Abstract

Planar noninvertible maps have been studied recently by several authors such as Mira , Gardini , Cathala. Much of their work has been concentrated on analyzing some examples and making some conclusions on the properties of the maps .

Our concern in this work is to study planar noninvertible continuously differentiable maps $T:\Re^2 \to \Re^2$, we have proved two new theorems that are concerned with nonempty unbounded critical sets, and we have given two conjectures : one characterizes the attractor of the map T when the critical set is parabola, and the other characterizes periodic points of the map T when the critical set is a line or a hyperbola.

We have studied some properties of such kind of maps in particular absorbing areas, invariant areas of such maps, and we have mentioned some of the known results of the subject and have given proofs of some of the results that have appeared in literature without a proof . Moreover we have studied some examples that show certain phenomena on absorbing areas .

In our work , we have made use of the Matlab version 6.1 software to solve the discussed examples .

الـــخــلاصــــــــــــة

في هذا البحث تم تناول تطبيقات المستوي الغير قابلة للانعكاس والتي درست من قبل العديد من الباحثين Cathala ،Gardini, Mira الذين تضمن عملهم بناء بعض الامثلة واعطاء استنتاجاتهم عليها مع بعض الصفات .

في بحثنا هذا تناولنا تطبيقات المستوي الغير قابلة للانعكاس والتي تكون مستمرة قابلة للاشتقاق و بصورة خاصة المساحات الماصة و المساحات اللامتغيرة كما تم اعطاء بعض النتائج التي تخص الموضوع والتي ظهرت بدون برهان لذلك تم اعطاء برهان لها مع اخذ بعض الامثلة التوضيحية لتطبيق هذه المفاهيم وكذلك تم برهان مبرهنتين جديدتين التي تخص المجموعة الحرجة الغير خالية و المقيدة . كما نقترح مظنونتين: الاولى تخص جاذب التطبيق عندما تكون مجموعة النقاط الحرجة على شكل قطع مكافىء والثانية تخص النقاط الدورية للتطبيق عندما تكون مجموعة النقاط الحرجة مستقيماً او قطع زائد .

في عملنا استخدمنا برنامج Matlab v.6.1 في حل الامثلة المعطاة .

Acknowledgement

My deepest thanks for Allah, for his generous by providing me the strength to accomplish this work .

*I would like to express my deep appreciation and gratitude to my supervisor **Dr.Adil G.Naoum** whose advice will live with me , and thanks to him cannot be expressed in words and **Dr. Alaudin N. Ahmed** for his help and advise during the preparation of this thesis .*

Also, I would like to express my thanks to all members and fellowship in the Department of Mathematics and Computer Applications .

My deep thank is for my family and friends who encourage me through my work .

ZAINAB AL-WA'LI

Appendix

The Inverse Function Theorem*: *[11, p.172]

Let $T:\Re^2 \to \Re^2$. Suppose $T(\mathbf{0})=\mathbf{0}$ and $J(T(\mathbf{0}))$ is an invertible matrix. Then there exists a neighborhood U of $\mathbf{0}$ and a C^∞ map $G:U\to\Re^2$ such that $T\circ G(X)=X$ for all $X\in U$.

That is, if the Jacobaian matrix of T is invertible at $\mathbf{0}$, then there exists a local inverse for T.

Definition*:

Let (X,d) be a metric space and let $S\subset X$ and $\varepsilon\geq 0$. Then the neighborhood of S of radius ε is the set:

$N_\varepsilon(S)=\{x\in X : d(x,s)<\varepsilon$ for some $s\in S\}$.

Definition: [22, p.159]

Let $T:\Re^2\to\Re^2$ be a map and let p be a fixed point of T with eigenvalues λ and μ such that $|\lambda|<1$ and $|\mu|>1$, then p is called a saddle point.

* Al-Sa'idi N., "On the Mulit-Fuzzy Fractal Space" Ph.D. thesis, Al-Nahrain University, 2002.

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم

المساحات الماصة لدوال المستوي التربيعية

رسالة مقدمة الى
قسم الرياضيات وتطبيقات الحاسوب–كلية العلوم – جامعة النهرين
كجزء من متطلبات الحصول على درجة دكتوراه فلسفة في الرياضيات
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Chapter one

"Preliminaries"

Introduction:

It is always useful to derive some consequences from a few bits of information. We start with the definition of a dynamical system. Let K be $\Re$ or Z. A dynamical system Φ on space X is a continuous map $\Phi : K \times X \to X$ satisfying. $\Phi(0, x) = x$ for all $x \in X$ and $\Phi(s, \Phi(r, x)) = \Phi(r + s, x)$, for all $x \in X, r, s \in K$.

If $K = \Re$ then Φ will be called a flow (continuous dynamical system). If $K = Z$ then the dynamical system will be described as discrete.

One can always construct a discrete dynamical system by iterating a given homeomorphism $f : X \to X$ in the following manner $\Phi : Z \times X \to X$ is determined by

$$\Phi(n, x) = f^n(x) \text{ if } n \in Z$$

Where $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n-times}$ is defined as the nth iteration of f and $f^{-n} = \underbrace{f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}}_{n-times}$ if $n > 0$ and $f^0 = I$. One often studies iteration for $n \geq 0$ when f is not a homeomorphism.

The purpose of this chapter is to recall several of the basic definitions from the dynamical systems.

Throughout this work, we shall focus our study on continuously differentiable maps, and discrete systems, moreover, our maps are from $\Re^2 \to \Re^2$ (i.e. planar maps).

1.1 Definitions and Notation

In this work, we shall need few notation: Let $T : X \to X$, the rank-r image of x is the image

$$x_r = T^r x$$

r is positive integer. Similarly, x is one of the rank$-r$ preimages of x_r.

If $X = \Re^n$, then the map $T : X \to X$ will be called diffeomorphism, if it is continuously differentiable function of x, and if T^{-1} exists, unique and continuously differentiable (in this case T is invertible) in the domain of definition of T. When T is such that T^{-1} may be multi-valued, or may not exist, then T will be called a noninvertible map.

Example(1.1.1):

Consider the one dimensional map T, i.e. $T:\Re \to \Re$ which is given by:

$$x' = x^2 - \lambda$$

Where λ is a parameter and T^{-1} is given by:

$$x = \pm\sqrt{x' + \lambda}$$

So the rank-one preimage of a point x' is double- value for $x' > -\lambda$, and is not real for $x' < -\lambda$.

A periodic point of period k is a point x which in the domain of T such that $T^k(x) = x$ and in addition $x, T(x), T^2(x), ..., T^{k-1}(x)$ are distinct .The orbit of $x \in X$ is the set $\{T^k(x) : k \geq 0\}$. If x is a periodic point of period k, then the orbit of x which is

$$\{x, T(x), T^2(x), ..., T^{k-1}(x)\}$$

will be a periodic orbit and is called a $k-$cycle.

If $k = 1$, then x will be called a fixed point of T. Every point of a $k-$cycle is a fixed point of T^k .

A periodic point of period k will be attracting if all the eigenvalues of the Jacobian matrix of T^k at the periodic point have their modului less than one, and if at least one of the eigenvalues is larger than one in modulus, the cycle is repulsive [22, p.146].

A periodic point of period k is expanding if all the eigenvalues of Jacobian matrix of T^k at the period point are larger than one in modulus, and there exists a neighborhood U of the periodic point such that the absolute values of the eigenvalues are larger than one, for any x belonging to U, this is in case all the eigenvalue are real [29, p.4].

If some of the eigenvalues of the Jacobian matrix of T^k are complex at the periodic point then a periodic point will be stable (attracting) if all the eigenvalues have negative real parts otherwise it is unstable(repeller)[33,p.109].

A fixed point x is called a snap-back repeller, or SBR if (a) it is expanding and (b) if in the neighborhood $U(x)$ there exists a point q such that $T^m(q) = x$ for some positive integer m [29, p.109].

Example (1.1.2):

Consider the map $T:\Re^2 \to \Re^2$ defined by

$$T: \begin{cases} x' = 2 - 2y^2 - x \\ y' = x \end{cases}$$

T^{-1} is given by

$$T^{-1}: \begin{cases} x = y' \\ y = \pm\sqrt{1 - 0.5(x' + y')} \end{cases}$$

So, T is noninvertible and has two fixed points $p_1 = (-1,-1)$ and $p_2 = (2/3, 2/3)$.
$J(T)$ has two eigenvalues $\lambda_{1,2} = 1 \pm 2.3606$ at the fixed point p_1, therefore p_1 is a repulsive fixed point since $|\lambda_{1,2}| > 1$, where J is the Jacobian matrix of T and there is a $U_\varepsilon(p_1)$, where $\varepsilon = 0.2$ such that for any $x \in U_\varepsilon(p_1)$ we have $|\lambda_{1,2}| > 1$, therefore p_1 is an expanding fixed point.

While $J(T)$ has two complex eigenvalues $\lambda_{1,2} = \frac{-2}{3} \pm \frac{\sqrt{20}}{3} i$ at fixed point p_2, since the real parts of eigenvalues are negative therefore p_2 is a stable (attracting)fixed point .
Let T be a $p-$ dimensional noninvertible map, defined in $\Re^p$.

Definition (1.1.1): [29, p.13]

A nonempty set A is said to be invariant by T if $T(A) = A$. The set A is a backward invariant by T if $T^{-1}(A) = A$, where T^{-1} represents all the rank-one preimages of T.

Definition (1.1.2):[29, p.18]

A closed invariant set A is an attracting set if an arbitrary small neighborhood U of A exists such that $T(U) \subset U$ and $T^n(x) \to A$, when $n \to \infty$, for any $x \in U$. An attractor is an attracting set which is topologically transitive, i.e. if for any two open sets $U, V \subset A$, a positive integer k exists such that $T^k(U) \cap V \neq \phi$, or equivalently a point $p \in A$ exists the orbit (iterated sequence) of which is dense in A. In this case T is called a transitive map.

Definition (1.1.3):[30]

The basin of attraction $D(A)$ (or simply the basin) of an attracting set A is the set of all the points x such that $T^n(x) \to y,\ y \in A$ when $n \to \infty$.

Definition (1.1.4):[15]

Let T be a $p-$ dimensional map, and p^* a repulsive fixed point, and U be a neighborhood of p^*. The local unstable set $W_\ell^u(p^*)$ of p^* in U, and the global unstable set of p^*, $W^u(p^*)$,are given by:

$$W_l^u(p^*) = \{x \in U : x_{-n} \in T^{-n}(x) \to p^* \text{ and } x_{-n} \in U, \forall \text{n}\}$$
$$W^u(p^*) = \{x \in \Re^p : x_{-n} \in T^{-n}(x) \to p^*\}$$

It can be shown that $W^u(p^*) = \bigcup_{n\geq 0} T^n(W_l^u(p^*))$.

In fact let $x \in W^u(p^*)$ i.e. $x \in \Re^p$ and there is a sequence of preimages of x say $\{x_{-n}\}_{n=1}^{\infty}$ that converges to p^* i.e. $|x_{-n} - p^*| < \varepsilon$ where ε can be any small real number greater than zero.

Since $x \in \Re^p$, then there is point $p \in W_l^u(p^*)$ such that x is the successive image of p i.e. there is $n \in N$ such that $T^n(p) = x$ so $x \in \bigcup_{n\geq 0} T^n(W_l^u(p^*))$

This implies $W^u(p^*) \subseteq \bigcup_{n\geq 0} T^n(W_l^u(p^*))$.

Let $x \in \bigcup_{n\geq 0} T^n(W_l^u(p^*))$ i.e. there is $n \in N$ such that $x \in T^n(W_l^n(p^*))$

So, x is the nth successive image of the point p which belongs to the neighborhood U of p^* and has a sequence of preimage which converges to p^*.

This implies $x \in \Re^p$ and $x \in W^u(p^*)$, so $\bigcup_{n\geq 0} T^n(W_l^u(p^*)) \subseteq W^u(p^*)$

$\therefore W^u(p^*) = \bigcup_{n\geq 0} T^n(W_l^u(p^*))$.

The following proposition gives some of the properties of the unstable set that appeared in [29, p.15] without a proof, we give a proof.

Proposition (1.1.1):

(P1) $T(W^u(p^*)) = W^u(p^*)$ i.e. it is invariant set.

(P2) For any map T, $T^{-1}(W^u(p^*)) \supseteq W^u(p^*)$.

Note that if T is noninvertible, then $W^u(p^*)$ may not be backward invariant.

(P3) Let $V(p^*)$ be a neighborhood of p^*. For any $x \in W^u(p^*)$ an integer N exists (which depends on x) such that a rank-N perimage x_{-N} of x belongs to V and a sequence of preimages of x_{-N} exists which belongs to V and converges to p^*.

Proof:

(P1) $T(W^u(p^*)) = T\{x \in \Re^p : x_{-n} \in T^{-n}(x) \to p^*\}$

$$= \{T(x) \in \Re^p : T(x_{-n}) \in T(T^{-n}(x)) \to T(p^*)\}$$

$$= \{T(x) \in \Re^p : x_{-n+1} \in T^{-n+1}(x) \to p^*\}$$
$$= W^u(p^*)$$

(P2) $T^{-1}(W^u(p^*)) = T^{-1}(\bigcup_{n\geq 0} T^n(W_l^u(p^*)))$

$$= \bigcup_{n\geq 0} T^{-1}(T^n(W_l^u(p^*)))$$
$$= \bigcup_{n\geq 0} T^{n-1}(W_l^u(p^*)) \supseteq \bigcup_{n\geq 0} T^n(W_l^u(p^*)) = W^u(p^*)$$

(P3) Let $V(p^*)$ be a neighborhood of p^*. Let $x \in W^u(p^*)$. Then from definition of $W^u(p^*)$ there exists a sequence of preimages x_{-N} such that $\{x_{-n}\} \to p^*$ i.e. $\{x_{-n}\} \in V(p^*)$.♦

Remark (1.1.1):[29, p.15]

$W^u(p^*)$ is a connected, self intersection that may occur (so that it may not be a manifold). If it is smooth then it may be called a local manifold. When T is invertible the global set is also a manifold while when T is noninvertible then the global set may not be a manifold. A self intersection of $W^u(p^*)$ is allowed.

Definition (1.1.5):[15]

Let p^* be a fixed point of T which may be attracting or repulsive. The local stable set of p^* in a neighborhood U, and the global stable set $W_l^s(p^*)$, $W^s(p^*)$ are given by:

$$W_l^s(p^*) = \{x \in U, x_n = T^n(x) \to p^* and\ x_n \in U,\ \forall n\},$$
$$W^s(p^*) = \{x \in \Re^p, x_n = T^n(x) \to p^*\}.$$

Also, it can be shown that $W^s(p^*) = \bigcup_{n\geq 0} T^{-n}(W_l^s(p^*))$ as we have proved previously.

The following proposition gives some of the properties of the stable sets, it has appeared in [29, p.16] without a proof, we give a proof.

Proposition (1.1.2):

(P1) $T^{-1}(W^s(p^*)) = W^s(p^*)$.

(P2) $T(W^s(p^*)) \subseteq= W^s(p^*)$.

(P3) Let $V(p^*)$ be a neighborhood of p^*. For any $x \in W^s(p^*)$ an integer N exists (which depends on x) such that a rank$-N$ image x_N of x belongs to V and converge to p^*.

***Proof**:-*

(P1) $T^{-1}(W^s(p^*)) = T^{-1}\{x \in \Re^p : x_n = T^n(x) \to p^*\}$

$$= W^s(p^*)$$

(P2) $T(W^s(p^*)) = T(\bigcup_{n\geq 0} T^{-n}(W_l^s(p^*)))$

$$= \bigcup_{n\geq 0} T^{-n+1}(W_l^s(p^*))$$

$$\subseteq \bigcup_{n\geq 0} T^{-n}(W_l^s(p^*))$$

(P3) Let $V(p^*)$ be a neighborhood of p^*. Let $x \in W^s(p^*)$ then from the definition of $W^s(p^*)$ there exists a sequence of images $\{x_n\}$such that $\{x_n\} \to p^*$ i.e. $\{x_n\} \subset V(p^*)$ and $T(x_n) \to T(p^*) = p^*$ therefore the image of $\{x_n\}$ belongs to V. ♦

Remark (1.1.2):

(1) $W^s(p^*)$ may be a connected manifold, or the union of disjoint connected components, also $W^s(p^*)$ may be smooth so that it is called a local manifold, and self intersections cannot occur.

(2) When T is noninvertible, then there exists point in $W^s(p^*)$ say x such that $T^m(x) = p^*$ for a suitable integer m. If T is a noninvertible two-dimensional map, $W^s(p^*)$ may be non-connected and made up of infinitely many closed curves. And this property also holds for higher dimensions.

(3) In invertible maps, an expanding fixed point has no stable sets, while in a noninvertible map when p^* is expanding the local stable set of p^* consist of p^* itself, and the global stable set is given by all the preimages (of any rank) of this point: $W_\ell^s(p^*) = \bigcup_{n\geq 0} T^{-n}(p^*)$. In this case property (2) works for all the points of the stable set, i.e. $\forall\ x \in W^s(p^*)$ an integer $m(x)$ exists such that $T^{m(x)}(x) = p^*$.

Definition (1.1.6):[29, p.17]

A point q is said to be homoclinic to a repulsive (or expanding)fixed point p^* (or homoclinic of p^*) if $q \in W^u(p^*) \cap W^s(p^*)$.

Definition (1.1.7):[29, p.18]

A point q is said to be heteroclinic from the repulsive (or expanding)fixed point p^* to the repulsive fixed point r^*, if $q \in W^u(p^*) \cap W^s(r^*)$.

1.2 Two Dimensional Noninvertible Maps: Properties of Critical Curves

This section examines a class of continuously differentiable two-dimensional noninvertible maps (endomorphism).
We shall start by giving the definition of the critical curve LC (from Ligne Critique in French), this concept was first introduced in 1964 by Mira [5&15].

Definition (1.2.1): [28]

Let T be a noninvertible map of $\Re^2$ into itself defined by:

$$T: \begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \qquad (1.2.1)$$

Where f and g are continuously differentiable functions. The curve LC_{-1} is defined to be the set of points at which the Jacobian of f and g vanish i.e. the set of critical points of the map T. Then the successive forward iterates LC_i, $i = 0,1,2,...$, $LC_0 = LC$ are called critical curves of rank $i+1$.

Note that the concept of critical curve is a generalization of the concept of a critical point for one dimensional map.
The rank-one critical curve LC and the curve LC_{-1} may be made up of several branches with respect to the inverse map T^{-1}, the plane can be considered to be made up of N sheets joining at the branches of the first rank critical curve LC which bound regions where the number of first rank preimages is constant , N being the maximum number of such preimages (N will be called the map degree). Then every sheet is associated with a well defined first rank preimage, which leads to a foliation of the plane directly related to fundamental properties of the map [29, p.114].

Remark(1.2.1): If T is invertible, then it is diffiemorphism which implies $J(T) \neq 0$, so T has no critical points.

1.2.1 Types of Noninvertible Maps with Critical Curves, their Symbolic Representation

It is convenient to classify noninvertible maps according to the number of solutions of the inverse map for all possible points $X \in \Re^2$, and the relative arrangement of regions with different numbers of preimages. We have observed that LC divide the plane into open regions Z_i $(\Re^2 = \bigcup_i \bar{Z}_i)$, the points that have i distinct preimges of rank one. One can classify the maps into types depending on the number of regions and the number of preimages [28]. We shall try to mention some of these types:

1- $(Z_0 - Z_2)$ map: LC is made up of only one branch separating $\Re^2$ into regions, one is Z_0 with no preimage, the other Z_2 with two first rank preimages.

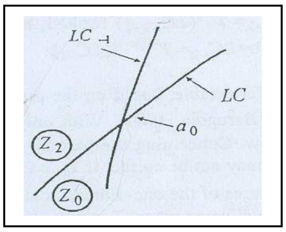
2- $(Z_1 - Z_3 - Z_1)$ map: LC consists of two branches separating $\Re^2$ into three regions, one Z_3 with three first rank preimages, and two Z_1 non connected regions, on both sides of Z_3, with only one first rank preimage.

3- Maps of type $(Z_0 - Z_2 - Z_4)$, … or more complex types that are generated by regions having a higher number of rank-one preimages, the branches of LC separating these regions.

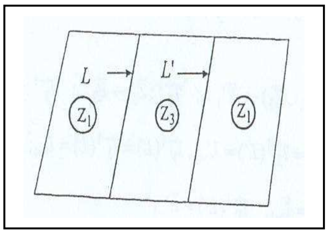
There are other complex kinds of maps that are related to the presence of one or more cusp points (cusp point is a point whose three first rank preimages coincide) on the critical curve LC, the symbolic representation of maps, may be refined by introducing the symbols "$<$", and "$>$" for the presence of such a point, some of these maps are:

1. $(Z_1 < Z_3)$ map: the curve LC has a cusp point corresponding to a "cape" of Z_3 "penetrating" into Z_1.
2. $(Z_1 < Z_3 >)$ map: LC is a closed curve with two cusps, forming a "lip" shape.
3. $(Z_0 - Z_2 << Z_4)$ map: the curve LC presents two cusps forming a dovetail figure. In this configuration, each cusp is a "cape" of Z_4 "penetrating" into Z_2 with a dovetail shape.

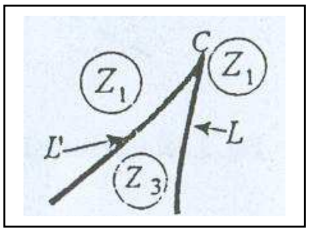
Figure (1.2.1) illustrates the above types of maps.



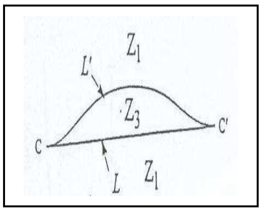
(a)



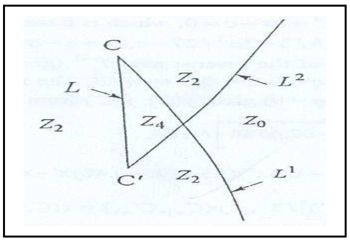
(b)



(c)



(d)



(e)

Fig(1.2.1) (a) $(Z_0 - Z_2)$ map $\overline{Z}_0 \bigcap \overline{Z}_2 = LC$ (b) $(Z_1 - Z_3 - Z_1)$ map, $LC = L \cup L'$, $\overline{Z}_1 \cap \overline{Z}_3 = L$, $\overline{Z}_3 \cap \overline{Z}_1 = L'$; (c) $(Z_1 < Z_3)$ map; $LC = L \cup L'$ cusp point $\equiv c = L \cap L'$; (d) $(Z_1 < Z_3 >)$ map; $LC = L \cup L'$, $\overline{Z}_1 \cap \overline{Z}_3 = L$, $L \cap L' = c, c'$;
(e) $(Z_0 - Z_2 << Z_4)$ map , $LC = L \cup L' \cup L''$, $c = L \cap L'$, $c' = L \cap L''$.

