})} t^{-\frac{t}{3}} & 0 \\ 0 & 0 & 0 & \frac{\Gamma(4)}{\Gamma(4-\frac{t}{3})} t^{-\frac{t}{3}} \end{bmatrix}$$

$$N_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\frac{t}{4})} t^{-\frac{t}{4}} & 0 & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\frac{t}{4})} t^{-\frac{t}{4}} & 0 \\ 0 & 0 & 0 & \frac{\Gamma(4)}{\Gamma(4-\frac{t}{4})} t^{-\frac{t}{4}} \end{bmatrix}$$

and

$$N_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\frac{t}{5})} t^{-\frac{t}{5}} & 0 & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\frac{t}{5})} t^{-\frac{t}{5}} & 0 \\ 0 & 0 & 0 & \frac{\Gamma(4)}{\Gamma(4-\frac{t}{5})} t^{-\frac{t}{5}} \end{bmatrix}$$

Substituting eqs. (3.18), (3.19), (3.20), (3.21) and (3.22) into eq. (3.16), we get:

$$c^T AMA^{-1}\Phi(t) + \sqrt{t}\, c^T AN_1A^{-1}\Phi(t) + t^{\frac{1}{3}} c^T AN_2A^{-1}\Phi(t) +$$
$$t^{\frac{1}{4}} c^T AN_3A^{-1}\Phi(t) + t^{\frac{1}{5}} c^T \Phi(t) = g(t) \qquad \ldots(3.23)$$

By taking $t_i = \frac{2i+1}{2(n+1)}$, for $i = 0,1,2,3$; we get a system of algebraic equations:

$$c^T AMA^{-1}\Phi(t_i) + \sqrt{t_i}\, c^T AN_1A^{-1}\Phi(t_i) + t_i^{\frac{1}{3}} c^T AN_2A^{-1}\Phi(t_i) +$$
$$t_i^{\frac{1}{4}} c^T AN_3A^{-1}\Phi(t_i) + t_i^{\frac{1}{5}} c^T \Phi(t_i) = g(t_i),\ i = 0,1,2,3 \qquad \ldots(3.24)$$

$$c^T \Phi(0) = 2$$

$$c^T A D^{(1)} A^{-1} \Phi(0) = 0$$

where:

$$D^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

Solving the obtained system, one can get the unknown vector c^T and as follows:

$$\mathrm{c} = [2,2.001,1.83,1.49]^T$$

and hence the approximate solution of equation (3.16) by substituting c into equation (3.18).

Table 3.1 represent the approximate solution of example (3.1) using the proposed method compared with the method [Liu,2016] and the exact solution.

Table 3.1

Comparison between the exact solution with the proposed method and the method given by [Liu,2016].

T	***Exact solution***	***The proposed method***	***Method [Liu,2016]***
0.125	1.992	1.992	1.992
0.25	1.969	1.969	1.969
0.37	1.932	1.931	1.932
0.5	1.875	1.875	1.875
0.75	1.719	1.718	1.719
1	1.5	1.499	1.5

Figure 3.1 represent the analytical solution and the numerical solution of equation (3.16).

