ABSTRACT

One of the aims of study the fuzzy set theory is to develop the methodology of the formulations and the solutions of problems that are too complicated or ill-defined to be acceptable to analysis by conventioal techniques. Therefore, fuzziness could be considered as a type of imprecision that steams from a grouping of elements into classes that do not have exact defined boundaries. Such classes, introduced by Zadeh L. A., in 1965 as a tool used to describe the ambiguity, vagueness and ambivalence in the mathematical models.

This thesis have three objectives. The first objective is to study fuzzy sets theory and presenting the proof of some well known results in this theory which are either not proof previously or the proofs are not given in details. The second objective is to study and proof the existence and uniqueness theorem of fuzzy differential equations [illegible] fixed point theorem. The third objective is to give an i[illegible] Boundary Value Problems of Fuzzy Differential Equations which had not been introduced previously, as well as, some methods of solution of such type of problems.

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Ammar Jaffar Al-Saedy

2006

SUPERVISOR CERTIFICATION---------------------

I certify that this thesis was prepared under my supervision at the Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University as a partial fulfillment of the requirements for the degree of Master of Science in mathematics

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We certify that we have read this thesis entitled **"Solution of Fuzzy Initial-Boundary Ordinary Differential Equations"** and as examining committee examined the student (Ammar Jaffar Muhesin) in its contents and in what it connected with, and that is in our opinion it meets the standards of a thesis for the degree of Master of Science in Mathematics.

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2 THEROTICAL RESULTS IN FUZZY SETS

Additional theoretical concepts in fuzzy set theory could be discussed concerning fuzzy mapping, differentiation and integration of fuzzy function, etc.; and therefore are presented in this chapter for completeness.

In section one, three types of fuzzy function, had been discussed with some basic properties of such type of functions. In section two, fuzzy mapping with [illegible] properties and [illegible] three, differentiation of fuzzy function [illegible] discussed, which are the differentiation of crisp function on a fuzzy point and differentiation of fuzzifying function at a non-fuzzy point.

Finally, in section four two types of fuzzy integration have been discussed, which are integration of real valued fuzzy function over closed interval and integration of crisp real valued function over fuzzy interval since of it's important in the existence and uniqueness theorem of fuzzy differential equations.

2.1 FUZZY FUNCTIONS [DUBOIS, 1980]

The term "fuzzy function" must be understood in several ways according to where fuzziness occurs. We start first with the first type:

2.1.1 Function with Fuzzy Constraint:

Let X and Y be two universal sets and let f be a classical function $f: X \rightarrow Y$ maps from a fuzzy domain $\tilde{A}$ in X into a fuzzy range $\tilde{B}$ in Y then f is a function with fuzzy constraint if for all $x \in X$, $\mu_{\tilde{B}}(f(x)) \geq \mu_{\tilde{A}}(x)$.

Example(2.1):

Let $X = Y = \mathrm{R}$, and consider two fuzzy sets:

$$\tilde{A} = \{(1,0.5),(2,0.8)\} \text{ and } \tilde{B} = \{(2,0.7),(4,0.9)\}$$

and a function $y = f(x) = 2x$ for all $x \in X$. It is clear that the function f satisfies the condition $\mu_{\tilde{B}}(f(x)) \geq \mu_{\tilde{A}}(x)$ for all $x \in X$.

2.1.2 Fuzzy Extension Function:

This type of fuzzy functions propagates the ambiguity of the independent variables to dependent variables, when f is a crisp function from X onto Y. Let $\tilde{A}$ be a fuzzy subset of X, then $\tilde{B} = f(\tilde{A})$ is a fuzzy subset of Y with membership function defined by:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x), & if \ f^{-1}(y) \neq \varnothing \\ 0, & if \ f^{-1}(y) = \varnothing \end{cases}$$

where $f^{-1}(y)$ is the inverse image of y.

Example (2.2):

Consider the universal sets $X = Y = R$ and consider a crisp function $f(x) = x^2$, with the domain given by the fuzzy set:

$\tilde{A}$ = {(-2, 0.9), (-1, 0.6), (0, 0.7), (1, 0.8), (2, 0.5)},

The independent variable x has an ambiguity and the fuzziness which is propagated to the fuzzy set $\tilde{B}$, then we can obtain $\tilde{B}$, as:

$\tilde{B}$ = {(4, 0.9), (0, 0.7), (1, 0.8)}.

2.1.3 Single Fuzzifying Function :

Fuzzifying function from X onto Y is a mapping from X into the fuzzy power set $\tilde{P}(Y)$(or I^X), i.e., $\tilde{F}: X \to \tilde{P}(Y)$, that is to say the fuzzifying function is a mapping from an ordinary domain to a fuzzy set of range, Fuzzifying function and fuzzy relations [illegible] mathematical manner. So, fuzzifying function [illegible] as a fuzzy relation $\tilde{R}$ defined as follows:

[illegible] where $y = \tilde{F}(x)$

2.1.4 Fuzzy Bunch Function:

Fuzzifying bunch of crisp functions from X onto Y is defined with fuzzy set of crisp function:

$$\tilde{f} = \left\{ (f_i, \mu_{\tilde{f}}(f_i)) \mid f_i : X \to Y, i \in N : N \text{ is the set of natural numbers} \right\}.$$

where $\mu_{\tilde{f}}(f_i)$ is the membership function of the crisp function f_i.

Example (2.3):

X = {1, 2, 3}, $\tilde{f} = \{(f_1, 0.4), (f_2, 0.7), (f_3, 0.5)\}$

where $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = 1 - x$.

2.2 FUZZY MAPPINGS

A fuzzy mapping is a generalization of the concept of a classical mapping which can be understood as follows:

Definition (2.1)[Dubois, 1982]:

A fuzzy mapping $\tilde{f}$ from a crisp set U onto a set V is a mapping from U to the power set of non-empty subsets V, namely $\tilde{P}(V) - \{\varnothing\}$.

In other words, to each element $u \in U$ corresponds a fuzzy set $\tilde{f}(u)$ defined on V, whose membership function is $\mu_{\tilde{f}(u)}$, and $\tilde{f}(u)$ is non-empty.

Other definitions of fuzzy mappings are given also in literatures, namely:

- An ordinary mapping f from U to V with a fuzzy domain [illegible] range [illegible] such that $\mu_{\tilde{A}}(f) \leq$ [illegible].

- An ordinary mapping f from $\tilde{P}(U)$ [illegible] mapping with the fuzzy argument which maps on fuzzy subset of V.

- A fuzzy set $\tilde{F}$ of V^U, i.e., a fuzzy set of ordinary mapping from U to V each mapping f: $U \longrightarrow V$ is assigned a membership grade $\mu_{\tilde{F}}(f)$.

Proposition(2.1)[Dubois, 1982]:

A fuzzy mapping is strictly equivalent to a fuzzy relation $\tilde{R}$ such that

$\forall u \in U, \exists v \in V, \quad \mu_{\tilde{R}}(u,v) = 0$.

Proof: See [Dubois, 1982]. ■

Remarks (2.1):

1. As a converse of proposition (2.1), a fuzzy relation can be viewed as a fuzzy mapping if $\mu_{\tilde{R}}(u,.)$ determines a nonempty fuzzy set $\tilde{f}(u)$.

2. Fuzzy mappings and fuzzy relations have different points of view on the same mathematical notion.

3. Fuzzy set of mappings (FSM's, for short) are not equivalent to fuzzy mapping. Indeed, a natural way of assigning membership grades $\mu(u,v)$ to possible images $v \in V$ of $u \in U$, given an FSM F, is to define $\mu(u,v) = \mu_F(f)$ whenever $v = f(u)$. Note that $\mu(u,v)$ is not uniquely defined since there may exist $f, g: U \to V$, $f \neq g$, such that $v = f(u) = g(u)$ and $\mu_F(f) \neq \mu_F(g)$.

[illegible] represent a fuzzy mapping in terms of the FSM's.

For the first case, there is at least [illegible] FSM from a fuzzy mapping.

Starting from the fuzzy mapping $\tilde{f}: U \to V$ for any $\alpha \in [0,1]$. We can define an ordinary multimapping f_α as follows:

$$f_\alpha(u) = \left\{ v \mid \mu_{\tilde{f}(u)}(v) \geq \alpha \right\} \subseteq V, \text{ for all } u \in U.$$

f_α is the α- cut of $\tilde{f}$.

Also, f_α can be viewed as a crisp subset of V^U, i.e., a set of mappings

$$f_\alpha = \{ f : U \to V \mid \forall u \in U, f(u) \in f_\alpha(u) \}$$

$$= \left\{ f : U \to V \mid \inf_{u \in U} \mu_{\tilde{f}(u)}(f(u)) \geq \alpha \right\}.$$

f_α is the α–cut of an FSM generated by f_α, denoted $\gamma(\tilde{f})$, such that

$$\mu_{\gamma(\tilde{f})}(f) = \inf_{u \in U} \mu_{\tilde{f}(u)}(f(u)),\ \forall f \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.1)}$$

Example(2.4) [Najeib S,W., 2002]:

Let X = {2, 3, 4,…,25}, a fuzzy mapping $\tilde{f}$ maps the elements in X to the power fuzzy set $\tilde{P}(X)$ in the following manner.

$$\tilde{f}(2) = \{(2,0.3),(3,0.5),(4,1),(5,0.5),(6,0.3),(9,0.2)\}$$

$$\tilde{f}(3) = \{(3,0.3),(5,0.5),(7,1),(9,0.5),(14,0.3),(16,0.2)\}$$

$$\tilde{f}(4) = \{(4,0.3),(8,0.5),(12,1),(16,0.5),(20,0.3),(25,0.2)\}$$

Now, given a function $f_1(x) = 2x$,

$$\mu_{\gamma(\tilde{f})}(f) = \inf\left\{\mu_{\tilde{f}(x)}(f_1(x)) \mid x \in X\right\}$$

$$= \inf\left\{\mu_{\tilde{f}(2)}(f_1(2)), \mu_{\tilde{f}(3)}(f_1(3)), \mu_{\tilde{f}(4)}(f_1(4))\right\}$$

$$= \inf\left\{\mu_{\tilde{f}(2)}(4), \mu_{\tilde{f}(3)}(6), \mu_{\tilde{f}(4)}(\text{[illegible]})\right\}$$

$$= \inf\{1,0,0.5\} = 0$$

For $f_2(x) = x^2$, we have

$$\mu_{\gamma(\tilde{f})}(f) = \inf\left\{\mu_{\tilde{f}(x)}(f_2(x)) \mid x \in X\right\}$$

$$= \inf\left\{\mu_{\tilde{f}(2)}(f_2(2)), \mu_{\tilde{f}(3)}(f_2(3)), \mu_{\tilde{f}(4)}(f_2(4))\right\}$$

$$= \inf\left\{\mu_{\tilde{f}(2)}(4), \mu_{\tilde{f}(3)}(9), \mu_{\tilde{f}(4)}(16)\right\}$$

$$= \inf\{1,0.5,0.5\} = 0.5$$

So, $\gamma(\tilde{f}) = \{(f_1,0),(f_2,0.5)\}$, where $f_1(x) = 2x$ and $f_2(x) = x^2$.

Now, to the second case which is the converse of the above construction which is also can be made as expressed in the following definition.

Definition(2.2) [Najeib S.W., 2002]:

Given a fuzzy set of mappings $\gamma(\tilde{f})$ with $\mu_{\gamma(\tilde{f})} : I^X \rightarrow [0,1]$, we can construct a fuzzy mapping $\tilde{f} : X \rightarrow \tilde{P}(X)$ such that $\tilde{f}(x)$ is a fuzzy set with membership function defined as follows :

$$\mu_{\tilde{f}(x)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_{\gamma(f)}(f), & when\, f^{-1}(y) \neq \emptyset \\ 0, & when\, f^{-1}(y) = \emptyset \end{cases} \qquad \text{.....................} (2.2)$$

Example(2.5) [Najeib S.W., 2002]:

In the case of crisp functions $f_1, f_2, f_3,$ [illegible] fuzzy set of mappings will be, for example:

$$\gamma(\tilde{f}) = \{(f_1, 0.2), (f_2, 0.5), (f_3, 0.7), (f_4, 0.3)\}$$

where

$f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = e^x$, $f_4(x) = x + 1$.

Then the image at $x = 1$, is:

$$\mu_{\tilde{f}(1)}(y) = \sup\left\{\mu_{\gamma(\tilde{f})}(f) \mid y = f(x)\right\}$$

The possible values of y is $\{f_1(1), f_2(1), f_3(1), f_4(1)\} = \{1, 1, e, 2\}$

$$\mu_{\tilde{f}(1)}(y) = \sup_{f_1, f_2}\left\{\mu_{\gamma(\tilde{f})}(f_1),\ \ \mu_{\gamma(\tilde{f})}(f_2)\right\}$$

$$= \sup_{f_1, f_2}\{0.2, 0.5\} = 0.5$$

$\mu_{\tilde{f}(1)}(e) = 0.7\,,\ \mu_{\tilde{f}(1)}(2) = 0.3$

So the fuzzy set $\tilde{f}(1) = \{(1,0.5),(2,0.3),(e,0.7)\}$

Similarly

$$\tilde{f}(2) = \{(2,0.2),(4,0.5),(e^2,0.7),(3,0.3)\}$$

$$\tilde{f}(3) = \{(3,0.2),(9,0.5),(e^3,0.7),(4,0.3)\}.$$

Hence:

$$\tilde{f}(x) = \left\{\left(f(x), \sup\{\mu_{\gamma(\tilde{f})}(f) \mid y = f(x), f \in \gamma(\tilde{f})\}\right)\right\},\ \text{for all } x \in X.$$

Lemma (2.1) [Dubois, 1982]:

$\gamma(\tilde{f} \oplus \tilde{g}) \supseteq \gamma(\tilde{f}) \oplus \gamma(\tilde{g})$ where $\tilde{f}$ and $\tilde{g}$ are fuzzy mappings [illegible]

Proof: We have

[illegible]

$$= \sup_{f,g:h(u)=f(u)+g(u)} \inf_{u\in U} \min\{\mu_{\tilde{f}(u)}(f(u)), \mu_{\tilde{g}(u)}(g(u))\}$$

$$\leq \inf_{u\in U} \sup_{f,g:h(u)=f(u)+g(u)} \min(\mu_{\tilde{f}(u)}(f(u)), \mu_{\tilde{g}(u)}(g(u))),$$

and since for any mapping $\varphi: A\times B \to R$.

$$\inf_x \sup_y \varphi(x,y) \geq \sup_x \inf_y \varphi(x,y),$$

hence,

$$\mu_{\gamma(f)\oplus\gamma(g)}(\mathrm{h(u)}) \leq \inf_{u\in U} \mu_{\gamma(f\oplus g)(u)}(h(u))$$

$$\leq \mu_{\gamma(\tilde{f}\oplus\tilde{g})(u)} h(u).\ \blacksquare$$

2.3 FUZZY DIFFERENTIATION [DUBOIS, 1982]

The fuzzy differentiation depends on the type of the considered function in section (2.1), i.e., differentiation of non-fuzzy function over fuzzy interval and that of fuzzifying function at non-fuzzy points, may be considered as a type of fuzzy differentiation.

2.3.1 Differentiation of Crisp Function on Fuzzy Points:

By the extension principle, differentiation $f'(\tilde{A})$ of a non-fuzzy function f at fuzzy point $\tilde{x}_0$ [Dubois, 1982b] is defined as:

$$\mu_{f'(\tilde{x}_0)}(y) = \max_{f'(x)=y} \mu_{\tilde{x}_0}(x)$$

The next example illustrate the above definition:

Example (2.6):

Let $f(x) = x^3$ and suppose the fuzzy poi

$\tilde{A} = \{(-1, 0.4), (0, 1), (1, 0.6)\}$

then $f'(x) = 3x^2$ and therefore:

$$f'(\tilde{A}) = \{(3, 0.4), (0, 1), (3, 0.6)$$

$$= \{(3, 0.6), ((0, 1)\}$$

2.3.2 Differentiation of Fuzzifying Function Over a Set of Non-Fuzzy Points:

For all x belongs to the ordinary domain D, we will define the differentiation of fuzzifying function $\tilde{f}$ at a non-fuzzy point. Let any α-cut of $\tilde{f}$ be differentiable for an arbitrary x in D, we define differentiation $(d\tilde{f}/dx)(x_0)$ at an ordinary point x_0 as:

$$\mu_{(d\tilde{f}/dx)(x_0)}(p) = \underset{(d\tilde{f}_\alpha/dx)(x_0)=p}{Max} \mu_{\tilde{f}}(\tilde{f}_\alpha)$$

The next example illustrates the above definition:

Example (2.7):

Consider the fuzzifying function:

$\tilde{f}$ ={(f_1, 0.4), (f_2, 0.7), (f_3, 0.4)}

where f_1, f_2 and f_3 are crisp functions defined by:

$f_1(x) = x$, $f_2(x) = x^2$ and $f_3(x) = x^3 + 1$

Then $f_1'(x) = 1, f_2'(x) = 2x$ and $f_3'(x) = 3x^2$.

Differentiation of this fuzzifying function at x_0 = 0.5 is obtained to be as:

$f_1'(0.5) = 1$, when $\alpha = 0.4$

$f_2'(0.5) = 1$, when $\alpha = 0.7$

$f_3'(0.5) = 0.75$, when $\alpha = 0.4$.

Hence:

$$\frac{d\tilde{f}}{dx}(x_0) = \max\{(1, 0.4), (1, 0.7), (0.75, 0.4)\}$$

$$= \{(1, 0.7), (0.75, 0.4)\}.$$

Another type of fuzzy differentiation which is called the ***L- R type*** could be used also in differentiating fuzzy functions (for more details, see, e.g., [Dubois, 1982]).

2.3.3 Algebraic Properties of Differentiation

As in non-fuzzy differentiation so many properties are given and proved successfully. Therefore, similarly several algebraic properties undertaking fuzzy differentiation could be given. The proofs will be given here for the sake of completeness.

We start first with the following theorem:

Theorem (2.1):

The extended sum ⊕ of the derivatives of the real valued functions f and g at the fuzzy point $\tilde{x}_0$ is defined by:

$$f'(\tilde{x}_0) \oplus g'(\tilde{x}_0) \supseteq (f' + g')(\tilde{x}_0)$$

Proof: See [Dubois, 1982]. ■

The equality, in the last theorem, holds also which is equipped if certain conditions are given on f and g, as it is ill[illegible] [illegible] the necessary condition for the other direction of theorem (2.1).

