ABSTRACT

Fractional Calculus is a branch of mathematical analysis that satisfies the possibility of considering the power of the differential operator as a real number. Several different families of fractional derivatives (such as, Riemann–Liouville, Caputo, Hadamard and others) are developed.

In this work, we are investigate the applications of the Laplace transform to construct the solution of homogenous and nonhomogeneous linear differential equations having multi–arbitrary fractional order derivatives involving the Riemann–Liouville fractional derivatives with constant coefficients in terms of special function called “Mittage–Leffler Function” by using Laplace transform formula for such special function and their derivatives.

Several examples are solved to demonstrate our constructed solutions formulas.

PREFACE

Arbitrary order's (real and complex number) derivatives and integrals are called generalized differentiation and integration or generalized differentiation for the sake of convenience. This generalized differentiation is commonly called as "Fractional Calculus" [14].

Although this subject can claim to be well upwards of 200 years old, and its foundations have been securely in place for more than a century, the first book – length account of the field did not appear until 1974, when Oldham and Spanier [16] published "The Fractional Calculus" [22]. This book provides brief, yet important, accounts of the history, definitions, properties, and some applications of the fractional calculus.

Many found, using their own notation and methodology, definitions that fit the concept of a fractional (non–integer) order integral or derivative, but the most famous of these definitions that have been popularized in the world of fractional calculus are the Riemann–Liouville and Grunwald–Letnikov definitions [16].

The importance and popularity of fractional calculus has been gained during the past three decades or so, due mainly to its demonstrated applications in many field of science and engineering, including fluid flow, diffusive transport theory, electrical networks, electromagnetic theory, and electrochemistry [22]. Also, applications of fractional calculus may be found in mechanical engineering and finance [8]. In signal processing Loverro [9] used a fractional order transfer function to provide a good prediction for the reaction of a certain algorithm planner where the input of this algorithm is a signal.

Moreover, classical calculus may be considered as a field of applications of fractional calculus [16] and [15], introducing a novel class of functions which have certain properties [6], have been made by means of fractional calculus.

Furthermore, fractional calculus has been used to solve some classes of differential, integro–differential equations and diffusion problems [16]. An application in probability was given in [21].

In this work, we constructing the solution of homogenous and nonhomogeneous linear fractional order differential equations with constant coefficients involving the Riemann–Liouville fractional derivatives

This work is organized as follows:

Chapter one provides some basic definitions and properties from such topics of Mathematical Analysis as functional spaces, special functions, integral transforms, generalized functions, and so on. The extensive modern–day usages of such special functions as the classical Mittag–Leffler functions. Moreover, this chapter contains the definitions and some potentially useful properties of Riemann–Liouville fractional derivatives.

In chapter two we investigate the applications of the Laplace integral transform with a view to constructing the solutions of homogeneous linear fractional order differential equations involving the Riemann–Liouville fractional derivatives with constant coefficients including some examples to demonstrate our constructed solutions formula.

In chapter three we investigate the applications of the Laplace integral transform with a view to constructing the solutions of nonhomogeneous linear fractional order differential equations involving the Riemann–Liouville fractional derivatives with constant coefficients including some examples to demonstrate our constructed solutions formula.

Finally, conclusions and future work have been presented.

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Farah A. Farjo

Supervisor Certification

I certify that this thesis was prepared under my supervision at the Department of Mathematics and Computer Applications, College of Science, Al–Nahrain University as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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CONTENTS

Chapter One

Preliminaries

Chapter One

PRELIMINARIES

The main purpose of this chapter is to make this work as self–contained as possible. So we shall give some definitions and properties from such Topics of Analysis as functional spaces, special functions, and integral transforms.

1.1 Space of Absolutely Continuous Functions:

In this section we present definitions of space of absolutely continuous that will be needed later. More detailed information may be found in [7], [18] and [13].

Let $[a,b]$ be a finite interval and let $f(x)$ be a functions which called absolutely continuous on $[a,b]$, if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any finite set of pairwise nonintersecting intervals $[a_k, b_k] \subset [a,b],\ k = 1,2,3,...,n$, such that $\sum_{k=1}^{n}(b_k - a_k) < \delta$, the inequality $\sum_{k=1}^{n}\left|f(b_k) - f(a_k)\right| < \varepsilon$ holds. The space of these functions is denoted by $AC[a,b]$. It is known in [7] that $AC[a,b]$ coincides with the space of primitives of Lebesgue summable functions:

$$f \in AC[a,b] \Leftrightarrow f(x) = c + \int_a^x \varphi(t)dt \qquad (1.1.1)$$

and therefore an absolutely continuous function $f(x)$ has a summable derivative $f'(x) = \varphi(x)$ almost everywhere on $[a,b]$. Thus (1.1.1) yields

$$\varphi(x) = f'(x) \qquad \text{and} \qquad c = f(a)$$

For $n \in N = \{1,2,3,...\}$ we denote by $AC^n[a,b]$ the space of complex–valued functions $f(x)$ which have continuous derivatives up of order $n-1$ on $[a,b]$ such that $f^{(n-1)} \in AC[a,b]$:

$$AC^n[a,b] = \left\{ f : [a,b] \to C \quad and \quad (D^{n-1} f) \in AC[a,b] \quad \left(D = \frac{d}{dx} \right) \right\}$$

C being the set of complex numbers. In particular, $AC^1[a,b] = AC[a,b]$.

<u>Lemma 1.1:</u> [18]

The space $AC^n[a,b]$ consists of those and only those functions $f(x)$ which can be represented in the form

$$f(x) = (I_{a+}^n \varphi)(x) + \sum_{k=0}^{n-1} c_k (x-a)^k \tag{1.1.2}$$

where $c_k \, (k = 0,1,...,n-1)$ are arbitrary constants, and

$$(I_{a+}^n \varphi)(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} \varphi(t) dt \tag{1.1.3}$$

It follows from (1.1.2) that

$$\varphi(x) = f^{(n)}(x), \quad c_k = \frac{f^{(k)}(a)}{k!} \qquad (k = 0,1,2,...,n-1)$$

1.2 The Gamma Function and Related Special Functions:

In this section we present the definitions and some properties of the Euler Gamma function and of some special functions connected with this function. More detailed information may be found in [5] and [1].

The Euler gamma function $\Gamma(z)$ is defined by the so-called Euler integral of the second kind:

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt \qquad (\mathrm{Re}(z)>0) \tag{1.2.1}$$

where $t^{z-1}=e^{(z-1)\log(t)}$. This integral is convergent for all complex $z\in C$ $(\mathrm{Re}(z)>0)$.

The Gamma function satisfies the recurrence relation

$$\Gamma(z+1)=z\Gamma(z) \qquad (\mathrm{Re}(z)>0) \tag{1.2.2}$$

It is obtained from (1.2.1) by integration by parts. Using this relation, the Euler gamma function is extended to the half–plane $\mathrm{Re}(z)\le 0$ by

$$\Gamma(z)=\frac{\Gamma(z+n)}{(z)_n} \qquad (\mathrm{Re}(z)>-n;\quad n\in N;\quad z\notin Z_0^-=\{0,-1,-2,...\}) \tag{1.2.3}$$

Here $(z)_n$ is defined for complex $z\in C$ and non–negative integer $n\in N=\{1,2,3,...\}$ by

$$(z)_0=1 \qquad \text{and} \qquad (z)_n=z(z+1)(z+2)...(z+n-1) \tag{1.2.4}$$

Equations (1.2.2) and (1.2.4) yield

$$\Gamma(n+1)=(1)_n=n! \qquad (n\in N_0=\{0,1,2,...\})$$

with $0!=1$.

It follows from (1.2.3) that the gamma function is analytic everywhere in the complex plane C except at $z=0,-1,-2,...$, where $\Gamma(z)$ has simple poles.

The beta function is defined by the Euler integral of the first kind [1]:

$$\beta(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}dt \qquad (\mathrm{Re}(z)>0;\ \ \mathrm{Re}(w)>0)$$

This function is connected with the gamma functions by the relation

$$\beta(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \qquad (z,w\notin Z^-)$$

The binomial coefficients are defined for $\alpha\in C$ and $n\in N$ by the formula

$$\binom{\alpha}{0}=1,\ \binom{\alpha}{n}=\frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-n+1)}{n!} \qquad (1.2.5)$$

In particular, when $\alpha = m \quad (m \in N_0)$, we have

$$\binom{m}{n} = \frac{m!}{n!(m-n)!} \qquad (n \in N_0; \quad m \geq n)$$

and

$$\binom{m}{n} = 0 \qquad (n \in N_0; \quad 0 \leq m < n)$$

If $\alpha \notin Z^- = \{-1,-2,-3,...\} = Z_0^- \setminus \{0\}$, (1.2.5) is represented via the gamma function by

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)} \qquad (\alpha \in C; \quad n \in N_0)$$

Such a relation can be extended from $n \in N_0$ to arbitrary complex $\beta \in C$ by

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)\Gamma(\beta+1)} \qquad (\alpha, \beta \in C; \quad \alpha \notin Z^-)$$

The incomplete gamma functions $\gamma(z,w)$ and $\Gamma(z,w)$ are defined for $z, w \in C$ by [1]

$$\gamma(z,w) = \int_0^w t^{z-1} e^{-t} dt \qquad (\mathrm{Re}(z) > 0)$$

and

$$\Gamma(z,w) = \int_w^\infty t^{z-1} e^{-t} dt$$

respectively. The following relation is evident:

$$\gamma(z,\infty) = \Gamma(z,0) = \Gamma(z) = \gamma(z,w) - \Gamma(z,w) \qquad (\mathrm{Re}(z) > 0)$$

1.3 Classical Mittag–Leffler Functions: [4]

In this section we present the definitions and some properties of two classical Mittag–Leffler functions.

The function $E_\alpha(z)$ defined by

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \qquad (z\in C;\quad \mathrm{Re}(\alpha)>0) \tag{1.3.1}$$

where $\Gamma(.)$ is the Gamma function.