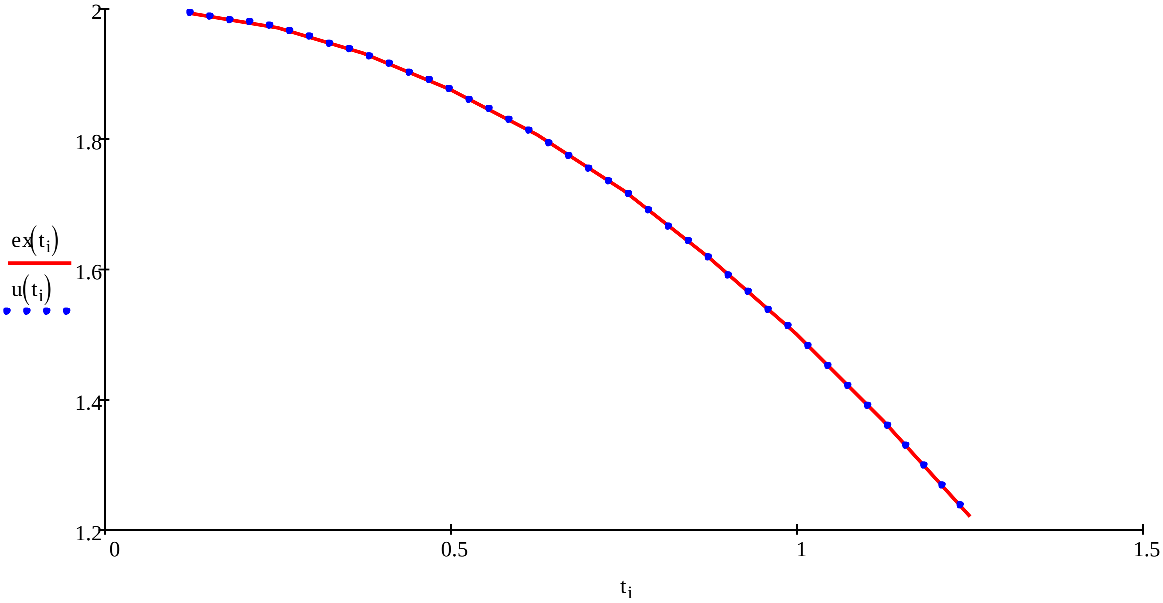


Figure 3.1: Analytical and numerical solution of example 3.1.

Example 3.2:

Consider the following linear fractional differential equation:

$$D^{(2)}u(t) + 3D^{(1)}u(t) + 2D^{0.1379}u(t) + D^{0.0159}u(t) + 5u(t) = g(t), t \in [0,1] \qquad \ldots(3.25)$$

$$u(0) = 0$$

$$u'(0) = 0$$

Where:

$$g(t) = 1 + 3t + \frac{2t^{2-0.1379}}{\Gamma(3-0.1379)} + \frac{t^{2-0.0159}}{\Gamma(3-0.0159)} + \frac{5t^2}{2}$$

The analytic solution is given in [El-Sayed,2004] by $u(t) = \frac{t^2}{2}$.

Consider:

$$u(t) = \sum_{i=0}^{3} c_i B_{i,3}(t) = c^T \Phi(t) \qquad \dots(3.26)$$

In this case n = 3, where:

$$\mathrm{c} = [\mathrm{c}_0, \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3]^T \quad \text{and} \quad \Phi(t) = [B_{0,3}(t), B_{1,3}(t), B_{2,3}(t), B_{3,3}(t)]^T$$

and

$$B_{0,3}(t) = 1 - 3t + 3t^2 - t^3$$

$$B_{1,3}(t) = 3t - 6t^2 + 3t^3$$

$$B_{2,3}(t) = 3t^2 - 3t^3$$

$$B_{3,3}(t) = t^3$$

According to the above, thus we have:

$$D^{(2)}u(t) = D^{(2)}(c^T\Phi(t)) = c^T D^{(2)}\Phi(t) = c^T A D^{(2)} A^{-1}\Phi(t) \qquad \dots(3.27)$$

Where:

$$A = \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$D^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix}$$

Similarly:

$$D^{(1)}u(t) = D^{(1)}\left(c^T\Phi(t)\right) = c^T D^{(1)}\Phi(t) = c^T A D^{(1)} A^{-1}\Phi(t) \qquad \dots(3.28)$$

$$D^{0.1379}u(t) = D^{0.1379}\left(c^T\Phi(t)\right) = c^T D^{0.1379}\Phi(t) = c^T A N_2 A^{-1}\Phi(t)$$

$$\dots(3.29)$$

$$D^{0.0159}u(t) = D^{0.0159}\left(c^T\Phi(t)\right) = c^T D^{0.0159}\Phi(t) = c^T A N_3 A^{-1}\Phi(t)$$

$$\dots(3.30)$$

where:

$$D^{(1)} = N_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$N_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(1.862)} t^{-0.1379} & 0 & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(2.862)} t^{-0.1379} & 0 \\ 0 & 0 & 0 & \frac{\Gamma(4)}{\Gamma(3.862)} t^{-0.1379} \end{bmatrix}$$

and

$$N_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(1.984)} t^{-0.0159} & 0 & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(2.984)} t^{-0.0159} & 0 \\ 0 & 0 & 0 & \frac{\Gamma(4)}{\Gamma(3.984)} t^{-0.0159} \end{bmatrix}$$

Substituting equations (3.26), (3.27), (3.28), (3.29) and (3.30) into equation (3.25), we get:

$$c^T A D^{(2)} A^{-1} \Phi(t) + 3c^T A D^{(1)} A^{-1} \Phi(t) + 2c^T A N_2 A^{-1} \Phi(t) + c^T A N_3 A^{-1} \Phi(t) + 5c^T \Phi(t) = g(t) \qquad (3.31)$$