Theorem (2.2):

If f' and g' are continuous and both non-decreasing (or non-increasing), then:

$$f'(\tilde{x}_0) \oplus g'(\tilde{x}_0) = (f' + g')(\tilde{x}_0)$$

Proof: See [Dubois, 1982]. ■

The next theorem illustrates the differentiation of product of two functions, which is given in [Dubois, 1982] and other literatures without proof (to the best of our knowledge); which will be presented here for completeness:

Theorem (2.3):

1. If f and g are crisp functions from X to Y and $\tilde{x}_0$ is a fuzzy point in X then:

$$(fg)'(\tilde{x}_0) = (f'g + fg')(\tilde{x}_0) \subseteq [f'(\tilde{x}_0) \odot g(\tilde{x}_0)] \oplus [f(\tilde{x}_0) \odot g'(\tilde{x}_0)].$$

2. If f, g, f' and g' are continuous, f and g are both positive, and f' and g' are both non-decreasing (f, g are negative and f', g' are non decreasing), then:

$$(fg)'(\tilde{x}_0) = [f'(\tilde{x}_0) \odot g(\tilde{x}_0)] \oplus [f(\tilde{x}_0) \odot g'(\tilde{x}_0)]$$

Proof:

1. According to the properties of fuzzy set, we must prove that:

$$\mu_{(fg)'(\tilde{x}_0)}(y) \le \mu_{[f'(\tilde{x}_0)\odot g(\tilde{x}_0)]\oplus[f(\tilde{x}_0)\odot g'(\tilde{x}_0)]}(y),$$

$$\mu_{[f'(\tilde{x}_0)\odot g(\tilde{x}_0)]\oplus[f(\tilde{x}_0)\odot g'(\tilde{x}_0)]}(y) = \sup_{u,v:y=u+v} \min\left\{ \sup_{u=f'(s)g(s)} \mu_{\tilde{x}_0}(s), \sup_{v=f(t)g'(t)} \mu_{\tilde{x}_0}(t) \right\}$$

$$= \sup_{s,t:y=f'(s)g(s)+f(t)g'(t)} \min\left\{ \mu_{\tilde{x}_0}(s), \; \mu_{\tilde{x}_0}(t) \right\} \text{..(2.3)}$$

Also, using the extension principle to the left hand side, one can get:

$$\mu_{(fg)'(\tilde{x}_0)}(y) = \mu_{[f'g+fg'](\tilde{x}_0)}(y) = \sup_{x:y=f'(x)g(x)+f(x)g'(x)} \min\left\{ \mu_{\tilde{x}_0}(x), \mu_{\tilde{x}_0}(x) \right\}$$

$$= \sup_{x:y=f'(x)g(x)+f(x)g'(x)} \mu_{\tilde{x}_0}(x) \text{ ....(2.4)}$$

Now, from equations (2.3) and (2.4), we have:

$$\sup_{x:y=f'(x)g(x)+f(x)g'(x)} \mu_{\tilde{x}_0}(x) \leq \sup_{s,t:y=f'(s)g(s)+f(t)g'(t)} \min\left\{\mu_{\tilde{x}_0}(s),\ \mu_{\tilde{x}_0}(t)\right\}$$

Since $\tilde{x}_0$ is a fuzzy point which has a supremum value therefore,

$$(fg)'(\tilde{x}_0) = (f'g + fg')(\tilde{x}_0) \subseteq [f'(\tilde{x}_0) \square\ g(\tilde{x}_0)] \oplus [f(\tilde{x}_0) \square\ g'(\tilde{x}_0)].$$

2. Since f' and g' are continuous on [a, b] and both non decreasing in [a, b], then:

$\forall\ s, \forall\ t > s, \exists\ x \in [s,t] \subseteq [a,b]$, such that:

$$f'(x)\ g(x) + f'(x)\ g'(x) = f'(s)\ g(s) + f(t)\ g'(t).$$

Hence in particular:

$$f'(x)\ g(x) + f(x)\ g'(x) \leq f'(s)\ g(s) + f(t)\ g'(t)$$

taking x = s, we have:

$$f'(s)\ g(s) + f(s)\ g'(s) \leq f'(s)\ g(s) + f(t)\ g'(t)$$

From the convexity of $\tilde{x}_0$:

$$\mu_{\tilde{x}_0}(x) \geq \min\{\mu_{\tilde{x}_0}(s),\ \mu_{\tilde{x}_0}(t)\}$$

Hence:

$$\mu_{[f'(\tilde{x}_0) \square\ g(\tilde{x}_0)] \oplus [f(\tilde{x}_0) \square\ g'(\tilde{x}_0)]}(y) =$$

$$\sup_{u,v:y=u+v} \min\left\{ \sup_{u=f'(s)g(s)} \mu_{\tilde{x}_0}(s),\ \sup_{v=f(t)g'(t)} \mu_{\tilde{x}_0}(t) \right\}$$

$$= \sup_{x:y=f'(x)g(x)+f(x)g'(x)} \min\left\{ \sup_{u=f'(x)g(x)} \mu_{\tilde{x}_0}(x),\ \sup_{v=f(x)g'(x)} \mu_{\tilde{x}_0}(x) \right\},\ x = s = t.$$

$$= \sup_{x:y=f'(x)g(x)+f(x)g'(x)} \mu_{\tilde{x}_0}(x).$$

Therefore;

$$\mu_{[f'(\tilde{x}_0)\odot g(\tilde{x}_0)]\oplus[f(\tilde{x}_0)\odot g'(\tilde{x}_0)]}(y) = \sup_{x: y=f'(x)g(x)+f(x)g'(x)} \mu_{\tilde{x}_0}(x).$$

But;

$$\mu_{(f.g)'(\tilde{x}_0)}(y) = \mu_{[f'.g+f.g'](\tilde{x}_0)}(y) = \sup_{x: y=f'(x).g(x)+f(x).g'(x)} \mu_{\tilde{x}_0}(y)$$

Then:

$$(f\ g)'(\tilde{x}_0) = [f'(\tilde{x}_0)\odot g(\tilde{x}_0)]\oplus[f(\tilde{x}_0)\odot g'(\tilde{x}_0)]. \quad \blacksquare$$

2.4 FUZZY INTEGRATION

The fuzzy integration is one of the most important part of the analysis of fuzzy set theory. Quite different suggestions have been made [illegible] integrals, so there are three types of fuzzy integration which are:

1. Integration of real fuzzy function over [illegible]
2. [illegible] of real function over fuzzy i[illegible]
3. Integration of real fuzzy function over fuzzy interval.

The first two of the above three types will be discussed next:

2.4.1 Integration of Real Fuzzy Function over Crisp Closed Interval [Dubois, 1982]

We shall now consider a fuzzy function $\tilde{f}$, which shall be integrated over the crisp interval $[a,b]$.

Definition(2.3):

Let $\tilde{f} : X \to \tilde{F}(\mathrm{R})$, the integral of $\tilde{f}$ over $X = [a,b]$ denoted by $\int_X \tilde{f}(t)dt$ is defined levelwise, as follows:

$$\left(\int_X \tilde{f}(t)dt\right)_\alpha = \int_X f_\alpha(t)dt \text{, for all } 0 \le \alpha \le 1$$

$$= \left(\int_X f_{\alpha^-}(t)dt, \int_X f_{\alpha^+}(t)dt\right) \text{..........................(2.5)}$$

Remark (2.2):

If the fuzzy integration over the interval X=[a,b] is reversed from b to a it is easily seen that :

$$\int_X \tilde{f} = \int_a^b \tilde{f} = -\int_b^a \tilde{f}$$

where $-\int_a \tilde{f}$ has for a membership function $\mu_{-\int_b^a \tilde{f}}(u) = \mu_{\int_a^b \tilde{f}}(-u)$.

Following, some properties of the integration of fuzzy function over crisp interval which are given in [Dubois, 1982].

1. $\int_X (\tilde{f} \oplus \tilde{g}) \supseteq \left(\int_X \tilde{f}\right) \oplus \left(\int_X \tilde{g}\right)$, where $\tilde{f}$ and $\tilde{g}$ are real fuzzy functions from the closed interval X to R, with bounded support.

2. Under the commutativity condition for $\int_X$ and $\oplus$,

$$\int_X (\tilde{f} \oplus \tilde{g}) = \left(\int_X \tilde{f} \right) \oplus \left(\int_X \tilde{g} \right) \quad \text{...(2.6)}$$

Example(2.8):

Consider the bunch fuzzy function given in (2.1.4), by:

$\tilde{f} = \{(f_1, 0.4), (f_2, 0.7), (f_3, 0.4)\}$

where

$f_1(x) = x, \;\; f_2(x) = x^2, \;\; f_3(x) = x+1$

and to integrate this bunch function over [1,2], we perform this as follows:

i) Integration at $\alpha = 0.7$, then

$f = f_2(x) = x^2$

and hence

$$I_\alpha(1,2) = \int_1^2 x^2 dx = \frac{1}{3} x^3 \Big]_1^2 = \frac{7}{3}.$$

Hence, the integration result is with possibility 0.7, is given by:

$\tilde{I}_{0.7}(1,2) = \{(\frac{7}{3}, 0.7)\}.$

ii) Integration at $\alpha = 0.4$, there are two functions

$f^+ = f_1(x) = x$ and $f^- = f_3(x) = x+1$, then for

$$I^+_\alpha (1,2) = \int_1^2 x dx = \frac{1}{2} x^2 \Big]_1^2 = \frac{3}{2}.$$

and

$$I_{\alpha}^{-}(1,2) = \int_{1}^{2}(x+1)dx = \frac{1}{2}x^{2} + x\Big]_{1}^{2} = \frac{5}{2}$$

The integration results are with possibility 0.4. Then,

$$\tilde{I}_{0.4}(1,2) = \{(\frac{3}{2}, 0.4), (\frac{5}{2}, 0.4)\}.$$

Finally, we have the total integration.

$$\tilde{I}(1,2) = \{(\frac{7}{3}, 0.7), (\frac{3}{2}, 0.4), (\frac{5}{2}, 0.4)\}$$

2.4.2 Integration of a (Crisp) Real Valued Function Over a Fuzzy Interval

[Klir, G. J., 2000]

A fuzzy domain $\tilde{D}$ of the real line R is ...

normalized convex fuzzy sets. The membership function of each are $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ respectively, where $\mu_{\tilde{a}}(x)$ and ... can be interpreted as the degree of confidence to which x can be c... of D. If $\underline{a}_0$ and $\overline{b}_0$ are the lower/upper limits of the supports of $\tilde{a}$ or $\tilde{b}$, then a_0 or b_0 are related to each other by $\underline{a}_0 = \text{Inf } S(\tilde{a}) \leq \text{Sup } S(\tilde{b}) = \overline{b}_0$.

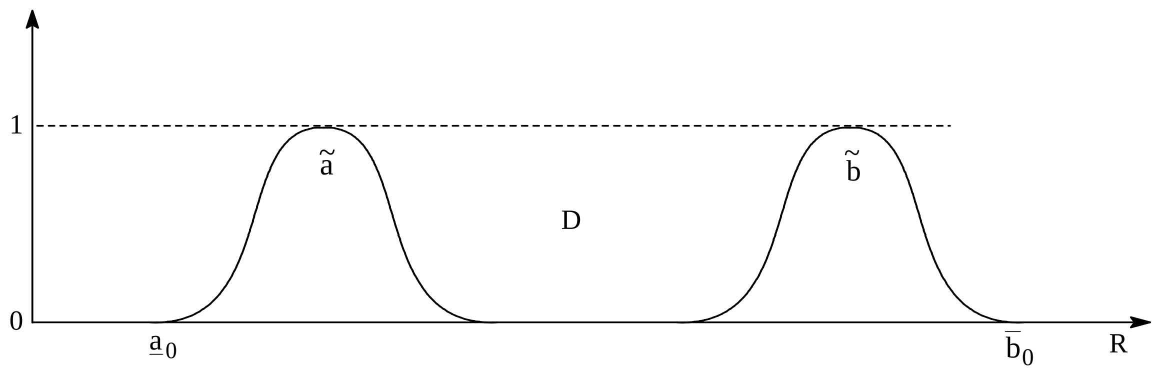

Fig.(2.6) Fuzzily bounded interval.

Definition (2.4)[Klir, G. J., 2000]:

Let f be a real valued function which is integrable in the interval J = [a_0, b_0], then according to the extension principle the membership function of the fuzzy integral $\int_{\tilde{D}} f$ is given by:

$$\mu_{\int_{\tilde{D}} f}(z) = \sup_{x,y \in J : z = \int_x^y f} Min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\}.$$

Some Properties of The Integration of Crisp Function over Fuzzy Interval [Klir, G. J., 2000]:

1. Let f be any function $f : D \to R$, which is integrable on D, then:

$$\int_{\tilde{D}} f = F(\tilde{b}) \ominus F(\tilde{a}).$$

where $\ominus$ denotes the extended subtraction.

2. Let f and g be any two functions $f, g : I \longrightarrow R$ which are integrable on I, then:

$$\int_{\tilde{a}}^{\tilde{b}} (f+g) \subseteq \int_{\tilde{a}}^{\tilde{b}} f \oplus \int_{\tilde{a}}^{\tilde{b}} g$$

where $\subseteq$ denotes the usual fuzzy set inclusion ($\tilde{A} \subseteq \tilde{B} \Leftrightarrow \mu_{\tilde{A}} \leq \mu_{\tilde{B}}$) and $\oplus$ denotes the extended addition.

3. If $f, g : I \longrightarrow R^+$ or $f, g : I \longrightarrow R^-$, then:

$$\int_{\tilde{a}}^{\tilde{b}} (f+g) = \int_{\tilde{a}}^{\tilde{b}} f \oplus \int_{\tilde{a}}^{\tilde{b}} g$$

The following examples illustrate fuzzy integration and its properties:

Example (2.9):

Let:

$\tilde{a} = \{(4, 0.8), (5, 1), (6, 0.4)\}$

$\tilde{b} = \{(6, 0.7), (7, 1), (8, 0.2)\}$

and, $f(x) = 2$, $x \in [a_0, b_0] = [4, 8]$

The problem is to find the fuzzy integration of $f(x)$ over J = [4, 8]. The following table illustrate these results.

Table (2.1)

Integration of f(x)=2, over an interval (a,b) with membership function

(a, b)	$\int_a^b 2\,dx = 2b - 2a$	$Min\{\mu_{\tilde{a}}(a), \mu_{\tilde{b}}(b)\}$
[illegible]	4	[illegible]
(4, 7)	6	0.8
(4, 8)	8	0.2
(5, 6)	2	0.7
(5, 7)	4	1.0
(5, 8)	6	0.2
(6, 6)	0	0.4
(6, 7)	2	0.4
(6, 8)	4	0.2

and by using the definition (2.4), then:

$$\int_{\tilde{D}} f = \{(0, 0.4), (4, 0.7), (4, 1), (6, 0.8), (8, 0.2)\}.$$

Example (2.10):

Let:

$f(x) = 2x - 3,\ g(x) = -2x + 5$

and

$\tilde{a} = \{(1, 0.8), (2, 1), (3, 0.4)\}$

$\tilde{b} = \{(3, 0.7), (4, 1), (5, 0.3)\}$

so:

$$\int_a^b f(x)\,dx = x^2 - 3x \,\Big|_a^b = (b^2 - 3b) - (a^2 - 3a).$$

$$\int_a^b g(x)\,dx = -x^2 + 5x \,\Big|_a^b = (-b^2 + 5b) - (-a^2 + 5a).$$

$$\int_a^b \big(f(x) + g(x)\big)\,dx = 2x \,\Big|_a^b = (2b - 2a).$$

Then:

$$\int_{\tilde{a}}^{\tilde{b}} f = \{(0, 0.4), (2, 0.7), (4, 0.4), (6, 1) \ldots$$ [illegible]

$$\int_{\tilde{a}}^{\tilde{b}} g = \{(-6, 0.3), (-4, 0.3), (-2, 1), (0, 0.8), (2, 0.7)\}$$

$$\int_{\tilde{a}}^{\tilde{b}} f \oplus \int_{\tilde{a}}^{\tilde{b}} g = \{(-6, 0.3), (-4, 0.3), (-2, 0.4), (0, 0.7), (2, 0.7), (4, 1), (6, 0.8), (8, 0.7), (10, 0.3), (12, 0.3), (14, 0.3)\}$$

$$\int_{\tilde{a}}^{\tilde{b}} (f + g) = \{(0, 0.4), (2, 0.7), (4, 1), (6, 0.8), (8, 0.3)\}$$

and it is clear that:

$$\int_{\tilde{a}}^{\tilde{b}} f \oplus \int_{\tilde{a}}^{\tilde{b}} g \supseteq \int_{\tilde{a}}^{\tilde{b}} f + \int_{\tilde{a}}^{\tilde{b}} g.$$

3 SOLUTION OF FUZZY DIFFERENTIAL EQUATIONS

Fuzzy differential equations with initial or boundary conditions are of great importance in applied mathematics, which have some difficulties in their solution. Henceforth, this chapter deals first with the statement and [illegible] of of the existence and uniqueness theorem of fuzzy differential equations using Schauder fuzzy fixed point theorem, which [illegible]

Therefore, this chapter consists, the s[illegible] problem, and modifying the approach to solve the non-homogenous and nonlinear fuzzy initial value problems.

Also, this chapter consists of an introduction to other types of fuzzy differential equations with boundary conditions which is solved by using the shooting method to solve numerically boundary value problems.

3.1 FUZZY DIFFERENTIAL EQUATIONS

Most dynamical real life problems could be formulated as a mathematical model, in which most of them are formulated either as system of ordinary or partial differential equations, especially in mathematical physics. Therefore, studies could be oriented toward two directions. The first direction is the evaluation of the solution and modifying methods to find such solution. While the second orientation is to study the stability of solutions without evaluating this solution explicitly.

In connection with these studies of differential equations a new field appeared recently in the late of 20-th century, which is the so called ***fuzzy differential equations***.

Consider the fuzzy differential equations:

$$y'(t) = f(t, y),\ y(t_0) \approx \tilde{y}_0\ , \forall t \in D \qquad \text{......(3.1)}$$

Where $t_0, \tilde{y}_0$ are given, It's clear that the solution will depend on the fuzzy initial and hence the solution $y(t)$ will be fuzzy and also f is a given function.

Before studying the solution of fuzzy differential equations, we will first study the existence and uniqueness theorem of fuzzy differential equations.

[illegible]

Definition (3.1) [Al-Hamaiwand S.M., 2001]:

For every mapping $T : X \to Y$, then [illegible] as follows:

$$T(\tilde{A}) = \sup\left\{\mu_{\tilde{A}}(w) \mid w \in T^{-1}(x)\right\}$$

where $\tilde{A}$ is a fuzzy set and $T(w) = x$. If X^* and Y^* are fuzzy Banach spaces, then T is called ***fuzzy operator***.

Also, if $T(c_1\tilde{A} + c_2\tilde{B}) = c_1T(\tilde{A}) + c_2T(\tilde{B})$, $\forall \tilde{A}, \tilde{B} \in I^X$ and $c_1, c_2 \in R$ or ***C***, then $\tilde{T}$ is called ***linear operator***.

Definition (3.2)[Najeib S.W., 2002]:

A fuzzy function $\tilde{F} : X \to F(R)$ is called levelwise continuous at $t_0 \in X$ if the mapping F_α is continuous at $t = t_0$ with respect to the Hausdorff metric D_H on *F(R)* for all $\alpha \in [0,1]$.

3.2 THE EXISTENCE AND UNIQUNESS THEOREM OF FUZZY DIFFERENTIAL EQUATIONS USING SCHAUDER FUZZY FIXED POINT THEOREM

In this section, the existence and uniqueness theorem of fuzzy differential equations is considered using Schauder fuzzy fixed point theorem. The proof of the theorem is given by transforming the fuzzy differential equation to an alternative Volterra fuzzy integral equation and then satisfying the conditions of the Schauder fuzzy fixed point theorem.

First, recall the fuzzy version of Schauder fixed point.

<u>Theorem (3.1) (Schauder Fuzzy Fixed Point Theorem) [Al-Hamaiwand S.M., 2001]:</u>

Let I^X be a [illegible] fuzzy Banach space B, and suppose that $T : I^X \to I^X$ is a compact fuzzy operator, then $\tilde{T}$ has a fuzzy fixed point.

