بِسْمِ اللهِ الرَّحْمَن الرَّحِيم

﴿اللَّهُ نُورُ السَّمَاوَاتِ وَالْأَرْضِ مَثَلُ نُورِهِ كَمِشْكَاةٍ فِيهَا مِصْبَاحٌ

الْمِصْبَاحُ فِي زُجَاجَةٍ الزُّجَاجَةُ كَأَنَّهَا كَوْكَبٌ دُرِّيٌّ يُوقَدُ مِن

شَجَرَةٍ مُّبَارَكَةٍ زَيْتُونِةٍ لَّا شَرْقِيَّةٍ وَلَا غَرْبِيَّةٍ يَكَادُ زَيْتُهَا يُضِيءُ

وَلَوْ لَمْ تَمْسَسْهُ نَار نُّورٌ عَلَى نُورٍ يَهْدِي اللَّهُ لِنُورِهِ مَن

يَشَاءُ وَيَضْرِبُ اللَّهُ الْأَمْثَالَ لِلنَّاسِ وَاللَّهُ بِكُلِّ شَيْءٍ عَلِيمٌ {٣٥}

﴾

صَدَقَ اللَّهُ ٱلعظيم

سورَةُ ٱلنور

وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم

حول استقرارية المعادلات التفاضلية لمتعددة الرتب الكسرية ذات معاملات ثابتة

رسالة

مقدمه إلى كلية العلوم في جامعة النهرين

وهي جزء من متطلبات نيل درجة ماجستير

علوم في الرياضيات

من قبل

صبا صادق مهدي شبيب

(بكالوريوس علوم ٢٠٠٤)

بإشراف

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Abstract

The use of fractional orders differential and integral operators in mathematical models has been increasingly widespread in recent years. The non-linear multi-term fractional (arbitrary) order differential equation has been considered. Its solution existences and uniqueness are proved by transform it into a linear system of equations. Also, stability theorem for such a differential equation is presented (by transform it into an ordinary differential equations, as well as, different examples, are presented to verify our stability results.

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Saba Sadiq Mahdi Shibeeb

July, 2007

1.1 Introduction

The concept of non-integer of integration can be traced back to the genesis of differential calculus itself: The philosopher and creator of modern calculus G.W.Leibniz made some remarks on the meaning and possibility of fractional derivative of order 1/2 in the late 17th century. However a rigorous investigation was first carried out by Liouville in a series of papers from 1832-1837, where he defined the first out cast of an operator of fractional integration. Later investigation and further developments by among others Riemann led to the construction of the integral-based Riemann-Liouville fractional integral operator, which has been a valuable cornerstone in fractional calculus ever since [Munkhammar, 2005].

Prior to Liouville and Riemann, Euler took the first step in the study of fractional integration when he studied the simple case of fractional integrals of monomials of arbitrary real order in the heuristic fashion of time; it have been said to have lead him to construct the gamma function for fractional powers of the fractional [Lavoie , 2000].

An early attempt by Liouville was later purified by the Swedish mathematician Holmagren, who in 1865 made important contributions to the growing study of fractional calculus.

Today there exist many different forms of fractional integral operators, ranging from divided-difference types to infinite-sum types, but the Riemann-Liouville Operator is still the most frequently used when fractional integration is performed [Samko, 1993].

There are many interesting applications of fractional calculus, for example in physics it is used to model anomalous diffusion and in Hamiltonian chaos fractional partial differential equations can be used [Meerschaert, 2004].

Other applications to physics involve fractional mechanics and fractional oscillators [Achar, 2001].

Applications of fractional calculus in general also appear in speculative option valuation in finance and are related to so call heavy tails in electrical engineering [Meerschaert, 2004].

In this chapter we give the definitions of the Riemann-Liouville fractional integrals and fractional derivatives on a finite interval of the real line and present some of their properties in space of summable and continuous functions. More detailed information may be found in [Munkhammar, 2005].

1.2 Fractional calculus

This section presents some of the most basic and important concepts in fractional calculus which are necessary for understanding the subject of fractional calculus.

Gamma and Beta Functions [Oldham, 1974]:

One of the basic notions in fractional order differential equations, which are necessary in calculating and proving some results in fractional derivatives are the gamma and beta functions. So, in addition, these functions play an important role in physical applications.

The basic definitions of gamma function of a positive integer n is defined by the following improper integral:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

It is easily seen that the following properties are satisfied on gamma function.

1. If $n = 1/2,$ then $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$ which is easily to prove by considering:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

And letting $x = t^2,$ then $dx = 2tdt,$ that lead's to

$$\Gamma(n) = \int_0^\infty t^{2n-2} e^{-t^2} 2tdt = 2\int_0^\infty t^{2n-1} e^{-t^2} dt$$

Put $n = \frac{1}{2}$, we have

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty t^0 e^{-t^2} dt = 2\int_0^\infty e^{-t^2} dt$$

Therefore:

$$[\Gamma(\tfrac{1}{2})]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dxdy$$

where

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dxdy = rdrd\theta$$
$$= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} rdrd\theta$$
$$= \pi$$

Which is equivalent to $\Gamma(1/2) = \sqrt{\pi}.$

2. Another form of gamma function is given by:

$$\Gamma(n) = 2\int_0^\infty y^{2n-1} e^{-y^2} dy$$

Which also can be proved easily by letting $x = y^2$ and substituting in general form of gamma function.

Another type of functions is called the beta function, which are defined by the following integral:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \quad n,m > 0$$

Similarly, as in gamma function, some properties can be seen in beta function, which can be summarized as follows:

1. $\beta(m,n)=\beta(n,m),$ Since if we let $x=1-t,$ then $dx=-dt$ and hence using the definition of beta functions:

$$\begin{aligned}\beta(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}dx \\ &= -\int_1^0 (1-t)^{m-1}t^{n-1}dt \\ &= \int_0^1 (1-t)^{m-1}t^{n-1}dt \\ &= \beta(n,m)\end{aligned}$$

i.e., beta function is symmetric.

2. Another form of beta function is given by:

$$\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

Which is easily to prove by considering the transformation $x=\sin^2\theta,$ and hence $dx=2\sin\theta\cos\theta\, d\theta$ and therefore:

$$\begin{aligned}\beta(m,n) &= \int_0^{\pi/2}\sin^{2m-2}\theta(1-\sin^2\theta)^{n-1}2\sin\theta\cos\theta\, d\theta \\ &= 2\int_0^{\pi/2}\sin^{2m-2}\theta\cos^{2n-2}\theta\sin\theta\cos\theta\, d\theta \\ &= 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta\end{aligned}$$

4. An important relationship connecting between gamma and beta functions is given by:

$$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Which also could be proved easily, since:

$$\Gamma(m)=2\int_0^\infty x^{2m-1}e^{-x^2}dx$$

$$\Gamma(n)=2\int_0^\infty y^{2n-1}e^{-y^2}dy$$

Therefore:

$$\Gamma(m)\Gamma(n)=4\int_0^\infty\int_0^\infty x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dxdy$$

And using polar transformation, the last double integral takes the form:

$$\begin{aligned}\Gamma(m)\Gamma(n)&=4\int_0^{\pi/2}\int_0^\infty r^{2m+2n-2}\cos^{2m-1}\theta\sin^{2n-1}\theta\, e^{-r^2}rdrd\theta\\&=4\int_0^{\pi/2}\int_0^\infty r^{2m+2n-1}\cos^{2m-1}\theta\sin^{2n-1}\theta\, e^{-r^2}rdrd\theta\\&=\left(2\int_0^{\pi/2}\cos^{2m-1}\theta\sin^{2n-1}\theta\, d\theta\right)\left(2\int_0^\infty r^{2(m+n)-1}e^{-r^2}dr\right)\\&=\beta(m,n)\Gamma(m+n)\end{aligned}$$

Hence:

$$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

1.3 Definitions and Theories [Samko, 1993]:

Riemann's modified form of Liouville's fractional integral operator is a direct generalization of Cauchy's formula for an n-fold integral

$$\int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} f(x_n)dx_1 dx_2 \cdots dx_n = \frac{1}{(n-1)!}\int_a^x \frac{f(t)}{(x-t)^{1-n}}dt. \tag{1.1}$$

By n-fold here means that the integration is deployed n-times. As an example let $f(x)=x,\ n=3$ and $a=0$ (To remove residue-terms) then (1.1) becomes

$$\int_0^x \int_0^{x_1} \int_0^{x_2} x_3 dx_3 dx_2 dx_1 = \frac{1}{2!}\int_0^x \frac{t}{(x-t)^{-2}}dt, \tag{1.2}$$

and by integration one gets

$$\frac{1}{2!}\int_0^x \frac{t}{(x-t)^{-2}}dt = \frac{x^4}{4!}, \tag{1.3}$$

Which gives the 3-fold integral of $f(x)=x$ which equals the LHS of (1.2). Since $(n-1)! = \Gamma(n)$, Riemann realized that the RHS of (1.1) might have meaning even when n takes non-integer values. Thus perhaps it was natural to define fractional integration, denoted by I_a^{α} as follows.

