# *ABSTRACT*

The main theme of this thesis is oriented toward three objectives.

The first objective, is a study to fuzzy set theory with some basic properties related to differential equations.

The second objective, is a study and prove of the existence and uniqueness theorem of fuzzy differential equations using two approaches, the first is by Brower fixed point theorem and the other by Schauder fixed point theorem. Furthermore, the analytical and numerical solutions of some namely fuzzy differentials equations are given.

The third objective, is to study the real life application, which is modeling and solution of the decay of the biochemical oxygen demand in water using fuzzy set theory, as well as, the numerical solution and compared with the exact solution.

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***Estapraq Mohammed,***

***August, 2005.***

# APPENDIX

# COMPUTER PROGRAMS

## *1. Program1.pas:*

```
program variablestepmethod;
uses
crt,dos;
label
10,20,30,40;
var
Uh0,e,Ut0,Ut00,Ucc,r,Ux0,Utf,UEest,Ux11,Uh00,Ux00,Ux12:real;
Lh0,Lt0,Lt00,Lcc,Lx0,Ltf,LEest,Lx11,Lh00,Lx00,Lx12,s:real;
i,j,ss:integer;
begin
clrscr;
Uh0:=0.02; Lh0:=0.02; Ucc:=0; Lcc:=0; e:=0.000001;
Ut0:=0; Lt0:=0; r:=0.2; Utf:=0.02; Ltf:=0.02;
Lx0:=0.75+0.25*r;
Ux0:=1.125-0.125*r;
writeln('Lx0=',Lx0,'.....','Ux0=',Ux0);
10:
if Ucc<=Utf then
begin
   Ux11:=Ux0+Uh0*Ux0;
   Uh00:=Uh0/2;
   Ux00:=Ux0;
     for i:=0 to 1 do
    begin
    Ut00:=Ut0+i*Uh00;
    Ux12:=Ux00+Uh00*Ux00;
    Ux00:=Ux12;
    end;
     UEest:=abs(Ux11-Ux12);
writeln('Ux11=',Ux11,'.....','Ux12=',Ux12);
writeln('Ut00=',Ut00,'.....','Uh00=',Uh00,'.....','UEest=',UEest)
end;
readln;
if Lcc<=Ltf then
begin
   Lx11:=Lx0+Lh0*Lx0;
   Lh00:=Lh0/2;
   Lx00:=Lx0;
    for j:=0 to 1 do
   begin
   Lt00:=Lt0+j*Lh00;
   Lx12:=Lx00+Lh00*Lx00;
```

```
   Lx00:=Lx12;
   end;
   LEest:=abs(Lx11-Lx12);
   writeln('Lx11=',Lx11,'.....','Lx12=',Lx00);
   writeln('Lh00=',Lh00,'.....','Lt00=',Lt00,'.....','LEest=',LEest)
end;
if (UEest>e) and (LEest>e) then
begin
 Lh0:=(sqrt((2*e)/(sqr(Lh0)*Lx12)))*Lh0;
 Uh0:=(sqrt((2*e)/(sqr(Uh0)*Ux12)))*Uh0;
  writeln('Lh0 new','.....','Lh0=',Lh0);
  writeln('Uh0 new','.....','Uh0=',Uh0);
  goto 10
  end
  else
   begin
 Lh0:=(sqrt((2*e)/(sqr(Lh0)*Lx12)))*Lh0;
 Uh0:=(sqrt((2*e)/(sqr(Uh0)*Ux12)))*Uh0;
  writeln('Lh0 new','.....','Lh0=',Lh0);
  writeln('Uh0 new','.....','Uh0=',Uh0);
  goto 20
  end;
20:
Ucc:=Ut0+Uh0;
 if Ucc>Utf  then
  begin
  Uh0:=Utf-Ut0;
  Ux0:=Ux12;
  writeln('Uh0N=',Uh0,'.....','Ucc=',Ucc)
  end;
Lcc:=Lt0+Lh0;
 if Lcc>Ltf then
 begin
 Lh0:=Ltf-Lt0;
 Lx0:=Lx12;
 writeln('Lh0N=',Lh0,'.....','Lcc=',Lcc)
 end;
if Ucc>Utf  then  goto 30;
if Lcc>Ltf  then goto 40;
Ut0:=Ut0+Uh0;
Lt0:=Lt0+Lh0;
Ux0:=Ux12;
Lx0:=Lx12;
writeln('Ut0=',Ut0,'.....','Uh0=',Uh0);
writeln('Lt0=',Lt0,'.....','Lh0=',Lh0);
goto 10;
30:
Ux11:=Ux0+Uh0*Ux0;
if Lcc<=Ltf then
begin
Lt0:=Lt0+Lh0;
Lx0:=Lx12;
writeln('Lt0=',Lt0);
goto 10
end;
writeln('the end Ux11(Ut0)','.....','Ux11(Ut0)=',Ux11);
goto 40;
40:
Lx11:=Lx0+Lh0*Lx0;
```

```
  if Ucc<=Utf then
begin
Ut0:=Ut0+Uh0;
Ux0:=Ux12;
writeln('Ut0=',Ut0);
goto 10
end;
writeln('the end','.....','Lx11(Lt0)=',Lx11);
readln;
end.
```

## *2. Program2.pas:*

```
program NumericaResultsofEX333;
uses
crt,dos;
var
h,r,Uy0,Uy1,Uyi,Ly0,Ly1,Lyi,Lyn,Uyn,Lz0,Uz0,Lzi,Uzi,Lzn,Uzn,Ly,Uy:real;
a,g,b,i,j,n,s:integer;
begin
clrscr;
a:=0;  b:=1;  n:=10;  s:=10;
h:=(b-a)/n;   r:=0;
 repeat
Ly0:=0.75+0.25*r;
 Uy0:=1.125-0.125*r;
    writeln('EXACT RESULT');
    Ly:=Ly0*exp(1);
    Uy:=Uy0*exp(1);
    writeln('Ly=',Ly:14:7,'Uy=',Uy:14:7);
 Ly1:=Ly0+h*Ly0+(Ly0)*(sqr(h)/2);
 Uy1:=Uy0+h*Uy0+(Uy0)*(sqr(h)/2);
      for i:=2 to (n) do
      begin
      Lyi:=Ly0+Ly0*(h/3)+Ly1*(4*h/3)+(Ly1+h*Ly1)*(h/3);
      Uyi:=Uy0+Uy0*(h/3)+Uy1*(4*h/3)+(Uy1+h*Uy1)*(h/3);
          Ly0:=Ly1;  Uy0:=Uy1;
          Ly1:=Lyi;  Uy1:=Uyi;
      end;
      writeln('SIMPSON RESULT');
      writeln('r=',r:6:2,'Ly10=',Ly1:14:7,'Uy10=',Uy1:14:7);
             writeln('EULER RESULT');
             Lz0:=0.75+0.25*r;
             Uz0:=1.125-0.125*r;
              for j:=1 to (s) do
                begin
                Lzi:=Lz0+h*Lz0;
                Uzi:=Uz0+h*Uz0;
                Lz0:=Lzi;   Uz0:=Uzi;
                end;
 writeln('r=',r:6:2,'Lz10=',Lz0:14:7,'Uz10=',Uz0:14:7);
readln;
 r:=r+0.2;
 until r>1;
end.
```

## *3. Program 3.pas:*

```
program NumericaResultsofEX34;
uses
crt,dos;
var
h,r,Uy0,Uy1,Uyi,Ly0,Ly1,Lyi,Lyn,Uyn,Lz0,Uz0,Lz1,Uz1,Lzi,Uzi,Lzn,Uzn,Ly,Uy
,x0,x1,x11:real;
h1,Uw0,Uw1,Uwi,Lw0,Lw1,Lwi,Lwn,Uwn,Lv0,Uv0,Uv1,Lv1,Lvi,Uvi,Lvn,Uvn,Lw,Uw,
x21,x20,x211:real;
a,b,c,d,i,j,i1,j1,n,s,n1,s1:integer;
begin
clrscr;
a:=-1;  b:=0;  n:=10;  s:=10;
h:=(b-a)/n;  r:=0;
c:=0; d:=1;   n1:=10; s1:=10;
h1:=(d-c)/n1;
 repeat
 Ly0:=sqrt(exp(1))-0.5*(1-r);
 Uy0:=sqrt(exp(1))+0.5*(1-r);
               writeln('EXACT RESULT TO x<0');
        Ly:=((exp(-1/2)+exp(1/2))/2)*Ly0+((exp(-1/2)-exp(1/2))/2)*Uy0;
        Uy:=((exp(-1/2)+exp(1/2))/2)*Uy0+((exp(-1/2)-exp(1/2))/2)*Ly0;
        writeln('Ly=',Ly:14:7,'Uy=',Uy:14:7);
  x1:=a+h;
 Ly1:=Ly0+h*a*Uy0+(Ly0)*(sqr(h)/2)*(1+sqr(a));
 Uy1:=Uy0+h*a*Ly0+(Uy0)*(sqr(h)/2)*(1+sqr(a));
              writeln('MID-POINT RESULT TO x<0');
        for i:=2 to (n) do
        begin
        Lyi:=Ly0+2*h*x1*Uy1;
        Uyi:=Uy0+2*h*x1*Ly1;
          Ly0:=Ly1;  Uy0:=Uy1;
          Ly1:=Lyi;  Uy1:=Uyi;
        x1:=x1+h;
        end;

writeln('x1=',x1:2:2,'r=',r:6:2,'Ly10=',Ly1:14:7,'Uy10=',Uy1:14:7);
                writeln('TRAPIZOIDAL RESULT TO x<0');
        Lz0:=sqrt(exp(1))-0.5*(1-r);
        Uz0:=sqrt(exp(1))+0.5*(1-r);
        Lz1:=Lz0+h*a*Uz0+(Lz0)*(sqr(h)/2)*(1+sqr(a));
        Uz1:=Uz0+h*a*Lz0+(Uz0)*(sqr(h)/2)*(1+sqr(a));
        x0:=a;  x11:=x0+h;
        for j:=1 to (s) do
        begin
        Lzi:=Lz0+h/2*(x11*Uz1+x0*Uz0);
        Uzi:=Uz0+h/2*(x11*Lz1+x0*Lz0);
        Lz0:=Lz1;   Uz0:=Uz1;
        Lz1:=Lzi;   Uz1:=Uzi;
        x0:=x11;
        x11:=x0+h;
        end;

writeln('x11=',x0:2:2,'r=',r:6:2,'Lz10=',Lzi:14:7,'Uz10=',Uzi:14:7);
 Lw0:=Ly;
 Uw0:=Uy;
                writeln('EXACT RESULT TO x>=0');
```

```
 Lw:=Lw0*exp(1/2);
 Uw:=Uw0*exp(1/2);
 writeln('Lw=',Lw:14:7,'Uw=',Uw:14:7);
 x21:=c+h1;
 Lw1:=Lw0+h1*c*Uw0+(Lw0)*(sqr(h1)/2)*(1+sqr(c));
 Uw1:=Uw0+h1*c*Lw0+(Uw0)*(sqr(h1)/2)*(1+sqr(c));
                  writeln('MID-POINT RESULT TO x>=0');
 for i1:=2 to (n1) do
 begin
 Lwi:=Lw0+2*h1*x21*Lw1;
 Uwi:=Uw0+2*h1*x21*Uw1;
 Lw0:=Lw1;  Uw0:=Uw1;
 Lw1:=Lwi;  Uw1:=Uwi;
 x21:=x21+h1;
 end;
 writeln('x21=',x21:2:2,'r=',r:6:2,'Lw10=',Lw0:14:7,'Uw10=',Uw0:14:7);
                 writeln('TRAPIZOIDAL RESULT TO x>=0');
 Lv0:=Lw0;
 Uv0:=Uw0;
 Lv1:=Lv0+h1*c*Lv0+(Lv0)*(sqr(h1)/2)*(1+sqr(c));
 Uv1:=Uv0+h1*c*Uv0+(Uv0)*(sqr(h1)/2)*(1+sqr(c));
 x20:=c;  x211:=x20+h1;
 for j1:=1 to (s1) do
 begin
 Lvi:=Lv0+h1/2*(x211*Uv1+x20*Uv0);
 Uvi:=Uv0+h1/2*(x211*Lv1+x20*Lv0);
 Lv0:=Lv1;   Uv0:=Uv1;
 Lv1:=Lvi;   Uv1:=Uvi;
 x20:=x211;
 x211:=x20+h1;
 end;
 writeln('x211=',x20:2:2,'r=',r:6:2,'Lv10=',Lvi:14:7,'Uv10=',Uvi:14:7);
 readln;
 r:=r+0.2;
 until r>1;
end.
```

## *4. Program4.pas:*

```
program NumericaResultsofEX35;
uses
crt,dos;
var
h,r,Uy0,Uy1,Uyi,Ly0,Ly1,Lyi,Lyn,Uyn,Lz0,Uz0,Lzi,Uzi,Lzn,Uzn,Ly,Uy:real;
Lk1,Lk2,Uk1,Uk2,Lz1,Uz1,Lz2,Uz2:real;
a,g,b,i,j,n,s:integer;
begin
clrscr;
a:=0;  b:=1;  n:=10;  s:=10;
h:=(b-a)/n;   r:=0;
 repeat
 Lk1:=0.5+0.5*r;
 Uk1:=1.5-0.5*r;
 Lk2:=0.75+0.25*r;
 Uk2:=1.25-0.25*r;
```

```
 writeln('...........EXACT RESULT............');
    Ly:=sqrt(Lk2/Lk1)*(sin(sqrt(Lk1*Lk2))/cos(sqrt(Lk1*Lk2)));
    Uy:=sqrt(Uk2/Uk1)*(sin(sqrt(Uk1*Uk2))/cos(sqrt(Uk1*Uk2)));
    writeln('Ly=',Ly:14:6,'Uy=',Uy:14:6);
 writeln('............FIRST EXPLIST METHOD..............');
 Ly0:=0;
 Uy0:=0;
  Ly1:=Ly0+h*(Lk2+Lk1*sqr(Ly0));
  Uy1:=Uy0+h*(Uk2+Uk1*sqr(Uy0));
      for i:=2 to (n) do
      begin
      Lyi:=2*Ly1-Ly0+h*(Lk1*sqr(Ly1)-Lk1*sqr(Ly0));
      Uyi:=2*Uy1-Uy0+h*(Uk1*sqr(Uy1)-Uk1*sqr(Uy0));
          Ly0:=Ly1;  Uy0:=Uy1;
          Ly1:=Lyi;  Uy1:=Uyi;
      end;
      writeln('r=',r:6:2,'Ly10=',Ly1:14:6,'Uy10=',Uy1:14:6);
writeln('...........SCOND EXPLICIT METHOD...........');
             Lz0:=0;
             Uz0:=0;
             Lz1:=Lz0+h*(Lk2+Lk1*sqr(Lz0));
             Uz1:=Uz0+h*(Uk2+Uk1*sqr(Uz0));
      for i:=2 to (s) do
      begin
Lzi:=Lz1+(h/2)*(3*(Lk2+Lk1*sqr(Lz1))-(Lk2+Lk1*sqr(Lz0)));
Uzi:=Uz1+(h/2)*(3*(Uk2+Uk1*sqr(Uz1))-(Uk2+Uk1*sqr(Uz0)));
          Lz0:=Lz1;  Uz0:=Uz1;
          Lz1:=Lzi;  Uz1:=Uzi;
      end;
 writeln('r=',r:6:2,'Lz10=',Lz1:14:6,'Uz10=',Uz1:14:6);
 readln;
 r:=r+0.2;
 until r>1;
end.
```

## *5. Program5.pas:*

```
program NumericaResultsofEX36;
uses
crt,dos;
var
h,r,Uy0,Uy1,Uyi,Ly0,Ly1,Lyi,Lyn,Uyn,Ly2,Uy2:real;
a,b,i,j,n:integer;
begin
clrscr;
a:=0;  b:=1;  n:=10;
h:=(b-a)/n;   r:=0;
writeln('when h=0.1');
for j:=1 to 2 do
begin
 repeat
 Ly0:=0.75+0.25*r;
 Uy0:=1.5-0.5*r;
 if sqr(abs(Uy0))>=sqr(abs(Ly0)) then
 begin
```

```
 Ly1:=Ly0+h*exp(-sqr(abs(Uy0)));
 Uy1:=Uy0+h*exp(-sqr(abs(Ly0)));
     if sqr(abs(Uy1))>=sqr(abs(Ly1)) then
     begin
     Ly2:=Ly1+h*exp(-sqr(abs(Uy1)));
     Uy2:=Uy1+h*exp(-sqr(abs(Ly1)))
     end;
    for i:=3 to (n) do
      begin
 if (sqr(abs(Uy2))>=sqr(abs(Ly2))) and (sqr(abs(Uy1))>=sqr(abs(Ly1)))
then
      begin
      Lyi:=Ly2+(h/12)*(23*exp(-sqr(abs(Uy2)))-16*exp(-
sqr(abs(Uy1)))+5*exp(-sqr(abs(Uy0))));
      Uyi:=Uy2+(h/12)*(23*exp(-sqr(abs(Ly2)))-16*exp(-
sqr(abs(Ly1)))+5*exp(-sqr(abs(Ly0))));
          Ly0:=Ly1;  Uy0:=Uy1;
          Ly1:=Ly2;  Uy1:=Uy2;
          Ly2:=Lyi;  Uy2:=Uyi
      end;
      end;
 end
 else
 begin
 Ly1:=Ly0+h*exp(-sqr(abs(Ly0)));
 Uy1:=Uy0+h*exp(-sqr(abs(Uy0)));
 Ly2:=Ly1+h*exp(-sqr(abs(Ly1)));
 Uy2:=Uy1+h*exp(-sqr(abs(Uy1)));
  for i:=3 to (n) do
      begin
      if (sqr(abs(Uy2))<=sqr(abs(Ly2))) and
(sqr(abs(Uy1))<=sqr(abs(Ly1))) then
      begin
      Lyi:=Ly2+(h/12)*(23*exp(-sqr(abs(Ly2)))-16*exp(-
sqr(abs(Ly1)))+5*exp(-sqr(abs(Ly0))));
      Uyi:=Uy2+(h/12)*(23*exp(-sqr(abs(Uy2)))-16*exp(-
sqr(abs(Uy1)))+5*exp(-sqr(abs(Uy0))));
          Ly0:=Ly1;  Uy0:=Uy1;
          Ly1:=Ly2;  Uy1:=Uy2;
          Ly2:=Lyi;  Uy2:=Uyi
      end;
 end;
 end;
if h=0.1 then
  begin
  end
  else
  begin
  end;
 writeln('r=',r:6:2,'Ly10=',Ly0:14:7,'Uy10=',Uy0:14:7);
 readln;
 r:=r+0.2;
 until r>1;
 n:=20;  h:=(b-a)/n;  r:=0;
 writeln('when h=0.05');
 end;
end.
```

# SUPERVISOR CERTIFICATION

I certify that this thesis was prepared under my supervision at the Al-Nahrain University, College of Science, in partial fulfillment of the requirements for the degree of master of science in mathematics

**Signature:**

**Name: Dr. Fadhel Subhi Fadhel**

**Data: / / 2005**

In view of the available recommendations I forward this thesis for debate by the examining committee.

**Signature:**

**Name: Prof. Dr. Akram M. Al-Abood**

**Head of the Department**

**Data: / / 2005**

# EXAMINING COMMITTEE CERTIFICATION

We certify that we have read this thesis entitled “***The Numerical Solution of Fuzzy Differential Equations Using Linear Multistep Methods***” and as examining committee examined the student (***Estapraq Mohammed Kahlil Al-Ani***) in its contents and in what it connected with, and that is in our opinion it meets the standards of a thesis for the degree of Master of Science in Mathematics.

***(Chairman)***

Signature:

Name: Dr. Ahlam J. Kahlil

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# CHAPTER ONE

# FUZZY SETS

## 1.1 INTRODUCTION

This chapter has the aim to introduce and illustrate fuzzy set theory, therefore, this chapter consists of nine sections, in section 2 we discuss in general fuzzy set theory, as well as, it historical background, while in section 3 some of the most important algebraic concepts related to fuzzy set theory are given. In section 4, a very strong notion which is related to fuzzy set theory will be introduced, which is the concept of $\alpha$-level sets which has the utility of expressing an element that belong to the fuzzy set. Section 5 introduces the concept of convex fuzzy sets which are related to the so called fuzzy number which is used commonly in fuzzy differential equations, as well as, section 6 illustrate. The extension principle which has the utility of extending non-fuzzy concepts to fuzzy set theory is introduced in section 7. Also, fuzzy function on fuzzy sets is discussed in section 8.

## 1.2 FUZZY SETS THEORY

Fuzzy set theory is a generalization of abstract set theory; it has a wider scope of applicability than abstract set theory in solving problems that involve to some degree subjective evaluation [Kandel, 1986].

Let X be a space of objects and x be the generic element of x, a classical set A, A $\subseteq$ X, is defined as a collection of elements or objects x $\in$ X, such that each element x can either belong or not to the set A. By defining a characteristic (or membership) function for each element x in X, we can represent a classical set A by a set of ordered pairs (x, 0) or (x, 1), which indicates x $\notin$ A or x $\in$ A, respectively. A fuzzy set express the degree to which an element belongs to a set. Hence, for simplicity, a membership function of a fuzzy set is allowed to have values between (0 and 1) which reflects the degree of membership of an element in the given set. In mathematical symbols, the membership function is given by; $\mu_{\tilde{A}} : X \longrightarrow [0, 1]$, and the fuzzy set (denoted by $\tilde{A}$) in X is defined as a set of ordered pairs [Zadeh, 1965]:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

***Remarks (1.1):***

1. Let X be a finite set, a fuzzy set on X is expressed as:

$$\tilde{A} = \mu_{\tilde{A}}(x_1)|x_1 + \mu_{\tilde{A}}(x_2)|x_2 + \ldots + \mu_{\tilde{A}}(x_n)|x_n$$

$$= \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)\big|x_i$$

When X is not finite, we write:

$$\tilde{A} = \mu_{\tilde{A}}(x_1)|x_1 + \mu_{\tilde{A}}(x_2)|x_2 + \ldots$$

$$= \int_X \mu_{\tilde{A}}(x) \mid x$$

or:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

where the slash ( | ) is employed link the elements of the support with their grades of membership in $\tilde{A}$, and the plus sign (+) or the integral ( $\int$ ) playing the role of "union" rather than arithmetic sum of integral [Zimmermann, 1985].

2. The difference between crisp and fuzzy sets is that the former always have unique membership, where as every fuzzy set has infinite number of memberships that may represented it.
3. Functions that maps X into the unit interval may be fuzzy sets, but they become fuzzy set when, and only when, they match some intuitively plausible semantic description of imprecise properties of the objects in X.

The following example illustrates this remark:

***Example (1.1) [Mahmood, 2004]:***

Suppose that:

X = {Aseel, Maha, Rula, Hadeel, Rana}

Is a set of women, and that $\tilde{A}$ is a fuzzy set of beautiful women in X. Then we may have:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

$\tilde{A}$ = beautiful

= Middle | Aseel + not low | Maha + low | Hadeel + very high | Rana + high | Rula where the fuzzy grades labeled middle, low, high are assumed to be fuzzy set in J = {0, 0.1, 0.2, .., 0.9, 1} $\subseteq$ [0, 1], for example, are expressed as follows:

$$\mu_{\tilde{A}}(x) = f(u_1) \mid u_1 + f(u_2) \mid u_2 + \ldots + f(u_n) \mid u_n$$

$$= \sum_{i=1}^{n} f(u_i) \mid u_i$$

where $u_i \in J$ and $f(u_i)$ stands for the membership function of $u_i$ in the fuzzy set $\mu_{\tilde{A}}(x)$.

Middle = 0.3 | 0.3 + 0.7 | 0.4 + 1 | 0.5 + 0.7 | 0.6 + 0.3 | 0.7

Low = 1 | 0 + 0.9 | 0.1 + 0.7 | 0.2 + 0.4 | 0.3

High = 0.4 | 0.7 + 0.7 | 0.8 + 0.9 | 0.9 + 1 | 1

Not low = 0.1 | 0.1 + 0.3 | 0.2 + 0.6 | 0.3

Very high = 0.16 | 0.7 + 0.49 | 0.8 + 0.81 | 0.9 + 1 | 1

***Example (1.2) (Fuzzy set with a discrete non-ordered universe):***

Let:

X = {Baghdad, Basra, Mousel}

be the set of certain cities any one that may choose to live in, the fuzzy set

$\tilde{A}$ = "Desirable city to live in"

may be described as follows:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

Then:

$\tilde{A}$ = {(Baghdad, 0.9), (Basra, 0.8), (Mousel, 0.6)}

***<u>Example (1.3) (Fuzzy set with a discrete ordered universe):</u>***

Let:

$X = \{0, 1, 2, 3, 4, 5, 6\}$

Be the set of number of children of a family may be choosed to have the fuzzy set:

$\tilde{B}$ = "desirable number of children in a family"

may be described as follows:

$\tilde{B} = \{(x, \mu_{\tilde{B}}(x)) \mid x \in X\}$

Implies that:

$\tilde{B} = \{(0, 0.1), (1, 0.3), (2, 0.7), (3, 1), (4, 0.7), (5, 0.3), (6, 0.1)\}$

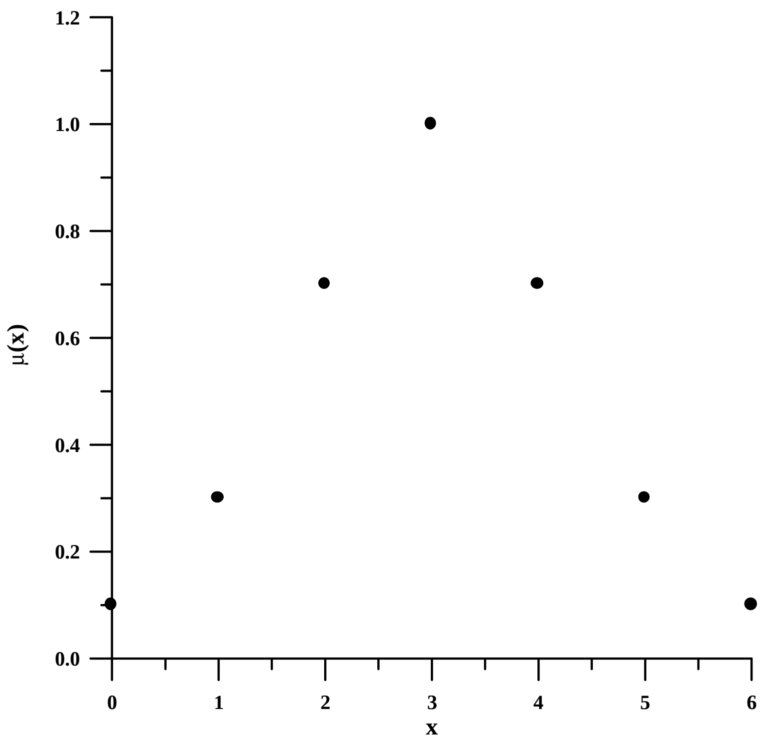


***Fig.(1.1) Membership function of discrete universe, where $\mu(x)$ stands for the membership function on discrete universe.***

***Example (1.4) (Fuzzy set with a continuous universe):***

Let $X = R^{+}$ be the set of possible ages for human beings, then the fuzzy set

$\tilde{C}$ = "About 50 years old"

may be expressed as:

$$\tilde{C} = \{(x, \mu_{\tilde{C}}(x)) \mid x \in X\}$$

where:

$$\mu_{\tilde{C}}(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^4}$$

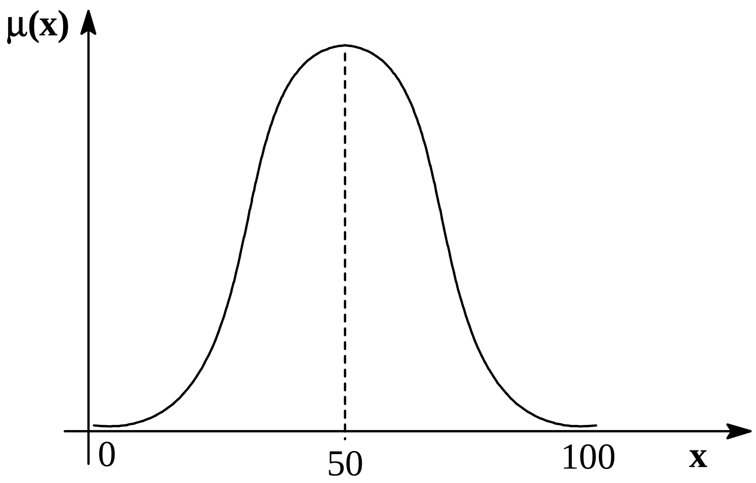


***Fig.(1.2) Membership function on a continuous universe.***

## *1.3 BASIC CONCEPTS OF FUZZY SET [DUBOIS, 1980], [ZIMERMANN, 1985]*

Let X be a space of objects, let $\tilde{A}$ be a fuzzy set in X, then one can define the following concepts:

1. The support of $\tilde{A}$ in the universal set X is a crisp set, denoted by:

   $$S(\tilde{A}) = \{x \mid \mu_{\tilde{A}}(x) > 0, \forall x \in X\}$$

2. The core (uncleus) of a fuzzy set $\tilde{A}$ is the set of all points $x \in X$, such that $\mu_{\tilde{A}}(x) = 1$.

3. The height of a fuzzy set $\tilde{A}$ is the largest membership grade over X, i.e., $\text{hgt}(\tilde{A}) = \sup_{x \in X} \mu_{\tilde{A}}(x)$.

4. The crossover point of a fuzzy set $\tilde{A}$ is the point in X, whose grade of membership in $\tilde{A}$ is 0.5.

5. Fuzzy singleton is a fuzzy set whose support is a single point in X, with $\mu_{\tilde{A}}(x) = \alpha, \alpha \in (0, 1]$.

6. A fuzzy set $\tilde{A}$ is called normalized, if its height is 1, otherwise it is subnormal., i.e., $\text{hgt}(\tilde{A}) < 1$.

   ***Note:***

   A non-empty fuzzy set $\tilde{A}$ can always be normalized by letting $\mu^*_{\tilde{A}}(x) = \dfrac{\mu_{\tilde{A}}(x)}{\sup_{x \in X} \mu_{\tilde{A}}(x)}$.

7. The empty fuzzy set $\tilde{\varnothing}$ and the universal set X are fuzzy sets, where $\forall x \in X$, $\mu_{\tilde{\varnothing}}(x) = 0$ and $\mu_X(x) = 1$, respectively.

8. If $\tilde{A}$ and $\tilde{B}$ are any two fuzzy subsets of X, then $\tilde{A} = \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x), \forall x \in X$.

9. If $\tilde{A}$ and $\tilde{B}$ are any two fuzzy subsets of X, then $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x), \forall\, x \in X$.

10. $\tilde{A}^c$ (the complement of fuzzy set $\tilde{A}$) is a fuzzy set whose membership function is defined by:

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x), \forall\, x \in X$$

11. Given two fuzzy sets, $\tilde{A}$ and $\tilde{B}$, their standard intersection, $\tilde{A} \cap \tilde{B}$, and standard union, $\tilde{A} \cup \tilde{B}$, are fuzzy sets and their membership functions are defined for simplicity for all $x \in X$, by the equations:

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \text{Max}\,\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$$

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = \text{Min}\,\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$$

***Note:***

It is important to notice that the only law of contradiction is $\tilde{A} \cup \tilde{A}^c =$ X and the law of excluded middle $\tilde{A} \cap \tilde{A}^c = \emptyset$. Both laws are broken for the fuzzy sets, since $\tilde{A} \cup \tilde{A}^c \neq X$ and $\tilde{A} \cap \tilde{A}^c \neq \emptyset$, indeed; $\forall\, x \in \tilde{A}$, such that $\mu_{\tilde{A}}(x) = \alpha$, $0 < \alpha < 1$, then according to point (7), we have:

$$\mu_{\tilde{A}\cup\tilde{A}^c}(x) = \text{Max}\,\{\alpha, 1 - \alpha\} \neq 1$$

$$\mu_{\tilde{A}\cap\tilde{A}^c}(x) = \text{Min}\,\{\alpha, 1 - \alpha\} \neq 0$$

## *1.4 $\alpha$-CUT SETS [GEOREGE, 1995]*

Among the basic concepts in fuzzy set theory is the concept of an α-cut and it's variant, a strong α-cut. Given a fuzzy set $\tilde{A}$ defined on X and any number $\alpha \in [0, 1]$, the α-cut, $A_\alpha$ (the strong α-cut, $A_{\alpha+}$) is the crisp set that contains all elements of the universal set X, whose membership grades in $\tilde{A}$ are greater than or equal to (only greater than) the specified value of α.

$$\tilde{A}_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha\}, \forall\, x \in X$$

$$\tilde{A}_{\alpha+} = \{x : \mu_{\tilde{A}}(x) > \alpha\}, \forall\, x \in X$$

The following properties are satisfied for all $\alpha \in [0, 1]$:

1. If $\alpha_1, \alpha_2 \in [0, 1]$, if $\alpha_1 \leq \alpha_2$, then $\tilde{A}_{\alpha_1} \supseteq \tilde{A}_{\alpha_2}$.
2. $(\tilde{A} \cup \tilde{B})_\alpha = \tilde{A}_\alpha \cup \tilde{B}_\alpha$.
3. $(\tilde{A} \cap \tilde{B})_\alpha = \tilde{A}_\alpha \cap \tilde{B}_\alpha$.
4. $\tilde{A} \subseteq \tilde{B}$ gives $\tilde{A}_\alpha \subseteq \tilde{B}_\alpha$.
5. $\tilde{A} = \tilde{B}$ if and only if $\tilde{A}_\alpha = \tilde{B}_\alpha$, $\forall\, \alpha \in [0, 1]$.