Note: In this work we have restricted the attention to the maps of type $(Z_0 - Z_2)$ unless otherwise stated.

1.2.2 Characterization of the Different Determinations of the Inverse Map

We can define different inverses in each region Z_i, with $i>0$ (for $i=0$, there is no inverse)[28]. Let $R_{i,j}$ be the range of one of the inverses of T defined in Z_i, $j=1,2,...,i$, then the corresponding inverse is:

$$T_{i,j}^{-1}:\overline{Z}_i \to \overline{R}_{i,j}$$

$$T_{1,1}^{-1} \equiv T_1^{-1} \quad , \quad \overline{R}_{1,1} \equiv \overline{R}_1$$

Where $\overline{Z}_i$, $\overline{R}_{i,j}$ are the closures of Z_i, $R_{i,j}$ respectively

The $R_{i,j}$'s are disjoint open regions bounded by arcs of LC_{-1} (the curve of rank-one merging preimages), since T possesses more than two first rank inverses, the rank-one preimages of LC consist of points at which the Jacobian of T does not vanish, these points are called extra preimages, i.e.

$T^{-1}(LC)=LC_{-1}\cup \overline{L}C_{-1}$, where $\overline{L}C_{-1}$ the extra set

Now, Let us have a little closer look at the extra set $\overline{L}C_{-1}$. Consider a branch $L\in LC$ separating the two regions Z_p, and Z_{p+2}, $p>0$. Then $p+2$ inverses, $T_{P+2,j}^{-1}(L)$, $j=1,2,...,p+2$ are defined in region Z_{p+2}. Similarly p inverses, $T_{P,j}^{-1}(L)$, $j=1,2,...,p$, are defined in Z_p. If $x\in L$, let $p+1$, $p+2$ are two of a first rank preimages of x merge into L_{-1}, $T(L_{-1})=L$ and $L_{-1}\subset T^{-1}(L)$. Thus L_{-1} is given by:

$$T_{p+2,p+1}^{-1}=T_{P+2,p+2}^{-1}(L)=L_{-1}$$

The other first rank preimages, those given by $T_{P,j}^{-1}(L)$, $j=1,2,...,p$, belong to the to the extra set $\overline{L}C_{-1}$, and are given by

$$\overline{(L_{-1})}_j = T_{P,j}^{-1}(L) \qquad , j=1,2,...,p.$$

Remark(1.2.2):-

We have noticed that by the inverse function theorem the inverses of T are continuously differentiable in the interior of their domains of definition, i.e. in each region Z_i. Moreover, LC_{-1} separates the plane into regions, inside which the Jacobian of T has a constant sign.

***Example (1.2.1)*: ** Let T be defined by

$$T: \begin{cases} x' = a-by-x^2 \\ y' = x^2 \qquad , b\neq 0 \end{cases}$$

T is noninvertible and has type $(Z_0 - Z_2)$.

T has two inverses

$$T^{-1}:\begin{cases} x = \pm\sqrt{y'} \\ y = (a - y' - x')/b \qquad ,b \neq 0 \end{cases}$$

The curve LC_{-1} is given by the equation $x = 0$ and the critical curve LC is given by $y = 0$. So LC_{-1} divides the plane $\Re^2$ into two regions R_1 with $x > 0$ and R_2 with $x < 0$. Also, the curve LC divides the plane $\Re^2$ into two regions: Z_0 satisfying $y < 0$ where each point has no preimage, and Z_2 satisfying $y > 0$ where each point has two first rank preimages. We can define different inverses in region Z_2, so let $T_{2,1}^{-1} : \overline{Z}_2 \to \overline{R}_{1,2}$ be defined by

$$x = -\sqrt{y'}$$
$$y' = \frac{a - y' - x'}{b}$$

and $T_{2,2}^{-1} : \overline{Z}_2 \to \overline{R}_{2,2}$ is defined by

$$x = \sqrt{y'}$$
$$y' = \frac{a - y' - x'}{b}$$

For particular case when $a = 1, b = 1$, T has two fixed points $p_1 = (-1,1)$ and $p_2 = (\frac{1}{2}, \frac{1}{4})$. Then

$$T_{2,1}^{-1}(-1,1) = (-1,1) \in \overline{R}_{2,1},\ T_{2,2}^{-1}(-1,1) = (1,-1) \in \overline{R}_{2,2}$$

Also, $T_{2,1}^{-1}(\frac{1}{2}, \frac{1}{4}) = (-\frac{1}{2}, \frac{1}{4}) \in \overline{R}_{2,1}$, $T_{2,2}^{-1}(\frac{1}{2}, \frac{1}{4}) = (\frac{1}{2}, \frac{1}{4}) \in \overline{R}_{2,2}$

Remark (1.2.3):

1- The preimages of $LC_k, k \geq 0$, which are points of $T^{-1}(LC_k)$ not belonging to $LC_{k-1} = T^{k-1}(LC)$ are called extra preimages.

2- When all the inverses of T are defined, their closed range $R_{i,j}$ give a finite cover of the plane $\Re^2$ with closed sets having disjoint interiors.

i.e. $\Re^2 = \bigcup_{i>0} \bigcup_{j=1}^{i} \overline{R}_{i,j}$.

1.2.3 Critical Set of A power of T

The critical set $EC(T^m)$ of T^m (EC comes from ensemble critique in French) $m>1$, is the locus of points $X=(x,y)$ having at least two coincident preimages $T^{-m}(x)$.
The following proposition can be found in [17]

***Proposition (1.2.1)*:**
Let T be continuously differentiable map.
(1) If T is a map without a Z_0 region, the critical set $EC(T^m)$, $m>1$ is given by:

$$EC(T^m)=\bigcup_{i=0}^{m-1} LC_i, \quad LC_0 \equiv LC \qquad (1.2.1)$$

A critical curve LC_i belonging to $EC(T^m)$, separates the $(x,y)-$ plane locally into two regions, one with points having p preimages of rank m, the other with points having q preimages of rank m, $p\geq 0$, $q\geq 0$. In the general case $q=p+2h,\ h=1,2,....$
(2) when a Z_0 region exists, the critical set $EC(T^m)$, $m>1$ is given by:

$$EC(T^m)=LC_{m-1}\cup T^m(LC_{-2})\cup\cdots\cup T^m(LC_{-m}) \qquad (1.2.2)$$

where

$$LC_{-2}=T^{-1}(LC_{-1}),\ LC_{-3}=T^{-1}(LC_{-2}) \text{ etc.}$$

(3) In both cases (1), (2), $EC_{-1}(T^m)$ is defined by

$$EC_{-1}(T^m)=LC_{-1}\cup LC_{-2}\cup\cdots\cup LC_{-m}$$

Proof:- Can be found in [17].

For understanding (1.2.2) contents consider the map T^2. Let $X_1\in LC_1$. Then at least one inverse T_i^{-1} of T exists such that $X_0=T_i^{-1}(X_1)\in LC$, X_0 having two merging first preimages $T_1^{-1}\circ T_i^{-1}(X_1)=T_2^{-1}\circ T_i^{-1}(X_1)\in LC_{-1}$.
i.e $X_1\in EC(T^2)$ because at least two rank one preimages of T^2 are merging, and $LC_1\subset EC(T^2)$. Now, Let $P\in LC$,

$$P_{-1}=T_1^{-1}(P)=T_2^{-1}(P)\in LC_{-1}.$$

Thus if $P_{-1}\in LC_{-1}\cap Z_0$, then $P\notin EC(T^2)$.
If $P_{-1}\notin LC_{-1}\cap Z_0$, then at least one inverse is defined in P_{-1}. When $P_{-1}\in Z_q$, it has q distinct inverse T_i^{-1}, $i=1,2,...,q$, with

$$P_{-2,i}=T_i^{-1}\circ T_1^{-1}(P)=T_i^{-1}\circ T_2^{-1}(P) \qquad , \ i=1,2,...,q$$

then $P\in EC(T^2)$ and q pairs of merging rank one preimages are defined in P.
From this it follows that:

1- The existence of more than one pair of merging first rank preimages is a generic case for maps T^k, $k>1$,
2- The inverses of the map T^k can be obtained by composition of the basic inverses of T.

Let T be a map with $LC_{-1}\cap Z_0 \neq \phi$ and:

$$LC_{-1}^{(a)} = LC_{-1} \setminus (LC_{-1}\cap Z_0)$$

which in the case of a $(Z_0 - Z_2)$ map becomes $LC_{-1}^{(a)} = LC_{-1}\cap Z_2$.
So one has (fig (1.2.2)):

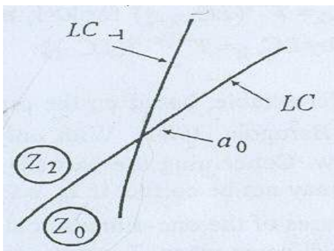
$$EC(T^2) = LC_1 \cup LC^{(a)} \ , \ LC^{(a)} = T(LC_{-1}^{(a)})$$

When Z_0 does not exists we have $EC(T^2) = LC_1 \cup LC$.
Remarking that $LC_{-2} = T^{-1}(LC_{-1} \setminus (LC_{-1}\cap Z_0))$, then $EC_{-1}(T^2) = LC_{-1}\cup LC_{-2}$
Note that :

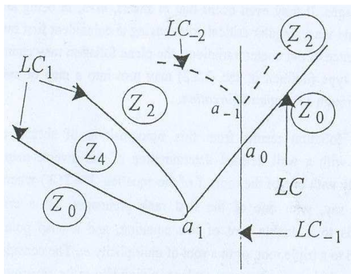
$$T(LC_{-2}) = LC_{-1} \text{ if } LC_{-1}\cap Z_0 = \phi$$

and $LC_{-1}^{(a)} = T(LC_{-2}) \subset LC_{-1}$ if $LC_{-1}\cap Z_0 \neq \phi$.

Note:-In figure (1.2.2), T is a $(Z_0 - Z_2)$ map and then T^2 is a $(Z_0 - Z_2 - Z_4)$ map.



(a)



(b)

fig(1.2.2) Z_i regions for : (a) map T & (b) map T^2, $a_0 = LC_{-1}\cap LC$, $a_1 = T(a_0)$, $a_{-1} = T^{-1}(a_0)$, $EC_{-1}(T^2) = LC_{-1}\cup LC_{-2}$.

Consequence:

If $Z_0 \neq \phi$ and $LC_{-1} \cap Z_0 \neq \phi$, $EC(T^m)$ contains wholly only LC_{m-1} and all the remaining critical curves $LC_{m-2},\ldots, LC$ are only partially contained in LC_{m-1}, due to their points belonging to $LC_{-1} \cap Z_0$.

1.2.4 Foliation of the Plane

Since the plane is divided by the critical curve LC into distinct regions Z_k, these regions are considered as the superposition of k sheets, each of them is associated with a given first rank preimage. We call this a foliation. Two such sheets may be connected in pairs by folds (the fold "projects" on one of the segments of the critical curve). Three sheets may join at a singular critical point, a cusp point at all the junction of two fold segments, which has three coincident first rank preimages [28].

Furthermore, it may occur that m sheets, $m > 0$ (m is odd integer) communicate at a singular critical point having m coincident first rank preimages.

The plane foliation may change; when a parameter is vanished, i.e. a map of a given type may turn into a map of another type as it passes through a foliation bifurcation.

Now, we shall give an example and an associated figure which illustrate this situation, this example of type $(Z_0 - Z_2)$ map where LC separates the plane into two open regions Z_0 and Z_2 such that $\overline{Z}_0 \cap \overline{Z}_2 = LC$, $\overline{Z}_0 \cup \overline{Z}_2 = \Re^2$. LC_{-1} separates the plane into two regions R_1 and R_2, $R_1 \cap R_2 = \phi$, $\overline{R}_1 \cap \overline{R}_2 = LC_{-1}$, $\overline{R}_1 \cup \overline{R}_2 = \Re^2$. Two distinct inverses are defined in Z_2 (fig 1.2.3)

$$T_1^{-1} : \overline{Z}_2 \to \overline{R}_1$$

$$T_2^{-1} : \overline{Z}_2 \to \overline{R}_2$$

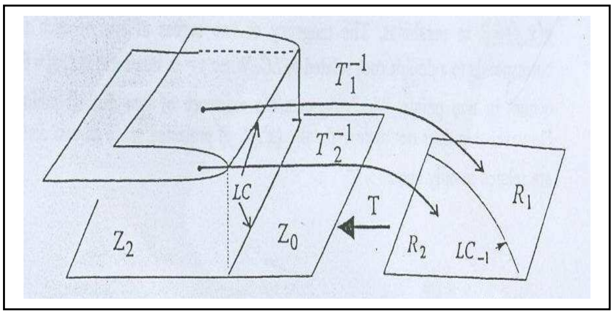


Fig. (1.2.3) Relation between the foliation (i.e. the two sheets) of a $(Z_0 - Z_2)$ map, the inverses determination T_1^{-1}, T_2^{-1} of T^{-1} and the regions R_1, R_2 with $T_i^{-1} : \overline{Z}_2 \to \overline{R}_i$, $\mathrm{i} = 1,2$.

Example (1.2.2): [28]

Let T be a map defined by :

$$T: \begin{cases} x' = y \\ y' = 0.5x + 0.1x^2 + \lambda xy \end{cases} \quad , X = (x, y)$$

T is continuously differentiable. LC_{-1} is the straight line given by $0.5 + 0.2x + \lambda\, y = 0$ and LC is the parabola $(0.5 + \lambda x)^2 + 0.4y = 0$. The parametric equation of $LC_1 = T(LC)$ is

$$x = -(0.5 + \lambda t)^2 / 0.4$$

$$y = 0.5\,t + 0.1\,t^2 - \lambda t((0.5 + \lambda t)^2 / 0.4$$

The inverse map $T^{-1}(X)$ is characterized by:

$$x = \left\{ -0.5 - \lambda x' \pm \left[(0.5 + \lambda x')^2 + 0.4y' \right]^{\frac{1}{2}} \right\} \Big/ 0.2$$

$$y = x'$$

Let R_1, R_2 be two open regions such that $LC_{-1} = \overline{R}_1 \cap \overline{R}_2$. For every $X \in Z_2$,
Let $T_1^{-1}(X) \in R_1$, $T_2^{-1}(X) \in R_2$ be the two determinations of $T^{-1}(X)$, i.e. the two first rank preimages of X . If $X \in LC$ then $T_1^{-1}(x) = T_2^{-1}(x) \in LC_{-1}$ with:

$$T_1^{-1}: \begin{cases} x = \left\{ -0.5 - \lambda x' + \left[(0.5 + \lambda x')^2 + 0.4y' \right]^{\frac{1}{2}} \right\} \Big/ 0.2 \\ y = x' \end{cases}$$

and

$$T_2^{-1}: \quad \begin{aligned} x &= \left\{-0.5-\lambda x' - \left[(0.5+\lambda x')^2 + 0.4y'\right]^{\frac{1}{2}}\right\} \Big/ 0.2 \\ y &= x' \end{aligned}$$

The critical curve LC of rank one separates the plane into the region $(0.5+\lambda x)^2 + 0.4y < 0$, each point of which has no preimages (region Z_0),and the region $(0.5+\lambda x)^2 + 0.4y > 0$, each point of which has first rank preimages (region Z_2) .

1.3 Chaos in Dynamical Systems

The study of orbits (periodicity, density, etc.) contributes to the knowledge of the behavior of the system. Systems with complicated behavior are usually called chaotic .Chaos is defined in several different ways through literature. In 1976 [22, p.91], Li and Yorke introduced the notion of chaos for a continuous map from I into itself, where I is a compact real interval. In Yorks paper, the term chaos appeared for the first time. In [11] , Devany defined the chaotic map as follows $T: X \to X$ is said to be chaotic if it is transitive , its periodic points are dense and it has sensitivity with respect to the initial conditions where X is any metric space. Gulick called the map chaotic if it is sensitive to initial conditions.

Definition (1.3.1): [34, p.21]

Let X be a metric space , d the distance on X and $T: X \to X$ has a sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood U of x there exists $y \in U$ and $n \geq 0$ such that $d(T^n(x), T^n(y)) > \delta$.

Example (1.3.1): [11, p. 52]

Let $B:[0,1] \to [0,1]$ be the baker map given by

$$B(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 0.5 \\ 2x-1 & \text{for } 0.5 < x \leq 1 \end{cases}$$

Notice that $B(\frac{1}{3}) = \frac{2}{3}$ *and* $B^2(\frac{1}{3}) = \frac{1}{3}$, so that the iterates of $\frac{1}{3}$ alternate between $\frac{1}{3}$ *and* $\frac{2}{3}$. To compare the iterates of $\frac{1}{3}$ and 0.333 we make the following table (where we use3-place approximations for the iterates of 0.333).

Iterates	1	2	3	4	5	6	7	8	9	10
1/3	2/3	1/3	2/3	1/3	2/3	1/3	2/3	1/3	2/3	1/3
0.333	0.666	0.332	0.664	0.328	0.656	0.312	0.624	0.248	0.496	0.992

Take $\delta = 0.5$, $n = 10$, so

$$d(B^n(0.333), B^n(\frac{1}{3})) = \left|0.992 - \frac{1}{3}\right| > \delta .$$

Thus, B has a sensitive dependence to initial condition on [0, 1].

We shall discuss in chapter three the chaotic area for a two-dimensional map.

Example (1.3.2): [11, p.50]

Let f be a continuous map from unit circle S^1, into itself i.e. $f : S^1 \to S^1$, given by $f(\theta) = 2\theta$. We shall use the definition of transitive map. Let U be any small open arc in S^1, there is $k \in \mathrm{N}$ such that $f^k(U)$ covers all of S^1, in particular, $f^k(U)$ intersects any other open arc V in S^1. This implies $f^k(U) \cap V \neq \phi$, thus f is transitive.

Theorem (1.3.1): [22, p.96]

Suppose that J is a closed subset of $\Re^m$, $f : J \to J$ and .Then f is transitive if and only if there is an x on J whose orbit is dense in J.

Proof: See [22, p.97].

Now, we shall turn to the main theme of this section, the notion of a chaotic dynamical system. There are many possible definitions of chaos in a dynamical system. We shall recall some of them and start by Gulick's definition before we give its definition; we need to mention the following:

Definition (1.3.2): [22, p.87]

Let J be a bounded interval, and $f : J \to J$ is continuously differential on J. Fix x in J , let $\lambda(x)$ be defined by $\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \ln\left|(f^n)'(x)\right|$ provided that the limit exists, $\lambda(x)$ is called the Lyapunov exponent of f at x . If $\lambda(x)$ is independent of x whenever $\lambda(x)$ is defined, then the common value of $\lambda(x)$ is denoted by λ, and is the Lyapunov exponent of f .

Definition (1.3.3): [22, p.90]

A map f is chaotic if it satisfies at least one of the following conditions:

(1) f has a positive Laypunov exponent at each point in its domain that is not eventually periodic .or
(2) f has sensitive dependence on initial conditions on its domain.

Also, Gulick defined strongly chaotic maps as follows:

Definition (1.3.4): [22, p.101]
A map f on an interval J is strongly chaotic if
(1) f is chaotic (in the sense of Gulick).
(2) f has a dense set of periodic points .
(3) f is transitive .
Now, we see another definition of chaos given by Devany .

Definition (1.3.5): [11, p.50]
Let J be a subset of $\Re$, $f : J \to J$ is said to be chaotic on J if f has a sensitive dependence on initial conditions, f is transitive and periodic points of f are dense in J .
By the comments above, f is chaotic equivalence to strongly chaotic in the sense of Gulick.

Definition (1.3.6): [22, p.104]
Let A and B be closed sets in $\Re^2$ and let $f : A \to A$ and $g : B \to B$ be two maps, f and g are said to be topological conjugate if there exists a homoeomorphism $h : A \to B$ such that $h \circ f = g \circ h$. The homeomorphism h is called a topological conjagacy. In this case, we write $f \approx_h g$.

Maps which are topological conjugate are completely equivalent in terms of their dynamics. For example, if f is topological conjugate to g via h and p is a fixed point for f , then $h(p)$ will be a fixed point for g .

The next result implies that the periodic points are inherited through conjugacy:-

Theorem (1.3.2): [22, p.106]
Let $f \approx_h g$ then
(1) $h \circ f^n = g^n \circ h$ for $n = 1,2,...$.

(2) If x is a periodic point for f of period n, then $h(x)$ is a periodic point of g of period n.

(3) If f has a dense set of periodic points, so does g.

Proof:-

$h \circ f = g \circ h$. For the purpose of an induction proof, let us assume that $h \circ f^{n-1} = g^{n-1} \circ h$. Then $h \circ f^{n} = (h \circ f) \circ f^{n-1} = (g \circ h) \circ f^{n-1} = g \circ (h \circ f^{n-1}) = g \circ (g^{n-1} \circ h) = g^{n} \circ h$, so that $h \circ f^{n} = g^{n} \circ h$ for $n = 1,2,...$. By induction , (1) is proved .

To proved (2), assume x is a periodic point of period n for f. That, is $f^{n}(x) = x$, (1) implies that $g^{n} \circ h(x) = (g^{n} oh)x = (hof^{n})x = h(f^{n}(x)) = h(x)$. Consequently $h(x)$ is a periodic point for g of period n.

Finally, the image $h(K)$ of a dense set K of periodic point of f contains only periodic points of g by (2), and is dense in the range of g by the definition of homeomorphism. ♦

The next result shows that transitivity is inherited through conjugacy.

Theorem (1.3.3): [22, p.107]

Let $f \approx_{h} g$, if f is transitive then g is transitive, too.