Definition (1.1) [Samko, 1993]:

If $f(x) \in C[a,b]$ "all continuous functions in $[a,b]$" then

$$I_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt \tag{1.4}$$

where $-\infty < \alpha < \infty$, is called the Riemann-Liouville fractional integral of order α. In the same fashion for $0 < \alpha < 1$, we let

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{\alpha}} dt, \tag{1.5}$$

Which is called the Riemann-Liouville fractional derivative of order α. As an example of fractional integration and differentiation one can take α=1/2 which is called the semi-integral if used in (1.4) and semi-derivative if used in (1.5). If taking $f(x) = x$ and letting α=1/2 in (1.4) then one obtains:

$$I_0^{1/2} x = \frac{1}{\Gamma(1/2)} \int_0^x \frac{t}{(x-t)^{1/2}} dt. \tag{1.6}$$

The integral may be found in [Samko,1993], hence the result will be:

$$I_0^{1/2} x = \frac{1}{\Gamma(5/2)} x^{3/2} = \frac{4}{3\sqrt{\pi}} x^{3/2} . \tag{1.7}$$

Since $\Gamma(5/2) = \frac{3\sqrt{\pi}}{4}$. The fractional derivative for $f(x) = x$ and $\alpha = 1/2$ (1.5) will be becomes:

$$D_0^{1/2} x = \frac{1}{\Gamma(1-1/2)} \int_0^x \frac{t}{(x-t)^{1-1/2}} dt = \frac{d}{dx} I_0^{1/2} x = \frac{2}{\sqrt{\pi}} x^{1/2}, \tag{1.8}$$

Since $\Gamma(3/2) = \sqrt{\pi}/2.$

The connection between the Riemann-Liouville fractional integral and derivative, as Riemann realized, be traced back to the solvability of Abel's integral equation for any $0 < \alpha < 1$,

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\Phi(t)}{(x-t)^{1-\alpha}} dt \qquad , x > 0. \tag{1.9}$$

Where $\Phi(t)=\begin{cases}\dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & ,t>0\\ 0 & ,t\leq 0\end{cases}$

Formally equation (1.9) can be solved by changing x to t and t to s respectively, further by multiplying both sides of the equation by $(x-t)^{-\alpha}$ and integrating of (1.9) with respect to we get

$$\int_a^x \frac{dt}{(x-t)^{\alpha}}\int_a^t \frac{\Phi(s)ds}{(t-s)^{1-\alpha}}=\Gamma(\alpha)\int_a^x \frac{f(t)dt}{(x-t)^{\alpha}}. \qquad (1.10)$$

Interchanging the order of integration in the left hand side by Fuibine's theorem we obtain

$$\int_a^x \Phi(s)ds\int_s^x \frac{dt}{(x-t)^{\alpha}(t-s)^{1-\alpha}}=\Gamma(\alpha)\int_a^x \frac{f(t)dt}{(x-t)^{\alpha}}. \qquad (1.11)$$

The inner integral is easily evaluated after the change of variable $t=s+\tau(x-s)$ and use of the formulae of the beta-function:

$$\begin{aligned}\int_a^x (x-t)^{-\alpha}(t-s)^{\alpha-1}dt &= \int_0^1 \tau^{\alpha-1}(1-\tau)^{-\alpha}d\tau\\ &=\beta(\alpha,1-\alpha)=\Gamma(\alpha)\Gamma(1-\alpha).\end{aligned} \qquad (1.12)$$

Therefore we get

$$\int_a^x \Phi(s)ds=\frac{1}{\Gamma(1-\alpha)}\int_a^x \frac{f(t)dt}{(x-t)^{\alpha}}. \qquad (1.13)$$

Hence after differentiation we have

$$\Phi(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^x \frac{f(t)dt}{(x-t)^{\alpha}}. \qquad (1.14)$$

Thus if (1.9) has a solution it is necessarily given by (1.14) for any $0<\alpha<1$. One observes that (1.9) is in a sense the α-order integral and the inversion (1.14) is the α-order derivative.

A very useful fact about the Riemann-Liouville operators is that they satisfy the following important properties of fractional integrals.

Theorem (1.1) [Munkhammar, 2005]

For any $f \in C[a,b]$ the Riemann-Liouville fractional integral satisfies

$$I_a^{\alpha} I_a^{\beta} f(x) = I_a^{\alpha+\beta} f(x) \tag{1.15}$$

for $\alpha > 0, \beta > 0.$

Proof

The proof is rather direct, we have by definition:

$$I_a^{\alpha} I_a^{\beta} f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \frac{dt}{(x-t)^{1-\alpha}} \int_a^t \frac{f(u)}{(t-u)^{1-\beta}} du,$$

and since $f(x) \in C([a,b])$ we can by Fubini's theorem interchange order of integration and by setting $t = u + s(x-u)$ we obtain

$$I_a^{\alpha} I_a^{\beta} f(x) = \frac{1}{\Gamma(\alpha+\beta)} \int_a^x \frac{f(u)}{(x-u)^{1-\alpha-\beta}} du = I_a^{\alpha+\beta} f(x).$$

The Riemann-Liouville fractional operator may in many cases be extended to hold for a larger set of α, and a rather technical detail is that we denote $\alpha = [\alpha] + \{\alpha\}$, where $[\alpha]$ denotes the integer part of α, and $\{\alpha\}$ denotes the remainder. This notation is used for convenience, observe the following definition.

Definition (1.2) [Samko, 1993]

If $\alpha > 0$ is not an integer then we define

$$D_a^{\alpha} f = \frac{d^{[\alpha]}}{dx^{[\alpha]}} D_a^{\{\alpha\}} f = \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} I_a^{1-\{\alpha\}} f, \tag{1.16}$$

Thus

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}, \tag{1.17}$$

for any $f \in C^{[\alpha]+1}([a,b])$ if $n=[\alpha]+1$. if on the other hand $\alpha < 0$ then the notation

$$D_a^{\alpha} f = I_a^{-\alpha} f, \tag{1.18}$$

may be used as definition.

Proposition [Munkhammar, 2005]

$D_a^{\alpha} f(x)$ exists for all $f \in C^{(k)}[a,b]$ and all $x \in [a,b]$(for all $f \in C^{[\alpha]+1}[a,b]$ and all $x \in [a,b]$) if $\alpha < 0$ ($\alpha > 0$), respectively.

Proof:

write $n=[\alpha]+1$ and apply Taylor's formula with remainder:

$$f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k + \frac{1}{(n-1)!}\int_a^t \frac{f^{(n)}}{(t-s)^{n-1}} ds, \qquad \forall t \in [a,b].$$

Substituting this into the definition of $D_a^{\alpha} f(x)$ and simplifying the integrals we obtain

$$D_a^{\alpha} f(x) = \frac{d^n}{dx^n}\left(\sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+2-\{\alpha\})}\cdot (x-a)^{k+1-\{\alpha\}}\right. \tag{1.19}$$
$$\left. + \frac{1}{\Gamma(n+1-\{\alpha\})}\int_a^x f^{(n)}(s)\cdot (x-s)^{n-\{\alpha\}} ds\right).$$

Clearly, this n-fold derivative can be carried out for all $x \in [a,b]$. in particular the integral is unproblematic since $f^{(n)} \in C([a,b])$ and since the

exponent $n-\{\alpha\}$ is larger than $n-1$, so that $\frac{d^k}{dx^k}(x-s)^{n-\{\alpha\}}$ is integrable for all $k = 0,1,2,...,n.$ This proves our claim.

For convenience in the later theorem we define the following useful space.

Definition (1.3) [samko, 1993]

For $\alpha > 0$ let $I_a^{\alpha}[a,b]$ denote the space of functions which can be represented by an Riemann-Liouville -integral of order α of some $C[a,b]$-function.

This gives rise to the following manifesting theorem

Theorem (1.2) [Munkhammar, 2005]

Let $f \in C[a,b]$ and $\alpha > 0$. In the order that $f(x) \in I_a^{\alpha} g(x)$ for some $g \in C[a,b]$, it is necessary and sufficient that

$$I_a^{n-\alpha} f \in C^n[a,b], \tag{1.20}$$

where $n = [\alpha]+1$, and that

$$\left(\frac{d^k}{dx^k} I_{a+}^{n-\alpha} f(x) \right)_{|x=a} = 0, \qquad k = 0,1,2,\dots,n-1 \tag{1.21}$$

Proof

$\Rightarrow$ First assume $f(x) \in I_a^\alpha[a,b]$; then $f(x) = I_a^\alpha g(x)$ for some $g \in C[a,b]$. Hence by (Theorem (1.1)) we have

$$I_a^{n-\alpha} f(x) = I_a^{n-\alpha} I_a^\alpha g(x) = I_a^\alpha g(x)$$

$$= \frac{1}{(n-1)!} \int_a^x \frac{g(t)}{(x-t)^{1-n}} dt$$

$$= \int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} g(x_n) dx_1 dx_2 \cdots dx_n$$

This implies that (1.20) holds, and by repeated differentiation we also see that (1.21) holds.

$\Leftarrow$ Conversely, assume that $f \in C[a,b]$ satisfies (1.20) and (1.21). Then by Taylor's formula applied to the function $I_a^{n-\alpha} f$, we have

$$I_a^{n-\alpha} f(t) = \int_a^t \frac{d^n}{ds^n} I_a^{n-\alpha} f(s) \cdot \frac{(t-s)^{n-1}}{(n-1)!} ds \qquad \forall t \in [a,b].$$

Let us write $\varphi(t) = \frac{d^n}{dt^n} I_a^{n-\alpha} f(t)$; then note that $\varphi \in C[a,b]$ by (1.20). Now, by Definition (1.1) and (Theorem (1.1)) the above relation implies

$$I_a^{n-\alpha} f(t) = I_a^n \varphi(t) = I_a^{n-\alpha} I_a^\alpha \varphi(t),$$

and thus

$$I_a^{n-\alpha}\left(f - I_a^{\alpha}\varphi\right) \equiv 0.$$

By a general fact about uniqueness of any solution to Abel's integral equation, and note that we have $n-\alpha > 0$, this implies $f \equiv I_a^{\alpha}\varphi$, and thus $f \in I_a^{\alpha}[a,b]$.

Theorem (1.3) [Munkhammar, 2005]

If $\alpha > 0$ then the equality

$$D_a^{\alpha} I_a^{\alpha} f(x) = f(x) \tag{1.22}$$

holds for any $f \in C^{[\alpha]+1}[a,b]$, however the equality

$$I_a^{\alpha} D_a^{\alpha} f(x) = f(x) \tag{1.23}$$

holds if f satisfies the condition in Theorem (1.2), otherwise

$$I_a^{\alpha} D_a^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \frac{d^{n-k-1}}{dx^{n-k-1}}\left(I_a^{n-\alpha} f(x)\right) \tag{1.24}$$

holds.