This equation is known as the Mittag–Leffler function [12]. We present some properties of this function. In particular, when $\alpha=1$ and $\alpha=2$, we have

$$E_1(z)=e^z \quad \text{and} \quad E_2(z)=\cosh(\sqrt{z})$$

When $\alpha=n\in N$, the following differentiation formulas hold for the function $E_n(\lambda z^n)$:

$$\frac{d^n}{dz^n}E_n(\lambda z^n)=\lambda E_n(\lambda z^n) \qquad (\lambda\in C) \tag{1.3.2}$$

and

$$\frac{d^n}{dz^n}\left[z^{n-1}E_n\left(\frac{\lambda}{z^n}\right)\right]=\frac{(-1)^n\lambda}{z^{n+1}}E_n\left(\frac{\lambda}{z^n}\right) \qquad (z\neq 0;\quad \lambda\in C) \tag{1.3.3}$$

The Mittag–Leffler function $E_{\alpha,\beta}(z)$, generalizing the one in (1.3.1), is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha\, k + \beta)} \qquad (z, \beta \in C; \quad \mathrm{Re}(\alpha) > 0) \tag{1.3.4}$$

When $\beta = 1$, $E_{\alpha,\beta}(z)$ coincides with the Mittag–Leffler function (1.3.1):

$$E_{\alpha,1}(z) = E_{\alpha}(z) \qquad (z \in C; \quad \mathrm{Re}(\alpha) > 0)$$

Like the Mittag–Leffler function $E_{\alpha}(z)$, $E_{\alpha,\beta}(z)$ is satisfies the following differentiation formulas generalized those in (1.3.2) and (1.3.3):

$$\frac{d^n}{dz^n}\left[z^{\beta-1} E_{n,\beta}\left(\lambda z^n\right)\right] = z^{\beta-n-1} E_{n,\beta-n}\left(\lambda z^n\right) \qquad (n \in N; \quad \lambda \in C)$$

and

$$\frac{d^n}{dz^n}\left[z^{n-\beta} E_{n,\beta}\left(\frac{\lambda}{z^n}\right)\right] = \frac{(-1)^n \lambda}{z^{n+\beta}} E_{n,\beta}\left(\frac{\lambda}{z^n}\right) \qquad (z \neq 0; \quad n \in N; \quad \lambda \in C)$$

Now, we present some properties of special functions defined in terms of the Mittag–Leffler $E_{\alpha,\alpha}(z)$

We consider a function defined for $z \in C \setminus \{0\}$ and $\alpha, \lambda \in C$ in terms of the Mittag–Leffler function (1.3.1) by

$$E_{\alpha}(\lambda z^{\alpha}) \qquad (\mathrm{Re}(\alpha) > 0) \tag{1.3.5}$$

The following differentiation formulas hold for this function with respect to z:

$$\frac{\partial^n}{\partial z^n}\left[E_\alpha\left(\lambda z^\alpha\right)\right]= z^{-n}E_{\alpha,1-n}\left(\lambda z^\alpha\right) \tag{1.3.6}$$

and with respect to λ:

$$\frac{\partial^n}{\partial \lambda^n}\left[E_\alpha\left(\lambda z^\alpha\right)\right]= n!\, z^{\alpha n}E_{\alpha,\alpha n+1}^{n+1}\left(\lambda z^\alpha\right) \tag{1.3.7}$$

where $E_\alpha(\lambda z^\alpha)$ is the generalized Mittag–Leffler function which is denoted by

$$E_{\alpha,\beta}^{p}(z)=\sum_{k=0}^{\infty}\frac{(p)_k\, z^k}{\Gamma(\alpha\, k+\beta)k!} \qquad (z\in C;\ \ \alpha,\beta,p\in C;\ \ \mathrm{Re}(\alpha)>0)$$

1.4 Riemann–Liouville Fractional Derivatives: [18]

In this section we give the definitions and some properties of the Riemann–Liouville fractional derivatives on a finite interval of the real line.

Let $\Omega=[a,b]$ be a finite interval on the real axis R. The Riemann–Liouville fractional derivatives $D^{\alpha}_{a+}y$ and $D^{\alpha}_{b-}y$ of order $\alpha\in C$ $(\mathrm{Re}(\alpha)\geq 0)$ are defined by

$$(D^{\alpha}_{a+}y)(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x\frac{y(t)dt}{(x-t)^{\alpha-n+1}} \tag{1.4.1}$$

$$(n=[\mathrm{Re}(\alpha)]+1;\quad x>a)$$

and

$$(D^{\alpha}_{b-}y)(x)=\frac{-1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b\frac{y(t)dt}{(t-x)^{\alpha-n+1}} \tag{1.4.2}$$

$$(n=[\mathrm{Re}(\alpha)]+1;\quad x<b)$$

respectively, where $[\mathrm{Re}(\alpha)]$ means the integral part of $\mathrm{Re}(\alpha)$. These derivatives are called the left–sided and the right–sided fractional derivatives. In particular, when $\alpha=n\in N_0$, then

$$(D^{0}_{a+}y)(x)=(D^{0}_{b-}y)(x)=y(x);\qquad \text{and}$$

$$(D^{n}_{a+}y)(x)=y^{(n)}(x)\qquad (D^{n}_{b-}y)(x)=(-1)^n y^{(n)}(x)\qquad (n\in N)$$

where $y^{(n)}(x)$ is the usual derivative of $y(x)$ of order n.

If $\mathbf{0 < Re(\alpha) < 1}$, then equations (1.4.1) and (1.4.2) becomes

$$(D_{a+}^{\alpha} y)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{y(t)dt}{(x-t)^{\alpha}} \qquad (x > a) \tag{1.4.3}$$

and

$$(D_{b-}^{\alpha} y)(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{y(t)dt}{(t-x)^{\alpha}} \qquad (x < b) \tag{1.4.4}$$

The Riemann–Liouville fractional derivatives (1.4.1) and (1.4.2), defined on a finite interval $[a,b]$ of the real line R, are naturally extended to the half–axis R^{+}. The fractional differentiation construction, corresponding to those in (1.4.1) and (1.4.2), are defined by

$$(D_{+}^{\alpha} y)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{y(t)dt}{(x-t)^{\alpha-n+1}} \tag{1.4.5}$$

and

$$(D_{-}^{\alpha} y)(x) = \frac{-1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^{\infty} \frac{y(t)dt}{(t-x)^{\alpha-n+1}} \tag{1.4.6}$$

with $n = [\mathrm{Re}(\alpha)] + 1$; $\mathrm{Re}(\alpha) \geq 0$; $x > 0$.

The above expressions for $D_{+}^{\alpha} y$ and $D_{-}^{\alpha} y$ are called the Riemann–Liouville left–sided and right–sided fractional derivatives on the half–axis R^{+}. In particular, when $\alpha = n \in N_0$, then

$$(D_{+}^{0} y)(x) = (D_{-}^{0} y)(x) = y(x);$$

$$(D_{+}^{n} y)(x) = y^{(n)}(x) \qquad (D_{-}^{n} y)(x) = (-1)^n y^{(n)}(x) \qquad (n \in N)$$

where $y^{(n)}(x)$ is the usual derivative of $y(x)$ of order n.

If $\mathbf{0 < Re(\alpha) < 1}$ and $\boldsymbol{x > 0}$, then equations (1.4.5) and (1.4.6) becomes

$$(D_+^{\alpha} y)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{y(t)dt}{(x-t)^{\alpha}} \tag{1.4.7}$$

and

$$(D_-^{\alpha} y)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^{\infty} \frac{y(t)dt}{(t-x)^{\alpha}} \tag{1.4.8}$$

1.5 Laplace Transform:

In this section we present the definitions and some properties of one–dimensional Laplace transform. More detailed information may be found in [19], [3], [2] and [20] (for the one–dimensional case).

1.5.1 Usual Laplace Transform:

The Laplace transform of a function f of a real variable $t \in R^+ = (0,\infty)$ is defined by

$$(\mathcal{L}\, f)(s) = \int_0^\infty e^{-st} f(t)dt \qquad (s \in C) \tag{1.5.1}$$

$$= \lim_{b\to\infty} \int_0^b e^{-st} f(t)dt = F(s)$$

whenever the limit exists (as a finite number). When it does, the integral (1.5.1) is said to be converge. If the limit does not exist, the integral (1.5.1) is said to be diverge and there is no Laplace transform defined for f.

Now, we want to consider the inverse problem, given a function $F(s)$, we want to find the function $f(t)$ whose Laplace transform is $F(s)$. We introduce the notation

$$(\mathcal{L}^{-1} F(s))(t) = f(t) \tag{1.5.2}$$

to denote such a function $f(t)$, and it is called the inverse Laplace transform of F.

The direct and inverse Laplace transforms are inverse to each other:

$$\mathcal{L}^{-1}\mathcal{L}\varphi = \varphi \qquad \text{and} \qquad \mathcal{L}\mathcal{L}^{-1}g = g$$

Now, we present some simple properties of the Laplace transform

$$\mathcal{L}\left[D^k \varphi(t)\right](s) = s^k(\mathcal{L}\,\varphi)(s) \qquad (k \in N) \tag{1.5.3}$$

$$D^k(\mathcal{L}\,\varphi)(s) = (-1)^k \mathcal{L}[t^k \varphi(t)](s) \qquad (k \in N)$$

The convolution operator of two functions $h(t)$ and $\varphi(t)$, given on R^+, is defined for $t \in R^+$ by the integral

$$h * \varphi = (h * \varphi)(t) = \int_0^t h(t-x)\varphi(x)dx \tag{1.5.4}$$

which has the commutative property

$$h * \varphi = \varphi * h \tag{1.5.5}$$

The Laplace transform of the convolution $h * \varphi$ is given by

$$(\mathcal{L}\,(h * \varphi))(s) = (\mathcal{L}\,h)(s)(\mathcal{L}\,\varphi)(s) \tag{1.5.6}$$

1.5.2 Laplace Transform for Fractional Ordinary Differential Equations: [18]

In this section we present the Laplace transform of the Riemann–Liouville fractional derivatives $D_{+}^{\alpha} y$ in (1.4.5).

Lemma 1.2: [18]

Let $\mathrm{Re}(\alpha) > 0$ and $n = [\mathrm{Re}(\alpha)] + 1;$ $y \in AC^{n}[0,b]$ for any $b > 0$. Also let the following estimate

$$|y(t)| \leq B e^{q_0 t} \qquad (t > b > 0) \tag{1.5.7}$$

hold, for constants $B > 0$ and $q_0 > 0$, and if $y^{(k)}(0) = 0$ $(k = 0,1,2,\ldots,n-1)$ then the relation

$$(\mathcal{L}\, D_{+}^{\alpha} y)(s) = s^{\alpha} (\mathcal{L}\, y)(s) \tag{1.5.8}$$

is valid for $\mathrm{Re}(s) > q_0$.