Proof: -

By theorem (1.3.1), a map is transitive if and only if it has a dense orbit. Suppose that the orbit of x for f is dense in the domain A. We shall show the orbit of $h(x)$ for g is dense in B. Let U be a nonempty open subset of B, because h is homeomorphism, it follows that $h^{-1}(U)$ is an open set of A. Since the orbit of x is dense in A, there is a positive integer in such that $f^{n}(x)$ is in $h^{-1}(U)$, so $h(x)$ is in $h(h^{-1}(U)) = U$. By (1) of theorem (1.3.2): $h(f^{n}(x)) = (hof^{n})x = (g^{n} oh)x = g^{n}(h(x))$. Therefore, $g^{n}(h(x))$ is in U. Since U is an arbitrary open interval in B, then we have succeeded in proving that the orbit of $h(x)$ for g is dense in B, so that by theorem (1.3.1), g is transitive. ♦

1.4 Planar Quadratic Maps

Our goal in this section is to give a brief description of the dynamics of planar quadratic maps that have nonempty critical set whatever (bounded or not).

Definition (1.4.1): [13]

A planar quadratic map is a map T that has the form:

$$T(x,y)=(t_1(x,y),t_2(x,y)) \qquad (1.4.1)$$

Where $t_1(x,y)=a_0x^2+a_1xy+a_2y^2+a_3x+a_4y+a$

$t_2(x,y)=b_0x^2+b_1xy+b_2y^2+b_3x+b_4y+b$

and where $a,b,a_i^{'}s$ and $b_i^{'}s$ are real constants.

The critical set or singular set $J(T)$ of a planar quadratic map (1.4.1) is the set:

$$J(T)=\{x\in\Re^2:\det(J(T(x))=0\}$$

Clearly the critical set $J(T)$ is a real planar algebraic curve of order not greater than two. This set may be bounded or not, we can show it is bounded when the following conditions are satisfied:

(1*a*) $a_0b_1-a_1b_0=a_1b_2-a_2b_1$ (We get circle) or

(1*b*) $a_0b_1-a_1b_0\neq a_1b_2-a_2b_1$ (We get ellipse)

(2) $a_0b_2-a_2b_0=0$

(3) $A^2+B^2-C\geq 0$, where $A=a_0b_4-a_4b_0+\frac{a_3b_1-a_1b_3}{4}$, $B=a_3b_2-a_2b_3-\frac{a_4b_1-a_1b_4}{4}$

and $C=a_3b_4-a_4b_3$

When one of the above conditions is not satisfied, the critical set is unbounded.

Remark (1.4.1):

The critical set $J(T)$ is empty when $\det(J(T))$ is constant i.e. when condition (1) is satisfied and equals zero, condition (2) is satisfied and $A=B=0$ and $C\neq 0$.

Now, we shall characterize maps with bounded critical set and maps with an unbounded critical set. So let

$\Im_1=\{T:J(T)$ is bounded and nonempty $\}$

$\Im_2=\{T:J(T)$ is unbounded and nonempty $\}$

When the critical set is bounded it can easily be shown that it is only an empty set, a point or an ellipse. Each map in $\Im_1$ or in $\Im_2$ can be brought into the standard form via an affine coordinate change, the standard form of a map which is in $\Im_1$, $\Im_2$ respectively are

$$T(x,y)=(x^2+a_1xy-y^2+a,b_1xy+b) \qquad (1.4.2a)$$

$$T(x,y)=(x^2+a_1xy+y^2+a_3x+a,b_1xy+b_3x+b) \qquad (1.4.2b)$$

And

$$T(x,y)=(a_0x^2+a_1xy+a_4y+a,b_1xy+b),\ \text{with}\ b_1\neq 0 \qquad (1.4.3a)$$

$$T(x,y)=(a_0x^2+a_2y^2+a_3x+a_4y+a,b_0x^2+b_2y^2+b_3x+b_4y+b) \qquad (1.4.3b)$$

Remark (1.4.2):

1- The critical set of the standard form (1.4.2a) is a point, while the form (1.4.2b) gives a critical set as an ellipse.

2- The critical set of the standard form (1.4.3a) is a parabola, while the form (1.4.3b) gives a critical set as a hyperbola or a straight line (we get a line in case $a_0 b_2 - a_2 b_0 = 0$).

3- The standard form is not unique. For example,
$T_1(x,y) = (x^2 - y^2, -xy)$ & $T_2(x,y) = (x^2 + \sqrt{2}xy - y^2, 3xy)$ are two maps in $\Im_1$ and T_1 is affine conjugate to T_2 via the affine conjugacy $h(x,y) = (\frac{-1}{3}x - \frac{\sqrt{2}}{3}y, \frac{\sqrt{2}}{3}x - \frac{1}{3}y)$.
Also, $T_1(x,y) = (ax + y, x^2 + b)$ & $T_2(x,y) = (y, ay - x^2 - b)$ are two maps in $\Im_2$ which are conjugate via $h(x,y) = (-x, -ax - y)$.

1.4.1 Some Properties of Planar Quadratic Maps

In this section we shall state and prove some basic properties of planar quadratic maps. Also we shall give evidence that support us to conjecture that any map in $\Im_1$ has infinitely many periodic points and for any map in $\Im_2$, if the critical set is a parabola then the map will have an attractor . On the other hand if the critical set is a straight line or hyperbola then the map will have a periodic points.
The degree-2 terms in a planar quadratic map play an important role as can be seen in the next lemma. But first, let us make the following definition.

Definition (1.4.2): [32]

Let T be a quadratic map defined by (1.4.1), then the mapping G, consisting of the quadratic terms of T, is called the initial form of T. i.e.
$G(x,y) = (a_0 x^2 + a_1 xy + a_2 y^2 \,,\, b_0 x^2 + b_1 xy + b_2 y^2)$.

Lemma (1.4.1):[32]

Let T be a quadratic map given by (1.4.1). If the origin is not in the image of the unit circle under the mapping G, the initial form of T, there is a positive real number K such that $|T(x,y)| > 2|(x,y)|$ whenever $|(x,y)| = \sqrt{x^2 + y^2} > K$. Hence, infinity is an attracting.

Proof:-

Write $x = r\cos\theta, y = r\sin\theta,\ \ \theta \in [0,2\pi]$, then the assumption on the image of the unit circle under the mapping G, implies that $\inf_{\theta\in[0,2\pi]} |G(\cos\theta,\sin\theta)| = \delta > 0$. Hence we have,

$$\begin{aligned}
|T(x,y)| &\geq |G(x,y)| - |(a_3x + a_4y + a, b_3x + b_4y + b)| \\
&\geq |G(x,y)| - |(a_3x + a_4y, b_3x + b_4y)| - |(a,b)| \\
&\geq r^2\delta - r|(a_3\cos\theta + a_4\sin\theta, b_3\cos\theta + b_4\sin\theta)| - |(a,b)| \\
&\geq r(r\delta - |(a_3\cos\theta + a_4\sin\theta, b_3\cos\theta + b_4\sin\theta)| - \frac{|(a,b)|}{r}
\end{aligned}$$

Since $\delta > 0$, we can choose $K > 0$ large enough such that $(K\delta - |(a_3\cos\theta + a_4\sin\theta, b_3\cos\theta + b_4\sin\theta)| - [|(a,b)|/K]) > 2,\ \forall\theta \in [0,2\pi]$. With such a $K,\ |T(x,y)| > 2|(x,y)|$ whenever $|(x,y)| > K$.♦

It is also straightforward to verify that maps either in $\Im_1$ or in $\Im_2$ have the property stated in the above lemma. Thus the only interesting dynamics of T occur on the set B_T, where
$B_T = \{x : O(x),$ forward orbit of x under T is bounded $\}$.
Since properties of set B_T when T is in $\Im_1$ are studied in [32], we shall give some properties of these without proof and then we shall try to look for these properties in a map which is in $\Im_2$. But first, let us recall the following definition.

Definition (1.4.3): [29, p.20]
A set Λ is said to have a Cantor–like structure if:
1- Λ is closed and invariant under T, i.e. $T(\Lambda) = \Lambda$,
2- $\Lambda = \{V_s, s \in \Sigma_2\} = \bigcup_s V_s$, $V_s \neq \phi, \forall s, V_s \cap V_{s'} = \phi$ for $s \neq s'$.
3- if $x \in V_s$, then $T(x) \in V_{\sigma(s)}$.
Where Σ_2 denotes the space of semi–infinite sequences on two symbols, $\Sigma_2 = \{s = (s_1s_2s_3...), s_i = 0 \text{ or } 1\}$ and σ denotes the shift map on Σ_2 : if $s = (s_1s_2s_3...)$ then $\sigma(s) = (s_2s_3s_4...)$.

Notice that condition (2) in the above definition says that the element of Λ can be considered as a collection of subsets which can be put into one–to–one correspondence with the elements of the space Σ_2. Condition (3) says that the

action of T on Λ is similar to that of the shift map. In particular, if each V_s consists of a single point then Λ is the usual Cantor set of points, and the restriction of T on Λ is conjugate to the shift map on Σ_2.
All properties of Σ_2 can be found in [11].
With the notion of Cantor–like structure, we shall give two theorems that characterize the nature of the set B_T in case T belongs to $\Im_1$.

Theorem (1.4.1): [32]
Let T be a map in $\Im_1$ with the standard form (1.4.2a). For any fixed a_1 and b_1 if $|(a,b)|$ is large enough then B_T has a Cantor–like structure.
Proof: See [32].

Conjecture (1.4.1): [32]
Let T be a map in $\Im_1$ with the standard form (1.4.2a). For any fixed a_1 and b_1 if $|(a,b)|$ is small enough then T has infinitely many periodic points. Numerical evidence suggests that it may not be dense in ∂B_T .

Theorem (1.4.2): [32]
Let T be a map in $\Im_1$ with the standard form (1.4.2b). For any fixed a_1, a_3, b_1 and b_3, if $|(a,b)|$ is large enough then B_T has a Cantor–like structure.
Proof: See [32].

Conjecture (1.4.2): [32]
Let T be a map in $\Im_1$ with the standard form (1.4.2b). For any fixed a_1 and b_1 if $|(a,b)|$ and $|(a_3,b_3)|$ are small enough, and the basin of the attracting periodic point near the origin is simply connected, and if the forward critical orbits $O(J)$ tend to the attracting periodic point, then we conclude that T has infinitely many periodic points on the basin boundary.

The set B_T is nonempty and bounded since it contains all accumulation points of the sequence of a preimage of any point that belongs to it i.e. accumulation points of the set $\bigcup_{n\geq 1} T^{-n}(p)$, where p is a point in B_T.

Also B_T is invariant i.e. $T(B_T) = B_T$.
In fact let $x \in T(B_T)$, then there is $p \in B_T$ such that $x = T(p)$. Since $p \in B_T$, therefore it has a bounded forward orbit, which implies that there exists $M > 0$

such that $\left|p^n\right| < M$, $\forall n$, i.e. $\{p^n\}_{n=1}^{\infty}$ is a bounded orbit of p, i.e. $\{T(p^n)\}_{n=1}^{\infty}$ is a bounded orbit of x therefore, $x \in B_F$ i.e. $T(B_T) \subseteq B_T$.

Let $x \in B_T$; this implies that x has a bounded orbit under T and $\{x^n, n \geq 1\} = \{T^n(x), n \geq 0\}$. Also, we conclude that $\{T^n(x), n \geq 1\}$ is a bounded orbit, but this is an orbit of $T(x)$, so $x \in T(B_T)$; this implies that $B_T \subseteq T(B_T)$
$\therefore B_T \subseteq T(B_T)$, i.e. B_T is invariant under T.

Next, the following theorems give us the nature of B_T when T is in $\Im_2$ and has the standard form (1.4.3a) or (1.4.3b). The following results are similar to (1.4.1) or (1.4.2) except that the set of critical points is not bounded. The result is new to the best of our knowledge.

Theorem (1.4.3):
Let T be a map in $\Im_2$ with the standard form (1.4.3a). For any fixed a_0, a_1, a_4 and b_1 if $|(a,b)|$ is large enough then B_T has a Cantor–like structure.

Proof:-
We claim that if $|(a,b)|$ is large enough then $|(a,b)| > K_{(a,b)}$ where $K_{(a,b)} = \inf\{K > 0 : K$ satisfies the property in lemma (1.4.1) $\}$.
We first estimate the number $K_{(a,b)}$. Let G be the initial form of T and write $x = r\cos\theta,\ y = r\sin\theta,\ \ \theta \in [0,2\pi]$, then

$$
\begin{aligned}
T(x,y) &\geq |G(x,y)| - |(a_4 y + a, b)| \\
&\geq |G(x,y)| - |(a_4 y, 0)| - |(a,b)| \\
&\geq r^2\delta - r|(a_4 \sin\theta, 0)| - |(a,b)|
\end{aligned}
$$

Where $\delta = \inf_{\theta\in[0,2\pi]} |G(\cos\theta, \sin\theta)|$. Note that δ is independent of a and b . Also note that if

$$r \geq (2 + |(\mathrm{a}_4 \sin\theta, 0)| + \sqrt{(2 + |(\mathrm{a}_4 \sin\theta, 0)|)^2 + 4|(a,b)|\delta})\big/2\delta$$

then

$$r^2\delta - r|(a_4 \sin\theta, 0)| - |(a,b)| \geq 2r$$

Hence

$$(2 + |(\mathrm{a}_4 \sin\theta, 0)| + \sqrt{(2 + |(\mathrm{a}_4 \sin\theta, 0)|)^2 + 4|(a,b)|\delta})\big/2\delta \ \geq K_{(a,b)}$$

by definition of $K_{(a,b)}$ .Hence if $|(a,b)|$ is large enough, $K_{(a,b)}$ is at most of order $|(a,b)|^{1/2}$, therefore $|(a,b)| > K_{(a,b)}$ if $|(a,b)|$ is sufficiently large.

Now choose a circle Γ centered at the origin with a radius between $K_{(a,b)}$ and $|(a,b)|$. Let D be the disk bounded by the circle Γ.Since $T^{-1}(D)$ consists of two disjoint areas inside D, therefore the set $B_T = \bigcap_{n\geq 1} T^{-n}(D)$ is nonempty and has a Cantor–like structure. ♦

From the numerical evidences that we obtained, can make the following conjecture:

Conjecture (1.4.3):
Let T be a map in $\Im_2$ with the standard form (1.4.3a). For any fixed a_0, a_1, a_4 and b_1 with $a_0 = b_1$ and small value of b, and not small value of a, T has attractor in the region $R = \{(x,y): -5.45 \leq x \leq 5.45 \ \ \& \ -2.9 \leq y \leq 1.5 \}$. Moreover, the iterates of each point in R will converge to $p = (1.8488,1.2267)$.

Theorem (1.4.4):
Let T be a map in $\Im_2$ with the standard form (1.4.3b). For any fixed a_0, a_2, a_3, a_4, b_0, b_2, b_3 and b_4 if $|(a,b)|$ is large enough then B_T has a Cantor–like structure.

Proof:
The idea of this proof is the same as theorem (1.4.3), the difference is only for choice of r, in this case we shall choose

$$r \geq (2 + |(\mathrm{a}_3\cos\theta + \mathrm{a}_4\sin\theta, \mathrm{b}_3\cos\vartheta + \mathrm{b}_4\sin\theta)| + \sqrt{(2 + |(\mathrm{a}_3\cos\theta + \mathrm{a}_4\sin\theta, \mathrm{b}_3\cos\theta + \mathrm{b}_4\sin\theta)|)^2 + 4|(a,b)|\delta)} \Big/ 2\delta \quad . \ ♦$$

Conjecture (1.4.4):
Let T be a map with a standard form (1.4.3b). For some values of $a, b, a_i^{'}s$ and $b_i^{'}s$, that give critical set as a line and the basin of the sink near the origin then we shall conclude that T has an attractor and periodic points on the basin boundary.

Chapter two

"Absorbing Areas and Invariant Areas of Two Dimensional Noninvertible Maps"

In this chapter, we shall study the structure of an absorbing areas and invariant areas generated by a noninvertible map of the plane. Here the term"area" only refers to a closed and bounded set not related to some measure.

The notion of absorbing was introduced at first by Gumowski & Mira in 1977, was developed by Kawakami & Kobayashi in 1979 and by Baraugola in 1982, 1985 [4& 29].

2.1 Definitions and General Properties

We shall start this section by definition of an absorbing area, and then we shall give some properties of this area and of an invariant areas, we shall give some properties and important results on a $(Z_0 - Z_2)$ map.

Definition (2.1.1):[24]

An area $d' \subset \Re^2$ is called an absorbing area of non-mixed type if it satisfies:

(i) $T(d') \subseteq d'$ i.e. it is either invariant, or strictly mapped into itself,

(ii) A neighborhood $U(d')$ exists such that $T(U(d')) \subseteq U(d')$, and any point $x \in U(d') \setminus d'$ has a finite rank image in the interior of d',

(iii) The boundary $\partial d'$ is made up of a finite number of segments of critical curves $LC, LC_1, \ldots, LC_k$.

From definition (2.1.1) we can conclude that an absorbing area d' is implicitly associated with the existence of an attracting set belonging to d'.

Definition (2.1.2):[29, p.187]

An area $\tilde{d}' \subseteq \Re^2$ is said to be absorbing of mixed type if it satisfies:

(i) $T(\tilde{d}') \subseteq \tilde{d}'$,

(ii) $\tilde{d}'$ is attracting, a neighborhood $U(\tilde{d}')$ exists such that $T(U(\tilde{d}')) \subseteq U(\tilde{d}')$, and almost all the points $x \in U(\tilde{d}') \setminus \tilde{d}'$ have a finite rank image in the interior of $\tilde{d}'$,

(iii) The boundary $\partial\tilde{d}'$ is made up of segments of critical curves and segments of the unstable set W^u of a saddle fixed point, or a saddle cycle (periodic point), or even segments of several an stable sets associated with different cycles.

The notion of a mixed absorbing areas was first introducing by Borugola & Cathala in 1992. In definition (2.1.2), almost all points in $U(\tilde{d}')\setminus\tilde{d}'$ have a finite rank image in the interior of $\tilde{d}'$ means any point $x \in U(\tilde{d}')\setminus\tilde{d}'$ has a finite rank image in the interior of $\tilde{d}'$ except the point x of the external segments of the stable sets W^s of the saddle points tend toward the boundary saddle point by infinite iteration.

Let s be a non-mixed area (d'), or a mixed area $(\tilde{d}')$. If s is non invariant absorbing area $T(s) \subseteq s$, then an invariant set can be obtained as:

(a) either an integer k exists such that $S = \bigcap_{j=0}^{k} T^j(s)$ is an invariant absorbing area.

or (b) $S = \bigcap_{j\geq 0} T^j(s)$ is a closed invariant absorbing set .

Definition (2.1.3):[19]

A mixed (non mixed)chaotic area d is an invariant mixed (non mixed) absorbing area with chaotic dynamics in the whole area d.

Let T be a $(Z_0 - Z_2)$ map of the plane. For such continuously differentiable maps T it is recalled that the critical curve LC separates the plane in two regions Z_0 and Z_2 such that $\bar{Z}_0 \cap \bar{Z}_2 = LC$, $\bar{Z}_0 \cup \bar{Z}_2 = \Re^2$. Like LC the curve LC_{-1} separates the plane in two open regions R_1, R_2 such that $R_1 \cap R_2 = \phi$, $\bar{R}_1 \cap \bar{R}_2 = LC_{-1}$, $\bar{R}_1 \cup \bar{R}_2 = \Re^2$. For every point $X \in Z_2$, let $T_1^{-1}(X) \in R_1$, $T_2^{-1}(X) \in R_2$ be the two first rank preimages of X.

Before we give some properties of a $(Z_0 - Z_2)$ map, we shall give two basic propositions which are preparatory of these properties.

Proposition (2.1.1):[29, p.208]

Let A be a closed subset of the plane. Then the points internal to A which can be mapped on the boundary of $T(A)$ belongs to $A \cap LC_{-1}$.

Proof:

Let p be an interior point of A, $p \notin LC_{-1}$. Since p is an interior point, there exists a neighborhood U of p such that $U(p) \subset A$, by the inverse mapping theorem we assume $T: U \to T(U)$ is one-one, U is open then $T(U) \subset T(A)$ is open i.e. $T(p)$ is interior point of $T(U)$.♦

Remark (2.1.1):

If $A \cap LC_{-1} = \phi$, then $\partial T(A) = T(\partial A)$, i.e. only points of the boundary of A are mapped on the boundary of $T(A)$.

Proposition (2.1.2):[29, p.208]

Let A be a closed subset of the plane. If ∂A is made up of points of critical segments, then also $\partial T(A)$ is made up of points of critical segments

Proof:

Let $z \in \partial T(A)$. Then z is an image of a point belonging either to the boundary of A or to the interior of A.

If z is an image of a point belonging to ∂A, then z belongs to a critical curve (as the image of critical points are critical points). If z is an image of a point belonging to the interior of A, then from prop. (2.1.1), z is the image of a point belonging to LC_{-1}, thus z belongs to $LC \cap \partial T(A)$. ♦

Remark (2.1.2):

The segments of the unstable set of a cycle are mapped by T into segments of the same unstable set.

The following proposition is a consequence of prop.(2.1.2) with remark (2.1.2). It appeared in[29, p.208] without a proof, we shall give a proof.

Proposition (2.1.3):

Let s be an absorbing area, mixed or not. Then also $T(s)$ is an absorbing area of the same type as s.

Proof:

Let s be non-mixed absorbing area. To show $T(s)$ is a non-mixed absorbing area.

Since s is an absorbing area, $T(T(s)) = T^2(s) \subseteq T(s)$ and a neighborhood $U(T(s))$ exists such that $T(U(T(s))) \subseteq U(T(s))$.

Let $x \in U(T(s)) \setminus T(s)$, to show x has a finite rank image in the interior of $T(s)$.

x is an image of a point in $U(s) \setminus s$ i.e. $\exists$ $p \in U(s) \setminus s$ such that $x = T(p)$, since s is an absorbing area, then there exists an integer k such that $T^k(p) \in s$, hence

$T^k(x) = T^k(T(p)) = T^{k+1}(p) \in s$

i.e. x has finite rank image in the interior of $T(s)$.

By using prop. (2.1.2) the boundary of $T(s)$ is made up of a finite number of segments of critical curves.

A similar proof can be used to prove the other case. ♦

The following interesting proposition is stated in [30] without proof, we shall give a proof.

