ABSTRACT

In this thesis the gamma distribution is considered for the reason of it's appearance in many statistical fields of applications. Some mathematical and statistical properties of the distribution are collected and unified. Moments and higher moments are illustrated and two methods of estimation for the distribution parameters are discussed theoretically and assessed practically.

A new proposed method of approximation to the cumulative distribution function is derived and it showed practically a high performance in comparison with four well known methods of approximation.

Finally five procedure for generating random variates from gamma distribution are discussed and their efficiencies are compared theoretically and practically by Monte-Carlo simulation.

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Huda

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CHAPTER 1

THE GAMMA DISTRIBUTION

1.1 INTRODUCTION

Gamma distribution is one of the important classes of continuous distribution, and it is useful for modeling times between events such as reliability and queuing studies [37]. The gamma distribution uses in the problem of modeling atmospheric aerosol (such as haze and fog) [7].

Gamma distribution has an important applications in the study of life time models, such as stops of a machine, failure or breakdowns of an equipment (e.g. Personal computer), air or road accidents, coal mining disasters, telephone call, etc., are examples of such events the occur in a real time and have properties exported for gamma case [37], and also the gamma distribution has been found to fit the precipitation distribution quite well [16]. Another useful use of gamma distn. is capable of modeling a variety of different probability density functions and hazard functions[9].

In hydrology [30] the gamma distn. has the advantage of having only positive values since hydrological variables such as rainfall and runoff are always positive. More detailed information about gamma distn. can be found in [5], [42]. The following literature review appearance the useful of the gamma distn. in many applied fields.

Greenwood and Durand (1960) [11] presented approximations for estimating the maximum likelihood parameters for the gamma distribution.

Mooley and Crutcher (1968) [32] investigated the number of years of record needed to stabilize the gamma parameters in a study of rainfall in India.

Shenton land Bowman (1970) [39] discuss some properties of Thom's estimators for gamma distribution.

Bridges and Haan (1972) [4] developed a technique for evaluating the reliability of precipitation estimates, where he present tables for the probabilities of the errors of various magnitudes in precipitation estimate as a function of record length.

A.Marazzi, C. Ruffieux (1995) [2] discussed the implementation (the numerical computation) for estimation and determination of some measures such as bias, variance which characterize their distribution.

David W. Coit and Tongdan Jin (2000) [9] consider the maximum likelihood estimation of the parameters of gamma distribution for reliability data with missing failure data (censored data).

Hafzulla Aksoy (2000) [12] used the 2-parameter gamma distribution in a mounts of daily rainfall data and the ascension curve of the hydrograph.

J. Belmont and M.Canela (2004) [20] presented a simple proposal for classification the pollen types, based on the shape parameter of gamma distribution, which could reduce the study of the tree pollen data of network to three selected types.

This thesis involves three chapters. In chapter one we gave some basic concepts of gamma distn. Moments and higher moments properties of the distn. are illustrated. Some related theorems concern the distn. are proved. Two methods of parameters estimation namely moment method and maximum likelihood are discussed theoretically and assessed practically by Monte-Carlo simulation. Where as in chapter two we gave a full discussion for the approximation to the c.d.f of gamma distn. by five numerical methods

namely Trapezoidal Rule, Simpson Rule, Guass-Quadrature Rule, Proposition Rule and Hit or Miss Monte-Carlo Rule, where the approximation results and compared inclusively and exclusively with the result of chi-square distn. table.

Finally in chapter three we discuss five procedures for generating random variates from gamma distn. One procedure depends on the independent sum theorem, and the other procedures depend on Acceptance-Rejection method. Efficiencies of the method are calculated theoretically and assessed by Monte-Carlo simulation.

1.2 SOME BASIC CONCEPTS OF GAMMA DISTRIBUTION

1.2.1 Definition: [17]

A continuous r.v. X is said to have gamma distribution, denoted by $X \sim G(\alpha, \beta)$ if X has p.d.f

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \cdot x^{\alpha-1} e^{-x/\beta},\ 0 < x < \infty$$

$$= 0, \text{ e.w, where } \alpha > 0,\ \beta > 0 \text{ and} \qquad \text{.....(1.1)}$$

$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t}\, dt$ is known as gamma function.

To verify that the Eq.(1.1) is valid p.d.f, we note that $f(x) > 0$, $\forall\ x \in (0, \infty)$ and the integral $\int_0^{\infty} f(x)\, dx$ is unity. Viz

$$\text{Let } I = \int_0^{\infty} f(x)\, dx = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \cdot x^{\alpha-1} e^{-x/\beta}\, dx$$

$$\text{Set } y = \frac{x}{\beta} \text{ or equivalently } x = \beta y \text{ by with } dx = \beta dy$$

So

$$I = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} y^{\alpha-1} e^{-y} dy = \frac{1}{\Gamma(\alpha)} \cdot \Gamma(\alpha) = 1$$

when $\alpha = 1$, the exponential distribution arise as a special case of the gamma distribution and the p.d.f of (1.1) becomes

$$f(x;\beta) = \frac{1}{\beta} \cdot e^{-x/\beta} \quad , 0 < x < \infty$$

.....(1.2)

$$= 0, \text{ e.w.}$$

Also a special case of gamma distribution that plays an extremely important role in both theoretical and applied statistics when the r.v $X \sim G(\frac{r}{2},2)$ and the p.d.f of (1.2) in this case is

$$f(x) = \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}} \cdot x^{\frac{r}{2}-1} e^{-x/2}, \quad 0 < x < \infty$$

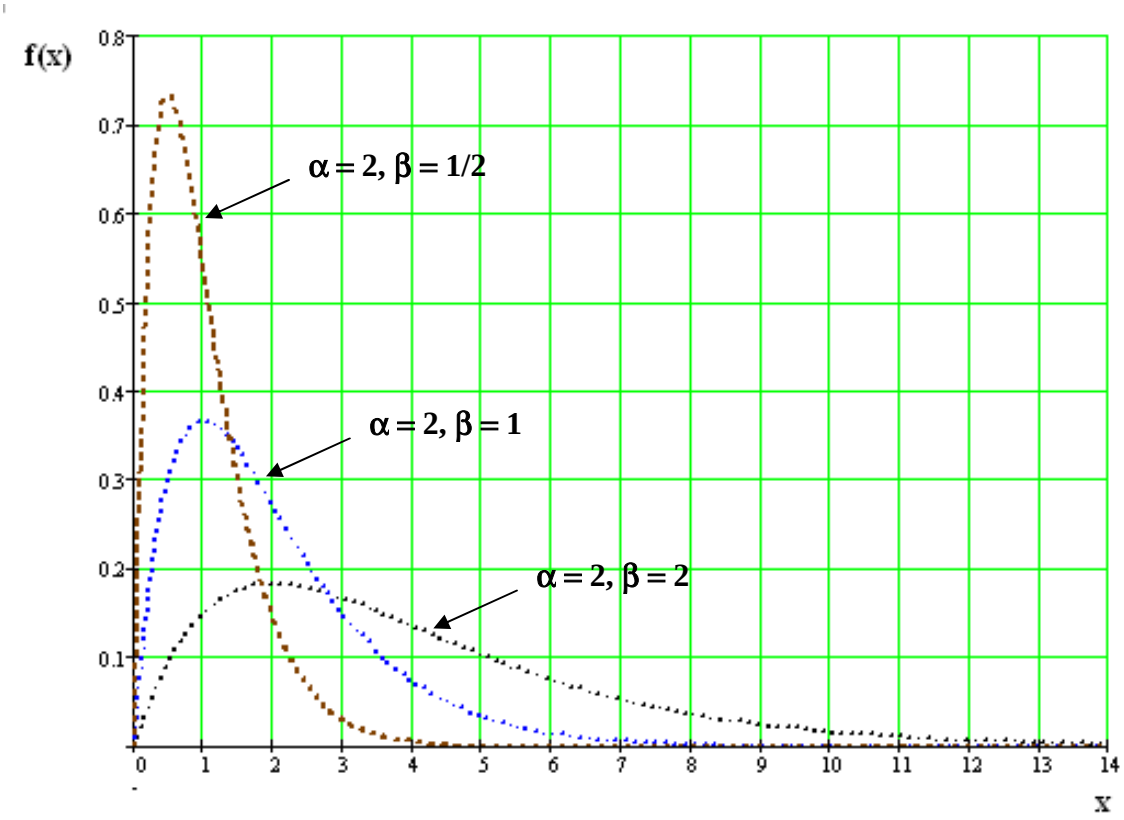
.....(1.3)

$$= 0 \text{ ,e.w.}$$

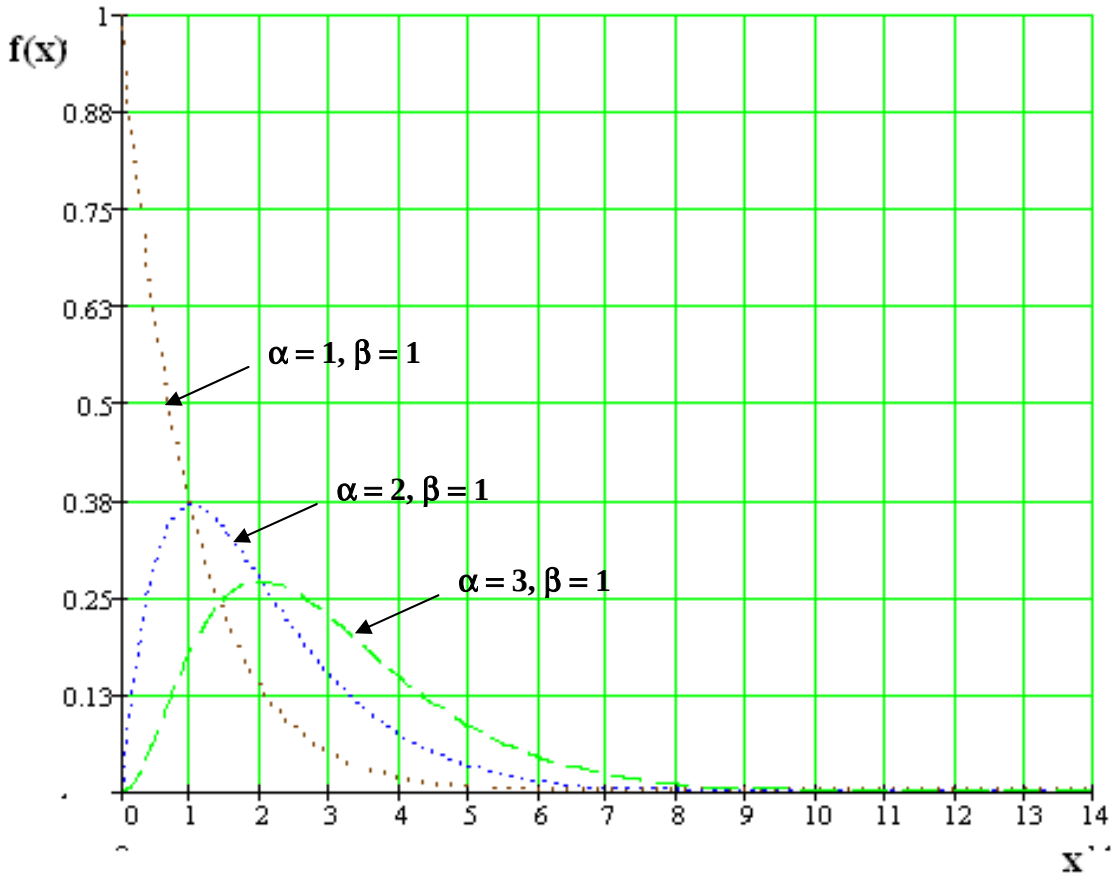
where r is positive integer

The r.v X that has p.d.f of Eq.(1.3) is said to have chi-square distribution with r degrees of freedom and denoted by $X \sim \chi^2_{(r)}$. The gamma distribution depends on two parameters α and β which are often referred to as shape and scale parameters.

By varying the values of α and β, a wide range of distribution shape can be generated. The professional MATHCAD, 2001, computer software is used to give a graphical representation of gamma p.d.f ó s when α is an integer



Fig(1): Gamma pdf 's with fixed α and β vary.



Fig(2): Gamma pdf's with fixed β and α vary.

Figures (1) and (2) show respectively some p.d.f's for fixed α and β varying and for fixed β and α varing. Both figures show that the curve increasing to locate its maximum and then decreasing to have the x-axis as an asymptote. Furthermore there is a sever skewness to right as α or β increasing except when $\beta = 1$ where the exponential distribution arise as a special case of gamma distribution.

1.2.2 The Cumulative Distribution Function:

The gamma c.d.f is known by the following incomplete gamma function

$$F(x) = Pr(X \le x) = \int_0^x \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \omega^{\alpha-1} e^{\frac{-\omega}{\beta}} d\omega \quad \text{.....(1.4)}$$

More discussion is given in chapter two about several suggested procedures for approximating the integral side of equation (1.4). Most books of mathematical statistics [17] involve a specific table for special case of gamma distribution know as chi-square distribution when $\alpha = \frac{r}{2}$ and $\beta = 2$ as shown in table (1).

Table (1) The chi-square distribution.

r \ *Exact*	Pr(X ≤ x)					
	0.01	**0.025**	**0.050**	**0.95**	**0.975**	**0.99**
1	0.000	0.001	0.004	3.84	5.02	6.63
2	0.020	0.051	0.103	5.99	7.38	9.21
3	0.115	0.216	0.352	7.81	9.35	11.3
4	0.297	0.484	0.711	9.49	11.1	13.3
5	0.554	0.831	1.15	11.1	12.8	15.1
6	0.872	1.24	1.64	12.6	14.4	16.8
7	1.24	1.69	2.17	14.1	16.0	18.5
8	1.65	2.18	2.73	15.5	17.5	20.1
9	2.09	2.70	3.33	16.9	19.0	21.7
10	2.56	3.25	3.94	18.3	20.5	23.2
11	3.05	3.82	4.57	19.7	21.9	24.7
12	3.57	4.40	5.23	21.0	23.3	26.2
13	4.11	5.01	5.89	22.4	24.7	27.7
14	4.66	5.63	6.57	23.7	26.1	29.1
15	5.23	6.26	7.26	25.0	27.5	30.6
16	5.81	6.91	7.96	26.3	28.8	32.0
17	6.41	7.56	8.67	27.6	30.2	33.4
18	7.01	8.23	9.39	28.9	31.5	34.8
19	7.63	8.91	10.1	30.1	32.9	36.2
20	8.26	9.59	10.9	31.4	34.2	37.6
21	8.90	10.3	11.6	32.7	35.5	38.9
22	9.54	11.0	12.3	33.9	36.8	40.3
23	10.2	11.7	13.1	35.2	38.1	41.6
24	10.9	12.4	13.8	36.4	39.4	43.0
25	11.5	13.1	14.6	37.7	40.6	44.3
26	12.2	13.8	15.4	38.9	41.9	45.6
27	12.9	14.6	16.2	40.1	43.2	47.0
28	13.6	15.3	16.9	41.3	44.5	48.3
29	14.3	16.0	17.7	42.6	45.7	49.6
30	15.0	16.8	18.5	43.8	47.0	50.9

1.2.3 Formulation of Gamma Distribution:

To introduce the gamma distribution, we consider its use as a waiting time distribution where the interest in the number of events occurring randomly in time according to Poisson process with constant rate λ which can be described as follows:-

If the number of events occurring during time $(x, \Delta x)$ is independent of the number occurring during time $(0, x)$, then the process satisfied the following postulates

(i) Pr [no events during $(x, \Delta x)$] $= 1 - \lambda\Delta x + u(\Delta x)$.

(ii) Pr [one event during $(x, \Delta x)$] $= \lambda\Delta x + u(\Delta x)$

(iii) Pr [two or more events during $(x, \Delta x)$] $= u(\Delta x)$.

where $u(\Delta x)$ be any function of Δx such that

$$\lim_{\Delta x \to 0} \frac{u(\Delta x)}{\Delta x} = 0$$

It has been shown [31] that if Y is a r.v representing the number of events occurring in fixed time (0, x], then Y has Poisson distribution with p.d.f

$$\Pr(Y = y) = \frac{e^{-\lambda x}(\lambda x)^y}{y!}, \quad y = 0, 1, 2$$

$$= 0, \text{ e.w.; } \lambda > 0 \qquad \text{.....(1.5)}$$

Now, suppose that our interest in the waiting time X from the occurrence of one event until r further events have occurred. Then the c.d.f of X when $x > 0$ is given by

$$F(x) = \Pr(X \le x) = 1 - \Pr(X > x)$$

$$= 1 - \Pr[(r-1) \text{ or fewer events occur during time } (0, x]]$$

$$= 1 - \sum_{y=0}^{r-1} \Pr(Y = y) = 1 - \sum_{y=0}^{r-1} \frac{e^{-\lambda x}(\lambda x)^y}{y!} \qquad \text{.....(1.6)}$$

Equation (1.6) is interesting in showing that the gamma c.d.f can be expressed as a cumulative sum of Poisson probabilities. Due to [40] the p.d.f of X is

$$f(x) = \frac{dF(x)}{dx} = \frac{d}{dx}\left\{1 - \sum_{y=0}^{n-1} \frac{e^{-\lambda x}(\lambda x)^y}{y!}\right\}$$

$$= -\sum_{y=0}^{r-1}\left[\frac{-\lambda e^{-\lambda x}(\lambda x)^y + y\lambda e^{-\lambda x}(\lambda x)^{y-1}}{y!}\right]$$

$$= \lambda\sum_{y=o}^{r-1}\frac{e^{-\lambda x}(\lambda x)^y}{y!} - \lambda\sum_{y=0}^{r-1}\frac{ye^{-\lambda x}(\lambda x)^{y-1}}{y!}$$

$$\lambda\sum_{y=0}^{r-1}\frac{e^{-\lambda x}(\lambda x)^y}{y!} - \lambda\sum_{y=1}^{r-1}\frac{e^{-\lambda x}(\lambda x)^{y-1}}{(y-1)!}$$

$$= \lambda\frac{e^{-\lambda x}(\lambda x)^{r-1}}{(r-1)!} = \frac{\lambda^r}{\Gamma(r)}x^{r-1}e^{-\lambda x},\ x > 0 \quad \text{.....(1.7)}$$

$$= 0, \text{ e.w.}$$

Which is as given by equation (1.1) with $\alpha = r$ and $\beta = \frac{1}{\lambda}$ the p.d.f. given by (1.7) is a special mumber of the gamma family of distributions and is sometimes referred to as a special Erlangian distribution after the Swedish scientist who used the distribution in early studies of queuing problems.