Proof

By Definition we have

$$D_a^{\alpha} I_a^{\alpha} f = \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{dt}{(x-t)^{\alpha-n+1}} \int_a^t \frac{f(s)ds}{(t-s)^{1-\alpha}}. \tag{1.25}$$

Since the integrals are absolutely convergent we deploy Fubini's Theorem and interchange the order of integration and after evaluating the inner integral we obtain

$$D_a^{\alpha} I_a^{\alpha} f = \frac{1}{\Gamma(n)} \frac{d^n}{dx^n} \int_a^x \frac{f(s)}{(x-s)^{n-1}} ds. \tag{1.26}$$

Then (1.22) follows from (1.26) by Cauchy's formula (1.1). Since f in (1.23) satisfies the conditions in Theorem (1.2) and $f \in C^{[\alpha]+1}([a,b])$ it follows immediately by (1.21) that (1.22) will hold (Because the residue terms of integration will vanish). If on the other hand any function $f \in C^{[\alpha]+1}([a,b])$ does not satisfy the condition (1.21) given in Theorem (1.2)

the residue terms out side the integral will not disappear like in (1.22), but as integration is deployed (1.24) is obtained by induction.

Perhaps the second part of Theorem (1.3) is somewhat surprising, and this gives rise to the following interesting corollary.

Corollary [Munkhammar, 2005]

Let $\alpha > 0,\ n \in Z^{+}$ and $f(x) \in C^{[\alpha]+n+1}([a,b])$. Then

$$f(x) = \sum_{k=-n}^{n-1} \frac{D_{a+}^{\alpha+k} f(x_0)}{\Gamma(\alpha+k+1)} (x-x_0)^{\alpha+k} + R_n(x), \qquad (1.27)$$

for all $a \le x_0 \le x \le b,$ where

$$R_n(x) = I_a^{\alpha+n} D_a^{\alpha+n} f(x) \qquad (1.28)$$

is the remainder.

One obtains (1.27) by deploying I_a^{α} to D_a^{α} in (1.19) and rearrange some. Heuristically when letting n and m tend to infinity, and if f is a sufficiently good function one obtains the Taylor-Riemann expansion which is a fractional generalization of Taylor's theorem. The concept of studying the

Riemann-Liouville operator for $\alpha \geq 1$ leads us to the following useful theorem

An interesting property of the Riemann-Liouville operators is that certain non-differentiable functions such as Weierstrass-function and Riemann-function seem to have fractional derivative of all orders[0,1], see [Faycal, 2001] [Samko, 1993] for investigations on non-differentiability and its relation to fractional calculus. This adds to the problem that the relation between the ordinary derivative and the fractional derivative is not entirely obvious, but the following theorem might give a picture on some of their covariance.

Theorem (1.4) [Munkhammar, 2005]

If $f \in C^1[a,b]$, $f(a) \geq 0$ and $\alpha \in [0,1]$, then $D_a^\alpha f(x)$ is non-negative if f is increasing on $[a,x]$.

Proof

Since $f \in C^1[a,b]$ we can deploy (1.19) if one lets $n = [\alpha] + 1 = 1$, then it reduces to two terms and appears likes:

$$D_a^\alpha f(x) = \frac{f(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha} + \frac{1}{\Gamma(1-\alpha)}\int_a^x f'(s)(x-s)^{-\alpha} ds. \qquad (1.29)$$

Since $\Gamma(1-\alpha) > 0$ for all $\alpha \in [0,1]$ and $x > a$ and then since the fact that $f(a) \geq 0$ was given, we conclude that the first term is non-negative. This leaves us to prove that the integral in the second term is non-negative. Observe that $f'(s) \geq 0$ on $[a,x]$ since f was increasing. Further $(x-s)^{-\alpha} > 0$

for $s \in [a, x]$ which implies that the integral is non-negative. This completes the proof.

1.4 General Properties [Samko, 1993]:

In this subsection, those properties of differ integral operators will be examined which we might expect to generalize classical formulas for derivatives and integrals. It is those properties, which will be provide our primary means of understanding and utilizing the fractional order differential equations:

Upon those properties are the following:

(1) $D_x^{\alpha} cf(x) = cD_x^{\alpha} f(x)$.

Proof:

$$D_x^{\alpha} cf(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-y)^{n-\alpha-1} f(t) c dt$$

$$= c \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-y)^{n-\alpha-1} f(t) dt$$

$$= cD_x^{\alpha} f(x), \ when\ \alpha > 0.$$

(2) $D_x^{\alpha}\{ f_1(x) + f_2(x)\} = D_x^{\alpha} f_1(x) + D_x^{\alpha} f_2(x)$.

Proof

$$D_x^{\alpha}\{ f_1(x) + f_2(x)\} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-y)^{n-\alpha-1} (f_1(t) + f_2(t)) dt$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x ((x-y)^{n-\alpha-1} f_1(t) + (x-y)^{n-\alpha-1} f_2(t)) dt$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-y)^{n-\alpha-1} f_1(t)dt +$$

$$\frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-y)^{n-\alpha-1} f_2(t)dt$$

$$= D_x^\alpha f_1(x) + D_x^\alpha f_2(x).$$

So, from properties ***(1)*** and ***(2)***, we can generalize the fractional differential to be:

$$D_x^\alpha \sum_{i=1}^{n} c_i f_i(x) = \sum_{i=1}^{n} c_i D_x^\alpha f_i(x)$$

Which is called the linear property.

3.1 Introduction

Stability is an asymptotic qualitative criterion of the control circuit and is the primary and necessary condition for the correct functioning of every control circuit. The existing methods developed so far for stability check are mainly for integer-order systems. However, for fractional order systems, it is difficult to evaluate the stability of linear system by simply examining its characteristic equation either by finding its dominant roots or by using other algebraic methods. At the moment, direct check of the stability of fractional order system using polynomial criteria (e.g., Routh's or Jury's type) is not possible, because the characteristic equation of the system is, in general, not a polynomial but a pseudopololynomial function of fractional powers of the complex variable s.

Problems of stability appeared for the first time in mechanics during the investigation of an equilibrium state of a system. A simple reflection may show that some equilibrium state of a system are stable with respect to small perturbations, where as other balanced states, although available in principle, cannot be realized in practice.

In this chapter, some important basic concepts of the stability of the system of integer order differential equations are presented. A theorem on identifies the stability of the system of fractional order differential equation (2.1) and (2.2) is presented with its proof.

3.2 Basic Concept and Definitions [Rao, 1980]

Consider the following ordinary differential equations

$$x' = F(t, x) \tag{3.1}$$

With the initial data $x(t_0) = x_0$, where $F(t, x)$ with n components is a real continuous vector-valued function defined on a domain D in $R^{(n+1)}$ plane.

Let $x(t) = x(t, t_0, x_0)$ be a solution of (3.1) through the initial point (t_0, x_0), existing to the right of $t_0 \geq 0$. Now we define the various of stability for the solution $x(t, t_0, x_0)$ of (3.1) [Rao, 1980] [Sanchez, 1983].

Definition (3.1) [Matignon, 1996]

A solution $x(t)$ of (3.1) is said to be stable if for each $\varepsilon > 0$ there exists a positive number $\delta = \delta(x_0, \varepsilon)$ such that any solution $\bar{x} = x(t, t_0, \bar{x}_0)$ of (3.1) satisfies that $\|\bar{x}(t) - x(t)\| < \varepsilon, t \geq t_0$ whenever $\|\bar{x}_0 - x_0\| < \delta$. For example in 2-deminsional space R^2, our can see the behavior of stable trajectory in (Fig. (3.1)).

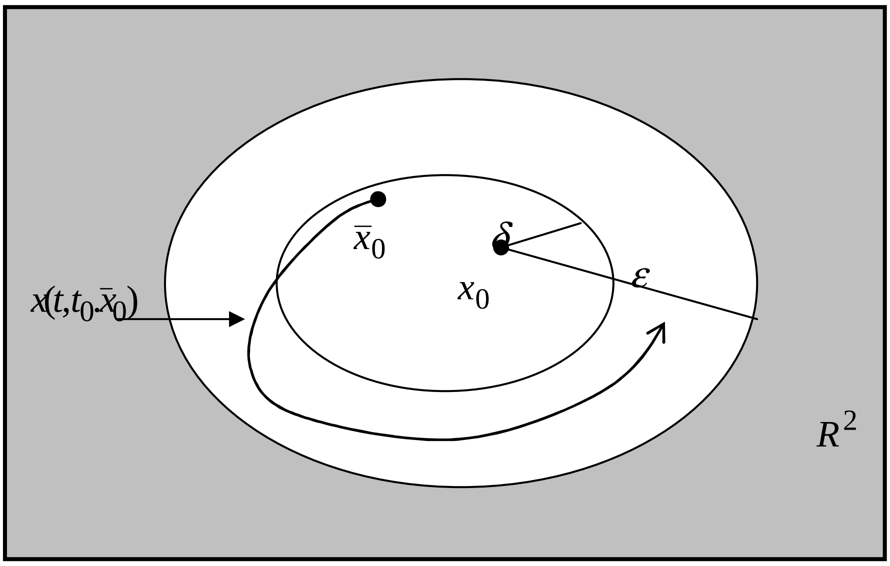

Fig. (3.1)

Otherwise the solution $x(t)$ is said to be unstable. For example in 2-deminsional space R^2, our can see the behavior of unstable trajectory in (Fig. (3.2)).