<u>Remark 1.1:</u> [18]

If $\mathbf{Re}(\alpha) > 0, \;\; n = [\mathbf{Re}(\alpha)] + 1; \;\; y(x) \in AC^n[0, b]$, for any $b > 0$, the condition in (1.5.6) is satisfied and there exist the finite limits

$$\lim_{x \to 0+} \left[D^k I_+^{n-\alpha} y(x) \right] \quad \text{and} \quad \lim_{x \to \infty} \left[D^k I_+^{n-\alpha} y(x) \right] = 0$$

$$(D = d / dx; \qquad k = 0, 1, 2, \ldots, n-1)$$

where $(I_+^{n-\alpha} y)$ is defined in (1.1.3).

Then from (1.4.7) and

$$\mathcal{L}[D^k \varphi(t)](s) = s^\alpha (\mathcal{L}\, \varphi)(s) - \sum_{j=0}^{k-1} s^{k-j-1} (D^j \varphi)(0) \qquad (k \in N)$$

we derive a relation, more general than that in (1.5.8), of the form

$$(\mathcal{L}\, D_+^\alpha y)(s) = s^\alpha (\mathcal{L}\, y)(s) - \sum_{k=0}^{n-1} s^{n-k-1} D^k (I_+^{n-\alpha} y)(0+) \qquad (1.5.9)$$

$$\left(\mathbf{Re}(s) > q_0 \right)$$

In particular, when $0 < \mathbf{Re}(\alpha) < 1$ and $y(x) \in AC[0, b]$ for any $b > 0$, then

$$(\mathcal{L}\, D_+^\alpha y)(s) = s^\alpha (\mathcal{L}\, y)(s) - (I_+^{1-\alpha} y)(0+) \qquad (1.5.10)$$

Putting $p = \beta = 1$ in

$$\mathcal{L}\left[t^{\beta-1} E_{\alpha,\beta}^{p}(\lambda t^{\alpha})\right](s) = \frac{s^{\alpha p-\beta}}{(s^{\alpha}-\lambda)^{p}}$$

where $\mathrm{Re}(s) > 0$, $\mathrm{Re}(\beta) > 0$, $\lambda \in C$, and $\left|\lambda s^{-\alpha}\right| < 1$ and taking $E_{\alpha,\beta}^{1}(z) = E_{\alpha,\beta}(z)$ $(z \in C)$ and $E_{\alpha,1}(z) = E_{\alpha}(z)$ $(z \in C;\ \mathrm{Re}(\alpha) > 0)$ into account we obtain the Laplace transform of the function (1.3.5):

$$\mathcal{L}\left[E_{\alpha}(\lambda t^{\alpha})\right](s) = \frac{s^{\alpha-1}}{s^{\alpha}-\lambda} \qquad (1.5.11)$$

$$(\mathrm{Re}(s) > 0, \lambda \in C, \text{and } \left|\lambda s^{-\alpha}\right| < 1)$$

and differentiating (1.5.11) n times with respect to λ leads to the relation

$$\mathcal{L}\left[t^{\alpha n} \frac{\partial^{n}}{\partial \lambda^{n}} E_{\alpha}(\lambda t^{\alpha})\right](s) = \frac{n!\, s^{\alpha-1}}{(s^{\alpha}-\lambda)^{n+1}} \qquad (n \in N) \qquad (1.5.12)$$

Next we consider a function, more general than that in (1.3.5), defined by

$$z^{\beta-1} E_{\alpha,\beta}(\lambda z^{\alpha}) \qquad (z \in C \setminus \{0\};\ \alpha, \beta, \lambda \in C;\ \mathrm{Re}(\alpha) > 0) \qquad (1.5.13)$$

The following relations, analogous to those in (1.3.6), (1.3.7), (1.5.11) and (1.5.12), are valid for the function in (1.5.13), $(n \in N)$:

$$\frac{\partial^n}{\partial z^n}\left[z^{\beta-1}E_{\alpha,\beta}(\lambda z^{\alpha})\right]= z^{\beta-n-1}E_{\alpha,\beta-n}(\lambda z^{\alpha}) \tag{1.5.14}$$

$$\frac{\partial^n}{\partial \lambda^n}\left[z^{\beta-1}E_{\alpha,\beta}(\lambda z^{\alpha})\right]= n!\, z^{\alpha n+\beta-1}E^{n+1}_{\alpha,\alpha n+\beta}(\lambda z^{\alpha}) \tag{1.5.15}$$

$$\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\right](s) = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda} \tag{1.5.16}$$

$$(\mathrm{Re}(s) > 0; \;\; \lambda \in C; \;\; \left|\lambda s^{-\alpha}\right| < 1)$$

$$\mathcal{L}\left[t^{\alpha n+\beta-1}\frac{\partial^n}{\partial \lambda^n}E_{\alpha,\beta}(\lambda t^{\alpha})\right](s) = \frac{n!\, s^{\alpha-\beta}}{(s^{\alpha}-\lambda)^{n+1}} \tag{1.5.17}$$

$$(\left|\lambda s^{-\alpha}\right| < 1)$$

Chapter Two

Integral Transform Method for the Solutions to Homogenous Fractional Order Differential Equations with Constant Coefficients

Chapter Two

INTEGRAL TRANSFORM METHOD FOR THE SOLUTIONS TO HOMOGENEOUS FRACTIONAL ORDER DIFFRENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The present chapter is devoted to the application of Laplace integral transform to construct the solutions to linear and homogeneous fractional order differential equations involving the Riemann–Liouville fractional derivatives with constant coefficients of the form

$$\sum_{k=1}^{m} A_k (D_+^{\alpha_k} y)(x) + A_0 y(x) = 0 \tag{2.1}$$

$$(x > 0;\ \ m \in N;\ \ 0 < \alpha_1 < \alpha_2 < \alpha_3 < ... < \alpha_m)$$

with the Riemann–Liouville fractional derivatives $D_+^{\alpha_k} y\ \ (k = 1,2,3,...,m)$, given by (1.4.5). Here $A_k \in R\ \ (k = 0,1,2,...,m)$ are real constants, and, generally speaking, we can take $A_m = 1$. We give the conditions when the solutions $y_1(x), y_2(x), y_3(x),..., y_l(x)$ of the equation (2.1) with $l - 1 < \alpha = \alpha_m \le l\ \ (l \in N)$ will be linearly independent, and when these linearly independent solutions form the fundamental system of solutions, which (by analogy with the ordinary case) is defined by

$$(D_+^{\alpha-k} y_j)(0) = 0 \qquad (k, j = 1,2,3,...,l;\ \ k \ne j)$$

$$(D_+^{\alpha-k} y_k)(0) = 1 \qquad (k = 1,2,3,...,l) \tag{2.2}$$

The Laplace transform method is based on the relation (1.5.9) which, in accordance with (1.4.5), is equivalent to the following one:

$$(\mathcal{L}\, D_+^{\alpha} y)(s) = s^{\alpha}(\mathcal{L}[y(x)])(s) - \sum_{j=1}^{l} d_j s^{j-1} \tag{2.3}$$

$$(l-1 < \alpha \le l; \quad l \in N)$$

$$d_j = (D_+^{\alpha-j} y)(0) \qquad (j = 1,2,3,\ldots,l) \tag{2.4}$$

Now, we give three theorems for finding the solutions of equation (2.1) in case $m = 1,\ \ m = 2$ and $m \in N$

The idea of these proves based on the implemented of Laplace Transform Method.

The Laplace transform was used by [17], [11] and [16] to solve simple and special cases of fractional order differential equations.

First we derive the solutions to equation (2.1) with $m=1$ in the form

$$(D_{+}^{\alpha}y)(x)-\lambda y(x)=0 \qquad (x>0;\ \ l-1<\alpha\le l;\ \ l\in N;\ \ \lambda\in R) \quad (2.5)$$

in terms of the Mittag–leffler functions (1.3.4). There holds the following statement.

Theorem 2.1:

Let $l-1<\alpha\le l \quad (l\in N)$ and $\lambda\in R$. Then the functions

$$y_j(x)=x^{\alpha-j}E_{\alpha,\alpha+1-j}(\lambda x^{\alpha}) \qquad (j=1,2,3,...,l)$$

yield the fundamental system of solutions to equation (2.5).

Proof:

Applying the Laplace transform formula to equation (2.5)

$$(\mathcal{L}\, D_{+}^{\alpha}y)(x)-(\mathcal{L}\,\lambda y)(x)=0$$

From (2.3) we have

$$s^{\alpha}(\mathcal{L}\,y)(s)-\sum_{j=1}^{l}d_j s^{j-1}-\lambda(\mathcal{L}\,y)(s)=0$$

Therefore, we have

$$(\mathcal{L}y)(s) = \sum_{j=1}^{l} d_j \frac{s^{j-1}}{s^{\alpha} - \lambda} \tag{2.6}$$

where $d_j \quad (j = 1,2,3,...,l)$ are given by (2.4). Formula (1.5.16) with $\beta = \alpha + 1 - j$ yields

$$\mathcal{L}\left[t^{\alpha-j} E_{\alpha,\alpha+1-j}(\lambda t^{\alpha})\right](s) = \frac{s^{j-1}}{s^{\alpha} - \lambda} \qquad (\left|\lambda\, s^{-\alpha}\right| < 1)$$

Thus from (2.7), we derive the following solution to equation (2.5):

$$y(x) = \sum_{j=1}^{l} d_j y_j(x) \qquad y_j(x) = x^{\alpha-j} E_{\alpha,\alpha+1-j}(\lambda x^{\alpha})$$

It is easily verified that the functions $y_j(x)$ are solutions to equation (2.5)

$$(D_+^{\alpha}\left[t^{\alpha-j} E_{\alpha,\alpha+1-j}(\lambda t^{\alpha})\right](x) = \lambda x^{\alpha-j} E_{\alpha,\alpha+1-j}(\lambda x^{\alpha}) \qquad (j = 1,2,3,...,l)$$

and, moreover,

$$(D_+^{\alpha-k} y_j)(x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma(\alpha\, n + k + 1 - j)} x^{\alpha\, n + k - j} \tag{2.8}$$

It follows from (2.8) that

$$\begin{gathered} (D_+^{\alpha-k} y_j)(0) = 0 \qquad (k, j = 1,2,3,...,l; \quad k > j) \\ (D_+^{\alpha-k} y_k)(0) = 1 \qquad (k = 1,2,3,...,l) \end{gathered} \tag{2.9}$$

If $k < j$, then

$$(D_{+}^{\alpha-k} y_j)(x) = \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma(\alpha n + k + 1 - j)} x^{\alpha n + k - j}$$

$$= \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{\Gamma(\alpha n + \alpha + k + 1 - j)} x^{\alpha n + \alpha + k - j} \qquad (2.10)$$

and since $\alpha + k - j \geq \alpha + 1 - l > 0$ for any $k, j = 1,2,3...,l$. The following relations hold

$$(D_{+}^{\alpha-k} y_j)(0) = 0 \qquad (k, j = 1,2,3,...,l; \quad k < j) \qquad (2.11)$$

By (2.9) and (2.11) the result of this theorem follows from (2.2).