1.3 MOMENTS AND HIGHER MOMENTS PROPERTIES OF GAMMA DISTRIBUTION

Moments are set of constants used for measuring a distribution properties and under certain circumstances they specify the distribution. The

moments of r.v. X (or distn.) are defined in terms of the mathematical expectation of certain power of X when they exist. For instance

$\mu'_r = E(X^r)$ is called the r^{th} moment of X about the origin and $\mu_r = E[(X-\mu)^r]$ is called the r^{th} central moment of X. That is

$$\mu'_r = E(X^r) = \begin{cases} \sum_X x^r f(x), & X \text{ is discrete r.v.} \\ \int_x x^r f(x)dx, & X \text{ is continuous r.v.} \end{cases}$$

and

$$\mu_r = E[(X-\mu)^r] = \begin{cases} \sum_x (x-\mu)^r f(x), & X \text{ is discrete r.v.} \\ \int_x (x-\mu)^r f(x)dx, & X \text{ is continuous r.v.} \end{cases}$$

Provided the sum or integral converges absolutely. The generating functions reflect certain properties of the distribution, they could be used to generate moments. Sometimes they are definition the distributions, and also have a particular usefulness in connection with sums of independent, r.v's.

First, we shall consider a function of a real t called the moment generating function, denoted by M(t), which can be used to generate moments of r.v X.

For continuous r.v X, the m.g.f is defined by

$M(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$, provided the integral converge absolutely.

When r.v X~G (α, β) with p.d.f given by (1.1), we have

$$M(t) = \int_0^{\infty} e^{tx} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{\frac{-x}{\beta}} dx$$

$$= \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{\frac{-1}{\beta}(1-\beta t)x} dx$$

set $y = (1 - \beta t)\, x$ implies $dy = (1 - \beta t)dx$

$$M(t) = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \left(\frac{y}{1-\beta t}\right)^{\alpha-1} e^{\frac{-y}{\beta}} \cdot \frac{dy}{1-\beta t}$$

$$= \frac{1}{(1-\beta t)^{\alpha}} \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha-1} e^{\frac{-y}{\beta}} dy$$

where the integral $\int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha-1} e^{\frac{-y}{\beta}} dy$ is unity

Hence

$$M(t) = (1-\beta t)^{-\alpha}, t < \frac{1}{\beta} \quad \text{.....(1.8)}$$

The theory of mathematical analysis show that the existence of M(t) for $t < \frac{1}{\beta}$

Implies that the derivatives of M(t) of all orders exist at t = 0.

Thus the r^{th} moment of X about the origin is

$$\mu'_r = E(X^r) = \frac{d^r M(t)}{dt^r}\Big|_{t=0}, r = 1,2,3,\ldots..$$

$$= \alpha(\alpha + 1)(\alpha + 2) \ldots (\alpha + r - 1)\, \beta^r (1 - \beta t)^{-(\alpha+r)}\Big|_{t=0}$$

$$= \beta^r \prod_{i=0}^{r-1} (\alpha + i)$$

$$= \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)} \beta^r, r = 1, 2, \ldots \quad \text{.....(1.9)}$$

The following useful recurrence relation between the r^{th} central moment and moments about origin is given by Kendal and Stuart [23].

$$\mu_r = E[(X - \mu)^r] = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \mu^i \mu'_{r-i} \quad \text{.....(1.10)}$$

(i) Mean

$E(X) = \mu = \mu'_1$ is called the mean of r.v X. It is a measure of central tendency. Use of equation (1.9) with r = 1, we have

$$\mu = \alpha\beta \quad \text{.....(1.11)}$$

(ii) Variance

$Var(X) = \sigma^2 = E[(X-\mu)^2]$ is called the variance of r.v X. It is a measure of dispersion. Use of equation (1.10) with r = 2.

$$\sigma^2 = \alpha\beta^2 \quad \text{.....(1.12)}$$

(iii) Coefficient of Variation

The variational coefficient of r.v X is defined by the ratio $\frac{\sigma}{\mu}$. It is a measure of dispersion. It is independent of scale measurement and denoted by V. For gamma case

$$V = \frac{\sigma}{\mu} = \frac{\sqrt{\alpha\beta}}{\alpha\beta} = \alpha^{\frac{-1}{2}} \quad \text{.....(1.13)}$$

which is independent of the scale parameter β.

(iv) Coefficient of Skewness

$\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}}$ is called the coefficient of skewness.

It is a measure of the departure of the frequency curve from symmetry. If $\gamma_1 = 0$, the curve is not skewed, $\gamma_1 > 0$, the curve is positively skewed, and $\gamma_1 < 0$, the curve is negatively skewed [34]. Use of equation (1.10) with r = 3, we have $\mu_3 = E[(X-\mu)^3] = 2\alpha\beta^3$ Thus:

$$\gamma_1 = \frac{2\alpha\beta^3}{\left(\alpha\beta^2\right)^{3/2}} = 2\alpha^{-1/2} \quad \text{.....(1.14)}$$

(v) Coefficient of Kurtosis

$\gamma_2 = \frac{\mu_4}{\mu_2^2} - 3$ is called the coefficient of kurtosis. It is a measure of the degree of flattening of the frequency curve. If $\gamma_2 = 0$, the curve is called mesokurtic, if $\gamma_2 > 0$, the curve is called leptokurtic, and if $\gamma_2 < 0$, the curve is called platykurtic [34] .

Use of equation (1.10) with r = 4, we have $\mu_4 = E[(X-\mu)^4] = 3\alpha(\alpha+2)\beta^4$. Thus

$$\gamma_2 = \frac{3\alpha(\alpha+2)\beta^4}{\left(\alpha\beta^2\right)^2} - 3 = 6\alpha^{-1} \quad \text{.....(1.18)}$$

(vi) Mode

A mode of a distribution is the value x of r.v X that maximize the p.d.f f(x). For continuous distributions, the mode x is a solution of $\frac{df(x)}{dx} = 0$ and $\frac{d^2f(x)}{dx^2} < 0$.

The mode is a measure of location. Also we note that the mode may not exist or we may have more than one mode.

For gamma case with p.d.f of (1.1), the logarithm of f(x) is

$$Lnf(x) = -Ln\Gamma(\alpha) - \beta Ln\alpha + (\alpha-1)Lnx - \frac{x}{\beta}$$

$$\frac{dLnf(x)}{dx} = \frac{\alpha-1}{x} - \frac{1}{\beta}$$

For maximum

$$\frac{dLnf(x)}{dx}=0\Rightarrow\frac{\alpha-1}{x}-\frac{1}{\beta}=0$$

$$x=(\alpha-1)\beta,\ \ \alpha\geq 1 \quad \text{.....(1.19)}$$

(vii) Median

A median of a distribution is defined to be the value x of r.v X such that $F(x) = Pr(X \leq x) = \frac{1}{2}$. The median is measure of location.

For gamma case, the c.d.f given by equation (1.4), we have

$$\frac{1}{2}=\int_0^x\frac{1}{\Gamma(\alpha)\beta^{\alpha}}w^{\alpha-1}e^{\frac{-w}{\beta}}dw \quad \text{.....(1.20)}$$

where the right hand side of Eq. (1.20) is the incomplete gamma function.

1.4 SOME RELATED THEOREM

1.4.1 Independent Sum Theorem:

If $X_1, X_2, \ldots, X_n$ is a r.s of size n from Exp(β). Then the r.v $Y=\sum_{i=1}^{n}X_i\sim G(n,\beta)$.

Proof:

Using m.g.f technique, the m.g.f for each r.v X_i~Exp(β) is $M_X(t)=(1-\beta t)^{-1}$

Let $M_Y(t)$ be the m.g.f of r.v $Y=\sum_{i=1}^{n}X_i$

$$M_Y(t) = E\left(e^{tY}\right) = E\left(e^{t\sum_{i=1}^{n} X_i}\right) = E\left(\prod_{i=1}^{n} e^{tX_i}\right)$$

$$= \prod_{i=1}^{n} E\left(e^{tX_i}\right) = \prod_{i=1}^{n} M_X(t) = \prod_{i=1}^{n} (1-\beta t)^{-1} = (1-\beta t)^{-n}$$

which show that Y has m.g.f of G(n, β).

1.4.2 Gamma–Beta Theorem:

If X and Y are two independent r.v's with X~Be($\alpha, 1-\alpha$) and Y~Exp(β), then the r.v Z = XY~G(α, β)

Proof:

The joint p.d.f of X and Y is

$$f(x,y) = \frac{1}{\beta\Gamma(\alpha)\Gamma(1-\alpha)} x^{\alpha-1}(1-x)^{-\alpha} e^{\frac{-y}{\beta}}, \quad 0 < x < 1,\ 0 < y < \infty$$

$$= 0, \text{ e.w.}$$

with the transformation Z = XY, we set W = Y.

The function z = xy, w = y define one-to-one transformation that maps the space A = {(x, y) : 0 < x < 1, 0 < y < ∞} onto the space B = {(z, w) : 0 < z < w < ∞} with inverse $x = \frac{z}{w}$, w = y and the Jacobian of this transformation is

$$J = \frac{\partial(x,y)}{\partial(z,w)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{w} & \frac{-z}{wz} \\ 0 & 1 \end{vmatrix} = \frac{1}{w}$$

Then joint p.d.f of r.v's Z and W is

$$g(z,w) = f\left(\frac{z}{w}, w\right)|J|$$

$$= \frac{1}{\beta\Gamma(\alpha)\Gamma(1-\alpha)}\left(\frac{z}{w}\right)^{\alpha-1}\left(1-\frac{z}{w}\right)^{-\alpha} e^{\frac{-w}{\beta}}\frac{1}{w}$$

$$= \frac{1}{\beta\Gamma(\alpha)\Gamma(1-\alpha)} z^{\alpha-1}(w-z)^{-\alpha} e^{\frac{-w}{\beta}},\ 0 < z < w < \infty$$

$$= 0, \text{ e.w.}$$

The marginal p.d.f of Z is

$$g_1(z) = \int_w g(z,w)dw = \frac{z^{\alpha-1}}{\beta\Gamma(\alpha)\Gamma(1-\alpha)} \int_{w=z}^{\infty} (w-z)^{-\alpha} e^{\frac{-w}{\beta}} dw$$

Let $t = w - z \Rightarrow dt = dw$

$$g_1(z) = \frac{z^{\alpha-1}}{\beta\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^{\infty} t^{-\alpha} e^{\frac{-t+z}{\beta}} dt$$

$$= \frac{z^{\alpha-1} e^{\frac{-z}{\beta}} \beta^{1-\alpha}}{\beta\Gamma(\alpha)} \int_0^{\infty} \frac{1}{\Gamma(1-\alpha)\beta^{1-\alpha}} t^{(1-\alpha)-1} e^{\frac{-t}{\beta}} dt$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} z^{\alpha-1} e^{\frac{-z}{\beta}},\ 0 < z < \infty$$

$$= 0, \text{ e.w.}$$

That is Z~G(α, β).

1.5 POINT ESTIMATION

Point estimation is concerned with inference about the unknown parameters of a distribution from a sample. It provides a single value for each unknown parameter.

1.5.1 Definition:

Any statistic whose value are used to estimate the unknown parameter θ for some function of θ say $\tau(\theta)$ is called point estimator.

Point estimation admits two problems:

First, developing methods of obtaining a statistics whose values could be used to estimate the unknown parameters of the distribution, such statistics are called point estimators.

Second, selecting criteria and technique to obtain a best estimator among possible estimators.

1.5.2 Methods of Finding Estimators:

Many techniques have been proposed in the literatures of finding estimators for the distn. parameters such as moments, maximum likelihood, minimum chi-square, minimum distance, least square, and Bayesian method. These methods provide a single value for each unknown parameter of the distribution

For gamma case we shall discuss two methods the oritically and practically the method of moments and the maximum likelihood method.

1.5.2.1 Moments Method:

Let $X_1, X_2, \ldots, X_n$ be a r.s of size n from a distribution whose p.d.f $f(x; \theta)$, $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ is a vector of unknown parameters, let $\mu_r' = E\left(X^r\right)$ be the r^{th} moment about origin of the population and $M_r = \frac{1}{n}\sum_{i=1}^{n} X_i^r$ be the r^{th} moment about origin of the sample. The method of moments can be described follows:

Since, we have k unknown parameters, equate

μ_r' to M_r at $\underset{\sim}{\theta} = \underset{\sim}{\hat{\theta}}$ .That is

$\mu_r' = M_r$ at $\underset{\sim}{\theta} = \underset{\sim}{\hat{\theta}}$, $r = 1, 2, \ldots, k$

For these k equations, we find a unique solution for $\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_k$ and we say that $\hat{\theta}_r (r = 1, 2, ..., k)$ is an estimate of θ_r obtained by method of Moments and the corresponding statistic $\hat{\Theta}_r$ is the method of moments estimator of θ_r.

Now, to estimate α and β by method of moments we let $X_1, X_2, \ldots, X_n$ be a r.s. of size n from $G(\alpha, \beta)$ is taken.

Since $G(\alpha, \beta)$ distribution involve two unknown parameters

We set $\mu_r' = M_r$ at $\alpha = \hat{\alpha}$, $\beta = \hat{\beta}$,$r = 1,2$

$r = 1$ implies

$$\mu_1' = E(X) = \alpha\beta,\ M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$$

$r = 2$ implies

$$\mu_2' = E(X^2) = \alpha\beta^2 + \alpha^2\beta^2,\ M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$= \frac{n-1}{n}S^2 + \bar{X}^2$$

where:

$$S^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right]$$

r = 1 implies $\mu_1' = M_1$ at $\alpha = \hat{\alpha}$, $\beta = \hat{\beta}$, we obtain

$$\hat{\alpha}\hat{\beta} = \bar{x} \quad \text{.....(1.21)}$$

r = 2 implies $\mu_2' = M_2$ at $\alpha = \hat{\alpha}$, $\beta = \hat{\beta}$, we obtain

$$\hat{\alpha}\hat{\beta}^2 + \hat{\alpha}^2\hat{\beta}^2 = \frac{(n-1)}{n}S^2 + \bar{X}^2 \quad \text{.....(1.22)}$$

Solving equations (1.21) and (1.22), we get

$$\hat{\alpha} = \frac{n\bar{X}^2}{(n-1)S^2} \text{ and } \quad \hat{\beta} = \frac{(n-1)S^2}{n\bar{X}} \quad \text{.....(1.23)}$$

are respectively the estimates of α and β obtained by method of moments and the corresponding estimators are $\frac{n\bar{X}^2}{(n-1)S^2}$, $\frac{(n-1)S^2}{n\bar{X}}$

Our conclusion about these estimators are

(i) Both estimators are functions of $\bar{X}$ and S^2.

(ii) The joint distribution of the estimators can not be found easily because the dependency existence among these estimators

(iii) For known $\beta, \hat{\alpha} \sim G(n\alpha, \frac{1}{n})$

(iv) For known $\alpha, \hat{\beta} \sim G(n\alpha, \frac{\beta}{n\alpha})$.

Definition (Likelihood function):

The likelihood function of a r.s $X_1,X_2,\ldots,X_n$ of size n from a distribution having p.d.f $f(x,\underset{\sim}{\theta})$ where $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ is a vector of unknown parameters is defined to be the joint p.d.f of the n r.v's $X_1, X_2,\ldots, X_n$ which is considered as a function of $\underset{\sim}{\theta}$ and denoted by $L(\underset{\sim}{\theta}, \underset{\sim}{x})$, that is

$$L = L(\underset{\sim}{\theta},\underset{\sim}{x}) = f(\underset{\sim}{x},\underset{\sim}{\theta}) = \prod_{i=1}^{n} f(x_i,\underset{\sim}{\theta}).$$

1.5.2.2 Maximum Liklihood Method

Let $L(\underset{\sim}{\theta},\underset{\sim}{x})$ be the likelihood function of a r.s $X_1, X_2,..., X_n$ of size n from a distribution whose p.d.f $f(x;\underset{\sim}{\theta})$, $\underset{\sim}{\theta}=(\theta_1, \theta_2, \ldots, \theta_k)$ is a vector of unknown parameters Let:

$$\hat{\underset{\sim}{\theta}} = \underset{\sim}{u}(\underset{\sim}{x})$$

$$= (u_1(\underset{\sim}{x}), u_2(\underset{\sim}{x}),..., u_k(\underset{\sim}{x}))$$

be a vector function of the observations $\underset{\sim}{x} = (x_1, x_2,..., x_n)$

If $\hat{\underset{\sim}{\theta}}$ have the value of $\underset{\sim}{\theta}$ which maximizes $L(\hat{\underset{\sim}{\theta}},\underset{\sim}{x})$ then $\hat{\underset{\sim}{\theta}}$ is the m.ℓ.e of $\underset{\sim}{\theta}$ and the corresponding statistic $\hat{\underset{\sim}{\Theta}}$ is the M.L.E of $\underset{\sim}{\theta}$ we note that

(i) Many likelihood function satisfy the condition that the m.ℓ.e is a solution of the likelihood eq's

$$\frac{\partial L(\underset{\sim}{\theta},\underset{\sim}{x})}{\partial \theta_r} = 0 \quad \text{, at } \underset{\sim}{\theta} = \hat{\underset{\sim}{\theta}} \quad r=1,2,\ldots,k$$

(ii) Since $L(\underset{\sim}{\theta}, \underset{\sim}{x})$ and $\ln L(\underset{\sim}{\theta}, \underset{\sim}{x})$ have their maximum at the same value of $\underset{\sim}{\theta}$ so sometimes it is easier to find the maximum of the logarithm of the likelihood.