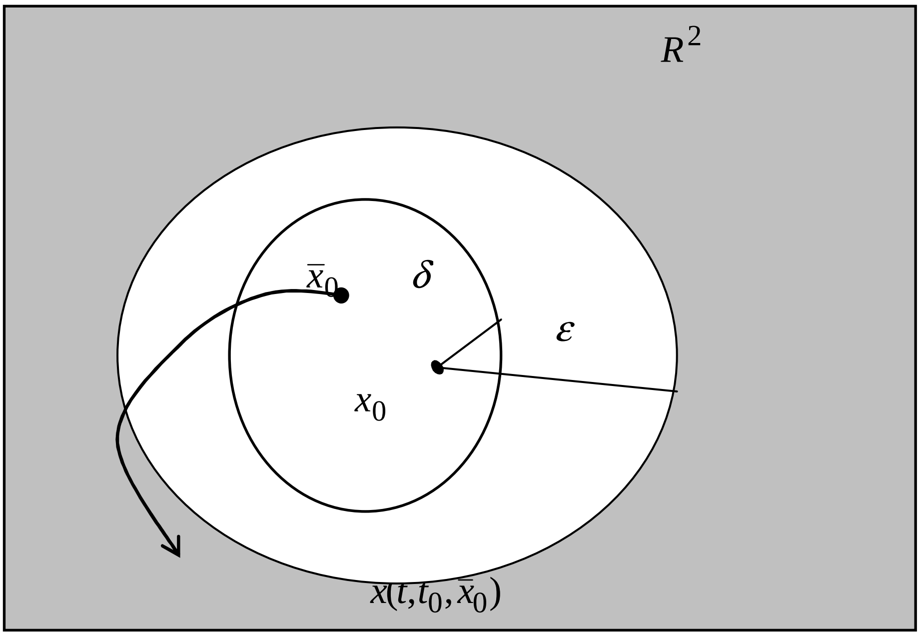

Fig. (3.2)

Definition (3.2) [Matignon, 1996]

A solution $x(t)$ is said to be asymptotically stable if it is stable and if there exist a number $\delta_0 > 0$ such any solution $\bar{x}(t)$ of (3.1) satisfies the condition that $\|\bar{x}(t) - x_0(t)\| \to 0 \, as \, t \to \infty$ whenever $\|\bar{x}_0 - x_0\| < \delta$. For example in 2-deminsional space R^2, our can see the behavior of asymptotically stable trajectory in (Fig. (3.3)).

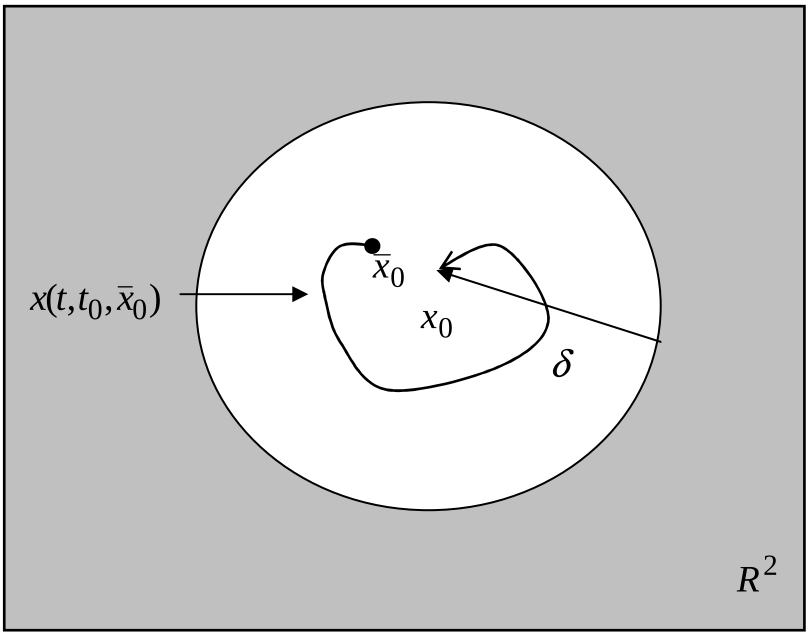

Fig. (3.3)

Geometrically this definition has the following interpretation [Matignon, 1996]:

We consider the sphere $\sum_{j=1}^{3} x_j^2 = \varepsilon$ with the arbitrary small radius $\sqrt{\varepsilon}$. If the motion is stable then one can find another sphere $\sum_{j=1}^{3} x_{0j}^2 = \delta$, with radius $\sqrt{\delta}$,such that starting at any point M_0 inside or on the surface of the δ-sphere, the image point M will always remain inside the ε-sphere, never reaching its external surface (Fig. 3.4)

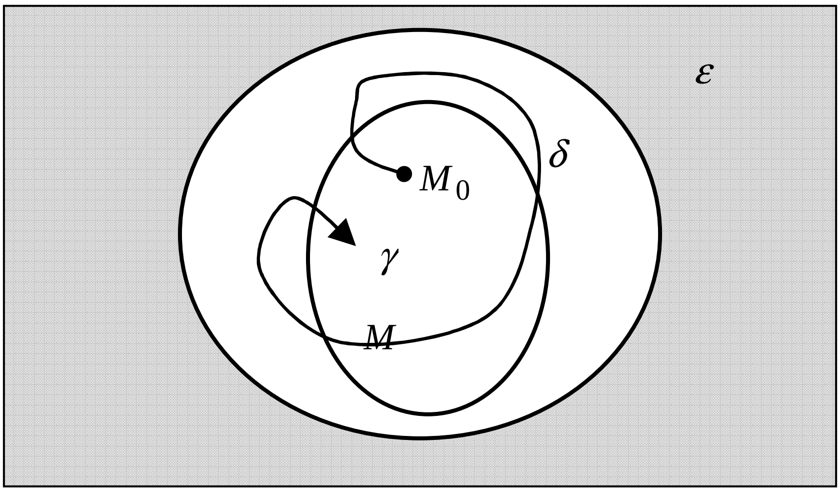

Fig. (3.4)

If the perturbed motion is unstable, then irrespective of how close the reference origin the point M_0 may be, in time, at least one trajectory of the reprentative point M will cross the δ-sphere from inside to out side. From practical point of view the stability of the unperturbed motion means that when the initial perturbations are small enough, the perturbed motion will defer from unperturbed motion by Avery slight amount. However, if the

Unperturbed motion is unstable then the perturbed motion will deviate from it, no matter how small the initial perturbation may be.

3.3 Stability of Linear System [Braur, 1970]

First it's often a useful trade-off to replace a differential equation of order higher than one by a first-order system at the expense of increasing the number of unknown functions. This is done in a useful way; for example, consider the scalar nth order nonhomogeneous linear ordinary differential equation

$$x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_{n-1}(t)x' + p_n(t)x = g(t) \tag{3.2}$$

here $x = x(t)$ is unknown scalar function, $g(t)$ and $p_i(t)$ for ($i = 1,\ldots,n$) are given continuous functions.

By letting

$$\begin{aligned} &x_1 = x \quad \to \; x_1' = x' = x_2 \\ &x_2 = x' \; \to \; x_2' = x'' = x_3 \\ &x_3 = x'' \to \; x_3' = x''' = x_4 \\ &\vdots \\ &x_n = x^{(n-1)} \to \; x_n' = x^{(n)} = -p_1(t)x_n - \ldots - p_{n-1}(t)x_2 - p_n(t)x_1(t) + g(t) \end{aligned}$$

We can easily show that equation (3.2) is equivalent to the 1^{st} order nonhomogeneous linear system

$$x' = A(t)x + G(t) \tag{3.3}$$

Where

$$A(t)=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & 0 & 1 \\ -p_n(t) & -p_{n-1}(t) & -p_{n-2}(t) & \cdots & -p_1(t) \end{pmatrix}$$

and

$$G(t)=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix}$$

And equivalently system (3.3) could be written as follows

$$x'=(A+B(t))x+G(t) \tag{3.4}$$

$$A=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$G(t)=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ -p_n(t) & -p_{n-1}(t) & -p_{n-2}(t) & \cdots & -p_1(t) \end{pmatrix}$$

Where A is $n\times n$ constant matrix, $B(t)$ is $n\times n$ continuous matrix for $0\le t<\infty$ and $G(t)$ is $n\times 1$ continuous vector function for $0\le t<\infty$

If $G(t)$ equal to zero then we get the following homogenous linear system

$$x' = (A+B(t))x \tag{3.5}$$

3.4Theorem(3.1)

The solution $y(t)$ of

$D^n x(t) = f(t, x(t), D^{\alpha_1}x(t), D^{\alpha_2}x(t),..., D^{\alpha_m}x(t)), \quad t>0$ and

$D^j x(0) = 0, \qquad j = 0,1,2,...,n-1$

is stable if $[\alpha_m]+1<n$.

$[\alpha_i]+1 = k_i, \quad i = 1,2,...,m$

Proof

The non linear multi fractional (arbitrary) orders differential equation (2.1) can be rewritten as

$$D^n y(t) = c_1 D^{\alpha_m} y(t) + c_2 D^{\alpha_{m-1}} y(t) + ... + c_n D^{\alpha_1} y(t) + y(t) \tag{3.6}$$

Using the definition:

$$\begin{aligned} D^{\alpha_i} y(t) &= I^{k_i-\alpha_i} y^{(k_i)}(t), \\ &= \frac{t^{k_i-\alpha_i-1}}{\Gamma(k_i-\alpha_i)} y^{(k_i)}(t) \\ &= \frac{1}{\Gamma(k_i-\alpha_i)} \frac{1}{t^{\alpha_i+1-k_i}} y^{(k_i)}(t), \qquad ((k_i-\alpha_i-1)<0) \end{aligned}$$

Since$[\alpha_m]+1<n$, we can rename $k_i=1,\dots,n-1$, then (3.6) becomes

$$y^{(n)}(t)+a_1(t)y^{(n-1)}(t)+a_2(t)y^{(n-2)}(t)+\dots+a_n(t)y'(t)-y(t)=0 \tag{3.7}$$

Where

$$a_{k_i}(t)=-\frac{1}{\Gamma(k_i-\alpha_i)}\frac{1}{t^{\alpha_i+1-k_i}}$$

The substitution

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ x_3 &= y'' \\ &\vdots \\ x_{n-1} &= y^{(n-2)} \\ x_n &= y^{(n-1)} \end{aligned}$$

Transforms (3.7) into an n-dimensional first order linear homogeneous system

$$x'=A(t)x, \tag{3.8}$$

Where

$$A(t)=\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & a_{n-1}(t) & a_{n-2}(t) & a_{n-3}(t) & \cdots & a_1(t) \end{pmatrix} \tag{3.9}$$

Is a given matrix function and

$$x(t)=(x_1(t),\dots,x_n(t))=(y(t),y'(t),\dots,y^{(n-1)}(t))$$

is an unknown vector function.

first, we must prove that all solutions of (3.8) are bounded. The system (3.8) can be written as

$$x' = (G + H(t))x \qquad (3.10)$$

Where

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$H(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_{n-1}(t) & a_{n-2}(t) & a_{n-3}(t) & \cdots & a_1(t) \end{pmatrix}$$

Treating $H(t)x(t)$ an inhomogeneous term and applying the variation of constants formula, we find that every solution $x(t)$ of equation (3.10) satisfies the linear integral equation

$$x(t) = y(t) + \int_0^t \Phi(t-s)B(s)x(s)\,ds\,, \qquad (3.11)$$

where $y(t)$ is the solution of $x' = Gx$ such that $y(0) = x(0) = x_0$ and Φ is the matrix solution of $\Phi' = G\Phi$ with $\Phi(0) = I$.