***<u>Remarks (1.2) [Georege, 1995]:</u>***

1. The set of all levels $\alpha \in [0, 1]$, that represent distinct α-cuts of a given fuzzy set $\tilde{A}$ is called a level set of $\tilde{A}$.

$$\Lambda(\tilde{A}) = \{\alpha \mid \mu_{\tilde{A}}(x) = \alpha, \text{ for some } x \in X\}$$

2. The support of $\tilde{A}$ is exactly the same as the strong α-cut of $\tilde{A}$ for $\alpha = 0$, $\tilde{A}_{0+} = S(\tilde{A})$.

3. The core of $\tilde{A}$ is exactly the same as the α-cut of $\tilde{A}$ for α = 1 (i.e., $A_1 = \text{core}(\tilde{A})$).

4. The height of $\tilde{A}$ may also be viewed as the supremum of α-cut for which $A_\alpha \neq \emptyset$.

## *1.5 CONVEX FUZZY SETS [GEORGE, 1995]*

An important property of fuzzy sets defined on $R^n$ (for some $n \in N$) is their convexity; this property is viewed as a generalization of the classical concept of convexity of crisp sets. The definition of convexity for fuzzy set does not necessarily mean that the membership function of a convex fuzzy set is also convex function.

***<u>Definition (1.1) [George, 1995]:</u>***

A fuzzy set $\tilde{A}$ on R is convex if and only if:

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \text{Min}\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\} \qquad (1.1)$$

for all $x_1, x_2 \in R$, and all $\lambda \in [0, 1]$.

***<u>Remark (1.3) [George, 1995]:</u>***

Assume that $\tilde{A}$ satisfies equation (1.1), we need to prove that for any $\alpha \in [0, 1]$, $\tilde{A}_\alpha$ is convex. Now, for any $x_1, x_2 \in \tilde{A}_\alpha$ and for any $\lambda \in [0, 1]$, by equation (1.1).

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \text{Min}\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\} \geq \text{Min}\ \{\alpha, \alpha\} = \alpha$$

i.e., $\lambda x_1 + (1-\lambda)x_2 \in \tilde{A}_\alpha$, therefore $\tilde{A}_\alpha$ is convex for any $\alpha \in [0, 1]$, $\tilde{A}$ is convex.

## *1.6 FUZZY NUMBER [ZIMMERMAN, 1988], [KANDEL, 1986]*

A fuzzy number $\tilde{M}$ is a convex normalized fuzzy set $\tilde{M}$ of the real line R, such that:

1. There exists exactly one $x_0 \in$ R, with $\mu_{\tilde{M}}(x_0) = 1$ ($x_0$ is called the mean value of $\tilde{M}$).
2. $\mu_{\tilde{M}}(x)$ is piecewise continuous.

Simple examples of fuzzy numbers are approximately 8, very close to 5, more or less large, etc. A special case of a fuzzy number is an interval.

## *1.7 THE EXTENSION PRINCIPLE OF FUZZY SETS*

One of the most basic concepts of fuzzy set theory, which can be used to generalize crisp mathematical concepts to fuzzy sets, is the extension principle.

***<u>Definition (1.2) (The extension principle) [Zimmermann, 1988]:</u>***

Let X be a Cartesian product of universes $X_1, X_2, \ldots, X_r$ and $\tilde{A}_1$, $\tilde{A}_2$, …, $\tilde{A}_r$ be r-fuzzy sets in $X_1, X_2, \ldots, X_r$, respectively, f is a mapping from X to a universe Y ($y = f(x_1, x_2, \ldots, x_r)$). Then a fuzzy set $\tilde{B}$ in Y is defined by:

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y = f(x_1, x_2, \ldots, x_r), (x_1, x_2, \ldots, x_r) \in X\}$$

where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \displaystyle\sup_{(x_1,x_2,\ldots,x_r)\in f^{-1}(y)} \mathrm{Min}\left\{\mu_{\tilde{A}_1}(x_1),\ldots,\mu_{\tilde{A}_r}(x_r)\right\}, & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

where $f^{-1}$ is the inverse image of f.

For r = 1, the extension principle, of course, reduces to:

$$f(\tilde{A}) = f\{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

where:

$$\mu_{\tilde{A}}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

***Example (1.5):***

Let X = {1, 2, …, 6}, $\tilde{A}$ be the comfortable type of house for two persons and the crisp set Y be the set of type of house, the contact function is given by f in Fig.(1.4), defined by:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\}$$

$$= \{(1, 0.8), (2, 1), (3, 0.6), (4, 0.2)\}$$

$Y = \{a_1, a_2, a_3, a_4\}$

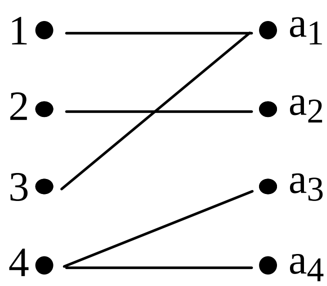


***Fig.(1.4).***

First for $a_1$:

$$f^{-1}(a_1) = \{(1, 0.8), (3, 0.6)\}$$

$$\mu_{f_A}(a_1) = \sup\{0.8\ 0.6\} = 0.8$$

Now, for $a_2$:

$$f^{-1}(a_2) = \{(2, 1), (4, 0.2)\}$$

$$\mu_{f_A}(a_2) = \sup\{1, 0.2\} = 1$$

Similarly, for $a_3$:

$$f^{-1}(a_3) = \{(4, 0.2)\}$$

$$\mu_{f_A}(a_3) = 0.2$$

Arranging all:

$$\tilde{f}(A) = \{x, \mu_y(f(A))\}$$

$$= \{(a_1, 0.8), (a_2, 1), (a_3, 0.2)\}$$

## *1.8 FUZZY FUNCTION ON FUZZY SETS [ZIMMERMANN, 1988]*

A fuzzy function is a generalization of the concept of the classical function. A classical function f is a mapping (correspondence) from the domain D of definition of the function into a space S; $f(D) \subseteq S$ is called the range of f. Different features of the classical concept of a function can be considered to be fuzzy rather than crisp. Therefore, different "degrees" of fuzzification of the classical notion of a function are conceivable:

1. There can be a crisp mapping from a fuzzy set, which carries along the fuzziness of the domain and therefore generates a fuzzy set. The image of a crisp argument would again be crisp.
2. The mapping itself can be fuzzy, thus blurring the image of a crisp argument. Thus, we shall call a fuzzy function. These are called “fuzzifying functions”, [Dubois and Prade, 1980].

3. Ordinary functions can have fuzzy properties or be constrained by fuzzy constraints.

Naturally, hybrid types can be considered. We shall focus our consideration, however, only on frequently used pure cases.

***Definition (1.3) [Zimmermann, 1985]:***

A classical function $f : X \longrightarrow Y$ maps from a fuzzy domain $\tilde{A}$ in X into a fuzzy range $\tilde{B}$ in Y if and only if:

$$\forall\, x \in X,\ \mu_{\tilde{B}}(f(x)) \geq .\mu_{\tilde{A}}(x)$$

Given a classical function $f : X \longrightarrow Y$ and a fuzzy domain $\tilde{A}$ in X, the extension principle yields the fuzzy range $\tilde{B}$ with the membership function:

$$\mu_{\tilde{B}}(y) = \underset{x \in f^{-1}(y)}{\text{Sup}}\ \mu_{\tilde{A}}(x)$$

Hence f is a function according to definition (1.3).

***Definition (1.4) [Zimmermann, 1985]:***

Let X and Y be universes and $\tilde{P}(Y)$ be the set of all fuzzy sets in Y (power set), $\tilde{f} : X \longrightarrow \tilde{P}(Y)$ is a mapping, then $\tilde{f}$ is a fuzzy function if and only if:

$$\mu_{\tilde{f}(x)}(y) = \mu_{\tilde{R}}(x, y),\ \forall\, (x, y) \in X \times Y$$

where $\mu_{\tilde{R}}(x, y)$ is the membership function of a fuzzy relation.

***Example (1.6):***

1. Let X be the set of all workers of a plant, $\tilde{f}$ the daily output, and y be the number of processed work pieces. A fuzzy function could then be:

   $$\tilde{f}(x) = y$$

2. $\tilde{a}, \tilde{b} \in R$, $X = R$, $\tilde{f} : x \longrightarrow \tilde{a}x \oplus \tilde{b}$, is a fuzzy function.

3. X = set of all 1-mile runners, $\tilde{f}$ = possible recorded times, $\tilde{f}(x) = \{y \mid y$ : achieved record times$\}$

# CHAPTER TWO

# THE EXISTENCE AND UNIQUENESS THEOREM OF FUZZY DIFFERENTIAL EQUATIONS

## 2.1 INTRODUCTION

In this chapter, we shall study mathematical models defined by:

$$\left.\begin{array}{l}\dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0\end{array}\right\} \quad \text{.......} \quad (2.1)$$

This first-odder fuzzy differential equation with x is a fuzzy mapping of t, f(t, x) a fuzzy function of the crisp variable t and the fuzzy variable x, and $\dot{x}$ is the fuzzy derivative of x.

Equation (2.1) is called fuzzy Cauchy problem. We introduce in section 2 some preliminary concepts related to this thesis and in section 3, we prove the existence and uniqueness theorem of the fuzzy Cauchy problem (2.1), where $f = I \times E^n$ is levelwise continuous function satisfies a generalized Lipschitz condition.

## *2.2 PRELIMINARIES [SONG AND WU, 2000], [PARK AND HAN, 1999]*

Let $P_k(R^n)$ denote the family of all non-empty compact convex subsets of $R^n$ and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Let A and B be two non-empty closed and bounded subsets of $R^n$. The distance between A and B is defined by the following Hausdorff metric:

$$d(A, B) = \max \{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\|\} \quad \text{.......................... (2.2)}$$

where $\|.\|$ denotes the usual Euclidean norm in $R^n$ defined by:

$$\|a - b\| = \left( \sum_{i=1}^{n} (a_i - b_i)^n \right)^{1/n}$$

Then it is clear that $(P_k(R^n), d)$ becomes a metric space.

***<u>Theorem (2.1) [Park and Han, 1999]:</u>***

The metric space $(P_k(R^n), d)$ is complete and separable.

***<u>Remark (2.1):</u>***

Some literatures and textbooks use the following symbolism in fuzzy set theory, $\tilde{u}$ fro the fuzzy set and $\tilde{u}(x)$ for its membership function, therefore this symbolism will be used here in this chapter for comparison with the existence and uniqueness of non-fuzzy set theory.

Now, we denote:

$$E^n = \{u : R^n \longrightarrow [0, 1] \mid u \text{ satisfies (i)-(iv) below}\} \quad \text{........................ (2.3)}$$

Where:

(i) $\tilde{u}$ is normal, i.e., there exists an $x_0 \in R^n$ such that $\tilde{u}(x_0) = 1$.

(ii) $\tilde{u}$ is fuzzy function convex, i.e., $\tilde{u}$ is convex if it's $\alpha$-cuts are convex, $\forall\, \alpha \in (0, 1]$.

(iii) $\tilde{u}$ is upper semicontinuous, i.e., its $\alpha$-cuts are closed, $\forall\, \alpha$.

(iv) $[\tilde{u}]^0 = \text{cl}\, \{x \in R^n \mid \tilde{u}(x) > 0\}$ is compact, since $[\tilde{u}]^0$ is the smallest closed set containing $\{x \in R^n \mid \tilde{u}(x) > 0\}$.

For $0 < \alpha \leq 1$, denote $[\tilde{u}]^\alpha = \{x \in R^n \mid \tilde{u}(x) \geq \alpha\}$, it follows that the $\alpha$-level set $[\tilde{u}]^\alpha \in P_k(R^n)$, for all $0 \leq \alpha \leq 1$, where for all $\tilde{u} \in E^n$, $\tilde{u}$ is levelwise continuous.

Now, if $g : R^n \times R^n \longrightarrow R^n$ is a function, then according to the Zadeh's extension principle, we can extend g to $E^n \times E^n \longrightarrow E^n$ by the equation:

$$g(\tilde{u}, \tilde{v})(z) = \sup_{z=g(x,y)} \min\{\tilde{u}(x), \tilde{v}(y)\} \quad \text{.......................................... (2.4)}$$

where g is any relation between $\tilde{u}$ and $\tilde{v}$. It is well known that:

$$g(\tilde{u}, \tilde{v})(z) = \sup_{z=g(x,y)} \min\{\tilde{u}(x), \tilde{v}(y)\}$$

Then:

$$[g(\tilde{u}, \tilde{v})(z)]^\alpha = [\sup_{z=g(x,y)} \min\{\tilde{u}(x), \tilde{v}(y)\}]^\alpha$$

$$= [\sup_{z=g(x,y)} \min\{[\tilde{u}(x)]^\alpha, [\tilde{v}(y)]^\alpha\}]$$

$$= g([\tilde{u}(x)]^\alpha, [\tilde{v}(y)]^\alpha)$$

Hence:

$$[g(\tilde{u}, \tilde{v})(z)]^\alpha = g([\tilde{u}(x)]^\alpha, [\tilde{v}(y)]^\alpha) \quad \text{.............................................. (2.5)}$$

for all $\tilde{u}, \tilde{v} \in E^n$, $0 \le \alpha \le 1$ and g is continuous, especially for addition and scalar multiplication.

We have:

$$[(\tilde{u} + \tilde{v})(z)]^\alpha = [\sup_{z=g(x,y)} \min\{\tilde{u}(x), \tilde{v}(y)\}]^\alpha$$

$$= [\sup_{z=g(x,y)} \min\{[\tilde{u}(x)]^\alpha, [\tilde{v}(y)]^\alpha\}] = [\tilde{u}(x)]^\alpha + [\tilde{v}(y)]^\alpha$$

Also, for $k \in R$

$$[(k\tilde{u})(z)]^\alpha = [\sup_{z=g(x)} \{k\tilde{u}(x)\}]^\alpha$$

$$= [\sup_{z=g(x)} \{k\tilde{u}(x)\}^\alpha]$$

$$= [\sup_{z=g(x)} [k]^\alpha[\tilde{u}(x)]^\alpha] = k[\tilde{u}(x)]^\alpha$$

Hence, as a result:

$$[\tilde{u} + \tilde{v}]^\alpha = [\tilde{u}]^\alpha + [\tilde{v}]^\alpha, \ [k\tilde{u}]^\alpha = k[\tilde{u}]^\alpha \qquad (2.6)$$

where $\tilde{u}, \tilde{v} \in E^n$, $k \in R$, $0 \le \alpha \le 1$.

***Theorem (2.2) [Park and Han, 1999]:***

If $\tilde{u} \in E^n$, then:

(1) $[\tilde{u}]^\alpha \in P_k(R^n)$, for all $0 \le \alpha \le 1$.

(2) $[\tilde{u}]^{\alpha_2} \subseteq [\tilde{u}]^{\alpha_1}$, for all $0 \le \alpha_1 \le \alpha_2 \le 1$.

(3) If $\{\alpha_k\} \in [0, 1]$ is a non-decreasing sequence converging to $\alpha > 0$, then:

$$[\tilde{u}]^\alpha = \bigcap_{k \ge 1} [\tilde{u}]^{\alpha_k} \qquad (2.7)$$

Conversely, if $\{A^{\alpha} : 0 \le \alpha \le 1\}$ is a family of subsets of $R^n$ satisfying (1)-(3), then there exists $u \in E^n$, such that:

$$[\tilde{u}]^{\alpha} = A^{\alpha}, \text{ for } 0 \le \alpha \le 1 \quad \ldots\ldots (2.8)$$

$$\text{and } [\tilde{u}]^{0} = \overline{\bigcup_{0<\alpha\le 1} A^{\alpha}} \subset A^{0} \quad \ldots\ldots (2.9)$$

Define $D : E^n \times E^n \longrightarrow [0, \infty)$ by the following:

$$D(\tilde{u}, \tilde{v}) = \sup_{0\le\alpha\le 1} d([\tilde{u}]^{\alpha}, [\tilde{v}]^{\alpha}) \quad \ldots\ldots (2.10)$$

Where d is the Hausdorff metric defined in (2.2). Then it is easy to see that D is a metric in $E^n$. Then D satisfies the following:

(1) $(E^n, D)$ is a complete metric space.

(2) $D(\tilde{u} + \tilde{w}, \tilde{v} + \tilde{w}) = D(\tilde{u}, \tilde{v})$, for all $\tilde{u}, \tilde{v}, \tilde{w} \in E^n$.

(3) $D(k\tilde{u}, k\tilde{v}) = |k| D(\tilde{u}, \tilde{v})$, for all $\tilde{u}, \tilde{v} \in E^n$ and $k \in R$.

***<u>Definition (2.1) [Park and Han, 1999]:</u>***

Suppose $T = [c, d] \subset R$ be a compact interval, then a mapping $F : T \longrightarrow E^n$ is called levelwise continuous at $t_0 \in T$ if the set-valued mapping $F_{\alpha}(t) = [F(t)]^{\alpha}$ is continuous at $t = t_0$ with respect to the Hausdorff metric d for all $\alpha \in [0, 1]$.

***<u>Definition (2.2) [Park and Han, 1999]:</u>***

Suppose that $T = [c, d] \subset R$ be a compact interval, then a mapping $F : T \longrightarrow E^n$ is strongly measurable if for all $\alpha \in [0, 1]$, the set-valued mapping $F_{\alpha}:T \longrightarrow P_k(R^n)$ defined by $F_{\alpha}(t) = [F(t)]^{\alpha}$ is Lebesegue measurable functions, when $P_k(R^n)$ is endowed with the topology generated by the Hausdorff metric d.

***Definition (2.3) [Park and Han, 1999]:***

A mapping $F : T \longrightarrow E^n$ is called integrably bounded if there exists an integrable function h, such that $\|x\| \leq h(t)$, for all $x \in F_0(t)$, where $T = [c, d] \subset R$ be a compact interval.

***Definition (2.4) [Park and Han, 1999]:***

Let $F : T \longrightarrow E^n$. The integral of F over T, denoted by $\int_T F(t)$ or $\int_c^d F(t)\,dt$, is defined levelwise for all $0 < \alpha \leq 1$, by:

$$\left(\int_c^d F(t)\,dt\right)^{\alpha} = \int_c^d F_{\alpha}(t)\,dt$$

$$= \{\int_T f(t)\,dt \mid f : T \longrightarrow R^n \text{ is Lebesegue measurable selection for } F_{\alpha}\} \quad \ldots\ldots (2.11)$$

A strongly measurable and integrably bounded mapping $F : T \longrightarrow E^n$ is said to be integrable over T if $\int_T F(t)\,dt \in E^n$.

The proof of the next theorem will be given which is appeared in [Park and Han, 1999] without proof:

***Theorem (2.3):***

If $F : T \longrightarrow E^n$ is strongly measurable and integrably bounded, then F is integrable.

***Proof:***

Since F is strongly measurable, then $F_\alpha$ is Lebesegue measurable functions, when $P_k(R^n)$ is endowed with the topology generated by the Hausdorff metric d.

Since F is integrably bounded, then there exists an integrable function h, such that $||x|| \leq h(t)$, for all $x \in F_0(t)$.

Then by theorem that say (Let $F_\alpha : S \longrightarrow R$ bounded measurable function, then $F_\alpha$ is integrable)

Then Since $F_\alpha$ is Lebesegue measurable function and bounded, then $F_\alpha$ is integrable, i.e.,

$$\int_T F_\alpha(t)\,dt = \left( \int_T F(t)\,dt \right)^\alpha$$

The last equation is levelwise formula of the integral F over T

Hence F is integrable over T. ■

It is known that, in particular,

$$\left( \int_T F(t)\,dt \right)^0 = \int_T F_0(t)\,dt$$

Similarly, the next theorem appears in [Park and Han, 1999] without proof, we give the proof for completeness:

***<u>Theorem (2.4):</u>***

Let F, G : T $\longrightarrow E^n$, be integrable and $\lambda \in R$. Then:

(i) $\int_T (F(t)+G(t))\,dt = \int_T F(t)\,dt + \int_T G(t)\,dt$.

(ii) $\int_T \lambda F(t)\,dt = \lambda \int_T F(t)\,dt$.

(iii) D(F, G) is integrable.

(iv) $D(\int_T F(t)\,dt, \int_T G(t)\,dt) \le \int_T D(F,G)(t)\,dt$.

***Proof:***

(i) To prove $\int_T (F(t)+G(t))\,dt = \int_T F(t)\,dt + \int_T G(t)\,dt$

By definition (2.5), we have:

$$\left[\int_T (F(t)+G(t))\,dt\right]^\alpha = \left[\int_T F(t)\,dt\right]^\alpha + \left[\int_T G(t)\,dt\right]^\alpha$$

$$= \int_T F_\alpha(t)\,dt + \int_T G_\alpha(t)\,dt$$

By equation (2.6), we have:

$$\int_T (F_\alpha(t)+G_\alpha(t))\,dt = \int_T F_\alpha(t)\,dt + \int_T G_\alpha(t)\,dt$$

In the given theorem, the given idea that $\int_T F_\alpha(t)\,dt$ and $\int_T G_\alpha(t)\,dt$ are integrable of $F_\alpha(t)$ and $G_\alpha(t)$, respectively, so we know that $\frac{d}{dt}\int_T F_\alpha(t)\,dt =$ $F_\alpha(t)$ and $\frac{d}{dt}\int_T G_\alpha(t)\,dt = G_\alpha(t)$. Thus:

$$\frac{d}{dt}\left[\int_T F_\alpha(t)\,dt + \int_T G_\alpha(t)\,dt\right] = \frac{d}{dt}\int_T F_\alpha(t)\,dt + \frac{d}{dt}\int_T G_\alpha(t)\,dt$$

$$= F_\alpha(t) + G_\alpha(t)$$

$$= \frac{d}{dt}\left[\int_T (F_\alpha(t)+G_\alpha(t))\,dt\right]$$

Hence:

$$\frac{d}{dt}\left[\int_T F(t)\,dt + \int_T G(t)\,dt\right] = F(t) + G(t)$$

$$= \frac{d}{dt}\left[\int_T (F(t)+G(t))\,dt\right]$$

(ii) To prove $\int_T \lambda F(t)\,dt = \lambda \int_T F(t)\,dt$

By definition (2.5), we have:

$$\left[\int_T \lambda F(t)\,dt\right]^{\alpha} = \left[\lambda \int_T F(t)\,dt\right]^{\alpha}$$

By equation (2.6), we have:

$$\int_T \lambda F_{\alpha}(t)\,dt = \lambda \int_T F_{\alpha}(t)\,dt$$

Then taking the right hand side

$$\frac{d}{dt}\left[\lambda \int_T F_{\alpha}(t)\,dt\right] = \lambda \frac{d}{dt}\int_T F_{\alpha}(t)\,dt$$

$$= \lambda F_{\alpha}(t) = \frac{d}{dt}\int_T \lambda F_{\alpha}(t)\,dt$$

Hence:

$$\frac{d}{dt}\left[\lambda \int_T F(t)\,dt\right] = \lambda F(t) = \frac{d}{dt}\int_T \lambda F(t)\,dt$$

(iii) To prove D(F, G) is integrable.

By equation (2.10),

$$D(F, G) = \sup_{0\le\alpha\le1} d([F]^\alpha, [G]^\alpha)$$

$$= \sup_{0\le\alpha\le1} \{\max\{ \sup_{t_1\in[F]^\alpha} \inf_{t_2\in[G]^\alpha} \|t_1 - t_2\|, \sup_{t_2\in[G]^\alpha} \inf_{t_1\in[F]^\alpha} \|t_2 - t_1\|\}\}$$

Now, since F and G are integrable, then $[F]^\alpha$ and $[G]^\alpha$ are also integrable

Therefore, $[F]^\alpha - [G]^\alpha$ or $[G]^\alpha - [F]^\alpha$ is integrable

Which implies that $\sum_{i=1}^{n}(t_{1i} - t_{2i})^n$ is integrable, for each $t_{1i} \in [F]^\alpha$ and $t_{2i} \in [G]^\alpha$

Hence $\|t_2 - t_1\|$ is integrable, for each $t_1 \in [F]^\alpha$ and $t_2 \in [G]^\alpha$

Then D(F, G) is integrable.

(iv) To prove that $D(\int_T F(t)\,dt, \int_T G(t)\,dt) \le \int_T D(F,G)(t)\,dt$

Now:

$$D(\int_T F(t)\,dt, \int_T G(t)\,dt) = \sup_{0\le\alpha\le1} d\left(\int_T F_\alpha(t)\,dt, \int_T G_\alpha(t)\,dt\right)$$

$$= \sup_{0\le\alpha\le1} \{\max\{ \sup_{t_1\in[F]^\alpha} \inf_{t_2\in[G]^\alpha} \| \int_T t_1\,dt - \int_T t_2\,dt \|,$$

$$\sup_{t_2\in[G]^\alpha} \inf_{t_1\in[F]^\alpha} \| \int_T t_2\,dt - \int_T t_1\,dt \| \}\}$$

$$= \sup_{0\le\alpha\le1} \{\max\{ \sup_{t_1\in[F(t)]^\alpha} \inf_{t_2\in[G(t)]^\alpha} \| \int_T (t_1 - t_2)\,dt \|,$$

$$\sup_{t_2\in[G(t)]^\alpha} \inf_{t_1\in[F(t)]^\alpha} \| \int_T (t_2 - t_1)\,dt \| \}\}$$

$$\leq \sup_{0\leq\alpha\leq 1} \{\max\{ \sup_{t_1\in[F(t)]^\alpha} \inf_{t_2\in[G(t)]^\alpha} \int_T \| t_1 - t_2 \| dt ,$$

$$\sup_{t_2\in[G(t)]^\alpha} \inf_{t_1\in[F(t)]^\alpha} \int_T \| t_2 - t_1 \| dt \}\}$$

$$\leq \int_T \sup_{0\leq\alpha\leq 1} d(F_\alpha(t), G_\alpha(t))\, dt$$

$$= \int_T D(F(t), G(t))\, dt$$

Hence:

$$D(\int_T F(t)\, dt , \int_T G(t)\, dt ) \leq \int_T D(F,G)(t)\, dt . \quad \blacksquare$$

***Theorem (2.5):***

Let $F : T \longrightarrow E^n$, be integrable and $c \in T$, then:

$$\int_{t_0}^{t_0+p} F(t)\, dt = \int_{t_0}^{c} F(t)\, dt + \int_{c}^{t_0+p} F(t)\, dt$$

***Note (2.1):***

Suppose that $A \in E^n$ and define $F : T \longrightarrow E^n$, by $F(s) = A$, for all $s \in T$, then:

$$\int_{t_0}^{t_0+p} F(t)\, dt = pA$$

***Theorem (2.6) [Song and Wu, 2000]:***

If $F : T \longrightarrow E^n$ is levelwise continuous, then it is strongly measurable.

***Proof:***

By the levelwise continuity of F, $F^{\alpha}$ is continuous with respect to the Hausdorff metric d for all $\alpha \in [0, 1]$

Therefore, $F_{\alpha}^{-1}(U)$ is open for each open U in $P_k(R^n)$

Since every open set in $R^n$ is an interval, then it is measurable

$\Rightarrow$ for any real number a, the set $\{t \in T : f(t) > a\}$ is measurable set

$\Rightarrow F_{\alpha}$ is Lebesegue measurable function $\Rightarrow$ F is strongly measurable. ■

***<u>Definition (2.5) [Bed and Cal, 2004]:</u>***

A function $F_{\alpha} : E^n \longrightarrow P_k(R^n)$ is called Hukuhara differentiable at a point $t_0 \in R^n$ if for h > 0 sufficiently small, we have:

$$F'_{\alpha}(t_0) = \lim_{h \to 0^+} \frac{F_{\alpha}(t_0 + h) - F_{\alpha}(t_0)}{h}$$

$$= \lim_{h \to 0^-} \frac{F_{\alpha}(t_0) - F_{\alpha}(t_0 - h)}{h}$$

$$= \lim_{h \to 0} \frac{F_{\alpha}(t_0 + h) - F_{\alpha}(t_0)}{h}$$

where the limits of Hukuhara derivative are taken in the metric space ($P_k(R^n)$, d), and $F_{\alpha}(t_0 + h) - F_{\alpha}(t_0) = (\overline{a} - \overline{b}, \underline{a} - \underline{b})$.

Also, the next theorem appears in [Song and Wu, 2000] without proof:

***<u>Theorem (2.7):</u>***

If $F : T \longrightarrow E^n$ is level wise continuous. Then it is integrable.

***Proof:***

Since F is a levelwise continuous, then we have $F_\alpha$ is continuous on a compact interval T

Hence $F_\alpha$ is bounded and by theorem (2.6), F is strongly measurable

Then $F_\alpha$ is integrable function, i.e., $\int_T F_\alpha(t)\,dt$ exist

Therefore, by definition (2.6) we have F is integrable function. ■

***Definition (2.6) [Song and Wu, 2000]:***

A mapping $F : T \longrightarrow E^n$ is called differentiable at $t_0 \in T$, if for any $\alpha \in [0, 1]$, the set valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is Hukuhara differentiable at a point $t_0$ with $DF_\alpha(t_0)$ and the family $\{DF_\alpha(t_0) \mid \alpha \in [0, 1]\}$ define a fuzzy number $F'(t_0) \in E^n$, which is called the differentiation of F at $t_0$.

If $F : T \longrightarrow E^n$ is differentiable at $t_0 \in T$, then we say that $F'(t_0)$ is the fuzzy derivative of $F(t_0)$ at the point $t_0$.

The next theorem appears in [Song and Wu, 2000], without proof:

***Theorem (2.8):***

Let $F : T \longrightarrow E^1$ be differentiable. Denote $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$, then $f_\alpha$ and $g_\alpha$ are differentiable and $[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]$.

***Proof:***

For h > 0, since

$F_\alpha(t + h) - F_\alpha(t) = [f_\alpha(t + h) - f_\alpha(t), g_\alpha(t + h) - g_\alpha(t)]$

$F_\alpha(t) - F_\alpha(t - h) = [f_\alpha(t) - f_\alpha(t - h), g_\alpha(t) - g_\alpha(t - h)]$

Then since F is differentiable, then:

$$F'_\alpha(t) = \lim_{h\to 0^+} \frac{F_\alpha(t+h) - F_\alpha(t)}{h}$$

$$= \lim_{h\to 0^-} \frac{F_\alpha(t) - F_\alpha(t-h)}{h}$$

$$\Rightarrow \lim_{h\to 0^+} \frac{[f_\alpha(t+h) - f_\alpha(t), g_\alpha(t+h) - g_\alpha(t)]}{h} =$$

$$\lim_{h\to 0^-} \frac{[f_\alpha(t) - f_\alpha(t-h), g_\alpha(t) - g_\alpha(t-h)]}{h}$$

$$\Rightarrow \left[\lim_{h\to 0^+} \frac{f_\alpha(t+h) - f_\alpha(t)}{h}, \lim_{h\to 0^+} \frac{g_\alpha(t+h) - g_\alpha(t)}{h}\right] =$$

$$\left[\lim_{h\to 0^-} \frac{f_\alpha(t) - f_\alpha(t-h)}{h}, \lim_{h\to 0^-} \frac{g_\alpha(t) - g_\alpha(t-h)}{h}\right]$$

$$\Rightarrow \left[\lim_{h\to 0^+} \frac{f_\alpha(t+h) - f_\alpha(t)}{h} = \lim_{h\to 0^-} \frac{f_\alpha(t) - f_\alpha(t-h)}{h},\right.$$

$$\left.\lim_{h\to 0^+} \frac{g_\alpha(t+h) - g_\alpha(t)}{h} = \lim_{h\to 0^-} \frac{g_\alpha(t) - g_\alpha(t-h)}{h}\right]$$

$$= [f'_\alpha(t), g'_\alpha(t)]$$

Hence $F'_\alpha(t) = [F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]$. ■

***<u>Theorem (2.9) [Song and Wu, 2000]:</u>***

If $F : T \longrightarrow E^n$ is differentiable, then it is levelwise continuous.