In such case, the m.ℓ.e $\hat{\underset{\sim}{\theta}}$ of $\underset{\sim}{\theta}$ which maximizes $L(\underset{\sim}{\theta}, \underset{\sim}{x})$ may be given the solution of the likelihood eq's

$$\frac{\partial \ln L(\underset{\sim}{\theta}, \underset{\sim}{x})}{\partial \theta_r} = 0 \quad \text{at} \quad \underset{\sim}{\theta} = \hat{\underset{\sim}{\theta}} \quad ,r=1,2,\ldots,k$$

For gamma case

Let $X_1, X_2, \ldots, X_n$ be a r.s. of size n from $G(\alpha, \beta)$ where the distribution p.d.f is given by (1.1). The likelihood function is

$$L(\alpha,\beta,\underset{\sim}{x}) = f(\underset{\sim}{x},\alpha,\beta)$$

$$= \prod_{i=1}^{n} f(x_i,\alpha,\beta)$$

$$= \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x_i^{\alpha-1} e^{\frac{-x_i}{\beta}}$$

$$= \left[\Gamma(\alpha)\right]^{-n} \beta^{-n\alpha} \left(\prod_{i=1}^{n} x_i \right)^{\alpha-1} e^{-\sum_{i=1}^{n} \frac{x_i}{\beta}}$$

$$LnL = -nLn\left[\Gamma(\alpha)\right] - n\alpha Ln\beta + (\alpha-1)\sum_{i=1}^{n} Lnx_i - \frac{1}{\beta}\sum_{i=1}^{n} x_i$$

$$\frac{\partial LnL}{\partial \alpha} = -n\psi(\alpha) - nLn\beta + \sum_{i=1}^{n} Lnx_i - 0 \quad \text{.....(1.24)}$$

where

$$\psi(\alpha) = \frac{d}{d\alpha} Ln\Gamma(\alpha) \quad \text{.....(1.25)}$$

$$\frac{\partial LnL}{\partial \beta} = 0 - \frac{n\alpha}{\beta} + 0 + \frac{1}{\beta^2}\sum_{i=1}^{n} x_i \quad \text{.....(1.26)}$$

We set $\frac{\partial LnL}{\partial \alpha} = 0$ and $\frac{\partial LnL}{\partial \beta} = 0$ at $\alpha = \hat{\alpha}, \beta = \hat{\beta}$,

we have:

$$-n\psi(\hat{\alpha}) - nLn\hat{\beta} + \sum_{i=1}^{n} Lnx_i = 0 \quad \text{.....(1.27)}$$

and

$$-n\frac{\hat{\alpha}}{\hat{\beta}} + \frac{1}{\hat{\beta}^2}\sum_{i=1}^{n} x_i = 0 \Rightarrow n\hat{\alpha} = \frac{1}{\hat{\beta}}\sum_{i=1}^{n} x_i$$

This implies that:

$$\hat{\alpha}\hat{\beta} = \bar{x} \quad \text{.....(1.28)}$$

From (1.27) and (1.28), we have:

$$Ln\hat{\alpha} - \psi(\hat{\alpha}) = Ln\bar{x} - \frac{1}{n}\sum_{i=1}^{n} Lnx_i \quad \text{.....(1.29)}$$

Analytic solution for $\hat{\alpha}$ can not be found from Eq. (1.29). In such case numerical method is needed for, finding an approximate value for $\hat{\alpha}$ and $\hat{\beta}$.

In practice, we choose Newton-Raphson method which can be described by the formula

$$\hat{\alpha}_{i+1} = \hat{\alpha}_i - \frac{f(\hat{\alpha}_i)}{f'(\hat{\alpha}_i)} \quad \text{.....(1.30)}$$

we set

$$f(\hat{\alpha}) = Ln\hat{\alpha} - \psi(\hat{\alpha}) - Ln\bar{X} + \frac{1}{n}\sum_{i=o}^{n} Lnx_i = 0$$

and

$$f'(\hat{\alpha}) = \frac{1}{\hat{\alpha}} - \psi'(\hat{\alpha})$$

where $\psi(\hat{\alpha})$ and $\psi'(\hat{\alpha})$ are the digamma and trigamma are tabulated. If table of $\psi(\hat{\alpha})$ and $\psi'(\hat{\alpha})$ are not available, excellent approximation [1], [8] is given by

$$\psi(\hat{\alpha}) \approx Ln\hat{\alpha} - (2\hat{\alpha} - \frac{1}{3} + \frac{1}{16\hat{\alpha}})^{-1}$$

where:

$$\psi(\hat{\alpha}) = \frac{d\text{Ln}\Gamma(\hat{\alpha})}{d\hat{\alpha}} \quad \text{.....(1.31)}$$

$$\psi'(\hat{\alpha}) \approx (\hat{\alpha} - \frac{1}{2} + \frac{1}{10\hat{\alpha}})^{-1} \quad \text{.....(1.32)}$$

In order to esses the results of the theoretical estimation practically for the two methods given by (1.5.2.1) and (1.5.2.2), we use theorem (1.4.1) for observing random sample of size n = 6(1)10(2)20 from G(α, β = 1) and the run size 20 is used. Table (A-2) of Appendix A gives the values of the estimation methods. Comparison between the two methods of estimation is made by using the measure of the mean square error and the results are given in table (A-3).

Table (A-2) The estimation values for $\hat{\alpha}$ ***and*** $\hat{\beta}$.

Sample size ***(α,β)***	***Estimation of*** $(\hat{\alpha},\hat{\beta})$	
	Moments Method	***M.L.E***
6	(7.0950,0.9281)	(6.5341,0.9236)
7	(7.7950,0.9454)	(7.2086,0.8747)
8	(8.9067,0.9241)	(7.4543, 1.3001)
9	(9.5340,0.9651)	(9.6331,0.5699)
10	(10.4594,1.0337)	(10.3425,1.0104)
12	(12.3094,1.0483)	(11.0386,1.1692)
14	(13.9410,1.0309)	(14.2693,0.9191)
16	(18.0570,0.9048)	(18.0538,0.9485)
18	(17.5400,1.1230)	(18.3712,0.9658)
20	(20.6581,1.0079)	(21.0235,0.9466)

Table (A-3) Mean Square Error

Sample size	*Moments Method*		*M.L.E*	
	MSE of α	**MSE of β**	**MSE of α**	**MSE of β**
6	3.2865	3.6438	3.4167	3.4396
7	5.4746	5.5794	8.3198	9.3263
8	6.7206	8.7993	4.2503	7.2376
9	8.6000	8.9825	8.5414	8.8706
10	14.7845	15.0317	10.8968	11.1170
12	15.7106	16.5951	18.6491	18.6542
14	15.3273	16.5296	5.8164	13.5831
16	12.8150	12.8768	20.9762	14.9359
18	19.3429	19.3572	7.0787	15.5042
20	13.6652	13.6631	14.8213	1.0939

CHAPTER 3

GENERATING RANDOM VERTICES FROM GAMMA DISTRIBUTION

3.1 INTRODUCTION

The first step in studying a certain problem under consideration is building a mathematical model, the next step is deriving a solution from this model. The solution may be obtained analytically or numerically. The analytic solution is usually obtained directly from it's mathematical representation in the form of the formula, while a numerical solution is generally an approximate solution obtained as a result of substitution of numerical values for the variables and parameters of the model. Many numerical methods are iterative, that is, each successive step in the solution uses the results from the previous step, such as Newton-Raphson method for approximating the roots of a non-linear equation. Two special types of numerical methods are simulation and Monte Carlo designed for a solution of deterministic and stochastic problems.

Simulation "in a wide scene" is defined as numerical technique for conducting experiments on a digital computer, which involve certain types of mathematical and logical models that describe the system behavior over extended periods of time, for examples, simulating football game, supersonic jet flight, a telephone communication system, a wind tunnel, a large scale

military battle (to evaluate defensive or offensive weapon system), or a maintenance operations (to evaluate the optimal size of repair crews).

Simulation is often viewed as a " Method of Last Resort" to be used when everything else has failed, software building and technical developments have made simulation one of the most widely used and accepted tools for designer in system analysis and operational research.

Simulation "in a narrow sence" (also called stochastic simulation) is defined as experimenting with the model over time, it includes sampling stochastic varieties from probability distribution. Because sampling from a particular distribution involve the use of random numbers, stochastic simulation sometimes called Monte Carlo Simulation.

Historically, the Monte Carlo method was considered as a technique using random or pseudorandom numbers for solution of a model. These random numbers are essentially independent random variables uniformly distributed over unit interval [0, 1].

Actually there are arithmetic codes available at computer center (0 to 9) occurs with approximately equal probability "imagine flips of a fair ten-side die". Such codes are called random number generators.

In the beginning of the 20^{th}-century the Monte Carlo was used to examine the Boltzmann equation.

In 1908 the famous statistician Gosset (student) uses the Monte Carlo method for estimating the correlation coefficient in his t-distribution, [13].

One of the earliest problems connected with Monte Carlo method is the famous Buffon's needle problem, who found the probability of a needle of length L thrown randomly onto a floor composed of parallel planks of width

D > L is $P = \frac{2L}{\pi D}$ which can be estimated as the ratio of the number of throws hitting the crack to the total number of throws.

A. N. Kolmogororv (1931) apply Monte Carlo method and showed the relationship between Markov stochastic processes and certain Integro-differential equations, [14].

In 1948 S.Ulam used Monte Carlo method for estimation of the eigenvalues of Schrodinger equation, [15].

The terms “Monte Carlo” was introduced by Von Neumann and Ulam during World War II, as a code word for secret work at Los Alamos, it was suggested by the gambling casinos at the city of Monte Carlo in Monaco. The Monte Carlo method was then applied to problems related to the atomic bomb [3] where the work involve direct simulation of behavior concerned with random neutron diffusion in fissionable material. Shortly thereafter Monte Carlo method were used to evaluate complex multidimensional integrals, solution of certain differential and integral equations stochastic problems, deterministic problems if they have the same formal expression as some stochastic process, evaluating parameters of queues and networks, sampling random varieties from probability distributions, and analyzing complex problems. A useful reference related to Monte Carlo simulation is given by Rubinstein [38] and Norman [33].

3.2 RANDOM NUMBER GENERATION

Many techniques for generating random numbers on digital computer by Monte Carlo method and simulation have been suggested tested and used in recent years. Some of these methods are based on random phenomena, others on deterministic recurrence procedures.

Initially manual methods were used to generate a sequence of numbers such as coin flipping, dice rolling, card shuffling, and roulette wheels, but these methods were to slow for general use, moreover the generated sequence not reproduced.

Shortly following with the computer aid it become possible to obtain random numbers. In 1951 Von Neumann [41] suggests the mid-square method using the arithmetic operations of a computer. His idea is to take the square of the preceding random number and entract the middle digits. For instance, suppose we wish to generate 4-digits numbers

1- Choose any 4-digit number, say 5232.

2- Square it, we have 27373824.

3- The next 4-digit number is the middle 4-digits of step2, that is, 3738.

4- Repeat the process.

Von Neumann's method proved slow and awkward for statistical analysis, furthermore the sequences tend to cyclicity and once a zero is encountered the sequence terminates. One method of generating random numbers on a digital computer is published in 1951 by RAND Corporation [35], the method consists of preparing a well known table of a million digits and storing it in the memory of the computer. The advantage of this method is reproducibility and it's disadvantage is its lack of speed and risk of exhausting the table.

It is noted in the literature that the random numbers generated by any method is good one if the random numbers are uniformly distributed, statistically independent, reproducible, fast, and requires minimum capacity in the computer memory.

The congruential methods for generating pseudorandom numbers are designed specifically to satisfy as many of these requirements as possible.

These methods produce a nonrandom sequence of numbers according to some recursive formula based on calculating the residues modulo of some integer m of a linear transformation Although these processes are completely deterministic, Knuth in 1969 [25] show that the numbers generated by such sequence appear to be uniformly distributed and statistically independent.

The congruential methods [27] are based on a fundamental congruence relationship, which may be formulated as:

$X_{i+1} = (aX_i + c) \pmod{m}, i = 1, 2, ..., m$ ..... (3.1)

where a is a multiplier, c is the increment, and m is the modulus (a, c, m are non-negative integers), (mod m) means that equation (3.1) can be written as:

$$X_{i+1} = aX_i + c - m\left[\frac{aX_i + c}{m}\right] \quad \text{..... (3.2)}$$

where [Z] is the largest positive integer in z.

Given an initial starting value X_1 with fixed values of a, c, and m, then equation (3.2) yields congruence relationship (modulo m) for any value i of the sequence $\{X_i\}$. The sequence $\{ X_i \}$ will repeat itself in at most m steps and will be therefore periodic. For example,

Let $a = c = x_1 = 3$, and $m = 5$, then the sequence obtained from the recursive formula

$X_{i+1} = (3X_i + 3) \pmod{5}$ is $X_i = 3, 2, 4, 0, 3, \ldots$

The random numbers on the unit interval [0, 1] can be obtained by:

$$U_i = \frac{X_i}{m} \quad \text{..... (3.3)}$$

It follow's from equation (3.3) that $X_i \le m$, $\forall$ i, this inequality mean that the period of the generator can not exceed m, that is, the sequence $\{X_i\}$ contains at most m distinct numbers. So m must be chosen as large as possible to ensure. A sufficiently large sequence of distinct numbers in the cycle.

It is noted in the literature [18], [26], [29] that good statistical results can be achieved from a computer by choosing $a = 2^7+1$, $c = 1$, and $m = 2^{35}$.

3.3 RANDOM VARIETIES GENERATION

Two well-known methods for generating random varieties form continuous distribution, namely the inverse transform method and acceptance-rejection method.

3.3.1 *The Inverse Transform Method*:

Recall the properties of the c.d.f

$Pr(X \le x) = F(x)$ of r.v. X

(i) $0 \le F(x) \le 1$.

(ii) $F(-\infty) = 0$, $F(\infty) = 1$.

(iii) $F(x)$ is non-decreasing function of x.

(iv) $F(x)$ is continuous function to the right at each x.

The inverse transform method is based on the following theorem:

Theorem (3.1) [26]:

The r.v. $U = F(X) \sim U(0, 1)$ if and only if the r.v $X = F^{-1}(U)$ has c.d.f. $Pr(X \le x) = F(X)$.

Proof:

$\Rightarrow$ Consider the r.v $U=F(X) \sim U(0, 1)$ then the c.d.f of U is

$$G(x) = Pr(U \le u) = \begin{cases} 0, u \le 0 \\ u, 0 < u < 1 \\ 1, u \ge 1 \end{cases}$$

$$Pr(X \le x) = Pr(F^{-1}(U) \le x) = Pr(U \le F(x)) = F(x)$$

$\Leftarrow$ conversely, consider the r.v X has c.d.f $F(x) = Pr(X \le x)$

$$G(u) = Pr(U \le u) = Pr(F(X) \le u)$$

$$= Pr[X \le F^{-1}(u)]$$

$$= F[F^{-1}(u)] = u$$

The IT algorithm describe the necessary steps for generating r.v by Inverse Transform Method

1- Generate U from U (0, 1).

2- Set $X = F^{-1}(U)$.

3- Deliver X as a r.v. generated from the p.d.f. f(x).

We note that, this method is valid when the c.d.f. F(x) exists in a form for which the corresponding inverse transform can be solved analytically.

3.3.2 The Acceptance-Rejection Method [41]:

This method consists of sampling a r.v from an appropriate distn and subjecting it to a test to determine whether or not it will be acceptable for use.

To carry out the method, the p.d.f. f(x) of the generated r.v X represented as

$f(x) = ch(x)g(x)$, where $c \ge 1$, h(x) is also p.d.f. and $0 < g(x) \le 1$.Then we generate two r.v.'s U and Y from U (0, 1) and h(y) respectively and test to see whether or not the inequality $U \le g(Y)$ hold:-

1- If the inequality hold, then accept Y = X as a r.v generated from f(x)

2- If the inequality violated, then reject the pair (U, Y) and try again

The theory behind this method is based on the following theorem.

Theorem (3.2) [38]:

Let the p.d.f. of r.v X represented as f (x) =c h(x) g(x) where $c \geq 1$, h(x) as also p.d.f., and $0<g(x) \leq 1$. Let U and Y be distributed U(0, 1) and h(y) respectively then $\Pr[Y = x \mid U \leq g(Y)] = f(x)$

Proof:

$$\Pr[Y = x \mid U \leq g(Y)] = \frac{\Pr[Y = x, U \leq g(Y)]}{\Pr[U \leq g(Y)]} \quad \text{..... (3.1)}$$

$$= \frac{\Pr[Y = x, U \leq g(Y)]}{\int_x \Pr[Y = x, U \leq g(Y)]dx} \quad \text{..... (3.2)}$$

Using Bayes theorem [21], we have:

$$\Pr[Y = x \mid U \leq g(Y)] = \frac{\Pr(U \leq g(Y) \mid Y = x)\Pr(Y = x)}{\int_x \Pr[U \leq g(Y) \mid Y = x]\Pr(Y = x)dx}$$

Since $\Pr[U \leq g(y) \mid Y = x] = \Pr[U \leq g(x)] = g(x)$ and $\Pr(Y = x) = h(x)$. Then:

$$\Pr[Y = x \mid U \leq g(y)] = \frac{g(x)h(x)}{\int_x g(x)h(x)dx} = \frac{g(x)h(x)}{\int_x \frac{f(x)}{c}dx}$$

$$= \frac{g(x)h(x)}{\frac{1}{c}} = cg(x)h(x) = f(x)$$

Q.E.D.

The efficiency of Acceptance- Rejection is determined by the inequality

$U \leq g(Y)$, where the efficiency = $\Pr[U \leq g(Y)] = \frac{1}{c}$.

Since the trials are independent, the probability of success in each trial is $P=\frac{1}{c}$.

If N is a random variable represent the number of trials before a successful pair (U, Y), then N has geometric distribution with P.d.f.

$$\Pr[N = n] = p(1-p)^{n-1} \quad , n = 1, 2, 3\ldots$$

$$= 0 \qquad\qquad , \text{e.w.}$$

and the expected number of trials is

$$E(N) = \frac{1}{p} = c$$

The AR-Algorithm describes the necessary steps of generating random varieties by Acceptance-Rejection method.

AR-Algorithm:

1- Generate U from U(0, 1).

2- Generate Y from h(y).

3- If $U \le g(Y)$, deliver (we accept) Y = X as a random variates generated from f(x).

4- Go to step (1).

5- Stop.

Remark:

For Acceptance-Rejection method to be of practical interest, the following conditions must be satisfied

(1) It should be easy to generate a r.v. from h(x)

(2) The efficiency (probability) of the procedure $\frac{1}{c}$ should be large, that is c should close to one.

Illustration of the Acceptance-Rejection Method we choose $c \geq 1$ such that

$$f(x) \leq ch(x) = \phi(x), \text{ say.}$$

The problem then is to find the function $\phi(x)$ and the function $h(x) = \frac{1}{c}\phi(x)$ from which the random variable can be easily generated.