[illegible] following definitions and remarks permit us to establish the existence of the solution of the fuzzy initial value problems by proving the existence of a solution of the fuzzy integral equation. This is important, because integrals are in general easier to estimate than derivatives, also integral equations could carry iterations rather than differentiation.

<u>Definition(3.3) [Park and Han, 1999]:</u>

A mapping $y : I \longrightarrow E^n$ is a solution to the problem $y'(t) = f(t, y(t))$, $y(t_0) \square \tilde{y}_0$, , if it is levelwise continuous and satisfy the integral equation:

$$y(t) = \tilde{y}_0 + \int_{t_0}^{t} f(s, y(s))ds, \ \forall t \in [t_0, t] \quad \text{.................................}(3.2)$$

Definition (3.4) [Park and Han, 1999]:

A function $[f]^\alpha$ which satisfies an inequality of the form

$d([f(t, x_2)]^\alpha, [f(t, x_1)]^\alpha) \leq L d([x_2]^\alpha, [x_1]^\alpha)$ ……………………….(3.3)

for all (t, x_1), (t, x_2) in a region D is said to satisfy a Lipschitz condition to $[f]^\alpha$ on D, where d represent the Hausdorff distance

Remark (3.1) [Park and Han, 1999]:

Define δ to be the smaller of the two positive numbers of a and b/M, then the fuzzy integral equation (3.2) is defined on the interval

$$I_\delta = \{t : |t - t_0| \leq \delta\}$$

and on this interval $\|y(t) - \tilde{y}_0\| \leq M|t - t_0| \leq b$ …………………(3.4)

[illegible] (3.2) (The Existence Theorem):

Consider the fuzzy differential equation

$y'(t) = f(t, y)$, $y(t_0)$ [illegible] $\tilde{y}_0$, $t_0 \leq t \leq T$, [illegible]

and suppose $\tilde{A}$ be an open fuzzy subset of $D \equiv R \times C[R^+ \times S(\alpha), R^n]$, where $S(\alpha) = \{\mu \in R^n : d^*(\tilde{y}, \tilde{y}_0) < \alpha\}$ and $f : \tilde{A} \to R^n$ be levelwise continuous and bounded function for any $(t_0, y(t)) \in \tilde{A}$, then there exist a solution to (3.5) which passes through (t_0, y).

Proof:

The proof is based on Schauder fuzzy fixed point theorem (3.1). First of all, in order to fix our symbols, define the following sets:

$I_T = \{t \in R : 0 \leq t \leq T\}$ and $B_\beta = \{\tilde{\psi} \in C : \|\tilde{\psi}\| \leq 1 + \beta\}$.

and suppose that f is bounded function at t_0, that is, there exist $M \in R^+$, such that:

$\|f(t_0, y)\| \leq M$ ……………………………………………………….… (3.6)

Since f is levelwise continues, then there exist $\delta, \beta > 0$, such that:

$\|f(t_0 + t, y + \tilde{\psi})\| \leq M$, where $(t, \tilde{\psi}) \in I_T \times B_\beta$ …………………….. (3.7)

Now, since our proof depends on the Schauder fuzzy fixed point theorem, then it sufficient to prove that I^X is a non empty, closed, bounded, and convex fuzzy subset of a fuzzy Banach space B and then the fuzzy operator $\tilde{T} : I^X \to I^X$ is a compact fuzzy operator.

Another set which will be constructed that contains all fuzzy subsets, such that, for any $\alpha^*, \beta^* \in R$ let:

$$A(\alpha^*, \beta^*) = \left\{ \tilde{\zeta}(t) \in C[R^+ \times S(\alpha^*), R^n]; \tilde{\zeta}(0) \square 0, \tilde{\zeta}(t) \in B_{\beta^*}, t \in I_{\alpha^*} \right\}$$

Let $\tilde{\varphi}$ be another fuzzy function satisfying $\tilde{\varphi}(t) \in C[R^+ \times S(\alpha^*), R^n]$

Such that $\tilde{\varphi}(0) = \tilde{\varphi}(t_0), \tilde{\varphi}(t_0 + t) = \tilde{\varphi}(t_0), t \in I_{\alpha^*}$

Suppose that we have chosen $\beta^* < \beta$, and

$\|\tilde{\varphi}(t_0 + t) - \tilde{\varphi}(t)\| \leq \beta - \beta^*, t \in I_{\alpha^*}, M\alpha^*$ [illegible]

Hence

$$\|\tilde{\zeta}(t) + \tilde{\varphi}(t_0 + t) - \tilde{\varphi}(t)\| \leq \|\tilde{\zeta}(t)\| + \|\tilde{\varphi}(t_0 + t) - \tilde{\varphi}(t)\|$$

$$\leq \beta^* + \beta - \beta^* = \beta < 1 + \beta$$

Now, using (3.7) we have

$$\left\| f\left(t_0 + t, \tilde{\zeta}(t) + \tilde{\varphi}(t_0 + t)\right) \right\| \leq M, t \in I_{\alpha^*}, \tilde{\zeta} \in A(\alpha^*, \beta^*).$$

Now, we will define a fuzzy operator $\tilde{T}$, as follows:

$$\tilde{T} : A(\alpha^*, \beta^*) \to C[R^+ \times S(\alpha), R^n] \subseteq A(\alpha^*, \beta^*).$$

and

$$\tilde{T}(\tilde{\zeta}(t)) = \begin{cases} \tilde{y}_0 + \int_0^t f(s, \tilde{\zeta}(s))ds & , if\ t \in I_{\alpha^*} \\ 0 & , if\ t = 0 \end{cases}$$

Since the fuzzy fixed points of $\tilde{T}$ in $A(\alpha^*, \beta^*)$ is a solution to the fuzzy differential equation with the following constraint:

$$y(t_0 + t) = \tilde{\zeta}(t) + \tilde{\varphi}(t_0 + t).$$

Now, to prove that $A(\alpha^*, \beta^*)$ is a closed, bounded and convex fuzzy subset of $C[R^+ \times S(\alpha^*), R^n]$.

It is clear that $A(\alpha^*, \beta^*)$ is closed and bounded fuzzy set (by construction).

To prove that $A(\alpha^*, \beta^*)$ is convex fuzzy [illegible]

Let $\tilde{\zeta}_1(t), \tilde{\zeta}_2(t) \in A(\alpha^*, \beta^*)$, then

$\tilde{\zeta}_1 \in C[R^+ \times S(\alpha^*), R^n]$, $\tilde{\zeta}_1(0) \square 0$, $\tilde{\zeta}_1$ [illegible]

$\tilde{\zeta}_2 \in C[R^+ \times S(\alpha^*), R^n]$, $\tilde{\zeta}_2(0) \square 0$, $\tilde{\zeta}_2(t) \in B_{\beta^*}$.

To prove

$$\tilde{\zeta}^*(t) = \lambda\tilde{\zeta}_1(t) + (1-\lambda)\tilde{\zeta}_2(t) \in A(\alpha^*, \beta^*).$$

i.e., to prove $\tilde{\zeta}^*(t) \in C[R^+ \times S(\alpha^*), R^n]$, $\tilde{\zeta}^*(0) \square 0$, $\tilde{\zeta}^*(t) \in B_{\beta^*}$.

Now, since $\tilde{\zeta}_1(t), \tilde{\zeta}_2(t) \in C[R^+ \times S(\alpha^*), R^n]$ and since the linear combination of levelwise continuous functions are also levelwise continuous, hence:

$$\tilde{\zeta}^*(t) \in C[R^+ \times S(\alpha^*), R^n].$$

and also

$$\tilde{\zeta}^*(0) = \lambda\tilde{\zeta}_1(0) + (1-\lambda)\tilde{\zeta}_2(0)$$
$$= \lambda.0 + (1-\lambda).0 = 0.$$

Moreover, to prove that $\left\|\tilde{\zeta}^*(t)\right\| \in A(\alpha^*, \beta^*)$ i.e., to prove $\left\|\tilde{\zeta}^*(t)\right\| \leq 1+\beta^*$

$$\left\|\tilde{\zeta}^*(t)\right\| = \left\|\lambda\tilde{\zeta}_1(t)+(1-\lambda)\tilde{\zeta}_2(t)\right\|$$

$$\leq \left\|\lambda\tilde{\zeta}_1(t)\right\| + \left\|(1-\lambda)\tilde{\zeta}_2(t)\right\| = |\lambda|\left\|\tilde{\zeta}_1(t)\right\| + |1-\lambda|\left\|\tilde{\zeta}_2(t)\right\|$$

$$\leq \lambda\beta^* + (1-\lambda)\beta^* = \beta^* < 1+\beta^*.$$

So, $\tilde{\zeta}^*(t) = \lambda\tilde{\zeta}_1(t)+(1-\lambda)\tilde{\zeta}_2(t) \in A(\alpha^*, \beta^*)$

Hence, $A(\alpha^*, \beta^*)$ is a convex set.

Now, since the composition of two levelwise continuous functions is levelwise continuous, hence $\tilde{T}(\tilde{\zeta}(t))$ is also levelwise continuous and since $\tilde{T}(\tilde{\zeta}(0)) \,\square\, 0$ then $\tilde{T}(\tilde{\zeta}(t)) \in C[R^+ \times S(\alpha^*), R^n]$.

Now, to prove $\tilde{T}(\tilde{\zeta}(t)) \in B_{\alpha^*}$; we have:

$$\left\|\tilde{T}(\tilde{\zeta}(t))\right\| = \left\|\tilde{y}_0(t) + \int_0^t f(s,\tilde{\zeta}(s))ds\right\|$$

$$\leq \left\|\tilde{y}_0(t)\right\| + \left\|\int_0^t f(s,\tilde{\zeta}(s))ds\right\|$$

$$\leq 1 + \int_0^t \left\|f(s,\tilde{\zeta}(s))\right\|ds$$

$$\leq 1 + Mt$$

$$\leq 1 + M\alpha^* \leq 1+\beta^*$$

Hence, $\tilde{T}(\tilde{\zeta}(t)) \in B_{\beta^*}$, i.e., $\tilde{T}(\tilde{\zeta}(t)) \in A(\alpha^*, \beta^*)$.

Now, to prove $\tilde{T}$ is levelwise continuous on $A(\alpha^*, \beta^*)$.

Let $t, t_1 \in I_{\alpha^*}$ such that $|t-t_1| < \delta$, then:

$$\left\|\tilde{T}(\tilde{\zeta}(t)) - \tilde{T}(\tilde{\zeta}(t_1))\right\| = \left\|\tilde{y}_0(t) + \int_0^t f\,ds - \tilde{y}_0 - \int_0^{t_1} f\,ds\right\|$$

$$= \left\| \int_{t_1}^{0} f\,ds + \int_{0}^{t} f\,ds \right\|$$

$$= \left\| \int_{t_1}^{t} f\,ds \right\| \leq \int_{t_1}^{t} \| f \| ds \leq \int_{t_1}^{t} M ds$$

$$= M(t - t_1) \leq M\,|t - t_1| \leq \varepsilon$$

If $|t - t_1| < \dfrac{\varepsilon}{M} = \delta(\varepsilon)$.

Hence, $\tilde{T}(A(\alpha^*, \beta^*))$ is contained in a compact subset of $C[R^+ \times S(\alpha^*), R^n]$.

Now, let $\{ \tilde{\zeta}_k \}$ be a sequence in $A(\alpha^*, \beta^*)$ such that $\{ \tilde{\zeta}_k \} \to \tilde{\zeta}$, since f is levelwise continuous on a compact set, then f is uniformly continuous on a compact subset of it's domain. Hence:

$$f(t_0 + t, \mu_{\tilde{\zeta}_k}(t) + \tilde{\varphi}(t_0 + t)) \to f(t_0 + t, \mu_{\tilde{\zeta}}(t) + \tilde{\varphi}(t_0 + t)) \ [illegible]$$

and the convergence is uniformly in [illegible]

[illegible] $(\tilde{\zeta}_k(t))$ is convergent uniformly f[illegible]

i.e., the sequence $\tilde{T}(\tilde{\zeta}_k(t))$ convergent to a point $x^* \in A(\alpha^*, \beta^*)$.

i.e., $\tilde{T}$ is a pointwise convergent in $A(\alpha^*, \beta^*)$.

i.e., $\tilde{T}$ is levelwise continuous and maps compactly, the closed bounded and convex subset of $A(\alpha^*, \beta^*)$ into a subset of it, i.e., $\tilde{T} : A(\alpha^*, \beta^*) \to A(\alpha^*, \beta^*)$ is compact

Hence using Schauder fuzzy fixed point theorem, $\tilde{T}$ has a fixed point which shows that the fuzzy fixed point x^* is the desired solution of the fuzzy differential equation. ■

Under the same conditions of the last theorem additional considerations are given in order to guarantee the uniqueness of the solutions as it is seen in the next theorem:

Theorem (3.3) (The Uniqueness Theorem):

Let $\tilde{A}$ be an open fuzzy subset of $R \times C[R^+ \times S(\alpha^*), R^n]$ and suppose that $f : \tilde{A} \to R^n$ be levelwise continuous and $f(t, \tilde{\varphi})$ be Lipschitzian with respect to $\tilde{\varphi}$ in every compact fuzzy subset of $\tilde{A}$ with Lipschitz constant K. If $(x_0, \tilde{\varphi}) \in \tilde{A}$, then the fuzzy differential equation (3.1) has a unique solution passes through $(x_0, \tilde{\varphi})$ where $KT^* < 1$.

Proof:

Consider I_T and B_β which are defined as in theorem (3.1), and let $x(t), y(t)$ be any two functions related to the solution of the fuzzy differential equation on $[R^+ \times S(\alpha^*), R^n]$ with $x(x_0) = \tilde{\varphi}(t) = y(x_0)$, hence:

$$\|x(t) - y(t)\| = \left\| \int_{t_0}^{t} f(s, x(s)) ds - \int_{t_0}^{t} f(s, y(s)) ds \right\|$$

$$= \left\| \int_{t_0}^{t} f(s, x(s)) - f(s, y(s)) ds \right\|$$

$$\leq \int_{t_0}^{t} \| f(s, x(s)) - f(s, y(s)) \| ds$$

$$\leq \int_{t_0}^{t} K \| x(s) - y(s) \| ds$$

$$\leq KT^* \sup_{x_0 \leq s \leq 1} \| x(s) - y(s) \| < \sup_{x_0 \leq s \leq 1} \| x(s) - y(s) \| < \varepsilon$$

We can choose T^* small as necessary to insure that $K\alpha^* < 1$, for all $t \in I_{\alpha^*}$.

Hence $x(t) = y(t), \forall t \in I_{\alpha^*}$.

This completes the proof of the theorem. ■

3.3 SOLUTION OF LINEAR HOMOGENOUS FUZZY SYSTEM [PEARSON D.W., 1997]

Consider the following fuzzy differential equation

$$x' = f(x),\ x(0) \square\ \tilde{x}_0$$

where the structure of the equation is known, represented by a given vector field *f*, but the model parameters on the initial value $\tilde{x}_0$ are not known exactly and the initial condition is a fuzzy number.

3.3.1 Solution of Fuzzy Differential Equations [Pearson D.W., 1997]:

In this subsection, we shall study, as a survey, a method for solving linear system of fuzzy differential equations.

Consider the problem of solving the fuzzy linear homogenous differential equation:

$$x' = Ax,\ x(0) \square\ \tilde{x}_0 \quad \ldots\ldots (3.9)$$

where $x \in R^n$, A is $n \times n$ matrix and $\tilde{x}$ [illegible] described by a vector made up of n-fuzzy n[illegible]

A fuzzy number $\tilde{x}_0$, can be prescribed easily by its α-level sets, as:

$$[x_0]_\alpha = \{s : \tilde{x}_0(s) \geq \alpha\},\ 0 \leq \alpha \leq 1.$$

Due to the properties of the so defined fuzzy numbers, corresponds to an interval for each given value of α:

$$[x_0]_\alpha = [\underline{x}_0, \overline{x}_0]$$

where $\underline{x}_0$ and $\overline{x}_0$ represents the lower and upper bounds of the fuzzy number $\tilde{x}_0$.

Suppose that each element of the vector *x* in (3.9) at time *t* is a fuzzy number, where:

$$x^k(t) = [\underline{x}^k_\alpha(t),\ \overline{x}^k_\alpha(t)],\ k = 1, 2, \ldots, n \qquad \text{.......................................(3.10)}$$

it is shown that the evolution of the system (3.9) can be described by 2n-differential equations for the end points of the intervals, this is for each given time instant t and value of α. These equations for the end points of the intervals are:

$$\left.\begin{aligned} \dot{\underline{x}}^k_\alpha(t) &= \text{Min}\{\sum_{j=1}^{n} a_{kj}u^j \ : u^i \in [\underline{x}^i_\alpha(t),\ \overline{x}^i_\alpha(t)]\} \\ \dot{\overline{x}}^k_\alpha(t) &= \text{Max}\ \{\sum_{j=1}^{n} a_{kj}u^j \ : u^i \in [\underline{x}^i_\alpha(t),\ \overline{x}^i_\alpha(t)]\} \end{aligned}\right\} \qquad \text{......................(3.11)}$$

with initial conditions $\underline{x}^k_\alpha(0) = \underline{x}^k_{\alpha 0}$, $\overline{x}^k_\alpha(0) = \overline{x}^k_{\alpha 0}$.

The vector in (3.9) is linear, then equation (3.11) could be written as:

$$\underline{x}'^k_\alpha(t) = \sum_{j=1}^{n} a_{kj}u^j \qquad \text{......................(3.12)}$$

where:

$$u^j = \begin{cases} \underline{x}^j_\alpha(t), & if \quad a_{kj} \geq 0 \\ \overline{x}^j_\alpha(t), & if \quad a_{kj} < 0 \end{cases}$$

and

$$\overline{x}'^k_\alpha(t) = \sum_{j=1}^{n} a_{kj}u^j \qquad \text{..(3.13)}$$

where:

$$u^j = \begin{cases} \overline{x}^j_\alpha(t), & if \quad a_{kj} \geq 0 \\ \underline{x}^j_\alpha(t), & if \quad a_{kj} < 0 \end{cases}$$

This means, for example, for any $\alpha \in [0,1]$ and k = 1, 2, 3, (i.e., 3×3 system), then six differential equations will be obtained where two of them are for each k and each α related to one of end points, in other words:

$$x_\alpha^k(t) = [\,\underline{x}_\alpha^k(t)\,,\ \overline{x}_\alpha^k(t)\,] = \begin{bmatrix} \underline{x}_\alpha^1(t) & \overline{x}_\alpha^1(t) \\ \underline{x}_\alpha^2(t) & \overline{x}_\alpha^2(t) \\ \underline{x}_\alpha^3(t) & \overline{x}_\alpha^3(t) \end{bmatrix}$$

The method for solving directly linear fuzzy system is meaningless; therefore an introduction of the representation of the fuzzy system using complex numbers is necessary

In order to solve the fuzzy system of differential equations:

$$x' = Ax,\ x(0) \square \tilde{x}_0$$

the utility of complex numbers will be used.

Recall that, there are two equations of the type (3.12) and (3.13) which could easily be written out explicitly.