***Proof:***

Let t, t + h ∈ T, with h > 0. Then for any α ∈ [0, 1], to prove that F is levelwise continuous, such that:

$$d(F_\alpha(t + h), F_\alpha(t)) = d(F_\alpha(t + h) - F_\alpha(t), \{0\})$$

$$= hd\left(\frac{F_\alpha(t+h) - F_\alpha(t)}{h}, \{0\}\right)$$

$$\leq hd\left(\frac{F_\alpha(t+h) - F_\alpha(t)}{h}, DF_\alpha(t)\right) + hd(DF_\alpha(t), \{0\})$$

$$\leq hd\left(\frac{F_\alpha(t+h) - F_\alpha(t)}{h}, DF_\alpha(t)\right) + hd(F'(t), \hat{0})$$

where h is so small that the H-difference $F_\alpha(t + h) - F_\alpha(t)$ exists. By the differentiability, we know that the right hand side goes to zero as $h \longrightarrow 0^+$ and hence F is right continuous levelwise.

The left continuity levelwise is proved similarly. ■

***Theorem (2.10) [Song and Wu, 2000]:***

Let $F : T \longrightarrow E^n$ be a levelwise continuous. Then for every t ∈ T, the integral $G(t) = \int_a^t F(s)\,ds$ is differentiable and $G'(t) = F(t)$.

***Proof:***

Let α ∈ [0, 1] be fixed. Since F is levelwise continuous, then for arbitrary ε > 0, t, t + h ∈ T and h > 0, then there exists an δ(α, ε) > 0, such that:

$$d(F_\alpha(t+h), F_\alpha(t)) < \varepsilon$$

whenever $0 < h < \delta(\alpha, \varepsilon)$. According to theorem (2.7), it is known that F is integrable, and by theorem (2.5) gives:

$$G_\alpha(t+h) - G_\alpha(t) = \int_t^{t+h} F_\alpha(s)\,ds$$

Consequently, by note (2.1) and the property of $(P_k(R^n), d)$, we have:

$$d\left(\frac{G_\alpha(t+h)-G_\alpha(t)}{h}, F_\alpha(t)\right) = \frac{1}{h} d\left(\int_t^{t+h} F_\alpha(s)\,ds, \int_t^{t+h} F_\alpha(s)\,ds\right)$$

$$\leq \frac{1}{h}\int_t^{t+h} d(F_\alpha(s), F_\alpha(s))\,ds$$

$$< \varepsilon \longrightarrow 0 \text{ as } h \longrightarrow 0$$

This implies:

$$\lim_{h\to 0^+} \frac{G_\alpha(t+h)-G_\alpha(t)}{h} = F_\alpha(t)$$

and similarly,

$$\lim_{h\to 0^-} \frac{G_\alpha(t)-G_\alpha(t-h)}{h} = F_\alpha(t)$$

which complete the proof. ■

The next theorem appears in [Song and Wu, 2000] without proof:

***Theorem (2.11):***

Let $F : T \longrightarrow E^n$ be differentiable and assume that the derivative $F'(t)$ is integrable over T. Then for each $s \in T$, we have:

$$F(s) = F(t_0) + \int_{t_0}^{s} F'(t)\,dt \quad \text{.......} (2.12)$$

***Proof:***

Since F is differentiable, then by definition (2.6), we have:

$$F'_\alpha(t_0) = \lim_{h\to 0} \frac{F_\alpha(t_0+h) - F_\alpha(t_0)}{h}$$

Then we can write this formula as:

$$F'_\alpha(t_0) = \frac{F_\alpha(t_0+h) - F_\alpha(t_0)}{h} \quad \text{as } h \longrightarrow 0$$

i.e.,

$$hF'_\alpha(t_0) = F_\alpha(t_0 + h) - F_\alpha(t_0) \quad \text{as } h \longrightarrow 0$$

Then by note (2.1), we have:

$$F_\alpha(t_0 + h) - F_\alpha(t_0) = \int_{t_0}^{t_0+h} F'_\alpha(t)\,dt \text{, as } h \longrightarrow 0$$

Let $s = t_0 + h$, we have:

$$F_\alpha(s) - F_\alpha(t_0) = \int_{t_0}^{s} F'_\alpha(t)\,dt$$

$$\Rightarrow \quad F_\alpha(s) = F_\alpha(t_0) + \int_{t_0}^{s} F'_\alpha(t)\,dt$$

$$\Rightarrow \quad [F(s)]^\alpha = [F(t_0)]^\alpha + \left[\int_{t_0}^{s} F'(t)\,dt\right]^\alpha$$

Then by definition (2.5), we have:

$$F(s) = F(t_0) + \int_{t_0}^{s} F'(t)\,dt\,. \quad \blacksquare$$

***Definition (2.7) [Park and Han, 1999]:***

A mapping $f : T\times E^n \longrightarrow E^n$ is called a levelwise continuous at a point $(t_0, x_0) \in T\times E^n$ provided that for any fixed $\alpha \in [0, 1]$ and arbitrary $\varepsilon > 0$, there exists a $\delta(\varepsilon, \alpha) > 0$, such that

$$d([f(t, x)]^\alpha, [f(t_0, x_0)]^\alpha) < \varepsilon \quad \text{.......... (2.13)}$$

whenever $|t - t_0| < \delta(\varepsilon, \alpha)$, and $d([x]^\alpha, [x_0]^\alpha) < \delta(\varepsilon, \alpha)$, for all $t \in T$, $x \in E^n$, that is for all $\alpha \in [0, 1]$.

## *2.3 THE EXISTENCE AND UNIQUENESS THEOREM OF FUZZY DIFFERENTIAL EQUATIONS*

Assume that $f : I\times E^n \longrightarrow E^n$ is a levelwise continuous function, where the interval $I = \{t : |t - t_0| \le \delta \le a\}$.

Consider, the fuzzy Cauchy problem (2.1), where $\tilde{x}_0 \in E^n$. We denote, $J_0 = I\times B(\tilde{x}_0, b)$, where $a > 0$, $b > 0$, $x_0 \in E^n$,

$$B(\tilde{x}_0, b) = \{\tilde{x} \in E^n \mid D(\tilde{x}, \tilde{x}_0) \le b\} \quad \text{.......... (2.14)}$$

***Definition (٢.8):***

A mapping $\tilde{x} : I \longrightarrow E^n$ is a solution to the problem (٢.1) if it is levelwise continuous and satisfies the integral equation:

$$\tilde{x}(t) = \tilde{x}_0 + \int_{t_0}^{t} f(s, \tilde{x}(s))\,ds, \text{ for all } t \in I \quad \text{.......... (2.15)}$$

Returning to the main question of proving the existence and uniqueness of solutions of (2.15), we outline a plausible method of attacking this problem. We start by using the constant function $\tilde{x}_0(t) = \tilde{x}_0$ as an approximation to the solution. We substitute this approximation into the right hand side of (2.15) and use the result:

$$\tilde{x}_1(t) = \tilde{x}_0 + \int_{t_0}^{t} f(s, \tilde{x}_0(s))\,ds$$

as a next approximation to a solution. Then after substituting this approximation $\tilde{x}_1(t)$ again into the right hand side of (2.15) to obtain what we hope is a still better approximation $\tilde{x}_2(t)$, given by:

$$\tilde{x}_2(t) = \tilde{x}_0 + \int_{t_0}^{t} f(s, \tilde{x}_1(s))\,ds$$

and so on continuing in this process. The goal is to find a mapping $\tilde{x}$ with the property that when it is substituted in the right hand side of (2.15) the result is the same mapping x. If we continue in our approximation procedure, we may hope that the sequence of functions $\{\tilde{x}_k(t)\}$, called successive approximations, converges to a limit function with this property. Under suitable hypotheses this is the case, and precisely this approach is used to prove the existence of the solution of the integral equation (2.15).

We will consider the problem (2.1) first with f continuous on a rectangle:

$$R = \{(t, \tilde{x}) \mid |t - t_0| \le a, D(\tilde{x}, \tilde{x}_0) \le b\}$$

centered at $(t_0, \tilde{x}_0)$. We assume that f is bounded on R, that is, there exists a constant M > 0 and L > 0, such that:

$$d([f(t, \tilde{x})]^{\alpha}, 0) \le M,\ d([\frac{\partial}{\partial x} f(t, \tilde{x})]^{\alpha}, 0) \le L \quad \text{.................................. (2.16)}$$

for all $\alpha \in [0, 1]$ and for all points $(t, \tilde{x})$ in R. If $(t, \tilde{x}_1)$ and $(t, \tilde{x}_2)$ are two points in R, then by the mean-value theorem, there exists a number $\eta$ between $x_1$ and $x_2$, such that:

$$[f(t, \tilde{x}_2)]^\alpha - [f(t, \tilde{x}_1)]^\alpha = \left[\frac{\partial}{\partial x} f(t,\eta)\right]^\alpha ([\tilde{x}_2]^\alpha - [\tilde{x}_1]^\alpha)$$

Since the point $(t, \eta)$ is also in R, $d\left(\left[\frac{\partial}{\partial x} f(t,\eta)\right]^\alpha, 0\right) \le L$, and we obtain:

$$d([f(t, \tilde{x}_2)]^\alpha, [f(t, \tilde{x}_1)]^\alpha) \le Ld([\tilde{x}_2]^\alpha, [\tilde{x}_1]^\alpha) \qquad (2.17)$$

valid whenever $(t, \tilde{x}_1)$ and $(t, \tilde{x}_2)$ are in R.

***Definition (2.9):***

A function $[f]^\alpha$ satisfies an inequality of the form (4.17) for all $(t, \tilde{x}_1)$, $(t, \tilde{x}_2)$ in a region D is said to satisfy a Lipschitz condition in D.

We have already indicated that the use of approximation procedure to establish the existence of solutions. Now, let us define the successive approximations in the general case by the equations:

$$\tilde{x}_0(t) = \tilde{x}_0$$

$$\tilde{x}_j(t) = \tilde{x}_0 + \int_{t_0}^{t} f(s, \tilde{x}_{j-1}(s))\,ds, \; j = 0, 1, 2, \ldots \qquad (2.18)$$

Before we can do anything with these successive approximations, we must show that they are defined properly. This means that in order to define $\tilde{x}_j$ on some interval I. We must first know that the point $(s, \tilde{x}_j(s))$ remains in the rectangle R for every s in I.

***Lemma (2.1):***

Define $\alpha$ to be the smaller of the two positive numbers a and b/M. Then the successive approximations $\tilde{x}_j$ given by (2.18) are defined on the interval I given by $|t - t_0| \leq \delta$, an on this interval, we have:

$$D(\tilde{x}_j(t), \tilde{x}_0) \leq M|t - t_0| \leq b,\ j = 0, 1, 2, \ldots \qquad (2.19)$$

where $M = D(f(t, x), \hat{0})$, $\hat{0} \in E^n$, such that $\hat{0}(t) = 1$, for t = 0 and 0 otherwise and for any $(t, \tilde{x}) \in J_0$.

***Proof:***

We shall prove this lemma by induction. It is obvious for j = 0, let $t \in I$, from (2.18), it follows that, for j = 1:

$$\tilde{x}_1(t) = \tilde{x}_0 + \int_{t_0}^{t} f(s, \tilde{x}_0(s))\,ds \qquad (2.20)$$

which proves $x_1(t)$ is levelwise continuous on $|t - t_0| \leq \delta \leq a$, since $\tilde{x}_0$ and f are levelwise continuous.

Moreover, for any $\alpha \in [0, 1]$, we have:

$$d([\tilde{x}_1]^\alpha, [\tilde{x}_0]^\alpha) = d\left(\left[\int_{t_0}^{t} f(s, \tilde{x}_0(s))\,ds\right]^\alpha, 0\right)$$

$$\leq \int_{t_0}^{t} d\left([f(s, \tilde{x}_0(s))]^\alpha, 0\right) ds$$

$$\leq \left|\int_{t_0}^{t} d\left([f(s, \tilde{x}_0(s))]^\alpha, 0\right) ds\right| \qquad (2.21)$$

We take the supremum to the two sides of (2.21), we have:

$$\sup_{0\le\alpha\le1} d([\tilde{x}_1(t)]^\alpha, [\tilde{x}_0]^\alpha) \le \sup_{0\le\alpha\le1} \left| \int_{t_0}^{t} d\left([f(s,\tilde{x}_0(s))]^\alpha, 0\right) ds \right|$$

$$\le \left| \int_{t_0}^{t} \sup_{0\le\alpha\le1} d\left([f(s,\tilde{x}_0(s))]^\alpha, 0\right) ds \right|$$

Then:

$$D(\tilde{x}_1(t), \tilde{x}_0(t)) \le \left| \int_{t_0}^{t} D\left(f(s,\tilde{x}_0(s)), \hat{0}\right) ds \right|$$

$$= \left| \int_{t_0}^{t} M\, ds \right| \le M|t - t_0|$$

$$\le M\delta \le b \quad \text{.......... (2.22)}$$

If $|t - t_0| \le \delta$, where $M = D(f(t, x), \hat{0})$, $\hat{0} \in E^n$ and for any $(t, \tilde{x}) \in J_0$.

Now, assume that $1 < j - 1 < j$ is levelwise continuous on $|t - t_0| \le \delta$ and that:

$$D(\tilde{x}_{j-1}(t), \tilde{x}_0) \le M|t - t_0| \le M\delta \le b \quad \text{.......... (2.23)}$$

If $|t - t_0| \le \delta$, where $M = D(f(t, \tilde{x}), \hat{0})$, $\hat{0} \in E^n$ and for any $(t, \tilde{x}) \in J_0$.

From (2.18), we deduce that $x_j(t)$ is levelwise continuous on $|t - t_0| \le \delta$, since x0 and f are levelwise continuous. Then:

$$d([\tilde{x}_j(t)]^\alpha, [\tilde{x}_0]^\alpha) = d\left( \left[ \int_{t_0}^{t} f(s, \tilde{x}_{j-1}(s))\, ds \right]^\alpha, 0 \right)$$

$$\leq \int_{t_0}^{t} d\left(\left[f(s, \tilde{x}_{j-1}(s))\right]^{\alpha}, 0\right) ds$$

$$\leq \left| \int_{t_0}^{t} d\left(\left[f(s, \tilde{x}_{j-1}(s))\right]^{\alpha}, 0\right) ds \right|$$

Upon taking the supremum on two sides, yields:

$$\sup_{0\leq\alpha\leq1} d([\tilde{x}_j(t)]^{\alpha}, [\tilde{x}_0]^{\alpha}) \leq \left| \int_{t_0}^{t} \sup_{0\leq\alpha\leq1} d\left(\left[f(s, \tilde{x}_{j-1}(s))\right]^{\alpha}, 0\right) ds \right|$$

By definition of D, we have:

$$D(x_n(t), \tilde{x}_0) \leq M|t - t_0| \leq M\delta \leq b \qquad (2.24)$$

If $|t - t_0| \leq \delta$, where $M = D(f(t, \tilde{x}), \hat{0})$, $\hat{0} \in E^n$, for any $(t, \tilde{x}) \in J_0$.

This establishes the lemma. ■

Now, in order to explain the choice of $\delta$ in lemma (2.1), we observe that the condition $d([f(t, \tilde{x})]^{\alpha}, 0) \leq M$ for $\alpha \in [0, 1]$ implies that a solution $\tilde{x}$ of (2.1) can not cross the lines of slope M and –M through the initial point $(t_0, \tilde{x}_0)$. The relation (2.19) established in the above lemma say that the successive approximations $x_j$ do not cross these lines either. The length of the interval I depends on where these lines meet the rectangle R. If they meet the vertical sides of the rectangle (Fig.(2.1)), then we define $\delta = a$, while if they meet the top and bottom of the rectangle (Fig.(2.2)), then we define $\delta = b/M$. In either case, all the successive approximations remain in the triangles indicated in the figures.

One can now state and prove the existence and uniqueness theorem for a solution of a fuzzy differential equation.

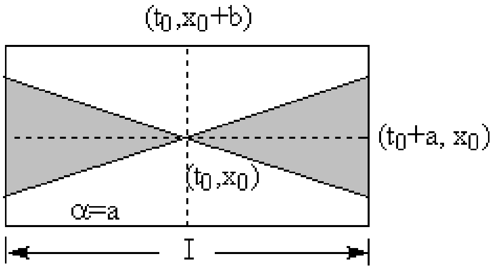


***Fig.(2.1)***

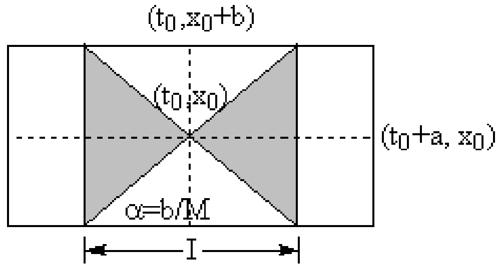


***Fig.(2.2)***

***<u>Theorem (2.12):</u>***

Assume that:

(i) A mapping $f : J_0 \longrightarrow E^n$ is level wise continuous.

(ii) For any pair $(t, \tilde{x}), (t, \tilde{y}) \in J_0$, f satisfied contraction condition, such that:

$$d([f(t, \tilde{x})]^\alpha, [f(t, \tilde{y})]^\alpha) \leq Ld([\tilde{x}]^\alpha, [\tilde{y}]^\alpha) \qquad (2.25)$$

where $0 \leq L < 1$ is a given constant and for any $\alpha \in [0, 1]$.

Then there exists a unique solution $\tilde{x} = \tilde{x}(t)$ of (2.1) defined on the interval:

$$|t - t_0| \leq \delta = \min\{a, b/M\} \qquad (2.26)$$

where $M = D(f(t, \tilde{x}), \hat{0})$, $\hat{0} \in E^n$, such that $\hat{0}(t) = 1$, for $t = 0$ and 0 otherwise for any $(t, \tilde{x}) \in J_0$.

Moreover, there exists a fuzzy set-valued mapping $\tilde{x} : I \longrightarrow E^n$, such that $D(\tilde{x}_n(t), \tilde{x}(t)) \longrightarrow 0$ on $|t - t_0| \leq \delta$, as $n \longrightarrow \infty$, with $[\tilde{x}_0]^\alpha = 0$, $\alpha \in [0, 1]$.

***Proof:***

Lemma (2.1) shows that, consequently, we conclude that $\{\tilde{x}_n(t)\}$ consists of levelwise continuous mappings on $|t - t_0| \leq \delta$, such that:

$(t, \tilde{x}_j(t)) \in J_0, |t - t_0| \leq \delta, j = 1, 2, \ldots$ ........................................... (2.27)

Let us prove that there exists a fuzzy set-valued mapping $x : I \longrightarrow E^n$ such that $D(\tilde{x}_n(t), \tilde{x}(t)) \longrightarrow 0$ uniformly on $|t - t_0| \leq \delta$, as $n \longrightarrow \infty$.

Now, for j = 2, from (2.18)

$$\tilde{x}_2(t) = \tilde{x}_0 + \int_{t_0}^{t} f(s, \tilde{x}(s))\,ds \quad \text{(2.28)}$$

From (2.20) and (2.28), we have:

$$d([\tilde{x}_2(t)]^\alpha, [\tilde{x}_1(t)]^\alpha) = d\left(\left[\int_{t_0}^{t} f(s, \tilde{x}_1(s))\,ds\right]^\alpha, \left[\int_{t_0}^{t} f(s, \tilde{x}_0(s))\,ds\right]^\alpha\right)$$

$$\leq \int_{t_0}^{t} d([f(s, \tilde{x}_1(s))]^\alpha, [f(s, \tilde{x}_0(s))]^\alpha)\,ds \quad \text{(2.29)}$$

for any $\alpha \in [0, 1]$.

According to the condition (2.25), we obtain:

$$\int_{t_0}^{t} d([f(s, \tilde{x}_1(s))]^\alpha, [f(s, \tilde{x}_0(s))]^\alpha)\,ds \leq \int_{t_0}^{t} Ld([\tilde{x}_1(s)]^\alpha, [\tilde{x}_0]^\alpha)\,ds$$

which implies to:

$$d([\tilde{x}_2(t)]^\alpha, [\tilde{x}_1(t)]^\alpha) \leq \int_{t_0}^{t} Ld([\tilde{x}_1(s)]^\alpha, [\tilde{x}_0]^\alpha)\,ds$$

and hence:

$$\sup_{0\leq\alpha\leq1} d([\tilde{x}_2(t)]^\alpha, [\tilde{x}_1(t)]^\alpha) \leq \sup_{0\leq\alpha\leq1} \int_{t_0}^{t} Ld([\tilde{x}_1(s)]^\alpha, [\tilde{x}_0]^\alpha)\,ds$$

$$\leq L \int_{t_0}^{t} \sup_{0\leq\alpha\leq1} d([\tilde{x}_1(s)]^\alpha, [\tilde{x}_0]^\alpha)\,ds$$

By the definition of D, we obtains:

$$D(\tilde{x}_2(t), \tilde{x}_1(t)) \leq L \int_{t_0}^{t} D(\tilde{x}_1(s), \tilde{x}_0(s))\, ds \quad \text{.......................................} (2.30)$$

Now, we can apply inequality (2.22) in the right-hand side of (2.30) to give:

$$D(\tilde{x}_2(t), \tilde{x}_1(t)) \leq ML \int_{t_0}^{t} |s - t_0|\, ds$$

$$= ML \frac{|t - t_0|^2}{2!} \leq ML \frac{\delta^2}{2!} \quad \text{.......................................} (2.31)$$

Starting from (2.22) and (2.31), assume that:

$$D(\tilde{x}_n(t), \tilde{x}_{n-1}(t)) \leq ML^{n-1} \frac{|t - t_0|^n}{n!}$$

$$\leq ML^{n-1} \frac{\delta^n}{n!} \quad \text{.......................................................} (2.32)$$

and let us prove that such an inequality holds for $D(\tilde{x}_{n+1}(t), \tilde{x}_n(t))$.

Indeed, from (2.18) and condition (2.25), it follows that:

$$d([\tilde{x}_{n+1}(t)]^\alpha, [\tilde{x}_n(t)]^\alpha) = d\left(\left[\int_{t_0}^{t} f(s, \tilde{x}_n(s))\, ds\right]^\alpha, \left[\int_{t_0}^{t} f(s, \tilde{x}_{n-1}(s))\, ds\right]^\alpha\right)$$

$$\leq \int_{t_0}^{t} d([f(s, \tilde{x}_n(s))]^\alpha, [f(s, \tilde{x}_{n-1}(s))]^\alpha)\, ds$$

$$\leq \int_{t_0}^{t} L d([\tilde{x}_n(s)]^\alpha, [\tilde{x}_{n-1}(s)]^\alpha)\, ds \quad \text{.....................} (2.33)$$

Hence:

$$\sup_{0\le\alpha\le 1} d([\tilde{x}_{n+1}(t)]^{\alpha}, [\tilde{x}_n(t)]^{\alpha}) \le L\int_{t_0}^{t} \sup_{0\le\alpha\le 1} d([\tilde{x}_n(s)]^{\alpha}, [\tilde{x}_{n-1}(s)]^{\alpha})\,ds$$

then the definition of D yields:

$$D(\tilde{x}_{n+1}(t), \tilde{x}_n(t)) \le L\int_{t_0}^{t} D(\tilde{x}_n(s), \tilde{x}_{n-1}(s))\,ds \quad \text{.................................... (2.34)}$$

According to (2.32)

$$D(\tilde{x}_{n+1}(t), \tilde{x}_n(t)) \le L^n M\int_{t_0}^{t} \frac{|s-t_0|^n}{n!}ds = ML^n\frac{|t-t_0|^{n+1}}{(n+1)!}$$

$$\le ML^n\frac{\delta^{n+1}}{(n+1)!} \quad \text{.................................................. (2.35)}$$

for $n = 1, 2, \ldots$ and $|t - t_0| \le \delta$.

It follows from (2.35) that the series $\sum_{n=0}^{\infty}\frac{M}{L}\frac{(L\delta)^{n+1}}{(n+1)!}$ is dominated on the interval $|t - t_0| < \alpha$ by the series of positive constants $\frac{M}{L}\sum_{n=0}^{\infty}\frac{(L\delta)^{n+1}}{(n+1)!}$, which is converges to $(M/L)(\exp(L\delta) - 1)$.

By the comparison test, the series $\sum_{n=0}^{\infty}\frac{M}{L}\frac{(L\delta)^{n+1}}{(n+1)!}$ converges (in fact, uniformly) on $|t - t_0| \le \delta$.

In view of the left hand side of (2.35), this implies the supremum distance (and uniform) convergence on $|t - t_0| \le \delta$ of the series $\sum_{n=0}^{\infty}[\tilde{x}_{n+1}(t) - \tilde{x}_n(t)]$

Since we can replace the sequence $\{\tilde{x}_n(t)\}$ and the series:

$$x_0 + \sum_{n=1}^{\infty}[\tilde{x}_n(t) - \tilde{x}_{n-1}(t)] \quad \text{.............................................................} (2.36)$$

by:

$$\tilde{x}_n(t) = \tilde{x}_0 + [\tilde{x}_1(t) - \tilde{x}_0] + \ldots + [\tilde{x}_n(t) - \tilde{x}_{n-1}(t)] \quad \text{.......................} (2.37)$$

because they have the same convergence properties.

From (2.37), $\{\tilde{x}_n(t)\}$ is converge uniformly to every $t \in I$ to a function of t, we call x(t).

Now, from (2.37) according to the convergence criterion of Weierstrass, it follows that (2.37) having the general term $\tilde{x}_n(t) - \tilde{x}_{n-1}(t)$, so $D(\tilde{x}_n(t), \tilde{x}_{n-1}(t)) \longrightarrow 0$ uniformly on $|t - t_0| \leq \delta$ as $n \longrightarrow \infty$.

Since $\tilde{x}_{n-1}$ is levelwise continuous on the closed interval I

Hence, there exists a fuzzy set-valued mapping $\tilde{x} : I \longrightarrow E^n$, where x is levelwise continuous on closed interval I, such that $D(\tilde{x}_n(t), x(t)) \longrightarrow 0$ uniformly on $|t - t_0| \leq \delta$, as $n \longrightarrow \infty$

From (2.25), we get:

$$d([f(t, \tilde{x}_n(t))]^\alpha, [f(t, \tilde{x}(t))]^\alpha) \leq Ld([\tilde{x}_n(t)]^\alpha, [\tilde{x}(t)]^\alpha) \quad \text{...................} (2.28)$$

since $(t, \tilde{x}(t)) \in J_0$ and for any $\alpha \in [0, 1]$, we take the supremum to the two sides of equation (2.38), to get:

$$\sup_{0 \leq \alpha \leq 1} d([f(t, \tilde{x}_n(t))]^\alpha, [f(t, \tilde{x}(t))]^\alpha) \leq L \sup_{0 \leq \alpha \leq 1} d([\tilde{x}_n(t)]^\alpha, [\tilde{x}(t)]^\alpha)$$

By the definition of D, we get $D(f(t, \tilde{x}_n(t)), f(t, \tilde{x}(t))) \leq LD(\tilde{x}_n(t), \tilde{x}(t)) \longrightarrow 0$ uniformly on $|t - t_0| \leq \delta$ as $n \longrightarrow \infty$. Hence:

$$D(f(t, \tilde{x}_n(t)), f(t, \tilde{x}(t))) \longrightarrow 0 \quad \text{....................................................} (2.39)$$

Taking (2.39) into account, from (2.18), we obtain, for $n \longrightarrow \infty$

$$\tilde{x}(t) = \tilde{x}_0 + \int_{t_0}^{t} f(s, \tilde{x}(s))\, ds \quad \text{.......... (2.40)}$$

Consequently, there is at least one levelwise continuous solution of (2.1)

We want to prove now that this solution is unique, that is, from:

$$\tilde{y}(t) = \tilde{x}_0 + \int_{t_0}^{t} f(s, \tilde{y}(s))\, ds \quad \text{.......... (2.41)}$$

on $|t - t_0| \le \delta$, it follows that $D(\tilde{x}(t), \tilde{y}(t)) \equiv 0$.

Indeed, from (2.18) and (2.41), we obtain:

$$d([\tilde{y}(t)]^{\alpha}, [\tilde{x}_n(t)]^{\alpha}) = d\left(\left[\int_{t_0}^{t} f(s, \tilde{y}(s))\, ds\right]^{\alpha}, \left[\int_{t_0}^{t} f(s, \tilde{x}_{n-1}(s))\, ds\right]^{\alpha}\right)$$

$$\le \int_{t_0}^{t} d([f(s, \tilde{y}(s))]^{\alpha}, [f(s, \tilde{x}_{n-1}(s))]^{\alpha})\, ds$$

$$\le \int_{t_0}^{t} L d([\tilde{y}(s)]^{\alpha}, [\tilde{x}_{n-1}(s)]^{\alpha})\, ds \quad \text{.......... (2.42)}$$

for any $\alpha \in [0, 1]$, $n = 1, 2, \ldots$ Hence:

$$\sup_{0 \le \alpha \le 1} d([\tilde{y}(t)]^{\alpha}, [\tilde{x}_n(t)]^{\alpha}) \le \int_{t_0}^{t} L \sup_{0 \le \alpha \le 1} d([\tilde{y}(s)]^{\alpha}, [\tilde{x}_{n-1}(s)]^{\alpha})\, ds$$

Then from the definition of D, one get:

$$D(\tilde{y}(t), \tilde{x}_n(t)) \le L \int_{t_0}^{t} D(\tilde{y}(s), \tilde{x}_{n-1}(s))\, ds, \; n = 1, 2, \ldots \quad \text{.......... (2.43)}$$

But, since $(t, \tilde{y}(t)) \in J_0$, then $y(t) \in B(\tilde{x}_0, b)$, we have $D(\tilde{y}(t), \tilde{x}_0) \leq b$ on $|t - t_0| \leq \delta$, $\tilde{y}(t)$ being solution of (2.41). It follows from (2.43), when n = 1, that:

$$D(\tilde{y}(t), \tilde{x}_1(t)) \leq L\int_{t_0}^{t} D(\tilde{y}(s), \tilde{x}_0(s))\,ds$$

$$\leq Lb\int_{t_0}^{t} ds = Lb|t - t_0| \quad \text{............................................. (2.44)}$$

on $|t - t_0| \leq \delta$. Now, assume hat it is true for:

$$D(\tilde{y}(t), \tilde{x}_n(t)) \leq bL^n \frac{|t - t_0|^n}{n!} \quad \text{...................................................... (2.45)}$$

On the interval $|t - t_0| \leq \delta$. From:

$$D(\tilde{y}(t), \tilde{x}_{n+1}(t)) \leq L\int_{t_0}^{t} D(\tilde{y}(s), \tilde{x}_n(s))\,ds$$

$$\leq L\int_{t_0}^{t} bL^n \frac{|s - t_0|^n}{n!}\,ds$$

$$= bL^{n+1}\frac{|t - t_0|^{n+1}}{(n+1)!} \quad \text{................................................ (2.46)}$$

Consequently, (2.45) holds for any n, which leads to the conclusion that:

$$D(\tilde{y}(t), \tilde{x}_n(t)) = D(\tilde{x}(t), \tilde{x}_n(t)) \longrightarrow 0 \quad \text{....................................... (2.47)}$$

On the interval $|t - t_0| \leq \delta$ as $n \longrightarrow \infty$, which implies that:

$$D(\tilde{y}(t), \tilde{x}(t)) \equiv 0, \text{ on } |t - t_0| \leq \delta \text{ as } n \longrightarrow \infty, \text{ i.e., } \tilde{y}(t) = \tilde{x}(t), \forall t. \text{(2.48)}$$

From (2.48) we have (2.1) have a unique solution. ■

## *2.4 EXISTENCE AND UNIQUENESS THEOREM USING SCHAUDER FIXED POINT THEOREM*

In this section, we shall take the fuzzy initial value problem as defined in section (2.1) by equation (2.1). Here, we shall state the existence theorem in different approach using Schauer fixed point theorem.