■

A Special Case of Theorem (2.1):

The equation

$$(D_{+}^{\alpha} y)(x) - \lambda\, y(x) = 0 \qquad (x > 0; \quad 0 < \alpha \leq 1; \quad \lambda \in R)$$

has its solution given by

$$y(x) = x^{\alpha-1} E_{\alpha,\alpha}(\lambda x^{\alpha})$$

while the equation

$$(D_{+}^{\alpha} y)(x) - \lambda\, y(x) = 0 \qquad (x > 0; \quad 1 < \alpha \leq 2; \quad \lambda \in R)$$

has the fundamental system of solutions given by

$$y_1(x) = x^{\alpha-1} E_{\alpha,\alpha}(\lambda x^{\alpha}), \qquad y_2(x) = x^{\alpha-2} E_{\alpha,\alpha-1}(\lambda x^{\alpha})$$

Example 2.1:

The equation

$$(D_{+}^{l-1/2}y)(x)-\lambda\, y(x)=0 \qquad (x>0;\ \ l\in N;\ \ \lambda\in R)$$

has the fundamental system of solutions given by

$$y_j(x)=x^{l-j-1/2}E_{l-1/2,l-j+1/2}(\lambda x^{l-1/2}) \qquad (j=1,2,3,...,l)$$

Example 2.2:

The following ordinary differential equation of order $l\in N$

$$y^{(l)}(x)-\lambda\, y(x)=0 \qquad (x>0)$$

has the fundamental system of solutions given by

$$y_j(x)=x^{l-j}E_{l,l+1-j}(\lambda x^{l}) \qquad (j=1,2,3,...,l)$$

Next we derive the solutions to equation (2.1) with $m = 2$ of the form

$$(D_{+}^{\alpha} y)(x) - \lambda (D_{+}^{\beta} y)(x) - \mu\, y(x) = 0 \qquad (2.12)$$

$$(x > 0;\;\; l-1 < \alpha \le l;\;\; l \in N;\;\; \alpha > \beta > 0)$$

with $\lambda, \mu \in R$.

Theorem 2.2:

Let $l-1 < \alpha \le l;\; l \in N;\; 0 < \beta < \alpha$ and $\lambda, \mu \in R$. Then equation (2.12) is solvable and the functional system

$$y_j(x) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} x^{\alpha n + \alpha - j} \frac{\partial^n}{\partial \lambda^n} E_{\alpha-\beta, n\beta+\alpha+1-j}(\lambda x^{\alpha-\beta}) \qquad (2.13)$$

$$(j = 1,2,3,\ldots,l)$$

are its solutions.

In particular, the equation

$$(D_{+}^{\alpha} y)(x) - \lambda (D_{+}^{\beta} y)(x) = 0 \qquad (2.14)$$

$$(x > 0;\;\; l-1 < \alpha \le l;\;\; l \in N;\;\; \alpha > \beta > 0)$$

has its solution given by

$$y_j(x) = x^{\alpha-j} E_{\alpha-\beta, \alpha+1-j}(\lambda x^{\alpha-\beta}) \qquad (j = 1,2,3,\ldots,l) \qquad (2.15)$$

If $\alpha - l + 1 \geq \beta$, then $y_j(x)$ in (2.13) and (2.15) are linearly independent solutions to equations (2.12) and (2.14), respectively. In particular, for $\alpha - l + 1 > \beta$ they yield the fundamental system of solutions.

Proof:

Let $m - 1 < \beta \leq m$ $(m \in N;\ m \leq l)$. Applying the Laplace transform to (2.12) and using (2.3) as in (2.7), we obtain

$$(\mathcal{L}y)(s) = \sum_{j=1}^{l} d_j \frac{s^{j-1}}{s^{\alpha} - \lambda s^{\beta} - \mu} \tag{2.16}$$

where, for all $j = m+1, m+2, m+3, ..., l$

$$d_j = (D_+^{\alpha-j} y)(0) + (D_+^{\beta-j} y)(0)$$

and for all $j = 1,2,3,...,m$

$$d_j = (D_+^{\alpha-j} y)(0)$$

For $s \in C$ and $\left| \frac{\mu s^{-\beta}}{s^{\alpha-\beta} - \lambda} \right| < 1$, using [9], we have

$$\frac{1}{s^{\alpha} - \lambda s^{\beta} - \mu} = \frac{s^{-\beta}}{(s^{\alpha-\beta} - \lambda)} \frac{1}{\left(1 - \frac{\mu s^{-\beta}}{s^{\alpha-\beta} - \lambda}\right)} = \sum_{n=0}^{\infty} \frac{\mu^n s^{-\beta-n\beta}}{\left(s^{\alpha-\beta} - \lambda\right)^{n+1}} \tag{2.17}$$

and hence (2.16) has the following representation:

$$(\mathcal{L}\, y)(s) = \sum_{j=1}^{l} d_j \sum_{n=0}^{\infty} \mu^n \frac{s^{j-1-\beta-n\beta}}{\left(s^{\alpha-\beta} - \lambda\right)^{n+1}} \tag{2.18}$$

Using (1.5.17), with α replaced by $\alpha-\beta$ and β by $\alpha+n\beta+1-j$, for $s\in C$ and $\left|\lambda s^{\beta-\alpha}\right|<1$, we have

$$\frac{s^{j-1-\beta-n\beta}}{\left(s^{\alpha-\beta}-\lambda\right)^{n+1}}=\frac{s^{(\alpha-\beta)-(\alpha+n\beta+1-j)}}{\left(s^{\alpha-\beta}-\lambda\right)^{n+1}}$$

$$=\frac{1}{n!}\left(\mathcal{L}\left[t^{\alpha n+\alpha-j}\frac{\partial^n}{\partial\lambda^n}E_{\alpha-\beta,\alpha+n\beta+1-j}(\lambda t^{\alpha-\beta})\right]\right)(s) \quad (2.19)$$

From (2.18) and (2.19) we derive the solution to equation (2.12)

$$y(x)=\sum_{j=1}^{l}d_j y_j(x) \quad (2.20)$$

where $y_j(x)$ $(j=1,2,3,...,l)$ are given by (2.13). It is readily verified that these functions are solutions to equation (2.12), which proves the first assertion of Theorem 2.2.

For $j,k=1,2,3,...,l$, the direct evaluation yields

$$(D_+^{\alpha-k}y_j)(x)=\sum_{n=0}^{\infty}\frac{\mu^n}{n!}x^{\alpha n+k-j}\frac{\partial^n}{\partial\lambda^n}E_{\alpha-\beta,n\beta+k+1-j}(\lambda x^{\alpha-\beta}) \quad (2.21)$$

It follows from (2.21) that the relations in (2.9) hold for $k\geq j$. If $k<j$, then, we rewrite (2.21) as follows:

$$(D_+^{\alpha-k}y_j)(x)=\sum_{q=0}^{\infty}\frac{\lambda^{q+1}}{\Gamma[(\alpha-\beta)(q+1)+k+1-j]}x^{(\alpha-\beta)q+\alpha-\beta+k-j}+$$

$$\sum_{n=1}^{\infty}\frac{\mu^n}{n!}x^{\alpha n+k-j}\frac{\partial^n}{\partial\lambda^n}E_{\alpha-\beta,n\beta+k+1-j}(\lambda x^{\alpha-\beta})=I_1(x)+I_2(x) \quad (2.22)$$

If $\alpha - l + 1 \geq \beta$, then $(\alpha-\beta)q + \alpha - \beta + k - j \geq \alpha - \beta + 1 - l \geq 0$ for any $j,k = 1,2,3,...,l$ and $q \in N_0$. Thus $\lim_{x\to 0+} I_1(x) = 0 \;\; (j,k = 1,2,3,...,l; \;\; k < j)$ except for the case $\alpha - l + 1 = \beta$ with $k = 1$ and $j = l$, for which $\lim_{x\to 0+} I_1(x) = \lambda$. Moreover, since $\alpha n + k - j \geq \alpha + 1 - l > 0$ for any $j,k = 1,2,3,...,l$ and $n \in N$, then $\lim_{x\to 0+} I_2(x) = 0 \;\; (k,j = 1,2,3,...,l; \;\; k < j)$. Thus (2.22) yields the relation (2.11) for any solution $y_j(x)$ in (2.13), except for the case $\alpha - l + 1 = \beta$ with $k = 1$ and $j = l$, for which

$$(D_+^{\alpha-1} y_l)(0) = \lambda \tag{2.23}$$

It follows from (2.9), (2.11) and (2.23) that $y_j(x)$ in (2.13) are linearly independent solutions to the equation (2.12).

If $\alpha - l + 1 > \beta$, then the relations (2.2) are valid, and hence $y_j(x)$ in (2.13) yield the fundamental system of solutions to equation (2.12).