Now, define new complex variables as follows:

$$z_\alpha^k = \underline{x}_\alpha^k(t) + i\ \overline{x}_\alpha^k(t) \quad \text{...(3.14)}$$

and the two operations carried on the complex numbers as:

(a) Identity operation which is given by e, such that:

$$e\,z_\alpha^k = z_\alpha^k \quad \text{...(3.15)}$$

(b) The operator g corresponding to a flip about the diagonal in the complex plane, i.e.

$$g(z_\alpha^k) = g(\underline{x}_\alpha^k(t) + i\ \overline{x}_\alpha^k(t)) = \overline{x}_\alpha^k(t) + i\,\underline{x}_\alpha^k(t) \quad \text{....................(3.16)}$$

where $g^2 = e$ and $g^k = e$ if k is even and $g^k = g$ if k is odd, and therefore:

$$(ug)\,z_\alpha^k = (gu)\,z_\alpha^k \text{ for } u \in R \quad \ldots\ldots(3.17)$$

Using (3.14), (3.15) and (3.16), yields:

$$z_\alpha^k = \underline{x}_\alpha^k(t) + i\,\overline{x}_\alpha^k(t)$$

and hence:

$$z'^k_\alpha = \underline{x}'^k_\alpha(t) + i\,\overline{x}'^k_\alpha(t)$$

but $\underline{x}'^k_\alpha(t) = \sum_{j=1}^{n} a_{kj} u^j$ and $i\,\overline{x}'^k_\alpha(t) = i\sum_{j=1}^{n} a_{kj} u^j$. Then:

$$\underline{x}'^k_\alpha(t) + i\,\overline{x}'^k_\alpha(t) = \sum_{j=1}^{n} a_{kj} u^j + i\sum_{j=1}^{n} a_{kj} u^j$$

Hence:

$$z'^k_\alpha = a_{kj}(u^j + i\,u^j)$$

$$= \begin{cases} a_{kj}(\underline{x}_\alpha^k + i\,\overline{x}_\alpha^k), & if \quad a_{kj} \ge 0 \\ a_{kj}(\overline{x}_\alpha^k + i\,\underline{x}_\alpha^k), & if \quad a_{kj} < 0 \end{cases}$$

$$= \begin{cases} a_{kj} z_\alpha^k, & if \quad a_{kj} \ge 0 \\ a_{kj}(g z_\alpha^k), & if \quad a_{kj} < 0 \end{cases}$$

Now, using equation (3.17), whenever:

$$z'^k_\alpha = \begin{cases} a_{kj} z_\alpha^k, & if \quad a_{kj} \ge 0 \\ g a_{kj} z_\alpha^k, & if \quad a_{kj} < 0 \end{cases}$$

and in order to simplify the last formula, let:

$$b_{ij} = \begin{cases} e a_{ij}, & if \quad a_{ij} \ge 0 \\ g a_{ij}, & if \quad a_{ij} < 0 \end{cases} \quad \ldots\ldots(3.18)$$

then:

$$z_{\alpha}'^{k} = \begin{cases} b_{ij} z_{\alpha}^{k}, & if \quad a_{kj} \geq 0 \\ b_{ij} z_{\alpha}^{k}, & if \quad a_{kj} < 0 \end{cases}$$

or in matrix form:

$$z_{\alpha}'^{k} = B\, z_{\alpha}^{k}$$

with initial condition $z_{\alpha}(0) = z_{\alpha 0}$.

Now, $x' = Ax$, which has the solution $x = ce^{At}$ and since $x(0) = x_0$, then $x(t) = x_0\, e^{At}$. Similarly:

$$z_{\alpha}(t) = z_{\alpha 0}\, e^{Bt} \quad \text{...} (3.19)$$

but since the problem is to evaluate the exponential of the matrix B, then certain elements are multiplied by the flip operators (e and g, where b_{ij} [illegible] a_{ij} if a_{ij} [illegible] and $b_{ij} = ga_{ij}$ if $a_{ij} < 0$). This can be achieved for small values of t writing the matrix B as the sum of two mat[illegible] the operator e and the other by g, for examp[illegible]

$$B = eC + gD$$

and for small t, we have:

$$\exp(tB) z_{\alpha 0} = \exp(t(eC + gD)) z_{\alpha 0}$$

$$= \exp(teC)\, \exp(tgD)\, z_{\alpha 0} + O(t)$$

where $O(t)$ is a function of t, such that $O(t)/t = 0$ as $t \longrightarrow 0$. The first part $\exp(teC)$ is simply the standard matrix exponential, because e is the identity operator. For the second part $\exp(tgD)$, noting that $g^k = e$ if k is even and $g^k = g$ if it is odd and then proceed to calculate the formal power series of $\exp\,(tgD)$ as follows:

$$\exp(tgD)z_0 = \left(I + tgD + \frac{t^2}{2!}g^2D^2 + \frac{t^3}{3!}g^3D^3 + \ldots \right) z_0$$

$$= \left(I + \frac{t^2}{2!}g^2D^2 + \ldots \right) z_0 + \left(tgD + + \frac{t^3}{3!}g^3D^3 + \ldots \right) z_0$$

$$= \left(I + \frac{t^2}{2!}D^2 + \ldots \right) z_0 + \left(tD + + \frac{t^3}{3!}D^3 + \ldots \right) gz_0$$

$$= \cosh(tD)z_0 + \sinh(tD)gz_0$$

Hence:

$$z_{\alpha 0}(t) = \exp(tC)(\cosh(tD)z_{\alpha 0} + \sinh(tD)gz_{\alpha 0})$$

Let $\varphi(t) = \exp(tC)\cosh(tD)$ and $\psi(t) = \exp(tC)\sinh(tD)$. Then:

$$z_\alpha^k = \varphi_{kj}(t)\, z_{\alpha 0}^j + \psi_{kj}(t) g\, z_{\alpha 0}^j$$

but $z_\alpha^k = \underline{x}_\alpha^k(t) + i\,\overline{x}_\alpha^k(t)$, one gets:

$$\underline{x}_\alpha^k(t) + i\,\overline{x}_\alpha^k(t) = \varphi_{kj}(t)\, z_{\alpha 0}^j + \psi_{kj}(t)\, z_{\alpha 0}^j$$

$$= \varphi_{kj}(t)(\underline{x}_{\alpha 0}^j(t) + i\,\overline{x}_{\alpha 0}^j(t)) + \psi_{kj}(t)(\overline{x}_{\alpha 0}^j(t) + i\,\underline{x}_{\alpha 0}^j(t))$$

Therefore:

$$\left.\begin{aligned} \underline{x}_\alpha^k(t) &= \sum_{j=1}^{n} \varphi_{kj}(t)\underline{x}_{\alpha_0}^j(t) + \psi_{kj}(t)\overline{x}_{\alpha_0}^j(t) \\ \overline{x}_\alpha^k(t) &= \sum_{j=1}^{n} \varphi_{kj}(t)\overline{x}_{\alpha_0}^j(t) + \psi_{kj}(t)\underline{x}_{\alpha_0}^j(t) \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.20)$$

Example(3.1) [Pearson D.W.,1997]:

Consider the linear system $x' = Ax$, where $A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$ with initial values to be $x_1(0)$ about 1 and $x_2(0)$ about -1, which are fuzzy numbers and using the membership functions defined by setting, for example,

$$x_0^1(s) = \begin{cases} 0, & s < 0 \\ 2s - s^2, & 0 \le s < 2 \\ 0, & s > 2 \end{cases}$$

and

$$x_0^2(s) = \begin{cases} 0, & s < -2 \\ -2s - s^2, & -2 \le s < 0 \\ 0, & s > 0 \end{cases}$$

Thus, for $\alpha \in [0, 1]$, we can represent the initial conditions as:

$$[x_0^1]_\alpha = [\underline{x}_{0_\alpha}^1, \overline{x}_{0_\alpha}^1] = [1 - \sqrt{1-\alpha}, 1 + \sqrt{1-\alpha}]$$

$$[x_0^2]_\alpha = [\underline{x}_{0_\alpha}^2, \overline{x}_{0_\alpha}^2] = [-1 - \sqrt{1-\alpha}, -1 + \sqrt{1-\alpha}]$$

The approximate solution given by equation (3.20), which takes the form:

$$\underline{x}_\alpha^k(t) = \sum_{j=1}^{n} \varphi_{kj}(t)\underline{x}_{\alpha_0}^j(t) + \psi_{kj}(t)\overline{x}_{\alpha_0}^j(t)$$

$$\overline{x}_\alpha^k(t) = \sum_{j=1}^{n} \varphi_{kj}(t)\overline{x}_{\alpha_0}^j(t) + \psi_{kj}(t)\underline{x}_{\alpha_0}^j(t)$$

So, if we let for simplicity

$$a = 1 - \sqrt{1-\alpha}\ ,\ b = 1 + \sqrt{1-\alpha}, c = -1 - \sqrt{1-\alpha}, d = -1 + \sqrt{1-\alpha}$$

Then the approximate solution could be evaluated as follows:

To find B, recall that $b_{ij} = ea_{ij}$ if $a_{ij} \ge 0$ and $b_{ij} = ga_{ij}$ if $a_{ij} < 0$, then

$a_{11} = -1$ implies that $b_{11} = g(-1) = -i$, $a_{12} = 1 \geq 0$ implies $b_{12} = e(1) = 1$ and so on. $B = \begin{bmatrix} ga_{11} & ea_{12} \\ ea_{21} & ga_{22} \end{bmatrix} = \begin{bmatrix} -i & 1 \\ 0 & -2i \end{bmatrix}$ and we can write the matrix B as the sum of two matrices, the first matrix is multiplied by the operator e and the other is multiplied by g, hence:

$$B = \begin{bmatrix} -i & 1 \\ 0 & -2i \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= e\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + g\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = eC + gD$$

It is easy to find e^{Ct}, which is $e^{Ct} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, and

$$\cosh(tD) = \begin{bmatrix} 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots & 0 \\ 0 & 1 + 2t^2 + \frac{2}{3}t^4 + \ldots \end{bmatrix}$$

Therefore:

$$\varphi(t) = e^{Ct}\cosh(tD)$$

$$= \begin{bmatrix} 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots & t + 2t^3 + \frac{2}{3}t^5 + \ldots \\ 0 & 1 + 2t^2 + \frac{2}{3}t^4 + \ldots \end{bmatrix}$$

Similarly, one can find sinh(tD), which takes the form:

$$\mathrm{Sinh}(tD) = \begin{bmatrix} -t - \frac{t^3}{3!} - \ldots & 0 \\ 0 & -2t - \frac{4}{3}t^3 - \ldots \end{bmatrix}$$

Hence:

$$\psi(t) = e^{Ct}\sinh(tD)$$

$$= \begin{bmatrix} -t - \frac{t^3}{3!} - \dots & -2t^2 - \frac{4}{3}t^4 - \dots \\ 0 & -2t - \frac{4}{3}t^3 - \dots \end{bmatrix}$$

Now, letting $t = 0.2$, we have:

$$\varphi(0.2) = \begin{bmatrix} 1.020066 & 0.216213 \\ 0 & 1.08166 \end{bmatrix}$$

$$\psi(0.2) = \begin{bmatrix} -0.201333 & -0.082133 \\ 0 & -0.410606 \end{bmatrix}$$

and therefore the approximate solution of equation (3.20) is given by:

$$\underline{x}^1_\alpha(0.2) = 1.020066a + 0.216231c - 0.201333b - 0.082133d$$

$$\overline{x}^1_\alpha(0.2) = 1.020066b + 0.216213d - 0 \text{[illegible]}$$

$$\underline{x}^2_\alpha(0.2) = 1.0810666c - 0.410666d$$

$$\overline{x}^2_\alpha(0.2) = -0.4106666c + 1.080666d$$

For example, if $\alpha = 0.1$, then

$$\underline{x}^1_{0.1}(0.2) = -0.7570682,\ \overline{x}^1_{0.1}(0.2) = 2.126394$$

$$\underline{x}^2_{0.1}(0.2) = -2.085526 \text{ and } \overline{x}^2_{0.1}(0.2) = 0.744773$$

and so on for any value of $\alpha \in [0,1]$.

Remarks(3.2):

1- In order to check the accuracy of the results, a comparison have been made between the crisp solution and the approximate solution of a fuzzy system at $\alpha = 1$, in which this comparison will be given in the following graph:

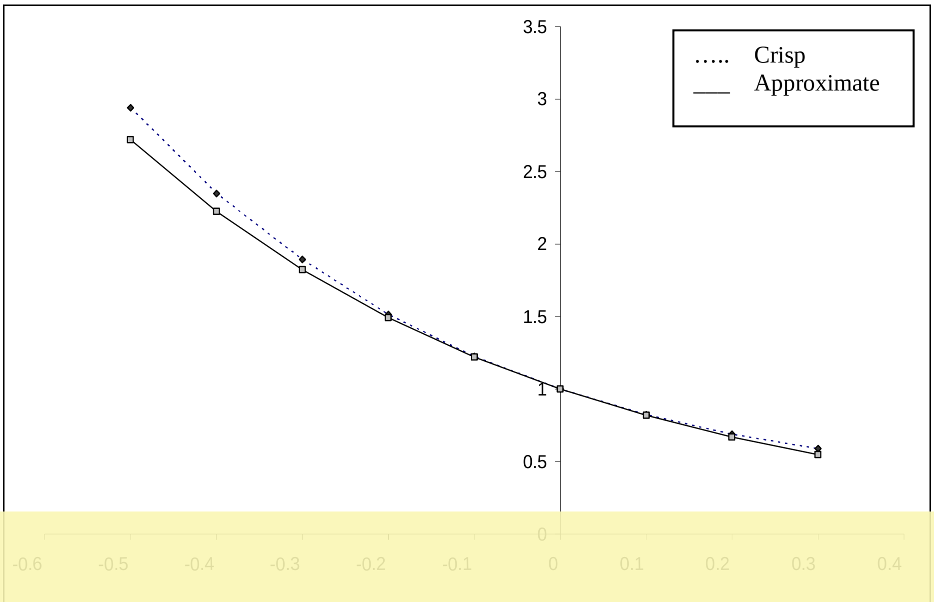

Fig (3.1) Compare between the first solution of the system (the crisp solution and the fuzzy solution with α=1).

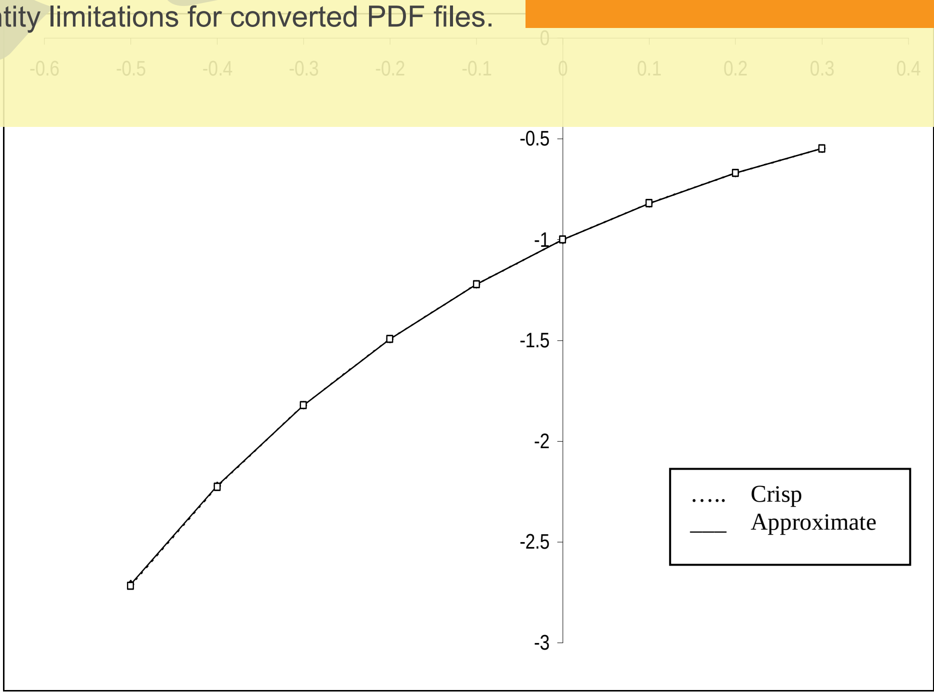

Fig (3.2) Compare between the second solution of the system (the crisp solution and the fuzzy solution with α=1).

2- The next two figures presents the membership functions for the solutions x^1 and x^2 at $t = 0.2$ for each $0 \le \alpha \le 1$ with step size 0.1.

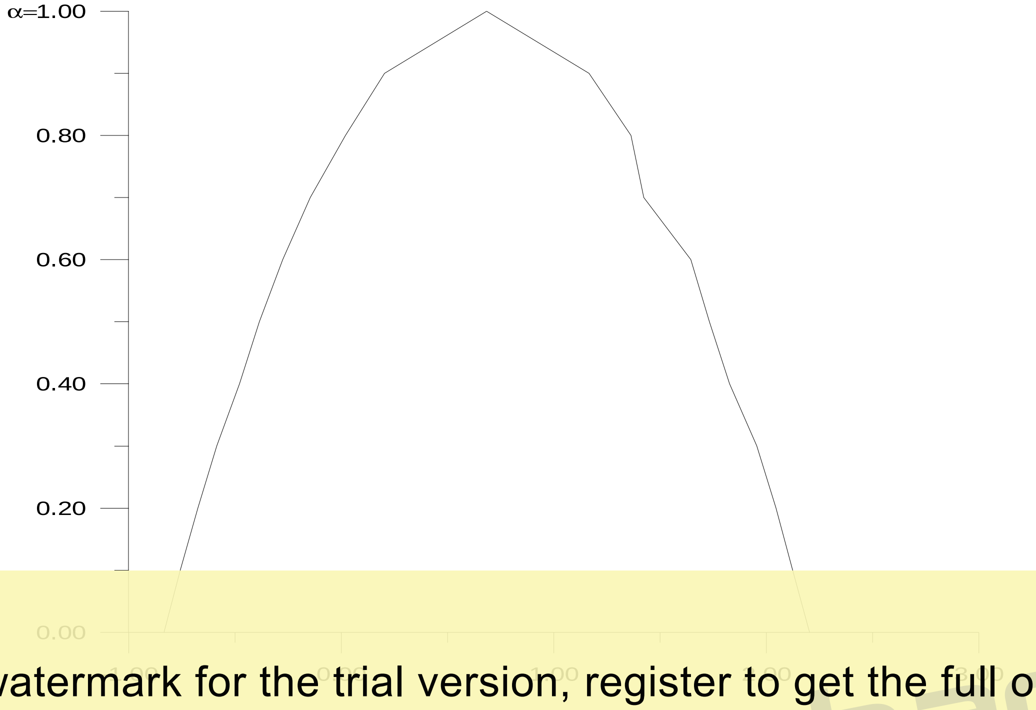

Fig (3.3) represents the upper and lower solution of x^1 at $t = 0.2$ at each $0 \le \alpha \le 1$ with step size 0.1.