Now, as it is previously defined in section (2.3), consider a rectangular region R which is a subset of $I \times E^n$ consisting of all (t, $\tilde{x}$ (t)), such that:

$$R = \{(t, \tilde{x}(t)) \mid |t - t_0| \le a, D(\tilde{x}, c) \le b\} \quad \text{.......................................} (2.49)$$

Where c is a point of $E^n$ (n-space), b > 0, each fixed throughout this section. Let a fuzzy function f(t, $\tilde{x}$ (t)) be a levelwise continuous in R, and consider the following fuzzy differential equation as in section one.

$$\frac{d\tilde{x}(t)}{dt} = f(t, \tilde{x}(t)) \quad \text{..........................................................................} (2.50)$$

we seek about a solution of (2.50) subject to the fuzzy initial condition:

$$\tilde{x}(t) = c, \text{ for } t = 0 \quad \text{...................................................................} (2.51)$$

which exists for t in I = $|t - t_0| \le a$, for some a > 0 and for which (s, x(s)) remains in the rectangular region R for each s in I. We shall call this problem the forward problem. In the backward problem, one seeks a solution on an interval $-a \le |t - t_0|$ for a > 0. However, replacing t by –t in (2.50) and converts a backward problem to a forward problem with the right hand member –f(t, $\tilde{x}$(t)). Thus it is sufficient to direct all our attention to the forward problem. We remark that there is no loss of generality in assuming that the initial value c is given for t = 0.

The forward fuzzy initial value problem is equivalent to the problem of evaluating the solution of the integral equation as obviously done in definition (2.9)

$$\tilde{x}(t) = c + \int_0^t f(s, \tilde{x}(s))\,ds, \text{ for } t \in I \quad \text{.......... (2.52)}$$

It is instructive to consider a solution of (2.52) as a fixed point of a certain transformation. To this end, define for each levelwise continuous mapping x(t) (defined on I, with values in the rectangular region R), the transformation:

$$T(\tilde{x}(t)) = c + \int_0^t f(s, \tilde{x}(s))\,ds \quad \text{.......... (2.53)}$$

Then:

$$\tilde{y} = T(\tilde{x}(t)) \quad \text{.......... (2.54)}$$

is a levelwise continuous (set-valued) function of t and for sufficiently small t remains in R. Since the fuzzy function f(t, x(t)) is levelwise continuous, then it is necessarily bounded in R as in equation (2.16). Hence, let M be such that

$$D(f(t, x(t)), 0) \leq M \quad \text{.......... (2.55)}$$

for each $(t, \tilde{x}) \in R$ and for all $\alpha \in [0, 1]$. Then from (5.4) and (6.4), we obtain

$$D(\tilde{y}, c) = D(T(\tilde{x}(t)), c) \leq \int_0^t D(f(s, \tilde{x}(s)), 0)\,ds$$

$$\leq \int_0^t M\,ds = Mt \quad \text{.......... (2.56)}$$

so long as (s, $\tilde{x}$ (s)) is in R, and letting:

$$a = \frac{b}{M} \quad \text{......} \quad (2.57)$$

Then (2.56) implies that D($\tilde{y}$, c) ≤ b for $|t - t_0| \le a$, so long as $\tilde{x}(t)$ satisfies D($\tilde{x}(t)$, c) ≤ b, for $|t - t_0| \le a$. This is to say that the set S, consisting of all levelwise continuous fuzzy functions $\tilde{x} = (t)$ with values in R for $|t - t_0| \le b$, is mapped into itself by the transformation (2.54), S is called an invariant set of transformation.

A solution of the integral equation (2.52) is a fixed point of T, i.e., a solution of the differential equation is a fixed point of $\tilde{x}(s) = \tilde{T}(\tilde{x}(s))$ and clearly that belongs to S. If it can be shown that there exists at least one fixed point of $\tilde{T}$, then a solution of the fuzzy initial value problem exists. If $\tilde{T}$ possesses exactly one fixed point, the solution of the fuzzy initial value problem is unique.

It is typical in fixed point problems to consider iterative procedures. One selects some member of S (arbitrary), say $\tilde{x}^{(0)}$, and defines recursively $\tilde{x}^{(1)} = \tilde{T}(\tilde{x}^{(0)})$, $\tilde{x}^{(2)} = \tilde{T}(\tilde{x}^{(1)})$, …, $\tilde{x}^{(k+1)} = \tilde{T}(\tilde{x}^{(k)})$, … The sequence $\tilde{x}^{(0)}$, $\tilde{x}^{(1)}$, …, generally wanders about in S and may or not "converges". Numerous devise might be employed to increase the changes of convergence, improve on the starting point $\tilde{x}^{(0)}$, average at each step or over several steps, etc.

In the present case, the iterates are known as successive approximations, from which the process receives its name. Typically, one chooses as a first approximation $\tilde{x}^{(0)} \equiv c$, i.e., the constant initial value itself. Although, this is not essential and often represents a very poor over-all approximation.

For completeness purpose, the statement of Schauder fixed point theorem will be given.

***<u>Theorem (2.13) (Schauer Fixed Point Theorem) [Al-Hamaiwand, 2001]:</u>***

Let $\tilde{M}$ be a non-empty, closed, bounded and convex subset of a Banach space X, and suppose $\tilde{T} : \tilde{M} \longrightarrow \tilde{M}$ is a compact operator, then $\tilde{T}$ has a fixed point.

Since the region R is non-empty, closed, bounded and convex subset from $I \times E^n$ and since $(E^n, D)$ is complete metric space, then $(E^n, D)$ is complete normed space from that we have $(E^n, D)$ is a Banach space.

Also, since $(P_k(R^n), d)$ is metric space and $[\tilde{x}(t)]^\alpha \in P_k(R^n)$, $\forall\ \alpha \in [0, 1]$, $\forall\ \tilde{x}(t) \in E^n$, then $[R]_\alpha \subset P_k(R^n)$, $\forall\ \alpha \in [0, 1]$. Clearly from the definition of the region R that $[R]_\alpha$ is bounded, $\forall\ \alpha \in [0, 1]$, which implies that $[R]_\alpha$ is relatively compact for all $\alpha \in [0, 1]$. Hence the fuzzy region R is relatively compact.

Also, the operator $\tilde{T} : R \longrightarrow R$ is compact operator since it maps bounded region into relatively compact region. Then by theorem (2.13) we have $\tilde{T}$ has a fixed point from the definition of S, we can write $\tilde{T}$ such that $\tilde{T} : S \longrightarrow S$.

After that, the next theorem can be stated which is the existence of a solution of fuzzy differential equation.

***<u>Theorem (2.14):</u>***

Let $\tilde{x}^{(0)}$ be in S, i.e., let $x^{(0)}$ be a levelwise continuous and satisfy $D(\tilde{x}^{(0)}, c) \leq b$, i.e., $(t, \tilde{x}^{(0)}) \in R$, for $|t - t_0| \leq a$. Define inductively for k = 0, 1, …:

$$\tilde{x}^{(k+1)}(t) = \tilde{T}(\tilde{x}^{(k)}) = c + \int_0^t f(s, \tilde{x}^{(k)}(s))\, ds \qquad (2.58)$$

If f(t, $\tilde{x}$ (t)) is a Lipschitz function, i.e., if there exists a constant L > 0, such that:

$$D(f(t, \tilde{x}(t)), f(t, \tilde{y}(t))) \leq LD(\tilde{x}(t), \tilde{y}(t)) \quad \text{.......... (2.59)}$$

For (t, $\tilde{x}$ (t)) and (t, $\tilde{y}$ (t)) in R and for any $\alpha \in [0, 1]$. Then the sequence $\tilde{x}^{(0)}$, $\tilde{x}^{(1)}$, … converges to a solution of fuzzy initial value problem (2.50)-(2.51).

***Proof:***

First, we note from (2.58) for k = 0 that:

$$\tilde{x}^{(1)}(t) = c + \int_0^t f(s, \tilde{x}^{(0)}(s))\, ds$$

Hence:

$$(\tilde{x}^{(1)} - \tilde{x}^{(0)})(t) = c - \tilde{x}^{(0)}(t) + \int_0^t f(s, \tilde{x}^{(0)}(s))\, ds$$

Then by (2.57),

$$D(\tilde{x}^{(1)}, \tilde{x}^{(0)}) \leq D(c, \tilde{x}^{(0)}) + \int_0^t D(f(s, \tilde{x}^{(0)}(s)), 0)\, ds$$

$$\leq b + Mt \leq 2b \quad \text{.......... (2.60)}$$

for $t \in I$, from (2.58) for k = 1, we obtain:

$$\tilde{x}^{(2)}(t) = c + \int_0^t f(s, \tilde{x}^{(1)}(s))\, ds$$

Then subtracting $\tilde{x}^{(1)}(t)$ from $\tilde{x}^{(2)}(t)$ gives:

$$(\tilde{x}^{(2)} - \tilde{x}^{(1)})(t) = \int_0^t f(s, \tilde{x}^{(1)}(s))\,ds - \int_0^t f(s, \tilde{x}^{(0)}(s))\,ds$$

therefore, from (2.57), we obtain:

$$D(\tilde{x}^{(2)}, \tilde{x}^{(1)}) = D\left(\int_0^t f(s, \tilde{x}^{(1)}(s))\,ds, \int_0^t f(s, \tilde{x}^{(0)}(s))\,ds\right)$$

$$\leq \int_0^t D\Big(f(s, \tilde{x}^{(1)}(s)), f(s, \tilde{x}^{(0)}(s))\Big)ds$$

Since each (s, $\tilde{x}^{(1)}$(s)) and (s, $\tilde{x}^{(2)}$(s)) is in R, for s $\in$ I, we have using (2.59):

$$D(\tilde{x}^{(2)}, \tilde{x}^{(1)}) \leq \int_0^t LD\Big(\tilde{x}^{(1)}(s), \tilde{x}^{(0)}(s)\Big)ds$$

With together with (2.60), yields:

$$D(\tilde{x}^{(2)}, \tilde{x}^{(1)}) \leq L\int_0^t 2b\,ds = 2a(mt) \qquad (2.61)$$

for t $\in$ I.

More generally, we have that for k > 0,

$$D(\tilde{x}^{(k+1)}, \tilde{x}^{(k)}) = D\left(\int_0^t f(s, \tilde{x}^{(k)}(s))\,ds, \int_0^t f(s, \tilde{x}^{(k-1)}(s))\,ds\right)$$

$$\leq \int_0^t D\Big(f(s, \tilde{x}^{(k)}(s)), f(s, \tilde{x}^{(k-1)}(s))\Big)ds$$

and since $\tilde{x}^{(k)}$(s) and $\tilde{x}^{(k-1)}$(s) lying in R for s $\in$ I, w have using (2.59), that:

$$D(\tilde{x}^{(k+1)}, \tilde{x}^{(k)}) \le L\int_0^t D\left(\tilde{x}^{(k)}(s), \tilde{x}^{(k-1)}(s)\right)ds \quad \text{.................................} (2.62)$$

For $t \in I$. Now if:

$$D(\tilde{x}^{(k)}(t), \tilde{x}^{(k-1)}(t)) \le 2b\frac{(Lt)^{k-1}}{(k-1)!} \quad \text{..................................................} (2.63)$$

For $t \in I$, then:

$$D(\tilde{x}^{(k+1)}(s), \tilde{x}^{(k)}(s)) \le \int_0^t \frac{2bL^k}{(k-1)!}s^{k-1}ds$$

$$= 2b\frac{(Lt)^k}{k!} \quad \text{...............................................................} (2.64)$$

for $t \in I$, but (2.64) is (2.63) with k replaced by k + 1 and since (2.61) is (2.63) for k = 2, we have (by mathematical induction) that (2.63) holds for all $k \ge 2$. Now for $k \ge 2$, $p > 0$, we have:

$$D(\tilde{x}^{(k+p)}, \tilde{x}^{(k)}) = D(\tilde{x}^{(k+p)} + \tilde{x}^{(k+p-1)} + \ldots + \tilde{x}^{(k+1)}, \tilde{x}^{(k+p-1)} + \ldots + \tilde{x}^{(k)})$$

$$\le D(\tilde{x}^{(k+p)}, \tilde{x}^{(k+p-1)}) + D(\tilde{x}^{(k+p-1)}, \tilde{x}^{(k+p-2)}) + \ldots + D(\tilde{x}^{(k+1)}, \tilde{x}^{(k)})$$

which together with (2.63) implies that:

$$D(\tilde{x}^{(k+p)}, \tilde{x}^{(k)}) \le 2b\left[\frac{(Lt)^{k+p-1}}{(k+p-1)!} + \frac{(Lt)^{k+p-2}}{(k+p-2)!} + \ldots + \frac{(Lt)^k}{k!}\right]$$

$$< 2b\left[\frac{(La)^k}{k!} + \frac{(La)^{k+1}}{(k+1)!} + \ldots\right]$$

$$< 2be^{La}\frac{(La)^k}{k!} \quad \text{.........................................................} (2.65)$$

with $|La| < 1$, then for $|t - 0| \leq b$. But $\lim\limits_{k\to\infty}\left[2be^{La}\dfrac{(La)^k}{k!}\right] = 0$, and so (2.65) implies that the sequence $\tilde{x}^{(0)}, \tilde{x}^{(1)}, \ldots$, converges (in the sense of Cauchy) uniformly on $|t - 0| \leq b$. Clearly, then, if we denote by x the limit, we have from (2.58):

$$\tilde{x}(t) = \lim_{k\to\infty}\tilde{x}^{(k+1)} = c + \lim_{k\to\infty}\int_0^t f(s,\tilde{x}^{(k)}(s))\,ds$$

$$= c + \int_0^t f(s,\lim_{k\to\infty}\tilde{x}^{(k)}(s))\,ds$$

$$= c + \int_0^t f(s,\tilde{x}(s))\,ds \qquad \text{(2.66)}$$

Since the convergence is uniform for $t \in I$, and f is uniformly continuous.

Thus, the theorem is proved.

And by using Lipshcitz condition, then uniqueness is satisfied in the same manner. ■

# CHAPTER THREE

# LINEAR MULTISTEP METHODS FOR SOLVING FUZZY DIFFERENTIAL EQUATIONS

## 3.1 INTRODUCTION

Numerical and approximate methods, may be sometimes, the most suitable methods for solving differential equations, and in particular fuzzy differential equations. Therefore, this chapter consists of studying the general linear multistep methods for solving fuzzy differential equations. This consists of seven sections. In section 2, we discuss linear multistep methods, while in section 3, the order of linear multistep methods is discussed. In section 4, the theory of general linear multistep method is presented. Also, in section 5, a modified approach for solving fuzzy deferential equations numerically, which is the variable step size method which has the utility of improving the accuracy of the results? Section 6 presents the analytical methods for solving fuzzy differential equations. Finally, section 7 presents some examples to illustrate the numerical solution of fuzzy differential equation.

## *3.2 GENERAL LINEAR MULTISTEP METHODS*

Consider the fuzzy initial value problem for a single first order fuzzy differential equation:

$$\tilde{y}'(x) = f(x, \tilde{y}(x)), \quad \tilde{y}(a) = \eta$$

then the r-level equation, such that:

$$[\tilde{y}']_r = [f(x, \tilde{y})]_r, [\tilde{y}(a)]_r = [\eta]_r \quad \text{.......} \quad (3.1)$$

We seek the solution in the range $a \le x \le b$, where a and b are finite, and assume that the differential equation satisfies the existence an uniqueness theorem that illustrated in chapter two, i.e., that the problem has a unique continuously differentiable solution, which we shall indicate it by $[\tilde{y}(x)]_r = [\underline{\tilde{y}}(x; r), \overline{\tilde{y}}(x; r)]$, $r \in [0, 1]$.

Consider the sequence of points $\{x_n\}$ defined by $x_n = a + nh$, $n = 0, 1, \ldots$ The parameter h, which will always be regarded as constant, except where otherwise indicated, is called the step length.

An essential property of the majority computational methods for the solution of (3.1) is that of discretization; that is, we seek an approximate solution not on the continuous interval $a \le x \le b$, but on the discrete point set $\{x_n \mid n = 0, 1, \ldots, (b - a)/h\}$. Let $[\tilde{y}_n]_r$ be an approximation to the theoretical solution at $x_n$, that is, to $[\tilde{y}(x_n)]_r$, and let $[\tilde{f}_n]_r \equiv [\tilde{f}(x_n, \tilde{y}_n)]_r$. If a computational method for determining the sequence $\{[\tilde{y}_n]_r\}$ takes the form of a linear relationship between $[\tilde{y}_{n+j}]_r$, $[\tilde{f}_{n+j}]_r$, $j = 0, 1, \ldots, k$. We call it a linear multistep method of step number k, or a linear k-step method.

The general linear multistep method may thus be written as:

$$\sum_{j=0}^{k} \alpha_j [\tilde{y}_{n+j}]_r = h \sum_{j=0}^{k} \beta_j [\tilde{f}_{n+j}]_r \quad \ldots\ldots (3.2)$$

In the case of lower and upper solutions, equation (3.2) can be decomposed into:

$$\sum_{j=0}^{k} \alpha_j \overline{y}_{n+j}(r) = h \sum_{j=0}^{k} \beta_j F_{n+j}(r)$$

and $\quad \ldots\ldots (3.3)$

$$\sum_{j=0}^{k} \alpha_j^{*} \underline{y}_{n+j}(r) = h \sum_{j=0}^{k} \beta_j^{*} G_{n+j}(r)$$

where $\alpha_j$, $\alpha_j^{*}$, $\beta_j$ and $\beta_j^{*}$ are constants to be determined. It is assumed that $\alpha_k \neq 0$ and $\alpha_k^{*} \neq 0$ and that not both of $\alpha_0$ and $\beta_0$ are zero in the same time. Also, not both of $\alpha_0^{*}$ and $\beta_0^{*}$ are zero. The arbitrariness will be removed by assuming throughout this chapter by letting $\alpha_k = 1$ and $\alpha_k^{*} = 1$. Hence, (3.3) can be written equivalently as:

$$\overline{y}_{n+k}(r) = h \sum_{j=0}^{k} \beta_j F_{n+j}(r) - \sum_{j=0}^{k-1} \alpha_j \overline{y}_{n+j}(r)$$

and $\quad \ldots\ldots (3.4)$

$$\underline{y}_{n+k}(r) = h \sum_{j=0}^{k} \beta_j^{*} G_{n+j}(r) - \sum_{j=0}^{k-1} \alpha_j^{*} \underline{y}_{n+j}(r)$$

***<u>Remarks (3.1):</u>***

1. Such equations are so difficult to handle theoretically than are non-linear fuzzy differential equations, but they have practical advantage of

permitting us to compute the sequence $\{[\tilde{y}_n]_r\}$ numerically. In order to do this, one must first supply a set of starting values, $[\tilde{y}_0]_r$, $[\tilde{y}_1]_r$, …, $[\tilde{y}_{k-1}]_r$ (supply by using any one step method).

2. The two equations in (3.4) are explicit if $\beta_k = 0$ and $\beta_k^* = 0$, and implicit if $\beta_k \neq 0$ and $\beta_k^* \neq 0$.

3. For a given step number k, implicit methods can be made more accurate than explicit ones, moreover, enjoy more favorable stability properties.

4. In (3.4) each equation is a k-step method and each one contain 2k + 1 unknowns.

## *3.3 THE ORDER OF LINEAR MULTISTEP METHOD*

From here, and for simplicity, the discussion will be given for the upper solution $\bar{\tilde{y}}(x; r)$, in which a similar manner is satisfied for the lower case.

Associated with the linear multistep method (3.3), define respectively the linear difference operator L by:

$$L(\bar{\tilde{y}}(x; r); h) = \sum_{j=0}^{k}[\alpha_j \bar{\tilde{y}}(x + jh; r) - h\beta_j \bar{\tilde{y}}'(x + jh; r)] \qquad \text{.......................} \ (3.5)$$

where $\bar{\tilde{y}}(x; r)$ is an arbitrary function, continuously differentiable on [a, b].

Expanding the test function $\bar{\tilde{y}}(x + jh; r)$ and its derivative $\bar{\tilde{y}}'(x + jh; r)$ as a Taylor series about x, and collecting similar terms in (3.5), gives:

$$L(\bar{\tilde{y}}(x; r); h) = C_0 \bar{\tilde{y}}(x; r) + C_1 h \bar{\tilde{y}}^{(1)}(x; r) + \dots + C_q h^q \bar{\tilde{y}}^{(q)}(x; r) + \dots \qquad \text{................................} \ (3.6)$$

where $C_i$, i = 0, 1, …, are constants.

A simple calculation yields the following formula for the constant $C_q$ in terms of the coefficients $\alpha_j$ and $\beta_j$:

$$C_q = \begin{cases} \sum_{j=0}^{k} \alpha_j, & q = 0 \\ \sum_{j=0}^{k} \left[ \frac{(-j)^q}{q!} \alpha_j + \frac{(-j)^{q-1}}{(q-1)!} \beta_j \right], & q > 0 \end{cases} \quad \text{.......................................... (3.7)}$$

***Definition (3.1):***

The difference operators (3.5) and the associated linear multistep method (3.3) are said to be of order p if in (3.7) $C_0 = C_1 = \ldots = C_p = 0$, but $C_{p+1} \neq 0$.

***Remarks (3.2):***

1. The upper local truncation error at $x_{n+k}$ of the method (3.3) is defined to be the expression $L(\bar{\tilde{y}}(x_n; r); h)$. This is given by (3.5) where $\bar{\tilde{y}}(x; r)$ is the upper theoretical solution of the initial value problem (3.1).
2. The terms $C_{p+1}h^{p+1}\bar{\tilde{y}}^{(p+1)}(x_n; r)$ is called the principle upper local truncation error., where p is the order of the method and $C_{p+1}$ is called the upper error constant.
3. The error $\bar{\tilde{Y}}(x_{n+k}; r) - \bar{\tilde{y}}_{n+k}(r) = \bar{e}_{n+k}$ is the upper global truncation error or accumulated upper local truncation error.

The necessary and sufficient conditions for LMM to have an order p can be studied by using two associated polynomials, which are given in the next definition.

***Definition (3.2):***

The first characteristic polynomial of the LMM in upper case in (3.3) is given by:

$$\rho(s) = \sum_{j=0}^{k} \alpha_j s^j = \alpha_k s^k + \alpha_{k-1} s^{k-1} + \ldots + \alpha_0$$

while the second characteristic polynomial is given by:

$$\sigma(s) = \sum_{j=0}^{k} \beta_j s^j = \beta_k s^k + \beta_{k-1} s^{k-1} + \ldots + \beta_0$$

Also, it is important to notice that if $\sigma(s)$ is given, then one can find a unique polynomial $\rho(s)$ of degree k, such that the method has an order $p \geq k$. Again, reminding that the last results and definitions are also satisfied and could be similarly defined and discussed for the lower case solution $\underline{\tilde{y}}(x; r)$ of the fuzzy differential equation (3.1).

## *3.4 THEORY OF GENERAL LINEAR MULTISTEP METHOD*

### *3.4.1 Classification of LMM:*

Consider the LMM in upper case, and according to the roots of the first characteristic polynomial ρ(s), and whether it is explicit or implicit.

i. If the roots of ρ(s) equals 1 and 0, then the methods are called of upper Adam's type, and if the LMM is explicit, then it is called of upper Adam Bashforth type, while if it is implicit, then i is called of upper Adam Moulton type, i.e., in upper Adam's methods, we have the following:

$$\rho(s) = s^k - s^{k-1}$$

$$= s^{k-1}(s - 1) = 0$$

Similarly, one can make this also on lower Adam's methods.

ii. If the roots of ρ(s) equals −1, 0 and 1, then the methods are called of upper Nystrom type, if it is explicit and if the method is implicit, then it is called of upper Milne-Simpson type, and therefore:

$$\rho(s) = s^k - s^{k-2}$$

$$= s^{k-2}(s^2 - 1)$$

$$= s^{k-2}(s - 1)(s + 1)$$

One can make this similarly in lower case.

### *3.4.2 Derivation of Some Linear Multistep Methods:*

Any specific linear multistep method may be derived in a number of different ways, most important of way will be given next in details:

### *(i) Derivation Through Taylor Expansion:*

This method could be considered as most simplest method for derivation of some LMM.

Consider the Taylor series expansion for $\bar{\tilde{y}}(x_n + h; r)$ about $x_n$:

$$\bar{\tilde{y}}(x_n + h; r) = \bar{\tilde{y}}(x_n; r) + h\bar{\tilde{y}}^{(1)}(x_n; r) + \frac{h^2}{2!}\bar{\tilde{y}}^{(2)}(x_n; r) + \ldots \text{ ...........} (3.8)$$

if we truncate this expansion after two terms and substitute for $\bar{\tilde{y}}^{(1)}(x; r)$ from the fuzzy differential equation (3.1), we have:

$$\bar{\tilde{y}}(x_n + h; r) \cong \bar{\tilde{y}}(x_n; r) + hF(x_n, \bar{\tilde{y}}(x_n; r), \underline{\tilde{y}}(x_n; r)) \text{.......................} (3.9))$$

a relation which is in truncation error given by:

$$\frac{h^2}{2!}\bar{\tilde{y}}^{(2)}(x_n; r) + \frac{h^3}{3!}\bar{\tilde{y}}^{(3)}(x_n; r) + \ldots \text{ .............................................} (3.10)$$

Equation (3.9) express an approximate relation between exact values of the solution of equation (3.1). Also, we can interpret it as an exact relation between approximate value of the solution of equation (3.1). By replacing $\bar{\tilde{y}}(x_n; r)$ and $\bar{\tilde{y}}(x_n + h; r)$ by $\bar{\tilde{y}}_n(r)$ and $\bar{\tilde{y}}_{n+1}(r)$, respectively, yielding:

$$\bar{\tilde{y}}_{n+1}(r) = \bar{\tilde{y}}_n(r) + hF_n(r) \text{...............................................................} (3.11)$$

which is an upper explicit linear one step method. It is, in fact, upper Euler's method.

In the same way, one can get lower Euler's method, that can be considered as the simplest of all LMM's. The error associated with eq.(3.11) as expressed in eq.(3.10) (multiplied by +1 or −1 according to the sense of definition of error) and is called the upper local truncation error or upper local discretization error.

Consider now Taylor series expansion $\overline{\tilde{y}}(x_n + h; r)$ and $\overline{\tilde{y}}(x_n - h; r)$ about $x_n$, and subtracting the two expansions, we get:

$$\overline{\tilde{y}}(x_n + h; r) - \overline{\tilde{y}}(x_n - h; r) = 2h\overline{\tilde{y}}^{(1)}(x_n; r) + \frac{h^3}{3!}\overline{\tilde{y}}^{(3)}(x_n; r) + \ldots$$

Arguing as previously yields the associated LMM

$$\overline{\tilde{y}}_{n+1}(r) - \overline{\tilde{y}}_{n-1}(r) = 2hF_n(r)$$

This can be brought into the standard form (3.3) by replacing n by n + 1, to give:

$$\overline{\tilde{y}}_{n+2}(r) - \overline{\tilde{y}}_{n}(r) = 2hF_{n+1}(r)$$

This is the upper Mid-Point rule, and in the same way getting lower Mid-Point rule.

The upper local truncation error is:

$$\pm\frac{h^3}{3!}\overline{\tilde{y}}^{(3)}(x_n) + \ldots$$

Similar techniques can be used to derive any LMM of given specification. Thus in order to find the most accurate one-step implicit method,

$$\overline{\tilde{y}}_{n+1}(r) + \alpha_0\overline{\tilde{y}}_{n}(r) = h(\beta_1F_{n+1}(r) + \beta_0F_n(r))$$

then the associated approximate relationship:

$$\overline{\tilde{y}}(x_n + h; r) + \alpha_0\overline{\tilde{y}}(x_n; r) \sqcup h[\beta_1\overline{\tilde{y}}^{(1)}(x_n + h; r) + \beta_0\overline{\tilde{y}}^{(1)}(x_n; r)] \text{.... (3.12)}$$

and choosing α0, β1, β0 so as to make the approximation as accurate as possible.

Now, expansion $\bar{\tilde{y}}(x_n + h; r)$ and $\bar{\tilde{y}}^{(1)}(x_n + h; r)$ and substituting in (3.12) and collecting the terms on the left hand side gives:

$$C_0\bar{\tilde{y}}(x_n;r)+C_1h\bar{\tilde{y}}^{(1)}(x_n;r)+C_2h^2\bar{\tilde{y}}^{(2)}(x_n;r)+C_3h^3\bar{\tilde{y}}^{(3)}(x_n;r)+\ldots \square\ 0 \quad .... (3.13)$$

where $C_0 = 1 + \alpha_0$, $C_1 = 1 - \beta_1 - \beta_0$, $C_2 = \frac{1}{2} - \beta_1$ and $C_3 = \frac{1}{6} - \frac{1}{2}\beta_1$.

Thus, in order to make the approximation in (3.13) as accurate as possible, one can choose $\alpha_0 = -1$, $\beta_1 = \beta_0 = \frac{1}{2}$. Then $C_3$ takes the value $-\frac{1}{12}$. Therefore, the implicit one-step method is now given by:

$$\bar{\tilde{y}}_{n+1}(r) - \bar{\tilde{y}}_n(r) = \frac{h}{2}(F_{n+1}(r) + F_n(r))$$

This is the upper Trapezoidal rule with upper local truncation error given by:

$$\pm\frac{1}{12}h^3\bar{\tilde{y}}^{(3)}(x_n) + \ldots$$

We can derive all of the above rules in lower case using similar approach.

The most difficulties of this method of derivation is that it leaves some unanswered questions.

If one take Taylor series expansion in upper and lower cases about $x_n + h$ and if we get the same values for the coefficients $\alpha_j$, $\beta_j$, $\alpha_j^*$ and $\beta_j^*$, respectively. Do we get the same values for the coefficients in the infinite series representing the upper local truncation error and the lower local truncation error ?. How to remove the ambiguity of the sign in the upper and lower truncation errors ?.

***(ii) Derivation through numerical integration:***

Consider the identity:

$$\tilde{\bar{y}}(x_{n+2}; r) - \tilde{\bar{y}}(x_n, r) \equiv \int_{x_n}^{x_{n+2}} \tilde{\bar{y}}'(x)\,dx \quad \ldots\ldots (3.14)$$

Hence, replacing $\tilde{\bar{y}}'(x; r)$ by $F(x, \tilde{\bar{y}}(x, r), \underline{\tilde{y}}(x, r))$ in the integrand. If our aim is to derive, say, a linear two-step method, then the available data for the approximation evaluation of the integral will be the values $F_n(r)$, $F_{n+1}(r)$, $F_{n+2}(r)$.

Let p(x) be the unique polynomial of degree two passing through the three points $(x, F_n(r))$, $(x_{n+1}, F_{n+1}(r))$ and $(x_{n+2}, F_{n+2}(r))$. By the Newton-Gregory forward interpolation formula:

$$p(x) = p(x_n + \alpha h) = F_n(r) + \alpha\Delta F_n(r) + \frac{\alpha(\alpha-1)}{2!}\Delta^2 F_n(r)$$

hence making the approximation:

$$\int_{x_n}^{x_{n+2}} \tilde{\bar{y}}'(x;r)\,dx \;\square\; \int_{x_n}^{x_{n+2}} p(x)\,dx$$

$$= \int_0^2 [F_n(r) + \alpha\Delta F_n(r) + \frac{1}{2}\alpha(\alpha-1)\Delta^2 F_n(r)]h\,dx$$

$$= h(2F_n(r) + 2\Delta F_n(r) + \frac{1}{3}\Delta^2 F_n(r))$$

and expanding $\Delta F_n(r)$ and $\Delta^2 F_n(r)$ in terms of $F_n(r)$, $F_{n+1}(r)$, $F_{n+2}(r)$ and substituting in (3.11), gives:

$$\tilde{\bar{y}}_{n+2}(r) - \tilde{\bar{y}}_n(r) = \frac{h}{3}(F_{n+2}(r) + 4F_{n+1}(r) + F_n(r))$$

which is the upper Simpson's rule. Thus replacing (3.11) by the identity:

$$\bar{\tilde{y}}(x_{n+2};r)-\bar{\tilde{y}}(x_{n+1};r)\equiv\int_{x_{n+1}}^{x_{n+2}}\bar{\tilde{y}}'(x;r)\,dx$$

and replacing $\bar{\tilde{y}}'(x)$ by p(x), defined as above, then the following method will be derived:

$$\bar{\tilde{y}}_{n+2}(r)-\bar{\tilde{y}}_{n+1}(r)=\frac{h}{12}(5F_{n+2}(r)+8F_{n+1}(r)-F_n(r))$$

which is the two-steps upper Adam-Moulton method.