■

A Special Case of Theorem (2.2):

The equation

$$(D_+^{\alpha} y)(x) - \lambda (D_+^{\beta} y)(x) - \mu \, y(x) = 0 \;\; (x > 0; \;\; 0 < \beta < \alpha \leq 1; \;\; \lambda, \mu \in R)$$

has its solution given by

$$y_1(x) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} x^{\alpha n + \alpha - 1} \frac{\partial^n}{\partial \lambda^n} E_{\alpha-\beta, n\beta+\alpha}(\lambda x^{\alpha-\beta}) \tag{2.24}$$

In particular,

$$y_1(x) = x^{\alpha-1} E_{\alpha-\beta,\alpha}(\lambda x^{\alpha-\beta}) \tag{2.25}$$

is the solution to the following equation:

$$(D_+^{\alpha} y)(x) - \lambda (D_+^{\beta} y)(x) = 0 \;\; (x > 0; \;\; 0 < \beta < \alpha \leq 1; \;\; \lambda \in R)$$

Another Special Case of Theorem (2.2):

The equation

$$(D_+^{\alpha} y)(x) - \lambda (D_+^{\beta} y)(x) - \mu\, y(x) = 0 \tag{2.26}$$

$$(x > 0;\ \ 1 < \alpha \le 2;\ \ 0 < \beta < \alpha;\ \ \lambda, \mu \in R)$$

has two solutions $y_1(x)$, given by (2.24), and

$$y_2(x) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!}\, x^{\alpha n + \alpha - 2}\, \frac{\partial^n}{\partial \lambda^n} E_{\alpha-\beta, n\beta+\alpha-1}(\lambda x^{\alpha-\beta})$$

In particular, the equation

$$(D_+^{\alpha} y)(x) - \lambda (D_+^{\beta} y)(x) = 0 \tag{2.27}$$

$$(x > 0;\ \ 1 < \alpha \le 2;\ \ 0 < \beta < \alpha;\ \ \lambda \in R)$$

has two solutions $y_1(x)$, given by (2.25), and

$$y_2(x) = x^{\alpha-2} E_{\alpha-\beta, \alpha-1}(\lambda x^{\alpha-\beta})$$

If $\alpha \ge \beta + 1$, then the above functions $y_1(x)$ and $y_2(x)$ are linearly independent solutions to the equation (2.26) and (2.27), respectively. In particular, for $\alpha > \beta + 1$ these functions provide the fundamental system of solutions.

Example 2.3:

The equation

$$y'(x) - \lambda(D_{+}^{\beta} y)(x) - \mu\, y(x) = 0 \ (x > 0; \quad 0 < \beta < 1; \quad \lambda, \mu \in R)$$

has its solution given by

$$y(x) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} x^n \frac{\partial^n}{\partial \lambda^n} E_{1-\beta, n+1}(\lambda x^{1-\beta})$$

In particular,

$$y(x) = \sum_{n=0}^{\infty} \frac{(\mu x)^n}{n!} \frac{\partial^n}{\partial \lambda^n} E_{1/2, n+1}(\lambda x^{1/2})$$

is the solution to the equation

$$y'(x) - \lambda(D_{+}^{1/2} y)(x) - \mu\, y(x) = 0 \quad (x > 0; \quad \lambda, \mu \in R)$$

Example 2.4:

The equation

$$y''(x) - \lambda(D_{+}^{\beta} y)(x) - \mu\, y(x) = 0 \quad (x > 0; \quad 0 < \beta < 2; \quad \lambda, \mu \in R)$$

has its two solutions given by

$$y_1(x) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} x^{2n+1} \frac{\partial^n}{\partial \lambda^n} E_{2-\beta, n\beta+2}(\lambda x^{2-\beta})$$

$$y_2(x) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} x^{2n} \frac{\partial^n}{\partial \lambda^n} E_{2-\beta, n\beta+1}(\lambda x^{2-\beta})$$

These solutions are linearly independent when $\beta \le 1$ and form the fundamental system of solutions when $\beta < 1$.

In particular, the equation

$$y''(x) - \lambda (D_{+}^{1/2} y)(x) - \mu\, y(x) = 0 \qquad (x > 0;\ \ \lambda, \mu \in R)$$

has the fundamental system of solutions given by

$$y_1(x) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} x^{2n+1} \frac{\partial^n}{\partial \lambda^n} E_{3/2,(1/2)n+2}(\lambda x^{3/2})$$

$$y_2(x) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} x^{2n} \frac{\partial^n}{\partial \lambda^n} E_{3/2,(1/2)n+1}(\lambda x^{3/2})$$

Example 2.5:

The following ordinary differential equation of order $l \in N$

$$y^{(l)}(x) - \lambda y^{(m)}(x) - \mu\, y(x) = 0 \qquad (x > 0;\ \ m \in N;\ \ m < l;\ \ \lambda, \mu \in R)$$

has l solutions given by

$$y_j(x) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} x^{nl+l-j} \frac{\partial^n}{\partial \lambda^n} E_{l-m,mn+l+1-j}(\lambda x^{l-m}) \quad (j = 1,2,3,\ldots,l)$$

when $m = 1$, these solutions are linearly independent.

In particular, the following ordinary second order differential equation

$$y''(x) - \lambda\, y'(x) - \mu\, y(x) = 0 \qquad (x > 0;\;\; \lambda, \mu \in R)$$

has two linearly independent solutions given by

$$y_1(x) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} x^{2n+1} \frac{\partial^n}{\partial \lambda^n} E_{1,n+2}(\lambda x)$$

$$y_2(x) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} x^{2n} \frac{\partial^n}{\partial \lambda^n} E_{1,n+1}(\lambda x)$$

Finally, we find the solutions to equation (2.1) with any $m \in N$ in the form

$$(D_+^{\alpha} y)(x) - \lambda(D_+^{\beta} y)(x) - \sum_{k=0}^{m} A_k (D_+^{\alpha_k} y)(x) = 0 \quad (x > 0) \tag{2.28}$$

$$(0 = \alpha_0 < \alpha_1 < \alpha_2 < ... < \alpha_m < \beta < \alpha; \quad \lambda, A_0, A_1, A_2, ..., A_m \in R)$$

Theorem 2.3:

Let $m \in N$, $l - 1 < \alpha \le l \quad (l \in N)$ and let β and $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_m$ be such that $\alpha > \beta > \alpha_m > \alpha_{m-1} > \alpha_{m-2} > ... > \alpha_1 > \alpha_0 = 0$, and let $\lambda, A_0, A_1, A_2, ..., A_m \in R$. Then equation (2.28) is solvable and the functional system

$$y_j(x) = \sum_{n=0}^{\infty} \left(\sum_{k_0 + ... k_m = n} \right) \frac{1}{k_0! ... k_m!} \left[\prod_{v=0}^{m} (A_v)^{k_v} \right]$$

$$\cdot\, x^{(\alpha-\beta)n + \alpha - j + \sum_{v=0}^{m} (\beta - \alpha_v) k_v} \frac{\partial^n}{\partial \lambda^n} E_{\alpha-\beta, \alpha + 1 - j + \sum_{v=0}^{m} (\beta - \alpha_v) k_v} (\lambda x^{\alpha-\beta}) \tag{2.29}$$

with $j = 1,2,3,...,l$, are its solutions. The inner sum is taken over all $k_0, k_2, k_3, ..., k_m \in N_0$ such that $k_0 + k_1 + k_2 + ... + k_m = n$.

If $\alpha - l + 1 \ge \beta$, then $y_j(x)$ in (2.29) are linearly independent solutions to equation (2.28) . In particular, for $\alpha - l + 1 > \beta$ they provide the fundamental system of solutions.

Proof:

Let $l_{m+1}-1<\beta\le l_{m+1},\ \ l_k-1<\alpha_k\le l_k$ $(k=1,2,3,...,m)$ $(0\le l_1\le l_2\le l_3\le ...\le l_{m+1}\le l)$. Applying the Laplace transform to (2.28) and using (2.3) as in (2.16), we obtain

$$(\mathcal{L}y)(s)=\sum_{j=1}^{l} d_j\,\frac{s^{j-1}}{s^{\alpha}-\lambda s^{\beta}-\sum_{k=0}^{m}A_k s^{\alpha_k}} \tag{2.30}$$

where

$$d_j=(D_+^{\alpha-j}y)(0)-\lambda(D_+^{\beta-j}y)(0)-\sum_{k=1}^{m}A_k(D_+^{\alpha_k-j}y)(0)\quad (j=1,2,3,...,l_1)$$

$$d_j=(D_+^{\alpha-j}y)(0)-\lambda(D_+^{\beta-j}y)(0)-\sum_{k=2}^{m}A_k(D_+^{\alpha_k-j}y)(0)\quad (j=l_1+1,l_1+2,...,l_2)...$$

$$d_j=(D_+^{\alpha-j}y)(0)-\lambda(D_+^{\beta-j}y)(0)\quad (j=l_m+1,l_m+2,l_m+3...,l_{m+1})$$

$$d_j=(D_+^{\alpha-j}y)(0)\quad (j=l_{m+1}+1,l_{m+1}+2,l_{m+1}+3,...,l)$$

here $\sum_{k=m}^{n} A_k = 0 \quad (m > n)$. For $s \in C$ and $\left| \frac{\sum_{k=0}^{m} A_k s^{\alpha_k - \beta}}{s^{\alpha-\beta} - \lambda} \right| < 1$, just as in (2.17), we have

$$\frac{1}{s^{\alpha} - \lambda s^{\beta} - \sum_{k=0}^{m} A_k s^{\alpha_k}} = \frac{s^{-\beta}}{(s^{\alpha-\beta} - \lambda)} \frac{1}{\left(1 - \frac{\sum_{k=0}^{m} A_k s^{\alpha_k - \beta}}{s^{\alpha-\beta} - \lambda}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{s^{-\beta}}{\left(s^{\alpha-\beta} - \lambda\right)^{n+1}} \left(\sum_{k=0}^{m} A_k s^{\alpha_k - \beta} \right)^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k_0 + \ldots k_m = n} \right) \frac{1}{k_0! \ldots k_m!} \left[\prod_{v=0}^{m} (A_v)^{k_v} \right] \frac{s^{-\beta - \sum_{v=0}^{m} (\beta - \alpha_v) k_v}}{(s^{\alpha-\beta} - \lambda)^{n+1}} \tag{2.31}$$

According to (1.5.17), just as in (2.19), for $s \in C$ and $\left| \lambda\, s^{\beta-\alpha} \right| < 1$, we have

$$\frac{s^{j-1-\beta-\sum_{v=0}^{m}(\beta-\alpha_v)k_v}}{\left(s^{\alpha-\beta} - \lambda\right)^{n+1}} = \frac{s^{(\alpha-\beta)-(\alpha+1-j+\sum_{v=0}^{m}(\beta-\alpha_v)k_v)}}{\left(s^{\alpha-\beta} - \lambda\right)^{n+1}}$$

$$= \frac{1}{n!} \left(\mathcal{L} \left[t^{(\alpha-\beta)n+\alpha-j+\sum_{v=0}^{m}(\beta-\alpha_v)k_v} \frac{\partial^n}{\partial \lambda^n} E_{\alpha-\beta,\alpha+1-j+\sum_{v=0}^{m}(\beta-\alpha_v)k_v} (\lambda t^{\alpha-\beta}) \right] \right)(s) \tag{2.32}$$

From (2.30), (2.31) and (2.32) we obtain the solution to equation (2.28) in the form (2.20), where $y_j(x)$ $(j=1,2,3,...,l)$ are given by (2.29). It is easily verified that these functions are solutions to (2.28), and thus the first assertion of the Theorem 2.3 is proved.