Proposition (2.1.4):Let T be a $(Z_0 - Z_2)$ map

(1) $X \in Z_2 \Rightarrow T^{-1}(X) = \{T_1^{-1}(X) \in R_1,\ T_2^{-1}(X) \in R_2\}$,
(2) $X \in LC \Rightarrow T_1^{-1}(X) = T_2^{-1}(X) \in LC_{-1}$,
(3) $X \in Z_0 \Rightarrow T^n(X) \in \overline{Z}_2$, $\quad n \geq 1$,
(4) $T(\overline{Z}_2) \subset \overline{Z}_2$.

Proof:

(1) Follows directly from the definition of $(Z_0 - Z_2)$ map.

(2) We know $LC = T(LC_{-1})$ and $T^{-1}(LC) \supseteq LC_{-1}$.
Since we have two determination of the inverse map T_1^{-1}, T_2^{-1} and each inverse is defined in different region R_1, R_2 respectively then $T_1^{-1}(X) = T_2^{-1}(X) \in LC_{-1}$ where $X \in LC$.

(3) Let $X \in Z_0$, $X_1 = T^n(X)$, $\quad n \geq 1$
Then X is a preimage of rank$-n$ of X_1 .Since each point in Z_0 has no preimage therefore X_1 can not belong to Z_0 and thus $X_1 \in \overline{Z}_2$.

(4) Let $X \in Z_2$, $X_1 = T(X)$.
Recall that X is a preimage of rank-1 of X_1 and then X_1 must belong to Z_2 because the plane consists of two regions Z_0, Z_2 each point in Z_0 has no preimage, while Z_2 contains all points having two distinct preimages.♦

The following proposition gives a way of constructing an absorbing area if it exists when R_2 contains a repulsive focus.

Proposition (2.1.5):[29, p.212]

If Δ_0 is a closed subset of $\overline{R}_2$, bounded by critical curves segments and segments of LC_{-1}, then:

(1) $\Delta = \bigcup_{i=1}^{k} T^i(\Delta_0)$ is bounded by critical curves segments $\forall\ k \geq 1$;

(2) $T^n(\Delta)$ is bounded by critical curves segments $\forall\ n \geq 1$.

Proof:

(1) For $k = 1$, $\Delta = T(\Delta_0)$

We know that no internal point of Δ_0 belongs to LC_{-1}, $\partial T(\Delta_0) = T(\partial \Delta_0)$.

Thus, $\partial\Delta$ consists of arcs of critical curves i.e. (1) is holds since $\partial T^i(\Delta_0)$ made up of arcs of critical curves (from prop. (2.1.2)) for all i, $\partial\Delta \subset \bigcup_{i=1}^{k} \partial T^i(\Delta_0)$, it follows that $\partial\Delta$ is made up of arcs of critical curves, therefore (1) holds for all $k \geq 1$.
Proof of (2) follows immediately from (1) and prop. (2.1.2).

Another proof we shall use induction to prove (1) First we will prove by induction if Δ_0 is a closed set bounded by critical curves, then $T^n(\Delta_0)$ is bounded by critical curves.
For $n=1$ the statement is true by prop.(2.1.2), suppose the statement is true for $n \geq 1$ i.e. if Δ_0 is closed bounded by critical curve then $T^n(\Delta_0)$ is bounded by critical curves
Now, to show $T^{n+1}(\Delta_0)$ is bounded by critical curves, since $T^n(\Delta_0)$ is bounded by critical curves, then by prop.(2.1.2) $T(T^n(\Delta_0))$ is bounded by critical curves.
$\therefore T^n(\Delta_0)$ is bounded by critical curve $\forall\ n \geq 0$.
Now, for $k=1$, $\Delta = T(\Delta_0)$ is bounded by critical curves as show above.
For $k>1$, $\partial\Delta \subset \bigcup_{i=1}^{k} \partial T^i(\Delta_0)$. Hence by prop.(2.1.2) $\partial\Delta$ consists of arcs of critical curves. ♦

Corollary(2.1.1):[16]

Let Δ_0 be a bounded area whose boundary consist of arcs of critical lines and $\Delta_k = \bigcup_{i=0}^{k} T^i(\Delta_0)$. If there exists an integer m such that $T(\Delta_m) \subseteq \Delta_m$, then Δ_m is an absorbing area.

Proof:

From the assumptions it follows that $\partial\Delta_k$ consists of arcs of critical segments because this is true for $\partial\Delta_0$, and the consequents of critical segments are critical segments ; moreover , there exists a basin of attraction D of Δ_k, as the points belonging to $\Delta_k \setminus T(\Delta_k)$ have antecedents outside Δ_k.♦

It is possible to restrict the analysis of asymptotic behaviors of a sequence of images with increasing rank, to the points of an absorbing region. This is shown by the next proposition which is stated in [29, p.213] without proof, we shall give a proof.

Proposition (2.1.6):[Elimination process]

Consider a $(Z_0 - Z_2)$ map then

(1) either $T^{k+1}(\overline{Z}_2) \subset T(\overline{Z}_2)$ for any $k > 0$, which permits to define

$$V = \bigcap_{k>0} T^k(\overline{Z}_2);$$

(2) or there exists a finite j such that $T^{j+1}(\overline{Z}_2) = T^j(\overline{Z}_2)$ which permits to define

$$V_j = T^j(\overline{Z}_2).$$

In both cases V and V_j are invariant absorbing regions. The boundary of each of V and V_j is made up of a finite number of critical curve segments

Proof:

Case (1) to show $T(V) \subseteq V$

$$T(V) = T(\bigcap_{k>0} T^k(\overline{Z}_2))$$

$$= \bigcap_{k>0} T^{k+1}(\overline{Z}_2) \subseteq \bigcap_{k>0} T^k(\overline{Z}_2)$$

i.e. V is an invariant region.

From prop. (2.1.2) V is bounded by segments of critical curves.

Case (2)

$$T(V_j) = T(T^j(\overline{Z}_2))$$

$$= T^{j+1}(\overline{Z}_2)$$

$$= T^j(\overline{Z}_2) = V_j$$

i.e. V_j is an invariant region, by prop. (2.1.2), ∂V_j is made up of segments of critical curves.

2.2 Properties of Absorbing Areas & of Invariant Areas

In this section we shall study some of the properties of absorbing and invariant areas for a $(Z_0 - Z_2)$ maps. The following proposition give some of the properties of the absorbing area s of either non mixed type, or mixed type, invariant, or non invariant it appeared in [29, p.220] without a proof, we shall give a proof.

Proposition(2.2.1):

Let s be an absorbing area for a $(Z_0 - Z_2)$ map T. If $Z_0 \neq \phi$ then $s \cap Z_0 = \phi$.

Proof:

It is known that each point in region Z_0 has no preimage, from the definition of s each point in s has a finite rank preimages therefore there is no common point between s and Z_0 i.e. $s \cap Z_0 = \phi$.♦

Proposition (2.2.2):[17]

Let T be as before, and let $T(S) = S$ then

(1) any point $p \in S$ has at least one infinite sequence of preimages in S;

(2) any point $p \notin S$ has all its preimages of rank $1,2,\ldots,$ out of S.

Proof:- The proof is immediate.

Let A be a closed subset of the plane. A can be simply connected set, or a multiply connected one, a non-connected one, invariant or not, absorbing or not, mixed or not.

If $A \cap LC_{-1} = \phi$, then the restriction $T|A$ of T is a map with a unique local inverse in A. Thus, to characterize properties due to the non uniqueness of the inverse of T in A, it will be assumed $A \cap LC_{-1} \neq \phi$.

Definition (2.2.1) :[4]

The largest subset of $A \cap LC_{-1}$ mapped by T on the boundary of $T(A)$, is denoted by g_{-1}, i.e. $g_{-1} \subseteq LC_{-1} \cap A$, thus $g = T(g_{-1}) \subseteq \partial T(A) \cap LC$.

The above definition is constructive, because from the set $A \cap LC_{-1}$ the points not mapped on the boundary of $T(A)$ are eliminated, i.e. the points mapped in the interior of $T(A)$. We shall refer to g as "arc g" whichever is its structure.

The following result appeared in [29, p.222] .

Proposition (2.2.3):

Let T be a $(Z_0 - Z_2)$ map, and A be a closed subset of the plane. Then $T(A \cap LC_{-1}) = T(A) \cap LC = \partial T(A) \cap LC$

Proof:

$T(A) \cap LC = \partial T(A) \cap LC$ is immediate.

Let $p \in A \cap LC_{-1}$, then $T(p) \in T(A) \cap LC$ is obvious .i.e. $T(A \cap LC_{-1}) \subseteq T(A) \cap LC$.

Let $p_1 \in T(A) \cap LC$ then p_1 is the image of at least one point of A, but it is also a point of LC, thus it has only two coincident rank-one preimages in a point belonging to LC_{-1}. Thus $p_1 = T(p)$, where $p \in A \cap LC_{-1}$, i.e. $T(A) \cap LC \subseteq T(A \cap LC_{-1})$

$\therefore T(A \cap LC_{-1}) = T(A) \cap LC$ ♦

Remark (2.2.1):

(1) Proposition (2.2.3) when applied to a closed invariant set A, it follows that $g = \partial A \cap LC$ is bounded by a finite number of critical segments, and g is called generating arc of ∂A.

(2) From prop. (2.1.1) we can deduce that the preimages of boundary points of $T(A)$ not belonging to g, are boundary points of A. because no interior points of A can be mapped on the boundary of $T(A)$ in points not belonging to LC. Thus another proposition can be stated (it appeared in [29, p.223] without proof), we shall give a proof.

Proposition (2.2.4):

let A be a closed and $p \in \partial T(A) \setminus g$ then all the rank-one preimages of p in A must belong to ∂A.

Proof:

Let $p \in \partial T(A) \setminus g$, then $p \notin LC$ {since $g = T(g_{-1}) \subseteq \partial T(A) \cap LC$ }

So $T^{-1}(p) \notin T^{-1}(LC) = LC_{-1} \cup \bar{L}C_{-1}$ and hence $T^{-1}(p) \notin A$, i.e. $T^{-1}(p)$ is not an interior point of A.

Thus by prop. (2.1.1) all rank-one preimages of p in A belong to ∂A i.e. $T^{-1}(p) \subseteq \partial A$.♦

We remark that propositions (2.2.3)-(2.2.4) hold for any closed area A and for any invariant area with a finite boundary or not, absorbing or not, mixed or not.

The next proposition may be used to characterize more explicitly the boundary point of an invariant area S which appeared in [15] without proof, we shall give a proof.

Proposition (2.2.5):

Let S be a closed set, $T(S) = S$, and $p \in \partial S$ then

(1) either a finite k , $k \geq 0$ exists such that $p \in T^k(g) \subset LC_k$;

(2) or $T^{-n}(p) \cap S \subset \partial S$, $\forall\ n \geq 0$.

Proof:

Let $p \in \partial S$ then either $p \in LC$ or $p \notin LC$.

(1) If $p \in LC$ then $p \in LC \cap \partial S$ i.e. $p \in g$ since g is generating arc of ∂S {remark (2.2.1)}, then there exists a finite integer k such that $p \in T^k(g) \subset LC_k$ if $p \notin LC$, the proof follows by prop.(2.2.4).

(2) The proof follows directly from prop.(2.2.2). ♦

The next proposition justifies the fact that g (or equivalently g_{-1}) is called "generated arc". It appeared in [29, p.224] without proof, we shall give the proof .

Proposition (2.2.6):

Let S be a closed area with finite boundary, $T(S) = S$ and $L_k \in \partial S$ a segment of critical curve LC_k, $k \geq 0$. Then its critical preimages $L_{k-1}, \dots, L_1,\ L_0 = L \subseteq g$ also belong to the boundary ∂S.

Proof:

We know $g = T(g_{-1}) \subseteq \partial T(S) \cap LC = \partial S \cap LC$ i.e. g belongs to the boundary ∂S. The critical preimages $L_{k-1}, \dots, L_1$ belong to the boundary ∂S by prop.(2.1.1).

Remark (2.2.2):[29, p.224]

When we apply prop.(2.2.6) to a non mixed invariant area S with a finite boundary an integer M exists such that $\partial S \subset \bigcup_{i=0}^{M} T^i(g) = \bigcup_{i=0}^{M} T^{i+1}(g_{-1})$, where $T^i(g)$ is a critical segment that belongs to LC_i. when we applying this proposition to a mixed invariant area (i.e. its boundary is made up of a finite number of critical segments and segments of saddle unstable set) a finite integer M exists such that all the critical segments on ∂S belong to $\bigcup_{i=0}^{M} T^i(g) = \bigcup_{i=0}^{M} T^{i+1}(g_{-1})$, while a segment of ∂S belonging to some saddle unstable set W^u has at least infinite sequence of its preimages on W^u belonging to the boundary of S which converge toward a saddle cycle.

The extra preimages of the critical segments $L_k, \dots, L_1, L_0$ (i.e. non critical preimages of critical segments) on the boundary of S cannot belong to the interior of S. Moreover when S is a non-mixed area, the extra preimages cannot belong to the boundary of S, thus the extra preimages of such segment must be out of S, except at most isolated points on ∂S which may be critical points. The following example illustrates this situation.

Example(2.2.1):[29, p.192]

Consider a map, predator prey model of two species

$$x' = xe^{b(1-.1x)-ay}$$
$$y' = x(1-e^{-ay})$$

with $a = 0.15b\left[1-e^{-.6b}\right]^{-1}$, theoretical studies of which can be found in [24].
The curve LC_{-1} is defined by $x = 0$ or $x = \frac{10}{b}e^{ay}$, the critical curve LC separates the (x, y) plane into two regions Z_0 and Z_2 and whose equations is:

$$y = 10\left[b-1-\ln(0.1bx)\right]/b$$

when $b = 2.63$, a self intersection of LC_1 occurs at a point h_1. The qualitative figure (2.2.1) represents the above situation.

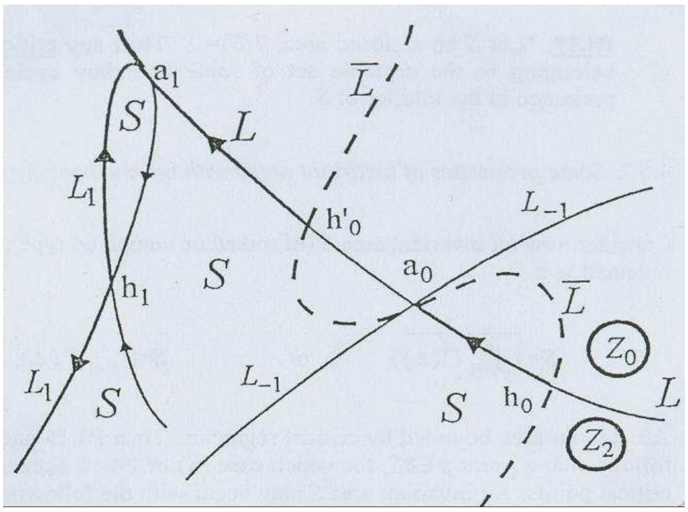


Fig.(2.2.1) $\overline{L}$ segment of extra preimages, L segment of critical curve, L_{-1} segment of preimages. $\overline{L}$ are two non-connected segments belonging to $\overline{Z}_0$ not belong to the boundary of S (the two segments of $\partial S \cap LC_1$ drawn with thicker lines), $LC \cap \overline{L} = a_0 \cup h_0 \cup h_0'$.

Proposition (2.2.7):[29, p.242]

Let S be an invariant area with a finite boundary. Then ∂S is not be an invariant.

Proof:

Let $p \in \partial S$ so $\exists\ q \in g$ such that $p = T^M(q)$ {from remark (2.2.2)}
$T(p) = T^{M+1}(q)$ which is in S i.e. $T(p) \notin T(\partial S)$
$\therefore T(\partial S) \neq \partial S$.♦

2.3 Construction of Absorbing Areas & of Invariant Areas

We studied in the previous section some properties of absorbing areas and invariant areas. In this section we shall study the construction of absorbing areas, invariant areas from bounded areas whose boundaries are made up of segments of the critical curves with finite rank.

Our goal of this section is to give an algorithms for constructions of absorbing areas, invariant areas for a noninvertible map of type a$(Z_0 - Z_2)$ or equivalent to such maps. This algorithm is described in [19, 29; p.191 & 35].

2.3.1 Construction Algorithm of Absorbing Areas

The structure of this algorithm depends on the use of the critical curves to obtain closed bounded regions (will be denoted by Δ) whose boundary consists of segments of critical curves LC_i, $i = 0,1,2,\ldots,N$ (N is finite integer), the such area is an absorbing .

First we suppose that the first rank critical curve LC and the curve LC_{-1} of merging preimages are made up of only one branch, these two curves having only one point of intersection say a_0. When these two curves intersection in more than one point, one of the them plays the rule of a_0.

We will adopt the following notation:

A segment of curve will be represented by $(\alpha\,\beta)$ where α , β are the two endpoints. The point a_n represents the nth iterate of a_0 i.e. $a_n = T^n(a_0)$.

Now, we are ready to describe the algorithm of construction as:

Let N be the first integer >0, such that the segment $(a_N a_{N+1})$ of LC_N (LC_N critical curve of rank$-N$) intersects LC_{-1} at a point say b_0, i.e. $b_0 \in (a_N a_{N+1}) \cap LC_{-1}$.

Then, define a simply connected area Δ bounded by

$$\partial\Delta = (b_1 a_1 a_2 ... a_N a_{N+1} b_1)$$

where

$(b_1 a_1)$ is a segment of LC , i.e. $(b_1 a_1) \subset LC$,

$(a_1 a_2)$ is a segment of LC_1, i.e. $(a_1 a_2) \subset LC_1$,

.

.

.

$(a_N a_{N+1})$ is a segment of LC_N , i.e. $(a_N a_{N+1}) \subset LC_N$

$(a_{N+1} b_1)$ is a segment of LC_{N+1}, i.e. $(a_{N+1} b_1) \subset LC_{N+1}$

$b_1 = T(b_0), a_i = T(a_{i-1})$

so, we get Δ is an absorbing areas.

Algorithm (2.3.1) may not work for certain examples, and it dose not include all possible cases of absorbing areas with a finite boundary (i.e. boundary made up of a finite number of critical segments) i.e. it may happen this is not absorbing area , that will be seen in the illustrative examples.

Remark (2.3.1):

1- If there are more than one point of intersection between the segment $(a_N a_{N+1})$ and LC_{-1} we choose the point b_0 that is farthest from a_0 .

2- When the above algorithm work, then we distinguish two possible cases for b_1:

(i) $b_1 \notin (a_0 a_1) \subset LC$ or equivalently $b_0 \notin (a_{-1} a_0) \subset LC_{-1}$.

(ii) $b_1 \in (a_0 a_1) \subset LC$ or equivalently $b_0 \in (a_{-1} a_0) \subset LC_{-1}$.

Let T be a map of type a$(Z_0 - Z_2)$, recall that LC_{-1} divides the plane $\Re^2$ into two open regions R_1, R_2 such that $R_1 \cap R_2 = \phi$, $\overline{R}_1 \cap \overline{R}_2 = LC_{-1}$, $\overline{R}_1 \cup \overline{R}_2 = \Re^2$. Let φ be a fixed point of T with $\varphi \in R_2$. $a_0 \in LC_{-1} \cap LC$. Then one of the following cases is possible:

(1) None of the successive images of the segment $(a_0 a_1)$ intersect LC_{-1} .

In this case we can obtain an absorbing area as follows:

In this situation one of the two inverse map T_2^{-1} gives rise to a rank$-m$ preimage $(a_{-m}a_{-m+1})$ of (a_0a_1) which intersects LC_{-1} a first time at the point say h_0 fig.(2.3.1a).

Necessarily $(a_0h_0)\supset(a_0a_1)$, thus $T^i(a_0h_0)=(a_ih_i)\supset T^i(a_0a_1)$ for $i>0$, so $h_i\in LC_i$, $h_{m-1}\in LC_{-1}$ and $h_m\in LC$ fig.(2.3.1b). Now applying construction algorithm (2.3.1) to obtain an absorbing area Δ boundary by the closed curve $\partial\Delta=(h_ma_1a_2...a_mh_m)\subset R_2$ where (h_ma_1) is a segment of critical curve LC, $(a_mh_m)\subset LC_m$, $(a_ia_{i+1})\subset LC_i$, $i=1,2,...,m-1$.

Figure (2.3.1) illustrate this situation.

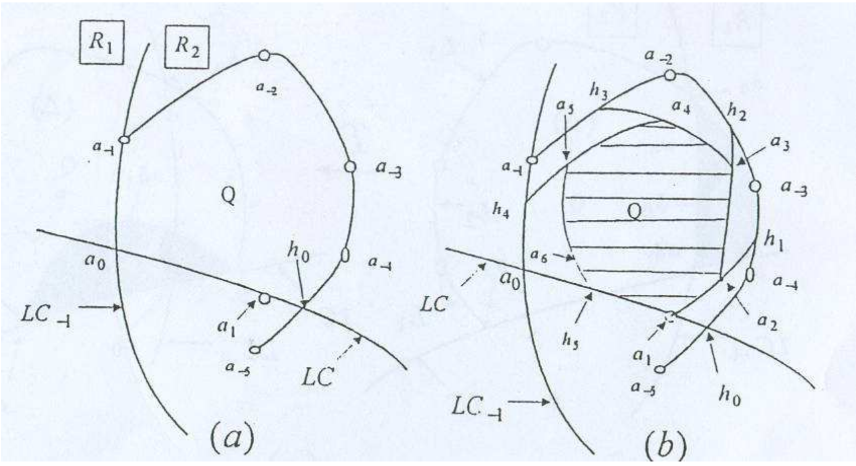


Fig.(2.3.1) Non of the successive image of the segment (a_0a_1) intersect LC_{-1}.

(2) One of the images of (a_0a_1) has a non transverse contact (order one, or order zero) with LC_{-1}.

In this case, let $b_0=(a_Na_{N+1})\cap LC_{-1}$ be the non transverse contact point. An absorbing area Δ is defined as in the construction algorithm and the boundary $\partial\Delta$ is $\partial\Delta=(b_1a_1a_2...a_Na_{N+1}b_1)\subset\overline{R}_2$ with $\partial\Delta\cap LC_{-1}=b_0$.

(3) An image of (a_0a_1) has a transverse intersection with LC_{-1} . Let N be the least integer such that (a_Na_{N+1}) intersects LC_{-1} .let $b_0\in(a_Na_{N+1})\cap LC_{-1}$ be the intersection point farthest from a_0 .So we apply the construction algorithm to obtain absorbing area , the two possible situation appear in remark (2.3.1) may occur .