Clearly, this technique can be used to derive only a subclass of LMM's consisting of those methods for which $\alpha_k = 1$, $\alpha_j = -1$, $\alpha_i = 0$, $i = 0, 1, \ldots, j - 1, j + 1, \ldots, k - 1$, and $j \neq k$.

The importance of such technique is such that it establishes a link between the concepts of polynomial interpolation and LMM's.

Other methods for derivations are also presented in literatures, such as the derivation through interpolation (see [Lambert 1973], [Atkinson, 1989]).

### *3.4.3 Consistency, Convergence and Zero Stability of FLMM's:*

A basic property which we shall demand of an acceptable LMM is that the numerical solution $\{\bar{\tilde{y}}_n(x;r)\}$ and $\{\underline{\tilde{y}}_n(x;r)\}$ generated by the method converges, in some sense to the theoretical solutions $\bar{\tilde{Y}}(x;r)$, $\underline{\tilde{Y}}(x;r)$, respectively, as the step length h tends to zero and $n \longrightarrow \infty$ and $x_n$ fixed. In converting this intuitive concept into a precise definition, the following points must be kept in mind:

i. It is inappropriate to consider n as remaining fixed while $h \longrightarrow 0$.

ii. The definition must take into account of the additional starting values for the upper and lower solutions of $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{k-1}$, which must be supplied by using any one-step method, when $k \geq 2$.

iii. If the term convergent is to be applied to the LMM, then the convergence property must hold for all fuzzy initial value must then fold for all fuzzy initial value problems:

$$\tilde{y}(x) = f(x, \tilde{y}(x)), \; \tilde{y}(x_0) = \tilde{y}_0$$

Subjected to the hypothesis of the existence and uniqueness theorem of fuzzy differential equation to be satisfied.

The next definition is of great importance in derivation of some LMM's.

***<u>Definition (3.3):</u>***

The LMM is said to be consistent with the fuzzy initial value problem:

$$\tilde{y}'(x) = f(x, \tilde{y}(x)), \; \tilde{y}(x_0) = \tilde{y}_0$$

if it has an order $p \geq 1$, to the two ordinary initial value problems:

$$\overline{\tilde{y}}'(x;r) = f(x, \overline{\tilde{y}}, \underline{\tilde{y}}), \; \overline{\tilde{y}}(x_0,r) = \overline{\tilde{y}}_{r_0}$$

$$\underline{\tilde{y}}'(x;r) = g(x, \overline{\tilde{y}}, \underline{\tilde{y}}), \; \underline{\tilde{y}}(x_0,r) = \underline{\tilde{y}}_{r_0}$$

is consistent methods implies $C_0 = C_1 = 0$, but $C_2 \neq 0$ in upper case and $C_0^* = C_1^* = 0$, but $C_2^* \neq 0$ in lower case, or

$$\sum_{j=0}^{k} \alpha_j = 0 \quad \text{and} \quad \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$$

$$\sum_{j=0}^{k} \alpha_j^* = 0 \quad \text{and} \quad \sum_{j=0}^{k} j\alpha_j^* = \sum_{j=0}^{k} \beta_j^*$$

***Lemma (3.1):***

Consider the upper LMM (3.3), with the first and second characteristic polynomials $\rho(\delta_j(u))$ and $\sigma(\delta_j(u))$, respectively, then:

$$\delta'_j(u) = \frac{\sigma(\delta_j(u))}{\rho'(\delta_j(u)) - u\sigma'(\delta_j(u))}$$

***Proof:***

Let $\delta_j(u)$, $\forall$ j = 1, 2, …, n; be the roots of the first characteristic polynomial, such that $\rho(\delta_j(u)) = 0$, then using Maclurian series expansion, we have:

$$\delta_j(u) = \delta_j(0) + u\frac{\delta'_j(0)}{1!} + u^2\frac{\delta''_j(0)}{2!} + \ldots$$

truncation after the second term, we have:

$$\delta_j(u) = \delta_j(0) + u\delta'_j(\xi),,\ 0 < \xi < u$$

Now, to find $\delta'_j(u)$, since the method is consistent, then:

$$\rho(\delta_j(u)) - u\sigma(\delta_j(u)) = 0$$

Then:

$$\rho'(\delta_j(u))\delta'_j(u) - u\sigma'(\delta_j(u))\delta_j(u) - \sigma(\delta_j(u)) = 0$$

$$\Rightarrow\ \delta'_j(u)\big(\rho'(\delta_j(u)) - u\sigma'(\delta_j(u))\big) = \sigma(\delta_j(u))$$

$$\Rightarrow\ \delta'_j(u) = \frac{\sigma(\delta_j(u))}{\rho'(\delta_j(u)) - u\sigma'(\delta_j(u))}$$

Also, in the same way we can prove for lower LMM (3.3) with the first and second characteristic polynomials $\rho^*(\delta^*_j(u))$ and $\sigma^*(\delta^*_j(u))$, respectively. ■

***Definition (3.4):***

The LMM is said to be zero-stable (0-stable) if all of the roots (zeros) $\delta_j$'s ($\delta^*_j$'s), j = 1, 2, …, k; of $\rho(s) = 0$ ($\rho^*(s^*) = 0$) satisfy $|\delta_j| \leq 1$ ($|\delta^*_j| \leq 1$), respectively, and $\delta_j$ ($\delta^*_j$) have a multiple zeros of $\rho$ (s) ($\rho^*(s^*)$), respectively, then $|\delta_j| < 1$ ($|\delta^*_j| < 1$).

***Theorem (3.1):***

Assume the consistency conditions, then the LMM is convergent if and only if the zero-stability conditions are satisfied.

***Proof:***

For the if direction.

Suppose that the method in upper and lower case convergent. To prove it is zero-stable in the upper and lower cases, i.e., $|\delta_j| \leq 1$ and $|\delta^*_j| \leq 1$

For simplicity, consider the problem $\tilde{y}' \equiv 0$, $\tilde{y}(0) = 0$, then for upper and lower cases, we have respectively:

$$\overline{\tilde{y}}'(r) = 0,\ \overline{\tilde{y}}(0; r) = 0$$

$$\underline{\tilde{y}}'(r) = 0,\ \underline{\tilde{y}}(0; r) = 0$$

which has the exact solutions $\overline{\tilde{y}}(x; r) = 0$ and $\underline{\tilde{y}}(x; r) = 0$, respectively.

The LMM takes the forms:

$$\sum_{j=0}^{k} \alpha_j \overline{\tilde{y}}_{n+j}(r) = 0, \text{ where } \alpha_k = 1$$

and

$$\sum_{j=0}^{k} \alpha_j^* \underline{\tilde{y}}_{n+j}(r) = 0, \text{ where } \alpha_k^* = 1$$

Then:

$$\overline{\tilde{y}}_k(r) = \sum_{j=0}^{k-1} \alpha_j \overline{\tilde{y}}_{n+j}(r) \qquad (3.15)$$

$$\underline{\tilde{y}}_k(r) = \sum_{j=0}^{k-1} \alpha_j^* \underline{\tilde{y}}_{n+j}(r) \qquad (3.16)$$

Then the solution $\overline{\tilde{y}}_k(r)$ and $\underline{\tilde{y}}_k(r)$ will depends on $\overline{\tilde{y}}_0(r)$, $\overline{\tilde{y}}_1(r)$, …, $\overline{\tilde{y}}_{k-1}(r)$ and $\underline{\tilde{y}}_0(r)$, $\underline{\tilde{y}}_1(r)$, …, $\underline{\tilde{y}}_{k-1}(r)$, respectively, which are chosen to satisfy:

$$\eta_1(h) = \operatorname*{Max}_{0\le n\le k-1} |\overline{\tilde{y}}_n(r)| \longrightarrow 0, \text{ as } h \longrightarrow 0 \qquad (3.17)$$

$$\eta_2(h) = \operatorname*{Max}_{0\le n\le k-1} |\underline{\tilde{y}}_n(r)| \longrightarrow 0, \text{ as } h \longrightarrow 0 \qquad (3.18)$$

For contrary, suppose that at least one of the roots of the characteristic equations $\rho(s)$ and $\rho^*(s^*)$ of equations (3.15) and (3.16) respectively are greater than one, i.e., $|s_j| > 1$ and $|s^*_j| > 1$

Then the solution of the finite difference equations (3.15) and (3.16) are given by:

$$\overline{\tilde{y}}_n(r) = h(\delta_j)^n \quad \text{and} \quad \underline{\tilde{y}}_n(r) = h(\delta^*_j)^n$$

Then conditions (3.17) and (3.18) are satisfied for this solution, i.e.,

$$\eta_1(h) = \operatorname*{Max}_{0\le n\le k-1} |\overline{\tilde{y}}_n(r)|$$

$$= \operatorname*{Max}_{0\le n\le k-1} |h(\delta_j)^n|$$

$$= h \operatorname*{Max}_{0\le n\le k-1} |\delta_j|^n$$

$$= h\, |\delta_j|^{k-1} \longrightarrow 0, \text{ as } h \longrightarrow 0$$

Similarly:

$$\eta_2(h) \longrightarrow 0 \text{ as } h \longrightarrow 0$$

But the sequence of numerical solution $\{\bar{\tilde{y}}_n(r)\}$ and $\{\underline{\tilde{y}}_n(r)\}$ does not converge, since:

$$\max_{0\le n\le k} |\bar{\tilde{y}}(x_n;r) - \bar{\tilde{y}}_n(r)| = h|\delta_j|^{N(h)} \quad \text{and} \quad \max_{0\le n\le k} |\underline{\tilde{y}}(x_n;r) - \underline{\tilde{y}}_n(r)| = h|\delta^*_j|^{N(h)}$$

Consider h = (b − a)/N(h) = b / N(h). Then by using L'Hospital's rule, we have:

$$\lim_{\substack{N\to\infty \\ h\to 0}} \frac{b}{N} |\delta_j|^N = \lim_{N\to\infty} bN|\delta_j|^{N-1} = \infty$$

and similarly:

$$\lim_{\substack{N\to\infty \\ h\to 0}} \frac{b}{N} |\delta^*_j|^N = \infty$$

which says that the numerical solutions converge to ∞ and does not converges to the exact solution $\bar{\tilde{Y}}(x;r) = 0$, $\underline{\tilde{Y}}(x;r) = 0$, respectively, which is a contradiction.

Hence the method is zero-stable.

Conversely, suppose that the LMM is zero-stable. To prove it is convergence.

For simplicity, consider the fuzzy differential equation $\tilde{y}' = \lambda\tilde{y}$, $\tilde{y}(0) = 1$

Then the problems related to this fuzzy differential equation in r-level sets of solutions are given by:

$$\bar{\tilde{y}}'(x;r) = \lambda\underline{\tilde{y}}(x;r), \ \bar{\tilde{y}}(0;r) = 1, \text{ if } \lambda < 0$$

and

$$\underline{\tilde{y}}'(x;r) = \lambda\, \overline{\tilde{y}}(x;r),\ \underline{\tilde{y}}(0;r) = 1, \text{ if } \lambda < 0$$

To show that the term $C_0[\delta_0(\lambda h)]^n$ and $C^*_0[\delta^*_0(\lambda h)]^n$ in its general solutions:

$$\overline{\tilde{y}}_n(r) = \sum_{j=0}^{k} C_j[\delta_j(\lambda h)]^n$$

$$\underline{\tilde{y}}_n(r) = \sum_{j=0}^{k} C^*_j[\delta^*_j(\lambda h)]^n$$

will be converge to the exact solutions $\overline{\tilde{Y}}(x;r) = e^{\lambda x}$, $\underline{\tilde{Y}}(x;r) = e^{\lambda x}$, respectively on [0, b]. The remaining terms $C_j[\delta_j(\lambda h)]^n$, and $C^*_j[\delta^*_j(\lambda h)]^n$, j = 1, 2, …, k – 1, converge to zero as $h \longrightarrow 0$.

Expanding $\delta_0(\lambda h)$, for simplicity the proof will be given for the lower case and similar argument could be carried for the upper case of solution, using Taylor’s theorem:

$$\delta_0(\lambda h) = \delta_0(0) + h\lambda\, \delta'_0(0) + O(h^2)$$

Then by lemma, we have:

$$\delta'_j(u) = \frac{\sigma(\delta_j(u))}{\rho'(\delta_j(u)) - u\sigma'(\delta_j(u))}$$

Hence:

$$\delta'_0(0) = \frac{\sigma(\delta_0(0))}{\rho'(\delta_0(0)) - 0\times\sigma'(\delta_0(0))}$$

$$= \frac{\sigma(\delta_0(0))}{\rho'(\delta_0(0))}$$

Since $\rho(\delta_0(0)) = \sum_{j=0}^{k} a_j \delta_0^j(0)$, if $\delta_0(0) = 1$, then by using consistency condition,

we have $\rho(1) = \sum_{j=0}^{k} a_j \delta_0^j(0) = C_0 = 0$.

Hence $\delta_0(0) = 1$ is a root

Now, since $\delta_0(0) = 1$. Then:

$$\delta_0'(0) = \frac{\sigma(1)}{\rho'(1)} = 1$$

Since $\sigma(1) = \sum_{j=0}^{k} b_j$ and $\rho'(\delta_0(0)) = \sum_{j=0}^{k} j a_j \delta_0^{j-1}(0)$

$$\rho'(1) = \sum_{j=0}^{k} j a_j$$

By consistency condition $C_0 = C_1 = 0$

Hence $C_1 = \sum_{j=0}^{k} j a_j - \sum_{j=0}^{k} b_j = 0$, then $\sum_{j=0}^{k} j a_j = \sum_{j=0}^{k} b_j$, which implies to $\rho'(1) = \sigma(1)$. Therefore $\delta_0'(0) = 1$

Hence:

$$\delta_0(\lambda h) = 1 + \lambda h + O(h^2)$$

$$= e^{\lambda h} - O(h^2) + O(h^2)$$

$$\approx e^{\lambda h}$$

Therefore:

$$[\delta_0(\lambda h)]^n \square \; [e^{\lambda h}]^n = e^{\lambda nh} = e^{\lambda x_n}$$

Hence:

$$\underset{0 \le x_n \le b}{\text{Max}} \left| [\delta_0(\lambda h)]^n - e^{\lambda x_n} \right| \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

Now, we have to show that $C_0 \longrightarrow 1$ as $h \longrightarrow 0$. The coefficients of $C_0(h)$, $C_1(h), \ldots, C_{k-1}(h)$ satisfy the linear system:

$$C_0 + C_1 + \ldots + C_{k-1} = \overline{\tilde{y}}_0(r)$$

$$C_0[\delta_0(\lambda h)] + C_1[\delta_1(\lambda h)] + \ldots + C_{k-1}[\delta_{k-1}(\lambda h)] = \overline{\tilde{y}}_1(r)$$

$$\vdots$$

$$C_0[\delta_0(\lambda h)]^{k-1} + C_1[\delta_1(\lambda h)]^{k-1} + \ldots + C_{k-1}[\delta_{k-1}(\lambda h)]^{k-1} = \overline{\tilde{y}}_{k-1}(r)$$

The initial values $\overline{\tilde{y}}_0(r), \overline{\tilde{y}}_1(r), \ldots, \overline{\tilde{y}}_{k-1}(r)$ depends on h and are assumed to satisfy the following:

$$\eta_1(h) = \underset{0 \le n \le k-1}{\text{Max}} \left| e^{\lambda x_n} - \overline{\tilde{y}}_n(r) \right| \longrightarrow 0 \text{ as } h \longrightarrow 0$$

which implies that $\lim_{h \to 0} \overline{\tilde{y}}_n(r) = 1$.

The coefficient $C_0$ can be obtained by using Cramer's rule to solve the above linear system.

$$C_0 = \frac{\begin{vmatrix} \bar{\tilde{y}}_0(r) & 1 & \cdots & 1 \\ \bar{\tilde{y}}_1(r) & \delta_1 & \cdots & \delta_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\tilde{y}}_{k-1}(r) & \delta_1^{k-1} & \cdots & \delta_{k-1}^{k-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \delta_0 & \delta_1 & \cdots & \delta_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_0^{k-1} & \delta_1^{k-1} & \cdots & \delta_{k-1}^{k-1} \end{vmatrix}}$$

and as $h \longrightarrow 0$, we have:

$$C_0 = \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \delta_0 & \delta_1 & \cdots & \delta_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_0^{k-1} & \delta_1^{k-1} & \cdots & \delta_{k-1}^{k-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \delta_0 & \delta_1 & \cdots & \delta_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_0^{k-1} & \delta_1^{k-1} & \cdots & \delta_{k-1}^{k-1} \end{vmatrix}} = 1$$

Hence $C_0 \longrightarrow 1$ as $h \longrightarrow 0$ and therefore the sequence $\{\bar{\tilde{y}}_n(r)\}$ converges to $\bar{\tilde{y}}(x;r) = e^{\lambda x}$.

Similarly, we can prove that the sequence of lower solutions $\{\underline{\tilde{y}}_n(r)\}$ converges to $\underline{\tilde{y}}(x;r) = e^{\lambda x}$ as $h \longrightarrow 0$.

Hence the method is converge. ■

### *3.4.4 Linear Multistep Methods for Solving Fuzzy Differential Equations:*

Consider a first order fuzzy initial value problem, given by:

$$\left.\begin{array}{l}\tilde{y}'(x) = f(x, \tilde{y}(x)),\ x \in [x_0, b] \\ \tilde{y}(x_0) = \tilde{y}_0\end{array}\right\} \quad \text{.......................................................} \quad (3.19)$$

where $\tilde{y}$ is a fuzzy function of x, f(x, $\tilde{y}$(x)) is a fuzzy function of the crisp variable x, and the fuzzy variable $\tilde{y}$ and $\tilde{y}'$ is the fuzzy derivative of $\tilde{y}$ and $\tilde{y}(x_0) = \tilde{y}_0$ is a triangular or a triangular shaped fuzzy number.

The r-level set of $\tilde{y}(x)$ for $x \in [x_0, b]$ is $[\tilde{y}(x)]_r = [\underline{\tilde{y}}(x; r), \overline{\tilde{y}}(x; r)]$. Also $[\tilde{y}'(x)]_r = [\underline{\tilde{y}}'(x; r), \overline{\tilde{y}}'(x; r)]$, and

$$[f(x, \tilde{y}(x))]_r = [\underline{f}(x, \tilde{y}(x); r), \overline{f}(x, \tilde{y}(x); r)]$$

$$= [G(x, \underline{\tilde{y}}, \overline{\tilde{y}}), F((x, \underline{\tilde{y}}, \overline{\tilde{y}})]$$

Because of $\tilde{y}' = f(x, \tilde{y})$, we have:

$$\underline{\tilde{y}}'(x; r) = \underline{f}(x, \tilde{y}(x); r) = G(x, \underline{\tilde{y}}(x; r), \overline{\tilde{y}}(x; r)) \text{..........................} (3.20)$$

$$\overline{\tilde{y}}'(x; r) = \overline{f}(x, \tilde{y}(x); r) = F(x, \underline{\tilde{y}}(x; r), \overline{\tilde{y}}(x; r)) \text{..........................} (3.21)$$

Also:

$$[\tilde{y}(x_0)]_r = [\tilde{y}_0]_r = [\underline{\tilde{y}}_0(r), \overline{\tilde{y}}_0(r)]$$

By using the extension principle, we have the membership function:

$$f(x, \tilde{y}(x))(s) = \sup_{s \in R} \{\mu_{\tilde{y}(x)}(\tau) \mid s = f(x, \tau)\} \quad (3.22)$$

so the fuzzy number $f(x, \tilde{y}(x))$. Hence, it follows that:

$$[f(x, \tilde{y}(x))]_r = [\underline{f}(x, \tilde{y}(x); r), \overline{f}(x, \tilde{y}(x); r)], \ r \in [0, 1] \quad (3.23)$$

where:

$$\underline{f}(x, \tilde{y}(x); r) = \min \{f(x, u) \mid u \in [\tilde{y}(x)]_r\} \quad (3.24)$$

$$\overline{f}(x, \tilde{y}(x); r) = \max \{f(x, u) \mid u \in [\tilde{y}(x)]_r\} \quad (3.25)$$

In equation (3.19), we take f to be continuous function satisfying the Lipschitz condition which is the sufficient condition for the existence of a unique solution of equation (3.19) as we previously explained this in chapter two.

Now, let $\tilde{Y} = [\underline{\tilde{Y}}, \overline{\tilde{Y}}]$ be the exact solution and $\tilde{y} = [\underline{\tilde{y}}, \overline{\tilde{y}}]$ be the approximate solution of the fuzzy initial value problem (3.19). Let:"

$[\tilde{Y}(x)]_r = [\underline{\tilde{Y}}(x; r), \overline{\tilde{Y}}(x; r)]$

$[\tilde{y}(x)]_r = [\underline{\tilde{y}}(x; r), \overline{\tilde{y}}(x; r)]$

Also, the value of r throughout each step is unchanged. The exact and approximate solution at $x_n$ denoted respectively for all n = 0, 1, …, N, by:

$[\tilde{Y}_n(x)]_r = [\underline{\tilde{Y}}_n(x; r), \overline{\tilde{Y}}_n(x; r)]$

$[\tilde{y}_n(x)]_r = [\underline{\tilde{y}}_n(x; r), \overline{\tilde{y}}_n(x; r)]$

The grid points at which the step of the solution is calculated through the step length $h = (x - x_0) / N$ and therefore, $x_i = x_0 + ih$, i = 0, 1, …, N. By using the general form of linear multistep method (3.4) it is obtained that:

$$\underline{\tilde{Y}}_{n+k}(r) = h\sum_{j=0}^{k}\beta_j^{*}G(x_{n+j},\underline{\tilde{Y}}_{n+j}(r),\overline{\tilde{Y}}_{n+j}(r)) - \sum_{j=0}^{k-1}\alpha_j^{*}\underline{\tilde{Y}}_{n+j}(r) + O(h^2) \quad \text{......} (3.26)$$

and

$$\overline{\tilde{Y}}_{n+k}(r) = h\sum_{j=0}^{k}\beta_j F(x_{n+j},\underline{\tilde{Y}}_{n+j}(r),\overline{\tilde{Y}}_{n+j}(r)) - \sum_{j=0}^{k-1}\alpha_j\overline{\tilde{Y}}_{n+j}(r) + O(h^2) \quad \text{......} (3.27)$$

Also, the approximate solution is given by:

$$\underline{\tilde{y}}_{n+k}(r) = h\sum_{j=0}^{k}\beta_j^{*}G(x_{n+j},\underline{\tilde{y}}_{n+j}(r),\overline{\tilde{y}}_{n+j}(r)) - \sum_{j=0}^{k-1}\alpha_j^{*}\underline{\tilde{y}}_{n+j}(r) \quad \text{......} (3.28)$$

and

$$\overline{\tilde{y}}_{n+k}(r) = h\sum_{j=0}^{k}\beta_j F(x_{n+j},\underline{\tilde{y}}_{n+j}(r),\overline{\tilde{y}}_{n+j}(r)) - \sum_{j=0}^{k-1}\alpha_j\overline{\tilde{y}}_{n+j}(r) \quad \text{......} (3.29)$$

The next lemma can be applied to study the convergence of our method.

***Lemma (3.2):***

Let a sequence of non-negative numbers $\{W_n\}_{n=0}^{N}$ satisfying:

$$\sum_{j=1}^{k}W_{n+j} \le \sum_{j=1}^{k}A_{j-1}W_{n+j-1} + B, \; n = 0, 1, \ldots, N-j, \; j = 1, 2, \ldots, k$$

for some given positive constants A and B. Then:

$$\sum_{j=1}^{k}W_{n+j-1} \le \sum_{j=1}^{k}A_{j-1}^{n+j-1}W_0 + B\sum_{j=1}^{k}\frac{A^{n+j-1}-1}{A-1} \quad \text{......} (3.30)$$

***Proof:***

With $A \neq 1$ and $B \neq 1$ and if we take $j = 1$, then:

$W_{n+1} \leq A_0 W_n + B$, $n = 0, 1, \ldots, N - 1$........................................(3.31a1)

Then using mathematical induction, it can be prove that:

$$W_n \leq A_0^n W_0 + B\frac{A^n - 1}{A - 1} \quad \text{.......................................................... (3.31b1)}$$

Suppose that the inequality is true for $n = 0$, then:

$$W_n \leq A_0^0 W_0 + B\left(\frac{1}{1 - A}\right)$$

$$= W_0 + B\left(\frac{1}{1 - A}\right)$$

and for $n = N - 2$, then with $A = A_0$:

$$W_{N-2} \leq A_0^{N-2} W_0 + B\left(\frac{A^{N-2} - 1}{A - 1}\right)$$

Now, to prove it is true for $n = N - 1$. Then:

$$W_N \leq A_0 W_{N-1} + B$$

$$W_{N-1} \leq A_0 W_{N-2} + B$$

$$\leq A_0(A_0^{N-2} W_0 + B\left(\frac{A^{N-2} - 1}{A - 1}\right)) + B$$

$$= A_0^{N-1} W_0 + B\left(\frac{A^{N-1} - 1}{A - 1}\right)$$

Then equation (3.31b1) is true for all $n = 0, 1, \ldots, N - 1$.

Similarly, in the same manner, we can get:

$W_{n+2} \leq A_2 W_{n+1} + B$, n = 0, 1, …, N–2 ........................................(3.31a2)

Then:

$$W_{n+1} \leq A_1^{n+1} W_0 + B\left(\frac{A^{n+1}-1}{A-1}\right) \quad (3.31b2)$$

and

$W_{n+3} \leq A_3 W_{n+2} + B$, n = 0, 1, …, N–3 ........................................(3.31a3)

Then:

$$W_{n+2} \leq A_2^{n+2} W_0 + B\left(\frac{A^{n+2}-1}{A-1}\right) \quad (3.31b3)$$

and so on until j = k. Hence, we have:

$W_{n+k} \leq A_k W_{n+k-1} + B$, n = 0, 1, …, N–k .....................................(3.31ak)

Then:

$$W_{n+k-1} \leq A_{k-1}^{n+k-1} W_0 + B\left(\frac{A^{n+k-1}-1}{A-1}\right) \quad (3.31bk)$$

Now, we summing all equations from (3.31a1) to (3.31ak) and summing all equations from (3.31b1) to (3.31bk), yields:

$$\sum_{j=1}^{k} W_{n+j} \leq \sum_{j=1}^{k} A_{j-1} W_{n+j-1} + B, \; n = 0, 1, \ldots, N-j, \; j = 1, 2, \ldots, k$$

Then:

$$\sum_{j=1}^{k} W_{n+j-1} \leq \sum_{j=1}^{k} A_{j-1}^{n+j-1} W_0 + B \sum_{j=1}^{k} \frac{A^{n+j-1}-1}{A-1}. \quad \blacksquare$$

***<u>Lemma (3.3):</u>***

Let a sequence of numbers $\{W_n\}_{n=0}^{N}$ and $\{V_n\}_{n=0}^{N}$ satisfying:

$$|W_{n+k}| \le \sum_{j=0}^{k} A_j \ \max\{|W_{n+j}|, |V_{n+j}|\} + \sum_{j=0}^{k-1} B_j \ |W_{n+j}| + C \ \ldots\ldots\ldots\ldots\ldots (3.32)$$

$$|V_{n+k}| \le \sum_{j=0}^{k} A_j \ \max\{|W_{n+j}|, |V_{n+j}|\} + \sum_{j=0}^{k-1} B_j \ |V_{n+j}| + C \ \ldots\ldots\ldots\ldots\ldots (3.33)$$

for some given positive constants $A_j$ and $B_j$ and C, and denote

$U_n = |W_n| + |V_n|$, $n = 0, 1, \ldots, N - j$

Then:

$$U_{n+j} \le \sum_{j=0}^{k} \left(\frac{2(A_j + B_j)}{1-2A_k}\right)^{n+j} U_0 + \left(\frac{2C}{1-2A_k}\right) \sum_{j=0}^{k} \frac{\left(\frac{2(A_j + B_j)}{1-2A_k}\right)^{n+j} - 1}{\left(\frac{2(A_j + B_j)}{1-2A_k}\right) - 1}$$

***Proof:***

Summing equations (3.32) with equation (3.33), we have:

$$|W_{n+k}|+|V_{n+k}| \le 2\sum_{j=0}^{k} A_j \max\{|W_{n+j}|, |V_{n+j}|\} + \sum_{j=0}^{k-1} B_j \ (|W_{n+j}|+|V_{n+j}|) + 2C$$

Then:

$$U_{n+k} \le 2\sum_{j=0}^{k-1} A_j \ \max\{|W_{n+j}|, |V_{n+j}|\} + 2\sum_{j=0}^{k-1} B_j \ (|W_{n+k}|+|V_{n+j}|) + 2A_k(U_{n+k}) + 2C$$

$$\le 2\sum_{j=0}^{k-1} (A_j + B_j) \ U_{n+j} + 2A_k(U_{n+k}) + 2C$$

$$\le \frac{2\sum_{j=0}^{k-1} (A_j + B_j) U_{n+j}}{1-2A_k} + \frac{2C}{1-2A_k}$$

Using lemma (3.2)

$$\sum_{j=0}^{k-1} U_{n+j} \leq \sum_{j=0}^{k-1}\left(\frac{2(A_j+B_j)}{1-2A_k}\right)^{n+j} U_0 + \left(\frac{2C}{1-2A_k}\right)\sum_{j=0}^{k-1}\frac{\left(\frac{2(A_j+B_j)}{1-2A_k}\right)^{n+j}-1}{\left(\frac{2(A_j+B_j)}{1-2A_k}\right)-1}$$

Hence:

$$U_{n+j} \leq \sum_{j=0}^{k-1}\left(\frac{2(A_j+B_j)}{1-2A_k}\right)^{n+j} U_0 + \left(\frac{2C}{1-2A_k}\right)\sum_{j=0}^{k-1}\frac{\left(\frac{2(A_j+B_j)}{1-2A_k}\right)^{n+j}-1}{\left(\frac{2(A_j+B_j)}{1-2A_k}\right)-1}$$

For all j = 0, 1, …, k – 1 and n = 0, 1, …, N. ■

***<u>Theorem (3.2):</u>***

Let F(x, u, v) and G(x, u, v) belong to $C^1(K)$ and let the partial derivative of F and G be bounded over K, where:

$K = \{(x, u, v) \mid x_0 \leq x \leq b, -\infty < v < \infty, -\infty < u < \infty\}$

Then, for arbitrary fixed r : $0 \leq r \leq 1$, the general linear multistep approximates $\tilde{y}_N$ converges to the exact solutions $\underline{\tilde{Y}}(x;r)$ and $\overline{\tilde{Y}}(x;r)$ uniformly in x.