For $j,k=1,2,3,...,l$, the direct evaluation leads to the following equations:

$$(D_+^{\alpha-k}y_j)(x)=x^{k-j}\frac{\partial^n}{\partial\lambda^n}E_{\alpha-\beta,k+1-j}(\lambda x^{\alpha-\beta})+$$

$$\sum_{n=1}^{\infty}\left(\sum_{k_0+...k_m=n}\right)\frac{1}{k_0!...k_m!}\left[\prod_{v=0}^{m}(A_v)^{k_v}\right]x^{(\alpha-\beta)n+k-j+\sum_{v=0}^{m}(\beta-\alpha_v)k_v}$$

$$\cdot\frac{\partial^n}{\partial\lambda^n}E_{\alpha-\beta,k+1-j+\sum_{v=0}^{m}(\beta-\alpha_v)k_v}(\lambda x^{\alpha-\beta}) \tag{2.33}$$

with $j=1,2,3,...,l$. If $k\geq j$, then the last formula yields the relations in (2.9). If $k<j$, then, the first term in the right – hand side of (2.33) takes the form

$$x^{k-j}\frac{\partial^n}{\partial\lambda^n}E_{\alpha-\beta,k+1-j}(\lambda x^{\alpha-\beta})=x^{k-j}\sum_{\mu=1}^{\infty}\frac{\lambda^{\mu}}{\Gamma[k+1-j+(\alpha-\beta)\mu]}x^{(\alpha-\beta)\mu}$$

$$=x^{k-j+\alpha-\beta}\sum_{\mu=0}^{\infty}\frac{\lambda^{\mu+1}}{\Gamma[k+1-j+(\alpha-\beta)(\mu+1)]}x^{(\alpha-\beta)\mu} \tag{2.34}$$

If $\alpha - l + 1 \geq \beta$, then for any $k < j$, we have $k - j + \alpha - \beta \geq \alpha - \beta + 1 - l \geq 0$ and $(\alpha - \beta)n + k - j + \sum_{v=0}^{m}(\beta - \alpha_v)k_v \geq \alpha - \beta + 1 - l > 0$ for any $n \in N_0$. Then from (2.33) and (2.34), we derive (2.11), except for the case $\alpha - l + 1 = \beta$ with $k = 1$ and $j = l$, for which the relation (2.23) holds. it follows from (2.9), (2.11) and (2.23) that the functions $y_j(x)$ in (2.29) are linearly independent solutions to equation (2.28). When $\alpha - l + 1 > \beta$, then the relations in (2.2) are valid, and thus $y_j(x)$ in (2.29) yield the fundamental system of solutions to the equation (2.28).

■

A Special Case of Theorem (2.3):

If $l \in N$ and $\lambda, A_0, A_1, A_2, ..., A_l \in R$, then the following ordinary differential equation of order l

$$y^{(l)}(x) - \lambda\, y^{(l-1)}(x) - \sum_{k=0}^{l} A_k y^{(k)}(x) = 0 \qquad (x > 0)$$

has l solutions given by

$$y_j(x) = \sum_{n=0}^{\infty}\left(\sum_{k_0+...k_l=n}\right)\frac{1}{k_0!...k_l!}\left[\prod_{v=0}^{l}(A_v)^{k_v}\right] x^{n+l-j+\sum_{v=0}^{l}(l-1-v)k_v}$$

$$\cdot \frac{\partial^n}{\partial \lambda^n} E_{1,l+1-j+\sum_{v=0}^{l}(l-1-v)k_v}(\lambda x) \qquad (j = 1,2,3,...,m)$$

Example 2.6:

The equation

$$(D_+^{\alpha} y)(x) - \lambda (D_+^{\beta} y)(x) - \delta (D_+^{\gamma} y)(x) - \mu\, y(x) = 0 \qquad (2.35)$$

with $l - 1 < \alpha \le l \ \ (l \in N)$ and $0 < \gamma < \beta < \alpha$, has l solutions given by

$$y_j(x) = \sum_{n=0}^{\infty} \left(\sum_{q+v=n} \right) \frac{\mu^q \delta^v}{q!v!} x^{(\alpha-\beta)n+\alpha-j+\beta q+(\beta-\gamma)v}$$

$$\cdot \frac{\partial^n}{\partial \lambda^n} E_{\alpha-\beta,\alpha+1-j+\beta q+(\beta-\gamma)v}(\lambda x^{\alpha-\beta}) \qquad (2.36)$$

$$(j = 1,2,3,\ldots,l)$$

If $\alpha - l + 1 \ge \beta$, then the functions $y_j(x)$ in (2.36) are linearly independent solutions to equation (2.35). In particular, for $\alpha - l + 1 > \beta$ these functions provide the fundamental system of solutions.

Example 2.7:

The ordinary differential equation of order $l \in N$

$$y^{(l)}(x) - \lambda\, y^{(m)}(x) - \delta\, y^{(k)}(x) - \mu\, y(x) = 0$$

where $x > 0$; $m, k \in N;\ \ k < m < l;\ \ \lambda, \delta, \mu \in R$

has l solutions given by

$$y_j(x) = \sum_{n=0}^{\infty} \left(\sum_{q+v=n} \right) \frac{\mu^q \delta^v}{q!v!}\, x^{(l-m)n+l-j+mq+(m-k)v}$$

$$\cdot \frac{\partial^n}{\partial \lambda^n} E_{l-m,l+1-j+mq+(m-k)v}(\lambda x^{l-m}) \qquad (j = 1,2,3,\ldots,l)$$

Chapter Three

Integral Transform Method for the Solutions to Nonhomogenous Fractional Order Differential Equations with Constant Coefficients

Chapter Three

INTEGRAL TRANSFORM METHOD FOR THE SOLUTIONS TO NONHOMOGENEOUS FRACTIONAL ORDER DIFFRENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The present chapter is to present a scheme for solving the fractional nonhomogeneous differential equation with constant coefficients of the form

$$\sum_{k=1}^{m} A_k (D_+^{\alpha_k} y)(x) + A_0 y(x) = f(x) \qquad (x > 0) \tag{3.1}$$

with $m \in N;\ 0 < \mathrm{Re}(\alpha_1) < \mathrm{Re}(\alpha_2) < ... < \mathrm{Re}(\alpha_m);\ A_0, A_1, A_2, ..., A_m \in R$, and involving the Riemann–Liouville fractional derivatives $D_+^{\alpha_k} y \ \ (k = 1,2,3,...,m)$, given by (1.4.5). By (1.5.10), for suitable functions y, the Laplace transform (2.1) of $D_+^{\alpha} y$ is given by

$$(\mathcal{L}\, D_+^{\alpha} y)(s) = s^{\alpha} (\mathcal{L}\, y)(s) \tag{3.2}$$

Taking the Laplace transform of (3.1) and taking (3.2) into account, we have

$$\left[A_0 + \sum_{k=1}^{m} A_k s^{\alpha_k} \right] (\mathcal{L}\, y)(s) = (\mathcal{L}\, f)(s)$$

Using the inverse Laplace transform $\mathcal{L}^{-1}$ given by (1.5.2), from here we obtain a particular solution to the equation (3.1) in the form

$$y(x) = \left(\mathcal{L}^{-1}\left[\frac{(\mathcal{L}f)(s)}{A_0 + \sum_{k=1}^{m} A_k s^{\alpha_k}} \right] \right)(x) \qquad (3.3)$$

Miller [11] introduced a fractional analog of the Green function $G_\alpha(x)$ defined, via the inverse Laplace transform (1.5.2), by

$$G_\alpha(x) = \left(\mathcal{L}^{-1}\left[\frac{1}{P(s^\alpha)} \right] \right)(x), \quad P(s) = \sum_{k=1}^{m} A_k s^k + A_0 \qquad (3.4)$$

represented a particular solution to the nonhomogeneous equation (3.1) in the form of the convolution of $G_\alpha(x)$ and $f(x)$:

$$y(x) = \int_0^x G_\alpha(x-t) f(t) dt$$

and proved that this formula yields a unique solution $y(x)$ to the equation (3.1) with the following initial conditions:

$$y(0) = y'(0) = y''(0) = \ldots = y^{(m-1)}(0) = 0$$

In Chapter 2 we applied the Laplace transform method to derive the solutions to the homogeneous equation (2.1) with the Riemann–Liouville fractional derivatives (1.4.1). Here we use this approach to find particular solutions to the corresponding nonhomogeneous equations.

$$\sum_{k=1}^{m} A_k (D_+^{\alpha_k} y)(x) + A_0 y(x) = f(x) \tag{3.5}$$

$$(x > 0;\ \ 0 < \alpha_1 < \alpha_2 < ... < \alpha_m;\ \ m \in N)$$

with real $A_k \in R\ \ (k = 0,1.2....,m)$ and a given function $f(x)$ on R^+. Our arguments are based on a scheme for deducing a particular solution (3.3) to equation (3.5), presented in Chapter 2. Using the Laplace convolution formula (1.4.6).

$$\left(\mathcal{L} \left(\int_0^x k(x-t) f(t) dt \right) \right)(s) = (\mathcal{L}\, k)(s)(\mathcal{L}\ f)(p)$$

just as in (3.4) we can introduce the Laplace fractional analog of the Green function as follows [10].

$$G_{\alpha_1,\alpha_2,\alpha_3,...,\alpha_n}(x) = \left(\mathcal{L}^{-1} \left[\frac{1}{P_{\alpha_1,\alpha_2,\alpha_3,...,\alpha_m}(s)} \right] \right)(x) \tag{3.6}$$

$$P_{\alpha_1,\alpha_2,\alpha_3,...,\alpha_m}(s) = A_0 + \sum_{k=1}^{m} A_k s^{\alpha_k}$$

and express a particular solution (3.3) of equation (3.5) in the form of the Laplace convolution of $G_{\alpha_1,\alpha_2,\alpha_3,...,\alpha_n}(x)$ and $f(x)$

$$y(x) = \int_0^x G_{\alpha_1,\alpha_2,\alpha_3,...,\alpha_n}(x-t)\, f(t)dt \tag{3.7}$$

Generally speaking, we can consider equation (3.5) with $A_m = 1$.