We know that any solution $y(t)$ of $x' = Gx$ can be expressed as

$$y(t) = \Phi(t)x_0.$$

Now, since all $a_{k_i}(t)$ be continuously differentiable function for t>0. Then all the solutions of $x' = Gx$ are bounded (by theorem 4.3.1 P (151) [Rao, 1980]). Let

$$c_1 = \max(\sup_{t\geq 0}\|y(t)\|, \sup_{t\geq 0}\|\Phi(t)\|).$$

Hence, from (3.11), we have

$$\|x(t)\| \leq c_1 + c_1\int_0^t \|H(s)\|\|x(s)\|ds.$$

By (Gronwall-Reid-Bellman inequality) theorem [Rao, 1980], we obtain for all $t \geq 0$

$$\|x(t)\| \leq c_1 \exp[c_1\int_0^t \|H(s)\|ds] \leq c_1 \exp[c_1\int_0^\infty \|H(s)\|ds]$$

It is clearly that $H(t)$ is continuous matrix for $t \geq 0,$ satisfying either of the conditions

$$\|H(t)\| \to 0 \quad as \quad t \to \infty \tag{3.12}$$

$$\int_0^t \|H(s)\|ds < \infty \tag{3.13}$$

Therefore, $\|x(t)\| = M < \infty.$

That means all solutions of (3.10) which is equivalent to (3.8) are bounded.

Then by the definition of the stability

If $\in > 0,$ then

$$\|x(t,t_0,\bar{x}_0)-x(t,t_0,x_0)\|$$

$$=\|\Phi(t)(\bar{x}_0-x_0)\|, if\ \|x_0-\bar{x}_0\|<\delta$$

$$\leq M\|\bar{x}_0-x_0\|$$

$$\leq M\delta\ ,\ let\ \in=M\delta$$

$$<\in$$

Hence all the solutions of (3.8) are stable.

3.5 Examples:-

We are consider 6-different examples according the different between the fractional order derivatives, and showing how we use the definition of Riemann-Liouville derivative to obtain an ordinary integer differential equation with variables coefficients.

Example (3.1)

Finds the stable solution of the equation

$$y^{(3)}(t)=y^{(1.7)}(t)+3y^{(0.7)}(t)+10y(t) \tag{3.14}$$

$$y^{(n)}(0)=0$$

with $\alpha_i-\alpha_{i-1}\equiv$ constant for all $i=0,1,2$

Solution:-

$$D^3y(t)=(D^{1.7}+3D^{0.7})y(t)+10y(t) \tag{3.15}$$

By using the definitions (2.1) and (2.2):

$$D^{\alpha_i} y(t) = \frac{t^{k_i-\alpha_i-1}}{\Gamma(k_i-\alpha_i)} y^{(k_i)}(t), \quad k_i = [\alpha_i]+1$$

$$i = 1,2,...,n-1$$
$$k_i < n, \; n = 3$$

$$D^{1.7} y(t) = \frac{t^{2-1.7-1}}{\Gamma(2-1.7)} y^{(2)}(t)$$

$$= \frac{t^{-0.7}}{\Gamma(0.3)} y^{(2)}(t)$$

$$= \frac{1}{\Gamma(0.3)t^{0.7}} y^{(2)}(t), \tag{3.16}$$

$$D^{0.7} y(t) = \frac{t^{1-0.7-1}}{\Gamma(1-0.7)} y'(t)$$

$$= \frac{t^{-0.7}}{\Gamma(0.3)} y'(t)$$

$$= \frac{1}{\Gamma(0.3)t^{0.7}} y'(t), \tag{3.17}$$

Substituting (3.16) and (3.17) in (3.14), we get

$$y^{(3)}(t) = \frac{1}{\Gamma(0.3)t^{0.7}} y^{(2)}(t) + \frac{3}{\Gamma(0.3)t^{0.7}} y'(t) + 10y(t) \tag{3.18}$$

The substitution

$$y_1 = y \quad \Rightarrow y_1' = y' = y_2$$
$$y_2 = y' \quad \Rightarrow y_2' = y'' = y_3$$
$$y_3 = y'' \quad \Rightarrow y_3' = y'''$$

Transform (3.18) into an 3-dimensional first order linear homogeneous system

$$y_3' = \frac{1}{\Gamma(0.3)t^{0.7}} y_3 + \frac{3}{\Gamma(0.3)t^{0.7}} y_2 + 10 y_1$$

$$Y' = A(t)Y \tag{3.19}$$

Where

$$A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & \dfrac{3}{\Gamma(0.3)t^{0.7}} & \dfrac{1}{\Gamma(0.3)t^{0.7}} \end{pmatrix}$$

The system (3.19) can be written as:

$$Y' = (G + H(t))Y\,, \quad \text{where } G + H(t) = A(t)$$

$$G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & 0 & 0 \end{pmatrix},$$

$$H(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \dfrac{3}{\Gamma(0.3)t^{0.7}} & \dfrac{1}{\Gamma(0.3)t^{0.7}} \end{pmatrix}$$

G is 3×3 constant matrix, $H(t)$ is continuous matrix at $t > 0$, and satisfies the condition

$$\|H(t)\| \to 0 \ \ as \ \ t \to \infty$$

So the solution of (3.19) is stable.

Example (3.2)

Finds the stable solution of the equation

$$y^{(4)}(t) = y^{(2.5)}(t) + y^{(1.5)}(t) + y(t) \tag{3.20}$$

$$y^{(i)}(0) = 0$$

with $\alpha_i - \alpha_{i-1} \equiv$ constant for all $i = 0,1,2,3$

Solution:-

$$D^4 y(t) = (D^{2.5} + D^{1.5})y(t) + y(t)$$

By using the definitions (2.1) and (2.2):

$$D^{\alpha_i} y(t) = \frac{t^{k_i - \alpha_i - 1}}{\Gamma(k_i - \alpha_i)} y^{(k_i)}(t), \quad k_i = [\alpha_i] + 1$$

$$i = 1,2,\ldots,n-1$$
$$k_i < n,\ n = 4$$

$$D^{2.5} y(t) = \frac{t^{3-2.5-1}}{\Gamma(3-2.5)} y^{(3)}(t)$$

$$= \frac{t^{0.5}}{\Gamma(0.5)} y^{(3)}(t)$$

$$= \frac{1}{\sqrt{\pi t}} y^{(3)}(t), \tag{3.21}$$

$$D^{1.5} y(t) = \frac{t^{2-1.5-1}}{\Gamma(2-1.5)} y^{(2)}(t)$$

$$= \frac{t^{-0.5}}{\Gamma(0.5)} y^{(2)}(t)$$

$$= \frac{1}{\sqrt{\pi t}} y^{(2)}(t), \tag{3.22}$$

Substituting (3.21) and (3.22) in (3.20), we get

$$y^{(4)}(t) = \frac{1}{\sqrt{\pi t}} y^{(3)}(t) + \frac{1}{\sqrt{\pi t}} y^{(2)}(t) + y(t) \tag{3.23}$$

The substitution

$$\begin{aligned}
y_1 &= y && \Rightarrow y_1' = y' = y_2 \\
y_2 &= y' && \Rightarrow y_2' = y'' = y_3 \\
y_3 &= y'' && \Rightarrow y_3' = y''' = y_4 \\
y_4 &= y''' && \Rightarrow y_4' = y^{(4)}
\end{aligned}$$

Transform (3.23) into an 4-dimensional first order linear homogeneous system

$$y_4' = \frac{1}{\sqrt{\pi t}} y_4 + \frac{1}{\sqrt{\pi t}} y_3 + y_1$$

$$Y' = A(t)Y \tag{3.24}$$

Where

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & \frac{1}{\sqrt{\pi t}} & \frac{1}{\sqrt{\pi t}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

The system (3.24) can be written as:

Where

$$Y' = (G + H(t))Y, \quad \text{where } G + H(t) = A(t)$$

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\pi t}} & \frac{1}{\sqrt{\pi t}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \right)$$

Where

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$H(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\pi t}} & \frac{1}{\sqrt{\pi t}} \end{pmatrix}$$

G is 4×4 constant matrix, $H(t)$ is continuous matrix at $t>0$, and satisfies the condition

$$\|H(t)\| \to 0 \ \ as \ \ t \to \infty$$

So the solution of (3.24) is stable.