2.3.2 Determination of Invariant Areas

Recall that when we apply the construction algorithm to construct an absorbing area Δ the two possible situation that appear in the remark (2.3.1) occur :

(i) $b_1 \notin (a_0 a_1) \subset LC$, or equivalently $b_0 \notin (a_{-1} a_0) \subset LC_{-1}$.

(ii) $b_1 \in (a_0 a_1) \subset LC$, or equivalently $b_0 \in (a_0 a_1) \subset LC_{-1}$.

The two situation are different because in case (i) $T(\Delta) \supseteq \Delta$ while in case (ii) $T(\Delta) \subset \Delta$, or $T(\Delta)$ is not comparable with Δ. i.e. $T(\Delta)$ is neither include in Δ nor includes Δ . In both cases (i) and (ii) ,Δ may be absorbing ,or not.

In fact, in the case (i) $\partial\Delta$ Intersects LC in two points b_0 and a_0,
i.e. $(b_0 a_0) = LC_{-1} \cap \Delta$ and $(b_0 a_0) \supset (a_{-1} a_0)$. Then under application of T the whole boundary $\partial\Delta$ is construct again and new parts may only come from $T(\Delta \cap \overline{R}_1) = T(\delta_0) = \delta_1$, thus if $T(\Delta \cap \overline{R}_1) \subset \Delta$ then the area Δ is invariant , $T(\Delta) = \Delta$ as in fig.(2.3.2a), while if $T(\Delta \cap \overline{R}_1)$ is not include in Δ, $T(\Delta) \supset \Delta$ fig.(2.3.2b), in this case $T^{m+1}(\Delta) \supseteq T^m(\Delta)$, $\forall\ m \geq 0$. Thus, either a finite integer M exists such that $T^{M+1}(\Delta) = T^M(\Delta)$, so we get $d'' = T^M(\Delta)$ is invariant areas, or a finite M does not exist , in which case we define

$$d'_\infty = \overline{U_{j=1}^{\infty} T^j(\Delta)} \qquad (2.3.1)$$

the area d'_∞ may be bounded or not. when it is bounded, it may be absorbing or not, and generally this situation denote a bifurcation resulting from the contact of the area boundary with its basin boundary [17, 20 &21].

In case (ii) $\partial\Delta$ intersects LC_{-1} in two points b_0 and c_0, where $c_0 \in (b_0 a_0)$ i.e. $(b_0 c_0) \subset (b_0 a_0)$ so that the boundary of Δ include the segment $(b_1 a_1)$, while $T(\Delta) \cap LC = (b_1 c_1) \subset (b_1 a_1)$, from which it appears that $T(\Delta) \supseteq \Delta$ is not possible . It follows that either $T(\Delta) \subset \Delta$ fig.(2.3.2c) or $T(\Delta)$ is not comparable with Δ fig.(2.3.2d).

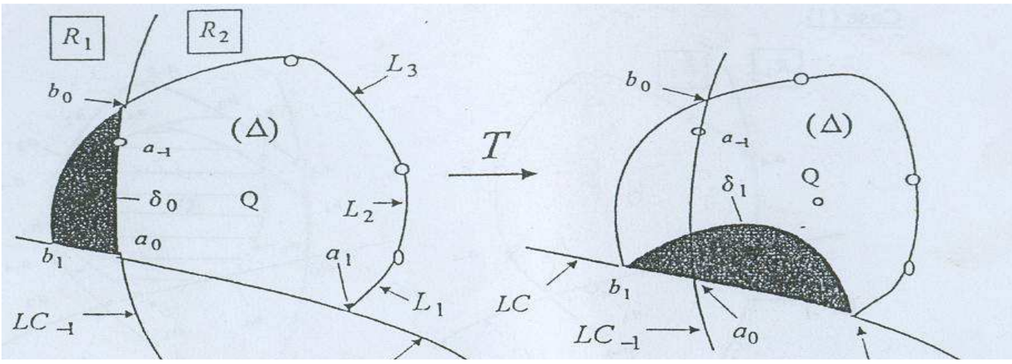
When $T(\Delta) \subset \Delta$ then $T^{m+1}(\Delta) \subseteq T^m(\Delta)$, $m \geq 0$, so that either a finite integer M exists such that $d'' = T^M(\Delta)$ is invariant or a finite M does not exists, in which case we define

$$d'_\infty = \bigcap_{j=1}^{\infty} T^j(\Delta) \qquad (2.3.2)$$

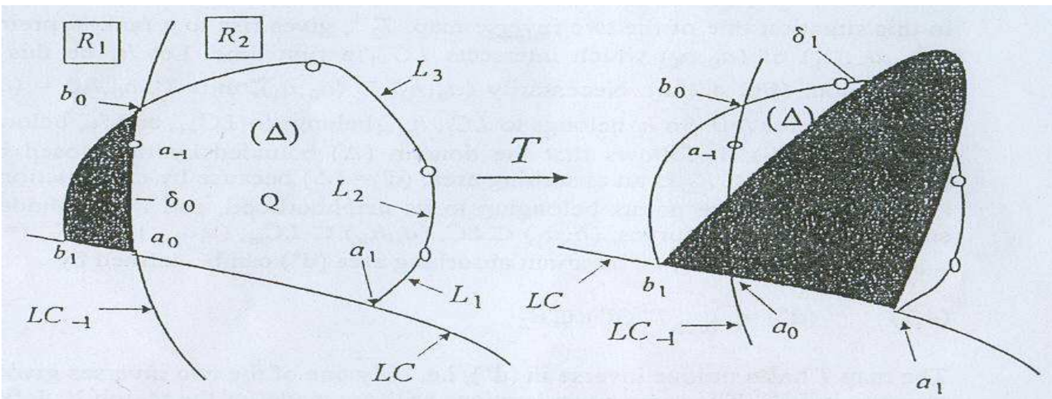
Since each area $T^k(\Delta)$, $k \geq 0$ is absorbing , then d'_∞ is bounded and absorbing. Case (ii) is more complex when $T(\Delta)$ is not comparable with Δ . In the simplest case a finite M exists such that $d'' = T^M(\Delta)$ is invariant . However it may occur that a finite M does not exist, this situation is more complex, it is possible to define

$$A_\infty = \overline{U_{j=1}^{\infty} T^j(\Delta)} \qquad (2.3.3)$$

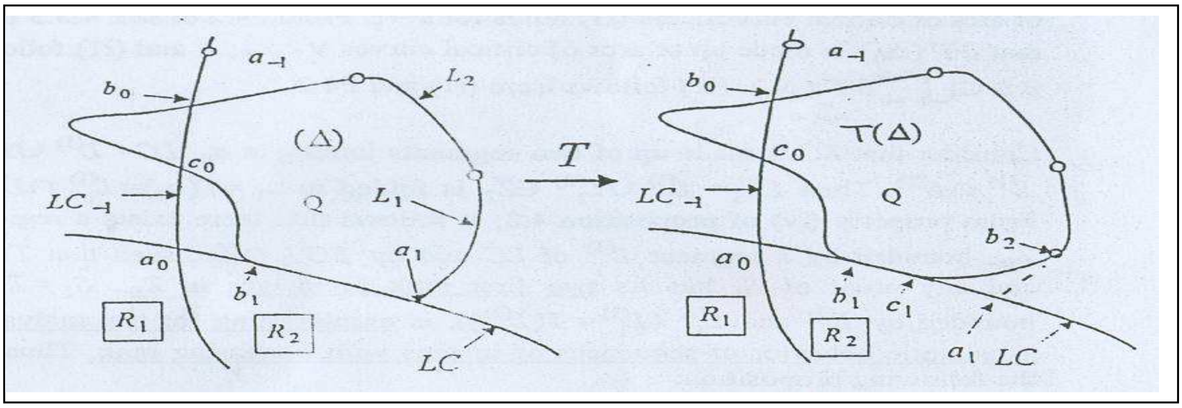
this area may be bounded or not. If it is bounded, it may be invariant or not . When it is not invariant , if there exist k such that $\bigcap_{j=1}^{k} T^j(A_\infty) = T^k(A_\infty)$, then this intersection is an invariant area. If there is no such k then $\bigcap_{j=1}^{\infty} T^j(A_\infty)$ is an invariant.



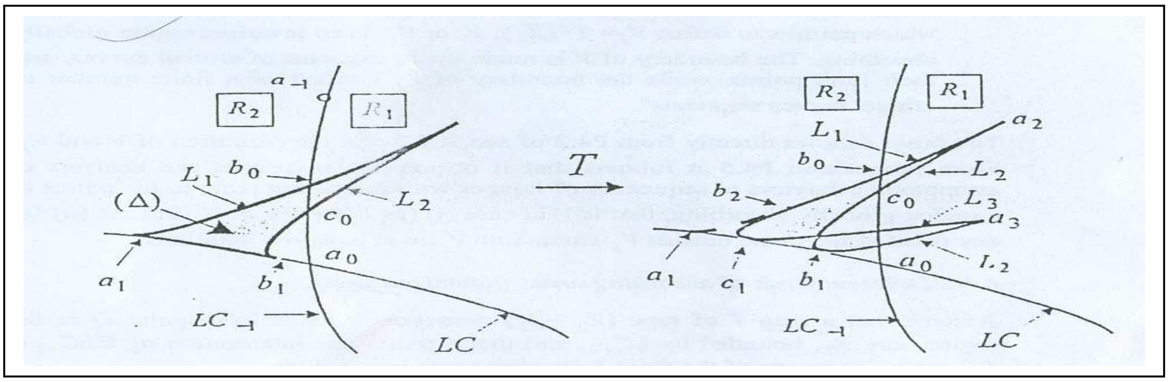
(a)



(b)



(c)



(d)

Fig(2.3.2) An absorbing area constructed by algorithm (2.3.1) (a): $b_1 \notin (a_0 a_1)$, $T(\Delta)=\Delta$, (b) $b_1 \notin (a_0 a_1)$, $T(\Delta)\supset\Delta$, (c) $b_1 \in (a_0 a_1)$, $T(\Delta)\subset\Delta$, (d) $b_1 \in (a_0 a_1)$, $T(\Delta)$ is not comparable with Δ.

2.4 Bifurcation

The term bifurcation generally refers to something " splitting a point " with general a system involving a parameter, it refers to change in the character of the solution as the parameter is changed continuously .

At the beginning of this section we shall give the definition of bifurcation followed by some types of bifurcation that be interest in our work , illustrative example will be given in chapter three .

Definition (2.4.1):[23]

Consider the system $x_{n+1} = f_\lambda(x_n); x \in \Re^n, \lambda \in \Re^k$ (2.4.1)

one is especially concerned how the phase portrait of (2.4.1) changes as λ varies. A value λ_0 where there is a basic structural change in this phase portrait is called a bifurcation point.

2.4.1 Some Types of Bifurcation

1-contad bifurcations: [17, 20& 25]

This basic bifurcation results from the contact of a basin boundary with a critical curve segment not belonging to a chaotic area boundary, also it occurs when a critical curve belonging to a chaotic area boundary. Such a bifurcation leads either to the chaotic area destruction, or to sudden and important modification of this area. Even if S is an absorbing area, the contact bifurcation may occur.

2- Bifurcation of non smoothness points on boundaries of invariant areas:[29, p.231]

Consider a smooth map T, area called Δ constructed by the algorithm (2.3.1), and there is an integer m such that $S = T^m(\Delta)$ is an invariant area with $S \cap LC_{-1} \neq \phi$, being finite i.e. ∂S made up of a finite number of critical segments. Let $p \in \partial S \setminus g$ the non smoothness of ∂S may correspond to one of the following cases:

Case 1: Before and after the bifurcation the contact between ∂S and $T(\partial S)$ on LC at an endpoint of the generating segment g is smooth. At the bifurcation the contact on LC in not smooth, due to cusp point of LC_1.

Case 2: A point of non smoothness of the S boundary may also born when a self intersection of a critical arc L_K occur, at $p \in L_K$, a point p is called a double point of L_K .

Case 3: A point of non smoothness of the S boundary may also created at p when two critical segments of different ranks are intersect i.e. $p \in LC_k \cap LC_j$, $k \neq j$, p is called an angular point .

When the boundary of S is smooth at p, p is said to be an ordinary point of ∂S.

Also, there is other types of bifurcations when the nature (stability) of a fixed point changes as some controls parameter change. The following definitions show these types.

Definition (2.4.2):[29, p.63]

Let T be a noninvertible map depending on a parameter λ. A SBR bifurcation value of a fixed point is a value λ^* of the parameter λ, such that $\lambda < \lambda^*$ for (respectively $\lambda > \lambda^*$) the fixed point is expanding but not SBR; for $\lambda > \lambda^*$ (respectively $\lambda < \lambda^*$) the fixed point is a SBR.

Definition(2.4.3):[14]

A homoclinic bifurcation (or homoclinic explosion) of a fixed point p^* occur for a parameter value $\lambda = \lambda^*$, if crossing the value λ^* infinitely many homoclinic points of p^* appear (or disappear).

2.5 Multiply Connected (non mixed) Invariant Absorbing Areas . Bifurcation of Annular Absorbing Area.

This section is concerned with the characterization of multiply connected invariant areas with holes surrounding a repulsive node or focus.

Let T be a $(z_0 - z_2)$ map and d'' be a connected non mixed invariant absorbing area with finite boundary such that $d'' \cap LC_{-1} \neq \phi$. Let φ be an expanding fixed point of T . $\varphi \in R_2$, so $T_2^{-1}(\varphi) = \varphi$ and $\varphi_{-1} = T^{-1}(\varphi) \in R_1$

Define

$$\delta_0 = d'' \cap \overline{R}_1 \text{ and } d'_a = \bigcup_{n=1}^{N} T^n(\delta_0) = \delta_1 \cup \delta_2 \cup .... \cup \delta_N \qquad (2.5.1)$$

where N is the least integer such that $\bigcup_{n=1}^{N+1} T^n(\delta_0) \subseteq \bigcup_{n=1}^{N} T^n(\delta_0)$ or equivalently $\delta_{N+1} \subset \bigcup_{n=1}^{N} \delta_n$ where $\delta_n = T^n(\delta_0)$ such a finite N exists because d'' is assumed to have a finite boundary .

So, there are three cases that may be distinguished :

(i) $T(d'_a) \subset d'_a$

(ii) $T(d'_a) = d'_a \subset d''$

(iii) $T(d'_a) = d'_a = d''$

When case (i) or (ii) occur , then the set defined by $W = d'' \setminus d'_a$ is nonempty. W is called the hole surrounding the repulsive fixed point φ and $d'' = W \cup d'_a$ [9 & 29, p.277].

The external boundary $\partial_e d'_a$ of d'_a is defined as the boundary of d'', $\partial_e d'_a = \partial d''$ and he internal boundary $\partial_i d'_a$ of $W : \partial_i d'_a = \partial d'_a \setminus \partial_e d'_a = \partial W$.

From the definition of d'_a ,one has:

$$\varphi_{-1} \notin d'' \Leftrightarrow \varphi_{-1} \notin \delta_0 \Leftrightarrow \varphi_{-1} \notin d'_a .$$

Here, we will characterize the bifurcation which leads to the transition between situation (i) and (ii) (given by props.(2.5.1)) and the transition between the situation (ii) and (iii) (given by props.(2.5.2)). But before we give these propositions we need the following definition:

Definition(2.5.1):[1, 10 &36]

An annular absorbing area is an absorbing area of annular shape, that is a simply area deprived of the point of a hole in it is interior.

Proposition (2.5.1):[29, p.281]

Consider d'_a defined as is (2.5.1) and $W = d'' \setminus d'_a$.

$$T(d'_a) \subset d'_a \text{ iff } W \cap LC_{-1} \neq \phi .$$

Proof:-

If $W \cap LC_{-1} \neq \phi$, then $W \cap \overline{R}_1$ belongs to δ_0 but not to $d'_a \cap \overline{R}_1$.

It follows that the generating segment of d'', $d'' \cap LC_{-1}$, is wider than $d'_a \cap LC_{-1}$ which implies $T(d'' \cap LC_{-1}) \supset T(d'_a \cap LC_{-1})$,

i.e. $[d'' \cap LC\ ] \supset T(d'_a) \cap LC$.

Observing that by the construction $d'' \cap LC = \delta_1 \cap LC = d'_a \cap LC$, therefore $(d'_a \cap LC) \supset T(d'_a) \cap LC$, which means $T(d'_a) \subset d'_a$.

Let $T(d'_a) \subset d'_a$, assume that $W \cap LC_{-1} = \phi$, then $d'_a \cap LC_{-1} = d' \cap LC_{-1}$
i.e. the generating segment of d'' belongs to d'_a . Then the external boundary of d'_a cannot be reduced, so that $T(d'_a) = d'_a$ which is contradicts the assumption .
So $W \cap LC_{-1} \neq \phi$. ♦

Recall: $\varphi_{-1} = T^{-1}(\varphi) \in \Re_1$ .The transition from situation (i) (annular invariant area d'_a) to (ii) (simply connected area d'') is characterized by the following proposition.

Proposition (2.5.2):[29, p.285]
A hole W containing φ exists $\Leftrightarrow \varphi_{-1} \notin d''$. Or equivalently a hole W containing φ does not exist $\Leftrightarrow \varphi_{-1} \in d''$.
Proof:
Let $\varphi_{-1} \in d''$, then $\varphi_{-1} \in \delta_0$ and $\varphi \in \delta_i$ for $i = 1,2,....,N$, which means no hole surrounding φ can exist; $W = \phi$ and $\varphi \in d'_a = d''$. C!
Let W does not exist, then φ must belong to some region δ_i for $i \geq 0$ and this can occur only if $\varphi_{-1} \in \delta_0$ i.e $\varphi_{-1} \in d''$.♦

When W exist, its points satisfy the following property stating that both the rank –one preimages of W are out of d'_a .

Proposition(2.5.3):[29, p.286]
Let $W \neq \phi$ be a hole. Then
(i) $T^{-1}(W) \cap d'_a = \phi$;
(ii) $T_2^{-1}(W) \subset \Re_2 \cap W$;
(iii) $T_2^{-1}(W) \cap W = \phi$.

Proof:
(i) Let $p \in W$ then

$$T^{-1}(p) = T_1^{-1}(p) \cup T_2^{-1}(p),\ T_1^{-1}(p) \in R_1,\ T_2^{-1}(p) \in R_2. \ .$$

If $T_1^{-1}(p) \in d'_a$ then $p \in T(d'_a) \subset d'_a$.
Also if $T_2^{-1}(p) \in d'_a$ then $p \in T(d'_a) \subset d'_a$. .
i.e. $p \in d'_a$ which is contradicts the assumption
None of the preimages of p belong to d'_a. Thus (i) is proved.

(ii) Let $p \in W$ and considering $T_2^{-1}(p)$.
Suppose $T_2^{-1}(p) \notin R_2 \cap W$, so $T_2^{-1}(W)$ must contain φ. Let η be a continuous path contained in W, connecting φ to p i.e. $\eta \cap d'_a = \phi$.
Then $T_2^{-1}(\eta)$ is a continuous path, connecting $T_2^{-1}(\varphi)$ and $T_2^{-1}(p)$, since $T_2^{-1}(\varphi) = \varphi \in W$ and $T_2^{-1}(p) \in R_2$ out of d'_a (thus out of d''), therefore $T_2^{-1}(\eta)$ must intersect d'_a, i.e. η must intersect d'_a which is contradicts the assumption . Thus (ii) is proved.
(iii) is proved similarly as (ii).
$T_1^{-1}(W)$ must contain φ_{-1}(external to d'') , which implies that $T_1^{-1}(W)$ cannot belong to $W \cap R_1$, and thus must be in R_1 out of d''. ♦

Remark(2.5.1):

1- From properties (i) and (ii) in propos.(2.5.3) we can conclude that $T_1^{-1}(w) \cap d'' = \phi$.
2- The global unstable set of φ can be defined by

$$W^u(\varphi) = \bigcup_{n \geq 0} T^n(U)$$

And the global stable set of φ is $W^s(\varphi) = \bigcup_{n \geq 0} T^{-n}(\varphi)$

where U is a neighborhood of φ .

The following propositions gives some properties of W.

Proposition(2.5.4):[17 &19]

$$W \subset W^u(\varphi) \Leftrightarrow \varphi = \bigcap_{n \geq 0} T_2^{-1}(W)$$

Proof:-
If $W \subset W^u(\varphi)$, then any point p of W must have at least a sequence of preimages tending toward φ .
From propos.(2.5.3) only the successive application of T_2^{-1} give this property. The converse is obvious. ♦

Proposition(2.5.5):[17 & 19]
Let $W \subset W^u(\varphi)$. Then no cycle, except φ , can belong to W .
Proof:-
The proof is immediate consequence of propos.(2.5.4). ♦

Proposition(2.5.6):[17 & 19]

Let $W \neq \phi$. Then no homoclinc orbit of φ can exist.

Proof:-

The unstable set of φ , $W^u(\varphi) = \bigcup_{n \geq 0} T^n(U)$ necessarily belongs to d''.

The stable set of φ consists of φ_{-1}, external to d'' (by props.(2.5.2)), and all the preimages of φ_{-1}, also external to d'' (by invariance of d'').

Then when W exists, $W^u(\varphi) \subseteq d''$ while $[W^s(\varphi) \setminus \varphi] \cap d'' = \phi$ so that $[W^u(\varphi) \cap W^s(\varphi)] \setminus \varphi = \phi$. ♦

The following proposition appeared in [29, p.288] without proof, we shall give the proof.

Proposition(2.5.7):

Let $d'' = W^u(\varphi)$. Then the bifurcation $\varphi_{-1} \in \partial d$ which causes the disappearance of a hole W surrounding φ , is the SBR bifurcation of φ.

Proof:-

The boundary of d'' is made up of a finite number of critical segments belonging to the image of generating segments $g_{-1} = LC_{-1} \cap \partial d''$. At the bifurcation $\varphi_{-1} \in \partial d''$ implies φ_{-1} is a critical point. Thus φ_{-1} has a finite rank preimage (say q) belonging to g_{-1}, i.e. $T^i(q) = \varphi_{-1}$ for integer i , $T^{i+1}(q) = \varphi$. At least one infinite sequence of preimages of q exist in d'' (since d'' is invariant)and if a sequence of preimages of q exist , tending toward φ, then a homoclinic orbit exist, i.e. $\varphi_{-1} \in W^u(\varphi) \cap W^s(\varphi)$. ♦

We will introduce the notion of area of branching points to distinguish the sequence of preimages in d'' related to a given point of d'' in the following definition.