3.4 GENERATING RANDOM VARIATES FROM GAMMA DISTRIBUTION

3.4.1 G1-Procedure:

This procedure is the most common use for generating random variates from gamma distn. When α is an integer, which utilize the independent sum theorem as discussed in chapter one, section (1.4.1). That is, if a sample $x_1, x_2, \ldots, x_\alpha$ of size α from Exp(β) is given, then the r.v $Y = \sum_{i=1}^{\alpha} X_i \sim G(\alpha, \beta)$. Generating independent exponential random variates of size α can be mode as if $U_i \sim U(0,1)$ then $-\beta \ln U_i \sim Exp(\beta)$, $i = 1,2,\ldots, \alpha$

Then G1-algorithm describe the necessary steps for generating gamma variates by this method

G1-Algorithm:

Step 1: Read β.

Step 2: Generate $U_1, U_2, \ldots, U_\alpha$ from U(0,1).

Step 3: Set $X_i = -\beta \ln U_i$, $i = 1,2, \ldots, \alpha$.

Step 4: Set $Y = \sum_{i=1}^{\alpha} X_i$.

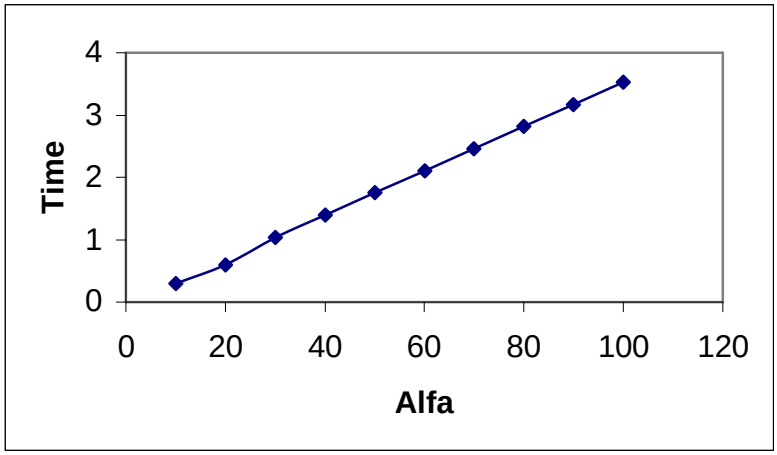
Step 5: Deliver Y as a random variate generated from G (α, β).

Step 6: Stop.

A computer program is made in appendix (C1) for generating a sample of size α from gamma distn. written in Pascal language use Microcomputer Pentium IV, CPU speed 1.7 GHz, when α = 10(10)100 and the run size 10 is taken. The average time is calculated for each gamma sample and the result is tabulated in table (8).

Table (8) The average time with different α-samples.

α	*Average time*
10	0.3
20	0.6
30	1.04
40	1.4
50	1.75
60	2.1
70	2.46
80	2.82
90	3.17
100	3.53



Fig(6) graph to show the linear tends between the average time and values of α.

Figure (6) show the graphical of the average time against the size α.

The figure (6) show that there is a linear trand among the observations. To find the best fitted line:

$$t_i = a + b\alpha_i + e_i \quad \text{..... (3.3)}$$

We utilize the least square method for estimating the parameters of linear equation(3.3) where we consider the error set of observations $\{e_i\}$ have zero mean, same variance, and uncorrelated. Viz

$$\text{Set } \Omega = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (t_i - a - b\alpha_i)^2$$

$$\frac{\partial \Omega}{\partial a} = -2\sum_{i=1}^{n} (t_i - a - b\alpha_i)$$

$$\frac{\partial \Omega}{\partial b} = -2\sum_{i=1}^{n} \alpha_i (t_i - a - b\alpha_i)$$

we set

$\frac{\partial \Omega}{\partial a}, \frac{\partial \Omega}{\partial b} = 0$ at $a = \hat{a}, b = \hat{b}$, we get

$$\hat{b} = \frac{\sum_{i=1}^{n} t_i \sum_{i=1}^{n} \alpha_i - n \sum_{i=1}^{n} \alpha_i t_i}{\left(\sum_{i=1}^{n} \alpha_i \right)^2 - n \sum_{i=1}^{n} \alpha_i^2}$$

$$\hat{a} = \frac{\sum_{i=1}^{n} t_i - \hat{b} \sum_{i=1}^{n} \alpha_i}{n}$$

3.4.2 G2-Procedure:

Johnk, M.D. [22] suggested a technique based on Gamma-Beta theorem given in section (1.4.2) of chapter one for generating random variates from G (α, β) when $0 < \alpha < 1$. His procedure dependent on generating two r.v.'s, the first generated with probability α from $\beta e(\alpha, 1 - \alpha)$ and the other from Exp(β) from which the G(α, β) can be generated. Johnk algorithm can be described as follows:

1- Read α and β

2- With probability α generated X from $\beta e(\alpha, 1 - \alpha)$

3- Generate r.v. Y from Exp (β)

4- Compute $z = xy$

5- Deliver z as a r.v generated from G (α, β)

The efficiency of Johnk algorithm depends on the value of α and it reaches to optimum when $\alpha = 1$.

Our technique is also based on the Gamma-Beta Theorem, but the generation from $\beta e(\alpha,1-\alpha)$, $0<\alpha<1$, is based on the acceptance-rejection method as follows:

Consider a r.v. $X\sim\beta e(\alpha, 1-\alpha)$where the distn. p.d.f.

$$f(x)=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}x^{\alpha-1}(1-x)^{-\alpha},0<x<1$$

$$=0,e.w;0<\alpha<1$$

$$=\frac{1}{\alpha(1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)}\alpha.x^{\alpha-1}(1-\alpha)(1-x)^{-\alpha}$$

As we describe in sec (3.3.2) that the acceptance-rejection method is based on writing this p.d.f. as:

$$f(x)=ch(x)g(x)$$

we take:

$$h(x)=\alpha.x^{\alpha-1},0<x<1$$

$$=0,e.w$$

$$c=\frac{1}{\alpha(1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)} \quad \text{..... (3.4)}$$

and

$$g(x)=(1-\alpha)(1-x)^{-\alpha}, \text{ where } 0<g(x)\le 1$$

The c.d.f of the p.d.f. h(x) is

$$H(x)=\begin{cases}0,x\le 0\\ x^{\alpha},0<x<1\\ 1,x\ge 1\end{cases}$$

Set $u_1=H(x)\Rightarrow u_1=x^{\alpha}\Rightarrow x=u_1^{1/\alpha}$

For r.v Y~Exp(β) the c.d.f is

$$H_2(y)=\begin{cases}0, y\le 0\\ 1-e^{\frac{-y}{\beta}}, 0<y<\infty\\ 1, y=\infty\end{cases}$$

set $u_2=H_2(y)=1-e^{\frac{-y}{\beta}} \Rightarrow e^{\frac{-y}{\beta}}=1-u_2=u_2$

$$\frac{-y}{\beta}=\ln u_2$$

$$y=-\beta\ln u_2$$

The G2-algorithm describe the steps for generating r.v from G (α, β), $0<\alpha<1$.

G2-Algorithm:

Step1: Read α, β

Step2: Generate U_2 from U(0, 1)

Step3: Set $Y=-\beta\ln U_2$

Step4: Generate U and U_1 from U(0, 1)

Step5: Set $X=U_1^{1/\alpha}$

Step6: If $U>g(x)=(1-\alpha)(1-x)^{-\alpha}$, go to step(4)

Step7: Set Z = XY

Step8 Deliver Z as a r.v generated from G(α, β)

Step9 Stop

The efficiency of the method is $\frac{1}{c} = \alpha(1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)$

Table (9) compute the true and simulated efficiencies for different configuration of α, $0 < \alpha < 1$.

A computer program is made in Appendix (C2) for computation the values of c when $\alpha = 0.1(0.2)0.9$ with run size of 100 is taken and the results are displayed in table (9) together with values of c as given by Eq.(3.4).

Table (9) Values of c with different α-samples using G2-procedure.

α	*Simul.effec.*	*Theo. effec.*	*error*	*Average error*
0.1	0.9150	0.8953	0.0197	0.039
0.3	0.8156	0.7852	0.0304	
0.5	0.7841	0.6640	0.1201	
0.7	0.8156	0.7831	0.0325	
0.9	0.9150	0.9231	0.0081	

3.4.3 G3-Procedure:

This procedure uses the acceptance-rejection method for generating random variates from G (α, 1) when $0 < \alpha < 1$.

Where we write the p.d.f.

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, 0 < x < \infty \quad \text{..... (3.5)}$$

$$= 0, \text{e.w} \qquad , 0 < \alpha < 1$$

as f(x) = ch(x)g(x) where c≥1, h(x) is also p.d.f. and 0<g(x)≤1 as described in section(3.3.2).

The theory behind this representation can be described as follows:

$$x^{\alpha-1}e^{-x} \le \begin{cases} x^{\alpha-1}, 0 < x \le 1 \\ e^{-x}, 1 < x < \infty \end{cases}$$

Then:

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1}e^{-x} \le \phi(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} x^{\alpha-1}, 0 < x \le 1 \\ \frac{1}{\Gamma(\alpha)} e^{-x}, 1 < x < \infty \end{cases} \quad \text{..... (3.6)}$$

$ch(x) = \phi(x)$, implies:

$$\int_0^{\infty} ch(x)dx = \int_0^{\infty} \phi(x)dx$$

$$c = \frac{1}{\Gamma(\alpha)}\left[\int_0^1 x^{\alpha-1}dx + \int_1^{\infty} e^{-x}dx\right]$$

which leads

$$c = \frac{1}{\Gamma(\alpha)}\left(\frac{1}{\alpha} + \frac{1}{e}\right) \text{..... (3.7)}$$

Since

$$h(x) = \frac{\phi(x)}{c} = \begin{cases} \frac{x^{\alpha-1}}{\left(\frac{1}{\alpha}+\frac{1}{e}\right)}, 0 < x \le 1 \\ \frac{e^{-x}}{\frac{1}{\alpha}+\frac{1}{e}}, 1 < x < \infty \end{cases} \quad \text{..... (3.8)}$$

From Eqs (3.5) and (3.6), we have

$$g(x)=\begin{cases} e^{-x}, 0<x\le 1 \\ x^{\alpha-1}, 1<x<\infty \end{cases} \quad \text{..... (3.9)}$$

The c.d.f of the p.d.f. h(x) is

$$H(x)=\begin{cases} 0, x\le 0 \\ \int_0^x h(t)dt, 0<x\le 1 \\ \int_0^1 h(t)dt+\int_1^x h(t)dt, 1<x<\infty \\ 1, x=\infty \end{cases}$$

$$\text{For } 0<x\le 1 \Rightarrow H(x)=\frac{1}{\frac{1}{\alpha}+\frac{1}{e}}\int_0^x t^{\alpha-1}dt=\frac{x^{\alpha}}{\alpha\left(\frac{1}{\alpha}+\frac{1}{e}\right)}$$

$$\text{For } 1<x<\infty \Rightarrow H(x)=\frac{1}{\frac{1}{\alpha}+\frac{1}{e}}\left[\int_0^1 t^{\alpha-1}dt+\int_1^x e^{-t}dt\right]=1-\frac{e^{-x}}{\frac{1}{\alpha}+\frac{1}{e}}$$

Therefore:

$$H(x)=\begin{cases} 0, x\le 0 \\ \frac{x^{\alpha}}{\alpha\left(\frac{1}{\alpha}+\frac{1}{e}\right)}, 0<x\le 1 \\ 1-\frac{e^{-x}}{\frac{1}{\alpha}+\frac{1}{e}}, 1<x<\infty \\ 1, x=\infty \end{cases}$$

$$\text{For } 0<y\le 1, \text{ set } u_2=H(y) \Rightarrow u_2=\frac{y^{\alpha}}{\alpha\left(\frac{1}{\alpha}+\frac{1}{e}\right)}$$

$$\Rightarrow y^{\alpha}=\alpha\left(\frac{1}{\alpha}+\frac{1}{e}\right)u_2$$

$$\Rightarrow y=\left[\left(1+\frac{\alpha}{e}\right)u_2\right]^{1/\alpha},0<u_2\le\left(1+\frac{\alpha}{e}\right)^{-1}$$

For $1<y<\infty$, set $u_2=H(y)\Rightarrow u_2=1-\frac{e^{-y}}{\left(\frac{1}{\alpha}+\frac{1}{e}\right)}\Rightarrow\frac{e^{-y}}{\frac{1}{\alpha}+\frac{1}{e}}=1-u_2=u_2$

$$\Rightarrow e^{-y}=\left(\frac{1}{\alpha}+\frac{1}{e}\right)u_2\Rightarrow y=-\ln\left[\left(\frac{1}{\alpha}+\frac{1}{e}\right)u_2\right],\left(1+\frac{\alpha}{e}\right)^{-1}<u_2<1$$

So:

$$y=\begin{cases}\left[\left(1+\frac{\alpha}{e}\right)u_2\right]^{1/\alpha} & ,0<u_2\le\left(1+\frac{\alpha}{e}\right)^{-1}\\ -\ln\left[\left(\frac{1}{\alpha}+\frac{1}{e}\right)u_2\right],\left(1+\frac{\alpha}{e}\right)^{-1}<u_2\le 1 & \end{cases}$$

and

$$g(y)=\begin{cases}e^{-y},0<y\le 1\\ y^{\alpha-1},1<y<\infty\end{cases}$$

G-3 algorithm described the necessary steps for generating random variates from G(α, 1) when 0<α<1.

G-3 Algorithm:

Step1: Read α

Step2: Generate U_1 and U_2 from U(0,1)

Step3: If $U_2\le\left(1+\frac{\alpha}{e}\right)^{-1}$ set $Y=\left[\left(1+\frac{\alpha}{e}\right)U_2\right]^{1/\alpha}$

else set $Y=-\ln\left[\left(\frac{1}{\alpha}+\frac{1}{e}\right)u_2\right]$

Step4: If $U_1 \le g(y) = \begin{cases} e^{-y}, 0 < y \le 1 \\ y^{\alpha-1}, 1 < y < \infty \end{cases}$

deliver Y = X as r.v generated from G(α,1)

Step5: otherwise go to step 2

Step6: Stop.

Note:

If a r.v. from G(α, β), 0<α<1 is required we add the statement Z = βX to G3-algorithm

The theoretical efficiency and the number of trials of G3-algorithm are respectively

$$\frac{1}{c} = \frac{\Gamma(\alpha)}{\frac{1}{\alpha}+\frac{1}{e}} \text{ and } c = \frac{\frac{1}{\alpha}+\frac{1}{e}}{\Gamma(\alpha)}$$

A computer program is made in Appendix (C3 and C4) for computation the values of c when $\alpha = 0.1(0.2)0.9$ with run size of 100 is taken and the results are displayed in tables (10) and (11) together with values of c as given by Eq.(3.7).

Table (10) Values of c with different α-samples using G3-procedure.

α	***Simul.effec.*** $g(y) = e^{-y}$	***Theo. effec.***	***error***	***Average error***
0.1	0.6803	0.9176	0.2373	0.123
0.3	0.6993	0.8157	0.1164	
0.5	0.6757	0.7483	0.0726	
0.7	0.6667	0.7225	0.0558	
0.9	0.7576	0.7282	0.0294	

Table (11) Values of c with different α-samples using G3-procedure.

α	***Simul. effec.*** $g(y) = y^{\alpha-1}$	***Theo. effec.***	***error***	***Average error***
0.1	0.8929	0.9176	0.0247	0.1852
0.3	0.9174	0.8157	0.1017	
0.5	0.9901	0.7483	0.2418	
0.7	0.9901	0.7225	0.2676	
0.9	1.0000	0.7282	0.2718	

3.4.4 G4-Procedure Fishman Procedure [10]

This procedure utilizes the acceptance-rejection method for generating r.v from $G(\alpha, 1)$ where the p.d.f.

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, 0 < x < \infty \quad \text{..... (3.10)}$$

$$= 0, \text{e.w}, \alpha \geq 1$$

by making use of inequalities

$$\frac{x}{\alpha} < e^{x/\alpha} \text{ and } e > 1$$

Using the acceptance-rejection method we write $f(x) = ch(x)g(x)$ where $c \geq 1$, $h(x)$ is also p.d.f., and $0 < g(x) \leq 1$.

The theory behind this representation can be described as follows:

Since

$$\frac{x}{\alpha} < e^{x/\alpha} \text{ and } e > 1 \Rightarrow \frac{x}{\alpha e} < e^{x/\alpha}$$

$$\Rightarrow x < \alpha e e^{x/\alpha}$$

$$\Rightarrow x^{\alpha-1} < (\alpha e)^{\alpha-1}\left(e^{x/\alpha}\right)^{\alpha-1} = (\alpha e)^{\alpha-1} e^{x} e^{-x/\alpha}$$

$$\Rightarrow x^{\alpha-1}e^{-x} < (\alpha e)^{\alpha-1} e^{-x/\alpha}$$

$$\Rightarrow f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1}e^{-x} < \phi(x) = \frac{1}{\Gamma(\alpha)}(\alpha e)^{\alpha-1} e^{-x/\alpha} \quad \text{..... (3.11)}$$

$$ch(x) = \phi(x) \text{ implies } \int_0^{\infty} ch(x)dx = \int_0^{\infty} \phi(x)dx$$

$$c = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} (\alpha e)^{\alpha-1} e^{-x/\alpha} dx$$

$$c = \frac{\alpha^{\alpha} e^{\alpha-1}}{\Gamma(\alpha)} \quad \text{..... (3.12)}$$

From Eqs.(3.11) and (3.12), we get

$$h(x) = \frac{\phi(x)}{c} \quad \text{..... (3.13)}$$

by using the above Eqs.(3.11) and (3.10) we get

$$g(x) = \frac{f(x)}{\phi(x)} = \frac{x^{\alpha-1} e^{-x\left(1-\frac{1}{\alpha}\right)}}{\alpha^{\alpha-1} e^{\alpha-1}}$$

also

$$H(x) = \int_0^{x} h(t)dt$$

$$= \int_0^{x} \frac{1}{\alpha} e^{\frac{-t}{\alpha}} dt = -e^{\frac{-t}{\alpha}} \Big|_0^{x}$$

Set $u_2 = 1 - e^{\frac{-x}{\alpha}}$

$$e^{-x/\alpha} = 1 - u_2 = u_2$$

$$\frac{-x}{\alpha} Lnu_2 \Rightarrow x = -Lnu_2$$

The G4-algorithm describe the steps for generating r.v from G (α, 1) when $\alpha \geq 1$.