***Proof:***

As in ordinary differential equations, it is sufficient to show that:

$$\lim_{h\to 0} \underline{\tilde{y}}_N(r) = \underline{\tilde{Y}}(x;r) \text{ and } \lim_{h\to 0} \overline{\tilde{y}}_N(r) = \overline{\tilde{Y}}(x;r).$$

Then using the general form of linear multistep method, to get an approximate solution, such that:

$$\bar{\tilde{y}}_{n+k}(r) = h\sum_{j=0}^{k}\beta_j F(x_n, \underline{\tilde{y}}_n(r), \bar{\tilde{y}}_n(r)) - \sum_{j=0}^{k-1}\alpha_j \bar{\tilde{y}}_{n+j}(r) \quad (3.34)$$

$$\underline{\tilde{y}}_{n+k}(r) = h\sum_{j=0}^{k}\beta_j^* G_{n+j}(x_n, \underline{\tilde{y}}_n(r), \bar{\tilde{y}}_n(r)) - \sum_{j=0}^{k-1}\alpha_j^* \bar{\tilde{y}}_{n+j}(r) \quad (3.35)$$

to get the exact solution, such that:

$$\bar{\tilde{Y}}_{n+k}(r) = h\sum_{j=0}^{k}\beta_j F(x_{n+j}, \underline{\tilde{Y}}_{n+j}(r), \bar{\tilde{Y}}_{n+j}(r)) - \sum_{j=0}^{k-1}\alpha_j \bar{\tilde{Y}}_{n+j}(r) + \frac{h^2}{2}\bar{\tilde{Y}}''(\bar{\xi}_n) \quad (3.36)$$

$$\underline{\tilde{Y}}_{n+k}(r) = h\sum_{j=0}^{k}\beta_j^* G_{n+j}(x_{n+j}, \underline{\tilde{Y}}_{n+j}(r), \bar{\tilde{Y}}_{n+j}(r)) - \sum_{j=0}^{k-1}\alpha_j^* \underline{\tilde{Y}}_{n+j}(r) + \frac{h^2}{2}\underline{\tilde{Y}}''(\underline{\xi}_n) \quad (3.37)$$

where, $x_n < \bar{\xi}_n, \underline{\xi}_n < x_{n+k}$, consequently, we subtract (3.3.6) from (3.34) and also (3.37) from (3.35), we have:

$$\bar{\tilde{Y}}_{n+k}(r) - \bar{\tilde{y}}_{n+k}(r) = h\left[\sum_{j=0}^{k}\beta_j F(x_{n+j}, \underline{\tilde{Y}}_{n+j}(r), \bar{\tilde{Y}}_{n+j}(r)) - \sum_{j=0}^{k}\beta_j F(x_{n+j}, \underline{\tilde{y}}_{n+j}(r), \bar{\tilde{y}}_{n+j}(r))\right] - \sum_{j=0}^{k-1}\alpha_j \bar{\tilde{Y}}_{n+j}(r) + \sum_{j=0}^{k-1}\alpha_j \bar{\tilde{y}}_{n+j}(r) + \frac{h^2}{2}\bar{\tilde{Y}}''(\bar{\xi}_n) \quad (3.38)$$

and

$$\underline{\tilde{Y}}_{n+k}(r) - \underline{\tilde{y}}_{n+k}(r) = h\left[\sum_{j=0}^{k}\beta_j^{*}G(x_{n+j}, \underline{\tilde{Y}}_{n+j}(r), \overline{\tilde{Y}}_{n+j}(r)) - \right.$$

$$\left.\sum_{j=0}^{k}\beta_j^{*}G(x_{n+j}, \underline{\tilde{y}}_{n+j}(r), \overline{\tilde{y}}_{n+j}(r))\right] - \sum_{j=0}^{k-1}\alpha_j^{*}\underline{\tilde{Y}}_{n+j}(r) +$$

$$\sum_{j=0}^{k-1}\alpha_j^{*}\underline{\tilde{y}}_{n+j}(r) + \frac{h^2}{2}\underline{\tilde{Y}}''(\underline{\xi}_n) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.39)$$

For simplicity, we let:

$$W_{n+k} = \overline{\tilde{Y}}_{n+k}(r) - \overline{\tilde{y}}_{n+k}(r) \quad \text{and} \quad V_{n+k} = \underline{\tilde{Y}}_{n+k}(r) - \underline{\tilde{y}}_{n+k}(r)$$

Then equations (3.38) and (3.39) takes the form:

$$|W_{n+k}| = h\sum_{j=0}^{k}\beta_j\left[\left|F(x_{n+j}, \underline{\tilde{Y}}_{n+j}(r), \overline{\tilde{Y}}_{n+j}(r)) - F(x_{n+j}, \underline{\tilde{y}}_{n+j}(r), \overline{\tilde{y}}_{n+j}(r))\right|\right] +$$

$$\sum_{j=0}^{k-1}\alpha_j\left|\overline{\tilde{Y}}_{n+j}(r) - \overline{\tilde{y}}_{n+j}(r)\right| + \frac{h^2}{2}\left|\overline{\tilde{Y}}''(\overline{\xi}_n)\right|$$

$$\leq Lh\sum_{j=0}^{k}\beta_j \; \max\{|V_{n+j}|, |W_{n+j}|\} + \sum_{j=0}^{k-1}\alpha_j \; |W_{n+j}| + \frac{h^2}{2}\overline{\tilde{M}}$$

and

$$|V_{n+k}| = h\sum_{j=0}^{k}\beta_j\left[\left|G(x_{n+j}, \underline{\tilde{Y}}_{n+j}(r), \overline{\tilde{Y}}_{n+j}(r)) - G(x_{n+j}, \underline{\tilde{y}}_{n+j}(r), \overline{\tilde{y}}_{n+j}(r))\right|\right] +$$

$$\sum_{j=0}^{k-1}\alpha_j\left|V_{n+j}\right| + \frac{h^2}{2}\left|\underline{\tilde{Y}}''(\underline{\xi}_n)\right|$$

$$\leq Lh\sum_{j=0}^{k}\beta_j \; \max\{|V_{n+j}|, |W_{n+j}|\} + \sum_{j=0}^{k-1}\alpha_j \; |V_{n+j}| + \frac{h^2}{2}\underline{\tilde{M}}$$

where:

$$\underline{\tilde{M}} = \max_{x_0 \le x \le b} \left| \underline{\tilde{Y}}''(x;r) \right|$$

$$\bar{\tilde{M}} = \max_{x_0 \le x \le b} \left| \bar{\tilde{Y}}''(x;r) \right|$$

and L > 0 is the Lipschitz constant. Then:

$$|W_{n+k}| \le Lh \sum_{j=0}^{k} \beta_j \ (|V_{n+j}| + |W_{n+j}|) + \sum_{j=0}^{k-1} \alpha_j \ |W_{n+j}| + \frac{h^2}{2} \bar{\tilde{M}} \ \text{..........} \ (3.40)$$

$$|V_{n+k}| \le Lh \sum_{j=0}^{k} \beta_j \ (|V_{n+j}| + |W_{n+j}|) + \sum_{j=0}^{k-1} \alpha_j \ |V_{n+j}| + \frac{h^2}{2} \underline{\tilde{M}} \ \text{............} \ (3.41)$$

By lemma (3.3), we have:

$$|W_{n+k}| \le \sum_{j=0}^{k-1} \left( \frac{2hLB_j + 2\alpha_j}{1 - 2hLB_k} \right)^{n+j} |U_0| + \left( \frac{2h^2 \bar{\tilde{M}} / 2}{1 - 2hLB_k} \right) \sum_{j=0}^{k-1} \frac{\left( \frac{2hLB_j + 2\alpha j}{1 - 2hLB_k} \right)^{n+j} - 1}{\left( \frac{2hLB_j + 2\alpha_j}{1 - 2hLB_k} \right) - 1}$$

$$|V_{n+k}| \le \sum_{j=0}^{k-1} \left( \frac{2hLB_j + 2\alpha_j}{1 - 2hLB_k} \right)^{n+j} |U_0| + \left( \frac{2h^2 \underline{\tilde{M}} / 2}{1 - 2hkB_k} \right) \sum_{j=0}^{k-1} \frac{\left( \frac{2hLB_j + 2\alpha_j}{1 - 2hLB_k} \right)^{n+j} - 1}{\left( \frac{2hLB_j + 2\alpha_j}{1 - 2hLB_k} \right) - 1}$$

where $|U_0| = |W_0| + |V_0|$. In particular, we take n =N, with $N = (b - x_0)/h$, and since $W_0 = V_0 = 0$, then we obtain:

$$|W_{N+k}| \le \frac{h^2\overline{\tilde{M}}}{1-2hLB_k} \sum_{j=0}^{k-1} \frac{\left(\dfrac{2hLB_j+2\alpha_j}{1-2hLB_k}\right)^{\left(\frac{b-x_0}{h}\right)+j} - 1}{\left(\dfrac{2hLB_j+2\alpha_j}{1-2hLB_k}\right) - 1}$$

$$|V_{N+k}| \le \frac{h^2\underline{\tilde{M}}}{1-2hLB_k} \sum_{j=0}^{k-1} \frac{\left(\dfrac{2hLB_j+2\alpha_j}{1-2hLB_k}\right)^{\left(\frac{b-x_0}{h}\right)+j} - 1}{\left(\dfrac{2hLB_j+2\alpha_j}{1-2hLB_k}\right) - 1}$$

and if $h \longrightarrow 0$, we get $W_{N+k} \longrightarrow 0$, $V_{N+k} \longrightarrow 0$, which completes the proof of the theorem. ■

## 3.5 VARIABLE STEP SIZE METHOD FOR SOLVING FUZZY DIFFERENTIAL EQUATIONS

As it is known from the usual methods of numerical analysis, the step size is fixed during the approach of solution, but still there are some methods for reducing the local truncation error such as the variable step-size methods.

In all fixed step-size methods the local truncation error will depends on x and on the numerical method used. But, in variable step-size methods, we shall find an approximation to the solution at a point $x_f$ for the initial value problem:

$$\tilde{y}'(x) = f(x, \tilde{y}(x)), \; x \in I = [x_0, b]$$

$$\tilde{y}(x_0) = \tilde{y}_0$$

those are accurate to within a specified tolerance.

Therefore, it turns out the for reasonable effective estimates of the step-size it is required to attain a specified upper local truncation error and lower local truncation error (tolerance) ε, which can be found that uses only the order of the upper local truncation error and lower local truncation error. And do not require further knowledge about of the error term. The variable step-size method which will be considered here, is based upon comparison of the estimates of the one and two step methods of solution (this application to upper and lower α-level values) of the value of x at some time obtained by the numerical method with upper local truncation error term and lower local truncation error term. Those are of the form $\overline{c}\,h^{p+1}$ and $\underline{c}\,h^{p+1}$, where $\overline{c}$ and $\underline{c}$ are unknown constant and p is the order of the method.

Suppose that we started with initial conditions $\overline{\tilde{y}}_{\alpha_0}$ and $\underline{\tilde{y}}_{\alpha_0}$, with step-size $h_0$. Using the numerical method to find the solutions $\overline{\tilde{y}}_{\alpha}^{(1)}(x_0+h_0)$ and $\overline{\tilde{y}}_{\alpha}^{(2)}(x_0+h_0)$ using the step-size $h_0$ and $\frac{h_0}{2}$, respectively.

That application to lower α-level case in the same way, similarly, let:

$$\overline{E}_{est} = |\,\overline{\tilde{y}}_{\alpha}^{(1)}(x_0+h_0) - \overline{\tilde{y}}_{\alpha}^{(2)}(x_0+h_0)\,|,\ h_0 = h_{old.}$$

If $\overline{E}_{est} \le \varepsilon$, then there is no problem and we can consider $\overline{\tilde{y}}_{\alpha}^{(2)}(x_0+h_0)$ as the solution at $x_0$ + $h_0$. Otherwise if $\overline{E}_{est} > \varepsilon$, then we need to find another estimation of the step- size say ($h_{new}$) which will produce an $\overline{E}_{est} < \varepsilon$. If this approximation was accepted then this value of $h_{new}$ will be used as $h_0$ in the next step; if not, then it will be used as $h_0$ and we repeat similarly as above. We apply as above to the lower α-level case similarly.

A common question may arise, which is how to find $h_{new}$? In this work, a new criterion has been developed for estimating the upper local truncation error and lower local truncation error, which control the step- size. The problem of error estimation is the most important problem that faces the user while using variable step-size method. To understand the concept of the error especially, the upper local truncation error and lower local truncation error, Taylor series expansion can be used, such that,

$$\bar{\tilde{y}}_\alpha(x+h) = \bar{\tilde{y}}_\alpha + h\bar{\tilde{y}}'_\alpha + \frac{h^2}{2!}\bar{\tilde{y}}''_\alpha + \ldots + \frac{h^p}{p!}\bar{\tilde{y}}^{(p)}_\alpha + \frac{h^{p+1}}{(p+1)!}\bar{\tilde{y}}^{(p+1)}_\alpha + O(h^{p+2})$$

Therefore, the upper local truncation error of order p must satisfy the condition:

$$\left|\frac{h^{p+1}}{(p+1)!}\bar{\tilde{y}}^{(p+1)}_\alpha(x_n)\right| \le \varepsilon$$

So, to estimate this quantity, we need to evaluate $\bar{\tilde{y}}^{(p+1)}_\alpha(x_n)$ which is a higher order derivative at the point $x_n$. Using the same analysis with the new step-size $\alpha h$ Taylor series expansion may be written as:

$$\bar{\tilde{y}}_\alpha(x+\alpha h) = \bar{\tilde{y}}_\alpha + \alpha h\bar{\tilde{y}}'_\alpha + \frac{(\alpha h)^2}{2!}\bar{\tilde{y}}''_\alpha + \ldots + \frac{(\alpha h)^p}{p!}\bar{\tilde{y}}^{(p)}_\alpha + \frac{(\alpha h)^{p+1}}{(p+1)!}\bar{\tilde{y}}^{(p+1)}_\alpha + O(h^{p+2})$$

which must satisfy the condition that:

$$\left|\frac{(\alpha h)^{p+1}}{(p+1)!}\bar{\tilde{y}}^{(p+1)}_\alpha(x_n)\right| \le \varepsilon \quad \ldots\ldots (3.42)$$

using some elementary manipulation. Equation (3.42) may be rewritten as:

$$\alpha \le \left[\frac{\varepsilon}{\frac{h^{p+1}}{(p+1)!}\bar{\tilde{y}}^{(p+1)}_\alpha(x_n)}\right]^{\frac{1}{p+1}}$$

Clearly, we can find $\alpha$ and get the new step-size such that $h_{new}=\alpha h_{old}$, i.e.,

$$h_{new} = \left[\frac{\varepsilon(p+1)!}{h_0^{p+1}\overline{\tilde{y}}_\alpha^{(p+1)}(x_n)}\right]^{\frac{1}{p+1}} h_0 \qquad (3.43)$$

Similarly, we apply the above criteria to the lower $\alpha$-level case in the same approach.

***Example (3.1):***

The above process will be illustrated by evaluating an approximation to tolerance error $[\tilde{y}(0.02)]_r$ to within an accuracy of e = 0.000001, for the first order fuzzy differential equation:

$$\tilde{y}'(x) = \tilde{y}(x),\ \tilde{y}(0) = (0.75 + 0.25r,\ 1.125 - 0.125r),\ x \in [0, 1],\ r = 0.2$$

The Euler method will be used and hence p =1 in equation (3.43) in the upper and lower cases, with $h_0 = h_0^* = 0.02$. In order to solve this problem, we evaluate $\overline{\tilde{y}}''(x)$ and $\underline{\tilde{y}}''(x)$, in upper and lower cases, which are respectively:

$$\overline{\tilde{y}}'(x) = \overline{\tilde{y}}(x) \Rightarrow \overline{\tilde{y}}''(x) = \overline{\tilde{y}}'(x) = \overline{\tilde{y}}(x)$$

and

$$\underline{\tilde{y}}'(x) = \underline{\tilde{y}}(x) \Rightarrow \underline{\tilde{y}}''(x) = \underline{\tilde{y}}'(x) = \underline{\tilde{y}}(x)$$

one can also notice that:

$$\nabla^k \overline{\tilde{y}}(x_n, r) = (-1)^k h^k \overline{\tilde{y}}^{(k)}(x_n, r)$$

$$\nabla^k \underline{\tilde{y}}(x_n, r) = (-1)^k h^k \underline{\tilde{y}}^{(k)}(x_n, r)$$

which implies that:

$$h^2\,\bar{\tilde{y}}''(x_n,r) = \bar{\tilde{y}}_n(r) - 2\,\bar{\tilde{y}}_{n-1}(r) + \bar{\tilde{y}}_{n-2}(r)$$

$$h^{*2}\,\underline{\tilde{y}}''(x_n,r) = \underline{\tilde{y}}_n(r) - 2\,\underline{\tilde{y}}_{n-1}(r) + \underline{\tilde{y}}_{n-2}(r)$$

Upon executing the (prog1.pas) program, we get the results presented in table (3.1):

***Table (3.1).***

***Numerical Results of Example (3.1).***

| $x_i$ | $\bar{\tilde{y}}_1$ | $\bar{\tilde{y}}_2$ | $\bar{E}_{est}$ | $H_{old}$ | $h_{new}$ |
|---|---|---|---|---|---|
| 0 | 1.12200E+00 | – | – | 0.02 | 1.33504E-03 |
| 1.33504E-03 | 1.10146E+00 | 1.10146E+00 | 4.90148E-07 | 1.33504E-03 | 1.34750E-03 |
| 1.34750E-03 | 1.10295E+00 | 1.10295E+00 | 5.00000E-07 | 1.34750E-03 | 1.34659E-03 |
| 2.69409E-03 | 1.10443E+00 | 1.10443E+00 | 5.00000E-07 | 1.34659E-03 | 1.34568E-03 |
| 4.03978E-03 | 1.10592E+00 | 1.10592E+00 | 5.00001E-07 | 1.34568E-03 | 1.34478E-03 |
| 5.38456E-03 | 1.10741E+00 | 1.10741E+00 | 5.00000E-07 | 1.34478E-03 | 1.34387E-03 |
| 6.72844E-03 | 1.10890E+00 | 1.10890E+00 | 4.99998E-07 | 1.34387E-03 | 1.34297E-03 |
| 8.07141E-03 | 1.11039E+00 | 1.11039E+00 | 4.99998E-07 | 1.34297E-03 | 1.34207E-03 |
| 9.413491-03 | 1.11188E+00 | 1.11188E+00 | 4.99998E-07 | 1.34207E-03 | 1.34117E-03 |
| 1.07546E-02 | 1.11337E+00 | 1.113377E+00 | 5.00000E-07 | 1.34117E-03 | 1.34027E-03 |
| 1.20949E-02 | 1.11486E+00 | 1.114867E+00 | 5.00000E-07 | 1.34027E-03 | 1.33937E-03 |
| 1.34343E-02 | 1.11636E+00 | 1.116361E+00 | 5.00000E-07 | 1.33937E-03 | 1.33848E-03 |
| 1.47728E-02 | 1.11785E+00 | 1.117856E+00 | 5.00000E-07 | 1.33848E-03 | 1.33758E-03 |
| 1.61103E-02 | 1.11939E+00 | 1.119351E+00 | 5.00000E-07 | 1.33758E-03 | 1.33669E-03 |
| 1.74470E-02 | 1.12084E+00 | 1.120848E+00 | 5.00001E-07 | 1.33669E-03 | 1.33580E-03 |
| 1.87828E-02 | 1.12234E+00 | 1.12234E+00 | 5.00001E-07 | 1.33580E-03 | 1.21711E-03 |
| 0.02 | 1.12371E+00 | 1.1237E+00 | 4.157E-07 | 1.21711E-03 | – |

| $x_i$ | $\underline{\tilde{y}}_1$ | $\underline{\tilde{y}}_2$ | $\underline{E}_{est}$ | $H_{old}$ | $h_{new}$ |
|---|---|---|---|---|---|
| 0 | 8.16000E-01 | – | – | 0.02 | 1.56548E-03 |
| 1.565483E-03 | 8.01252E-01 | 8.01252E-01 | 4.90148E-07 | 1.56548E-03 | 1.57990E-03 |
| 1.57990E-03 | 8.02518E-01 | 8.025192E-01 | 5.000001E-07 | 1.579902E-03 | 1.57865E-03 |
| 3.15855E-03 | 8.03786E-01 | 8.037866E-01 | 5.00000E-07 | 1.57865E-03 | 1.57741E-03 |
| 4.73596E-03 | 8.05054E-01 | 8.05055E-01 | 5.00000E-07 | 1.57741E-03 | 1.57616E-03 |
| 6.31213E-03 | 8.06323E-01 | 8.06324E-01 | 5.00000E-07 | 1.57616E-03 | 1.57492E-03 |
| 7.88705E-03 | 8.07594E-01 | 8.07594E-01 | 4.99999E-07 | 1.57492E-03 | 1.57368E-03 |
| 9.46074E-03 | 8.08865E-01 | 8.08866E-01 | 4.99999E-07 | 1.57368E-03 | 1.57244E-03 |
| 1.10331E-02 | 8.10138E-01 | 8.10138E-01 | 5.00000E-07 | 1.572449E-03 | 1.57121E-03 |
| 1.26044E-02 | 8.11411E-01 | 8.11412E-01 | 5.00000E-07 | 1.571213E-03 | 1.56998E-03 |
| 1.41743E-02 | 8.12685E-01 | 8.12686E-01 | 5.00001E-07 | 1.56998E-03 | 1.56874E-03 |
| 1.57431E-02 | 8.13961E-01 | 8.13961E-01 | 5.00000E-07 | 1.56874E-03 | 1.56751E-03 |
| 1.73106E-02 | 8.152377E-01 | 8.15238E-01 | 4.99999E-07 | 1.567519E-03 | 1.566291E-03 |
| 1.8876E-02 | 8.16515E-01 | 8.16515E-01 | 5.00000E-07 | 1.56629E-03 | 1.12304E-03 |
| 0.02 | 8.17432E-01 | 8.174E-01 | 4.794E-07 | 1.1230494E-03 | – |

## 3.6 ANALYTIC SOLUTION OF LINEAR FUZZY DIFFERENTIAL EQUATIONS [WUHAIB, 2005], [PEARSON, 1997]

Consider the system:

$$\tilde{y}'(x) = A\,\tilde{y}(x),\ \tilde{y}(0) = \tilde{y}_0 \quad \ldots\ldots (3.44)$$

where $\tilde{y}'(x) = \frac{d\tilde{y}}{dx}$, $A : R^n \longrightarrow R^n$, $\tilde{y}$ is a fuzzy mapping $\tilde{y} : R^n \longrightarrow [0, 1]$, where $\tilde{y}$ is a vector made up of n-fuzzy mapping.

Each element of the vector $\tilde{y}$ in (3.44) at the intervals x written as:

$$\tilde{y}_r^k(x) = [\,\underline{\tilde{y}}_r^k(x),\ \overline{\tilde{y}}_r^k(x)],\ k = 1, 2, \ldots, n \quad \ldots\ldots (3.45)$$

It is shown that the evaluation of the system (3.44) can be described by 2n differential equations for the end points of the variables (3.45), this for each given x and value of r of course, the complete overview have to be built up numerically by interpolation or some other means. The equation for the end points of the intervals are given by:

$$\left.\begin{array}{l} \underline{\tilde{y}}_r'^{k}(x) = \min\left\{(A\tilde{u})_k : \tilde{u}^i \in [\underline{\tilde{y}}_r^i(x), \overline{\tilde{y}}_r^i(x)]\right\} \\ \overline{\tilde{y}}_r'^{k}(x) = \max\left\{(A\tilde{u})_k : \tilde{u}^i \in [\underline{\tilde{y}}_r^i(x), \overline{\tilde{y}}_r^i(x)]\right\} \\ \underline{\tilde{y}}_r^k(0) = \underline{\tilde{y}}_{r_0}^k,\ \overline{\tilde{y}}_r^k(0) = \overline{\tilde{y}}_{r_0}^k \end{array}\right\} \qquad (3.46)$$

where $(A\tilde{u})_k := \sum_{j=1}^{n} a_{kj}\tilde{u}^j$ is the $k^{th}$ row of $A\tilde{u}$.

The vector field in (3.44) is linear, and so the following rule applies in (3.46):

$$\underline{\tilde{y}}_r'^{k}(x) = \sum_{j=1}^{n} a_{kj}\tilde{u}^j \qquad (3.47)$$

where:

$$\tilde{u}^j = \begin{cases} \underline{\tilde{y}}_r^j(x), & \text{if} \quad a_{kj} \geq 0 \\ \overline{\tilde{y}}_r^j(x), & \text{if} \quad a_{kj} < 0 \end{cases}$$

and

$$\overline{\tilde{y}}_r'^{k}(x) = \sum_{j=1}^{n} a_{kj}\tilde{u}^j \qquad (3.48)$$

where:

$$\tilde{u}^j = \begin{cases} \overline{\tilde{y}}_r^j(x), & \text{if} \quad a_{kj} \geq 0 \\ \underline{\tilde{y}}_r^j(x), & \text{if} \quad a_{kj} < 0 \end{cases}$$

Equations (3.47) and (3.48) are called the parametric equations.

***Representation by complex numbers:***

This approach is based on the following criteria, for each variable in (3.44), therefore, there are two equations of the type (3.47) and (3.48), which could be written out explicity.

However, we propose a slightly more compact representation of the same thing by passing to the filed of complex numbers.

Define new variables n complex form:

$$\tilde{z}_r^k = \underline{\tilde{y}}_r^k + i\ \overline{\tilde{y}}_r^k \qquad (3.49)$$

where i := $\sqrt{-1}$ and the two operations carried on the complex numbers as:

$$\begin{aligned} e\,\tilde{z}_r^k &= \tilde{z}_r^k \\ g(\tilde{z}_r^k) &= \overline{\tilde{y}}_r^k + i\,\underline{\tilde{y}}_r^k \end{aligned} \qquad (3.50)$$

e is just the identity operation and g corresponds to a flip about the diagonal in the complex plane. We notice that $g^2 = e$ and $g^k = e$ if k is even and $g^k = g$ if k is odd. It is easily verified that:

$$(\eta g)\ \tilde{z}_r^k = (g\eta)\,\tilde{z}_r^k \ \text{ for } \eta \in R$$

and we extend g to vectors via:

$$g\,\tilde{z}_r = \begin{bmatrix} g\tilde{z}_r^1 \\ g\tilde{z}_r^2 \\ \vdots \\ g\tilde{z}_r^n \end{bmatrix}$$

Using (3.49) and the two operators (3.50), it is fairly easy to se that equations (3.47) and (3.48) can be written as:

$$\tilde{z}_r' = B\,\tilde{z}_r\,,\ \tilde{z}_r^k(0) = \tilde{z}_{r_0} \qquad (3.51)$$

where the elements of the matrix B are determined from those of A as follows:

$$b_{ij} = \begin{cases} ea_{ij}, & a_{ij} \geq 0 \\ ga_{ij}, & a_{ij} < 0 \end{cases} \quad \ldots\ldots (3.52)$$

The solution of (3.51) is then given by:

$$\tilde{z}_r(x) = \exp(xB)\ \tilde{z}_{r_0} \quad \ldots\ldots (3.53)$$

The problem is therefore to calculate the exponential of the matrix B, where certain elements are multiplied by the flip operator (3.52). This can be done for small values of x by first of all writing the matrix B as the sum of two matrices, one of which is multiplied by the operator e, and the other by g, for example:

$$B = eC + gD$$

Now, for small x, we have:

$$\exp(xB)\ \tilde{z}_{r_0} = \exp(x(eC + gD))\ \tilde{z}_{r_0}$$

$$= \exp(xeC)\ \exp(xgD)\ \tilde{z}_{r_0} + O(x) \quad \ldots\ldots (3.54)$$

where O(x) is a function of x, such that $\lim_{x \to 0} O(x)/x = 0$.

The first part on the right hand side exp(xeC) is simply the standard matrix exponential, because e is the identity operator. For the second part exp(xgD), we note that $g^k = e$ if k is even and $g^k = g$ if it is odd and then proceed to calculate the formal series of exp (xgD):

$$\exp(xgD)\,\tilde{z}_{r_0} = \left( I + xgD + \frac{x^2}{2!}D^2 + \frac{x^3}{3!}gD^3 + \ldots \right)\tilde{z}_{r_0}$$

$$= \left( I + \frac{x^2}{2!}D^2 + \ldots \right)\tilde{z}_{r_0} + \left( xD + \frac{x^3}{3!}D^3 + \ldots \right) g\,\tilde{z}_{r_0}$$

$$= \cosh(xD)\ \tilde{z}_{r_0} + \sinh(xD)g\,\tilde{z}_{r_0} \quad \ldots\ldots (3.55)$$

Combining (3.54) with (3.55), the solution (3.53) for small x is then given by:

$$\tilde{z}_r(x) = \exp(xC)(\cosh(xD)\ \tilde{z}_{r_0} + \sinh(xD)g\,\tilde{z}_{r_0})$$

letting:

$$\varphi(x) = \exp(xC)\cosh(xD) \qquad (3.56)$$

$$\psi(x) = \exp(xC)\sinh(xD) \qquad (3.57)$$

and so in component form, one have:

$$\tilde{z}_r(x) = \varphi_{kj}(x)\tilde{z}^j_{r_0} + \psi_{kj}(x)g\,\tilde{z}^j_{r_0}$$

Then by (3.49) this reduces to:

$$\underline{\tilde{y}}^k_r(x) + i\,\overline{\tilde{y}}^k_r(x) = \varphi_{kj}(x)(\underline{\tilde{y}}^j_{r_0}(x) + i\,\overline{\tilde{y}}^j_{r_0}(x)) + \psi_{kj}(x)(\overline{\tilde{y}}^j_{r_0}(x) + i\,\underline{\tilde{y}}^j_{r_0}(x))$$

and in other words:

$$\underline{\tilde{y}}^k_r(x) = \varphi_{kj}(x)\ \underline{\tilde{y}}^j_{r_0}(x) + \psi_{kj}(x)\ \overline{\tilde{y}}^j_{r_0}(x) \qquad (3.58)$$

$$\overline{\tilde{y}}^k_r(x) = \varphi_{kj}(x)\overline{\tilde{y}}^j_{r_0}(x) + \psi_{kj}(x)i\,\underline{\tilde{y}}^j_{r_0}(x) \qquad (3.59)$$

***Example (3.2):***

Consider the linear system $\tilde{y}' = A\tilde{y}$, where $A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$ with initial values to be $\tilde{y}^1(0)$ about 1 and $\tilde{y}^2(0)$ about −1, which are fuzzy numbers and using membership function defined by setting, for example,

$$\mu_{\tilde{y}^1_0}(x) = \begin{cases} 0, & x < 0 \\ 2x - x^2, & 0 \le x < 2 \\ 0, & x > 2 \end{cases}$$

and

$$\mu_{\tilde{y}_0^2}(x) = \begin{cases} 0, & x > -2 \\ -2x - x^2, & -2 \le x < 0 \\ 0, & x > 0 \end{cases}$$

Thus, for r $\in$ [0, 1], we can represent the initial condition as:

$$[\,\tilde{y}_0^1\,]_r = [\,\underline{\tilde{y}}^1_{0_r}\,,\ \overline{\tilde{y}}^1_{0_r}\,] = [1 - \sqrt{1-r}\,,\ 1 + \sqrt{1-r}\,]$$

$$[\,\tilde{y}_0^2\,]_r = [\,\underline{\tilde{y}}^2_{0_r}\,,\ \overline{\tilde{y}}^2_{0_r}\,] = [-1 - \sqrt{1-r}\,,\ -1 + \sqrt{1-r}\,]$$

Now, to solve the above system, first we use equations (3.52) to have:

$$B = \begin{bmatrix} g(-1) & e(1) \\ e(0) & g(-2) \end{bmatrix} = \begin{bmatrix} -i & 1 \\ 0 & -2i \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & -2i \end{bmatrix}$$

$$= e\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + g\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= eC + gD$$

Using equations (3.56), (3.57) at x =0.2, we have:

$$\varphi(0.2) = \begin{bmatrix} 1.020066 & 0.216213 \\ 0 & 1.08166 \end{bmatrix}$$

$$\psi(0.2) = \begin{bmatrix} -0.201333 & -0.082133 \\ 0 & -0.410606 \end{bmatrix}$$

So, if we let for simplicity:

$$a = 1 - \sqrt{1-r}\,,\ b = 1 + \sqrt{1-r}\,,\ c = -1 - \sqrt{1-r}\ \text{ and } d = -1 + \sqrt{1-r}$$

Then using equations (3.58), (3.59) to have the true solutions given by:

$$\underline{\tilde{y}}_r^1(0.2) = 1.020066a + 0.216231c - 0.201333b - 0.082133d$$

$$\overline{\tilde{y}}_r^1\ (0.2) = 1.020066b + 0.216213 - 0.201333a - 0.082133c$$

$$\underline{\tilde{y}}_r^2\ (0.2) = 1.0810666c - 0.410666d$$

$$\overline{\tilde{y}}_r^2\ (0.2) = -0.4106666c + 1.080666d$$

For example, if r = 0.1, then a = 0.0513167, b = 1.94867, c = −1.94868 and d = −0.051317.

$$\underline{\tilde{y}}_{0.1}^1\ (0.2) = -0.7570682$$

$$\overline{\tilde{y}}_{0.1}^1\ (0.2) = 2.126394$$

$$\underline{\tilde{y}}_{0.1}^2\ (0.2) = -2.085526$$

$$\overline{\tilde{y}}_{0.1}^2\ (0.2) = 0.744773$$

## 3.7 NUMERICAL RESULTS

In this section, some numerical examples are presented as an illustration to the numerical methods discussed in the last sections.