Definition(2.5.2):[29, p.288]

Let S be an invariant area generated by a $(z_0 - z_2)$ map. The area of branching point δ is the subset of S made up of points having both the rank- one preimages distinct in S , that is

$q \in S \Leftrightarrow T_1^{-1}(q) \neq T_2^{-1}(q)$ both belongs to S .

Remark(2.5.1):

From definition (2.5.2) we see that :

1- $\delta = [T(S \cap \overline{R}_1)] \cap [T(S \cap R_2)]$ and $\delta \in \delta_1$, where $\delta_1 = T(\delta_0)$, with $\delta_0 = S \cap \overline{R}_1$ defined as in (2.5.1).

2- If $S = d'', q \in d'' \setminus \delta$,then only $T_1^{-1}(q)$ or $T_2^{-1}(q)$ belongs to d''. In particular if $q \in \delta_1 \setminus \delta$, then only $T_1^{-1}(q) \in d''$ otherwise $T_2^{-1}(q) \in d''$.

3- $T_2^{-1}(S) \subset \overline{R}_2 \cap S$ (2.5.2)

Proposition(2.5.8):[29, p.289]

$$T_2^{-1}(S) \subset \overline{R}_2 \cap S \Leftrightarrow \overline{\delta} = \delta_1 = T(S \cap \overline{R}_1).$$

Proof:-

Assume that $T_2^{-1}(S) \subset \overline{R}_2 \cap S$, then any point of δ_1 (which are the only points of S having T_1^{-1} in S) also has T_2^{-1} in S , thus $\overline{\delta} = \delta_1$.

Assume that $\overline{\delta} = \delta_1$. The invariant area S is such that $\delta = W \cup (\delta_1 \cup .... \cup \delta_N)$, where W may be empty or not. It was already proved that , if W is not empty, then $T_2^{-1}(W) \subset \overline{R}_2 \cup W$. By assumption $T_2^{-1}(\delta_1) \subset S$ and thus $T_2^{-1}(\delta_1) \subset S \cap \overline{R}_2$. Consider $T_2^{-1}(\delta_i), i = 1,2,..., N$. Writing , $\delta_i = T(\delta_{i-1} \cap \overline{R}_2) \cup T(\delta_{i-1} \cap \overline{R}_1)$.

Thus,

$$T_2^{-1}(\delta_i) = (\delta_{i-1} \cap \overline{R}_2) \cup T_2^{-1}(T(\delta_i \cap \overline{R}_1)),$$

if $(\delta_{i-1} \cap \overline{R}_2) \subset (S \cap \overline{R}_2)$ not empty and if $T(\delta_{i-1} \cap \overline{R}_1) \subset \delta_1$ not empty, so that $T_2^{-1}(T(\delta_{i-1} \cap \overline{R}_1)) \subset R_2 \cap S$. Thus $T_2^{-1}(\delta_i) \subset (\overline{R}_2 \cap S)$. ♦

The following proposition shows that such a bifurcation is the SBR bifurcation of φ with less effort required.

Proposition(2.5.9):[29, P.290]

Let d'' be invariant absorbing area

$$T_2^{-1}(g_{-1}) \to \varphi \text{ as } n \to \infty \qquad (2.5.3)$$

Then the bifurcation $\varphi_{-1} \in \partial d''$ is the SBR homoclinic bifurcation of φ .

Proof:-

Since, $\varphi_{-1} \in \partial d''$ therefore a finite rank preimage of φ belongs to $g_{-1} = LC_{-1} \cap d''$, $T^k(q) = \varphi_{-1}$, for integer $k > 0$ and by assumption φ_{-1} has a sequence of preimages in d'' which converge toward φ. Thus φ_{-1} which belongs to the unstable set of φ, also belongs to the unstable set of φ . ♦

The property (2.5.3) is not sufficient to conclude that $d'' = W^u(\varphi)$. The following proposition gives sufficient conditions for $d'' = \overline{W^u(\varphi)}$.

Proposition(2.5.10): [29,p.291]

Let d'' be an invariant area which satisfies(2.5.2). Then

(2.5.3) holds $\Leftrightarrow \varphi = \bigcap_{n\geq 0} T_2^{-n}(d'')$.

Proof:-

If $\varphi = \bigcap_{n\geq 0} T_2^{-n}(d''),$ then $T_2^{-n}(g_{-1}) \to \varphi$ is obvious . We prove now the converse.

Denote $B = d'' \cap \overline{\Re}_2$. By assumption (2.5.2) holds, thus $T_2^{-1}(d'') \subset B$. T is uniquely invertible in B by T_2^{-1} and has $T_2^{-1}(d'') \subset B$

and $\partial T_2^{-1}(B) = T_2^{-1}(\partial B)$.

By successive iterations of T_2^{-1} a sequence of embedded areas are obtained ,

$T_2^{-(n+1)}(B) \subset T_2^{-n}(B)$, $\forall n \geq 0$ the boundaries of which satisfy $\partial T_2^{-(n+1)}(B) = T_2^{-1}(\partial T_2^{-n}(B))$.

The critical segments of the boundary of B having necessarily a finite rank preimage by T_2^{-1} on g_{-1} embedded areas with boundaries made up of points belonging to some $T_2^K(g_{-1})$ are got , and such points converge toward φ under T_2^{-1} by assumption , thus $\varphi = \bigcap_{n\geq 0} T_2^{-n}(B)$. ♦

Chapter three

"Illustrative Examples"

As we observed in the introduction, much of the work in the field of dynamics of planar maps, is done on particular examples, and then observing certain phenomena. For example see [١٠, 16, 18 & 20].
In this chapter, several examples will be considered to illustrate the concepts defined in the first two chapters, and some observation will be made on the dynamics of the maps, particularly on the absorbing areas.

Recall that algorithm (2.3.1) does not guarantee that a closed area Δ will be constructed is absorbing therefore we shall verify that a closed area Δ is an absorbing area by satisfying the conditions of the definition of absorbing area numerically and if we succeed in doing that, we shall try to apply what has previously been mentioned in the preceding chapters, (i.e. chaotic area, invariant area, bifurcation types).

It is worth nothing that results presented here were essentially obtained via a numerical method, but guided by fundamental considerations stated in chapter two and using the critical curve tool.

In all examples we shall use Matlab version 6.1 Software for numerical computations and for plotting figures.

3.1 Examples of Absorbing Areas

In this section we shall give some examples that illustrate some phenomena on absorbing areas.

Example (3.1.1): Consider the map T defined by

$$T: \begin{cases} x' = a - by - x^2 \\ y' = x^2 \end{cases}, \text{ with } b \neq 0 \qquad (3.1.1)$$

It is easily seen that T is not invertible. T has two fixed points (x, y) where $x = a - (b+1)y$, $y = (2a(b+1) + 1 \pm \sqrt{4a(b+1)+1}) / 2(b+1)^2$ .Since T is real map, it must have real fixed points. Thus x and y are real if $a \geq -1/4(b+1)$ and $b \neq -1$.
The curve LC_{-1} is given by $x = 0$, the equation of the critical curve LC is given by $y = 0$. Recall that LC divides the plane into two regions: Z_0

satisfies $y<0$ where each point has no preimages and Z_2 with $y>0$ where each point has two first rank preimages. LC_{-1} divides the plane into two regions R_1, R_2. R_1 is the region $x<0$, R_2 with $x>0$.

Now we shall take some values of the parameters a and b to study the dynamical behavior of the map (3.1.1).

For $a=1$, $b=1$, the fixed points of T are :
$p_1=(-1,1)$ with eigenvalues $\lambda_1=2.73051$ and $\lambda_2=-0.73051$, therefore p_1 is a repulsive fixed point.
$p_2=(\frac{1}{2},\frac{1}{4})$ with eigenvalues $\lambda_{1,2}=\frac{-1\pm\sqrt{3}i}{2}$, since the eigenvalues have negative real parts then p_2 is stable fixed point . Notice that $p_1\in R_1$ and $p_2\in R_2$.

Now, we shall apply algorithm (2.3.1) with $a_0=(0,0)$. We look for integer N such that LC_N intersects LC_{-1} , so we shall find that $b_0\in LC_1\cap LC_{-1}$ i.e. $N=1$, $b_0=(0,1)$ and $b_1=(0,0)\in LC_2\cap LC$. We construct the closed area Δ whose boundary $\partial\Delta=(b_1a_1a_2b_1)$. Before we represent Δ in a figure let us compute the equations of LC_1 and LC_2.
The equation of LC is $y=0$, by substituting this in map (3.1.1) we get :

$x=a-y$ which the equation is of LC_1.

Now, to compute the equation of LC_2 , substitute equation of LC_1 in map (3.1.1) we get :

$$x=a-ab\pm b\sqrt{y}-y$$

which is the equation of LC_2. So in fig.(3.1.1) the closed area Δ is shown. This area $\Delta=d'$ is an absorbing area since it satisfies the conditions of the definition of an absorbing area (non mixed). In fact, by construction $\partial\Delta$ consists of critical curves of finite rank, numerical computations shows that the successive iterates of any points which either belong to Δ or to $U(\Delta)\setminus\Delta$, enter Δ after a finite number of iteration and can not get away after entering. Also we notice that if we take any point in Δ, the successive iterates of this point will bifurcate into three subsequences, each of them converges to vertex of the region Δ (represented in a fig.(3.1.1) by a thicker line , with point $p=(0.5,0.5)$).

We notice that from fig.(3.1.1) $b_1 \in (a_0 a_1)$. Moreover, $b_1 = a_0$ in this case either $T(\Delta) \subseteq \Delta$ or Δ is not comparable with $T(\Delta)$. We have seen that $\partial\Delta$ intersects LC_{-1} in two points b_0 and a_2 i.e. $\partial\Delta \cap LC_{-1} = \{b_0, a_2\}$, so that $\partial\Delta$ includes the segment $(b_1 a_1)$ and $T(\Delta) \subset \Delta$, therefore $T^{m+1}(\Delta) \subseteq T^m(\Delta)$, $\forall\ m \geq 0$ and we have fuond $M = 4$ which satisfies $T^M(\Delta) = T^{M+1}(\Delta)$. So $d'' = T^M(\Delta)$ is the invariant absorbing area as shown in fig.(3.1.2).
Also, we note T has $3-$cycle $\{b_1, a_1, a_2\}$. d' is a chaotic area in the sense of Gulick since T is sensitive to initial condition on d'.

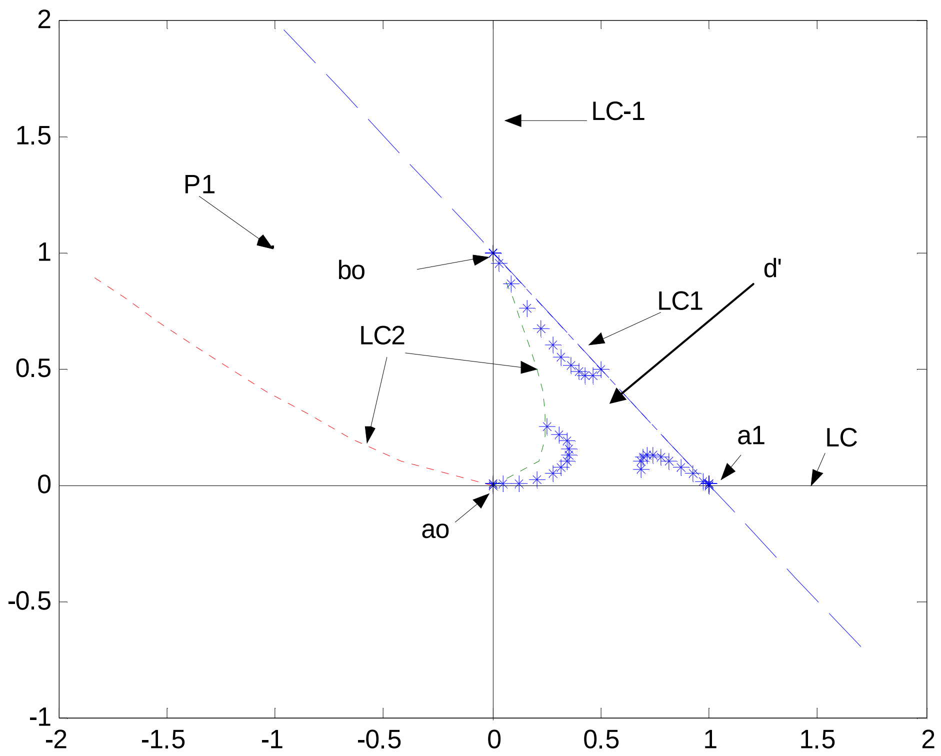


Fig.(3.1.1) d' absorbing area of the map (3.1.1) with $a = 1$, $b = 1$.

Fixed points: $p_1 = (-1,1)$, $p_2 = (\frac{1}{2}, \frac{1}{4})$, $a_0 = b_1 = (0,0)$, $a_1 = (1,0)$ & $a_2 = b_0 = (0,1)$

thicker line is the orbit of the point $p=(0.5,0.5)$ which lies on the boundary.

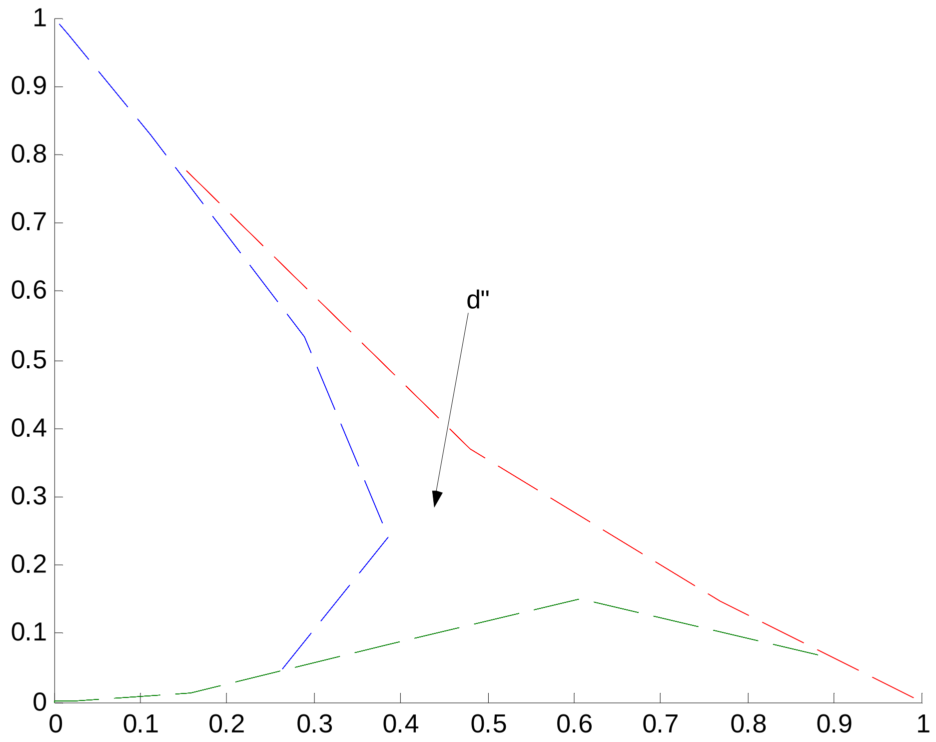


Fig.(3.1.2) $d''=T^{M}(\Delta)$ invariant absorbing areas of the map(3.1.1) with $a=1$, $b=1$.

Now, we shall take other values of the parameters a and b, to see the dynamics of the map (3.1.1), we shall take $a=1$, $b=2$. $p_1=(-0.767591879,0.589197293)$ and $p_1=(0.434258445,0.188580484)$ are two fixed points of T. Numerical computations show that p_1, is an expanding fixed point while p_2 is a stable fixed point. Again applying algorithm (2.3.1) produces a closed area as shown in fig.(3.1.3) and a numerical simulation shows that this area is absorbing $\Delta=d'$.

In such a case the situation $b_1\notin(a_0a_1)$ occurs, then we note that $\overline{R}_1\cap\Delta$ is the region whose boundary is $b_1a_0a_2b_1$ and $T(\overline{R}_1\cap\Delta)\subset\Delta$ then $T(\Delta)=\Delta=d''$ is invariant.

T has $2-$cycle $\{p,\ q\}$where $p=(0,1)$, $q=(-1,0)$.

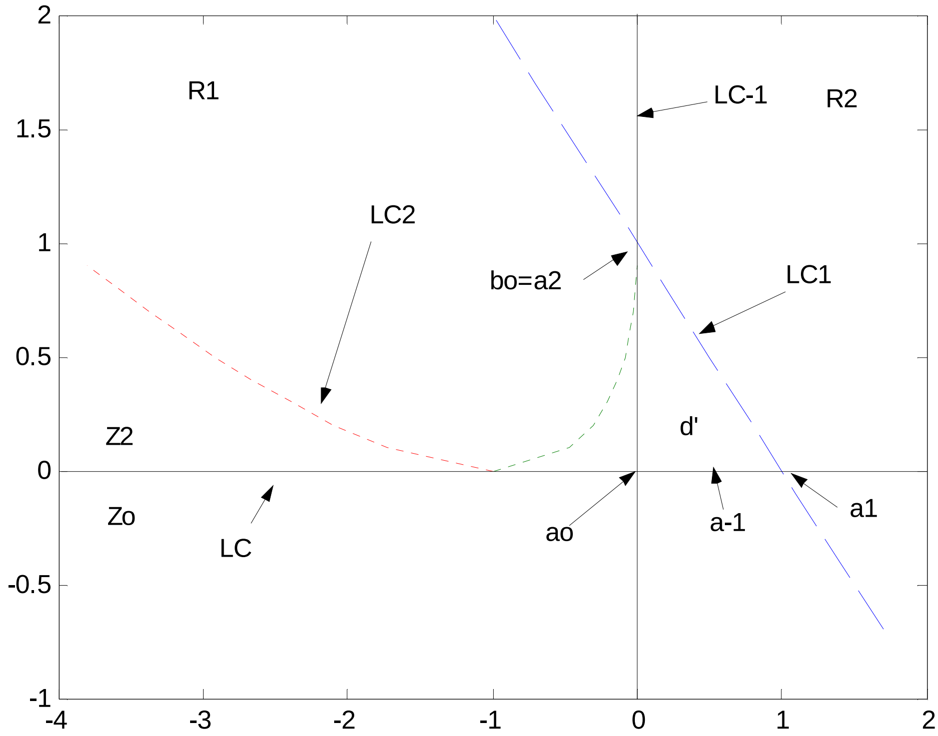


Fig.(3.1.3) The absorbing area d' for the map (3.1.1) with $a=1$, $b=2$, $a_0=(0,0)$, $a_1=(1,0)$, $a_2=b_0=(0,1)$ & $b_1=(1,0)$.

Again we shall take another value of a and b, for example $a=-1$, $b=-10$, we have noticed that the shape of a closed region constructed by algorithm(2.3.1) is not changed, but $\Delta\cap R_2=\phi$ shown in fig.(3.1.4).

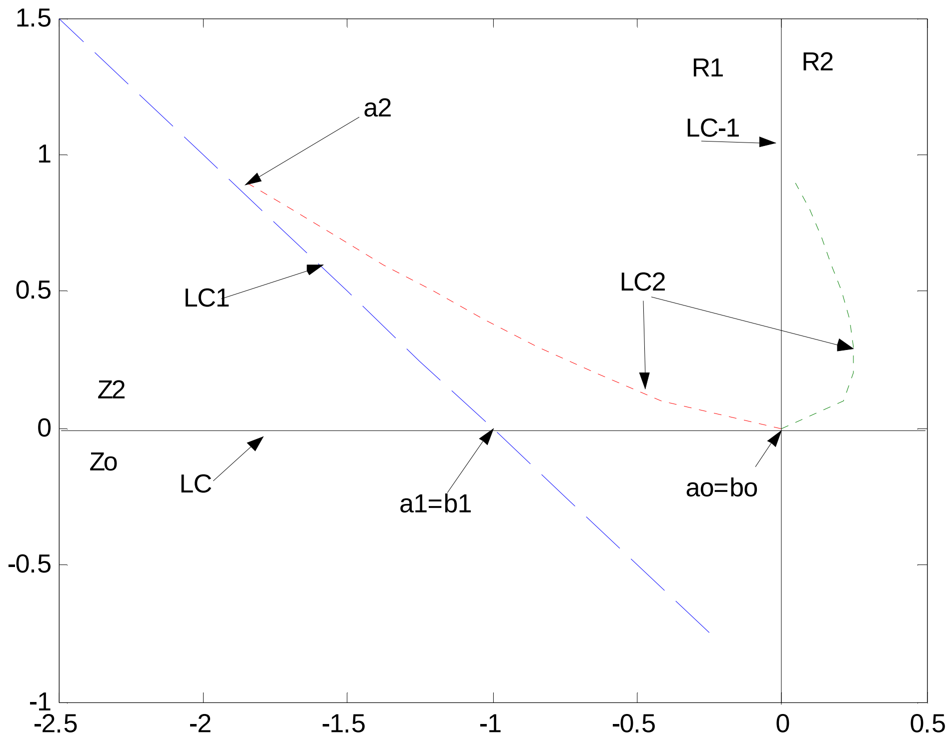


Fig.(3.1.4) Map (3.1.1) with $a=-1,\ b=-10$.

Apply algorithm (2.3.1) again for the map(3.1.1) with $a=1,\ b=-0.5$, we get a closed bounded area whose boundary $\partial\Delta=(b_1a_1a_2b_1)$ as is shown in fig.(3.1.5). T has two fixed points $p_1=(0.732050807, 0.535898384)$ and $p_1=(-2.732050808, 7.464101615)$, both p_1 and p_2 do not belong to Δ. It can easily be shown that p_1 is an unstable fixed point, p_2 is an expanding fixed point. Numerical computations show that this area is not absorbing.