G4-Algorithm:

Step1: Read α

Step2: Generate U_1 and U_2 from U (0, 1)

Step3: Set $Y = -\alpha \ln U_2$

Step4: If $U_1 \leq g(y) = \frac{y^{\alpha-1} e^{-y\left(1-\frac{1}{\alpha}\right)}}{\alpha^{\alpha-1} e^{\alpha-1}}$ then deliver Y = X

Step5: Otherwise go to step2

Step6: Stop

The theoretical efficiency and the number of trials of G4-algorithm are respectively $\frac{1}{c} = \frac{\Gamma(\alpha)}{\alpha^{\alpha} e^{\alpha-1}}$ and $c = \frac{\alpha^{\alpha} e^{\alpha-1}}{\Gamma(\alpha)}$

A computer program is made in Appendix (C5) for computation the values of c when α = 1.1(0.2)1.9 with run size of 100 is taken and the results are displayed in table (12) together with values of c as given by equation (3.12)

Table (12) Values of c with different α-samples using G4-procedure.

α	*Theo. efficiency*	*Simul. efficiency*	*Error*	*Average error*
1.1	0.7748	0.8621	0.0873	0.18312
1.3	04724	0.6132	0.1408	
1.5	0.2925	0.4808	0.1883	
1.7	0.1831	0.4132	0.2301	
1.9	0.1155	0.3846	0.2691	

3.4.5 G5-Procedure Gamma Procedure:

This procedure due to Cheng [6] which describes gamma generation $G(\alpha,1)$ for $\alpha > 1$.

Which utilize the acceptance-rejection method

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x},\ 0 < x < \infty$$

$$= 0, \text{ e.w.}$$

$$f(x) = ch(x)g(x)$$

where $h(x) = \dfrac{\lambda\theta.x^{\lambda-1}}{\left(\theta + x^{\lambda}\right)^2},\ 0 < x < \infty$

$$= 0, \text{ e.w.}$$

$$c = \frac{4\alpha^{\alpha}}{\lambda\Gamma(\alpha)e^{\alpha}}$$

$$g(x) = x^{\alpha-\lambda}(\theta + x^{\lambda})^2 \frac{e^{\alpha-x}}{4\alpha^{\alpha+\lambda}}$$

where $\theta = \alpha^{\lambda}$, and $\lambda = (2\alpha - 1)^{1/2}$

$$H(x)=\int_0^x h(t)dt=\frac{-\lambda\theta}{\theta+e^{\lambda}}\Big|_0^x$$

$$=-\lambda\theta\left[\frac{1}{\theta+x^{\lambda}}-\frac{1}{\theta}\right]=\lambda\left[1-\frac{\theta}{\theta+x^{\lambda}}\right]$$

$$\text{Set } u_2=H(x)\Rightarrow u_2=\lambda\left[1-\frac{\theta}{\theta+x^{\lambda}}\right]\Rightarrow\frac{u_2}{\lambda}=1-\frac{\theta}{\theta+x^{\lambda}}$$

$$\Rightarrow\frac{\theta}{\theta+x^{\lambda}}=1-\frac{u_2}{\lambda}=\frac{\lambda-u_2}{\lambda}$$

$$\theta+x^{\lambda}=\frac{\theta\lambda}{\lambda-u_2}\Rightarrow x^{\lambda}=\frac{\theta\lambda}{\lambda-u_2}-\theta=\frac{\theta u_2}{\lambda-u_2}$$

$$\therefore x=\left(\frac{\theta u_2}{\lambda-u_2}\right)^{1/\lambda}$$

G5-Algorithm:

Step1: Read α

Step2: Set $\lambda=(2\alpha-1)^{1/2}$, $\theta=\alpha^{\lambda}$

Step3: Generate U_1 and U_2 from U(0, 1)

Step4: $v=\lambda\ln\left[\frac{U_1}{1-U_2}\right]$ and $x=\alpha e^{v}$

Step5: If $x\geq g(y)=y^{\alpha-\lambda}\left(\theta+y^{\lambda}\right)^2\frac{e^{\alpha-y}}{4\alpha^{\alpha+\lambda}}$

Step6: Go to step 3

Step7: Stop

The theoretical efficiency and the number of trials of G5-algorithm are respectively:

$$\frac{1}{c} = \frac{\lambda\Gamma(\alpha)e^{\alpha}}{4\alpha^{\alpha}} \quad \text{and} \quad c = \frac{4\alpha^{\alpha}}{\lambda\Gamma(\alpha)e^{\alpha}}$$

A computer program is made in Appendix (C6) for computation the values of c when $\alpha = 1.1(0.2)1.9$ with run size of 100 is taken and the results are displayed in table (13) together with values of c as given by equation (3.14).

Table (13) Values of c with different α-samples using G5-procedure.

α	*Theo. efficiency*	*Simul. efficiency*	*Error*	*Average error*
1.1	0.7046	0.9009	0.1963	0.193
1.3	0.7400	0.9259	0.1859	
1.5	0.7642	0.9524	0.1882	
1.7	0.7819	0.9709	0.1890	
1.9	0.7947	1.0000	0.2053	

CHAPTER 2

APPROXIMATION TO THE C.D.F OF GAMMA DISTRIBUTION

2.1 INTRODUCTION

The importance of good numerical integration schemes is evident. There are many deterministic quadrature formulas can be found throughout the literature for computation of ordinary integrals with well behaved integrands. It is often more convenient to compute such integrals by Monte-Carlo method, which, although less accurate than conventional quadrature formulas, but is much simpler to use. In this chapter we consider five numerical procedures namely Trapezoidal Rule, Simpson Rule, Guass-Quadrature Rule, Proposition Rule and Hit or Miss Monte-Carlo Rule for approximating the gamma c.d.f of equation (1.4)

$$F(x) = \int_0^x \frac{1}{\Gamma(\alpha)\beta^{\alpha}} t^{\alpha-1} e^{\frac{-t}{\beta}} dt \quad \text{..... (2.1)}$$

In order to obtain the efficiencies of the five procedures we take $\alpha = \frac{r}{2}$, $\beta = 2$, in order to make comparison with a well-known table (1) of chi-square distribution. Furthermore, if r is even, then $\Gamma(\alpha) = (\alpha - 1)!$ and if r is odd, we use the relation:

$$\Gamma\left(\alpha + \frac{1}{2}\right) = \frac{1.3.5...(2\alpha - 1)}{2\alpha}\sqrt{\pi}, \text{ where } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \text{ [31].}$$

2.2 APPROXIMATION BY TRAPEZOIDAL RULE [19]

Trapezoidal method is used for approximating the area under a curve by series of trapezoids. It has been shown theoretically that using an infinite number of trapezoids give perfect accuracy, but rounding of error will give us problems.

The trapezoidal rule procedure can be illustrated as follows:

Suppose we wish to approximate the integral

$$I = \int_a^b f(x)dx \text{, by using trapezoidal rule} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.2)$$

We divide the interval from a to b into n equal parts as shown in figure (3), where the boundaries of the trapezoids are $x_0, x_1, \ldots, x_n$.

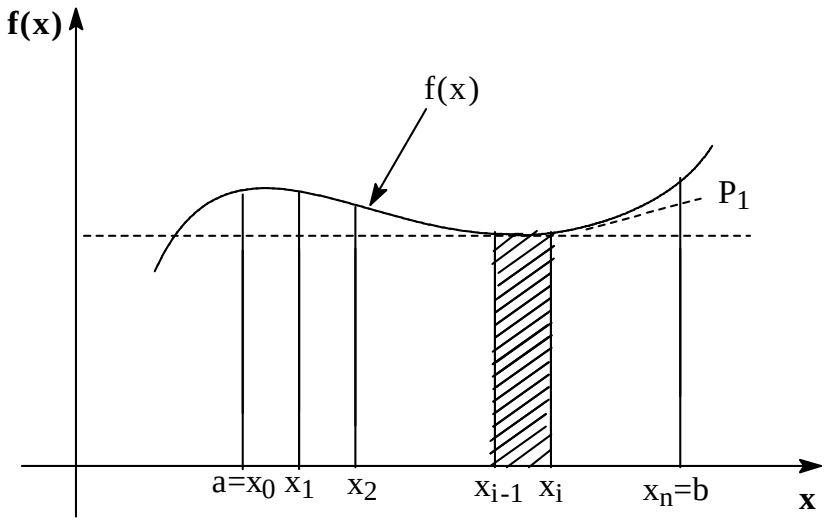


Figure (3) Integration by the trapezoidal rule.

Let $h = \dfrac{b-a}{n}$ be the width of the ith trapezoid that lies between x_{i-1} and x_i whose heights at the left and right side are respectively $f(x_{i-1})$and $f(x_i)$.

The area of the ith trapezoid is:

$$A_i = \frac{h}{2}\left[f(x_{i-1}) + f(x_i)\right]$$

The total area of all n trapezoids is the trapezoidal approximation to the integral I. That is:

$$I \approx A_1 + A_2 + \ldots + A_n.$$

$$= \frac{h}{2}[f(x_0)+f(x_1)+\frac{h}{2}[f(x_1)+f(x_2)]+\frac{h}{2}[f(x_2)+f(x_3)+\ldots+\frac{h}{2}[f(x_{n-1})+f(x_n)]$$

$$= \frac{h}{2}[f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.3)$$

TRAPEZOIDAL RULE ALGORITHM

Step1: Input a, b (Interval of integration) n (Number of subintervals)

Step2: Define f(x) (integrand)

Step3: Set h = (b – a)/n

Step4: sum = 0

Step5: x = a + ih

Step6: For i = 1 to n – 1, sum = sum + 2f(x)

Step7: $g = \frac{h}{2}(f(a) + sum + f(b))$

Step8: Output g

Step9: Stop

Appendix (B1) involve a computer program written in Pascal language using trapezoidal rule for approximating the integral side of equation (2.1). The x-values of the upper limit of the integral is taken from chi-square table (1). Table (3) gives together the approximate and the errors values of F(x).

Table (3): Approximation by trapezoidal method.

Exact / r	0.01		0.025		0.05		0.95		0.975		0.99	
	Approx.	Error	Approx.	Error	Approx.	Error	Approx.	Error	Approx.	Error	Approx.	Error
1	٠,٠٠٠٠٠٠٠	0.0000000	0.0226215	0.0023785	0.0452177	0.0047823	0.7893737	0.1606263	0.7916502	0.1833498	0.7798132	0.2101868
2	٠,٠٠٩٨٥٠٢	٠,٠٠٠١٤٩٨	0.0249226	0.0000774	0.0496814	0.0003186	0.9202974	0.0297026	0.9385705	0.0364295	0.9446480	0.0453520
3	٠,٠١٠٠١٢٥	٠,٠٠٠٠١٢٥	0.0250110	0.0000110	0.0499823	0.0000177	0.9448038	0.0051962	0.9683749	0.0066251	0.9809966	0.0090034
4	٠,٠٠٩٩٩٢٨	٠,٠٠٠٠٠٧٢	0.0249594	0.0000406	0.0500322	0.0000322	0.9492723	0.0007277	0.9734929	0.0015071	0.9886172	0.0013828
5	٠,٠٠٩٩٩٨٦	٠,٠٠٠٠١١٥	0.0249878	0.0000122	0.0504197	0.0004197	0.9504569	0.0004569	0.9745350	0.0004650	0.989659	0.0001341
6	٠,٠١٠٠٢٥٣	٠,٠٠٠٠٠١٤	0.0251400	0.0001400	0.0503460	0.0003460	0.9501196	0.0001196	0.9745006	0.0004994	0.9899354	0.0000646
7	٠,٠١٠٠٧٤١	٠,٠٠٠٠٢٥٣	0.0250095	0.0000095	0.0501714	0.0001714	0.9505346	0.0005346	0.9748607	0.0001393	0.9900839	0.0000839
8	٠,٠١٠٠٤٠٠	٠,٠٠٠٠٧٤١	0.0250148	0.0000148	0.0498652	0.0001348	0.9498377	0.0001623	0.9746677	0.0003323	0.9900207	0.0000207
9	٠,٠١٠٠٣١٤	٠,٠٠٠٠٤٠٠	0.0249938	0.0000062	0.0502463	0.0002463	0.9496500	0.0003500	0.9747750	0.0002250	0.9901018	0.0001018
10	٠,٠٠٩٩٥٥٧	٠,٠٠٠٠٣١٤	0.0250939	0.0000939	0.0499986	0.0000014	0.9498422	0.0001578	0.9751024	0.0001024	0.9899478	0.0000522
11	٠,٠٠٩٩٩٧٦	٠,٠٠٠٠٤٤٣	0.0251190	0.0001190	0.0498205	0.0001795	0.9503191	0.0003191	0.9748029	0.0001971	0.9898940	0.0001060
12	٠,٠١٠٠٤١٥	٠,٠٠٠٠٠٢٤	0.0249195	0.0000805	0.0501634	0.0001634	0.9495632	0.0004368	0.9746754	0.0003246	0.9899211	0.0000789
13	٠,٠١٠٠٠١٦	٠,٠٠٠٠٤١٥	0.0250387	0.0000387	0.0499526	0.0000474	0.9504748	0.0004748	0.9746882	0.0003118	0.9900129	0.0000129
14	٠,٠١٠٠١٣٤	٠,٠٠٠٠٠١٦	0.0250387	0.0000387	0.0499988	0.0000012	0.9501457	0.0001457	0.9748154	0.0001846	0.9898442	0.0001558
15	٠,٠٠٩٩٨٦٥	٠,٠٠٠٠١٣٤	0.0249721	0.0000279	0.0499923	0.0000077	0.9499894	0.0000106	0.9750356	0.0000356	0.9900401	0.0000401
16	٠,٠١٠٠٢٨٤	٠,٠٠٠٠١٣٥	0.0250580	0.0000580	0.0499752	0.0000248	0.9499788	0.0000212	0.9746295	0.0003705	0.9899715	0.0000285
17	٠,٠٠٩٩٦٧٢	٠,٠٠٠٠٢٨٤	0.0249433	0.0000567	0.0499763	0.0000237	0.9500911	0.0000911	0.9750093	0.0000093	0.9899443	0.0000557
18	٠,٠٠٩٩٨٧٥	٠,٠٠٠٠٣٢٨	0.0250053	0.0000053	0.0500155	0.0000155	0.9503074	0.0003074	0.9747664	0.0002336	0.9899532	0.0000468
19	٠,٠١٠٠٠٧٣	٠,٠٠٠٠١٢٥	0.0250747	0.0000747	0.0495940	0.0004060	0.9493787	0.0006213	0.9752566	0.0002566	0.9899931	0.0000069
20	٠,٠١٠٠٣٢٠	٠,٠٠٠٠٠٧٣	0.0250087	0.0000087	0.0512577	0.0012577	0.9497885	0.0002115	0.9751373	0.0001373	0.9900599	0.0000599
21	٠,٠٠٩٩٩٤١	٠,٠٠٠٠٣٢٠	0.0252743	0.0002743	0.0502417	0.0002417	0.9502605	0.0002605	0.9750726	0.0000726	0.9898756	0.0001244
22	٠,٠١٠٠٤٢١	٠,٠٠٠٠٠٥٩	0.0252748	0.0002748	0.0491658	0.0008342	0.9496217	0.0003783	0.9750565	0.0000565	0.9899920	0.0000080
23	٠,٠١٠٣٠٦٣	٠,٠٠٠٠٤٢١	0.0251818	0.0001818	0.0502503	0.0002503	0.9502203	0.0002203	0.9750836	0.0000836	0.9898588	0.0001412
24	٠,٠٠٩٨٦١٢	٠,٠٠٠٣٠٦٣	0.0250110	0.0000110	0.0490010	0.0009990	0.9497309	0.0002691	0.9751489	0.0001489	0.9900136	0.0000136
25	٠,٠١٠٠٢٦٤	٠,٠٠٠١٣٨٨	0.0247755	0.0002245	0.0498037	0.0001963	0.9504245	0.0004245	0.9746474	0.0003526	0.9899225	0.0000775
26	٠,٠١٠١٤٥٩	٠,٠٠٠٠٢٦٤	0.0244865	0.0005135	0.0504721	0.0004721	0.9500571	0.0000571	0.9747882	0.0002118	0.9898505	0.0001495
27	٠,٠١٠٢٢٤٨	٠,٠٠٠١٤٥٩	0.0253546	0.0003546	0.0510204	0.0010204	0.9497471	0.0002529	0.9749556	0.0000444	0.9900500	0.0000500
28	٠,٠١٠٢٦٧٧	٠,٠٠٠٢٢٤٨	0.0249396	0.0000604	0.0495112	0.0004888	0.9494900	0.0005100	0.9751466	0.0001466	0.9900096	0.0000096
29	٠,٠٠٩٨٥٠٢	٠,٠٠٠٢٦٧٧	0.0244972	0.0005028	0.0498937	0.0001063	0.9503350	0.0003350	0.9747916	0.0002084	0.9899839	0.0000161
30	0.0102788	0.0002788	0.0251387	0.0001387	0.0501899	0.0001899	0.9501606	0.0001606	0.9750331	0.0000331	0.9899718	0.0000282

2.3 APPROXIMATION BY SIMPSON RULE [28]

In Simpson method we try to approximate $\int_a^b f(x)\,dx$ by a series of parabolic segments hoping that parabola will more closely much to a given curve of f(x) than it would be straight line in the trapezoidal method.