For each example, numerical and theoretical results are presented in tables in order to give a good comparison between the results.

### *3.7.1 Examples of Linear Fuzzy Differential Equations:*

First, in this subsection, we will present some examples in linear case, i.e., examples of LFDE.

***<u>Example (3.3):</u>***

Consider the fuzzy initial value problem:

$\tilde{y}'(x) = \tilde{y}(x),\ x \in [0, 1]$

$\tilde{y}_r(0) = (0.75 + 0.25r,\ 1.125 - 0.125r),\ r \in [0, 1]$

The parametric form of $\tilde{y}(x)$ in this case is given by:

$$\underline{\tilde{y}}'(x; r) = \underline{\tilde{y}}(x; r) \ \text{ and } \ \overline{\tilde{y}}'(x; r) = \overline{\tilde{y}}(x; r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (3.60)$$

with initial condition are given for all $r \in [0, 1]$ by:

$\underline{\tilde{y}}(0; r) = 0.75 + 0.25r$

$\overline{\tilde{y}}(0; r) = 1.125 - 0.125r$

The exact solution could be evaluated easily as:

$\underline{\tilde{Y}}_r(x) = e^{(x - x_0)}(0.75 + 0.25r)$

$\overline{\tilde{Y}}_r(x) = e^{(x - x_0)}(1.125 - 0.125r)$

at $x = 1$ and $x_0 = 0$, we get:

$\underline{\tilde{Y}}(1; r) = (0.75 + 0.25r)e$

$\overline{\tilde{Y}}(1; r) = (1.125 - 0.125r)e$

Using Euler method given by:

$$\begin{aligned} \underline{\tilde{y}}_{n+1}(r) &= \underline{\tilde{y}}_n(r) + hG[x_n,\ \underline{\tilde{y}}_n(r),\ \overline{\tilde{y}}_n(r)] \\ &\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (3.61) \\ \overline{\tilde{y}}_{n+1}(r) &= \overline{\tilde{y}}_n(r) + hF[x_n,\ \underline{\tilde{y}}_n(r),\ \overline{\tilde{y}}_n(r)] \end{aligned}$$

Using equations (3.60) with (3.61), we get:

$\underline{\tilde{y}}_{n+1}(r) = \underline{\tilde{y}}_n(r) + h\,\underline{\tilde{y}}_n(r)$

$\overline{\tilde{y}}_{n+1}(r) = \overline{\tilde{y}}_n(r) + h\,\overline{\tilde{y}}_n(r)$

and Simpson method

$$\underline{\tilde{y}}_{n+2}(r) - \underline{\tilde{y}}_{n}(r) = \frac{h}{3}[G[x_{n+2}, \underline{\tilde{y}}_{n}(r) + 2hG(x_n, \underline{\tilde{y}}_{n}(r), \overline{\tilde{y}}_{n}(r)), \overline{\tilde{y}}_{n}(r) + 2hF(x_n, \underline{\tilde{y}}_{n}(r), \overline{\tilde{y}}_{n}(r))] + 4G(x_{n+1}, \underline{\tilde{y}}_{n+1}(r), \overline{\tilde{y}}_{n+1}(r)) + G(x_n, \underline{\tilde{y}}_{n}(r), \overline{\tilde{y}}_{n}(r))]$$

$$\overline{\tilde{y}}_{n+2}(r) - \overline{\tilde{y}}_{n}(r) = \frac{h}{3}[F[x_{n+2}, \underline{\tilde{y}}_{n}(r) + 2hG(x_n, \underline{\tilde{y}}_{n}(r), \overline{\tilde{y}}_{n}(r)), \overline{\tilde{y}}_{n}(r) + 2hF(x_n, \underline{\tilde{y}}_{n}(r), \overline{\tilde{y}}_{n}(r))] + 4F(x_{n+1}, \underline{\tilde{y}}_{n+1}(r), \overline{\tilde{y}}_{n+1}(r)) + F(x_n, \underline{\tilde{y}}_{n}(r), \overline{\tilde{y}}_{n}(r))]$$

.............................. (3.62)

Using (3.60) with (3.62), to have:

$$\underline{\tilde{y}}_{n+2}(r) - \underline{\tilde{y}}_{n}(r) = \frac{h}{3}(\underline{\tilde{y}}_{n}(r) + 2h\underline{\tilde{y}}_{n}(r) + 4\underline{\tilde{y}}_{n+1}(r) + \underline{\tilde{y}}_{n}(r))$$

$$\overline{\tilde{y}}_{n+2}(r) - \overline{\tilde{y}}_{n}(r) = \frac{h}{3}(\overline{\tilde{y}}_{n}(r) + 2h\overline{\tilde{y}}_{n}(r) + 4\overline{\tilde{y}}_{n+1}(r) + \overline{\tilde{y}}_{n}(r))$$

with:

$$\underline{\tilde{y}}_{1} = \underline{\tilde{y}}_{0} + h\underline{\tilde{y}}_{0} + \frac{h^2}{2}\underline{\tilde{y}}_{0} \quad \text{and} \quad \overline{\tilde{y}}_{1} = \overline{\tilde{y}}_{0} + h\overline{\tilde{y}}_{0} + \frac{h^2}{2}\overline{\tilde{y}}_{0}$$

as the initial value with $h = 0.1$.

The results are presented in table (3.2) after carrying our the computer program (program2.pas).

***Table (3.2)***

***Results of example (3.3) with h = 0.1.***

| r | Euler results $(\underset{\sim}{y}(r), \overline{\tilde{y}}(r))$ | Simpson results $(\underset{\sim}{y}(r), \overline{\tilde{y}}(r))$ | Exact results $(\underset{\sim}{y}(r), \overline{\tilde{y}}(r))$ |
|---|---|---|---|
| 0 | (1.945307, 2.91796) | (2.036928, 3.055392) | (2.038711, 3.058067) |
| 0.2 | (2.074994, 2.853117) | (2.172723, 2.987494) | (2.174625, 2.99011) |
| 0.4 | (2.204681, 2.788273) | (2.308518, 2.919597) | (2.310539, 2.922153) |
| 0.6 | (2.334368, 2.72343) | (2.444314, 2.851699) | (2.446454, 2.854196) |
| 0.8 | (2.464055, 2.658586) | (2.580109, 2.783802) | (2.582368, 2.786239) |
| 1.0 | (2.593742, 2.593742) | (2.715904, 2.715904) | (2.718282, 2.718282 |

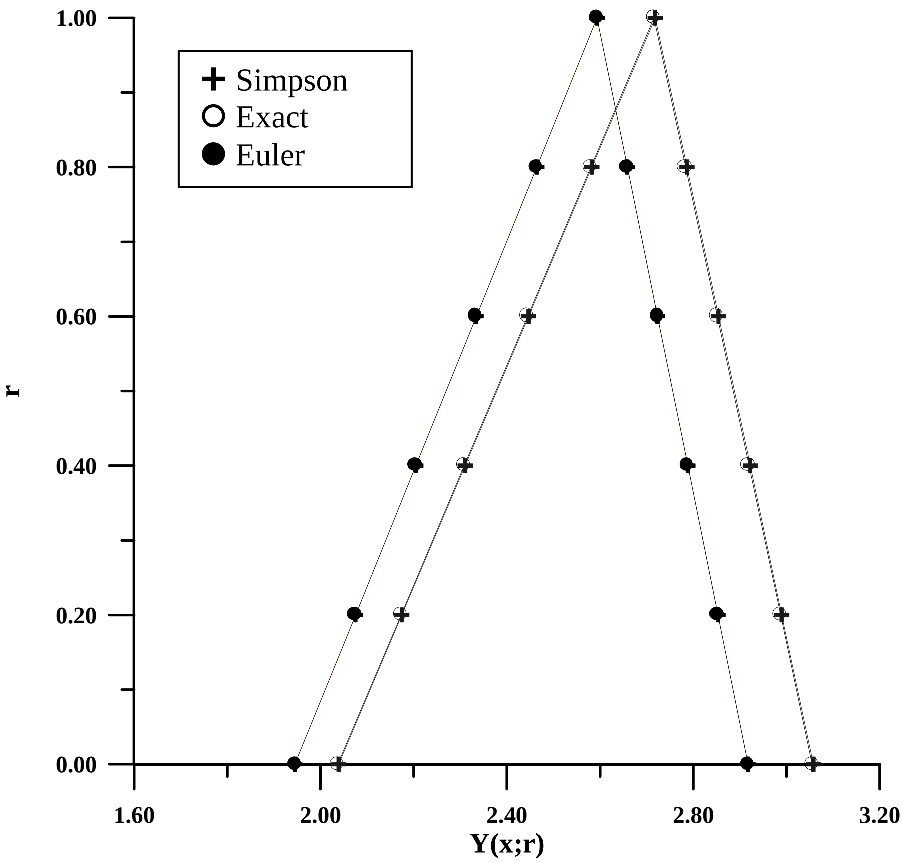


***Figure (3.1) Analytical and Numerical Results of example (3.3).***

***Example (3.4):***

In order to solve the FIVP:

$$\tilde{y}'(x) = x\,\tilde{y}(x),\ x \in [-1, 1]$$

with initial condition in parametric form:

$$\tilde{y}(-1) = (\underline{\tilde{y}}_{r0},\ \overline{\tilde{y}}_{r0})$$

$$= (\sqrt{e} - 0.5(1 - r),\ \sqrt{e} + 0.5(1 - r))$$

The method of solution will be discussed for x > 0 and then for x ≤ 0 as the following cases show:

**a. If x < 0:** The parametric form in this case:

$$\underline{\tilde{y}}'(x; r) = x\,\overline{\tilde{y}}(x; r) \quad \text{and} \quad \overline{\tilde{y}}'(x; r) = x\,\underline{\tilde{y}}(x; r)$$

with initial conditions are:

$$\underline{\tilde{y}}(-1; r) = \underline{\tilde{y}}_{r0} \quad \text{and} \quad \overline{\tilde{y}}(-1; r) = \overline{\tilde{y}}_{r0}$$

The unique solution is given by:

$$\underline{\tilde{Y}}(x; r) = \cosh\left(\frac{x^2 - x_0^2}{2}\right)\underline{\tilde{y}}_{r0} + \sinh\left(\frac{x^2 - x_0^2}{2}\right)\overline{\tilde{y}}_{r0}$$

$$= \frac{A+B}{2}\underline{\tilde{y}}_{r0} + \frac{A-B}{2}\overline{\tilde{y}}_{r0}$$

$$\overline{\tilde{Y}}(x; r) = \cosh\left(\frac{x^2 - x_0^2}{2}\right)\overline{\tilde{y}}_{r0} + \sinh\left(\frac{x^2 - x_0^2}{2}\right)\underline{\tilde{y}}_{r0}$$

where $A = e^{\frac{x^2 - x_0^2}{2}}$, $B = \frac{1}{A}$

For $x = 0$, $x_0 = -1$, one get:

$$\underline{\tilde{Y}}(0; r) = \frac{e^{-1/2} + e^{1/2}}{2}\, \underline{\tilde{y}}_{r0} + \frac{e^{-1/2} - e^{1/2}}{2}\, \overline{\tilde{y}}_{r0} = \underline{\tilde{Y}}_{r0}$$

$$\overline{\tilde{Y}}(0; r) = \frac{e^{-1/2} + e^{1/2}}{2}\, \overline{\tilde{y}}_{r0} + \frac{e^{-1/2} - e^{1/2}}{2}\, \underline{\tilde{y}}_{r0} = \overline{\tilde{Y}}_{r0}$$

and using these numbers as initial conditions for the next step.

**b. If $x \geq 0$:** The parametric are given by:

$$\underline{\tilde{y}}'(x; r) = x\,\underline{\tilde{y}}(x; r) \quad \text{and} \quad \overline{\tilde{y}}'(x; r) = x\,\overline{\tilde{y}}(x; r)$$

with initial conditions are $\underline{\tilde{y}}_{r0}$, $\overline{\tilde{y}}_{r0}$.

The unique solution at $x \geq 0$ is:

$$\underline{\tilde{Y}}(x; r) = \underline{\tilde{Y}}_{r0}\, e^{\frac{x^2 - x_0^2}{2}}$$

$$\overline{\tilde{Y}}(x; r) = \overline{\tilde{Y}}_{r0}\, e^{\frac{x^2 - x_0^2}{2}}$$

Substituting the arguments of $\underline{\tilde{Y}}_{r0}$, $\overline{\tilde{Y}}_{r0}$ in the last result, these leading to:

$$\underline{\tilde{Y}}(1; r) = \frac{1-e}{2}\, \overline{\tilde{y}}_{r0} + \frac{1+e}{2}\, \underline{\tilde{y}}_{r0}$$

$$\overline{\tilde{Y}}(1; r) = \frac{1+e}{2}\, \overline{\tilde{y}}_{r0} - \frac{e-1}{2}\, \underline{\tilde{y}}_{r0}$$

To get method approximation, we divide $[-1, 1]$ into (even number) N with equally spaced subintervals. In the numerical calculations, we use:

$$\underline{\tilde{y}}_1(r) = \underline{\tilde{y}}_0 + hx_0\,\overline{\tilde{y}}_0 + \frac{h^2}{2}\,(1 + x_0^2)\,\underline{\tilde{y}}_0$$

$$\overline{\tilde{y}}_1(r) = \overline{\tilde{y}}_0 + hx_0\,\underline{\tilde{y}}_0 + \frac{h^2}{2}\,(1 + x_0^2)\,\overline{\tilde{y}}_0$$

as initial value of two step method.

Now, upon carrying the midpoint method, such that:

$$\underline{\tilde{y}}_{n+2}(r) = \underline{\tilde{y}}_{n}(r) + 2hG(x_{n+1}, \underline{\tilde{y}}_{n+1}(r), \overline{\tilde{y}}_{n+1}(r))$$

$$\overline{\tilde{y}}_{n+2}(r) = \overline{\tilde{y}}_{n}(r) + 2hF(x_{n+1}, \underline{\tilde{y}}_{n+1}(r), \overline{\tilde{y}}_{n+1}(r))$$

when $x_n \geq 0$. Then:

$$\underline{\tilde{y}}_{n+2}(r) = \underline{\tilde{y}}_{n}(r) + 2hx_{n+1}\underline{\tilde{y}}_{n+1}(r)$$

$$\overline{\tilde{y}}_{n+2}(r) = \overline{\tilde{y}}_{n}(r) + 2hx_{n+1}\overline{\tilde{y}}_{n+1}(r)$$

In case of $x_n < 0$. Then:

$$\underline{\tilde{y}}_{n+2}(r) = \underline{\tilde{y}}_{n}(r) + 2hx_{n+1}\overline{\tilde{y}}_{n+1}(r)$$

$$\overline{\tilde{y}}_{n+2}(r) = \overline{\tilde{y}}_{n}(r) + 2hx_{n+1}\underline{\tilde{y}}_{n+1}(r)$$

and trapizodal method, given by:

$$\underline{\tilde{y}}_{n+1}(r) = \underline{\tilde{y}}_{n}(r) + \frac{h}{2}[G(x_{n+1}, \underline{\tilde{y}}_{n+1}(r), \overline{\tilde{y}}_{n+1}(r)) + G(x_n, \underline{\tilde{y}}_{n}(r), \overline{\tilde{y}}_{n}(r))]$$

$$\overline{\tilde{y}}_{n+1}(r) = \overline{\tilde{y}}_{n}(r) + \frac{h}{2}[F(x_{n+1}, \underline{\tilde{y}}_{n+1}(r), \overline{\tilde{y}}_{n+1}(r)) + F(x_n, \underline{\tilde{y}}_{n}(r), \overline{\tilde{y}}_{n}(r))]$$

when $x_n \geq 0$:

$$\underline{\tilde{y}}_{n+1}(r) = \underline{\tilde{y}}_{n}(r) + \frac{h}{2}[x_{n+1}\underline{\tilde{y}}_{n+1}(r) + x_n\underline{\tilde{y}}_{n}(r)]$$

$$\overline{\tilde{y}}_{n+1}(r) = \overline{\tilde{y}}_{n}(r) + \frac{h}{2}[x_{n+1}\overline{\tilde{y}}_{n+1}(r) + x_n\overline{\tilde{y}}_{n}(r)]$$

when $x_n < 0$, one have:

$$\underline{\tilde{y}}_{n+1}(r) = \underline{\tilde{y}}_{n}(r) + \frac{h}{2}[x_{n+1}\overline{\tilde{y}}_{n+1}(r) + x_n\overline{\tilde{y}}_{n}(r)]$$

$$\overline{\tilde{y}}_{n+1}(r) = \overline{\tilde{y}}_{n}(r) + \frac{h}{2}[x_{n+1}\underline{\tilde{y}}_{n+1}(r) + x_n\underline{\tilde{y}}_{n}(r)]$$

The results are presented in table (3.3), when x < 0 and in table (3.4) when x ≥ 0 using the computer program (program3.pas).

***Table (3.3)***

***Results of example (3.4) with h =0.1, for $x_n$ < 0***

| r | Mid point results | Trapizoidal results | Exact results |
|---|---|---|---|
| 0 | (0.1741345, 1.831501) | (0.4982622, 1.876977) | (0.1756394, 1.824361) |
| 0.2 | (0.3398712, 1.665765) | (0.6361336, 1.739105) | (0.3405115, 1.659489) |
| 0.4 | (0.5056078, 1.500028) | (0.7740051, 1.601234) | (0.5053836, 1.494616) |
| 0.6 | (0.6713445, 1.334291) | (0.9118765, 1.463362) | (0.6702557, 1.329744) |
| 0.8 | (0.8370812, 1.168555) | (1.049748, 1.325491) | (0.8351279, 1.164872) |
| 1.0 | (1.002818, 1.002818) | (1.187619, 1.187619) | (1, 1) |

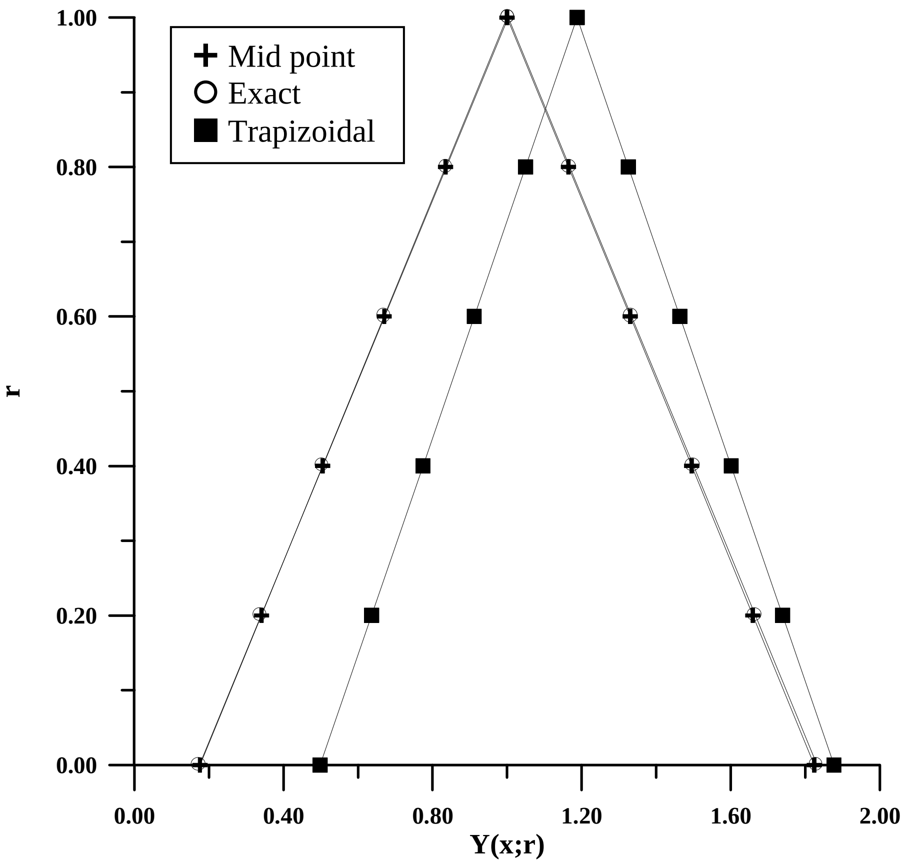


***Figure (3.2) Analytical and Numerical Results of example (3.4).***

***Table (3.4)***

***Results of example (3.4) with h =0.1, for $x_n \geq 0$***

| r | Mid point results | Trapizoidal results | Exact results |
|---|---|---|---|
| 0 | (0.2627423, 2.729096) | (1.031197, 2.90632) | (0.2895803, 3.007862) |
| 0.2 | (0.5093776, 2.48246) | (1.218709, 2.718808) | (0.5614085, 2.736034) |
| 0.4 | (0.756013, 2.235825) | (1.406222, 2.531295) | (0.8332367, 2.464206) |
| 0.6 | (1.002648, 1.98919) | (1.593734, 2.343783) | (1.105065, 2.192378) |
| 0.8 | (1.249284, 1.742554) | (1.781246, 2.156271) | (1.376893, 1.92055) |
| 1.0 | (1.495919, 1.495919) | (1.968758, 1.968758) | (1.648721, 1.648721) |

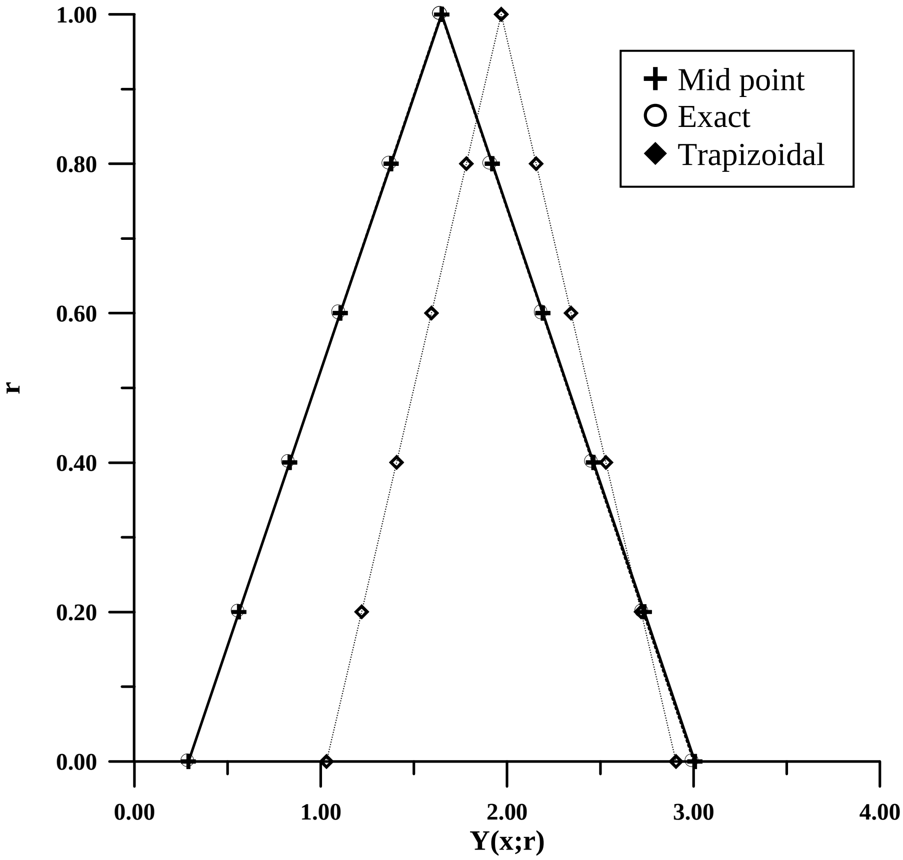


***Figure (3.3) Analytical and Numerical Results of example (3.4).***

### *3.7.2 Non-Linear Fuzzy Differential Equations [Buckly and Feuring, 2000]:*

In some cases, it may be happen that a fuzzy differential equation appears in non-linear form as:

$$\tilde{y}'(x) = f(x, \tilde{y}(x), \tilde{k}), \; \tilde{y}(0) = \tilde{c} \quad \text{.................................................} \quad (3.63)$$

where $\tilde{k} = (\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_n)$ be a vector of triangular fuzzy numbers, and x is in some interval I (closed and bounded) containing zero. Therefore, in order to find the solution (3.63), the following remark and lemma will be introduced first:

***Remark (3.3) [Buckly and Feuring, 2000]:***

For $\tilde{y}(x)$ to be a solution to the fuzzy initial value problem, we need that $\tilde{y}'(x)$ to be exists with the restriction equation (3.63) must be hold.

To check equation (3.63), we must first compute f(t, $\tilde{y}$, $\tilde{k}$). The r-level of f(x, $\tilde{y}$, $\tilde{k}$) which can be found as follows:

$$[f(x, \tilde{y}, \tilde{k})]_r = [G(x; r), F(x; r)] \quad \text{.................................................} \quad (3.64)$$

with:

$$G(x; r) = \min \{f(x, y, k) \mid y \in [\tilde{y}(x)]_r, k \in [\tilde{k}]_r\} \quad \text{..........................} \quad (3.65)$$

$$F(x; r) = \max \{f(x, y, k) \mid y \in [\tilde{y}(x)]_r, k \in [\tilde{k}]_r\} \quad \text{..........................} \quad (3.66)$$

For $x \in I$, $r \in [0, 1]$. It is said that $\tilde{y}(x)$ is a solution of equation (3.63) if $\tilde{y}'(x)$ exists and

$$\underline{\tilde{y}}'(x; r) = G(x; r) \quad \text{and} \quad \overline{\tilde{y}}'(x; r) = F(x; r) \quad \text{..................................} \quad (3.67)$$

$$\underline{\tilde{y}}(0; r) = \tilde{C}_1(r) \quad \text{and} \quad \overline{\tilde{y}}(0; r) = \tilde{C}_2(r) \quad \text{........................................} \quad (3.68)$$

where $[\tilde{C}]_r = [\tilde{C}_1(r), \tilde{C}_2(r)]$.

***Lemma (3.4) [Buckly and Feuring, 2000]:***

Assume that $\tilde{y}'(x)$ exists for $x \in I$. Then the solution is $\tilde{y}(x)$ if:

$$\frac{\partial f}{\partial y} > 0, \ \frac{\partial g}{\partial c} > 0 \qquad (3.69)$$

and

$$\left(\frac{\partial g}{\partial k_i}\right)\left(\frac{\partial f}{\partial k_i}\right) > 0 \qquad (3.70)$$

for all i =1, 2, …, n. If equation (3.69) does not hold or equation (3.70) does not hold for some i, then $\tilde{y}(x)$ does not solve the FIVP, where f is the first order ordinary differential equation, and g is the unique solution of f.

***Proof:***

Let us assume that there is only one $k_i = k$ and that $\frac{\partial g}{\partial k} < 0$ and $\frac{\partial f}{\partial k} < 0$.

The proof $\frac{\partial g}{\partial k} > 0$ and $\frac{\partial f}{\partial k} > 0$ is similar and therefore omitted.

Since $\frac{\partial g}{\partial k} < 0$ and $\frac{\partial g}{\partial c} > 0$, we have:

$$\underline{\tilde{y}}(x; r) = g(x, \overline{\tilde{k}}(r), \tilde{C}_1(r)), \ \overline{\tilde{y}}(x; r) = g(x, \underline{\tilde{k}}(r), \tilde{C}_2(r))$$

Also, because $\frac{\partial f}{\partial y} > 0$ and $\frac{\partial f}{\partial k} < 0$, we see that:

$$G(x; r) = f(x, \underline{\tilde{y}}(x; r), \overline{\tilde{k}}(r)), \ F(x; r) = f(x, \overline{\tilde{y}}(x; r), \underline{\tilde{k}}(r))$$

Now, $y = g(x, k, c)$ is the unique solution of $y' = f(x, y, k)$ and $y(0) = c$, which implies that:

$$g'(x, k, c) = f(x, g(x, k, c), k) \quad \text{.......} \quad (3.71)$$

and

$$g(0, k, c) = c \quad \text{.......} \quad (3.72)$$

Assuming $\tilde{y}'(x)$ exists, we see that:

$$\underline{\tilde{y}}'(x; r) = g'(x, \overline{\tilde{k}}(r), \tilde{C}_1(r))$$

$$= f(x, g(x, \overline{\tilde{k}}(r), \tilde{C}_1(r)), \overline{\tilde{k}}(r))$$

$$= G(x; r)$$

and $\underline{\tilde{y}}(0; r) = g(0, \overline{\tilde{k}}(r), \tilde{C}_1(r)) = \tilde{C}_1(r)$, and also

$$\overline{\tilde{y}}'(x; r) = g'(x, \underline{\tilde{k}}(r), \tilde{C}_2(r))$$

$$= f(x, g(x, \underline{\tilde{k}}(r), \tilde{C}_2(r)), \underline{\tilde{k}}(r))$$

$$= F(x; r)$$

and $\underline{\tilde{y}}(0; r) = g(0, \underline{\tilde{k}}(r), \tilde{C}_2(r)) = \tilde{C}_2(r)$, for all $r \in [0, 1]$ and $x \in I$.

Hence equations (3.67) and (3.68) in remark (3.3) hold

Now, consider the situation, where equation (3.69) or (3.70) does not hold. Let us only look at one case, where $\partial f/\partial y < 0$ (assume $\partial g/\partial c > 0$, $\partial f/\partial k > 0$, $\partial g/\partial k > 0$). Then, we have:

$$G(x; r) = f(x, \overline{\tilde{y}}(x; r), \underline{\tilde{k}}(r)),$$

$$F(x; r) = f(x, \underline{\tilde{y}}(x; r), \overline{\tilde{k}}(r)),$$

$$\underline{\tilde{y}}(x; r) = g(x, \underline{\tilde{k}}(r), \tilde{C}_1(r))$$

$$\overline{\tilde{y}}(x; r) = g(x, \overline{\tilde{k}}(r), \tilde{C}_2(r))$$

The part one of equation (3.67) becomes

$$\underline{\tilde{y}}'(x; r) = g'(x, \underline{\tilde{k}}(r), \tilde{C}_1(r)) = G(X; r)$$

$$= f(x, g(x, \overline{\tilde{k}}(r), \tilde{C}_2(r)), \underline{\tilde{k}}(r))$$

which is not true. ■

***Example (3.5):***

Consider the non-homogeneous, non-linear ordinary initial value problem:

$$y' = k_1 y^2 + k_2,\ y(0) = 0 \quad \text{....} \quad (3.73)$$

where $k_1, k_2 > 0$. The solution is given by:

$$y = g(x, k_1, k_2, c) = \lambda \tan(wx) \quad \text{....} \quad (3.74)$$

on $I = [0, 1]$, with $w = \sqrt{k_1 k_2}$ and $\lambda = \sqrt{k_2 / k_1}$.

Now, consider the corresponding fuzzy initial value problem with $\tilde{k}_1$, $\tilde{k}_2 > 0$. We calculate $\tilde{Y}(x)$ using lemma (3.4), i.e., since $\partial g/\partial k_1 > 0$ and $\partial g/\partial k_2 > 0$.

$$\underline{\tilde{y}}(x; r) = \underline{\tilde{\lambda}}(r) \tan(\underline{\tilde{w}}(r)x) \quad \text{....} \quad (3.75)$$

$$\overline{\tilde{y}}(x; r) = \overline{\tilde{\lambda}}(r) \tan(\overline{\tilde{w}}(r)x) \quad \text{....} \quad (3.76)$$

with $\underline{\tilde{\lambda}}(r) = \sqrt{\underline{\tilde{k}}_2(r) / \underline{\tilde{k}}_1(r)}$, $\overline{\tilde{\lambda}}(r) = \sqrt{\overline{\tilde{k}}_2(r) / \overline{\tilde{k}}_1(r)}$, $\underline{\tilde{w}}(r) = \sqrt{\underline{\tilde{k}}_1(r) \underline{\tilde{k}}_2(r)}$ and $\overline{\tilde{w}}(r) = \sqrt{\overline{\tilde{k}}_1(r) \overline{\tilde{k}}_2(r)}$, with:

$[\tilde{k}_1]_r = [0.5 + 0.5r, 1.5 - 0.5r]$

$[\tilde{k}_2]_r = [0.75 + 0.25r, 1.25 - 0.25r]$

The r-level sets of $\tilde{y}'(x)$ are:

$$\underline{\tilde{y}}'(x; r) = \underline{\tilde{k}}_2(r)\sec^2(\underline{\tilde{w}}(r)x) \quad (3.77)$$

$$\overline{\tilde{y}}'(x; r) = \overline{\tilde{k}}_2(r)\sec^2(\overline{\tilde{w}}(r)x) \quad (3.78)$$

which defines a fuzzy number.