Example (3.3)

Finds the stable solution of the equation

$$y^{(3)}(t) = y^{(1.5)}(t) + y^{(0.7)}(t) + y^{(0.3)}(t) + y(t) \qquad (3.25)$$

$$y^{(i)}(0) = 0$$

with $\alpha_i - \alpha_{i-1} \equiv$ different arbitraries constants for all $i = 0,1,2$

Solution:-

$$D^3 y(t) = (D^{1.5} + D^{0.7} + D^{0.3}) y(t) + y(t)$$

By using the definitions (2.1) and (2.2):

$$D^{\alpha_i} y(t) = \frac{t^{k_i - \alpha_i - 1}}{\Gamma(k_i - \alpha_i)} y^{(k_i)}(t), \quad k_i = [\alpha_i] + 1$$

$$i = 1,2,\dots, n-1$$
$$k_i < n, \; n = 3$$

$$\begin{aligned} D^{1.5} y(t) &= \frac{t^{2-1.5-1}}{\Gamma(2-1.5)} y^{(2)}(t) \\ &= \frac{t^{-0.5}}{\Gamma(0.5)} y^{(2)}(t) \\ &= \frac{1}{\sqrt{\pi t}} y^{(2)}(t). \end{aligned} \tag{3.26}$$

$$\begin{aligned} D^{0.7} y(t) &= \frac{t^{1-0.7-1}}{\Gamma(1-0.7)} y'(t) \\ &= \frac{t^{-0.7}}{\Gamma(0.3)} y'(t) \\ &= \frac{1}{t^{0.7}\Gamma(0.3)} y'(t) \end{aligned} \tag{3.27}$$

$$D^{0.3}y(t) = \frac{t^{1-0.3-1}}{\Gamma(1-0.3)}y'(t)$$

$$= \frac{t^{-0.3}}{\Gamma(0.7)}y'(t)$$

$$= \frac{1}{t^{0.3}\Gamma(0.7)}y'(t) \tag{3.28}$$

Substituting (3.26), (3.27) and (3.28) in(3.25), we get

$$y'''(t) = \frac{1}{\sqrt{\pi t}}y''(t) + \frac{1}{t^{0.3}\Gamma(0.7)}y'(t) + \frac{1}{t^{0.7}\Gamma(0.3)}y'(t) + y(t)$$

$$y'''(t) = \frac{1}{\sqrt{\pi t}}y''(t) + (\frac{1}{t^{0.3}\Gamma(0.7)} + \frac{1}{t^{0.7}\Gamma(0.3)})y'(t) + y(t) \tag{3.29}$$

The substitution

$$y_1 = y \quad \Rightarrow y_1' = y' = y_2$$
$$y_2 = y' \quad \Rightarrow y_2' = y'' = y_3$$
$$y_3 = y'' \quad \Rightarrow y_3' = y'''$$

Transform (3.29) into an 3-dimensional first order linear homogeneous system

$$y_3'(t) = \frac{1}{\sqrt{\pi t}}y_3(t) + (\frac{1}{t^{0.3}\Gamma(0.7)} + \frac{1}{t^{0.7}\Gamma(0.3)})y_2(t) + y_1(t)$$

$$Y' = A(t)Y \tag{3.30}$$

Where

$$A(t)=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & (\frac{1}{t^{0.3}\Gamma(0.7)}+\frac{1}{t^{0.7}\Gamma(0.3)}) & \frac{1}{\sqrt{\pi t}} \end{pmatrix}$$

The system (3.30) can be written as:

$$Y'=(G+H(t))Y\,,\quad \text{where } G+H(t)=A(t)$$

$$G=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$H(t)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & (\frac{1}{t^{0.3}\Gamma(0.7)}+\frac{1}{t^{0.7}\Gamma(0.3)}) & \frac{1}{\sqrt{\pi t}} \end{pmatrix},$$

G is 3×3 constant matrix, $H(t)$ is continuous matrix at $t>0$, and satisfies the condition

$$\|H(t)\|\to 0 \;\; as \;\; t\to\infty$$

So the solution of (3.30) is stable.

Example (3.4)

Finds the stable solution of the equation

$$y^{(4)}(t)=y^{(2.8)}(t)+5y^{(1.2)}(t)+6y^{(0.6)}(t)+2y(t) \qquad (3.31)$$

$$y^{(i)}(0)=0$$

with $\alpha_i-\alpha_{i-1}\equiv$ different arbitraries constants for all $i=0,1,2,3$

Solution:-

$$D^4 y(t) = (D^{2.8} + 5D^{1.2} + 6D^{0.6})y(t) + 2y(t)$$

By using the definitions (2.1) and (2.2):

$$D^{\alpha_i} y(t) = \frac{t^{k_i - \alpha_i - 1}}{\Gamma(k_i - \alpha_i)} y^{(k_i)}(t), \quad k_i = [\alpha_i] + 1$$

$$i = 1,2,...,n-1$$
$$k_i < n, \; n = 4$$

$$D^{2.8} y(t) = \frac{t^{3-2.8-1}}{\Gamma(3-2.8)} y^{(3)}(t)$$

$$= \frac{t^{-0.8}}{\Gamma(0.2)} y^{(3)}(t)$$

$$= \frac{1}{\Gamma(0.2)t^{0.8}} y^{(3)}(t). \tag{3.32}$$

$$D^{1.2} y(t) = \frac{t^{2-1.2-1}}{\Gamma(2-1.2)} y^{(2)}(t)$$

$$= \frac{t^{-0.2}}{\Gamma(0.8)} y^{(2)}(t)$$

$$= \frac{1}{t^{0.2}\Gamma(0.8)} y^{(2)}(t) \tag{3.33}$$

$$D^{0.6}y(t)=\frac{t^{1-0.6-1}}{\Gamma(1-0.6)}y'(t)$$
$$=\frac{t^{-0.6}}{\Gamma(0.4)}y'(t)$$
$$=\frac{1}{t^{0.6}\Gamma(0.4)}y'(t) \quad (3.34)$$

Substuting (3.32), (3.33) and (3.34) in (3.31), we get

$$y^{(4)}(t)=\frac{1}{\Gamma(0.2)t^{0.8}}y^{(3)}(t)+\frac{5}{t^{0.2}\Gamma(0.8)}y^{(2)}(t)+\frac{6}{t^{0.6}\Gamma(0.4)}y'(t)+10y(t) \quad (3.35)$$

The substitution

$$\begin{aligned} y_1 &= y && \Rightarrow y_1' = y' = y_2 \\ y_2 &= y' && \Rightarrow y_2' = y'' = y_3 \\ y_3 &= y'' && \Rightarrow y_3' = y''' = y_4 \\ y_4 &= y''' && \Rightarrow y_4' = y^{(4)} \end{aligned}$$

Transform (3.35) into an 4-dimensional first order linear homogeneous system

$$y_4'(t)=\frac{1}{\Gamma(0.2)t^{0.8}}y_4(t)+\frac{5}{\Gamma(0.8)t^{0.2}}y_3(t)+\frac{6}{\Gamma(0.4)t^{0.6}}y_2(t)+10y_1(t)$$

$$Y'=A(t)Y \quad (3.36)$$

Where

$$A(t)=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10 & \dfrac{6}{\Gamma(0.4)t^{0.6}} & \dfrac{5}{\Gamma(0.8)t^{0.2}} & \dfrac{1}{\Gamma(0.2)t^{0.8}} \end{pmatrix}$$

The system (3.36) can be written as:

$Y'=(G+H(t))Y$, where $G+H(t)=A(t)$

$$G=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10 & 0 & 0 & 0 \end{pmatrix},$$

$$H(t)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \dfrac{6}{\Gamma(0.4)t^{0.6}} & \dfrac{5}{\Gamma(0.8)t^{0.2}} & \dfrac{1}{\Gamma(0.2)t^{0.8}} \end{pmatrix},$$

G is 4×4 constant matrix, $H(t)$ is continuous matrix at $t>0$, and satisfies the condition

$$\|H(t)\| \to 0 \;\; as \;\; t\to\infty$$

So the solution of (3.36) is stable.

Example (3.5):-

Finds the stable solution of the equation

$$y^{(5)}(t) = 2y^{(3.4)}(t) + y^{(2.4)}(t) + 3y^{(1.5)}(t) + y(t) \qquad (3.37)$$

$$y^{(i)}(0) = 0$$

with $\alpha_i - \alpha_{i-1} \equiv$ different arbitraries constants for all $i = 0,1,2,3,4$

Solution:-

$$D^5 y(t) = (2D^{3.4} + D^{2.4} + 3D^{1.5})y(t) + y(t)$$

By using the definitions (2.1) and (2.2):

$$D^{\alpha_i} y(t) = \frac{t^{k_i - \alpha_i - 1}}{\Gamma(k_i - \alpha_i)} y^{(k_i)}(t), \quad k_i = [\alpha_i] + 1$$

$$i = 1,2,\ldots,n-1$$

$$k_i < n, \ n = 5$$

$$\begin{aligned} D^{3.4} y(t) &= \frac{t^{4-3.4-1}}{\Gamma(4-3.4)} y^{(4)}(t) \\ &= \frac{t^{-0.4}}{\Gamma(0.6)} y^{(4)}(t) \qquad (3.38) \\ &= \frac{1}{\Gamma(0.6)t^{0.4}} y^{(4)}(t). \end{aligned}$$

$$D^{2.4}y(t) = \frac{t^{3-2.4-1}}{\Gamma(3-2.4)}y^{(3)}(t)$$
$$= \frac{t^{-0.4}}{\Gamma(0.6)}y^{(3)}(t) \qquad (3.39)$$
$$= \frac{1}{\Gamma(0.6)t^{0.4}}y^{(3)}(t).$$

$$D^{1.5}y(t) = \frac{t^{2-1.5-1}}{\Gamma(2-1.5)}y^{(2)}(t)$$
$$= \frac{t^{-0.5}}{\Gamma(0.5)}y^{(2)}(t) \qquad (3.40)$$
$$= \frac{1}{\sqrt{\pi t}}y^{(2)}(t).$$

Substituting (3.38), (3.39), (3.40) in (3.37) we get:

$$y^{(5)}(t) = \frac{2}{\Gamma(0.6)t^{0.4}}y^{(4)}(t) + \frac{1}{\Gamma(0.6)t^{0.4}}y^{(3)}(t) + \frac{1}{\sqrt{\pi t}}y^{(2)}(t) + 3y(t) \qquad (3.41)$$

the substitution

$$y_1 = y \quad \Rightarrow y_1' = y' = y_2$$
$$y_2 = y' \quad \Rightarrow y_2' = y'' = y_3$$
$$y_3 = y'' \quad \Rightarrow y_3' = y''' = y_4$$
$$y_4 = y''' \quad \Rightarrow y_4' = y^{(4)} = y_5$$
$$y_5 = y^{(4)} \Rightarrow y_5' = y^{(5)}$$

Transform (3.41) into an 5-dimensional first order linear homogeneous system

$$y_5'(t) = \frac{2}{\Gamma(0.6)t^{0.4}} y_5(t) + \frac{1}{\Gamma(0.6)t^{0.4}} y_4(t) + \frac{1}{\sqrt{\pi t}} y_3(t) + 3y_1(t)$$

$$Y' = A(t)Y \tag{3.42}$$

Where

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & \frac{1}{\sqrt{\pi t}} & \frac{1}{\Gamma(0.6)t^{0.4}} & \frac{2}{\Gamma(0.6)t^{0.4}} \end{pmatrix}$$

The system (3.42) can be written as:

$$Y' = (G + H(t))Y, \quad \text{where } G + H(t) = A(t)$$

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$H(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\pi t}} & \frac{1}{\Gamma(0.6)t^{0.4}} & \frac{2}{\Gamma(0.6)t^{0.4}} \end{pmatrix}$$

G is 5×5 constant matrix, $H(t)$ is continuous matrix at $t>0$, and satisfies the condition

$$\|H(t)\| \to 0 \quad as \quad t \to \infty$$

So the solution of (3.42) is stable.