Simpson rule (or Simpson 1/3 rule) is given by the equation $A_1 = \frac{h}{3}(f_0 + 4f_1 + f_2) + o(h^5)$ where A_1 denotes the area under the graph of f(x) from the point x_0 to the point x_2 and $h = \frac{x_n - x_0}{n}$, n = 1, 2,…. This equation calculate the integral over two segments of integration. Repeated application of Simpson $\frac{1}{3}$ rule over segment pairs of segments, and summation of all the formulas over the total interval, gives the multiple segments Simpson $\frac{1}{3}$ Rule:

$$A = \sum_{i=1}^{n} A_i = \frac{h}{3}[f_0 + 4\sum_{i=1}^{n-1} f_i + 2\sum_{i=2}^{n-2} f_i + f_n] + o(h^5) \quad \text{..... (2.4)}$$

i = odd i = even

See Figure (4)

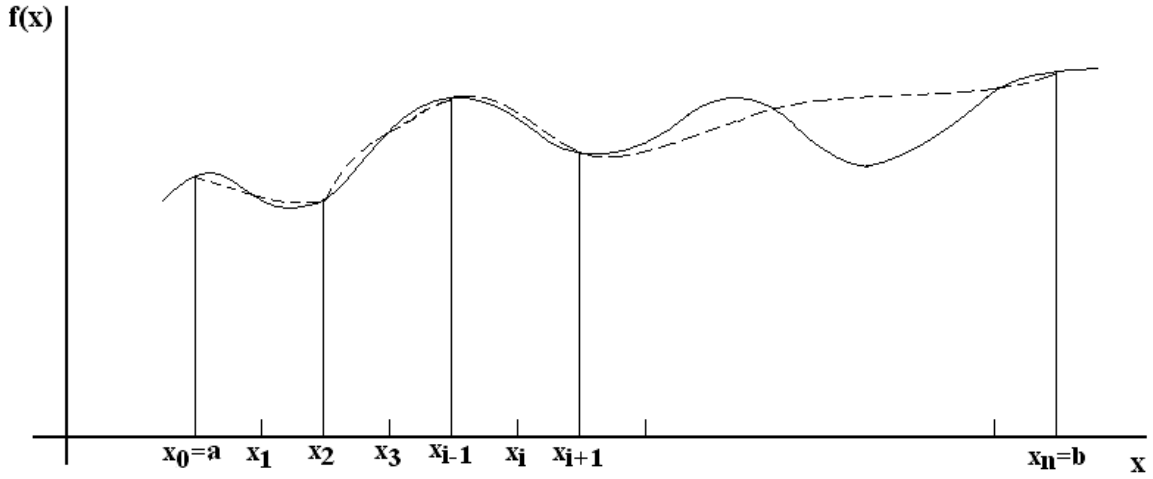


Figure (4): Integration by the Simpson rule.

Since Simpson $\frac{1}{3}$ Rule fits pair of segments, the total intervals must be divided into an even number of segments. The first summation term in equation (2.4) sums up to odd subscripted terms and the second summation odds up to the even-subscripted terms, the order of the error of the multiple-segment Simpson $\frac{1}{3}$ rule was reduced by one order of magnitude of $o(h^4)$ for the same reason as in the trapezoidal rule. For more details see [24], [36].

SIMPSON RULE ALGORITHM

Step1: Input a, b (Interval of integration) n (Number of subintervals)

Step2: Define f(x) (Integrand)

Step3: Set h= (b-a)/n

Step4: sum1 = 0, sum2 = 0

Step5: for i=1 to n-1

Step6: $x = a + i * h$

Step7: If i is even then $\text{sum1} = \text{sum1} + 2f(x)$

Else $\text{sum2} = \text{sum2} + 4 * f(x)$

Step8: $g = \frac{h}{3}[f(a) + \text{sum1} + \text{sum2} + f(b)]$

Step9: Out put g

Step10: Stop

Appendix (B2) involve a computer program written in Pascal language using Simpson rule for approximating the integral side of equation (2.1).The x-values of the upper limit of the integral is taken from chi-square table (1). Table (4) gives together the approximate and the errors values of F(x).

Table (4): Approximation by Simpson method

Exact / r	0.01		0.025		0.05		0.95		0.975		0.99	
	Approx.	Error	Approx.	Error	Approx.	Error	Approx.	Error	Approx.	Error	Approx.	Error
1	٠,٠٠٠٠٠٠٠	0.0000000	0.0229814	0.0020186	0.0459376	0.0040624	0.8111404	0.1388596	0.8163480	0.1586520	0.8079021	0.1820979
2	٠,٠٠٩٨٨٣٥	٠,٠٠٠١١٦٥	0.0250076	0.0000076	0.0498530	0.0001470	0.9299968	0.0200032	0.9504282	0.0245718	0.9592987	0.0307013
3	٠,٠١٠٠١٧٩	٠,٠٠٠٠١٧٩	0.0250249	0.0000249	0.0500114	0.0000114	0.9478659	0.0021341	0.9723538	0.0026462	0.9862372	0.0037628
4	٠,٠٠٩٩٩٣٠	٠,٠٠٠٠٠٧٠	0.024602	0.0000398	0.0500345	0.0000345	0.9500455	0.0000455	0.9745346	0.0004654	0.9900955	0.0000955
5	٠,٠٠٩٩٨٧٦	٠,٠٠٠٠١٢٤	0.0249860	0.0000140	0.0504169	0.0004169	0.9506171	0.0006171	0.9747459	0.0002541	0.9901690	0.0001690
6	٠,٠٠٩٩٩٧٢	٠,٠٠٠٠٠٢٨	0.0251371	0.0001371	0.0503413	0.0003413	0.9501577	0.0001577	0.9745336	0.0004664	0.9899660	0.0000340
7	٠,٠١٠٠٢٣٤	٠,٠٠٠٠٢٣٤	0.0250055	0.0000055	0.0501651	0.0001651	0.9505637	0.0005637	0.9748758	0.0001242	0.9900810	0.0000810
8	٠,٠١٠٠٧١٧	٠,٠٠٠٠٧١٧	0.0250098	0.0000098	0.0498572	0.0001428	0.9498748	0.0001252	0.9746911	0.0003089	0.9900271	0.0000271
9	٠,٠١٠٠٣٧٠	٠,٠٠٠٠٣٧٠	0.0249876	0.0000124	0.0502364	0.0002364	0.9496938	0.0003062	0.9748054	0.0001946	0.9901169	0.0001169
10	٠,٠١٠٠٢٧٨	٠,٠٠٠٠٢٧٨	0.0250865	0.0000865	0.0499869	0.0000131	0.9498908	0.0001092	0.9751369	0.0001369	0.9899675	0.0000325
11	٠,٠٠٩٩٥١٥	٠,٠٠٠٠٤٨٥	0.0251105	0.0001105	0.0498069	0.0001931	0.9503714	0.0003714	0.9748406	0.0001594	0.9899160	0.0000840
12	٠,٠٠٩٩٩٢٨	٠,٠٠٠٠٠٧٢	0.0249097	0.0000903	0.0501479	0.0001479	0.9496196	0.0003804	0.9747157	0.0002843	0.989945	0.0000555
13	٠,٠١٠٠٣٦٠	٠,٠٠٠٠٣٦٠	0.0250277	0.0000277	0.0499352	0.0000648	0.9505344	0.0005344	0.9747310	0.0002690	0.9900375	0.0000375
14	٠,٠٠٩٩٩٥٤	٠,٠٠٠٠٠٤٦	0.0250264	0.0000264	0.0499794	0.0000206	0.9502091	0.0002091	0.9748606	0.0001394	0.9898704	0.0001296
15	٠,٠١٠٠٠٦٦	٠,٠٠٠٠٠٦٦	0.0249584	0.0000416	0.0499708	0.0000292	0.9500565	0.0000565	0.9750831	0.0000831	0.9900673	0.0000673
16	٠,٠٠٩٩٧٩٠	٠,٠٠٠٠٢١٠	0.0250430	0.0000430	0.0499517	0.0000483	0.9500495	0.0000495	0.9746799	0.0003201	0.9900002	0.0000002
17	٠,٠١٠٠٢٠١	٠,٠٠٠٠٢٠١	0.0249269	0.0000731	0.0499508	0.0000492	0.9501653	0.0001653	0.9750617	0.0000617	0.9899744	0.0000256
18	٠,٠٠٩٩٥٨٣	٠,٠٠٠٠٤١٧	0.0249876	0.0000124	0.0499878	0.0000122	0.9503850	0.0003850	0.9748215	0.0001785	0.9899846	0.0000154
19	٠,٠٠٩٩٧٧٩	٠,٠٠٠٠٢٢١	0.0250555	0.0000555	0.0495643	0.0004357	0.9494605	0.0005395	0.9753136	0.0003136	0.9900258	0.0000258
20	٠,٠٠٩٩٩٦٩	٠,٠٠٠٠٠٣١	0.0249881	0.0000119	0.0512253	0.0012253	0.9498735	0.0001265	0.9751968	0.0001968	0.9900938	0.0000938
21	٠,٠١٠٠٢٠٨	٠,٠٠٠٠٢٠٨	0.0252522	0.0002522	0.0502075	0.0002075	0.9503486	0.0003486	0.9751346	0.0001346	0.9899112	0.0000888
22	٠,٠٠٩٩٨٢٢	٠,٠٠٠٠١٧٨	0.0252512	0.0002512	0.0491299	0.0008701	0.9497138	0.0002862	0.9751209	0.0001209	0.9900287	0.0000287
23	٠,٠١٠٠٢٩٥	٠,٠٠٠٠٢٩٥	0.0251569	0.0001569	0.0502117	0.0002117	0.9503154	0.0003154	0.9751503	0.0001503	0.9898971	0.0001029
24	٠,٠١٠٢٩٢٦	٠,٠٠٠٢٩٢٦	0.0249848	0.0000152	0.0489608	0.0010392	0.9498299	0.0001701	0.9752179	0.0002179	0.9900528	0.0000528
25	٠,٠٠٩٨٤٧٢	٠,٠٠٠١٥٢٨	0.0247480	0.0002520	0.0497609	0.0002391	0.9505262	0.0005262	0.974195	0.0002805	0.9899633	0.0000367
26	٠,٠١٠٠١١٥	٠,٠٠٠٠١١٥	0.0244577	0.0005423	0.0504267	0.0004267	0.9501626	0.0001626	0.9748624	0.0001376	0.9898928	0.0001072
27	٠,٠١٠١٣٠١	٠,٠٠٠١٣٠١	0.0253237	0.0003237	0.0509725	0.0009725	0.9498564	0.0001436	0.9750319	0.0000319	0.9900930	0.0000930
28	٠,٠١٠٢٠٨١	٠,٠٠٠٢٠٨١	0.0249075	0.0000925	0.0494619	0.0005381	0.9496029	0.0003971	0.9752249	0.0002249	0.9900540	0.0000540
29	٠,٠١٠٢٥٠١	٠,٠٠٠٢٥٠١	0.0244640	0.0005360	0.0498420	0.0001580	0.9504504	0.0004504	0.9748729	0.0001271	0.9900297	0.0000297
30	٠,٠١٠٢٦٠٤	0.0002604	0.0251035	0.0001035	0.0501357	0.0001357	0.9502796	0.0002796	0.9751162	0.0001162	0.9900189	0.0000189

2.4 APPROXIMATION BY GAUSSIAN QUADRATURE METHOD [36]

To introduce the ideas involved in Gaussian Quadrature, we consider the more general integral $\int_a^b w(x)f(x)dx$ where $w(x) > 0$ is a weight function. We are interested only in the case $w(x) = 1$ but different choices do play very important roles in numerical integration and a discussion of these can be found in [28]. The orthogonal polynomials corresponding to this weight function are known as the Legendre polynomials. Quadrature using these polynomials is called Gauss-Legendere quadrature or, simply, Gaussian quadrature which have the general formula

$$\int_{-1}^{1} f(x)dx = \sum_{i=0}^{n} w_i f(x) \quad \text{..... (2.5)}$$

The coefficients w_i ($i = 0, 1, \ldots, n$) could be calculated, but this is not necessary because they, and the points x_i, have already been tabulated for a large values of n see [19], [24]. Some of the roots of the Legendre polynomials and the corresponding weights are used. We need in this method transforming the interval [a, b] in to [−1, 1], by using the simple linear transformation

$T = [1/ (b - a)] (2x - a - b)$ which provided $b > a$, the Legendre polynomials reduce to approximating

$$\int_{-1}^{1} f\left(\frac{(b-a)t+b+a}{2}\right)\frac{(b-a)}{2}dt$$

where f is any function that can be evaluated at the required region of points.

GAUSSIAN QUADRATURE RULE ALGORITHM

Step1: Input a, b (Interval of integration)

n (Number of nodes)

Step2: Define f(x) (integrand)

Initialize Array x(n, i) and w(n, i) for the Gauss nodes and weights, x(n, i) is the ith nonnegative node for the Gauss n-point formula, and w(n,i) is the corresponding weight.

Step3: h := (b – a)/2

m := (a + b)/2

x := hx(n, i)

Step4: If n is odd then g := hw(n, 1)f(x)

Else g := hw(n,1)(f(–x + m) + f(x + m))

Step5:for i:=2 to $\left[\frac{n+1}{2}\right]$

x := hx(n, i)

Step6: g := g + hw(n, i)(f(–x + m) + f(x + m))

Step7: Output g

Step8: Stop

Appendix (B3) involve a computer program written in Pascal language using Gaussian Quadrature Rule for approximating the integral side of equation (2.1). The x-values of the upper limit of the integral is taken from chi-square table (1). Table (5) gives together the approximate and the errors values of F(x).

Table (5): Approximation by Gaussian quadrature method

Exact / r	0.01		0.025		0.05		0.95		0.975		0.99	
	Approx.	Error	Approx.	Error	Approx.	Error	Approx.	Error	Approx.	Error	Approx.	Error
1	٠,٠٠٠٠٠٠٠	0.0000000	0.0220841	0.0029159	0.0441428	0.0058572	0.8696013	0.0803987	0.8829769	0.0920231	0.8841374	0.1058626
2	٠,٠٠٩٩٥٠٢	٠,٠٠٠٠٤٩٨	0.0251776	0.0001776	0.0501964	0.0001964	0.9499634	0.0000366	0.9750280	0.0000280	0.9599983	0.0000017
3	٠,٠١٠٠٦٠٧	٠,٠٠٠٠٦٠٧	0.0251355	0.0001355	0.0502427	0.0002427	0.9514336	0.0014336	0.9770540	0.0020540	0.9925284	0.0025284
4	٠,٠٠٩٩٩٣٠	٠,٠٠٠٠٠٧٠	0.0249602	0.0000398	0.0500345	0.0000345	0.9500469	0.0000469	0.9745372	0.0004628	0.9901007	0.0001007
5	٠,٠٠٩٩٨١٥	٠,٠٠٠٠١٨٥	0.0249687	0.0000313	0.0503762	0.0003762	0.9504435	0.0004435	0.9744917	0.0005083	0.9897709	0.0002291
6	٠,٠٠٩٩٩٧٢	٠,٠٠٠٠٠٢٨	0.0251369	0.0001369	0.0503402	0.0003402	0.9501535	0.0001535	0.9745265	0.0004735	0.9899528	0.0000472
7	٠,٠١٠٠٢٦٧	٠,٠٠٠٠٢٦٧	0.0250166	0.0000166	0.0501951	0.0001951	0.9505923	0.0005923	0.9749229	0.0000771	0.9901660	0.0001660
8	٠,٠١٠٠٧٢٩	٠,٠٠٠٠٧٢٩	0.0250166	0.0000166	0.0498830	0.0001170	0.9498781	0.0001219	0.9746967	0.0003033	0.9900388	0.0000388
9	٠,٠١٠٠٣٣٤	٠,٠٠٠٠٣٣٤	0.0249796	0.0000204	0.0502294	0.0002294	0.9496847	0.0003153	0.9747869	0.0002131	0.9900748	0.0000748
10	٠,٠١٠٠١٩٦	٠,٠٠٠٠١٩٦	0.0250610	0.0000610	0.0499334	0.0000666	0.9498889	0.0001111	0.9751300	0.0001300	0.9899425	0.0000575
11	٠,٠٠٩٩٤١٦	٠,٠٠٠٠٥٨٤	0.0250715	0.0000715	0.0497089	0.0002911	0.9503801	0.0003801	0.9748567	0.0001433	0.9899424	0.0000576
12	٠,٠٠٩٩٨٤٥	٠,٠٠٠٠١٥٥	0.0248662	0.0001338	0.0500162	0.0000162	0.9496298	0.0003702	0.9747415	0.0002585	0.9900084	0.0000084
13	٠,٠١٠٠٣٣٣	٠,٠٠٠٠٣٣٣	0.0249889	0.0000111	0.0497908	0.0002092	0.9505363	0.0005363	0.9747457	0.0002543	0.9900993	0.0000993
14	٠,٠١٠٠٠١٨	٠,٠٠٠٠٠١٨	0.0250032	0.0000032	0.0498430	0.0001570	0.9501980	0.0001980	0.9748498	0.0001502	0.9898883	0.0001117
15	٠,٠١٠٠٢٤٩	٠,٠٠٠٠٢٤٩	0.0249607	0.0000393	0.0498650	0.0001350	0.9500355	0.0000355	0.9750447	0.0000447	0.9900212	0.0000212
16	٠,٠١٠٠١١٢	٠,٠٠٠٠١١٢	0.0250785	0.0000785	0.0498971	0.0001029	0.9500265	0.0000265	0.9746247	0.0003753	0.9898909	0.0001091
17	٠,٠١٠٠٦٧٥	٠,٠٠٠٠٦٧٥	0.0250024	0.0000024	0.0499657	0.0000343	0.9501503	0.0001503	0.9750057	0.0000057	0.9898252	0.0001748
18	٠,٠١٠٠٢٠٨	٠,٠٠٠٠٢٠٨	0.0251076	0.0001076	0.0500875	0.0000875	0.9503861	0.0003861	0.9747857	0.0002143	0.9898308	0.0001692
19	٠,٠١٠٠٥٥٦	٠,٠٠٠٠٥٥٦	0.0252234	0.0002234	0.0497620	0.0002380	0.9494825	0.0005175	0.9753114	0.0003114	0.9899061	0.0000939
20	٠,٠١٠٠٨٩٠	٠,٠٠٠٠٨٩٠	0.0252046	0.0002046	0.0515294	0.0015294	0.9499160	0.0000840	0.9752398	0.0002398	0.9900426	0.0000426
21	٠,٠١٠١٢٥٩	٠,٠٠٠١٢٥٩	0.0255199	0.0005199	0.0506248	0.0006248	0.9504074	0.0004074	0.9752239	0.0002239	0.9899577	0.0000423
22	٠,٠١٠٠٩٧٧	٠,٠٠٠٠٩٧٧	0.0255673	0.0005673	0.0496585	0.0003415	0.9497793	0.0002207	0.9752498	0.0002498	0.9901799	0.0001799
23	٠,٠١٠١٥٤٤	٠,٠٠٠١٥٤٤	0.0255176	0.0005176	0.0508655	0.0008655	0.9503795	0.0003795	0.9753059	0.0003059	0.9901512	0.0001512
24	٠,٠١٠٤٢٨٦	٠,٠٠٠٤٢٨٦	0.0253851	0.0003851	0.0497180	0.0002820	0.9498794	0.0001206	0.9753828	0.0003828	0.9903953	0.0003953
25	٠,٠٠٩٩٧٨٨	٠,٠٠٠٠٢١٢	0.0251819	0.0001819	0.0506422	0.0006422	0.9505539	0.0005539	0.9748683	0.0001317	0.9903643	0.0003643
26	٠,٠١٠١٤٧٤	٠,٠٠٠١٤٧	0.0249183	0.0000817	0.0514309	0.0014309	0.9501575	0.0001575	0.9749803	0.0000197	0.9903167	0.0003167
27	٠,٠١٠٢٦٧٠	٠,٠٠٠٢٦٧٠	0.0258340	0.0008340	0.0520961	0.0020961	0.9498133	0.0001867	0.9751009	0.0001009	0.9905129	0.0005129
28	٠,٠١٠٣٤٢٦	٠,٠٠٠٣٤٢٦	0.02545311	0.0004311	0.0506504	0.0006504	0.9495199	0.0004801	0.9752297	0.0002297	0.9904203	0.0004203
29	٠,٠١٠٣٧٨٧	٠,٠٠٠٣٧٨٧	0.0249927	0.0000073	0.0511323	0.0011323	0.9503282	0.0003282	0.9747979	0.0002021	0.9903051	0.0003051
30	0.0103796	0.0003796	0.0256648	0.0006648	0.0515198	0.0015198	0.9501217	0.0001217	0.9749600	0.0000400	0.9901712	0.0001712