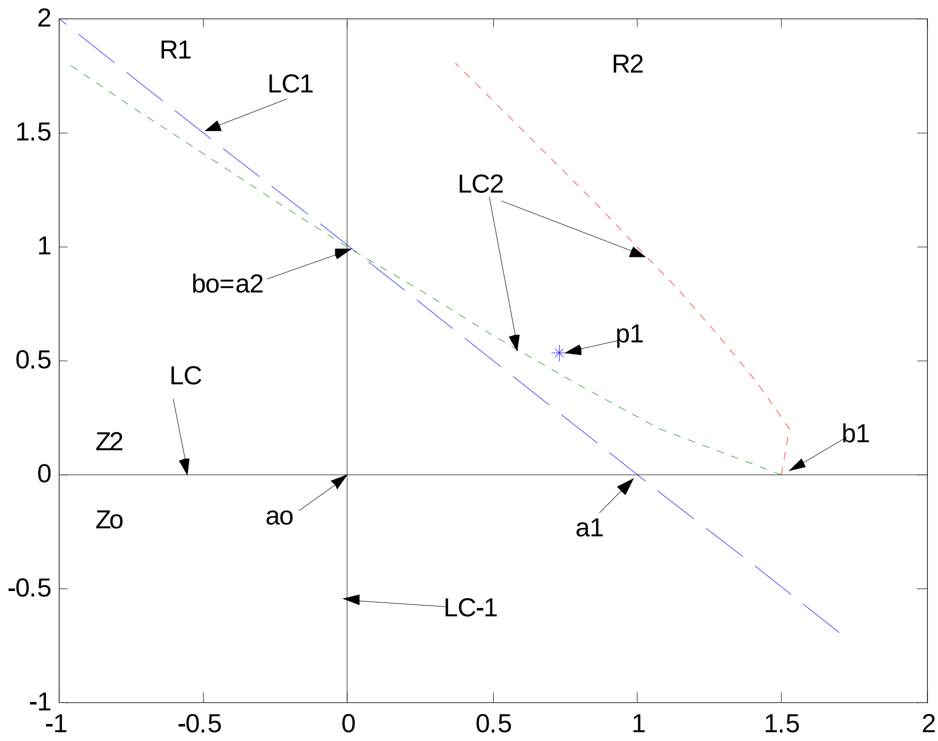


Fig.(3.1.5) d' closed bounded area which is not absorbing area for the map (3.1.1) with $a=1$, $b=-0.5$, $a_0=(0,0)$, $a_1=(1,0)$, $a_2=b_0=(0,1)$ & $b_1=(1.5,0)$.

Example (3.1.2): Consider the map T defined by

$$T: \begin{cases} x' = y^2 + x \\ y' = ax + 1 \end{cases}, \text{ with } a \neq 0 \qquad (3.1.2)$$

T is continuously differentiable and noninvertible map whose inverses are

$$T^{-1}: \begin{cases} x = \dfrac{y'-1}{a} \\ y = \pm\sqrt{x' - \dfrac{y'-1}{a}} \end{cases}$$

T has a fixed point $(\frac{-1}{a},0)$. The curve LC_{-1} is given by $y=0$ which divides the plane $\Re^2$ into two regions R_1 with $y<0$, R_2 with $y>0$, the equation of the critical curve LC is $y=ax+1$. LC separates the plane $\Re^2$ into two regions: Z_0 with $y<ax+1$, Z_2 with $y>ax+1$. The point of intersection of LC_{-1} and LC is $a_0=(\frac{-1}{a},0)$.

When algorithm (2.3.1) is applied, we shall get a point $a_0 = (-\frac{1}{a}, 0)$ i.e. an absorbing area is just the point $a_0 = (-\frac{1}{a}, 0)$ as is shown in figure(3.1.6).

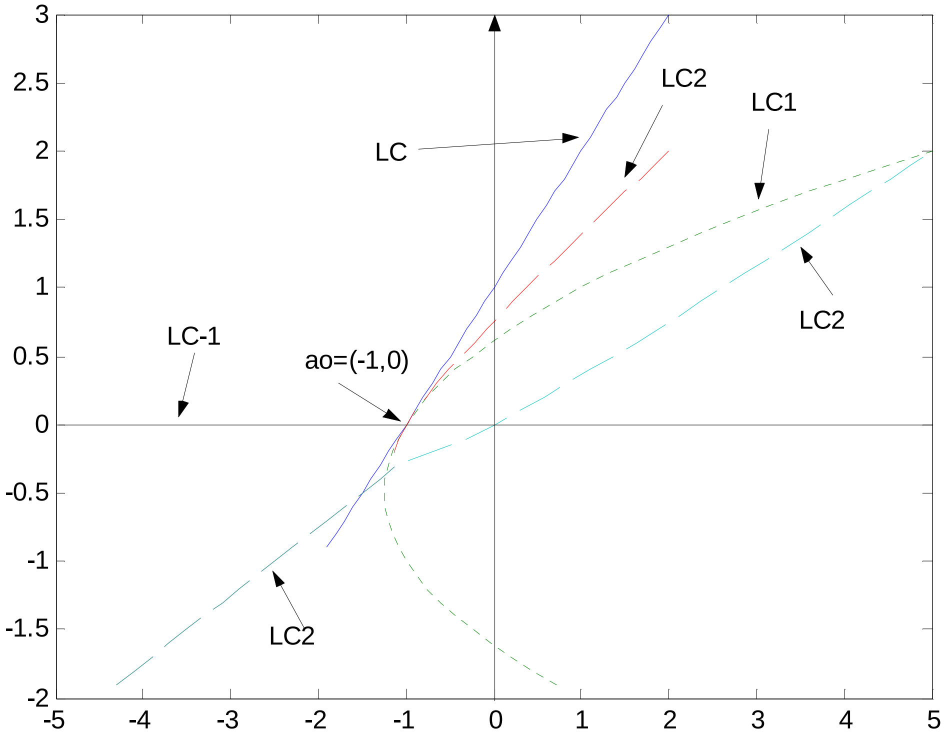


Fig.(3.1.6) the map (3.1.2) with $a = 1$, $a_0 = (-1, 0)$.

From figure(3.1.6) we note that there are two closed regions: one is bounded by segments of critical curves LC and LC_1, the other is bounded by segments of critical curves LC, LC_1 and LC_2. Numerical computations show that both regions are not absorbing.

Example(3.1.3): Consider a noninvertible map T defined by

$$T: \begin{cases} x' = a - by^2 - x^2 \\ y' = x \end{cases}, \text{ with } b \neq 0 \qquad (3.1.3)$$

T has inverses given by

$$T^{-1}: \begin{cases} x = y' \\ y = \pm\sqrt{\dfrac{a - y' - x'}{b}} \end{cases}$$

T has two fixed points (x, x) where $x = \dfrac{-1 \pm \sqrt{1 + 4a(b+1)}}{a(b+1)}$, if $b \geq \dfrac{-1}{4a} - 1$, $a \neq 0$ and $b \neq -1$.

The curve LC_{-1} is given by $y = 0$ and the equation of the critical curve LC is $x = a - y^2$, so the point of intersection of LC_{-1} and LC is $a_0 = (a, 0)$. Recall that LC divides the plane $\Re^2$ into two regions: Z_0 satisfying $x > a - y^2$ where each point has no preimages, and Z_2 with $x < a - y^2$ where each point has two first rank preimages. LC_{-1} divides the plane into two regions R_1, R_2. R_1 is the region $y < 0$, R_2 with $y > 0$.

For $a = 1$, $b = 1.4$ numerical computation shows that T has two fixed points : $p_1 = (0.469951, 0.469951)$ and $p_2 = (-0.886618, 0.886618)$. $J(p_1)$ has eigenvalues $\lambda_{1,2} = -0.46995 \pm 1.0464\, i$, thus p_1 is an attracting fixed point, while $J(p_2)$ has eigenvalues $\lambda_1 = 2.695$, $\lambda_1 = 0.9213$, thus p_2 is a saddle fixed point.

Now, we shall try to find an absorbing area by applying algorithm (2.3.1) with $a_0 = (1, 0)$, again we shall look for integer N such that LC_N intersects LC_{-1}, so we find $N = 1$, $b_0 = (-0.4, 1) \in LC_1 \cap LC_{-1}$ and $b_1 = T(b_0) = (0.84, \text{-}\,0.4) \in LC_2 \cap LC$, we construct a closed area Δ whose boundary $\partial\Delta = (b_1 a_1 a_2 b_1)$. To represent this area graphically let us compute the equations of LC_1 and LC_2 .

The equation is of $LC_1 : x = a - ab + by - y^2$.

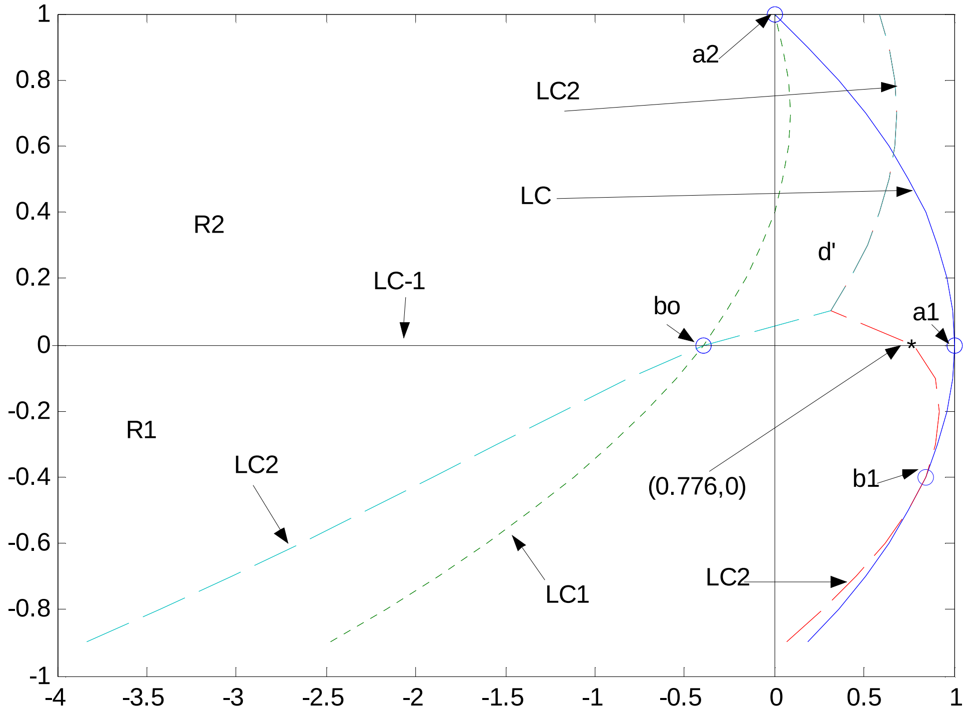
The equation of LC_2 : $x = a - \dfrac{b^3}{2} - ab + ab^2 + by - y^2 \pm b^2\sqrt{a - ab - y + \dfrac{b^2}{4}}$.

For the particular case ($a = 1$, $b = 1.4$), the equation of LC_1, LC_2 respectively :

$$x = -0.4 + 1.4y - y^2$$

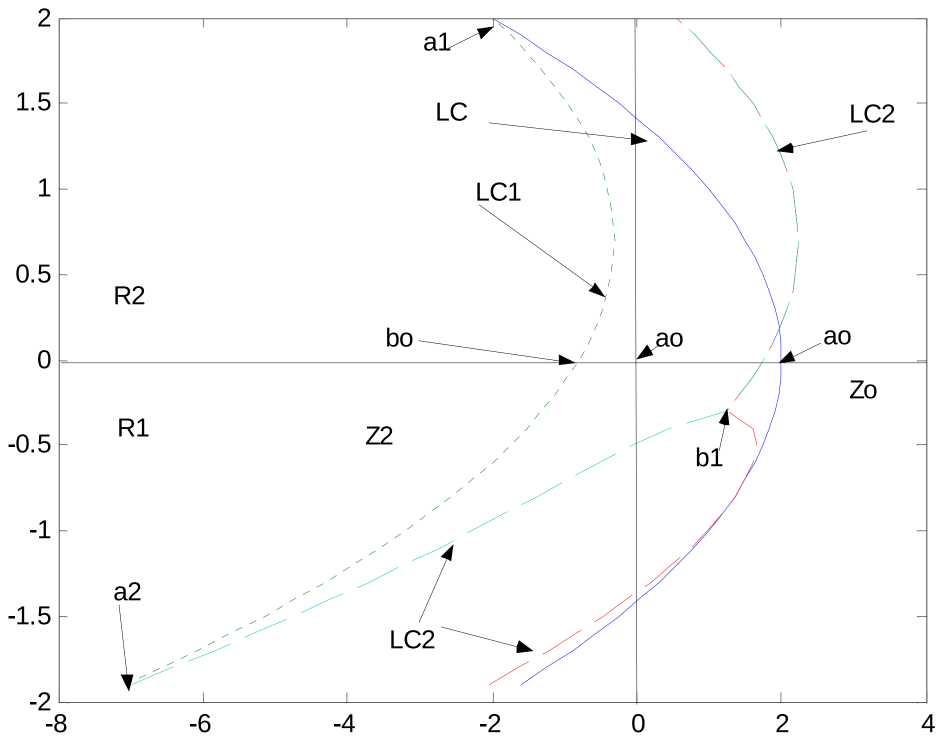
$$x = 0.188 + 1.4y - y^2 \pm 1.96\sqrt{0.09 - y}$$

Next, fig.(3.1.7) represents the closed Δ, this area is absorbing since it satisfies the conditions of absorbing as is suggested by numerical computations of iterates of any points which either belongs to Δ or to $U(\Delta) \setminus \Delta$.



Fig(3.1.7) Map (3.1.3) with $a = 1, b = 1.4$, the absorbing area $d' = \Delta$.

We need to find an integer M such that $T^M(\Delta)$ is invariant , from fig. (3.1.7) we notice that $b_1 \notin (a_0 a_1)$ i.e $b_0 \notin (a_{-1}, a_0)$, therefore $\partial\Delta$ intersect LC_{-1} at two points c_0, a_0, $c_0 = (0.776, 0)$. We notice that $\delta_0 = \overline{R}1 \cap \Delta$, $T(\delta_0) \subseteq \Delta$, therefore $T(\Delta) = \Delta$ i.e $M = 0$. & $\Delta = d' = d''$ is the invariant absorbing area .
Also numerical computations show that there is a cycle of period -6 inside region d', also we notice that it has an attractor, and any point inside d' has an orbit lies on the triangles, for example if we take the iterates of a point $p = (.2,.3) \in \Delta$, we see that the orbits of $\{p^n\}_{n=1}^{\infty}$ lie on the triangles , when we apply the conditions of chaos , we find in this region T is sensitive to initial conditions, therefore Δ is a chaotic area in the sense of Gulick



Fig(3.1.8) map(3.1.3) with $a = 2,\ b = 1.4$.

Now , we shall take the other values of the parameters $a \& b$, to see the dynamics of the map (3.1.3). For $a = 2, b = 1.4$, a closed area constructed by algorithm (2.3.1) is shown by fig (3.1.8) whose boundary $\partial\Delta = (a_1 a_2 b_1 a_1)$. The two fixed points $p_1 = (0.728, 0.728)$, $p_2 = (-1.145, -1.145)$ are stable, saddle fixed points respectively, with $N = 1,\ a_0 = (2,0)$, $b_0 = (-0.8,0)$, $b_1 = (1.36,-0.8)$ $a_1 = (-2,2)$, & $a_2 = (-7.6,-2)$.

Example (3.1.4): Consider the map T defined by

$$T: \begin{cases} x' = ax + y \\ y' = b + x^2 \end{cases} \qquad (3.1.4)$$

T has two fixed points (x, y) where $x = (1 - a \pm \sqrt{(a-1)^2 - 4b})/2$ and $y = (1-a)x$, if $a \geq 2\sqrt{b} + 1$ and $b \geq 0$.
LC_{-1} is defined by $x = 0$ and LC by $y = b$.
T is a noninvertible map whose inverses are

$$T^{-1}: \begin{cases} x = \pm\sqrt{y' - b} \\ y = x' \mp a\sqrt{y' - b} \end{cases}$$

For $a = 0.7$ and $b = -0.82$, T has two fixed points : $p_1 = (1.06788, 0.32036)$ & $p_2 = (-0.76788, -0.32036)$

$J(p_1)$ has eigenvalues $\lambda_1 = 1.8775$, $\lambda_2 = -1.152749$ therefore p_1 is an expanding fixed point while p_2 is an unstable fixed point since $J(p_2)$ has eigenvalues $\lambda_{1,2} = \frac{0.7 \pm 2.3776i}{2}$.

Moreover, p_1 is not a SBR since there is no q such that $T^k(q) = p_1$, for some integer k.

$$x = \frac{1 - a \pm \sqrt{(a-1)^2 - 4b}}{2}$$

Fig (3.1.9) represents the invariant absorbing area constructed by algorithm (2.3.1) $d'' = T^4(\Delta)$, this region includes an annular chaotic area $d_a \subset d''_a$ according to (2.5.1). The area d'' satisfies the properties (2.5.2) and (2.5.3) , The unstable set $W^u(p_1)$ of p_1 is such that $W^u(p_1) = d_a \cup W$.

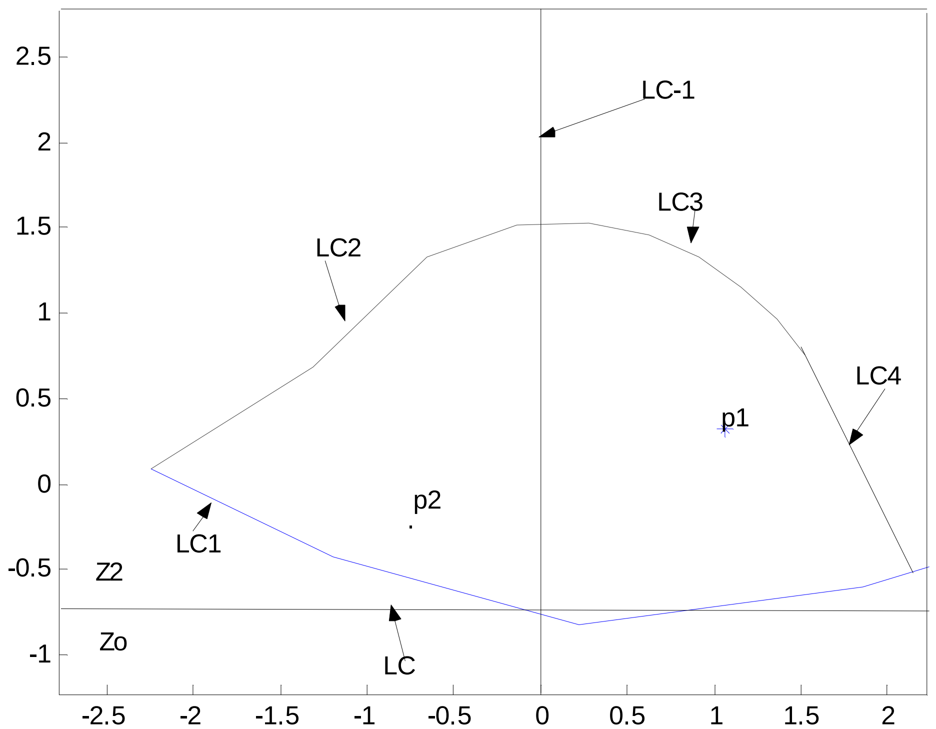


Fig.(3.1.9) Map (3.1.4) with $a = 0.7$, $b = -0.82$ invariant area obtained by algorithm (2.3.1) includes annular chaotic area $d_a \subset d''_a$.

3.2 Examples of Symmetric Maps

In this section we shall give examples that show the relation between the dynamics of a two-dimensional map and a certain associated one-dimensional map.

Example (3.2.1):- Consider a map T defined by

$$T: \quad \begin{aligned} x' &= a - by - x^2 \\ y' &= a - bx - y^2 \end{aligned} \qquad (3.2.1)$$

First we shall establish symmetric properties in the dynamics of T and identify its fixed points. The eigenvalues of the Jacobian matrix of T, $J(T)$ evaluated at these points, are determined via classical analysis. Also, the critical curve LC_{-1} will be evaluated.

Symmetry

Let $\rho : \Re^2 \to \Re^2$ be defined by

$$\rho(x, y) = (y, x)$$

ρ is the reflection through the diagonal $D = \{(x, x)\} \subset \Re^2$

T is symmetric, i.e. $T \circ \rho = \rho \circ T$ in fact,

let $f_{ab}(x, y) = a - by - x^2$. Then

$$\begin{aligned} \rho \circ T(x, y) &= \rho(T(x, y)) \\ &= \rho(f_{ab}(x, y), f_{ab}(y, x)) \\ &= (f_{ab}(y, x), f_{ab}(x, y)) \\ &= T(y, x) = T \circ \rho(x, y) \end{aligned}$$

Thus, T commutes with ρ. Also, T we have some properties such as:

(i) The diagonal D is invariant, i.e. $T(D) = D$.

(ii) If p is a fixed point of T, so is $\rho(p)$.

(iii) If $\{p_i, i \in N\}$ is an orbit of T, so is $\{\rho(p_i), i \in N\}$.

If we restrict the map to the one invariant diagonal D, we have on dimensional map, say

$$g_{ab}(x) = a - bx - x^2$$

For its graph, note that $g'_{ab}(x) = -b - 2x$ and $g''_{ab}(x) = -2 < 0$. Thus the local maximum (called a critical point of rank-0 of g_{ab}) exists at $c_{-1} = \frac{-b}{2}$, and the critical point of g_{ab} of rank-1 is the point $c = g_{ab}(c_{-1})$, and the critical points of g_{ab} of rank$-(i+1)$ for $i \geq 1$ are the forward images (or iterates) $c_i = g_{ab}^{i+1}(c_{-1}) = g_{ab}^{i}(c)$.

The fixed points of $g_{ab}(x)$ i.e. solutions of the equation $x = a - bx - x^2$ are $p_1 = \frac{-(b+1)+\sqrt{(b+1)^2+4a}}{2}$ and $p_2 = \frac{-(b+1)-\sqrt{(b+1)^2+4a}}{2}$ with $a<0$ and $b > \sqrt{-4a} - 1$.

About the multipliers, note that $|g'_{ab}(p_1)| = \left|1-\sqrt{(b+1)^2+4a}\right| < 1$, if $a<1$ and $b < 2\sqrt{1-a} - 1$, but we have $a<0$ i.e. p_1 is an attracting fixed point if $b < 2\sqrt{1-a} - 1$, and p_1 is a repelling fixed point if $b > 2\sqrt{1-a} - 1$, therefore $b = 2\sqrt{1-a} - 1$ is a bifurcation point.

Since $|g'_{ab}(p_2)| = \left|1+\sqrt{(b+1)^2+4a}\right| > 1$, then p_2 is a repelling fixed point.