Example (3.6):-

Finds the stable solution of the equation

$$y^{(6)}(t) = 2y^{(4.4)}(t) + y^{(3.4)}(t) + 7y^{(1.7)}(t) + 2y(t) \tag{3.43}$$

$$y^{(i)}(0) = 0$$

with $\alpha_i - \alpha_{i-1} \equiv$ different arbitraries constants for all $i = 0,1,2,3,4,5$

Solution:-

$$D^6 y(t) = (2D^{4.4} + D^{3.4} + 7D^{1.7})y(t) + 2y(t)$$

By using the definitions (2.1) and (2.2):

$$D^{\alpha_i} y(t) = \frac{t^{k_i - \alpha_i - 1}}{\Gamma(k_i - \alpha_i)} y^{(k_i)}(t), \quad k_i = [\alpha_i] + 1$$

$$i = 1,2,\dots,n-1$$
$$k_i < n, \; n = 6$$

$$\begin{aligned} D^{4.4} y(t) &= \frac{t^{5-4.4-1}}{\Gamma(5-4.4)} y^{(5)}(t) \\ &= \frac{t^{-0.4}}{\Gamma(0.6)} y^{(5)}(t) \qquad (3.44) \\ &= \frac{1}{\Gamma(0.6)t^{0.4}} y^{(5)}(t). \end{aligned}$$

$$D^{3.4}y(t) = \frac{t^{4-3.4-1}}{\Gamma(4-3.4)}y^{(4)}(t)$$

$$= \frac{t^{-0.4}}{\Gamma(0.6)}y^{(4)}(t) \tag{3.45}$$

$$= \frac{1}{\Gamma(0.6)t^{0.4}}y^{(4)}(t).$$

$$D^{1.7}y(t) = \frac{t^{2-1.7-1}}{\Gamma(2-1.7)}y^{(2)}(t)$$

$$= \frac{t^{-0.7}}{\Gamma(0.3)}y^{(2)}(t)$$

$$= \frac{1}{\Gamma(0.3)t^{0.7}}y^{(2)}(t). \tag{3.46}$$

Substituting (3.44), (3.45), (3.46) in (3.43) we get:

$$y^{(6)}(t) = \frac{2}{\Gamma(0.6)t^{0.4}}y^{(5)}(t) + \frac{1}{\Gamma(0.6)t^{0.4}}y^{(4)}(t) + \frac{7}{\Gamma(0.3)t^{0.7}}y^{(2)}(t) + 2y(t)$$

(3.47)

the substitution

$$y_1 = y \quad \Rightarrow y_1' = y' = y_2$$
$$y_2 = y' \quad \Rightarrow y_2' = y'' = y_3$$
$$y_3 = y'' \quad \Rightarrow y_3' = y''' = y_4$$
$$y_4 = y''' \quad \Rightarrow y_4' = y^{(4)} = y_5$$
$$y_5 = y^{(4)} \Rightarrow y_5' = y^{(5)} = y_6$$
$$y_6 = y^{(5)} \Rightarrow y_6' = y^{(6)}$$

Transform (3.47) into an 6-dimensional first order linear homogeneous system

$$y_6'(t) = \frac{2}{\Gamma(0.6)t^{0.4}} y_6(t) + \frac{1}{\Gamma(0.6)t^{0.4}} y_5(t) + \frac{7}{\Gamma(0.3)t^{0.7}} y_3(t) + 2y_1(t)$$

$$Y' = A(t)Y \tag{3.48}$$

Where

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & \dfrac{7}{\Gamma(0.3)t^{0.7}} & 0 & \dfrac{1}{\Gamma(0.6)t^{0.4}} & \dfrac{2}{\Gamma(0.6)t^{0.4}} \end{pmatrix}$$

The system (3.48) can be written as:

$$Y' = (G + H(t))Y, \quad \text{where } G + H(t) = A(t)$$

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$H(t)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{7}{\Gamma(0.3)t^{0.7}} & 0 & \dfrac{1}{\Gamma(0.6)t^{0.4}} & \dfrac{2}{\Gamma(0.6)t^{0.4}} \end{pmatrix}$$

G is 6×6 constant matrix, $H(t)$ is continuous matrix at $t>0$, and satisfies the condition

$$\|H(t)\| \to 0 \quad as \quad t \to \infty$$

So the solution of (3.48) is stable.

2.1 Introduction

The use of fractional orders differential and integral operators in mathematical models has become increasingly widespread in recent year [Diethelm , 2002] [Podlubny, 1999]. Several forms of fractional differential equations have been proposed in standard models, and there has been significant interest in developing their schemes solution [Edwards, 2002] [Mainardi, 1997]. In this chapter, we consider a non-linear multi-term fractional (arbitrary) order differential equation, in which its existence and uniqueness solution is proved.

Applications for such equations arise, e.g., in various areas of mechanics [Podlubny, 1999], the Bagely-Torvik equation [Diethelm , 2002] and the Basset equation [Mainardi, 1997].A theorem proving the existence and uniqueness of the solution will be proved.

2.2 problem and Definitions

Consider the nonlinear multi-term fractional (arbitrary) order differential equations

$$D^n x(t) = f(t, x(t), D^{\alpha_1} x(t), D^{\alpha_2} x(t), \ldots, D^{\alpha_m} x(t)), \quad t > 0 \qquad (2.1)$$

Subject to the initial values

$$D^j x(0) = 0, \qquad j = 0,1,2,\ldots, n-1 \qquad (2.2)$$

Where α_i are real numbers (i=1,2,…,m), such that

$$0 < \alpha_1 < \alpha_2 < \ldots < \alpha_m < n, \qquad \alpha_i \in (0,1), \quad i = 0,1,2,\ldots,n$$

and n is any positive integer number.

Now we give some definitions of the fractional order differential and integral operators, which are needed in our theory.

Definition (2.1) [Simpson, 2001]

Let $f(t) \in C[a,b], \beta \in R^{+}$.the fractional (arbitrary) order integral of the function $f(t)$ of order β which is defined by (1.4) can be written as

$$I_a^{\beta} f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds, \qquad (2.3)$$

When $a = 0$ we can write $I^{\beta} f(t) = I_0^{\beta} f(t) = f(t) * \Phi_{\beta}(t),\ where \Phi_{\beta}(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ for $t > 0, \Phi_{\beta}(t) = 0\ for\ t \le 0\ and\ \Phi(t)_{\beta} \to \delta(t)$ (the delta function) as $\beta \to 0$

Definition (2.2.) [Caputo, 1967], [Podlubny, 1996]

The fractional derivative D^{α} of order $\alpha \in (0,1]$ of the absolutely continuous functions $g(t)$ is which defined by (1.5) can be written as

$$D_a^{\alpha} g(t) = \frac{d}{dt} I_a^{1-\alpha} g(t), \qquad t \in [a,b]. \qquad (2.4)$$

2.3 Existence and Uniqueness Solution

By a solution of the initial value problem (2.1) and (2.2) we mean a function $x \in C(I)$ and all its derivative up to order (n-1)are vanishing at t =0. and $D = I \times C\ (I)\ where\ C\ (I)$ is the class of all continuous column vectors X(t) (defined by $X(t) = (x_0(t), x_1(t), ..., x_m(t))^T$) with the norm

$$\|X\| = \sum_{i=0}^{m} \|x_i\| = \sum_{i=0}^{m} \max_{t \in I} |x_i|,$$

$$X \in C\ (I)$$

By a solution of the system ($X(t) = A(t)\,X(t) + B(t, X(t)), t > 0)$ we mean a column vector $X(t) \in C\ \ (I)$ and X(0)=0.

Assuming that the function $f(t, x_0(t), x_1(t), ..., x_m(t))$ satisfies Lipschitz condition

$$\left| f(t, x_0, x_1, ..., x_m) - f(t, y_0, y_1, ..., y_m) \right| \le k \sum_{i=0}^{m} \left| x_i(t) - y_i(t) \right| \tag{2.5}$$

for $(t, x_0, x_1, ..., x_m)$ and $(t, y_0, y_1, ..., y_m) \in D, k > 0$.

Theorem

Let $f(t, x(t), D^{\alpha_1} x(t), D^{\alpha_2} x(t), ..., D^{\alpha_m} x(t)) \in C(D)$ and satisfies the Lipschitz condition (2.5). If

$$\sum_{i=0}^{m} T^{\alpha_{i+1}-\alpha_i} \le \frac{1}{(1+n)(1+k)}, \ (\alpha_{m+1} = n)$$

Now set I= [0, T], where T is a suitable positive number.

Then the nonlinear multi-term fractional (arbitrary) orders differential equations (2.1) and (2.2) has one and only one solution $X \in C(I)$, that satisfies $D^{\alpha_i} x \in C(I)$, i=0,1,2,…,m. where α_i are real numbers, s.t. $0 < \alpha_1 < \alpha_2 < ... < \alpha_m < n$ and n is any positive integer number.