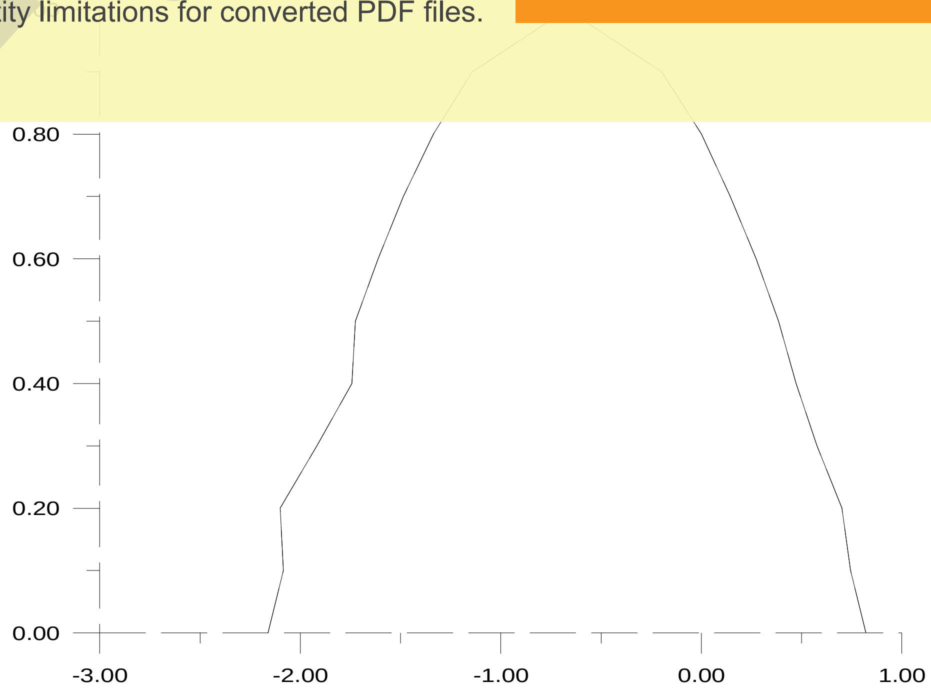

Fig (3.4) represents the upper and lower solutions of x^2 at $t = 0.2$ at each $0 \le \alpha \le 1$ with step size 0.1.

3.4 SOLUTION OF NON-HOMOGENOUS AND NONLINEAR SYSTEM OF FUZZY DIFFERENTIAL EQUATIONS

The last approach followed in section (3.3) is so difficult to modify for solving non-homogenous fuzzy systems.

Therefore, new approaches are given in this section for solving non-homogenous fuzzy differential equations. The approximate methods followed in this section are:

1- The method of successive approximation for solving non-homogenous fuzzy systems.

2- The method of linearization for solving nonlinear and non-homogenous fuzzy systems.

3.4.1 The Method of Successive Approximation for Solving Fuzzy Differential equations

Consider the fuzzy differential equation (homogenous or non-homogenous):

$$\tilde{y}' = \tilde{f}(t,\tilde{y}), \quad \tilde{y}(t_0) = \tilde{y}_0 \quad \ldots\ldots\ldots (3.21)$$

where $\tilde{f}(t,\tilde{y})$ takes the form $\tilde{K}(x,t)\tilde{y}(t) + h(t)$, and the derivative $\tilde{y}'$, the function $\tilde{f}$, the initial condition $\tilde{y}_0$ and the solution $\tilde{y}(t)$ are assumed to be fuzzy while the parameter t is considered to be crisp. where $h(t)$ represent the non-homogenous term.

Now, according to definition (3.3), then equation (3.21) is equivalent to the fuzzy integral equation (3.2). The problem now is to solve the fuzzy integral equation (3.2), and recalling the solution of this problem had been discussed by [Najieb S.W., 2002].

Now, the method of successive approximations for solving fuzzy integral equations is to consider the fuzzy integral equation:

$$\tilde{y}(t) = \tilde{y}_0 + \int_0^t \left[\tilde{K}(t,x)\tilde{y}(x) + \tilde{h}(x) \right] dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3.22)}$$

where $\tilde{y}(t) = \left((y_i(t), \alpha_i) \right)$, $\tilde{K}(t,x) = \left((K_i(t,x), \alpha_i) \right)$, i=1,2,…,n. Then:

$$\left((y_i(t), \alpha_i) \right) = \left(\tilde{y}_0 \right) + \int_o^t \left((K_i(t,x), \alpha_i) \right) \left((y_i(x), \alpha_i) \right) + \left((h_i(x), \alpha_i) \right) dx$$

Implies

$$\left(y_i(t), \alpha_i \right) = \left(\tilde{y}_0 + \int_0^t \tilde{K}_i(t,x)\tilde{y}_i(x) + \tilde{h}_i(x) dx, \alpha_i \right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3.23)}$$

Which implies that $\forall \alpha_i \in [0,1]$.

$$y_i(t) = y_0 + \int_0^t \left[K_i(t,x) y_i(x) + h_i(x) \right] dx, \quad \forall i = 1,2,\ldots,n \quad \ldots\ldots\ldots\ldots \text{(3.24)}$$

Now we can apply the method of successive approximations for equation (3.21), so we have

$$\tilde{y}_i^{(m+1)}(t) \Box \tilde{y}_0 + \int_0^t \left[\tilde{K}_i(t,x)\tilde{y}_i^{(m)}(x) + h_i(x) \right] dx$$

and for each $\alpha_i \in [0,1]$.

Example(3.2)

To solve the non-homogenous fuzzy differential equation

$\tilde{y}' = \tilde{f}(t, \tilde{y}) = \tilde{k}(t,x)\tilde{y}(t) + \tilde{h}(t)$ where

$\tilde{k}(t,x) = \{(k_1(t,x), 0.4), (k_2(t,x), 1.0)\}$

$\tilde{h}(t) = \{(h_1(t), 0.4), (h_2(t), 1.0)\}$

and, $k_1(t,x) = \frac{1}{2}t$, $k_2(t,x) = 1$, $h_1(t) = \frac{1}{2}t$ and $h_2(t) = 1$.

Applying equation (3.25), we get for $\alpha = 0.4$

$$\tilde{y}_1^{(m+1)}(t) \Box \tilde{y}_0 + \int_0^t \left[\tilde{K}_1(t,x)\tilde{y}_1^{(m)}(x) + h_1(x) \right] dx$$

$$\tilde{y}_1^{(m+1)}(t) \cong 1+\int_0^t \left[\tfrac{1}{2} x \tilde{y}_1^{(m)}(x)+\tfrac{1}{2} x\right] dx \quad \text{.............................}(3.26)$$

Since, $y_1^{(0)}(t)=1$ then from equation (3.25) we obtain:

$$y_1^{(1)}(t)=1+\int_0^t [\tfrac{1}{2}x+\tfrac{1}{2}x]dx=1+\frac{t^2}{2}$$

and hence:

$$y_1^{(2)}(t)=1+\int_0^t [\tfrac{1}{2}x(1+\frac{t^2}{2})+\tfrac{1}{2}x]dx=1+\frac{t^2}{2}+\frac{t^4}{4}$$

$$y_1^{(3)}(t)=1+\int_0^t [\tfrac{1}{2}x(1+\frac{t^2}{2}+\frac{t^4}{4})+\tfrac{1}{2}x]dx=1+\frac{t^2}{2}+\frac{t^4}{8}+\frac{t^6}{16}$$

$$y_1^{(4)}(t)=1+\int_0^t [\tfrac{1}{2}x(1+\frac{t^2}{2}+\frac{t^4}{8}+\frac{t^6}{16})+\tfrac{1}{2}x]dx=1+\frac{t^2}{2}+\frac{t^4}{8}+\frac{t^6}{32}+\frac{t^8}{64}$$

Continuing in this process for further iterations we obtain:

$$y_1(t)=1+\frac{t^2}{2}+\frac{t^4}{8}+\frac{t^6}{32}+\frac{t^8}{64}+\cdots$$

which is the Maclurian series of $cosh(t)$.

Now, for $\alpha=1$, we have:

$$\tilde{y}_2^{(m+1)}(t) \cong \tilde{y}_0+\int_0^t \left[\tilde{K}_2(t,x)\tilde{y}_2^{(m)}(x)+h_2(x)\right]dx$$

$$\tilde{y}_2^{(m+1)}(t) \cong 1+\int_0^t \left[1.\tilde{y}_2^{(m)}(x)+1\right]dx$$

and similarly:

$$\tilde{y}_2^{(1)}(t) \cong 1+\int_0^t [(1).(1)+1]dx=1+2t.$$

$$\tilde{y}_2^{(2)}(t) \cong 1+\int_0^t [(1).(1+2t)+1]dx=1+2t+t^2.$$

$$\tilde{y}_2^{(3)}(t) \square 1+\int_0^t \left[(1).(1+2t+t^2)+1\right]dx = 1+2t+t^2+\frac{t^3}{3}.$$

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if we continue in this process we get $y_2(t)$ also as

$$y_2(t) = 1+2t+t^2+\frac{t^3}{3}+\ldots$$

$$= 1+2(e^t-1)$$

So the solution of the fuzzy integral equation (which is equivalent to the solution of the fuzzy differential equation) is given by:

$$\tilde{y}(t) = \{(\cosh(t),0.4),(1+2(e^t-1),1.0)\}.$$

3.4.2 Linearization Technique for Solving Non-Linear and Non-Homogenous Fuzzy System of Differential Equations

In order to solve fuzzy differenti[illegible] non-homogenous the method of linearization is proposed to solve such type of problems.

The solution of fuzzy differential equations in this subsection deals with ordinary differential equations with fuzzy initial values.

The linearization approach depends on transforming the non-linear fuzzy system to a linear fuzzy system and then using the method of parametric equations (see section 3.3) to solve the resulting system.

Similar approach can be used to solve non-homogenous system of fuzzy differential equations.

The solution of fuzzy differential equations are also compared with the exact solution of ordinary differential equations, i.e., with crisp initial values when $\alpha = 1$.

3.4.2.1 Linearization Theorem of Non-Linear and Non-Homogenous Fuzzy Differential Equations

Suppose that the non-linear fuzzy system given by:

$$y' = f(x, y), \;\; y(x_0) \cong \tilde{y}_0$$

where f is a given function which is assumed to be differentiable, x_0 is fixed and $\tilde{y}_0$ is given fuzzy number which can be rewritten as:

$$y' \cong Ay + g(y), \;\; y(x_0) \cong \tilde{y}_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.27)$$

or

$$y_1' = a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n + g_1(y_1, y_2, \ldots, y_n)$$
$$y_2' = a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n + g_2(y_1, y_2, \ldots, y_n)$$
$$\vdots$$
$$y_n' = a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nn}y_n + g_n(y_1, y_2, \ldots, y_n)$$

where $\lim\limits_{\|y\| \to 0} \dfrac{\|g\|}{\|y\|} = 0$, and A is the Jacobian matrix evaluated at the equilibrium point. The [illegible] system

$$y' \cong Ay, \;\; y(x_0) \cong \tilde{y}_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.28)$$

is said to be the linearization (or the linearized system) of (3.27) at the equilibrium point.

The systematic approach of obtaining the linearization is by utilizing Taylor series expansion of a function $f(x_1, x_2, \ldots, x_n)$ in some neighborhood of a point $(\eta_1, \eta_2, \ldots, \eta_n)$ is given by:

$$f(x_1, x_2, \ldots, x_n) = f(\eta_1, \eta_2, \ldots, \eta_n) + (x_1 - \eta_1)\frac{\partial f}{\partial x_1}(\eta_1, \eta_2, \ldots, \eta_n) + (x_2 - \eta_2)\frac{\partial f}{\partial x_2}(\eta_1, \eta_2, \ldots, \eta_n)$$
$$+ \ldots + (x_n - \eta_n)\frac{\partial f}{\partial x_n}(\eta_1, \eta_2, \ldots, \eta_n) + r(x_1, x_2, \ldots, x_n).$$

where r is the reminder function satisfying

$$\lim_{r \to 0} \frac{r(x_1, x_2, \ldots, x_n)}{\square} = 0$$

where □ $= \sqrt{(x_1-\eta_1)^2+(x_2-\eta_2)^2+\ldots+(x_n-\eta_n)^2}$

therefore, if $(\eta_1,\eta_2,\ldots,\eta_n)$ is a fixed points of the system $y' = f(x_1,x_2,\ldots,x_n)$, then the linearized system is given by $Y' = AY$, where

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ . & . & \cdots & . \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{(x_1,x_2,\ldots,x_n)=(\eta_1,\eta_2,\ldots,\eta_n)}$$

The following examples illustrate the above discussion.

Example(3.3)

To solve the non-linear fuzzy system

$y_1' = e^{[illegible]} - y_2$, $y_1(0)$ □ $\tilde{1}$

$y_2' = [illegible]$, $y_2(0)$ □ $-\tilde{1}$.

Hence, the Jacobian at the critical point (-1,1):

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Therefore, the linearized fuzzy system is given by:

$$y' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad y_1(0) \text{ □ } \tilde{1}, \quad y_2(0) \text{ □ } -\tilde{1}.$$

Now, using the method discussed in section (3.3), to solve the linear homogenous fuzzy system

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

$$= e\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + g\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = eC + gD$$

Hence,

$$e^{Ct} = \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \cosh(tD) = \begin{bmatrix} 1 & 0 \\ 0 & \cosh(t) \end{bmatrix}$$

Therefore:

$$\varphi(t) = e^{Ct}\cosh(tD) = \begin{bmatrix} e^t & 0 \\ 0 & \cosh(t) \end{bmatrix}$$

Similarly:

$$\text{Sinh}(tD) = \begin{bmatrix} 0 & 0 \\ 0 & -\sinh(t) \end{bmatrix}$$

and hence:

$$\psi(t) = e^{Ct}\sinh(tD)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & -\sinh(t) \end{bmatrix}$$

Now, letting $t = 0.2$, we have:

$$\varphi(0.2) = \begin{bmatrix} 1.02 & 0 \\ 0 & 1.02 \end{bmatrix} \text{ and } \psi(0.2) = \begin{bmatrix} 0 & 0 \\ 0 & -0.2 \end{bmatrix}$$

and therefore the approximate solution is given by:

$$\underline{y}^1_\alpha(0.2) = (1.22)a\,,\ \overline{y}^1_\alpha(0.2) = (1.22)b\,,\ \underline{y}^2_\alpha(0.2) = (1.02)c + (-0.2)d$$

and $\overline{y}^2_\alpha(0.2) = (-0.2)c + (1.02)d$

where $a = 1 - \sqrt{1-\alpha}$, $b = 1 + \sqrt{1-\alpha}$, $c = -1 - \sqrt{1-\alpha}$, $d = -1 + \sqrt{1-\alpha}$.

Remark(3.3)

Now, To check the accuracy of the solution of the non-linear fuzzy system. We must find the solution of the non-linaer crisp system by letting $\alpha = 1$. Letting $\alpha = 1$ gives $\underline{y}_1^1(t) = \overline{y}_1^1$, $\underline{y}_1^2(t) = \overline{y}_1^2$, and hence

$$y^1(0.2) = 1.22, \quad y^2(0.2) = -0.82$$

Solving the crisp system using Euler method, gives:

$$y^1(0.2) = 1.31, \quad y^2(0.2) = -0.86$$

Example (3.4):

To solve the following non-homogenous fuzzy system using linearization technique,

$$y_1' = y_1 - 3y_2 + t^2 \qquad , y_1(0) \square \tilde{1} \qquad t \geq 0$$

$$y_2' = -2y_1 + 4y_2 + \cos t \quad , y_2(0) \square -\tilde{1}$$

The Jacobian matrix at $(2t^2 + 1.5\cos$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$$

So, we get the following homogenous fuzzy system

$$\dot{y} = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad y_1 \square \tilde{1}, \quad y_2 \square -\tilde{1}.$$

So, if we let for simplicity

$$\underline{Y}_{\alpha_0}^1 = a = 1 - \sqrt{1-\alpha} \qquad , \overline{Y}_{\alpha_0}^1 = b = 1 + \sqrt{1-\alpha}$$

$$\underline{Y}_{\alpha_0}^2 = c = -1 - \sqrt{1-\alpha} \qquad , \overline{Y}_{\alpha_0}^2 = d = -1 + \sqrt{1-\alpha}$$

Now, Proceeding similarly as in example (3.3), we have:

$$B=\begin{bmatrix}1 & -3i\\ -2i & 4\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}+\begin{bmatrix}0 & -3i\\ -2i & 0\end{bmatrix}$$

$$C=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix},\quad D=\begin{bmatrix}0 & -3\\ -2 & 0\end{bmatrix}$$

and therefore,

$$\varphi(t)=\begin{bmatrix}e^t(1+3t^2+\frac{3}{2}t^4) & 0\\ 0 & 1+4t+11t^2+12t^3+\frac{51}{2}t^4+6t^5+12t^6\end{bmatrix}$$

Also,

$$\psi(t)=\begin{bmatrix}0 & e^t(-3t^3-3t)\\ -2t-8t^2-32t^3-8t^4-16t^5 & 0\end{bmatrix}$$

In special, for $t=0$ we have:

$$\varphi(0)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \text{ and } \psi(0)=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Now, by equation's (3.20), the approximate solution for the linearized system at t is given by:

$\underline{Y}^1_\alpha(0)=a,\overline{Y}^1_\alpha(0)=b$

$\underline{Y}^2_\alpha(0)=c,\overline{Y}^2_\alpha(0)=d$

Also, for $t=1$ we have:

$$\varphi(1)=\begin{bmatrix}14.951 & 0\\ 0 & 71.5\end{bmatrix} \text{ and } \psi(1)=\begin{bmatrix}0 & -16.31\\ 66 & 0\end{bmatrix}$$

Also, by equation (3.20) we get:

$\underline{Y}^1_\alpha(1)=(14.951)a+(-16.31)d$

$\overline{Y}^1_\alpha(1)=(14.951)b+(-16.31)c$

$\underline{Y}^2_\alpha(1)=(71.5)c+(66)b$

$\overline{Y}^2_\alpha(1)=(71.5)d+(66)a$

Remark (3.4):

The solution of the non-homogenous fuzzy system (which is the linearized to the homogenous system) could be checked and compared with the exact non-homogenous crisp system;

Suppose $\alpha = 1$ for the solution of the non-homogenous fuzzy system, we have

$$Y(0) = \begin{bmatrix} Y_1(0) \\ Y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad Y(1) = \begin{bmatrix} Y_1(1) \\ Y_2(1) \end{bmatrix} = \begin{bmatrix} 31.261 \\ -5.5 \end{bmatrix}$$

Then the exact solution of the non-homogenous crisp system is:

$$Y\ (0) = \begin{bmatrix} Y_1(0) \\ Y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad Y\ (1) = \begin{bmatrix} Y_1(1) \\ Y_2(1) \end{bmatrix} = \begin{bmatrix} 31.332 \\ -5.467 \end{bmatrix}$$

We notice that the difference is negligible.

3.5 THE NUMERICAL SOLUTION OF FUZZY BOUNDARY VALUE PROBLEMS

In [illegible] study in deta[illegible] problems of fuzzy differential equations (linear, non-linear, homogenous and non-homogenous) using the shooting method with some illustrative examples, where the crisp solution for these examples is compared with the solution of the boundary value problem of crisp differential equations.

3.5.1 Boundary Value Problems of Fuzzy Differential Equations:

The boundary value problems whose equations are given with fuzzy initial conditions given at two or more points. When the fuzzy initial conditions are given at two points then the problem is called (a two point fuzzy boundary value problems).

We consider differential equations of order two with boundary fuzzy conditions at *a* and *b*.

The general problem of the second order is given by:

$$y'' = f(t, y, y'), \ a \le t \le b \qquad (3.29)$$

with boundary conditions

1- $y(a) \square \tilde{\alpha}, \ y(b) \square \tilde{\beta}$.