By taking $t_i = \frac{2i+1}{2(n+1)}, for\ i = 0,1,2,3$ we get a system of algebraic equations:

$$c^T A D^{(2)} A^{-1} \Phi(t_i) + 3c^T A D^{(1)} A^{-1} \Phi(t_i) + 2c^T A N_2 A^{-1} \Phi(t_i) + c^T A N_3 A^{-1} \Phi(t_i) + 5c^T \Phi(t_i) = g(t_i),\ i = 0,1,2,3 \qquad \ldots(3.32)$$

$$c^T \Phi(0) = 0$$

$$c^T A D^{(1)} A^{-1} \Phi(0) = 0$$

Solving the obtained system, one can get the unknown vector c^T, where:

$$\mathrm{c} = [8.408 \times 10^{-4}, -6.625 \times 10^{-4}, 0.166, 0.5]^T$$

and hence the approximate solution of equation (3.25) by substituting c^T into equation (3.26).

Table 3.2 represent the approximate solution of example (3.2) using the proposed method compared with the exact solution.

Table 3.2

Comparison between the exact solution with the proposed method

t	*Exact solution*	*The proposed method*
0.25	0.031	0.031
0.37	0.07	0.07
0.5	0.125	0.125
0.75	0.281	0.281
1	0.5	0.5

Figure 3.2 represent the analytical solution and the numerical solution of equation (3.16).

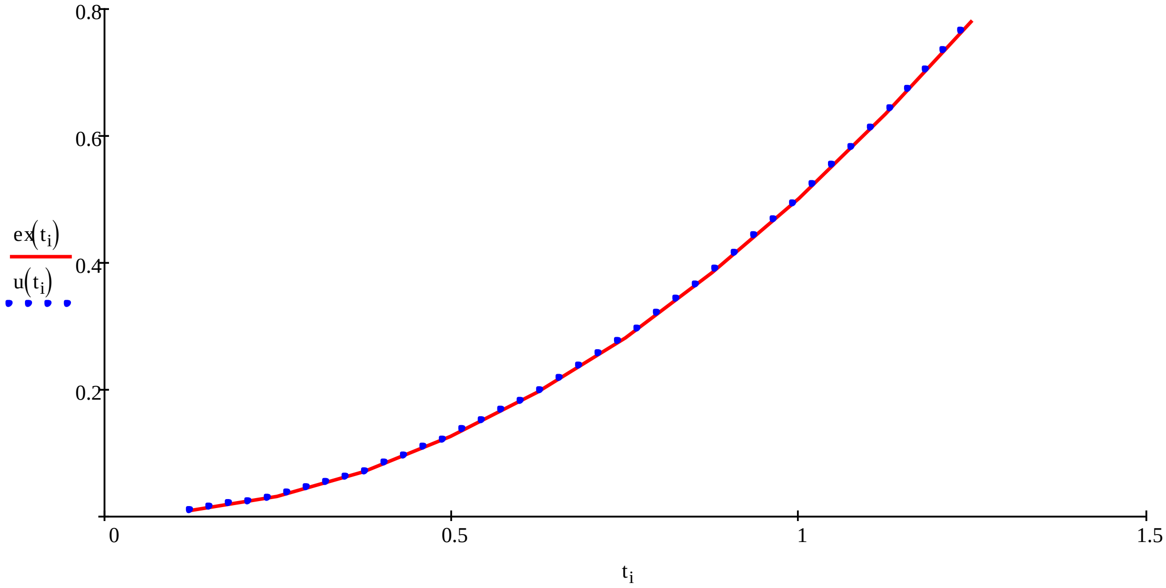


Figure 3.2: Analytical solution and numerical solution of example 3.2.

Conclusions and Future Works

From the study, we can conclude the following:

1. Bernstein operational matrices have been proved to be powerful method for solving linear variable-order and multiterm variable-order fractional differential equations.
2. In this thesis, different kinds of fractional operational matrices in terms of Bernstein polynomials are utilized to seek the numerical solution of the variable-order and multiterm variable-order fractional differential equations.
3. The proposed approach given in this thesis transformed the variable-order and multiterm variable-order differential equations into the product of some matrices which can also be viewed as the system of algebraic equations, solving the resulting system the numerical solution can be obtained.
4. The proposed method is simple in theory and easy in computation, so this method has wide applications in solving the various kinds of variable-order fractional differential equations.