Fixed point of T

T possesses four fixed points. From discussion of g_{ab} above it follows that $P_1 = (p_1, p_1)$ and $P_2 = (p_2, p_2)$ are two fixed points of T on D. We shall determine now the fixed points (x^*, y^*) with $x^* \neq y^*$ by direct computation. From the definition of T (putting $x' = x$ and $y' = y$) we shall get the following system:

$$x = a - by - x^2$$
$$y = a - bx - y^2$$

we get $P_3 = (x^*, y^*)$ and $P_4 = (y^*, x^*)$ where

$x^* = \frac{-(b+1)-\sqrt{-3b^2+2b+1+4a}}{2}$, $y^* = \frac{-(b+1)+\sqrt{-3b^2+2b+1+4a}}{2}$, with $a \geq \frac{-3b^2+2b+1}{4}$.

Eigenvalues of $J(T)$ ***at the fixed points:***

The Jacobian matrix of T is :

$$J(T) = \begin{pmatrix} -2x & -b \\ -b & -2y \end{pmatrix}$$

(1) At the fixed point $P_1 = (p_1, p_1)$ we have

$$J(T(P_1)) = \begin{pmatrix} b+1-\sqrt{(b+1)^2+4a} & -b \\ -b & b+1-\sqrt{(b+1)^2+4a} \end{pmatrix}$$

The eigenvalues of $J(T)$ are

$$\lambda_{1,2} = -(b+1) + \sqrt{(b+1)^2+4a} \pm |b|$$

Clearly $\lambda_{1,2}$ are real , then the stability of P_1 depends on the value of b.

(2) At the fixed points $P_2 = (p_2, p_2)$, the eigenvalues of $J(T)$ are $\lambda_{1,2} = -(b+1) - \sqrt{(b+1)^2 + 4a} \pm |b|$

$\lambda_{1,2}$ are real , again the stability of P_1 depends on the value of b .

(3) At the fixed point P_3, the eigenvalues of $J(T)$ are $\lambda_{1,2} = -2(b+1) \pm \sqrt{b^2 + 8b + 1 - 4a}$

Since $J(T)$ has real eigenvalues if $\mathrm{b} \geq \sqrt{4\mathrm{a} + 15} - 4$ and $-3.75 \leq \mathrm{a} \leq 0$.

(4) At the fixed point P_4 we shall have :

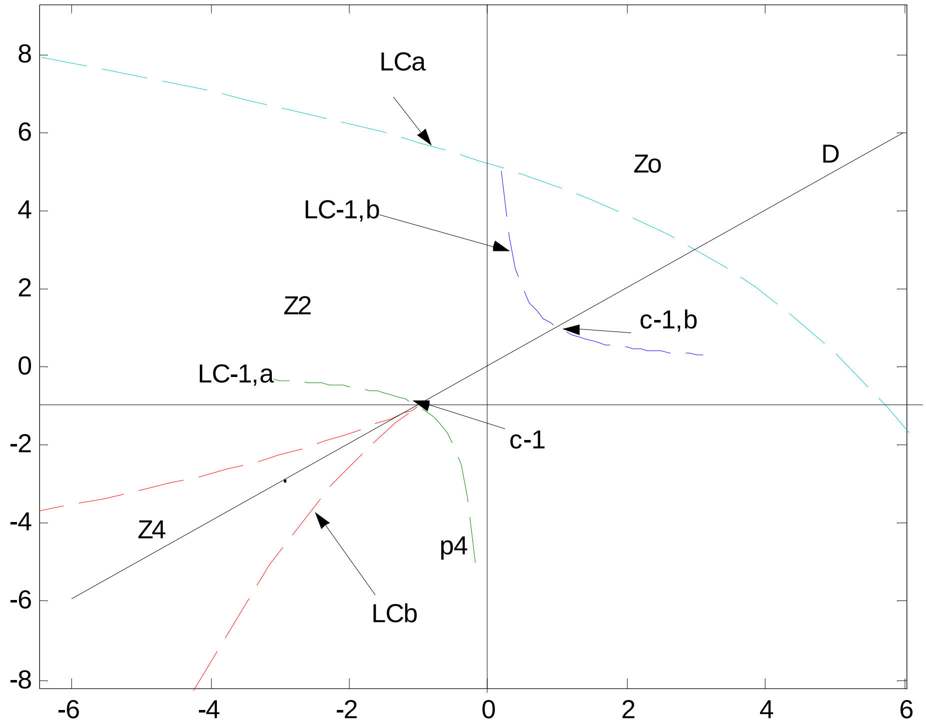
Since T is symmetric with respect to ω then the dynamics of T at P_4 is like the dynamics of T at P_3, therefore P_4 has a real eigenvalue if $\mathrm{b} \geq \sqrt{4\mathrm{a} + 15} - 4$ and $-3.75 \leq \mathrm{a} \leq 0$.

Critical curve

The map T defined in (3.2.1) is clearly a map with a nonunique inverse. The critical curve LC of T is the image of the locus of points in which the Jacobian J of T vanishes, i.e. $LC = T(LC_{-1})$ where LC_{-1} is the curve :

$$xy = \frac{b^2}{4}$$

LC_{-1} is a hyperbola of two branches. Let $LC_{-1} = LC_{-1,a} \cup LC_{-1,b}$ where $LC_{-1,b}$ shows the upper branch (for $x > 0$ and $y > 0$) and $LC_{-1,a}$ denotes the lower branch as shown in figure (3.2.1). It follows that the critical curve of rank -1, LC , consists of two branches, say $LC = LC_a \cup LC_b$ where $LC_a = T(LC_{-1,a})$ and $LC_b = T(LC_{-1,b})$. The two branches of LC_{-1} and LC are symmetric with respect to D . The qualitative shape of LC is shown in fig.(3.2.1). LC separates the plane into three open regions, named Z_0, Z_2 and Z_4, locus of points having $0, 2$ and 4 distinct preimages of rank -1 respectively.



Fig(3.2.1)The qualitative shape of $LC_{-1} = LC_{-1,a} \cup LC_{-1,b}$ and $LC = LC_a \cup LC_b$.

Example (3.2.2):- Consider the map T defined by

$$T: \begin{cases} x' = by + x^2 \\ y' = bx + y^2 \end{cases}$$

As in previous example we can show that T is symmetric and satisfies properties (i , ii and iii) that appeared in example (3.2.1) and if we restrict T to the invariant diagonal D, we shall get a one dimensional map:

$$g_b(x) = bx + x^2$$

g_b has fixed points $p_1 = 0$ and $p_2 = 1 - b$.
$|g_b'(p_1)| = |b|$, therefore p_1 is an attracting fixed point if $|b| < 1$.
$|g_b'(p_2)| = |2 - b|$, therefore p_2 is an attracting fixed point if $1 < b < 3$.

T has four fixed points :
$P_1 = (0,0)$, $P_2 = (-b+1, -b+1)$, $P_3 = (x^*, y^*)$ and $P_4 = (y^*, x^*)$

where $x^* = \dfrac{b+1-\sqrt{1-2b-3b^2}}{2}$, $y^* = \dfrac{b+1+\sqrt{1-2b-3b^2}}{2}$ with $1-2b-3b^2 \geq 0$.

As, we have mentioned in the previous example we can discuss the stability of each fixed points easily by simple computation.

3.3 Other Examples

In this section we shall give examples that illustrate the nature of the critical set.

Example (3.3.1):

Consider a standard form(1.4.3a) that appeared in chapter one

$T(x,y) = (a_0 x^2 + a_1 xy + a_4 y + a, b_1 xy + b)$; $b_1 \neq 0$

Recall that T has nonempty unbounded critical set which is a parabola.

Now, by using different values of the coefficient a_0, a_1, a_4, b_1, a and b, we find for $a = 3$, $a_0 = b_1 = 0.1$, $a_1 = a_4 = -1$ and $b = 1$, T has bounded trajectory and we find the region $R = \{(x,y) : -5.45 \leq x \leq 5.45,\ -2.9 \leq y \leq 1.5\}$ such that any points in R has an attractor $A^* = (1.848, 1.2267)$.

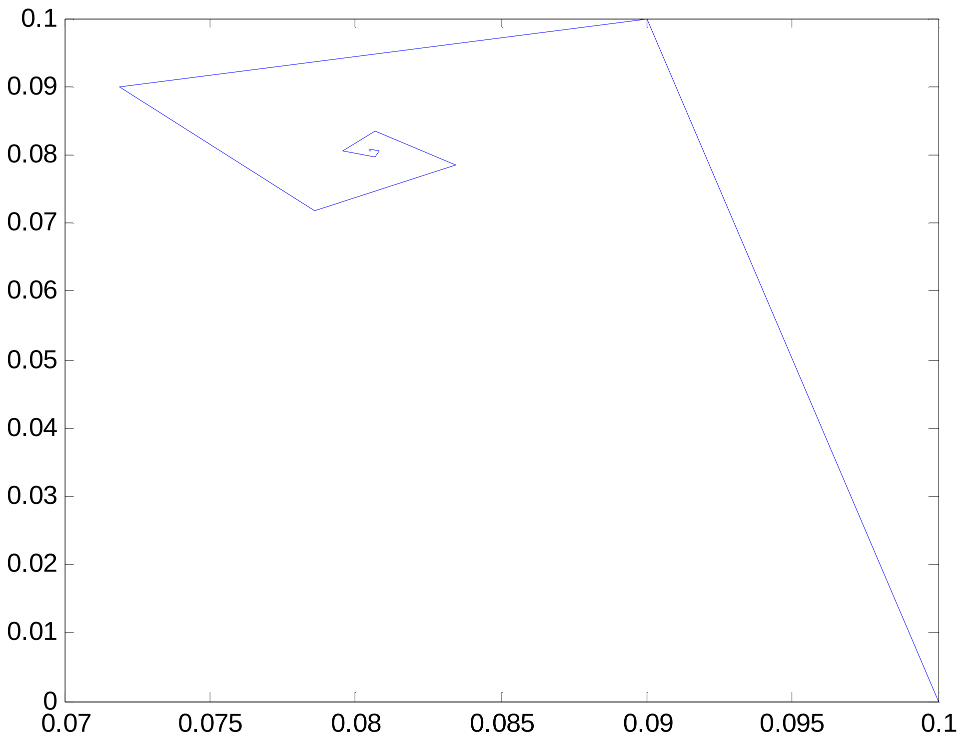
Also, if we take two points say: $p_1 = (-5.4887968245303, 1)$ and $p_2 = (-5.4887968245304, 1)$, we shall see that for $n \geq 10$, $T^n(p_1)$ will converge to A^* while $T^n(p_2)$ converge to infinity.

Example(3.3.2):- Consider a standard form(1.4.3b) that appeared in chapter one

$T(x,y) = (a_0 x^2 + a_2 y^2 + a x_3 + a_4 y + a, b_0 x^2 + b_2 y^2 + b_3 x + b_4 y + b)$;

Recall that T has a nonempty unbounded critical set which is either line or hyperbola .

In particular, let $a_0 = -1$, $a_2 = -2$, $a_3 = a_4 = 0$, $a = 0.1$, $b_0 = b_2 = b_4 = b = 0$, and $b_3 = 1$. we see that the iterate of any point which belongs to the region $D = \{(x,y) : -0.25 \leq x \leq 0.25.0.5 \leq y \leq 0.5\}$ converges to the point $(0.0805, 0.0805)$ as is shown in fig (3.3.1) .



Fig(3.3.1) Map (3.3.2) the orbit of the point (0.1,0) & (0.0805,0.0805) is the attractor.

3.4 Conclusions & Recommendations

1- We have seen in example (3.1.1) that if algorithm (2.3.1) succeeds to find an absorbing area d', the iterate of any points in d' bifurcate into three subsequences each one converges to a vertex of d' . And there is no point of non-smoothness in the boundary of d' .

2- In example (3.1.2), the point of intersection of LC_{-1} & LC is a fixed point $(-\frac{1}{a},0)$, therefore an absorbing area is just a point $(-\frac{1}{a},0)$.
If we try to find a closed area bounded by segments of critical curves, we shall find two closed areas: one bounded by segments of critical curves LC_1 & LC_2 , the other is bounded by segments of critical curves LC, LC_1 & LC_2 but both areas are not absorbing .

3- In example (3.1.3), we find an absorbing area d' for some values of the parameters a and b , and each point in such area has an orbit inside d' which lies in triangular shape .

4- Algorithm (2.3.1) may fail to find an absorbing area , therefore we suggest that one should examine the map, before we apply the algorithm by finding the successive image of $a_0 = LC_{-1} \cap LC$, if such iterates converge then the closed area obtained by (2.3.1) is absorbing .

Our recommendations are:

Much of the work on the subject have concentrated on investigating properties of particular examples and trying to make general observations so we suggest the work to be more theoretical.

Certification of the Examining Committee

We chairman and members of the examination committee, certify that we have studied the thesis entitled "***On Absorbing Areas of Planar Quadratic Maps***", presented by student ***Zainab Abdul-Naby Al-Wa'li*** and examined her in its contents and that we have found it worthy to be accepted for the requirements of the degree of Doctor of Philosophy in Science of Mathematics with (Excellence) grade.

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Republic of Iraq
Ministry of Higher Education &
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On Absorbing Areas of Planar Quadratic Maps

A Thesis
Submitted to the Department of Mathematics and Computer Applications of College of Science of Al-Nahrain University
in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics and Computer Applications

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August 2005 Rajab1426

بسم الله الرحمن الرحيم

((وعنده مفاتح الغيب لايعلمها الا هو ويعلم مافي البر والبحر وماتسقط من ورقة الا يعلمها ولاحبة في ظلمات الارض ولارطب ولا يابس إِلاّ في كتاب مبين))

صدق الله العلي العظيم

سورة الانعام (الآية ٥٩)

Introduction :

Dynamical systems as a mathematical discipline goes back to Poincare', who developed a qualitative approach to problems that arose from celestial mechanics.

The subject has expanded considerably in scope and has undergone some fundamental progress in the last three decades. Today, it stands at crossroads of several areas of mathematics, including analysis, geometry, topology and mathematical physics. It is generally regarded as a study of iteration of maps, or of time evaluation of differential equation.

The basic goal of a dynamical system is to the eventual or asymptotic behavior of an iteration process. If this process is a differential equation whose independent variable is time, then the theory will attempts to predicate the ultimate behavior of solution of the equation in either the distant future $(t \to \infty)$ or the distant past $(t \to -\infty)$. If the process is a discrete process such as the iteration of a function, then the theory will hopes to understand the eventual behavior of the points $x, T(x), T^2(x), \ldots, T^n(x)$ as n becomes large. That a dynamical system asks the some what nonmathematical sounding question: where do points go and what do they do when they get there ?
Function which determine dynamical systems are also called mappings, or maps [11].

The complex dynamical behavior of solutions of various mathematical models has been an object of study for a number of years . Point – mappings or recurrence, are especially of interest because they appear as natural descriptions of evolutionary phenomena in physics, ecology, biology and control systems [9, 27& 29].

A complex dynamical behavior called " chaos" is observed in mathematical models expressed in the form of recurrences with a non-unique inverse. The chaotic solution of such second -order point –mapping is located in bounded areas. All the attractive limit sets of an endomorphism (noninvertible map), whatever the nature may be, are located in phase plane designated by absorbing areas[2, 15& 32] .

Critical curves appear as the natural – two dimensional a generalization of the notion of critical point of a one-dimensional noninvertible map, they permit to define the essential notions of absorbing area , and chaotic area [15&24]. Roughly speaking an absorbing area d' is a region bounded by critical curves segments of finite rank, such that the successive images of all points of a neighborhood $U(d')$ enter into d' and cannot get away after entering, after a finite number of iterations [38]. A chaotic area is an invariant absorbing area , the points of which give rise to iterated sequences (orbits) having the property of sensitivity to initial conditions.

The role of critical curves also is fundamental in the definition of bifurcation leading either to the destruction , or to a sudden and important modification of absorbing areas .

Recently an extended notion of absorbing area, chaotic area, that of mixed absorbing area, mixed chaotic area was introduced by Barugola & Cathala in 1992. These last areas differ from the non mixed ones by the fact that their boundaries are made up of the union of critical curves segments, and segment of the unstable set of a saddle fixed point, or a saddle cycle (periodic point), or even segment of several unstable sets associated with different cycles. With respect to a ‘simple’ (non mixed) absorbing, or chaotic area, these are such that successive images, of almost all points of a neighborhood enter into the area and can not go a way after entering, after a finite number of iterations. The successive images of the points, which do not enter into the area, are those one of the two segments of the stable set of saddle points on the area boundary .

During the last few years the study of two-(and higher) dimensional noninvertible maps is becoming a subject of increasingly wider interest and research, and some of the results mentioned have been sporadically rediscovered by other authors. It is worth nothing that many systems in engineering, particularly in control theory and electronics, lead to models in the form of noninvertible maps. It is particularly the case in some control systems using either sampled data, or switching elements ,or pulse modulation and also in some adaptive controls .Moreover , modeling in economics and biology often given rise to noninvertible maps [3, 6& 12] .

To our knowledge , the notion of critical curves in the study of two – dimensional endomorphism was introduced in 1964 in relation to its role in the determination of basin boundaries by Mira & Gardini [1991, 1992, 1993, 1994] has recently studied in series of papers global bifurcation and invariant manifold interaction for the noninvertible case , in contrast to the corresponding invertible phenomena .In the same spirit Frouzzakis (1992) discussed the formation of self – intersecting loops of the unstable set of saddle fixed point in a model of an adaptively controlled system (in the form of a two – dimensional noninvertible map) .

Many researchers were interested in the field of noninvertible maps due to their importance. The following are some of them :

- Gardini L. in [16] studied the global dynamics and bifurcations of a croeconomic model which showed the interactions between “good market “ and “ the money market “ by using the role of critical curves .
- Gardini L. , Abraham R. , Record R. ,and Fourntier –Prunaret D. in [19] studied the dynamics occurring in logistic map and by use of critical curves , absorbing and invariant areas were determined inside which global bifurcation of the attracting sets (fixed points , closed invariant curves , cycles or chaotic attractors) take place . The basin of attraction of the absorbing areas are determined together with their bifurcation .
- Cathala J., in [10] examined chaotic areas and absorbing area without specifying the structures of the attractors that they contain for the map $(T: x \to ax + y, y \to bx + x^3, b = -1.9)$ also he defined some bifurcations that modify the nature of the chaotic areas .
- Mira C. , and Narayaninsamy T. ,in [25] determined dynamical properties and bifurcations for the map $(T: x \to x^2 - y^2 + \lambda + +\varepsilon x, y \to 2xy - \frac{5}{2}\varepsilon y)$ by using critical curves .
- Mira C., Gardini L., Fournier - Prunaret D., Kawaakami H. , and Cathala J., in [26] studied some properties of the basins of noninvertible map $(T: x \to ax + y, y \to x^2 + b)$ by using the method of critical curves , also they described different kinds of basin bifurcation , some of them were leading to basin boundary fractalization .

The aim of this thesis is to study noninvertible planar maps $T: \Re^2 \to \Re^2$, in particular absorbing areas of such maps , we shall give some examples that illustrate certain phenomena for such areas .

We shall try to make two conjectures:

1) When the critical set of a map $T: \Re^2 \to \Re^2$ is a parabola, then T has an attractor
2) If the critical set of the map is a line or a hyperbola, then the map will have periodic points .

It is important to remark that much of the work done on planar maps concentrated on presenting certain examples and pointing out certain phenomena .

The work is divided into three chapters , these chapters are organized as follows :

Chapter one introduces the mathematical background of the main notions and proposition on the theory of the dynamical system . Definition of critical curves and some different types of noninvertible maps related to their critical curves are presented , we shall also mention different definitions of chaos and some properties of topological conjugate maps .Also we shall give the definition and some properties of planar quadratic maps. Moreover, we shall prove two new theorems of planar quadratic maps with unbounded critical set that can be considered a generalization of theorems given in [31].

Chapter two deals with a special type of planar maps, namely $(z_0 - z_2)$ maps. The chapter includes:

- Proving some properties of absorbing areas and invariant areas .
- Proving some properties of a $(z_0 - z_2)$ maps .
- How to construct absorbing areas, invariant areas by using the critical curves.
- Recalling some bifurcation types.

In chapter three we shall illustrate the concepts of the two previous chapters by applications on some noninvertible maps .Since the algorithm construction of an absorbing areas which appeared in chapter two does not guarantee that constructed areas are absorbing, therefore we shall try to give conjection ensuring absorbing.

Our examples in this chapter illustrate certain phenomena that are different from the ones found in literature.

List of Symbols :

Symbols	Meaning
$W_l^u(p^*)$	Local unstable set of fixed point p^*
$W^u(p^*)$	Global unstable set of fixed point p^*
$W_l^s(p^*)$	Local stable set of fixed point p^*
$W^s(p^*)$	Global stable set of fixed point p^*
$D(A)$	Basin of attraction
A	Closed set
T	Planar quadratic map
LC_{-1} or $J(T)$	Critical set of T
LC_i	Critical curve of rank-i of the map T
$\overline{LC}_i$	Extra critical curve
$EC(T^m)$	Critical curve of T^m
$G(x,y)$	Initial form of the quadratic map
B_T	Set of all points x in $\Re^2$ with bounded orbits
$\Im_1$	Set of maps T with nonempty bounded critical set
$\Im_2$	Set of maps T with nonempty unbounded critical set
d'	Non-mixed absorbing area
$\tilde{d}'$	Mixed absorbing area
d	Chaotic area
S	Invariant area
Δ	Closed area
s	Mixed or non-mixed absorbing area noninvariant
d''	Connected non-mixed noninvariant absorbing area
Δ_0	Closed subset of $\overline{\Re}_2$
d'_a	Annular absorbing area
W	Hole

Contents :

Acknowledgement

My deepest thanks for **Allah** for his generous by providing me the strength me accomplish this work .

I would like to express my deep appreciation and gratitude to my supervisor **Dr. Adil G. Naoum** whose advice will live up with me , and thanks to him cannot be expressed in words , and **Dr. Aludin N. Ahmed** for their help and encouragement during the work .

Also, I would like to express my thanks to all members and fellowship in the Departement of Mathematics and Computer Applications .

My deep thank is for my family and friends who encourage my through my work .

Zainab Al-wa'li

إلى نبراس علوم الأولين والآخرين

إلى بدر الدجى وامام المتقين

إلى المبعوث رحمة للعالمين

إلى معلم البشرية وخاتم النبيين

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