2.5 APPROXIMATION BY HIT OR MISS MONTE-CARLO METHOD [38]

In this section we consider a simple technique for computing the one-dimensional integral:

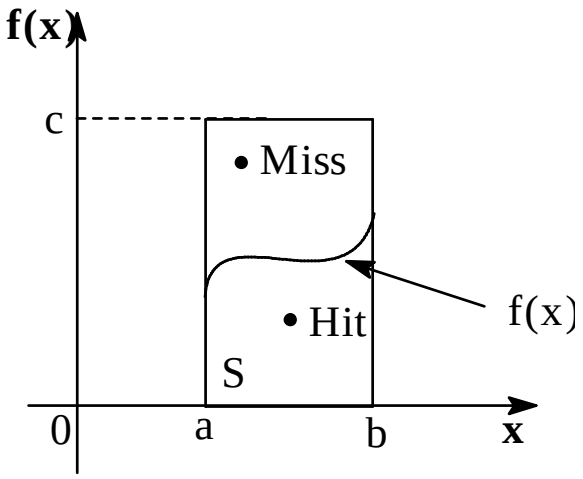
$$I = \int_a^b f(x)dx \quad \text{..... (2.6)}$$

by Monte-Carlo method. Viz

For simplicity we assume that the integrand f(x) is bounded $0 \le f(x) \le c$, $a \le x \le b$. Let $\Omega = \{(x, y) : a \le x \le b, 0 \le y \le c\}$ be a rectangle as shown in Figure (5).

Let (x, y) be a random vector uniformly distributed over the rectangle Ω with joint p.d.f.

$$g(x, y) = \begin{cases} \dfrac{1}{c(b-a)}, & (x,y) \in \Omega \\ 0, & \text{otherwise} \end{cases} \quad \text{..... (2.7)}$$



Fig(5): Graphical representation of the Hit or Miss Monte-Carlo Method.

Let p be the probability that the random vector (x, y) falls within the area under the curve f(x), and let S = {(x, y) : y ≤ f(x)}. The area under the curve f(x) is:

$$\text{Area } S = \int_a^b f(x)dx$$

$$P = \frac{\text{area}S}{\text{area}\Omega} = \frac{\int_a^b f(x)dx}{c(b-a)} = \frac{I}{c(b-a)} \quad \text{..... (2.8)}$$

Assume that N independent random vectors (x_1, y_1), (x_2, y_2), …, (x_n, y_n) are generated. Then the probability P can be estimated by:

$$\hat{p} = \frac{N_H}{N} \quad \text{..... (2.9)}$$

where N_H is the number of occasions on which $f(x_i) \geq y_i$, $i = 1,2,\ldots,N$, that is, the number of hits and $N\text{-}N_H$ is the number of misses, we score a miss if $f(x_i) < y_i$, $i = 1, 2, \ldots, N$.

It follows if $P \approx \hat{P}$ then from (2.8) and (2.9) the integral I can be estimated by:

$$\frac{I}{c(b-a)} \approx \frac{N_H}{N} \text{ implies } I \approx \frac{c(b-a)N_H}{N} \quad \text{..... (2.10)}$$

In other words we estimate the integral I by sampling N from the distn. of Eq.(2.7), count the number N_H of hits and apply Eq.(2.10).

HIT-MISS ALGORITHM

Step1: Input a, b and c and generate $\{U_j\}_1^{2N}$ of 2N random number.

Step2: Arrange the random numbers into N pairs $(U_1, U_1'),(U_2, U_2'),...,$ (U_N, U_N') in any fashion such that each random number U_i is used exactly once.

Step3: Set $x_i = a+(b-a)U_i$, calculate $f(x_i)$, i = 1, 2,…, N.

Step4: Count the number of cases for which $f(x_i) > c\,U_i'$

Step5: Estimate $I \approx c\,(b-a)\dfrac{N_H}{N}$

Appendix (B4) involve a computer program written in Pascal language using Hit or Miss Monte-Carlo Method for approximating the integral side of Eq. (2.1). The x-values of the upper limit of the integral is taken from chi-square table (1). Table (6) gives together the approximate and the errors values of F(x).

Table (6): Approximation by Hit and Miss Method.

Exact / r	0.01		0.025		0.05		0.95		0.975		0.99	
	Approx.	Error	Approx.	Error	Approx.	Error	Approx.	Error	Approx.	Error	Approx.	Error
1	٠,٠٠٩٨٥٠٢	0.0001498	0.0226215	0.0023785	0.0452177	0.0047823	0.8111404	0.1388596	0.7916502	0.1833498	0.7798132	0.2101868
2	٠,٠١٠٠١٢٥	٠,٠٠٠٠١٢٥	0.0249226	0.0000774	0.0496814	0.0003186	0.9299968	0.020032	0.9385705	0.0364295	0.9446480	0.453520
3	٠,٠٠٩٩٩٢٨	٠,٠٠٠٠٠٧٢	0.0250110	0.0000110	0.0499823	0.0000177	0.9478659	0.0021341	0.9683749	0.0066251	0.9809966	0.0090034
4	٠,٠٠٩٩٨٨٥	٠,٠٠٠٠١١٥	0.0249594	0.0000406	0.0500322	0.0000322	0.9500455	0.0000455	0.9734929	0.0015071	0.9886172	0.0013828
5	٠,٠٠٩٩٩٨٦	٠,٠٠٠٠٠١٤	0.0249878	0.0000122	0.0504197	0.0004197	0.9506171	0.0006171	0.9745350	0.0004650	0.9898659	0.0001341
6	٠,٠١٠٠٢٥٣	٠,٠٠٠٠٢٥٣	0.0251400	0.0001400	0.0503460	0.0003460	0.9501577	0.0001577	0.9745006	0.0004994	0.9899354	0.0000646
7	٠,٠١٠٠٧٤١	٠,٠٠٠٠٧٤١	0.0250095	0.0000095	0.0501714	0.0001714	0.9505637	0.0005637	0.9748607	0.0001393	0.9900839	0.0000839
8	٠,٠١٠٠٤٠٠	٠,٠٠٠٠٤٠٠	0.0250148	0.0000148	0.0498652	0.0001348	0.9498748	0.0001252	0.9746677	0.0003323	0.9900207	0.0000207
9	٠,٠١٠٠٣١٤	٠,٠٠٠٠٣١٤	0.0249938	0.0000062	0.0502463	0.0002463	0.9496938	0.0003062	0.9747750	0.0002250	0.9901018	0.0001018
10	٠,٠٠٩٩٥٥٧	٠,٠٠٠٠٤٤٣	0.0250939	0.0000939	0.0499986	0.0000014	0.9498908	0.0001092	0.9751024	0.0001024	0.9899478	0.0000522
11	٠,٠٠٩٩٩٧٦	٠,٠٠٠٠٠٢٤	0.0251190	0.0001190	0.0498205	0.0001795	0.9503714	0.0003714	0.9748029	0.0001971	0.9898940	0.0001060
12	٠,٠١٠٠٤١٥	٠,٠٠٠٠٤١٥	0.0249195	0.0000805	0.0501634	0.0001634	0.9496196	0.0003804	0.9746754	0.0003246	0.9899211	0.0000789
13	٠,٠١٠٠٠١٦	٠,٠٠٠٠٠١٦	0.0250387	0.0000387	0.0499526	0.0000474	0.9505344	0.0005344	0.9746882	0.0003118	0.9900129	0.0000129
14	٠,٠١٠٠١٣٤	٠,٠٠٠٠١٣٤	0.0250387	0.0000387	0.0499988	0.0000012	0.9502091	0.0002091	0.9748154	0.0001846	0.9898442	0.0001558
15	٠,٠٠٩٩٨٦٥	٠,٠٠٠٠١٣٥	0.0249721	0.0000279	0.0499923	0.0000077	0.9500565	0.0000565	0.9750356	0.0000356	0.9900401	0.0000401
16	٠,٠١٠٠٢٨٤	٠,٠٠٠٠٢٨٤	0.0250580	0.0000580	0.0499752	0.0000248	0.9500495	0.0000495	0.9746295	0.0003705	0.9899715	0.0000285
17	٠,٠٠٩٩٦٧٢	٠,٠٠٠٠٣٢٨	0.0249433	0.0000567	0.0499763	0.0000237	0.9501653	0.0001653	0.9750093	0.0000093	0.9899443	0.0000557
18	٠,٠٠٩٩٨٧٥	٠,٠٠٠٠١٢٥	0.0250053	0.0000053	0.0500155	0.0000155	0.9503850	0.0003850	0.9747664	0.0002336	0.9899532	0.0000468
19	٠,٠١٠٠٠٧٣	٠,٠٠٠٠٠٧٣	0.0250747	0.0000747	0.0495940	0.0004060	0.9494605	0.0005395	0.975266	0.0002566	0.9899931	0.0000069
20	٠,٠١٠٠٣٢٠	٠,٠٠٠٠٣٢٠	0.0250087	0.0000087	0.0512577	0.0012577	0.9498735	0.0001265	0.9751373	0.0001373	0.9900599	0.0000599
21	٠,٠٠٩٩٩٤١	٠,٠٠٠٠٠٥٩	0.0252743	0.0002743	0.0502417	0.0002417	0.9503486	0.0003486	0.9750726	0.0000726	0.9898756	0.0001244
22	٠,٠١٠٠٤٢١	٠,٠٠٠٠٤٢١	0.0252748	0.0002748	0.0491658	0.0008342	0.9497138	0.0002862	0.9750565	0.0000565	0.9899920	0.0000080
23	٠,٠١٠٣٠٦٣	٠,٠٠٠٣٠٦٣	0.0251818	0.0001818	0.0502503	0.0002503	0.9503154	0.0003154	0.9750836	0.0000836	0.9898588	0.0001412
24	٠,٠٠٩٨٦١٢	٠,٠٠٠١٣٨٨	0.0250110	0.0000110	0.0490010	0.0009990	0.9498299	0.0001701	0.9751489	0.0001489	0.9900136	0.0000136
25	٠,٠١٠٠٢٦٤	٠,٠٠٠٠٢٦٤	0.0247755	0.0002245	0.0498037	0.0001963	0.9505262	0.0005262	0.9746474	0.0003526	0.9899225	0.0000775
26	٠,٠١٠١٤٥٩	٠,٠٠٠١٤٥٩	0.0244865	0.0005135	0.0504721	0.0004721	0.9501626	0.0001626	0.9747882	0.0002118	0.9898505	0.0001495
27	٠,٠١٠٢٢٤٨	٠,٠٠٠٢٢٤٨	0.0253546	0.0003546	0.0510204	0.0010204	0.9498564	0.0001436	0.9749556	0.0000444	0.9900500	0.0000500
28	٠,٠١٠٢٦٧٧	٠,٠٠٠٢٦٧٧	0.0254355	0.0004355	0.0495112	0.0004888	0.9496029	0.0003971	0.9751466	0.0001466	0.9900096	0.0000096
29	٠,٠١٠٢٧٤٤	٠,٠٠٠٢٧٤٤	0.0245433	0.0005433	0.0498937	0.0001063	0.9500656	0.000656	0.9747916	0.0002087	0.9899839	0.0000165
30	0.010344	0.0000344	0.0253211	0.0003252	0.0501899	0.0001899	0.9500385	0.0000385	0.9752488	0.0002488	0.9900245	0.0000128

2.6 APPROXIMATION BY PROPOSITION METHOD

A new approximation method proposed for the c.d.f of gamma distn. function given by Eq.(1.4) with B = 2

$$F(x) = \Pr(X \le x) = \frac{1}{\Gamma(\alpha)2^{\alpha}} \int_0^x t^{\alpha-1} e^{\frac{-t}{2}} dt \quad \text{..... (2.11)}$$

Finding bound for the integral side of Eq.(2.11) is necessary, so we make the transformation $s = \frac{t}{x}$ implies t = xs implies dt = xds Then Eq.(2.11) becomes

$$F(x) = \frac{x^{\alpha}}{\Gamma(\alpha)2^{\alpha}} \int_0^1 s^{\alpha-1} e^{\frac{-x}{2}s} ds$$

$$= \frac{x^{\alpha}}{\Gamma(\alpha)2^{\alpha}} G(x,\alpha) \quad \text{..... (2.12)}$$

where

$$G(x,\alpha) = \int_0^1 s^{\alpha-1} e^{\frac{-x}{2}s} ds \quad \text{..... (2.13)}$$

We integrate Eq.(2.13) by parts by setting

$$u = e^{\frac{-x}{2}s} \quad , dv = s^{\alpha-1} ds$$

$$du = \frac{-x}{2} e^{\frac{-x}{2}s} ds \ , \ v = \frac{s^{\alpha}}{\alpha}$$

Then Eq.(2.13) have the following form

$$G(x,\alpha) = \frac{e^{\frac{-x}{2}s} s^{\alpha}}{\alpha} |_0^1 + \frac{x}{2\alpha} \int_0^1 s^{\alpha} e^{\frac{-x}{2}s} ds$$

$$= \frac{e^{\frac{-x}{2}}}{\alpha} + \frac{x}{2\alpha} \int_0^1 s^{(\alpha+1)-1} e^{\frac{-x}{2}s} ds$$

$$= \frac{e^{\frac{-x}{2}}}{\alpha} + \frac{x}{2\alpha} G(x,\alpha+1) \quad \text{..... (2.14)}$$

If we replace α by α+1, α+2, α+3, …; we have:

$$G(x, \alpha+1) = \frac{e^{\frac{-x}{2}}}{\alpha+1} + \frac{x}{2(\alpha+1)} G(x, \alpha+2)$$

$$G(x, \alpha+2) = \frac{e^{\frac{-x}{2}}}{\alpha+2} + \frac{x}{2(\alpha+2)} G(x, \alpha+3)$$

$$G(x, \alpha+3) = \frac{e^{\frac{-x}{2}}}{\alpha+3} + \frac{x}{2(\alpha+3)} G(x, \alpha+4) \text{ and so on …}$$

and we conclude that

$$G(x, \alpha+i) = \frac{e^{\frac{-x}{2}}}{\alpha+i} + \frac{x}{2(\alpha+i)} G(x, \alpha+i+1),\ i = 0,1,..... \quad (2.15)$$

Substitution of Eq.(2.15) into Eq.(2.12)

$$F(x) = \frac{1}{\Gamma(\alpha)} (\frac{x}{2})^{\alpha} \left[\frac{e^{\frac{-x}{2}}}{\alpha} + \frac{x}{2\alpha} \left\{ \frac{e^{\frac{-x}{2}}}{\alpha+1} + \frac{x}{2(\alpha+1)} (\frac{e^{\frac{-x}{2}}}{\alpha+2} + \frac{x}{2(\alpha+2)} (\frac{e^{\frac{-x}{2}}}{\alpha+3} + \frac{x}{2(\alpha+3)} (... \right\} \right]$$

$$= \frac{1}{\Gamma(\alpha)} \left(\frac{x}{2}\right)^{\alpha} \frac{e^{\frac{-x}{2}}}{\alpha} \left[1 + \frac{\left(\frac{x}{2}\right)}{\alpha+1} + \frac{\left(\frac{x}{2}\right)^2}{(\alpha+1)(\alpha+2)} + \frac{\left(\frac{x}{2}\right)^3}{(\alpha+1)(\alpha+2)(\alpha+3)} + ... \right]$$

$$= \frac{1}{\Gamma(\alpha+1)} \left(\frac{x}{2}\right)^{\alpha} e^{\frac{-x}{2}} \left[1 + \frac{\left(\frac{x}{2}\right)}{(\alpha+1)} + \frac{\left(\frac{x}{2}\right)^2}{(\alpha+1)(\alpha+2)} + \frac{\left(\frac{x}{2}\right)^3}{(\alpha+1)(\alpha+2)(\alpha+3)} + ... \right]$$

$$= \frac{1}{\Gamma(\alpha+1)} \left(\frac{x}{2}\right)^{\alpha} e^{\frac{-x}{2}} \sum_{n=0}^{\infty} J_n(x, \alpha)$$

where $J_n(x,\alpha) = \prod_{i=1}^{n} \frac{\left(\frac{x}{2}\right)^i}{\alpha+i}$, n = 1, 2, … and $J_0(x,\alpha) = 1$

The algorithm for approximating the incomplete gamma integral can be described by the following algorithm

PROPOSITION ALGORITHM

Step1: Read α and x

Step2: Find $\Gamma(\alpha + 1)$

Step3: Set $s = \frac{\left(\frac{x}{2}\right)^{\alpha} e^{\frac{-x}{2}}}{\Gamma(\alpha+1)}, z = 1, n = 1, J = 1$

Step4: i = 1

Step5: $J = J\frac{\left(\frac{x}{2}\right)}{\alpha + i}$

Step6: If i = n then z = z + J, J = 1

Step7: Otherwise i = i + 1 go to step (5)

Step8: If $J \leq 0.00001$, then n = n+1, go to step (5)

Step9: F = sJ

Step10: Stop.