For Future Works we recommend the following problems:

1. Bernstein operational matrices for solving nonlinear multiterm variable-order partial differential equations.
2. Numerical solution of variable-order delay fractional differential equations.
3. Numerical solution of variable-order fractional integro-differential equations.

References

1. Abdulkalik F., “Solution of Fractional Order Delay Differential Equation”, M.Sc. Thesis, Department of Mathematics and Computer Applications, Al-Nahrain University, (2008).
2. Bataineh A., Isik O., Aloushoush N. and Shawagfeh N., “Bernstein Operational Matrix with Error Analysis for Solving High Order Delay Differential Equations”, J. Appl. Comput. Math doi:10.1007/s40819-016-0212-5, (2016).
3. Bhrawy A. H., Baleanu D. and Assas L., “Efficient Generalized Laguerre-Spectral Methods for Solving Multi-Term Fractional Differential Equations on the Half Line”, J. Vib. Control; 20:973-85,(2013).
4. Bhrawy A. H., Doha E. H., Ezz-Eldien S. S. and Gorder R. A. V., “A New Jacobi Spectral Collocation Method for Solving 1+1fractional Schrödinger Equations and Fractional Coupled Schrödinger Systems”, Eur. Phys. J. Plus. 129(12):1-21, (2014).
5. Bhrawy A. H., Doha E. H., Baleanu D. and Ezz-Eldien S. S., “A Spectral Tau Algorithm Based on Jacobi Operational Matrix for Numerical Solution of Time Fractional Diffusion-Wave Equations”, J. Comput. Phys.; 293: 142-56, (2015).
6. Bhrawy A. H., Taha T. M. and Machado J. A., “A Review of Operational Matrices and Spectral Techniques for Fractional Calculus “, Volume 81, Issue 3, pp.1023-1052, August (2015).
7. Caponetto R., Dongola G., Fortuna L. and Petráš I., “Fractional Order Systems: Modeling and Control Applications”, World Scientific, Singapore, (2010).

8. Caputo M., "Linear Models of Dissipation Whose Q is Almost Frequency Independent Part II", J. Roy. Astral. Soc., 13:529-539, (1967).
9. Chechkin A. V., Gorenflo R. and Sokolov I. M., "Fractional Diffusion in Inhomogeneous Media", J. Phys., No. 42, L679-L684, A 38 (2005).
10. Chen C., Liu F., Anh V. and Turner I., "Numerical Schemes with High Spatial Accuracy for a Variable-Order Anomalous Subdiffusion Equation", SIAM J. Sci. Comput., 32(4):1740-60, (2010).
11. Chen Y., Wu Y., Cui Y., Wang Z. and Jin D., "Wavelet Method for a Class of Fractional Convection-Diffusion Equation with Variable Coefficients", Journal of Computational Science, L146, 149, (2010).
12. Chen Y.M., Wei Y.Q., Liu D.Y. and Yu H., "Numerical Solution for a Class of Nonlinear Variable-Order Fractional Differential Equations with Legendre Wavelets", Appl. Math. Lett. (2015), http://dx.doi.org/10.1016/j.aml. 2015.02.010.
13. Chen J., "Analysis of stability and convergence of numerical approximation for the Riesz fractional reaction-dispersion equation", Journal of Xiamen university (Natural Science) 46(5):16-619, (2007).
14. Coimbra C.F.M., "Mechanics with Variable-Order Differential Operators", Ann. Phys., 12(11-12):692-703 (2003).
15. Dina T., Ricardo A. and Delfim F. M., "Caputo Derivatives of Fractional Variable Order: Numerical Approximations", Communications in Nonlinear Science and Numerical Simulation, November (2015).
16. Doha E. H., Bhrawy A. H. and Saker M. A., "Integrals of Bernstein Polynomials an Application for the Solution of High Even-Order Differential Equations", Appl. Math. 559-565, Lett. 24 (2011).
17. El-Sayed A.M.A., El-Mesriy A.E.M. and El-Saka H.A.A., "Numerical solution for multi-term fractional (arbitrary) orders differential equations'', Computational and Applied Mathematics, Vol. 23, N. 1, pp. 33–54, (2004).