The results are presented in the following table upon using the following explicit method:

$$\underline{\tilde{y}}_n(r) - 2\underline{\tilde{y}}_{n+1}(r) + \underline{\tilde{y}}_{n+2}(r) = h[G(x_{n+1}, \underline{\tilde{y}}_{n+1}(r), \overline{\tilde{y}}_{n+1}(r)) - G(x_n, \underline{\tilde{y}}_n(r), \overline{\tilde{y}}_n(r))] \quad (3.79)$$

$$\overline{\tilde{y}}_n(r) - 2\overline{\tilde{y}}_{n+1}(r) + \overline{\tilde{y}}_{n+2}(r) = h[F(x_{n+1}, \underline{\tilde{y}}_{n+1}(r), \overline{\tilde{y}}_{n+1}(r)) - F(x_n, \underline{\tilde{y}}_n(r), \overline{\tilde{y}}_n(r))]$$

Then from equations (3.77) and (3.78), we have:

$$\underline{\tilde{y}}'(x; r) = \underline{\tilde{k}}_2(r) + \underline{\tilde{k}}_1(r)\,\underline{\tilde{y}}^2(r)$$

$$\overline{\tilde{y}}'(x; r) = \overline{\tilde{k}}_2(r) + \overline{\tilde{k}}_1(r)\,\overline{\tilde{y}}^2(r) \quad (3.80)$$

Now, from (3.80) and (3.79), one have:

$$\underline{\tilde{y}}_{n+2}(r) = 2\underline{\tilde{y}}_{n+1}(r) - \underline{\tilde{y}}_n(r) + h[(\underline{\tilde{k}}_2(r) + \underline{\tilde{k}}_1(r)\underline{\tilde{y}}^2_{n+1}(r)) - (\underline{\tilde{k}}_2(r) + \underline{\tilde{k}}_1(r)\underline{\tilde{y}}^2_n(r))]$$

$$= 2\underline{\tilde{y}}_{n+1}(r) - \underline{\tilde{y}}_n(r) + h[\underline{\tilde{k}}_1(r)\underline{\tilde{y}}^2_{n+1}(r) - \underline{\tilde{k}}_1(r)\underline{\tilde{y}}^2_n(r)]$$

$$\overline{\tilde{y}}_{n+2}(r) = 2\overline{\tilde{y}}_{n+1}(r) - \overline{\tilde{y}}_n(r) + h[(\overline{\tilde{k}}_2(r) + \overline{\tilde{k}}_1(r)\overline{\tilde{y}}^2_{n+1}(r)) - (\overline{\tilde{k}}_2(r) + \overline{\tilde{k}}_1(r)\overline{\tilde{y}}^2_n(r))]$$

$$= 2\overline{\tilde{y}}_{n+1}(r) - \overline{\tilde{y}}_n(r) + h[\overline{\tilde{k}}_1(r)\,\overline{\tilde{y}}^2_{n+1}(r) - \overline{\tilde{k}}_1(r)\overline{\tilde{y}}^2_n(r)]$$

and also, using the another explicit method given by:

$$\underline{\tilde{y}}_{n+2}(r) - \underline{\tilde{y}}_{n+1}(r) = \frac{h}{2}[3G(x_{n+1}, \underline{\tilde{y}}_{n+1}(r), \overline{\tilde{y}}_{n+1}(r)) - G(x_n, \underline{\tilde{y}}_n(r), \overline{\tilde{y}}_n(r))]$$

.............. (3.81)

$$\overline{\tilde{y}}_{n+2}(r) - \overline{\tilde{y}}_{n+1}(r) = \frac{h}{2}[3F(x_{n+1}, \underline{\tilde{y}}_{n+1}(r), \overline{\tilde{y}}_{n+1}(r)) - F(x_n, \underline{\tilde{y}}_n(r), \overline{\tilde{y}}_n(r))]$$

Then from (3.80) and (3.81), we get:

$$\underline{\tilde{y}}_{n+2}(r) = \underline{\tilde{y}}_{n+1}(r) + \frac{h}{2}[3(\underline{\tilde{k}}_2(r) + \underline{\tilde{k}}_1(r)\ \underline{\tilde{y}}^2_{n+1}(r)) - (\underline{\tilde{k}}_2(r) + \underline{\tilde{k}}_1(r)\ \underline{\tilde{y}}^2_n(r))]$$

$$\overline{\tilde{y}}_{n+2}(r) = \overline{\tilde{y}}_{n+1}(r) + \frac{h}{2}[3(\overline{\tilde{k}}_2(r) + \overline{\tilde{k}}_1(r)\ \overline{\tilde{y}}^2_{n+1}(r)) - (\underline{\tilde{k}}_2(r) + \underline{\tilde{k}}_1(r)\ \overline{\tilde{y}}^2_n(r))]$$

Using $\overline{\tilde{y}}_0(r) = 0$, $\underline{\tilde{y}}_0(r) = 0$, and

$$\overline{\tilde{y}}_1(r) = \overline{\tilde{y}}_0(r) + h(\overline{\tilde{k}}_2(r) + \overline{\tilde{k}}_1(r)\ \overline{\tilde{y}}^2_0(r))$$

$$\underline{\tilde{y}}_1(r) = \underline{\tilde{y}}_0(r) + h(\underline{\tilde{k}}_2(r) + \underline{\tilde{k}}_1(r)\ \underline{\tilde{y}}^2_0(r))$$

as an initial conditions in the above explicit methods.

The results are presented in table (3.5) using the computer program (program4.pas)

***Table (3.5. Numerical and Exact results of example (3.5).***

| **r (h = 0.1)** | **First explicit results** $(\underline{\tilde{y}}(r), \overline{\tilde{y}}(r))$ | **Second explicit results** $(\underline{\tilde{y}}(r), \overline{\tilde{y}}(r))$ | **Exact results** $(\underline{\tilde{y}}(r), \overline{\tilde{y}}(r))$ |
|---|---|---|---|
| 0 | (0.8398467, 2.597979) | (0.855959, 3.317767) | (0.8603294, 4.469125) |
| 0.2 | (0.926897, 2.258159) | (0.9512431, 2.739682) | (0.9585038, 3.285743) |
| 0.4 | (1.023628, 1.981418) | (1.059543, 2.30838) | (1.071439, 2.591944) |
| 0.6 | (1.132181, 1.752413) | (1.184381, 1.976965) | (1.203806, 2.133143) |
| 0.8 | (1.255272, 1.5601) | (1.330517, 1.715554) | (1.362381, 1.805155) |
| 1.0 | (1.396394, 1.396394) | (1.5045, 1.5045) | (1.557408, 1.557408) |

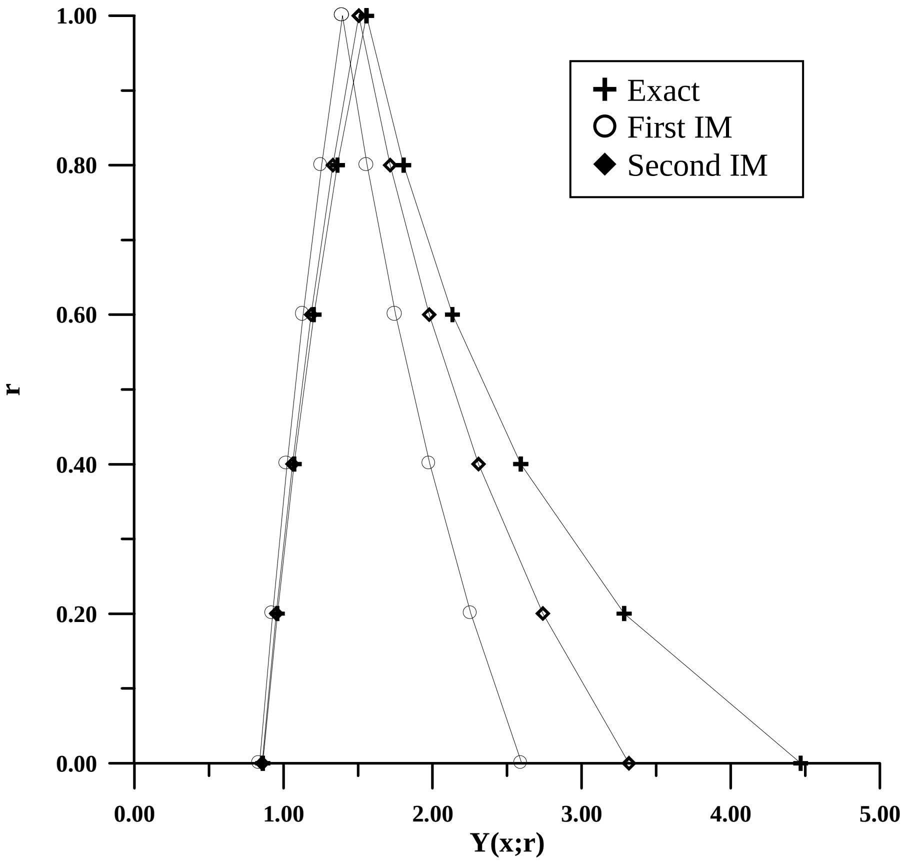


***Figure (3.4) Results of Example (3.5).***

***Example (3.6):***

Consider the non-linear FIVP:

$$\tilde{y}'(x) = e^{-\tilde{y}^2(x)},\ y(0) = (0.75 + 0.25r,\ 1.5 - 0.5r)$$

over the interval [0, 1].

The parametric equations are:

$$\underline{\tilde{y}}'(x; r) = \exp\{-[\max(|\underline{\tilde{y}}(x; r)|,\ \overline{\tilde{y}}(x; r)|)]^2\}$$

$$\overline{\tilde{y}}'(x; r) = \exp\{-[\min(|\underline{\tilde{y}}(x; r)|,\ |\overline{\tilde{y}}(x; r)|)]^2\} \qquad \text{.................................... (3.82)}$$

with the initial condition ($\underline{\tilde{y}}(0; r)$, $\overline{\tilde{y}}(0; r)$). This example could not be solved analytically, therefore numerical methods will be used for different values of step size h.

Using Adam-Bashforth method:

$$\underline{\tilde{y}}_{n+3}(r) - \underline{\tilde{y}}_{n+2}(r) = \frac{h}{12}[23G(x_{n+2}, \underline{\tilde{y}}_{n+2}(r), \overline{\tilde{y}}_{n+2}(r)) - 16G(x_{n+1}, \underline{\tilde{y}}_{n+1}(r), \overline{\tilde{y}}_{n+1}(r)) + 5\,G(x_n, \underline{\tilde{y}}_n(r), \overline{\tilde{y}}_n(r))]$$

$$\overline{\tilde{y}}_{n+3}(r) - \overline{\tilde{y}}_{n+2}(r) = \frac{h}{12}[23F(x_{n+2}, \underline{\tilde{y}}_{n+2}(r), \overline{\tilde{y}}_{n+2}(r)) - 16F(x_{n+1}, \underline{\tilde{y}}_{n+1}(r), \overline{\tilde{y}}_{n+1}(r)) + 5F(x_n, \underline{\tilde{y}}_n(r), \overline{\tilde{y}}_n(r))]$$

............................ (3.83)

Now, take equations (3.82) and (3.83), to have:

$$\underline{\tilde{y}}_{n+3}(r) = \underline{\tilde{y}}_{n+2}(r) + \frac{h}{12}[23\exp\{-[\max(|\underline{\tilde{y}}_{n+2}(x; r)|, |\overline{\tilde{y}}_{n+2}(x; r)|)]^2\} - 16\exp\{-[\max(|\underline{\tilde{y}}_{n+1}(x; r)|, |\overline{\tilde{y}}_{n+1}(x; r)|)]^2\} + 5\exp\{-[\max(|\underline{\tilde{y}}_n(x; r)|, |\overline{\tilde{y}}_n(x; r)|)]^2\}]$$

$$\overline{\tilde{y}}_{n+3}(r) = \overline{\tilde{y}}_{n+2}(r) + \frac{h}{12}[23\exp\{-[\min(|\underline{\tilde{y}}_{n+2}(x; r)|, |\overline{\tilde{y}}_{n+2}(x; r)|)]^2\} - 16\exp\{-[\min(|\underline{\tilde{y}}_{n+1}(x; r)|, |\overline{\tilde{y}}_{n+1}(x; r)|)]^2\} + 5\exp\{-[\min(|\underline{\tilde{y}}_n(x; r)|, |\overline{\tilde{y}}_n(x; r)|)]^2\}]$$

with using Eular method as an initial condition, such that:

$$\underline{\tilde{y}}_{n+1}(r) = \underline{\tilde{y}}_n(r) + h\exp\{-[\max(|\underline{\tilde{y}}_n(x; r)|, |\overline{\tilde{y}}_n(x; r)|)]^2\}$$

$$\overline{\tilde{y}}_{n+1}(r) = \overline{\tilde{y}}_n(r) + h\exp\{-[\min(|\underline{\tilde{y}}_n(x; r)|, |\overline{\tilde{y}}_n(x; r)|)]^2\}$$

for n = 0, 1.

The results are presented in table (3.6) using the computer program (program6.pas).

***Table (3.6) Numerical results of example(3.6) using Adam's method.***

| r | Adam-Bashforth results with h = 0.1 | | Adam-Bashforth results with h = 0.05 | |
|---|---|---|---|---|
| | $\underline{\tilde{y}}(r)$ | $\overline{\tilde{y}}(r)$ | $\underline{\tilde{y}}(r)$ | $\overline{\tilde{y}}(r)$ |
| 0.0 | 0.7962322 | 1.93692 | 0.7971708 | 1.989998 |
| 0.2 | 0.8666981 | 1.795828 | 0.8690056 | 1.842866 |
| 0.4 | 0.9441779 | 1.653868 | 0.9487784 | 1.694455 |
| 0.6 | 1.029957 | 1.511592 | 1.03812 | 1.545408 |
| 0.8 | 1.124981 | 1.369808 | 1.138298 | 1.396737 |
| 1.0 | 1.229553 | 1.229553 | 1.249807 | 1.249807 |

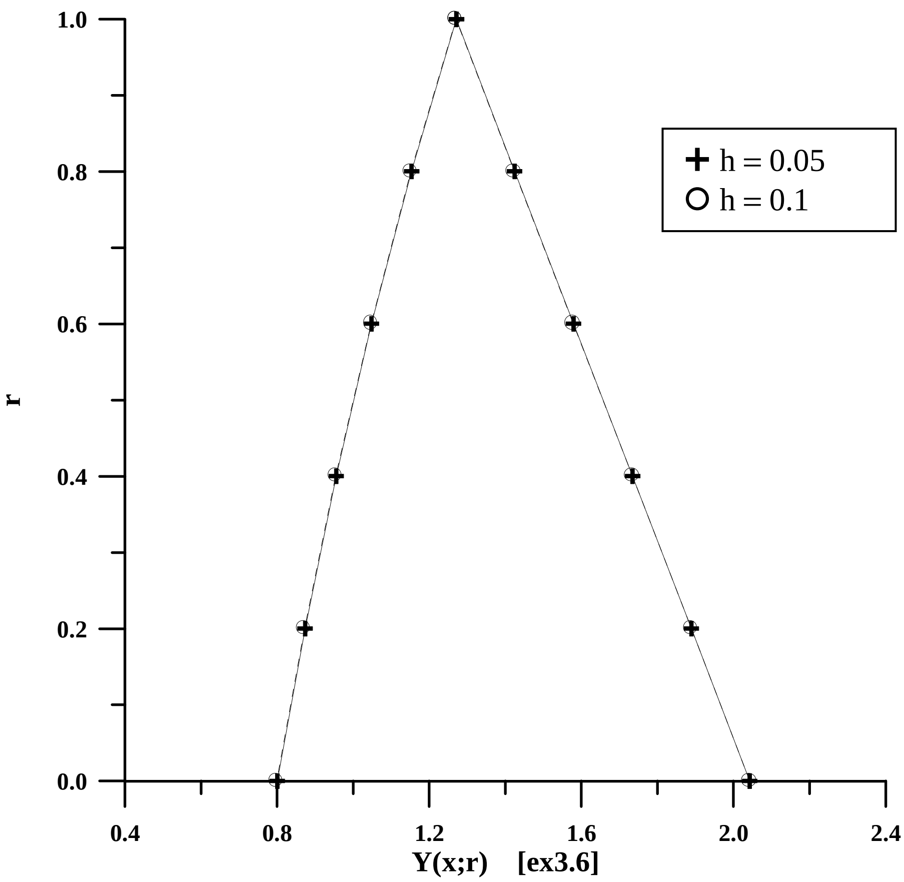


***Figure (3.5) Numerical results of example (3.6)***

# CHAPTER FOUR

# MODELING AND SOLUTION OF THE DECAY OF THE BIOCHEMICAL OXYGEN DEMAND IN WATER

## *3.1 INTRODUCTION*

A very important physical-chemical parameter of water is the concentration of dissolved oxygen necessary for all living equatic organisms. In this chapter, we have proposed a fuzzy model to describe the decay of the dissolved oxygen concentration in water using fuzzy differential equations, the classic analytic solution of which is well known. We use variable steps method to obtain an approximate solution of an initial value problem of a fuzzy linear ordinary differential equation modeling decay.

We compare numerical results with the fuzzy analytic solution for the similar fuzzy differential equations. With wide spread public interest in the quality of the environment, an quality in particular, many researchers have increased their work in the analysis and modeling of the more general problem of external effects or externalities. Usually, the external (i.e., cost or benefit) is transferred to the affected party through some complex series of physical, chemical or biological process.

Water quality is usually measured by the deviation of the concentration levels of certain chemicals or materials from desired levels. Some of the typical concentrations that are considered important are dissolved oxygen level, dissolved oxygen deficit (DOD), heat concentration (temperature level) and concentration of various chemicals. All rivers water concentration can be altered by mixing the river flow with influent flows of various concentration, the new concentration being a weighted average of the influent and mainstream concentrations.

Historically, the dissolved oxygen level has been the common measure of water quality. Recently, however, thermal pollution has become increasingly important not only because of its direct effects on equatic life but also because of its direct affects on the dissolved oxygen level.

However, these studies and models don’t take into account the vagueness and uncertainty present in parameters or on the measurement process. In this sense, we propose a fuzzy model in order to give this problem a mathematical treatment.

## *4.2 THE PROBLEM DISCRIPTION AND ITS MATHEMATICAL MODEL*

The model for the biochemical oxygen demand can be described as follows.

Assume that:

$$\frac{dB}{dt} = -kB,\ B(0) = B_0 \qquad (1.4)$$

where B is the biochemical oxygen demand (mg/L) at the fixed river section, k is the constant deoxygenation rate 20 $^{o}$C (days$^{-1}$). In this river section and $B_0$ is the initial condition at the biochemical oxygen demand. Data collection, as well as, the adopted method, is nearly always affected by vagueness and uncertainty, brought about by the measurement process, due to the subjectivity in the adopted method, as well as the uncertainty in the initial condition. In this way, in order to give a mathematical treatment to the problem, the use of fuzzy systems may be seem as an essential tool for the analysis and understanding of the studied phenomena. Therefore, we adopted the fuzziness process. For the first approach, considering the initial condition as a fuzzy set, we will adopt, for the model described by equation (4.1). The following fuzzy ordinary differential equation is obtained:

$$\tilde{y}'(x) = -k\,\tilde{y}(x),\ \tilde{y}(0) = \tilde{y}_0 \in E \quad \text{......} \quad (4.2)$$

where $\tilde{y}'(x)$ is Hukuhara's derivative, $\tilde{y} : R^{+} \longrightarrow E$ and $x, k \in R^{+}$. In other words, $[\tilde{y}]^{\alpha} = [[\tilde{y}_1]^{\alpha}, [\tilde{y}_2]^{\alpha}]$ is a closed interval.

The parametric form of equation (4.2) be such that:

$$\left.\begin{aligned} [\tilde{y}_1'(x)]^{\alpha} &= -k[\tilde{y}_2(x)]^{\alpha}, [\tilde{y}_1(0)]^{\alpha} = [\tilde{y}_{10}]^{\alpha},\ 0 \le \alpha \le 1 \\ [\tilde{y}_2'(x)]^{\alpha} &= -k[\tilde{y}_1(x)]^{\alpha}, [\tilde{y}_2(0)]^{\alpha} = [\tilde{y}_{20}]^{\alpha},\ 0 \le \alpha \le 1 \end{aligned}\right\} \quad \text{......} \quad (4.3)$$

where $[\tilde{y}_1(x)]^{\alpha}$ and $[\tilde{y}_2(x)]^{\alpha}$ are the $\alpha$-level of the solution for moment x, $[\tilde{y}_{10}]^{\alpha}$ and $[\tilde{y}_{20}]^{\alpha}$ are the $\alpha$-level of $u_0$.

For each $\alpha$, the analytic solution of the system described by (4.3) is given by:

$$\left.\begin{aligned}[\tilde{y}_1(x)]^{\alpha} &= \frac{[\tilde{y}_{10}]^{\alpha} - [\tilde{y}_{20}]^{\alpha}}{2} e^{kx} + \frac{[\tilde{y}_{10}]^{\alpha} + [\tilde{y}_{20}]^{\alpha}}{2} e^{-kx} \\ [\tilde{y}_2(x)]^{\alpha} &= \frac{[\tilde{y}_{20}]^{\alpha} - [\tilde{y}_{10}]^{\alpha}}{2} e^{kx} + \frac{[\tilde{y}_{20}]^{\alpha} + [\tilde{y}_{10}]^{\alpha}}{2} e^{-kx}\end{aligned}\right\} \quad \text{........................ (4.4)}$$

these solution are the $\alpha$-level of fuzzy solution $\tilde{y}(x)$.

For the system given by (4.4), we define for moment t, the diameter of

The interval $[[\tilde{y}_1(x)]^{\alpha}, [\tilde{y}_2(x)]^{\alpha}]$, for the $\alpha$-level, $\alpha \in (0, 1]$, as following:

$$\text{diam}[[\tilde{y}_1(x)]^{\alpha}, [\tilde{y}_2(x)]^{\alpha}] = [\tilde{y}_2(x)]^{\alpha} - [\tilde{y}_1(x)]^{\alpha}$$

$$= ([\tilde{y}_{20}]^{\alpha} - [\tilde{y}_{10}]^{\alpha})e^{kx} \quad \text{............................ (5.4)}$$

where $\alpha \in (0, 1]$ and $x \in [0, b]$.

Thus, we may note that the diameters. At each $\alpha$-level are increasing in time. This may be interpreted as the increasing of the uncertainty to go by the time, which is, in fact, reasonable.

## *4.3 NUEMRICAL METHOD*

In order to obtain a numerical approximation for the solution of each equation, given in system (4.3), we replace the interval [0, b] for a previously defined b > 0, by the set $\{0 = x_0 < x_1 < \ldots < x_n = b\}$ of discreet equally spaced grid points with fixed $\alpha$, we then use a numerical scheme for the approximation using variable step method.

In the simulation, we consider as the initial condition, the biochemical oxygen demand of about 100 mg/L described by the following fuzzy set:

$$\mu_0(\tilde{y}(x)) = \begin{cases} \dfrac{\tilde{y}(x)-90}{10}, & \text{if } 90 \le \tilde{y}(x) \le 100 \\ \dfrac{110-\tilde{y}(x)}{10}, & \text{if } 100 \le \tilde{y}(x) \le 110 \\ 0, & \text{otherwise} \end{cases}$$

for all $\tilde{y}(x) \in R$.

For the constant deoxygenating rate, we use $k = 0.038$ day$^{-1}$ and the time interval that we considered was [0, 30] days, with iterations h = 0.3 and the results are sketched in figure (4.1) below.

The results of the exact solution are presented in Figure (4.1) with using implicit Euler method with ε = 0.233, while the results of numerical results obtained using variable step size method are presented in Figure (4.2).

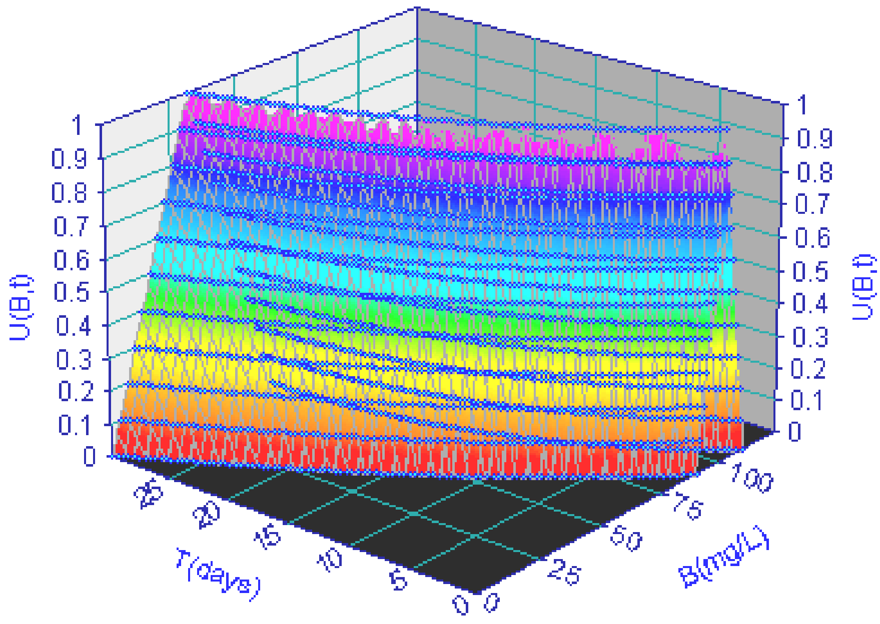


***Figure (4.1) Exact Results of example (4.1).***

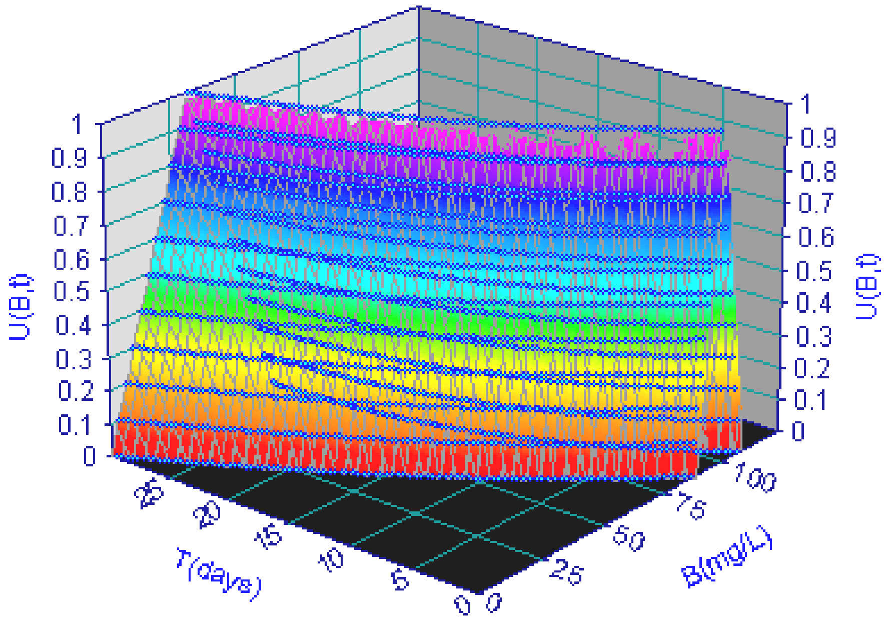


***Figure (4.2) Numerical results of example (4.1).***

# *CONCLUSIONS AND RECOMMINDATIONS*

From the present study, we can conclude the following:

1. Exact solution of fuzzy differential equations, may be some times so difficult to evaluated, especially in non-linear cases.
2. Parametric equations are so useful in solving fuzzy differential equations.
3. As it is expected, there is a very strong relationship between fuzzy differential equations and it level sets in ordinary form (parametric form).

Also, we can recommend the following for future work:

1. Studying existence and uniqueness theorem of fuzzy differential equations using, such as Burbakee fixed point theorem, Amann and Tersaki fixed point theorem, etc.
2. Extending the work of this thesis to study the solution of fuzzy partial differential equations, numerically and analytically.
3. Studying other real life problem, in which the governing mathematical modeling is fuzzy differential equations.
4. Studying fuzzy differential equations using other definition for differentiation, such as Goetschel-Voxman derivative, Seikkala derivative and Puri-Ralescu derivative, etc., (see [Buckley and Feuring, 2000].

# *Dedication*

*To My Family*

# INTRODUCTION

Most or our traditional tools for formal modeling reasoning and computing are crisp, deterministic, and precise in character, by crisp, we mean dichotomous, that is, yes or no type rather than more or less type. In conventional dual logic, for instance, statement can be true or false and nothing in between. In set theory, an element can either belong to a set or not, precision assumes that the parameters of a model represent exactly either our perception of a phenomena modeling or the futures of the real system that has been modeled. Generally, precision also implies that the model is unequivocal, that is, that it contains no ambiguities.

Certainty eventually indicates that we assume the structures and parameters of the model to be definitely known, and that there are no adopts about their values or their accuracy. If the model under consideration is a formal model [Zimmerman, 1980, p.127], that is, if it does not pretend to model reality adequately, then the model assumptions are in a sense arbitrary, that is, the model builder that can freely decide which model characteristics he chooses. If, however the model or theory asserts to be factual [Popper, 1959], [Zimmerman, 1980], that is, conclusions drawn from these model have a bearing or reality and they are supposed to model reality adequately, then the modeling language has to be suited to model the characteristics of the situation under study appropriately.

Zadeh in 1965, introduced the notion of fuzzy set provided a convenient point of departure for the construction of a conceptual frame work which parallels in many respects, the frame used in the case of ordinary system , but in more general than the later and, rotationally, many prove to nave a much reader scope of applicability, particularly in the field of pattern classification and information processing. Essentially, such a frame work provides a natural way of dealing with problems in which the source of imprecision of classical membership rather than the presence of random variables.

This thesis consists of four chapters.

Chapter on, entitled (Fuzzy set) introduces a basic concepts and definitions including definitions of fuzzy sets, basic properties and algebraic operations, membership function, level sets, fuzzy number.

Chapter two, entitled (The existence and uniqueness of fuzzy differential equations), which discussed in details, with proofs of existence and uniqueness theorem of fuzzy differential equation, in which the fuzziness occurs in the initial condition, and therefore in the solution of the fuzzy differential equation.

In chapter three, entitled (Linear multistep methods for solving fuzzy differential equations) discussion on the numerical solution of fuzzy differential equation using linear multistep method, as well as, derivation of some numerical methods and proving its convergence and stability, whenever the consistency condition is satisfied, in addition, variable step size method is also discussed for fuzzy set theory, the method efficiency is compared with the other results.

In chapter four, entitled (Modeling and solution of the decay of the biochemical oxygen demanding), a real life problem, which is the decay of biochemical oxygen demand in water in which it is introduced and discussed, as well as, its mathematical modeling using fuzzy set theory and therefore, solving the problem using the variable step method derived in chapter three, the results had proven its efficiency and compared with exact results.

The result of the numerical examples are given either in tabulated form or graphically and its comparison with the exact results, when it is necessary.

Computer programs are written in PASCAL language which are listed in appendix, while the computer software which are listed in appendix, while the computer software used to sketch the results the (GRAPH FOR WIN), (TABLE CURVE 3D, V.4.0).

All the results are executed in micro personal computer Pentium 4, processor (Celeron 2.4 MHz) located at the Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University.

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# *The Numerical Solution of Fuzzy Differential Equations Using Linear Multistep Methods*

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*By*

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جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة النهرين

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رسالة
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وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

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