Now, we give three theorems for finding the solutions of equation (3.1) in case $m = 1,\ \ m = 2$ and $m \in N$

The idea of these proves based on the implemented of the Green Function.

The Green function was used by [10] to solve nonhomogeneous fractional order differential equations.

First we derive a particular solution to equation (3.5) with $m = 1$ in the form

$$(D_{+}^{\alpha} y)(x) - \lambda\, y(x) = f(x) \qquad (x > 0; \;\; \alpha > 0) \tag{3.8}$$

in terms of the Mittag–Leffler function (1.3.4).

Theorem 3.1:

Let $\alpha > 0,\;\; \lambda \in R$ and let $f(x)$ be a given function defined on R^{+}. Then equation (3.8) is solvable, and its particular solution has the form

$$y(x) = \int_{0}^{x} (x-t)^{\alpha-1} E_{\alpha,\alpha}\left[\lambda (x-t)^{\alpha}\right] f(t)dt \tag{3.9}$$

Proof:

Its clear that equation (3.8) can be obtained from equation (3.5) in case if we take $m = 1,\;\; \alpha_1 = \alpha,\;\; A_1 = 1,\;\; A_0 = -\lambda$, and equation (3.6) takes the form

$$G_{\alpha}(x) = \left(\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha} - \lambda}\right]\right)(x)$$

By (1.5.16) with $\beta = \alpha$, we have

$$\mathcal{L}\left[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})\right](s) = \frac{1}{s^{\alpha}-\lambda} \qquad (\mathrm{Re}(s) > 0; \quad \left|\lambda s^{-\alpha}\right| < 1)$$

Hence

$$G_{\alpha}(x) = x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^{\alpha})$$

and thus (3.7), with $G_{\alpha_1,\alpha_2,\alpha_3\ldots,\alpha_m}(x) = G_{\alpha}(x)$, yields (3.9).

■

Example 3.1:

The equation

$$(D_{+}^{l-1/2}y)(x) - \lambda\, y(x) = f(x) \qquad (x > 0; \quad l \in N; \quad \lambda \in R)$$

has a particular solution given by

$$y(x) = \int_{0}^{x}(x-t)^{l-3/2}E_{l-1/2,l-1/2}[\lambda(x-t)^{l-1/2}]\, f(t)dt$$

Example 3.2:

The following ordinary differential equation of order $l \in N$

$$y^{(l)}(x) - \lambda\, y(x) = f(x) \qquad (x > 0)$$

has a particular solution given by

$$y(x) = \int_0^x (x-t)^{l-1} E_{l,l}[\lambda (x-t)^l]\, f(t)dt$$

Next we derive a particular solution to equation (3.5) with $m = 2$ of the form

$$(D_+^{\alpha} y)(x) - \lambda (D_+^{\beta} y)(x) - \mu\, y(x) = f(x) \qquad (x > 0;\ \ \alpha > \beta > 0) \tag{3.10}$$

Theorem 3.2:

Let $\alpha > \beta > 0$, $\lambda, \mu \in R$ and let $f(x)$ be a given function defined on R^{+}. Then equation (3.10) is solvable, and its particular solution has the form

$$y(x) = \int_0^x (x-t)^{\alpha-1} G_{\alpha,\beta;\lambda,\mu}(x-t) f(t) dt \tag{3.11}$$

where

$$G_{\alpha,\beta;\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} z^{\alpha n} \frac{\partial^n}{\partial \lambda^n} E_{\alpha-\beta, n\beta+\alpha}(\lambda z^{\alpha-\beta}) \tag{3.12}$$

In particular, the equation

$$(D_+^{\alpha} y)(x) - \lambda (D_+^{\beta} y)(x) = f(x) \qquad (x > 0;\ \ \alpha > \beta > 0) \tag{3.13}$$

has a particular solution given by

$$y(x) = \int_0^x (x-t)^{\alpha-1} E_{\alpha-\beta,\alpha}\left[\lambda (x-t)^{\alpha-\beta}\right] f(t) dt \tag{3.14}$$

Proof:

Its clear that equation (3.10) can be obtained from equation (3.5) in case if we take $m = 2$, $\alpha_0 = \alpha,\ \alpha_1 = \beta,\ A_2 = 1,\ A_1 = -\lambda,\ A_0 = -\mu$, and equation (3.6) is given by

$$G_{\alpha,\beta}(x) = \left(\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha} - \lambda s^{\beta} - \mu}\right]\right)(x)$$

According to (2.17) for $s \in C$ and $\left|\dfrac{\mu s^{-\beta}}{s^{\alpha-\beta} - \lambda}\right| < 1$, we have

$$G_{\alpha,\beta}(x) = \left(\mathcal{L}^{-1}\left[\sum_{n=0}^{\infty} \mu^{n} \frac{s^{-\beta-n\beta}}{\left(s^{\alpha-\beta} - \lambda\right)^{n+1}}\right]\right)(x) \quad (3.15)$$

By (1.5.17) with $\alpha = \alpha - \beta$ and $\beta = \alpha + n\beta$, for $s \in C$ and $\left|\lambda s^{\beta-\alpha}\right| < 1$, we have

$$\frac{s^{-\beta-n\beta}}{\left(s^{\alpha-\beta} - \lambda\right)^{n+1}} = \frac{1}{n!}\left(\mathcal{L}\left[t^{(\alpha-\beta)n+(\alpha+n\beta)-1} \frac{\partial^{n}}{\partial\lambda^{n}} E_{\alpha-\beta,\alpha+n\beta}(\lambda t^{\alpha-\beta})\right]\right)(s)$$

$$= \frac{1}{n!}\left(\mathcal{L}\left[t^{\alpha(n+1)-1} \frac{\partial^{n}}{\partial\lambda^{n}} E_{\alpha-\beta,\alpha+n\beta}(\lambda t^{\alpha-\beta})\right]\right)(s)$$

and hence (3.15) takes the following form:

$$G_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \mu^{n} x^{\alpha(n+1)-1} \frac{\partial^{n}}{\partial\lambda^{n}} E_{\alpha-\beta,\alpha+n\beta}(\lambda x^{\alpha-\beta})$$

Thus the result in (3.11) follows from (3.7) with $G_{\alpha_1,\alpha_2,\alpha_3...,\alpha_n}(x) = G_{\alpha,\beta}(x)$. (3.11) with $\mu = 0$ yields (3.14). Note that, in the limiting case $\beta = 0$, the solution (3.14) of equation (3.13) coincides with the solution (3.9) of equation (3.8).

■

Example 3.3:

The equation

$$y'(x) - \lambda(D_+^{\beta} y)(x) - \mu\, y(x) = f(x) \quad (x > 0;\;\; 0 < \mathrm{Re}(\beta) < 1;\;\; \lambda, \mu \in R)$$

has a particular solution given by

$$y(x) = \int_0^x G_{1,\beta;\lambda,\mu}(x-t) f(t) dt$$

where

$$G_{1,\beta;\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{(\mu z)^n}{n!} \frac{\partial^n}{\partial \lambda^n} E_{1-\beta, n\beta+1}(\lambda z^{1-\beta})$$

In particular, the equation

$$y'(x) - \lambda(D_+^{1/2} y)(x) - \mu\, y(x) = f(x) \qquad (x > 0;\;\; \lambda, \mu \in R)$$

has a particular solution given by

$$y(x) = \int_0^x G_{1,1/2;\lambda,\mu}(x-t) f(t) dt$$

Where

$$G_{1,1/2;\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{(\mu z)^n}{n!} \frac{\partial^n}{\partial \lambda^n} E_{1/2,(1/2)n+1}(\lambda z^{1/2})$$

Example 3.4:

The equation

$$y''(x) - \lambda(D_{+}^{\beta} y)(x) - \mu\, y(x) = f(x) \qquad (x > 0;\;\; 0 < \beta < 2;\;\; \lambda, \mu \in R)$$

has a particular solution given by

$$y(x) = \int_{0}^{x} (x-t) G_{2,\beta;\lambda,\mu}(x-t) f(t) dt$$

where

$$G_{2,\beta;\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} z^{2n} \frac{\partial^n}{\partial \lambda^n} E_{2-\beta, n\beta+2}(\lambda z^{2-\beta})$$

Example 3.5:

The following ordinary differential equation of order $l \in N$

$$y^{(l)}(x) - \lambda\, y^{(m)}(x) - \mu\, y(x) = f(x) \qquad (x > 0;\ \ m \in N;\ \ m < l;\ \ \lambda, \mu \in R)$$

has a particular solution

$$y(x) = \int_0^x (x-t)^{l-1} G_{l,m;\lambda,\mu}(x-t) f(t)dt$$

where

$$G_{l,m;\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} z^{ln} \frac{\partial^n}{\partial \lambda^n} E_{l-m,mn+l}(\lambda z^{l-m})$$

In particular,

$$y(x) = \int_0^x (x-t) G_{2,1;\lambda,\mu}(x-t) f(t)dt$$

where

$$G_{2,1;\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} z^{2n} \frac{\partial^n}{\partial \lambda^n} E_{1,n+2}(\lambda z)$$

is a particular solution to the equation

$$y''(x) - \lambda\, y'(x) - \mu\, y(x) = f(x) \qquad (x > 0)$$

Finally, we find a particular solution to equation (3.5) with any $m \in N$. It is convenient to rewrite (3.5), just as (2.28) in the form

$$(D_+^{\alpha} y)(x) - \lambda (D_+^{\beta} y)(x) - \sum_{k=1}^{m} A_k (D_+^{\alpha_k} y)(x) - A_0 y(x) = f(x) \quad (x > 0) \tag{3.16}$$

with $0 < \alpha_1 < \alpha_2 < \alpha_3 ... < \alpha_m < \beta < \alpha$ and $\lambda, A_0, A_1, A_2, ..., A_m \in R$.