18. Geng W., Chen Y., Li Y. and Wang D., “Wavelet Method for Nonlinear Partial Differential Equations of Fractional Order”, Computer and Information Science”, Vol.4, No.5: September (2011).
19. Guo S., Mei L. and Li Y., “Fractional Variational Homotopy Perturbation Iteration Method and its Application to a Fractional Diffusion Equation”, Applied Mathematics and Computation, Volume 219, Issue 11, Pages 5909-5917, 1 February (2013).
20. Gorenflo R., Mainardi F. and Podubny I., “Introduction to Fractional Calculus”, R. Vilela Mendes, July (2008).
21. Hashemizadeh E., Maleknejad K. and Moshsenyzadeh M., “Bernstein operational matrix method for solving physiology problems”, Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran, 34 A34: 92 C 30, (2013).
22. Heydari M. H., Hooshmandasl M. R., Mohammadi F. and Cattani C., “Wavelets Method for Solving Systems of Nonlinear Singular Fractional Volterra Integro Differential Equations”, Commun Nonlinear Sci Num Simul; 19(1): 37-48, (2014).
23. Jafari H. and Seifi S., “Solving a System of Nonlinear Fractional Partial Differential Equations Using Homotopy Analysis Method”, Commun. Nonlinear Sci. Numer. Simul. 14(5):1962, (2009).
24. Jinsheng W., Liqing L., Lechun L. and Yiming C., “Numerical Solution for the Variable Order Fractional Partial Differential Equation with Bernstein polynomials”, International Journal of Advancements in Computing Technology (IJACT) Volume 6, Number 3, May (2014).
25. Kenneth I. Joy , “BERNSTEIN POLYNOMIALS’’, Department of Computer Science University of California, Davis, (2000).
26. Khader M.M., “Introducing an Efficient Modification of the Homotopy Perturbation Method by Using Chebyshev Polynomials”, Arab J. Math. Sci.; 18:61-71 (2012).

27. Korovkin P. P., “Bernstein Polynomials", in Hazewinkel, Michiel, Encyclopedia of Mathematics”, Springer, ISBN 978-1-55608-010-4, (2001).
28. Lin R., Liu F., Anh V. and Turner I., “Stability and Convergence of a New Explicit Finite-Difference Approximation for the Variable-Order Nonlinear Fractional Diffusion Equation”, Appl. Math. Comput., 212:435-45 (2009).
29. Liu J., Xia L. and Limeng W., “An Operational Matrix of Fractional Differentiation of the Second Kind of Chebyshev Polynomial for Solving Multiterm Variable Order Fractional Differential Equation”, Mathematical Problems in Engineering Volume 2016, Article ID 7126080, 10 pages, (2016).
30. Lorenzo C. F. and Hartley T. T., “Initialization, Conceptualization, and Application in the Generalized Fractional Calculus”, Critical Reviews in Biomedical Engineering 35(6): 447-553. February (2007).
31. Ma J., Liu J. and Zhou Z., “Convergence Analysis of Moving Finite Element Methods for Space Fractional Differential Equations”, J. Comput. Appl. Math.; 255:661-70 (2014).
32. Magin R. L., “Fractional Calculus in Bioengineering”, Begell House Publishers, (2006).
33. Maleknejad K., Hashemizadeh E. and Ezzati R., “A New Approach to the Numerical Solution of Volterra Integral Equations by Using Bernstein Approximation”, Commun. Nonlinear Sci. Numer. Simul. 647-655, 16 (2011).
34. Maleknejad K., Hashemizadeh E. and Basirat B., “Computational Method Based on Bernstein Operational Matrices for Nonlinear Volterra-Fredholm Hammerste in Integral Equations”, Commun. Nonlinear Sci. Numer. Simul. 17(1) 52-61, (2012).
35. Meerschaert M. M. and Tadjeran C., “Finite Difference Approximations for Two-Sided Space-Fractional Partial Differential Equations”, Applied Numerical Mathematics 56, 80-90, (2006).

36. Miller K. S. and Ross B., ‘‘an Introduction to the Fractional Calculus and Fractional Differential Equations’’, John Wiley and Sons. Inc., New York, (1993).

37. Mittal R. C. and Nigam R., “Solution of Fractional Integro-Differential Equation by Adomian Decomposition Method”, Int. J Appl. Math. And Mech., 4(2): pp.87-94, (2008).

38. Minardi F., “Fractional Calculus: Some Basic Problems in Continuum and Statistical Mechanics”, Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien and New York, pp. 291-348, Volume 378, (1997).

39. Mohammed O. H., “A Direct Method for Solving Fractional Order Variational Problems by Hat Basis Functions”, Ain Shams Eng. J. (2016), http://dx.doi.org/10.1016/j.asej.2016.11.006.