2- $y'(a) \square \tilde{\alpha}, \ y'(b) \square \tilde{\beta}$.

3- $y(a) \square \tilde{\alpha}, \ y'(b) \square \tilde{\beta}$

4- $a_0 y(a) + a_1 y'(a) \square \tilde{\alpha}, \ b_0 y(b) + b_1 y'(b) \square \tilde{\beta}$.

where a_0, b_0, a_1, b_1 are given constants and $\tilde{\alpha}, \tilde{\beta}$ are fuzzy numbers.

When f is linear in y and y' then we get a fuzzy boundary problem of order two, The general form of a linear second order fuzzy boundary value problem:

$$y'' = P(t)y' + q(t)y + r(t)$$

With boundary conditions given by 1,2,3, or 4 above.

Remarks

1- ... problems with fuzzy boundary conditions of type (1) only.

2- A unique solution of the problem is assumed to be exist.

3.5.2 The Shooting Method for Solving Boundary Value Problems:

Consider the second order boundary value problems

$$y'' = P(t)y' + q(t)y + r(t) \ , \ a < t < b$$

$$y(a) \square \tilde{\alpha}, \quad y(b) \square \tilde{\beta}.$$

Consider the homogenous problem

(1) $u'' = p(t)u' + q(t)u$ with $u(a) \square \tilde{0}, u'(a) \square \tilde{1}$.

Consider the non homogenous problem

(2) $v'' = p(t)v' + q(t)v + r(t)$ with $v(a) \square \tilde{\alpha}, v'(a) \square \tilde{0}$.

Then the solution can be obtained using the previous discussed methods which are given by:

$$\underline{y}(t) = \underline{v}_1(t) + \underline{\lambda}\underline{u}_1(t)$$

$$\overline{y}(t) = \overline{v}_1(t) + \overline{\lambda}\overline{u}_1(t)$$

where

$$\underline{\lambda} = \frac{\beta - \underline{v}_1(b)}{\underline{u}_1(b)}, \quad \overline{\lambda} = \frac{\beta - \overline{v}_1(b)}{\overline{u}_1(b)}$$

Example (3.5):

To solve the homogenous fuzzy boundary value problem using the shooting method, where

$$y'' = -y, \quad y(0) \equiv \tilde{1}, y(1) \equiv -\tilde{1} \quad , t \in [0,1]$$

or in matrix form

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad y(0) \equiv \tilde{1}, y(1) \equiv -\tilde{1}$$

[illegible] the first problem:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad u_1(0) \equiv \tilde{0}, u_2(0) \equiv \tilde{1}, \ t \in [0,1]$$

Now, applying the method of parametric representation mentioned in earlier, one will find the solution to be as follows:

Hence at $t = 1$

$$\underline{U}_\alpha^1(1) = a + c - b, \quad \overline{U}_\alpha^1(1) = b + d - a$$

$$\underline{U}_\alpha^2(1) = c - b, \quad \overline{U}_\alpha^2(1) = d - a$$

Where $\quad a = -\sqrt{1-\alpha}, \quad b = \sqrt{1-\alpha}, \ c = 1 - \sqrt{1-\alpha}, \ d = 1 + \sqrt{1-\alpha}$

Now, consider now the second problem

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad v_1(0) = \tilde{1}, v_2(0) = \tilde{0},\ t \in [0,1]$$

and for $t = 1$

$$\underline{V}_\alpha^1(1) = a + c - b, \quad \overline{V}_\alpha^1(1) = b + d - a$$

$$\underline{V}_\alpha^2(1) = c - b, \quad \overline{V}_\alpha^2(1) = d - a$$

where $a = 1 - \sqrt{1-\alpha}$, $b = 1 + \sqrt{1-\alpha}$, $c = -\sqrt{1-\alpha}$, $d = \sqrt{1-\alpha}$

Now,

$$\underline{\lambda} = \frac{\beta - \underline{V}_1(b)}{\underline{U}_1(b)} = \frac{-1 - (a + c - b)}{(a + c - b)}$$

$$\overline{\lambda} = \frac{\beta - \overline{V}_1(b)}{\overline{U}_1(b)} = \frac{-1 - (b + d - a)}{(b + d - a)}$$

The general solution of the boundary value problem is given by:

$$\underline{Y}(t) = \underline{V}_1(t) + \underline{\lambda}\underline{U}_1(t), \ \overline{Y}(t) = \overline{V}_1(t) + \overline{\lambda}\overline{U}_1(t).$$

We can check the results by comparing with the crisp solution at $\alpha = 1$, and for t = 1, we have

$$\underline{Y}(1) = \underline{V}_1(1) + \underline{\lambda}\underline{U}_1(1) = -0.9992, \qquad \overline{Y}(1) = \overline{V}_1(1) + \overline{\lambda}\overline{U}_1(1) = -0.9992$$

Where the crisp solution at t = 1 is: $Y(1) = -1$.

Also, we could make a good comparison between the crisp solution and fuzzy solution with $\alpha = 1$in the next figure:

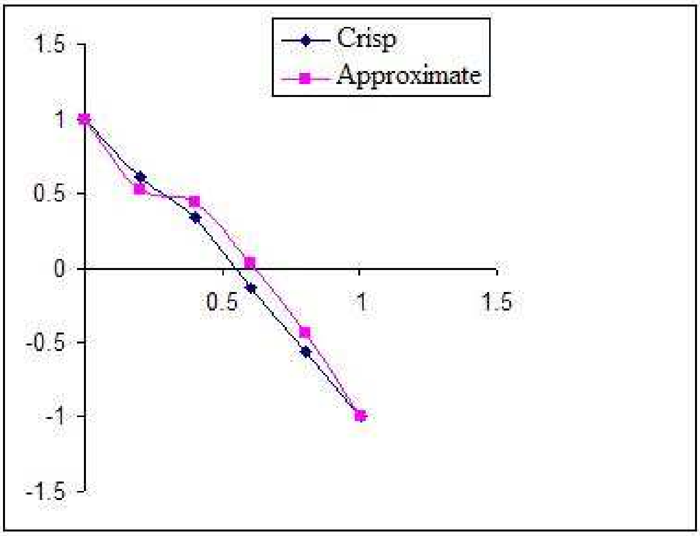

Fig (3.5) A comparison between the crisp solution and fuzzy solution at $\alpha = 1$

Example (3.6):

To solve the non-homogenous fuzzy boundary value problems

$$y'' = y + 2t\,,\; y(0) \cong 1,\; y(0.4) \cong 1.1, t \in [0, 0.4]$$

using the shooting method. This problem has the exact crisp solution

$y(t) =$ [illegible]

[illegible] homogenous problem [illegible]

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},\; u_1(0) \cong \tilde{0},\; u_2(0) \cong \tilde{1},\; t \in [0, 0.4]$$

So, the desired system is linear homogenous system of fuzzy initial value problem, and upon carrying out similar calculations in solving fuzzy differential equations, whenever for $t = 0.4$

$$\underline{U}^1_\alpha(0.4) = (1.08)a + (0.41)c,\;\; \overline{U}^1_\alpha(0.4) = (1.08)b + (0.41)d$$

$$\underline{U}^2_\alpha(0.4) = (0.41)a + (1.08)c,\;\; \overline{U}^2_\alpha(0.4) = (0.41)b + (1.08)d$$

where $a = -\sqrt{1-\alpha}$, $b = \sqrt{1-\alpha}$, $c = 1 - \sqrt{1-\alpha}$, $d = 1 + \sqrt{1-\alpha}$

Also the non-homogenous problem:

$$v'' = v + 2t\,,\ v(0) \square \tilde{1}\,,\ v'(0) \square \tilde{0}$$

or in matrix form

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2t \end{bmatrix} = \begin{bmatrix} v_2 \\ v_1 + 2t \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

with $v_1(0) \square \tilde{1},\ v_2(0) \square \tilde{0},\ t \in [0, 0.4]$

So, the desired non-homogenous system of differential equations with fuzzy initial conditions could also be solved.

Solving the non-homogenous system using the linearization method.

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2t \end{bmatrix},\ v_1(0) \square \tilde{1},\ v_2(0) \square \tilde{0},\ t \in [0, 0.4]$$

and applying the method of linearization one could get the following results.

for $t = 0.4$

$\underline{V}_\alpha^1(0.4) = (1.08)a + (0.41)c,\quad \overline{V}_\alpha^1(0.4) = (1.08)b + (0.41)d$

$\underline{V}_\alpha^2(0.4) = $ [illegible] (0.41) [illegible] $(1.08)e,\quad \overline{V}_\alpha(0.4) = $ [illegible]

where $a = 1 - \sqrt{1-\alpha}\,, b = 1 + \sqrt{1-\alpha}\,,\ c = -\sqrt{1-\alpha}\ ,\ d = \sqrt{1-\alpha}$

Now, to find the value of $\underline{\lambda}$ and $\overline{\lambda}$:

$$\underline{\lambda} = \frac{B - \underline{V_1}(b)}{\underline{U}_1(b)} = \frac{1.103 - (1.08a + 0.41c)}{(1.08a + 0.41c)},\ \overline{\lambda} = \frac{B - \overline{V_1(b)}}{\overline{U}_1(b)} = \frac{1.103 - (1.08b + 0.41d)}{1.08b + 0.41d}$$

Hence, the general solution of fuzzy boundary value problem using shooting method is given by:

$\underline{Y}(t) = \underline{V_1}(t) + \underline{\lambda}\underline{U}_1(t)$

$\overline{Y}(t) = \overline{V_1}(t) + \overline{\lambda}\overline{U}_1(t)$

and at $t = b = 0.4$ with $\alpha = 1$, then

$$\underline{Y}(0.4) = \underline{V_1}(0.4) + \underline{\lambda}\underline{U}_1(0.4) = 1.103$$

$$\overline{Y}(0.4) = \overline{V}_1(0.4) + \overline{\lambda}\overline{U}_1(0.4) = 1.103$$

Clearly, $\overline{y}(t)$ and $\underline{y}(t)$ are equal only when $\alpha = 1$ and when $t = b$.

Also, we can make a comparison between the crisp solution and the fuzzy solution with $\alpha = 1$, by the following figure:

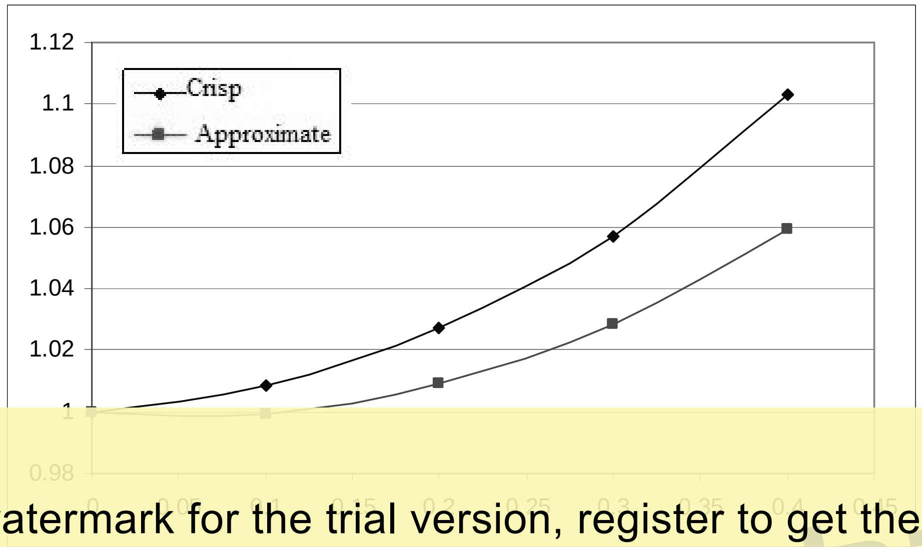

Fig (3.6) A comparison of results of example (3.6)

Example (3.7):

To solve the nonlinear fuzzy boundary value problem using the shooting method, where:

$$y'' = 2yy', \quad y(0) \cong -\tilde{1}, y(\pi/2) \cong -\tilde{1} \quad , t \in [0, \pi/2]$$

Hence the linearized system evaluated at (1/2, 0) is given by:

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad y(0) \cong -\tilde{1}, y(\pi/2) \cong -\tilde{1}$$

Now, consider the first problem:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad u_1(0) \cong \tilde{0}, u_2(0) \cong \tilde{1}, \; t \in [0, \pi/2]$$

and upon applying the method of parametric representation mentioned earlier, one will find the solution to be as follows:

Hence at $t = \pi/2$

$$\underline{U}^1_\alpha(\pi/2) = a + (3.811)c, \overline{U}^1_\alpha(\pi/2) = b + (3.811)d$$
$$\underline{U}^2_\alpha(\pi/2) = (4.811)c, \overline{U}^2_\alpha(\pi/2) = (4.811)d$$

Where $a = -\sqrt{1-\alpha}$, $b = \sqrt{1-\alpha}$, $c = 1-\sqrt{1-\alpha}$, $d = 1+\sqrt{1-\alpha}$

Now, consider now the second problem

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad v_1(0) \square -\tilde{1}, v_2(0) \square \tilde{0}, \; t \in [0, \pi/2]$$

and for $t = \pi/2$

$$\underline{V}^1_\alpha(\pi/2) = a + (3.811)c, \overline{V}^1_\alpha(\pi/2) = b + (3.811)d$$
$$\underline{V}^2_\alpha(\pi/2) = (4.811)c, \overline{V}^2_\alpha(\pi/2) = (4.811)d$$

where $a = 1 - \sqrt{1-\alpha}$, $b = -1 + \sqrt{1-\alpha}$, $c =$ [illegible]

Now,

$$\underline{\lambda} = \frac{\beta - \underline{V}_1(b)}{\underline{U}_1(b)} = \frac{1-(a+(3.811)c)}{(a+(3.811)c)}$$
$$\overline{\lambda} = \frac{\beta - \overline{V}_1(b)}{\overline{U}_1(b)} = \frac{1-(b+(3.811)d)}{(b+(3.811)d)}$$

The general solution of FBVP using the shooting method is given by:

$$\underline{Y}(t) = \underline{V}_1(t) + \underline{\lambda}\underline{U}_1(t), \overline{Y}(t) = \overline{V}_1(t) + \overline{\lambda}\overline{U}_1(t)$$

We can check the results by comparing with the crisp solution at $\alpha = 1$, and for $t = \pi/2$, we have

$$\underline{Y}(\pi/2) = \underline{V}_1(\pi/2) + \underline{\lambda}\underline{U}_1(\pi/2) = -1, \qquad \overline{Y}(\pi/2) = \overline{V}_1(\pi/2) + \overline{\lambda}\overline{U}_1(\pi/2) = -1$$

Where the crisp solution at $t = \pi/2$ is: $Y(\pi/2) = -1$.

Also, we could make a good comparison between the crisp solution and fuzzy solution with $\alpha = 1$ in the next figure:

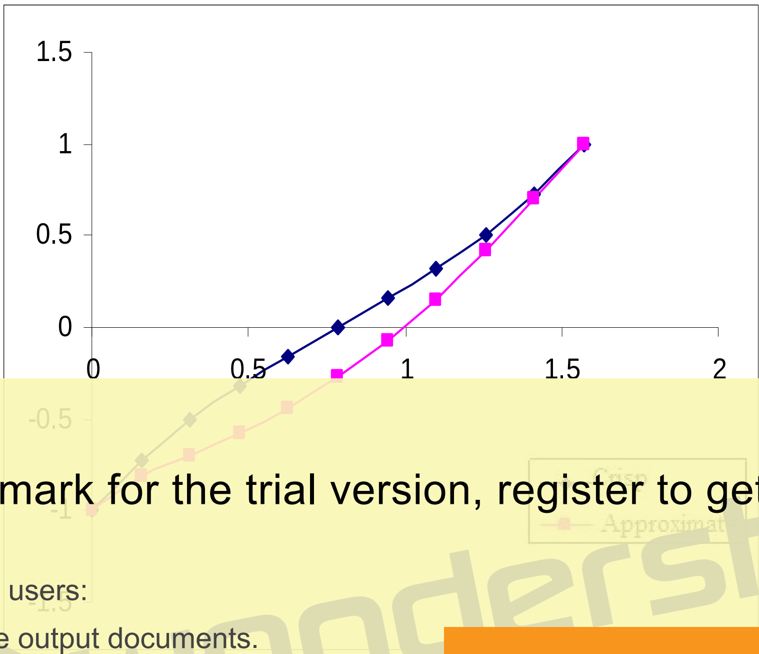

Fig. (3.7) A comparison between crisp

2 THEROTICAL RESULTS IN FUZZY SETS

Additional theoretical concepts in fuzzy set theory could be discussed concerning fuzzy mapping, differentiation and integration of fuzzy function, etc.; and therefore are presented in this chapter for completeness.

In section one, three types of fuzzy function, had been discussed with some basic properties of such type of functions. In section two, fuzzy mapping with some related properties and [illegible] three, differentiation of fuzzy function [illegible] discussed, which are the differentiation of crisp function on a fuzzy point and differentiation of fuzzifying function at a non-fuzzy point.

Finally, in section four two types of fuzzy integration have been discussed, which are integration of real valued fuzzy function over closed interval and integration of crisp real valued function over fuzzy interval since of it's important in the existence and uniqueness theorem of fuzzy differential equations.

2.1 FUZZY FUNCTIONS [DUBOIS, 1980]

The term "fuzzy function" must be understood in several ways according to where fuzziness occurs. We start first with the first type:

2.1.1 Function with Fuzzy Constraint:

Let X and Y be two universal sets and let f be a classical function $f: X \to Y$ maps from a fuzzy domain $\tilde{A}$ in X into a fuzzy range $\tilde{B}$ in Y then f is a function with fuzzy constraint if for all $x \in X$, $\mu_{\tilde{B}}(f(x)) \geq \mu_{\tilde{A}}(x)$.

Example(2.1):

Let $X = Y = R$, and consider two fuzzy sets:

$$\tilde{A} = \{(1,0.5),(2,0.8)\} \text{ and } \tilde{B} = \{(2,0.7),(4,0.9)\}$$

and a function $y = f(x) = 2x$ for all $x \in X$. It is clear that the function f satisfies the condition $\mu_{\tilde{B}}(f(x)) \geq \mu_{\tilde{A}}(x)$ for all $x \in X$.

2.1.2 Fuzzy Extension Function:

This type of fuzzy functions propagates the ambiguity of the independent variables to dependent variables, when f is a crisp function from X onto Y. Let $\tilde{A}$ be a fuzzy subset of X, then $\tilde{B} = f(\tilde{A})$ is a fuzzy subset of Y with membership function defined by:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

where $f^{-1}(y)$ is the inverse image of y.