Theorem 3.3:

Let $m \in N$, $\alpha > \beta > \alpha_m > \alpha_{m-1} > \alpha_{m-2} > ... > \alpha_1 > \alpha_0 = 0$, let $\lambda, A_0, A_1, A_2, ..., A_m \in R$, and let $f(x)$ be a given function defined on R^+. Then equation (3.16) is solvable, and its particular solution has the form

$$y(x) = \int_0^x (x-t)^{\alpha-1} G_{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_m, \beta, \alpha; \lambda}(x-t) f(t) dt \tag{3.17}$$

where

$$G_{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_m, \beta, \alpha; \lambda}(z) = \sum_{n=0}^{\infty} \left(\sum_{k_0 + ... k_m = n} \right) \frac{1}{k_0! ... k_m!} \left[\prod_{v=0}^{m} (A_v)^{k_v} \right]$$

$$\cdot z^{(\alpha-\beta)n + \sum_{v=0}^{m} (\beta - \alpha_v) k_v} \frac{\partial^n}{\partial \lambda^n} E_{\alpha-\beta, (\alpha-\beta)n + \alpha + \sum_{v=0}^{m} (\beta-\alpha_v) k_v} (\lambda z^{\alpha-\beta}) \tag{3.18}$$

The inner sum is taken over all $k_0, k_1, k_2, ..., k_m \in N_0$ such that $k_0 + k_1 + k_2 + ... + k_m = n$.

Proof:

Its clear that equation (3.16) can be obtained from equation (3.5) in case if we take $\alpha_m = \alpha,\ \alpha_{m-1} = \beta,\ A_m = 1,\ A_{m-1} = -\lambda$, and with $-A_k$ instead of A_k for $k = 0,1,2,...,m$. Since $\alpha_0 = 0$, equation (3.6) takes the form

$$G_{\alpha_1,\alpha_2,\alpha_3,...,\alpha_m}(x) = \left(\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha} - \lambda s^{\beta} - \sum_{k=0}^{m} A_k s^{\alpha_k}}\right]\right)(x)$$

For $s \in C$ and $\left|\frac{\sum_{k=0}^{m} A_k s^{\alpha_k - \beta}}{s^{\alpha-\beta} - \lambda}\right| < 1$, in accordance with (2.31), we have

$$G_{\alpha_1,\alpha_2,\alpha_3,...,\alpha_m}(x) = \left(\mathcal{L}^{-1}\left[\sum_{n=0}^{\infty}\left(\sum_{k_0+...k_m=n}\right)\frac{1}{k_0!...k_m!}\left[\prod_{v=0}^{m}(A_v)^{k_v}\right]\frac{s^{-\beta-\sum_{v=0}^{m}(\beta-\alpha_v)k_v}}{(s^{\alpha-\beta}-\lambda)^{n+1}}\right]\right)(x)$$

(3.19)

Using (1.5.17) with $\alpha = \alpha - \beta$ and $\beta = \alpha + \sum_{v=0}^{m}(\beta - \alpha_v)k_v$ just as in (2.32), for $s \in C$ and $\left|\lambda s^{\beta-\alpha}\right| < 1$, we have

$$\frac{s^{-\beta-\sum_{v=0}^{m}(\beta-\alpha_v)k_v}}{(s^{\alpha-\beta}-\lambda)^{n+1}} = \frac{1}{n!}\left(\mathcal{L}\left[t^{(\alpha-\beta)n+\alpha+\sum_{v=0}^{m}(\beta-\alpha_v)k_v-1}\frac{\partial^n}{\partial\lambda^n}E_{\alpha-\beta,\alpha+\sum_{v=0}^{m}(\beta-\alpha_v)k_v}(\lambda t^{\alpha-\beta})\right]\right)(s)$$

(3.20)

It follows from (3.20) that $G_{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_m}(x)$ in (3.19) is given by

$$G_{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_m}(x) = \sum_{n=0}^{\infty}\left(\sum_{k_0+\ldots k_m=n}\right)\frac{n!}{k_0!\ldots k_m!}\left[\prod_{v=0}^{m}(A_v)^{k_v}\right]$$

$$\cdot\, x^{(\alpha-\beta)n+\alpha+\sum_{v=0}^{m}(\beta-\alpha_v)k_v-1}\frac{\partial^n}{\partial\lambda^n}E_{\alpha-\beta,\alpha+\sum_{v=0}^{m}(\beta-\alpha_v)k_v}(\lambda x^{\alpha-\beta})$$

Hence (3. 7) yields the result in (3.17)

■

A Special Case of Theorem (3.3):

If $l \in N$ and $\lambda, A_0, A_1, A_2, ..., A_l \in R$, then the ordinary differential equation of order l

$$y^{(l)}(x) - \lambda\, y^{(l-1)}(x) - \sum_{k=0}^{l} A_k y^{(k)}(x) = f(x) \qquad (x > 0)$$

is solvable, and its particular solution has the form

$$y(x) = \int_0^x (x-t)^{l-1} G_{\lambda,l}(x-t) f(t) dt \tag{3.21}$$

where

$$G_{\lambda,l}(z) = \sum_{n=0}^{\infty} \left(\sum_{k_0 + ... k_l = n} \right) \frac{1}{k_0! ... k_l!} \left[\prod_{v=0}^{l} (A_v)^{k_v} \right]$$

$$\cdot\, z^{n + \sum_{v=0}^{l} v k_v} \frac{\partial^n}{\partial \lambda^n} E_{1, l + \sum_{v=0}^{l} v k_v}(\lambda z) \tag{3.22}$$

provided that the series in (3.22) and the integral (3.21) are convergent.

Example 3.6:

The equation

$$(D_+^{\alpha} y)(x) - \lambda (D_+^{\beta} y)(x) - \delta (D_+^{\gamma} y)(x) - \mu\, y(x) = f(x)$$

$$(x > 0; \quad \lambda, \delta, \mu \in R)$$

with $l - 1 < \alpha \leq 1 \quad (l \in N), \quad 0 < \gamma < \beta < \alpha$, has a particular solution given by

$$y(x) = \int_0^x (x-t)^{\alpha-1} G_{\gamma,\beta,\alpha;\lambda}(x-t)\, f(t)dt$$

where

$$G_{\gamma,\beta,\alpha;\lambda}(z) = \sum_{n=0}^{\infty} \left(\sum_{i+v=n} \right) \frac{\mu^i \delta^v}{i!v!}$$

$$\cdot\, z^{(\alpha-\beta)n+\beta\, i-(\beta-\gamma)v} \frac{\partial^n}{\partial \lambda^n} E_{\alpha-\beta,\alpha+\beta\, i-(\beta-\gamma)v}(\lambda z^{\alpha-\beta})$$

Example 3.7:

The following ordinary differential equation of order $l \in N$

$$y^{(l)}(x) - \lambda\, y^{(m)}(x) - \delta\, y^{(k)}(x) - \mu\, y(x) = f(x)$$

$$(x > 0;\quad m, k \in N;\quad k < m < l)$$

with $\lambda, \delta, \mu \in R$ has a particular solution given by

$$y(x) = \int_0^x (x-t)^{l-1} G_{k,m,l;\lambda}(x-t)\, f(t)dt$$

where

$$G_{k,m,l;\lambda}(z) = \sum_{n=0}^{\infty} \left(\sum_{q+v=n} \right) \frac{\mu^q \delta^v}{q!v!}$$

$$\cdot\, z^{(l-m)n+mq-(m-k)v} \frac{\partial^n}{\partial \lambda^n} E_{l-m,l+mq-(m-k)v}(\lambda z^{l-m})$$

Conclusions and Future work

CONCLUSIONS

We are obtained the solutions for ordinary multi–fractional order differential equations with constant coefficients for homogenous and nonhomogeneous fractional order differential equations making use of the Laplace transform formula of special function Mittage–Leffler function and their derivatives, by considering, simple one term with fractional order differential equation, extended to two terms with different arbitrary fraction order derivatives, then generalized to m–terms with different arbitrary fraction order derivatives.

FUTURE WORK

We are recommended the following future works for constructing the explicit solutions to homogeneous and non – homogeneous:

1. System of fractional order differential equations.
2. Special Types of fractional order differential equations with variable coefficients.
3. Fractional order differential equations using other integral transforms, such as Fourier, Mellin Integral Transforms.
4. Fractional order differential equations using other definitions, such as Caputo, Hadamard.

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Laplace Transform Method for Solving Ordinary Fractional Order Differential Equations with Constant Coefficients

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الخلاصة

حسبان التفاضل والتكامل ذات الرتب الكسرية هو أحد فروع الرياضيات التحليلية التي تمكن إمكانية إعتماد عدد حقيقي كرتبة للمعادلات التفاضلية. وإن هناك عدة أنواع مختلفة من المشتقات الكسرية مثل ريمان– لوفيل (Riemann–Liouville) و كبوتو (Caputo) و هادمرد (Hadamard) وغيرها قد طورت.

لقد قمنا بهذا البحث بتطوير تطبيقات تحويل لابلاس (Laplace Transform) لإستنباط حل للمعادلات التفاضلية الخطية المتجانسة والغير متجانسة التي تحتوي على رتب كسرية متعددة والتي تتضمن المشتقات ذات الرتب الكسرية لريمان– لوفيل (Riemann–Liouville) ذات المعاملات الثابتة بدلالة دوال خاصة تسمى دالة متيج لفلر Mittag–Leffler Function، وبإستخدام تحويل لابلاس لهكذا دوال ومشتقاتها.

وقد تم حل عدة امثلة خلال هذا البحث لتوضيح صيغ الحلول التي تم أستنباطها.

جمهورية العراق
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حل المعادلات التفاضلية الإعتيادية ذات الرتب الكسرية والمعاملات الثابتة بإستخدام تحويل لابلاس

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من

فرح انور فرجو
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بإشراف

د. علاء الدين نوري احمد

ذي القعدة ١٤٢٨ تشرين الثاني ٢٠٠٧