40. Momani S. and Odibat Z., “Generalized Differential Transform Method for Solving a Space and Time-Fractional Diffusion-Wave Equation”, Phys. Lett. A; 370: 379-87, (2007).

41. Momani S. and Shawagfeh N., “Decomposition Method for Solving Fractional Riccati Differential Equations”, Appl. Math. Comput.; 182: 1083-92, (2006).

42. Nishimoto K., “Fractional Calculus: Integrations and Differentiations of Arbitrary Order”, Koriyama, Japan: Descartes Press Co, (1983).

43. Odibat Z. and Momani S., “The Variational Iteration Method: An Efficient Scheme for Handing Fractional Partial Differential Equations in Fluid Mechanics”, Computers and Mathematics Applications, 58: 2199-2208, (2009).

44. Oldham K. B. and Spanier J., “The Fractional Calculus”, Academic Press, New York, (1974).

45. Podlubny, I., “Fractional Differential Equations”, Academic Press, San Diego, (1999).

46. Saadatmandi A., "Bernstein Operational Matrix of Fractional Derivatives and its Applications", Appl. Math. Modeling (2013), doi: http:// dx.doi.org/ 10.1016/j.apm. 2013.08.007.

47. Soon C. M., Coimbra F. M. and Kobayashi M. H., "The Variable Viscoelasticity Oscillator", Ann. Phys., 14(6): 378-89 (2005).

48. Suarez L. and Shokooh A., "An Eigenvector Expansion Method for the Solution of Motion Containing Fractional Derivatives", J. Appl. Mech.; 64:629-35, (1997).

49. Sun H.G., Chen W. and Chen Y.Q., "Variable Order Fractional Differential Operators in Anomalous Diffusion Modeling", Physica 4586-4592, A. 388 (2009).

50. Sweilam N. H., Khader M. M. and Al-Bar R. F., "Numerical Studies for a Multi-Order Fractional Differential Equation", Phys. Lett. A; 371: 26-33. (2007).

51. Tan Y., Abbasbandy S., "Homotopy Analysis Method for Quadratic Riccati Differential Equation", Commun. Nonlinear Sci. Num. Simul.; 13(3): 539-46, (2008).

52. Wu J. L., "A Wavelet Operational Method for Solving Fractional Partial Differential Equations Numerically", Applied mathematics and computation, Volume 214, Issue 1, 1 August (2009), Pages 31-40.

53. Zhuang P., Liu F., Anh V. and Turner I., "Numerical Methods for the Variable-Order Fractional Advection-Diffusion Equation with a Nonlinear Source Term", SIAM J. Numer. Anal., 47:1760-81 (2009).

الملخص

الغرض الرئيسي لهذه الرسالة يتمحور حول ثلاثة أهداف رئيسية:

الهدف الاول هو دراسة المبادئ الاساسية للحساب الكسري و المعادلات التفاضلية ذات الرتبة الكسرية المتغيرة.

الهدف الثاني هو حل المعادلات التفاضلية ذات الرتبة الكسرية المتغيرة عدديا بأستخدام مصفوفة العمليات لمتعددات حدود بيرنشتاين.

الطريقة المقترحة تعمل على تحويل المعادلات التفاضلية ذات الرتبة الكسرية المتغيرة الى حاصل ضرب مصفوفات والتي يمكن توضيحها على شكل منظومة من المعادلات الجبريه الخطية , الحل لهذه المنظومة يمكننا من أيجاد الحل العددي.

الهدف الثالث هو أيجاد الحل العددي للمعادلات التفاضلية ذات الرتب الكسرية المتغيرة المتعددة بأستخدام مصفوفة العمليات لمتعددات حدود بيرنشتاين, أيضا الطريقة المقترحة سوف تعمل على تحويل هذا النوع من المعادلات الى حاصل ضرب من المصفوفات بعبارة أخرى ألى منظومة من المعادلات الجبرية الخطية و بأمكاننا الوصول ألى الحل العددي عن طريق حل المنظومة الناتجة من المعادلات الجبرية.

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة النهرين
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قسم الرياضيات وتطبيقات الحاسوب

الحلول العددية للمعادلات التفاضلية الخطية ذات الرتبة الكسرية المتغيرة بأستخدام متعددات حدود بيرنشتاين

رسالة
مقدمة الى كلية العلوم – جامعة النهرين
و هي جزء من متطلبات نيل درجة ماجستيرعلوم
في الرياضيات

من قبل
الشيماء عبد الفتاح عمر
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