Example (2.2):

Consider the universal sets $X = Y = R$ and consider a crisp function $f(x) = x^2$, with the domain given by the fuzzy set:

$$\tilde{A} = \{(-2, 0.9), (-1, 0.6), (0, 0.7), (1, 0.8), (2, 0.5)\},$$

The independent variable x has an ambiguity and the fuzziness which is propagated to the fuzzy set $\tilde{B}$, then we can obtain $\tilde{B}$, as:

$$\tilde{B} = \{(4, 0.9), (0, 0.7), (1, 0.8)\}.$$

2.1.3 Single Fuzzifying Function :

Fuzzifying function from X onto Y is a mapping from X into the fuzzy power set $\tilde{P}(Y)$ (or I^X), i.e., $\tilde{F} : X \rightarrow \tilde{P}(Y)$, that is to say the fuzzifying function is a mapping from an ordinary domain to a fuzzy set of range, Fuzzifying function and fuzzy relations coincides with [illegible] mathematical manner. So, fuzzifying function can be interpreted as a fuzzy relation $\tilde{R}$ defined as follows:

$$\mu_{\tilde{R}}(x, y) = \mu_{[illegible]}(x, y), \text{ where } y = \tilde{F}(x)$$

2.1.4 Fuzzy Bunch Function:

Fuzzifying bunch of crisp functions from X onto Y is defined with fuzzy set of crisp function:

$$\tilde{f} = \left\{ (f_i, \mu_{\tilde{f}}(f_i)) \mid f_i : X \rightarrow Y, i \in N : N \text{ is the set of natural numbers} \right\}.$$

where $\mu_{\tilde{f}}(f_i)$ is the membership function of the crisp function f_i.

Example (2.3):

$$X = \{1, 2, 3\}, \quad \tilde{f} = \{(f_1, 0.4), (f_2, 0.7), (f_3, 0.5)\}$$

where $f_1(x) = x, \quad f_2(x) = x^2, \quad f_3(x) = 1 - x$.

2.2 FUZZY MAPPINGS

A fuzzy mapping is a generalization of the concept of a classical mapping which can be understood as follows:

Definition (2.1)[Dubois, 1982]:

A fuzzy mapping $\tilde{f}$ from a crisp set U onto a set V is a mapping from U to the power set of non-empty subsets V, namely $\tilde{P}(V) - \{\varnothing\}$.

In other words, to each element $u \in U$ corresponds a fuzzy set $\tilde{f}(u)$ defined on V, whose membership function is $\mu_{\tilde{f}(u)}$, and $\tilde{f}(u)$ is non-empty.

Other definitions of fuzzy mappings are given also in literatures, namely:

- An ordinary mapping f from U to V with a fuzzy domain [illegible] range $\tilde{B}$ such that $\mu_{\tilde{A}}(f) \leq \mu_{\tilde{}}(f)$ [illegible].
- An ordinary mapping f from $\tilde{P}(U)$ [illegible] mapping with the fuzzy argument which maps on fuzzy subset of V.
- A fuzzy set $\tilde{F}$ of V^U, i.e., a fuzzy set of ordinary mapping from U to V each mapping f: $U \longrightarrow V$ is assigned a membership grade $\mu_{\tilde{F}}(f)$.

Proposition(2.1)[Dubois, 1982]:

A fuzzy mapping is strictly equivalent to a fuzzy relation $\tilde{R}$ such that

$\forall u \in U, \exists v \in V, \quad \mu_{\tilde{R}}(u,v) = 0$.

Proof: See [Dubois, 1982]. ■

Remarks (2.1):

1. As a converse of proposition (2.1), a fuzzy relation can be viewed as a fuzzy mapping if $\mu_{\tilde{R}}(u,.)$ determines a nonempty fuzzy set $\tilde{f}(u)$.

2. Fuzzy mappings and fuzzy relations have different points of view on the same mathematical notion.

3. Fuzzy set of mappings (FSM's, for short) are not equivalent to fuzzy mapping. Indeed, a natural way of assigning membership grades $\mu(u,v)$ to possible images $v \in V$ of $u \in U$, given an FSM F, is to define $\mu(u,v) = \mu_F(f)$ whenever $v = f(u)$. Note that $\mu(u,v)$ is not uniquely defined since there may exist $f, g: U \to V$, $f \neq g$, such that $v = f(u) = g(u)$ and $\mu_F(f) \neq \mu_F(g)$.

[illegible] represent a fuzzy mapping in terms of the FSM's.

For the first case, there is at least [illegible] FSM from a fuzzy mapping.

Starting from the fuzzy mapping $\tilde{f}: U \to V$ for any $\alpha \in [0,1]$. We can define an ordinary multimapping f_α as follows:

$$f_\alpha(u) = \left\{ v \mid \mu_{\tilde{f}(u)}(v) \geq \alpha \right\} \subseteq V, \text{ for all } u \in U.$$

f_α is the α- cut of $\tilde{f}$.

Also, f_α can be viewed as a crisp subset of V^U, i.e., a set of mappings

$$f_\alpha = \{ f : U \to V \mid \forall u \in U, f(u) \in f_\alpha(u) \}$$

$$= \left\{ f : U \to V \mid \inf_{u \in U} \mu_{\tilde{f}(u)}(f(u)) \geq \alpha \right\}.$$

f_α is the α–cut of an FSM generated by f_α, denoted $\gamma(\tilde{f})$, such that

$$\mu_{\gamma(\tilde{f})}(f) = \inf_{u \in U} \mu_{\tilde{f}(u)}(f(u)),\ \forall f \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.1)$$

Example(2.4) [Najeib S,W., 2002]:

Let $X = \{2, 3, 4, \ldots, 25\}$, a fuzzy mapping $\tilde{f}$ maps the elements in X to the power fuzzy set $\tilde{P}(X)$ in the following manner.

$$\tilde{f}(2) = \{(2,0.3),(3,0.5),(4,1),(5,0.5),(6,0.3),(9,0.2)\}$$

$$\tilde{f}(3) = \{(3,0.3),(5,0.5),(7,1),(9,0.5),(14,0.3),(16,0.2)\}$$

$$\tilde{f}(4) = \{(4,0.3),(8,0.5),(12,1),(16,0.5),(20,0.3),(25,0.2)\}$$

Now, given a function $f_1(x) = 2x$,

$$\mu_{\gamma(\tilde{f})}(f) = \inf\{\mu_{\tilde{f}(x)}(f_1(x)) \mid x \in X\}$$

$$= \inf\{\mu_{\tilde{f}(2)}(f_1(2)), \mu_{\tilde{f}(3)}(f_1(3)), \mu_{\tilde{f}(4)}(f_1(4))\}$$

$$= \inf\{\mu_{\tilde{f}(2)}(4), \mu_{\tilde{f}(3)}(6), \mu_{\tilde{f}} \ldots$$ [illegible]

$$= \inf\{1, 0, 0.5\} = 0$$

For $f_2(x) = x^2$, we have

$$\mu_{\gamma(\tilde{f})}(f) = \inf\{\mu_{\tilde{f}(x)}(f_2(x)) \mid x \in X\}$$

$$= \inf\{\mu_{\tilde{f}(2)}(f_2(2)), \mu_{\tilde{f}(3)}(f_2(3)), \mu_{\tilde{f}(4)}(f_2(4))\}$$

$$= \inf\{\mu_{\tilde{f}(2)}(4), \mu_{\tilde{f}(3)}(9), \mu_{\tilde{f}(4)}(16)\}$$

$$= \inf\{1, 0.5, 0.5\} = 0.5$$

So, $\gamma(\tilde{f}) = \{(f_1, 0), (f_2, 0.5)\}$, where $f_1(x) = 2x$ and $f_2(x) = x^2$.

Now, to the second case which is the converse of the above construction which is also can be made as expressed in the following definition.

Definition(2.2) [Najeib S.W., 2002]:

Given a fuzzy set of mappings $\gamma(\tilde{f})$ with $\mu_{\gamma(\tilde{f})} : I^X \to [0,1]$, we can construct a fuzzy mapping $\tilde{f} : X \to \tilde{P}(X)$ such that $\tilde{f}(x)$ is a fuzzy set with membership function defined as follows :

$$\mu_{\tilde{f}(x)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_{\gamma(f)}(f), & when\, f^{-1}(y) \neq \emptyset \\ 0, & when\, f^{-1}(y) = \emptyset \end{cases} \quad \text{.....................} (2.2)$$

Example(2.5) [Najeib S.W., 2002]:

In the case of crisp functions $f_1, f_2, f_3,$

mappings will be, for example:

$$\gamma(\tilde{f}) = \{(f_1, 0.2), (f_2, 0.5), (f_3, 0.7), (f_4, 0.3)\}$$

where

$$f_1(x) = x,\ f_2(x) = x^2,\ f_3(x) = e^x,\ f_4(x) = x + 1.$$

Then the image at $x = 1$, is:

$$\mu_{\tilde{f}(1)}(y) = \sup\left\{\mu_{\gamma(\tilde{f})}(f) \mid y = f(x)\right\}$$

The possible values of y is $\{f_1(1), f_2(1), f_3(1), f_4(1)\} = \{1, 1, e, 2\}$

$$\mu_{\tilde{f}(1)}(y) = \sup_{f_1, f_2}\left\{\mu_{\gamma(\tilde{f})}(f_1),\ \mu_{\gamma(\tilde{f})}(f_2)\right\}$$

$$= \sup_{f_1, f_2}\{0.2, 0.5\} = 0.5$$

$\mu_{\tilde{f}(1)}(e) = 0.7$, $\mu_{\tilde{f}(1)}(2) = 0.3$

So the fuzzy set $\tilde{f}(1) = \{(1,0.5),(2,0.3),(e,0.7)\}$

Similarly

$$\tilde{f}(2) = \{(2,0.2),(4,0.5),(e^2,0.7),(3,0.3)\}$$

$$\tilde{f}(3) = \{(3,0.2),(9,0.5),(e^3,0.7),(4,0.3)\}.$$

Hence:

$$\tilde{f}(x) = \left\{\left(f(x), \sup\{\mu_{\gamma(\tilde{f})}(f) \mid y = f(x), f \in \gamma(\tilde{f})\}\right)\right\}, \text{ for all } x \in X.$$

Lemma (2.1) [Dubois, 1982]:

$\gamma(\tilde{f} \oplus \tilde{g}) \supseteq \gamma(\tilde{f}) \oplus \gamma(\tilde{g})$ where $\tilde{f}$ and $\tilde{g}$ are fuzzy mappings

Proof: We have

$$\mu_{\gamma(f)\oplus\gamma(g)}(h(u)) = \sup_{h(u)=f(u)+g(u)} \text{m}[illegible]$$

$$= \sup_{f,g:h(u)=f(u)+g(u)} \inf_{u\in U} \min\{\mu_{\tilde{f}(u)}(f(u)), \mu_{\tilde{g}(u)}(g(u))\}$$

$$\leq \inf_{u\in U} \sup_{f,g:h(u)=f(u)+g(u)} \min(\mu_{\tilde{f}(u)}(f(u)), \mu_{\tilde{g}(u)}(g(u))),$$

and since for any mapping $\varphi: A\times B \to R$.

$$\inf_x \sup_y \varphi(x,y) \geq \sup_x \inf_y \varphi(x,y),$$

hence,

$$\mu_{\gamma(f)\oplus\gamma(g)}(h(u)) \leq \inf_{u\in U} \mu_{\gamma(f\oplus g)(u)}(h(u))$$

$$\leq \mu_{\gamma(\tilde{f}\oplus\tilde{g})(u)} h(u). \blacksquare$$

2.3 FUZZY DIFFERENTIATION [DUBOIS, 1982]

The fuzzy differentiation depends on the type of the considered function in section (2.1), i.e., differentiation of non-fuzzy function over fuzzy interval and that of fuzzifying function at non-fuzzy points, may be considered as a type of fuzzy differentiation.

2.3.1 Differentiation of Crisp Function on Fuzzy Points:

By the extension principle, differentiation $f'(\tilde{A})$ of a non-fuzzy function f at fuzzy point $\tilde{x}_0$ [Dubois, 1982b] is defined as:

$$\mu_{f'(\tilde{x}_0)}(y) = \underset{f'(x)=y}{Max} \; \mu_{\tilde{x}_0}(x)$$

The next example illustrate the above definition:

Example (2.6):

Let $f(x) = x^3$ and suppose the fuzzy poi[illegible]

$\tilde{A}$ = {([illegible]1, 0.4), (0, 1), (1, 0.6)}

then $f'(x) = 3x^2$ and therefore:

$$f'(\tilde{A}) = \{(3, 0.4), (0, 1), (3, 0.6)$$

$$= \{(3, 0.6), ((0, 1)\}$$

2.3.2 Differentiation of Fuzzifying Function Over a Set of Non-Fuzzy Points:

For all x belongs to the ordinary domain D, we will define the differentiation of fuzzifying function $\tilde{f}$ at a non-fuzzy point. Let any α-cut of $\tilde{f}$ be differentiable for an arbitrary x in D, we define differentiation $(d\tilde{f}/dx)(x_0)$ at an ordinary point x_0 as:

$$\mu_{\left(d\tilde{f}\,/\,dx\right)(x_0)}(p) = \underset{\left(d\tilde{f}_\alpha\,/\,dx\right)(x_0)=p}{Max} \mu_{\tilde{f}}(\tilde{f}_\alpha)$$

The next example illustrates the above definition:

Example (2.7):

Consider the fuzzifying function:

$\tilde{f}$ ={(f_1, 0.4), (f_2, 0.7), (f_3, 0.4)}

where f_1, f_2 and f_3 are crisp functions defined by:

$f_1(x) = x$, $f_2(x) = x^2$ and $f_3(x) = x^3 + 1$

Then $f_1'(x) = 1, f_2'(x) = 2x$ and $f_3'(x) = 3x^2$.

Differentiation of this fuzzifying function at $x_0 = 0.5$ is obtained to be as:

$f_1'(0.5) = 1$, when $\alpha = 0.4$

$f_2'(0.5) = 1$, when $\alpha = 0.7$

$f_3'(0.5) = 0.75$, when $\alpha = 0.4$.

Hence:

$$\frac{d\tilde{f}}{dx}(x_0) = \max\ \{(1, 0.4), (1, 0.7), (0.75, 0.4)\}$$

$$= \{(1, 0.7), (0.75, 0.4)\}.$$

Another type of fuzzy differentiation which is called the ***L- R type*** could be used also in differentiating fuzzy functions (for more details, see, e.g., [Dubois, 1982]).

2.3.3 Algebraic Properties of Differentiation

As in non-fuzzy differentiation so many properties are given and proved successfully. Therefore, similarly several algebraic properties undertaking fuzzy differentiation could be given. The proofs will be given here for the sake of completeness.

We start first with the following theorem:

Theorem (2.1):

The extended sum ⊕ of the derivatives of the real valued functions f and g at the fuzzy point $\tilde{x}_0$ is defined by:

$$f'(\tilde{x}_0) \oplus g'(\tilde{x}_0) \supseteq (f' + g')(\tilde{x}_0)$$

Proof: See [Dubois, 1982]. ■

The equality, in the last theorem, holds also which is equipped if certain conditions are given on f' and g', as it is ill[illegible] [illegible]ed the necessary condition for the other direction of theorem (2.1).

Theorem (2.2):

If f' and g' are continuous and both non-decreasing (or non-increasing), then:

$$f'(\tilde{x}_0) \oplus g'(\tilde{x}_0) = (f' + g')(\tilde{x}_0)$$

Proof: See [Dubois, 1982]. ■

The next theorem illustrates the differentiation of product of two functions, which is given in [Dubois, 1982] and other literatures without proof (to the best of our knowledge); which will be presented here for completeness:

Theorem (2.3):

1. If f and g are crisp functions from X to Y and $\tilde{x}_0$ is a fuzzy point in X then:

$$(fg)'(\tilde{x}_0) = (f'g + fg')(\tilde{x}_0) \subseteq [f'(\tilde{x}_0) \square\ g(\tilde{x}_0)] \oplus [f(\tilde{x}_0) \square\ g'(\tilde{x}_0)].$$

2. If f, g, f' and g' are continuous, f and g are both positive, and f' and g' are both non-decreasing (f, g are negative and f', g' are non decreasing), then:

$$(fg)'(\tilde{x}_0) = [f'(\tilde{x}_0) \square\ g(\tilde{x}_0)] \oplus [f(\tilde{x}_0) \square\ g'(\tilde{x}_0)]$$

Proof:

1. According to the properties of fuzzy set, we must prove that:

$$\mu_{(f\ g)'(\tilde{x}_0)}(y) \le \mu_{[f'(\tilde{x}_0)\square\ g(\tilde{x}_0)]\oplus[f(\tilde{x}_0)\square\ g'(\tilde{x}_0)]}(y),$$

Now, using the extension principle to the ri

$$\mu_{[f'(\tilde{x}_0)\square\ g(\tilde{x}_0)]\oplus[f(\tilde{x}_0)\square\ g'(\tilde{x}_0)]}(y) = \sup_{u,v:y=u+v} \min\left\{ \sup_{u=f'(s)g(s)} \mu_{\tilde{x}_0}(s),\ \sup_{v=f(t)g'(t)} \mu_{\tilde{x}_0}(t) \right\}$$

$$= \sup_{s,t:y=f'(s)g(s)+f(t)g'(t)} \min\left\{\mu_{\tilde{x}_0}(s),\ \mu_{\tilde{x}_0}(t)\right\} \text{..(2.3)}$$

Also, using the extension principle to the left hand side, one can get:

$$\mu_{(f\ g)'(\tilde{x}_0)}(y) = \mu_{[f\acute{g}+fg'](\tilde{x}_0)}(y) = \sup_{x:y=f'(x)g(x)+f(x)g'(x)} \min\left\{\mu_{\tilde{x}_0}(x), \mu_{\tilde{x}_0}(x)\right\}$$

$$= \sup_{x:y=f'(x)g(x)+f(x)g'(x)} \mu_{\tilde{x}_0}(x) \text{ ....(2.4)}$$

Now, from equations (2.3) and (2.4), we have:

$$\sup_{x:y=f'(x)g(x)+f(x)g'(x)} \mu_{\tilde{x}_0}(x) \le \sup_{s,t:y=f'(s)g(s)+f(t)g'(t)} \min\left\{\mu_{\tilde{x}_0}(s),\ \mu_{\tilde{x}_0}(t)\right\}$$

Since $\tilde{x}_0$ is a fuzzy point which has a supremum value therefore,

$$(fg)'(\tilde{x}_0) = (f'g + fg')(\tilde{x}_0) \subseteq [f'(\tilde{x}_0) \odot g(\tilde{x}_0)] \oplus [f(\tilde{x}_0) \odot g'(\tilde{x}_0)].$$

2. Since f' and g' are continuous on [a, b] and both non decreasing in [a, b], then:

$\forall\, s, \forall\, t > s, \exists\, x \in [s,t] \subseteq [a,b]$, such that:

$$f'(x)\, g(x) + f'(x)\, g'(x) = f'(s)\, g(s) + f(t)\, g'(t).$$

Hence in particular:

$$f'(x)\, g(x) + f(x)\, g'(x) \le f'(s)\, g(s) + f(t)\, g'(t)$$

taking x = s, we have:

$$f'(s)\, g(s) + f(s)\, g'(s) \le f'(s)\, g(s) + f(t)\, g'(t)$$

From the convexity of $\tilde{x}_0$:

$$\mu_{\tilde{x}_0}(x) \ge \min\{\mu_{\tilde{x}_0}(s),\ \mu_{\tilde{x}_0}(t)\}$$

Hence:

$$\mu_{[f'(\tilde{x}_0) \odot g(\tilde{x}_0)] \oplus [f(\tilde{x}_0) \odot g'(\tilde{x}_0)]}(y) =$$

$$\sup_{u,v:y=u+v} \min\left\{\sup_{u=f'(s)g(s)} \mu_{\tilde{x}_0}(s),\ \sup_{v=f(t)g'(t)} \mu_{\tilde{x}_0}(t)\right\}$$

$$= \sup_{x:y=f'(x)g(x)+f(x)g'(x)} \min\left\{\sup_{u=f'(x)g(x)} \mu_{\tilde{x}_0}(x),\ \sup_{v=f(x)g'(x)} \mu_{\tilde{x}_0}(x)\right\},\ x = s = t.$$

$$= \sup_{x:y=f'(x)g(x)+f(x)g'(x)} \mu_{\tilde{x}_0}(x).$$

Therefore;

$$\mu_{[f'(\tilde{x}_0)\odot g(\tilde{x}_0)]\oplus[f(\tilde{x}_0)\odot g'(\tilde{x}_0)]}(y) = \sup_{x:y=f'(x)g(x)+f(x)g'(x)} \mu_{\tilde{x}_0}(x).$$

But;

$$\mu_{(f.g)'(\tilde{x}_0)}(y) = \mu_{[f'.g+f.g'](\tilde{x}_0)}(y) = \sup_{x:y=f'(x).g(x)+f(x).g'(x)} \mu_{\tilde{x}_0}(y)$$

Then:

$$(f\ g)'(\tilde{x}_0) = [f'(\tilde{x}_0)\odot g(\tilde{x}_0)]\oplus[f(\tilde{x}_0)\odot g'(\tilde{x}_0)]. \ \blacksquare$$

2.4 FUZZY INTEGRATION

The fuzzy integration is one of the most important part of the analysis of fuzzy set theory. Quite different suggestions have been made to define fuzzy integrals, so there are three types of fuzzy integration which are:

1. Integration of real fuzzy function over [illegible]
2. [illegible] tion of real function over fuzzy in[illegible]
3. Integration of real fuzzy function over fuzzy interval.