Appendix (B5) involve a computer program written in Pascal language using Proposition method for approximating the integral side of Eq. (2.1). The x-values of the upper limit of the integral is taken from chi-square table (1). Table (7) gives together the approximate and the errors values of F(x).

Table (7): Approximation by Proposition method.

Exact / r	0.01		0.025		0.05		0.95		0.975		0.99	
	Approx.	Error	Approx.	Error	Approx.	Error	Approx.	Error	Approx.	Error	Approx.	Error
1	٠,٠٠٩٩٥٠٢	0.0000498	0.0252271	0.0002271	0.0504290	0.0004290	0.9499561	0.0000439	0.9749437	0.0000563	0.9899724	0.0000276
2	٠,٠١٠٠٢١٥	٠,٠٠٠٠٢١٥	0.0251776	0.0001776	0.0501963	0.0001963	0.9499633	0.0000367	0.9750279	0.0000279	0.9899982	0.0000018
3	٠,٠٠٩٩٩٣٠	٠,٠٠٠٠٠٧٠	0.0250341	0.0000341	0.0500305	0.0000305	0.9498939	0.0001061	0.9750181	0.0000181	0.9897904	0.0002096
4	٠,٠٠٩٩٨٧٦	٠,٠٠٠٠١٢٤	0.0249602	0.0000398	0.0500345	0.0000345	0.9500467	0.0000467	0.9745370	0.0004630	0.9901007	0.0001007
5	٠,٠٠٩٩٩٧٢	٠,٠٠٠٠٠٢٨	0.0249859	0.0000141	0.0504167	0.0004167	0 .950566	0.0005668	0.9746730	0.0003270	0.9900564	0.0000564
6	٠,٠١٠٠٢٣٤	٠,٠٠٠٠٢٣٤	0.0251371	0.0001371	0.0503412	0.0003412	0.9501533	0.0001533	0.9745263	0.0004737	0.9899528	0.0000472
7	٠,٠١٠٠٧١٧	٠,٠٠٠٠٧١٧	0.0250055	0.0000055	0.0501651	0.0001651	0.9505687	0.0005687	0.9748836	0.0001164	0.9900933	0.0000933
8	٠,٠١٠٠٣٧٠	٠,٠٠٠٠٣٧٠	0.0250098	0.0000098	0.0498571	0.0001429	0.9498777	0.0001223	0.9746961	0.0003039	0.9900357	0.0000357
9	٠,٠١٠٠٢٧٨	٠,٠٠٠٠٢٧٨	0.0249876	0.0000124	0.0502364	0.0002364	0.9496946	0.0003054	0.9748069	0.0001931	0.9901200	0.0001200
10	٠,٠٠٩٩٥١٥	٠,٠٠٠٠٤٨٥	0.0250865	0.0000865	0.0499869	0.0000131	0.9498908	0.0001092	0.9751370	0.0001370	0.9899680	0.0000320
11	٠,٠٠٩٩٩٢٨	٠,٠٠٠٠٠٧٢	0.0251105	0.0001105	0.0498069	0.0001931	0.9503712	0.0003712	0.9748404	0.0001596	0.9899158	0.0000842
12	٠,٠١٠٠٣٦٠	٠,٠٠٠٠٣٦٠	0.0249097	0.0000903	0.0501479	0.0001479	0.9496193	0.0003807	0.9747156	0.0002844	0.9899443	0.0000557
13	٠,٠٠٩٩٩٥٤	٠,٠٠٠٠٠٤٦	0.0250276	0.0000276	0.0499352	0.0000648	0.9505342	0.0005342	0.9747309	0.0002691	0.9900374	0.0000374
14	٠,٠١٠٠٠٦٦	٠,٠٠٠٠٠٦٦	0.0250264	0.0000264	0.0499794	0.0000206	0.9502090	0.0002090	0.9748605	0.0001395	0.9898704	0.0001296
15	٠,٠٠٩٩٧٩٠	٠,٠٠٠٠٢١٠	0.0249584	0.0000416	0.0499708	0.0000262	0.9500563	0.0000563	0.9750830	0.0000830	0.9900673	0.0000672
16	٠,٠١٠٠٢٠١	٠,٠٠٠٠٢٠١	0.0250430	0.0000430	0.0499517	0.0000483	0.9500494	0.0000494	0.9746798	0.0003202	0.9900002	0.0000002
17	٠,٠٠٩٩٥٨٢	٠,٠٠٠٠٤١٨	0.0249269	0.0000731	0.049908	0.0000492	0.9501653	0.0001653	0.9750616	0.0000616	0.9899745	0.0000255
18	٠,٠٠٩٩٧٧٩	٠,٠٠٠٠٢٢١	0.0249876	0.0000124	0.0499878	0.0000122	0.9503848	0.0003848	0.9748215	0.0001785	0.9899864	0.0000154
19	٠,٠٠٩٩٩٦٩	0.0001301	0.0250555	0.0000555	0.0495643	0.0004357	0.9494604	0.0005396	0.9753136	0.0003136	0.9900258	0.0000258
20	٠,٠١٠٠٢٠٨	٠,٠٠٠٠٢٠٨	0.0249881	0.0000119	0.0512252	0.0012252	0.9498734	0.0001266	0.9751967	0.0001967	0.9900938	0.0000938
21	٠,٠٠٩٩٨٢٢	٠,٠٠٠٠١٧٨	0.0252521	0.0002521	0.0502074	0.0002074	0.9503485	0.0003485	0.9751346	0.0001346	0.9899112	0.0000888
22	٠,٠١٠٠٢٩٥	٠,٠٠٠٠٢٩٥	0.0252512	0.0002512	0.0491298	0.0008702	0.9497136	0.0002864	0.9751208	0.0001208	0.9900287	0.0000287
23	٠,٠١٠٢٩٢٦	٠,٠٠٠٢٩٢٦	0.0251569	0.0001569	0.0502117	0.0002117	0.9503153	0.0003153	0.9751503	0.0001503	0.9898971	0.0001\029
24	٠,٠٠٩٨٤٧٢	٠,٠٠٠١٥٢٨	0.0249847	0.0000153	0.0489607	0.0010393	0.9498297	0.0001703	0.9752179	0.0002179	0.9900528	0.0000528
25	٠,٠١٠٠١١٥	٠,٠٠٠٠١١٥	0.0247480	0.0002520	0.0497609	0.0002391	0.9505261	0.0005261	0.9747195	0.0002805	0.9899634	0.0000366
26	٠,٠١٠١٣٠١	٠,٠٠٠١٣٠١	0.0244577	0.0005423	0.0504267	0.0004267	0.9501625	0.0001625	0.9748624	0.0001376	0.989829	0.0001071
27	٠,٠١٠٢٠٨١	٠,٠٠٠٢٠٨١	0.0253237	0.0003237	0.0509725	0.0009725	0.9498563	0.0001437	0.9750319	0.0000319	0.9900931	0.0000931
28	0.0102081	0.0002081	0.0249075	0.0000925	0.0494618	0.0005382	0.9496028	0.0003972	0.9752248	0.0002248	0.9900541	0.0000541
29	0.0102501	0.0002501	0.0244640	0.0005360	0.0498419	0.0001581	0.9504503	0.0004503	0.9748729	0.0001271	0.9900298	0.000298
30	0.0102604	0.0002604	0.0251035	0.0001035	0.0501356	0.0001356	0.9502795	0.0002795	0.9751162	0.0001162	0.9900190	0.0000190

2.7 ERROR OF APPROXIMATION

In order to compare the method who gives the minimum error of pproximation we list the errors of all methods for $Pr(X \leq x)=0.05$ and 0.95 when r=1(1)10 where these errors are given in tables (B1) and (B2).

Table (B1)

Error of Approximation to the c.d.f when the exat value is 0.05

Errors / *r*	*Trapezoidal Rule*	*Simpson Rule*	*Gaussian-Quadrature Rule*	*Hit or Miss Rule*	*Proposition Rule*
1					
2					
3					
4			0.0000345		
5					
6					
7					
8					
9					
10	0.0000014	0.0000131		0.0000014	0.0000131

Table (B2)

Error of Approximation to the c.d.f when the exact value is 0.95

Errors / *r*	*Trapezoidal Rule*	*Simpson Rule*	*Gaussian-Quadrature Rule*	*Hit or Miss Rule*	*Proposition Rule*
1					0.0000439
2			0.0000366		
3					
4		0.000455		0.0000455	
5					
6	0.0001196				
7					
8					
9					
10	0.0000014	0.0000131		0.0000014	0.0000131

CONCLUSIONS AND FUTURE WORK

CONCLUSIONS

1. The board conclusion of the results of estimation given in table (2) as follows:

(i) The values of the estimators given by equation (1.23) become more accurate and tends to the true values of α and β as the sample size increase.

(ii) The values of the estimators given by equation (1.30) flachaute near the true values of α and β because in this method we use Newton-Raphson approach which is based on the choose of the initial values of starting and as expected there is also of approximation due to the randomness. Further more the MLE depends on the non linear equation and the best starting point is taken from a string of points produced by the computer. See table (A-2) and (A-3).

2. Trapezoidal rule is need more iterations relative to Simpson rule which make the trapezoidal rule more accurate, but the inverse is not true.

3. Trapezoidal and Simpson rules based on using linear and quadratic interpolation.

4. Simpson rule is more accurate than Trapezoidal rule because the function in Simpson rule is nearly quadratic on [a, b].

5. Gaussian integration is more efficient than the Trapezoidal and Simpson rule because if Gauss-Quadreture formula of degree n then the error will be $(2n - 1)$.

6. As we mention in section (2.1) that hit and miss Monte-Carlo method is less accurate than Quadrature formulas where the practical result shown in table (5) make this conclusion is certain, but the accuracy of hit and miss Monte-Carlo increase as the sample size increases.

7. The Proposition method gives results superior than all methods of approximation for the c.d.f F(x) because this method involve a recurrence formula which approximate the value of F(x) efficiently.

8. G1-procedure is the most efficient one in comparsion with the other procedures because this method deal with +ive integer values of α only and it provide a linear relationship between the average time and the sample size in evaluation of the parameters estimate.

9. For $0 < \alpha < 1$, the simulated and theoretical efficiency in G2-Procedure are very close as shown in table (9), where the error is 3×10^{-2},while in G3-Procedure there are two cases on calculating the efficiencies as shown tables (10) and (11)

(i) When $0 < y \leq 1$, the results of table (10) shown that the simulated and theoretical efficiencies become close as α increasing.

(ii) When $1 < y < \infty$, the results of table (11) show that efficiencies become close as α decreasing and that due to the dependent of y on α. In general G2-Procedure is more efficient than G3-Procedure for practical use.

10. For $\alpha > 1$, the simulated and theoretical efficiencies in G4-Procedure decreases as α increase while in G5-Procedure the simulated efficiency approach one as α increasing with stability of the theoretical efficiency between 0.7 and 0.8. From table (12) and (13) results we notice the performance of G5-Procedure is higher than G4-Procedure in view of practical use.

FUTURE WORK

1. The methods of approximation to the c.d.f. with two-parameters can be extended to the c.d.f. of generalized gamma.

2. The procedures for generating random variates from gamma distn. can be used as a simulated data for real applications.

Examining Committee's Certification

We certify that we read this thesis entitied "*Numerical Evaluation to the Gamma Cummulative Distribution Function With Random Varietes Generating By Using Monte-Carlo Simulation*" and as examining committee examined the student, *Huda Mahdi Ahmed* in its contents and in what it connected with, and that is in our opinion it meet the standard of thesis for the degree of Master of Science, in Mathematics.

(Chairman)

Signature:

Name:Dr.Alaudeen Noori Al-Baker,

Date:

(Member)

Signature:

Name:Dr.Fadhel Subhi Fadhel

Date:

(Member)

Signature:

Name: Dr. Iden Hassan Hussan

Date:

(Member and Supervisor)

Signature:

Name: Dr. Akram M.AL-Abood

Date:

Approved for the Dean of the College of Education

Signature:

Name: Assis. Prof. Dr. LAIYH ABDUL -AZIZ

Dean of College of Science of Al-Nahrain University

Date: / /2006

Republic of Iraq
Ministry of Higher Education
and Scientific Research,
Al-Nahrain University,
College of Science
Department of Mathematics and Computer Applications.

Numerical Evaluation To the Gamma Cumulative Distribution Function With Random Varietes Generation By Using Monte-Carlo Simulation

**A Thesis submitted to
The College of Science, Al-Nahrain University,
as a partial Fulfillment of the Requirements for
the Degree of Master of Science in Mathematics**

By
Huda Mahdi Ahmed AL-Aubaidy
(B.Sc 2003)

Supervised By
Dr. Akram Mohammed Al-Abood

April 2006 *Rabia'a Al-Awal 1427*

Supervisor certification

I certify that this thesis was prepared under my supervision at the Department of Mathematical and Computer Applications, College of Science, Al-Nahrain University as a partial requirements for the Degree of Master of Science in Mathematics.

Signature:

Advisor: Dr. Akram M. Al-Abood

Date: / /2006

In view of the available recommendation, I forward this thesis for debate by the Examining Committee.

Signature:

Name: Asist. Prof.Dr. Akram M. Al-Abood

Chairman of Mathematics Department

Date: / /2006

Examining Committee's Certification

We, the Examining Committee, certify that we read this thesis and have examined the student, ***Huda Mahdi Ahmed*** *in its contents and that, in our opinion; it is adequate as a thesis for the Degree of Master of Science, in Mathematics and Computer Applications.*

Chairman	*Member*
Signature:	*Signature:*
Name:	*Name:*
Date:	*Date:*
Member	*Member (Supervisor)*
Signature:	*Signature:*
Name:	*Name: Dr. Akram M.AL-Abood*
Date:	*Date:*

Approved for the Dean of the College of Science, Al-Nahrain University. Signature:

Name: Assis. Prof. Dr. LAIYH ABDUL AZIZ

Dean of College of Science of Al-Nahrain University

بِسمِ اللهِ الرحمن الرحيم

نرفع درجات من نشاء وفوق كل ذي علم عليم

سورة يوسف

جزء من آيه(٧٦

الإهداء

إلى...

من أحيا القلوب بعد مماتها وأنارها بعد ظلمتها وألف بينها بعد شتاتها الحبيب المصطفى محمد(صلى الله عليه وآله الغر الميامين).

من أفتخر بالإنتساب اليه...مثلي الأعلى — والدي العزيز

حضن الحنان والحب الساكن في صدري — والدتي العزيزه

من أحاطتني بالمحبه والدعاء — المرحومه جدتي

من يبعث وجودهم الأمن والطمأنينه في قلبي — أخوتي وأخواتي الأعزاء

من نوروا لي الطريق... — أساتذتي الأعزاء

رفاق الدرب... — زملائي وزميلاتي

من قدم لي نصيحه أو مساعده... — ولم أذكره بالأسم

هدى

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Finally to all my friends…….. I present my thanks.

Huda

April ,2006

Abstract

In this thesis the gamma distribution Is considered for the reason of it appearance in many statistical fields of applications. Some mathematical and statistical properties of the distribution. Are collected and unified. Moments and higher moments are illustrated and two methods of estimation for the distribution parameters are discussed theoretically and assessed practically.

A new method of approximation to the cumulative distn. Function is drived and compared with four well-known method of approximation and it shown a high performance.

Finally five procedure for generating random variates from gamma distn. Are discussed and their efficiencies are compared theoretically and pratically by Monte-Carlo simulation.

Notations And Abbreviations

r.v= random variable

r.s= random sample

Distn. =Distribution

p.d.f= Probability density function

c.d.f= Cumulative density function

m.l.e= maximum likelihood estimate

M.L.E= Maximum Likelihood Estimator

m.g.f= Moment generating function

MC= Monte-Carlo

IT= Inverse Transform

CHAPTER

ONE

CHAPTER

TWO

CHAPTER

THREE

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APPENDICES

LIST OF CONTENTS

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Supervisors Certification

I certify that this thesis was prepared under my supervision at the Al-Nahrain University,College of Science, in partial fulfillment of as of the requirements for the degree of Master of Science in Mathematics.

Signature:
Name: Dr. Fadhel Subhi Fadhel
Date: / /2006

In view of the available recommendations, I forward this thesis for debate by the examination committee.

Signature:
Name: Assist. Prof. Dr. Akram M. Al-Abood
Chairman of Mathematics Department
Date: / /2006

الإهداء

إلى...

من أحيا القلوب بعد مماتها وأنارها بعد ظلمتها وألف بينها بعد شتاتها الحبيب المصطفى محمد(صلى الله عليه وآله الغر الميامين).

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رفاق الدرب... **زملائي وزميلاتي**

من قدم لي نصيحه أو مساعده... **ولم أذكره بالأسم**

هدى

المستخلص

فـي هـذه الإطروحـه تطرقنـا الـى توزيـع گامـا لأهميتـه فـي مجـالات الإحصـاء التطبيقي.وتضمنت إستعراض وتوحيد لخواص التوزيع الرياضيه والإحصـائيه والعزوم والعزوم العليا وتم مناقشة طريقتين لتخمين معلمات التوزيع نظريا وأختبرت عمليا.

أقترحت طريقه جديده لتقريب دالة التوزيع التجميعية وقورنت نتائجها مع نتـائج أربعة طرق معروفه للتقريب وأظهرت المقارنه قوة الطريقه المقترحه.

وأخيرا تطرقنا الى خمس أسـاليب لتوليد متغيرات عشوائيه حيث قورنت كفاءة هذه الطرق نظرياً وعملياً بأستخدام محاكاة مونت كارلو.

بِسم اللّهِ الرَّحمنِ الرَّحيم

نَرْفَعُ دَرَجاتٍ مَنْ نَشاءْ وَفَوْقَ كُلِّ ذي عِلْمٍ عَليمُ

صَدَقَ اللّهُ العظيم

سورة يوسف

جزء من الآية (٧٦)

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم
قسم الرياضيات وتطبيقات الحاسوب

الحل العددي لدالة توزيع گاما التجميعية مع توليد متغيرات عشوائيه باستخدام طرائق محاكاة مونت كارلو

رسالة مقدمة الى
قسم الرياضيات وتطبيقات الحاسوب- كلية العلوم- جامعةالنهرين-
كجزء من متطلبات نيل شهادة الماجستير في علوم الرياضيات

من قِبَلْ
هدى مهدي احمد العبيدي
بكالوريوس ٢٠٠٣ (جامعة النهرين)

باشراف
د.أكرم محمد العبود

ربيع الاول ١٤٢٧ نيسان ٢٠٠٦