The first two of the above three types will be discussed next:

2.4.1 Integration of Real Fuzzy Function over Crisp Closed Interval

[Dubois, 1982]

We shall now consider a fuzzy function $\tilde{f}$, which shall be integrated over the crisp interval $[a,b]$.

Definition(2.3):

Let $\tilde{f}: X \to \tilde{F}(\mathrm{R})$, the integral of $\tilde{f}$ over $X = [a,b]$ denoted by $\int_X \tilde{f}(t)dt$ is defined levelwise, as follows:

$$\left(\int_X \tilde{f}(t)dt \right)_\alpha = \int_X f_\alpha(t)dt \text{, for all } 0 \le \alpha \le 1$$

$$= \left(\int_X f_{\alpha^-}(t)dt, \int_X f_{\alpha^+}(t)dt \right) \text{..........................(2.5)}$$

Remark (2.2):

If the fuzzy integration over the interval X=[a,b] is reversed from b to a it is easily seen that :

$$\int_X \tilde{f} = \int_a^b \tilde{f} = -\int_b^a \tilde{f}$$

where $-\int_a^b \tilde{f}$ has for a membership function $\mu_{-\int_b^a \tilde{f}}(u) = \mu_{\int_a^b \tilde{f}}(-u)$.

Following, some properties of the integration of fuzzy function over crisp interval which are given in [Dubois, 1982].

1. $\int_X (\tilde{f} \oplus \tilde{g}) \supseteq \left(\int_X \tilde{f} \right) \oplus \left(\int_X \tilde{g} \right)$, where $\tilde{f}$ and $\tilde{g}$ are real fuzzy functions from the closed interval X to R, with bounded support.

2. Under the commutativity condition for $\int_X$ and $\oplus$,

$$\int_X (\tilde{f} \oplus \tilde{g}) = \left(\int_X \tilde{f} \right) \oplus \left(\int_X \tilde{g} \right) \quad \text{..(2.6)}$$

Example(2.8):

Consider the bunch fuzzy function given in (2.1.4), by:

$\tilde{f} = \{(f_1, 0.4), (f_2, 0.7), (f_3, 0.4)\}$

where

$f_1(x) = x,\ f_2(x) = x^2,\ f_3(x) = x+1$

and to integrate this bunch function over [1,2], we perform this as follows:

i) Integration at $\alpha = 0.7$, then

$f = f_2(x) = x^2$

and hence

$I_{0.7}(1,2) = \int_1^2 x^2\, dx = \frac{1}{3} x^3 \Big]_1^2 = \frac{7}{3}$.

Hence, the integration result is with possibility 0.7, is given by:

$\tilde{I}_{0.7}(1,2) = \{(\frac{7}{3}, 0.7)\}$.

ii) Integration at $\alpha = 0.4$, there are two functions

$f^+ = f_1(x) = x$ and $f^- = f_3(x) = x+1$, then for

$I^+_{\alpha}(1,2) = \int_1^2 x dx = \frac{1}{2} x^2 \Big]_1^2 = \frac{3}{2}$.

and

$$I_{\alpha}^{-}(1,2) = \int_{1}^{2}(x+1)dx = \frac{1}{2}x^{2} + x\Big]_{1}^{2} = \frac{5}{2}$$

The integration results are with possibility 0.4. Then,

$$\tilde{I}_{0.4}(1,2) = \{(\frac{3}{2}, 0.4), (\frac{5}{2}, 0.4)\}.$$

Finally, we have the total integration.

$$\tilde{I}(1,2) = \{(\frac{7}{3}, 0.7), (\frac{3}{2}, 0.4), (\frac{5}{2}, 0.4)\}$$

2.4.2 Integration of a (Crisp) Real Valued Function Over a Fuzzy Interval [Klir, G. J., 2000]

A [illegible] $\tilde{D}$ of the [illegible] R is [illegible] normalized convex fuzzy sets. The membership function of each are $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ respectively, where $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ can be interpreted as the degree of confidence to which x can be c[illegible] of D. If $\underline{a}_0$ and $\overline{b}_0$ are the lower/upper limits of the supports of $\tilde{a}$ or $\tilde{b}$, then a_0 or b_0 are related to each other by $\underline{a}_0 = \text{Inf } S(\tilde{a}) \leq \text{Sup } S(\tilde{b}) = \overline{b}_0$.

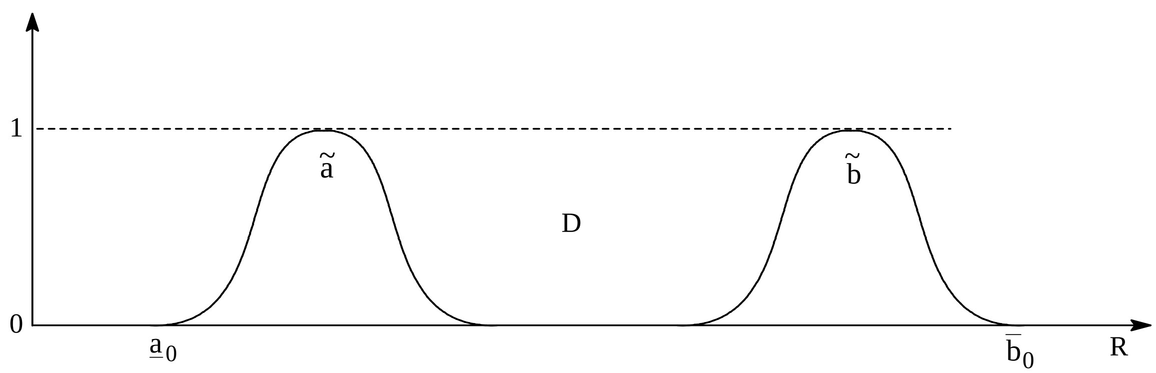

Fig.(2.6) Fuzzily bounded interval.

Definition (2.4)[Klir, G. J., 2000]:

Let f be a real valued function which is integrable in the interval J = [a_0, b_0], then according to the extension principle the membership function of the fuzzy integral $\int_{\tilde{D}} f$ is given by:

$$\mu_{\int_{\tilde{D}} f}(z) = \sup_{x, y \in J : z = \int_x^y f} Min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\}.$$

Some Properties of The Integration of Crisp Function over Fuzzy Interval [Klir, G. J., 2000]:

1. Let f be any function $f : D \to R$, which is integrable on D, then:

$$\int_{\tilde{D}} f = F(\tilde{b}) \ominus F(\tilde{a}).$$

where $\ominus$ denotes the extended subtraction.

2. Let f and g be any two functions $f, g : I \to R$ which are integrable on I, then

$$\int_{\tilde{a}}^{\tilde{b}} (f + g) \subseteq \int_{\tilde{a}}^{\tilde{b}} f \oplus \int_{\tilde{a}}^{\tilde{b}} g$$

where $\subseteq$ denotes the usual fuzzy set inclusion ($\tilde{A} \subseteq \tilde{B} \Leftrightarrow \mu_{\tilde{A}} \le \mu_{\tilde{B}}$) and $\oplus$ denotes the extended addition.

3. If $f, g : I \longrightarrow R^+$ or $f, g : I \longrightarrow R^-$, then:

$$\int_{\tilde{a}}^{\tilde{b}} (f + g) = \int_{\tilde{a}}^{\tilde{b}} f \oplus \int_{\tilde{a}}^{\tilde{b}} g$$

The following examples illustrate fuzzy integration and its properties:

Example (2.9):

Let:

$\tilde{a} = \{(4, 0.8), (5, 1), (6, 0.4)\}$

$\tilde{b} = \{(6, 0.7), (7, 1), (8, 0.2)\}$

and, $f(x) = 2$, $x \in [a_0, b_0] = [4, 8]$

The problem is to find the fuzzy integration of $f(x)$ over J = [4, 8]. The following table illustrate these results.

Table (2.1)

Integration of f(x)=2, over an interval (a,b) with membership function

(a, b)	$\int_a^b 2\,dx = 2b - 2a$	$Min\{\mu_{\tilde{a}}(a), \mu_{\tilde{b}}(b)\}$
(4, 6)	4	[illegible]
(4, 7)	6	0.8
(4, 8)	8	0.2
(5, 6)	2	0.7
(5, 7)	4	1.0
(5, 8)	6	0.2
(6, 6)	0	0.4
(6, 7)	2	0.4
(6, 8)	4	0.2

and by using the definition (2.4), then:

$$\int_{\tilde{D}} f = \{(0, 0.4), (4, 0.7), (4, 1), (6, 0.8), (8, 0.2)\}.$$

Example (2.10):

Let:

$f(x) = 2x - 3, \; g(x) = -2x + 5$

and

$\tilde{a} = \{(1, 0.8), (2, 1), (3, 0.4)\}$

$\tilde{b} = \{(3, 0.7), (4, 1), (5, 0.3)\}$

so:

$$\int_a^b f(x)\,dx = x^2 - 3x \Big|_a^b = (b^2 - 3b) - (a^2 - 3a).$$

$$\int_a^b g(x)\,dx = -x^2 + 5x \Big|_a^b = (-b^2 + 5b) - (-a^2 + 5a).$$

$$\int_a^b \big(f(x) + g(x)\big)\,dx = 2x \Big|_a^b = (2b - 2a).$$

Then:

$$\int_{\tilde{a}}^{\tilde{b}} f = \{(0, 0.4), (2, 0.7), (4, 0.4), (6, 1$$

$$\int_{\tilde{a}}^{\tilde{b}} g = \{(-6, 0.3), (-4, 0.3), (-2, 1), (0, 0.8), (2, 0.7)\}$$

$$\int_{\tilde{a}}^{\tilde{b}} f \oplus \int_{\tilde{a}}^{\tilde{b}} g = \{(-6, 0.3), (-4, 0.3), (-2, 0.4), (0, 0.7), (2, 0.7), (4, 1), (6, 0.8), (8, 0.7), (10, 0.3), (12, 0.3), (14, 0.3)\}$$

$$\int_{\tilde{a}}^{\tilde{b}} (f + g) = \{(0, 0.4), (2, 0.7), (4, 1), (6, 0.8), (8, 0.3)\}$$

and it is clear that:

$$\int_{\tilde{a}}^{\tilde{b}} f \oplus \int_{\tilde{a}}^{\tilde{b}} g \supseteq \int_{\tilde{a}}^{\tilde{b}} f + \int_{\tilde{a}}^{\tilde{b}} g\,.$$

CONCLUSIONS AND RECOMMENDATIONS ------

From the present work, the following conclusions are drawn:

1. All observed examples are defined on intervals which are belong to R^+.
2. Up to our knowledge boundary value problems haven't been discussed before, especially methods of solution.
3. The method of successive approximations can be used to solve those problems in which the fuzziness appears in the solution and the kernel functions as a bunch fuzzy function.

Also, we can recommend the following open problems for future work:

1. Studying eigenvalue problems of boundary value problems of Sturm-Liouville fuzzy differential equations as well as the relationship between the eigenvalue and the eigen[illegible]
2. Solving fuzzy partial differential equ[illegible]
3. Using the collocation or the finite difference method for solving fuzzy boundary value problems.
4. Studying the variational formulation of fuzzy boundary value problems.
5. Studying new types of fuzzy differential equations with fractional derivatives.

CONTENTS

INTRODUCTION --

Every day life, we use so many properties which cannot be dealt with satisfactory on a simple "Yes" or "No" answers, i.e., mathematically either belongs or not. Such properties perhaps are best indicated by shade of gray, rather than by in the black or white. Assigning each individual in a set (called the universal set and is denoted by X) on a "Yes" or "No" values as in ordinary set theory, is not an adequate way for dealing with such type of problems [Zadeh L.A., 1965].

Zadeh L.A., in 1965 introduced the subject of fuzzy set theory in which he considered the class of objects with continuum grads of membership, such a set is characterized by a membership or characteristic function which assigns to each object from the set a grade of membership that ranging between 0 and 1 [Zadeh L.A., 1965], [Bellman R.E., 1964].

The elements of real world problems are perturbed by imperfection and thus there exists no elements that perfectly [illegible] exact concepts corresponding to the so[illegible] mathematics, while inexact structures encounter us in real life problems [Zadeh L.A., 1965].

This thesis consists of three chapters.

Chapter one entitled (Fundamental Concepts in Fuzzy Sets) consist of seven sections. Section one consists of basic concepts and definitions related to fuzzy set theory which are necessary for the completeness of this thesis. Section two stands for studying different methods for constructing membership function numerically and analytically. In section three a study to the extension principle is given which is necessary for extending non-fuzzy concepts to fuzzy logic. In section four, and because of their importance in solving fuzzy differential equations, we study the α- level sets, as well as, some of it's properties. Finally, in sections five, six and seven we study convex fuzzy sets, fuzzy relations and fuzzy number, respectively, which are necessary for the study of initial conditions of fuzzy differential equations.

Chapter two, entitled (Theoretical Results in Fuzzy Sets) consist of four sections. In section one, three types of fuzzy function have been discussed with some basic properties of such type of functions. In section two, we introduce the concept of fuzzy mapping with some related properties and propositions. In section three, differentiation of fuzzy function have been discussed with some basic algebraic properties. In section four we study the concept of fuzzy integration since of it's important in the existence and uniqueness theorem of fuzzy differential equations.

Finally, chapter three entitled (Solution of Fuzzy Differential Equations) consist of five sections. In section one, we introduce the concept of fuzzy differential equations with some related definitions. In section two, we study in details the statement and proof of the existence and uniqueness theorem of fuzzy differential equations using Schauder fuzzy fixed point theorem. In section three we discuss the solution of line [illegible] [illegible] examples. In section fou [illegible] solve the non-homogenous and nonlinear fuzzy initial value problems. In section five an introduction to the solution of fuzzy differential equations with boundary conditions is presented using the shooting method to solve numerically boundary value problems.

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SOLUTION OF FUZZY INITIAL-BOUNDARY ORDINARY DIFFERENTIAL EQUATIONS

A Thesis

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as a Partial Fulfillment of the Requirements for the Degree of
Master of Science in Mathematics

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جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة النهرين

حلول المعادلات التفاضلية ذات الشروط الابتدائية–الحدودية الضبابية

رسالة

مقدمة إلى كلية العلوم – جامعة النهرين

وهي جزء من متطلبات نيل درجة ماجستير علوم

الرياضيات

من قبل

عمار جعفر محيسن الساعدي

(بكالوريوس، جامعة النهرين، ٢٠٠٣)

إشراف

د. فاضل صبحي فاضل

أذار ٢٠٠٦ م

الاهداء

الى من أرسله الله منقذاً للبشرية ... الى هبة السماء رسول الله **محمد** (صلى الله عليه وآله)

الى الأئمة الميامين ... آل بيت محمد الطيبين الطاهرين

الى وطني الحبيب ... عراق الصابرين

الى أغلى ما في الوجود ... الي التي جعل الله الجنة تحت قدميها ... منبع الدفء والحنان ... عَلّني اجازيها خيراً

أُمي العزيزة

الى معلمي الأول ومثلي الأعلى في الحياة

والدي العزيز

الى من أشد بهم أزري .. الى قرة العين

أحوتي ألاعزاء

الى كل من علمني حرفاً .. الى شموع العطاء ومناهل العلم والضياء

أساتذتي الأفاضل

الى أجمل ما يمكن أن يحصل عليهِ إنسان في الحياة

أصدقائي الأوفياء

الى التي أتمنى أن يجعلها الله شريكة عمري وتوأم روحي

عمار جعفر الساعدي

بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

اللَّهُ نُورُ السَّمَاوَاتِ وَالْأَرْضِ مَثَلُ نُورِهِ
كَمِشْكَاةٍ فِيهَا مِصْبَاحٌ الْمِصْبَاحُ فِي زُجَاجَةٍ
الزُّجَاجَةُ كَأَنَّهَا كَوْكَبٌ دُرِّيٌّ يُوقَدُ مِن شَجَرَةٍ
مُبَارَكَةٍ زَيْتُونَةٍ لَّا شَرْقِيَّةٍ وَلَا غَرْبِيَّةٍ يَكَادُ زَيْتُهَا
يُضِيءُ وَلَوْ لَمْ تَمْسَسْهُ نَارٌ نُّورٌ عَلَىٰ نُورٍ يَهْدِي
اللَّهُ لِنُورِهِ مَن يَشَاءُ وَيَضْرِبُ اللَّهُ الْأَمْثَالَ لِلنَّاسِ
وَاللَّهُ بِكُلِّ شَيْءٍ عَلِيمٌ

صدق الله العلي العظيم

سورة النور، الآية ٣٥

المستخلص ---

إن احد اهداف دراسة موضوع نظرية المجموعات الضبابية هو لتطوير اساليب صياغة و حل المسائل التي تكون على درجة كبيرة من التعقيد او تلك التي تكون ذات تعريف غير دقيق وذلك لكي تكون مقبولة عند التعامل معها بالطرق التحليلية المألوفة. ولذلك يمكن اعتبار الضبابية على انها نوع من انواع اللادقة التي تواجهنا عند ايجاد الصياغة الرياضية لمسألة عملية والتي يكون فيها نوع من الغموض. هذا النوع من المجموعات استحدث من قبل العالم زاده في عام ١٩٦٥ كاسلوب لمعالجة هذا الغموض او اللادقة في النماذج الرياضية.

لهذه الاطروحة ثلاثة اهداف.

الهدف الاول هو دراسة المجموعات الضبابية و برهنة بعض النتائج المشهوره التي اما ان تكون غير [illegible] البراهين التي تفتقر الى التفاصيل.

الهدف الثاني هو [illegible] للمعادلات التفاضلية الضبابية باستخدام مبرهنة [illegible]

[illegible] ضوع في المعادلات التفاضلية الضبابية [illegible] والتي لم تتم مناقشتها مسبقا بالاضافة الى استعراض عدد من طرائق الحل.