Proof

Let

$$x_i(t) = D^{\alpha_i} x(t), \qquad i = 1,2,...,m \tag{2.6}$$

and

$$x_0(t) = x(t) \tag{2.7}$$

From (2.6) we get

$$x_i(t) = I^{\alpha_{i+1}-\alpha_i} x_{i+1}(t), \quad i = 0,1,2,...,m-1, \qquad \alpha_0 = 0 \tag{2.8}$$

and

$$x_m(t) = I^{n-\alpha_m} D^n x(t) \tag{2.9}$$

In which it can be written as

$$x_m(t) = I^{n-\alpha_m} f(t, x_0(t), x_1(t),..., x_m(t)) \tag{2.10}$$

Using the definition of fractional derivative and equations (2.6), (2.7), (2.8) and (2.9), we can easily obtain the following transforming system

$$X(t) = A(t) X(t) + B(t, X(t)), \quad t > 0 \tag{2.11}$$

Where

$$X(t) = (x_0(t), x_1(t),..., x_m(t))^T \tag{2.12}$$

$$B(t, X(t)) = (0,0,..., I^{n-\alpha_m} f(x_0(t), x_1(t),..., x_m(t))^T \tag{2.13}$$

$$A(t) = \begin{pmatrix} 0 & A_1(t) & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_2(t) & 0 & \cdots & 0 \\ \vdots & \vdots & . & . & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & A_m(t) \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}_{(m+1)\times(m+1)} \tag{2.14}$$

and $A_{i+1}(t)x_{i+1}(t) = I^{\alpha_{i+1}-\alpha_i} x_{i+1}(t),\ i = 0,1,2,...,m-1,\ \alpha_0 = 0$

if we write

$$FX(t) = A(t)X(t) + B(t, X(t)). \tag{2.15}$$

where F is a mapping from $C(I)$ to $C(I)$.

Then for $(t, x_0, x_1,..., x_m)$ and $(t, y_0, y_1,..., y_m) \in D$, We get

$$\|FX(t)-FY(t)\| \le \|A(t)(X(t)-Y(t)\| + \left\|I^{n-\alpha_m}(f(x_0,x_1,...,x_m)-f(y_0,y_1,...,y_m))\right\|. \tag{2.16}$$

since for $t>0$ and $i=0,1,2,...,m-1$

$$\left\|I^{\alpha_{i+1}-\alpha_i}x_{i+1}(t)\right\| = \left|\frac{1}{\Gamma(\alpha_{i+1}-\alpha_i)}\int_0^t (t-u)^{\alpha_{i+1}-\alpha_i-1}x_{i+1}(u)du\right|$$
$$\le \frac{\|x_{i+1}\|}{\Gamma(\alpha_{i+1}-\alpha_i+1)}T^{\alpha_{i+1}-\alpha_i}$$

further,

$$\frac{1}{\Gamma(\alpha_{i+1}-\alpha_i+1)} < \alpha_{i+1}-\alpha_i+1 < \alpha_{i+1}+1 < n+1 \quad \text{hence}$$

$$\left|I^{\alpha_{i+1}-\alpha_i}x_{i+1}(t)\right| \le (n+1)T^{\alpha_{i+1}-\alpha_i}\|x_{i+1}\|, \tag{2.17}$$

$$\|A(t)(X(t)-Y(t)\| = \sum_{i=0}^{m-1}\|A_{i+1}(t)(x_{i+1}(t)-y_{i+1}(t))\|$$

$$= \sum_{i=0}^{m-1}\left\|I^{\alpha_{i+1}-\alpha_i}(x_{i+1}(t)-y_{i+1}(t))\right\|$$

$$= \sum_{i=0}^{m-1}\max\left|I^{\alpha_{i+1}-\alpha_i}(x_{i+1}(t)-y_{i+1}(t))\right|$$

$$\le \sum_{i=0}^{m-1}(n+1)T^{\alpha_{i+1}-\alpha_i}\|x_{i+1}-y_{i+1}\|$$

$$\leq (n+1)\|X(t)-Y(t)\| \sum_{i=0}^{m-1} T^{\alpha_{i+1}-\alpha_i} \qquad (2.18)$$

$$\leq (n+1)\|X(t)-Y(t)\| \sum_{i=0}^{m} T^{\alpha_{i+1}-\alpha_i}$$

since

$$\left\| I^{n-\alpha_m}(f(x_0,x_1,\ldots,x_m)-f(y_0,y_1,\ldots,y_m)) \right\|$$

$$\leq k(n+1)\|X(t)-Y(t)\| \; T^{n-\alpha_m}$$

$$\leq k(n+1)\|X(t)-Y(t)\| \sum_{i=0}^{m} T^{n-\alpha_m} \qquad (2.19)$$

then

$$\|FX(t)-FY(t)\| \leq (n+1)(k+1)\sum_{i=0}^{m} T^{\alpha_{i+1}-\alpha_i} \|X-Y\| \qquad (2.20)$$

Hence the map $F : C \ (I) \to C \ (I)$ is a contracting (and then, it the fixed point $X = FX$) providing

$$\sum_{i=0}^{m} T^{\alpha_{i+1}-\alpha_i} \leq \frac{1}{(n+1)(k+1)}, \qquad (2.21)$$

and hence, there exists a unique column vector $X(t) \in C \ (I)$, which is the solution of the system (2.10). Therefore, from (2.11) and the definition of $C \ (I)$, we deduce that there exists one and only one solution $x(t) \in C(I)$, and this solution satisfies $D^{\alpha_i} x(t) \in C(I), i = 1,2,\ldots,m.$

Contents

DISCUSSION AND FUTURE WORK

We discusses the stability of multi fractional order (arbitrary) differential equation with constant coefficient by transform it into a system of integer order, We are consider 6-different examples according the different between the fractional order derivatives, and showing how we use the definition of Riemann-Liouville derivative to obtain an ordinary integer differential equation with variables coefficients.

For future works we consider the following problems:

1. A multi fractional arbitrary order differential equation with variable coefficients.

2. A system of multi fractional arbitrary order differential equations with constants and variable coefficients.

3. using, other conditions(such as liapunove function) to study the stability solution for fractional order differential equations.

Examining Committee's Certification

We certify that we read this thesis entitled ***"On Stability of Multi Fractional order Differential Equations with Constant Coefficient "*** and as examining committee examined the student, ***Saba Sadiq Mahdi*** in its contents and in what it connected with, and that is in our opinion it meet the standard of thesis for the degree of Master of Science in Mathematics.

(Chairman)

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Introduction

The subject of fractional calculus (that is, calculus of integrals and derivative of any arbitrary real order) has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly divers and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extension and generalizations in one and more variables [Trujillo, 2006].

the concept of fractional calculus is popularly believed to have stemmed from a question raised in the year 1695 by Marquis de L'Hopital (1661-1704) to Gottfried Wilhelm Leibniz (1646-1716), which sought the meaning of Leibniz's (currently popular) notation $\frac{d^n y}{dx^n}$ for the derivative of order $n \in N_0 := \{0,1,2,...\}$ when $x = \frac{1}{2}$. In his reply, dated 30 September 1695, Leibniz wrote to L'Hopital as follows: "… This is an apparent paradox from which, one day, useful consequences will be drawn…"

Subsequent mention of fractional derivative was made, in some context or the other, by (for example) Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Grunwald in 1867, Letnikov in 1868, Sonin in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917. In fact, in [Trujillo, 2006], entitled " Traite du Calculus Differential et du Calculus Integral " (Second edition; Courcier, Paris, 1819), S. F. Lacroix

devoted two pages (pp. 409-410) to fractional calculus, showing eventually that

$$\frac{d^{\frac{1}{2}}}{dv^{\frac{1}{2}}} v^{\frac{1}{2}} = \frac{2\sqrt{v}}{\sqrt{\pi}}.$$

In addition, of course, to the theories of differential, integral, and integro-differential equations, and special functions of mathematical physics as well as their extensions and generalizations in one and more variables, some of the areas of present-day applications of fractional calculus include Fluid Flow, Rheology, Dynamical Processes in Self-Similar and Porous Structures, Electrical Networks, Probability and Statistics, Control Theory of Dynamical System, Viscoelasticity, Electrochemistry of Corrosion, Chemical Physics, Optics and Signal Processing, and so on [Trujillo, 2006].

The first work, devoted exclusively to the subject of fractional calculus, is the book by Oldham and Spanier [Oldham, 1974] published in 1974. One of the most recent works on the subject of fractional calculus is the book of Podlubny [Podlubny, 1999] published in 1999, which deals principally with fractional differential equations.

Stability is an asymptotic qualitative criterion of the control circuit and is the primary and necessary condition for the correct functioning of every control circuit. The existing methods developed so far for stability check are mainly for integer-order systems. However, for fractional order systems, it is difficult to evaluate the stability by simply examining its characteristic equation either by finding its dominant roots or by using other algebraic methods. We try to have the papers, [Matignon, 1996] [Matignon, 1998] concerning the stability of fractional order differential equations but unfortunately, we couldn't have them.

This work consists three chapters, in addition to an introduction that we display the development of fractional calculus, while their definitions and theories are presented in chapter one.

Chapter two present the problem of non linear multi fractional order differential equations with constant coefficient, in which its existence and uniqueness solution is proved, while chapter three discussed its solution stability, by presenting a stability theorem for non-linear multi fractional order differential equations with constant coefficient.

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Supervisors Certification

I certify that this thesis was prepared under my supervision at the department of mathematics and computer applications, ***College of Science, Al-Nahrain University*** as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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On

Stability of Multi Fractional Order Differential Equations with Constant Coefficient

A Thesis

Submitted to the Department of Mathematics, College of Science, Al-Nahrain University, as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

Saba Sadiq Shibeeb

(B.Sc. 2004)

Supervision by

Assist. Prof. Dr. Alaudin N. Ahmed

July 2007 *Jamada I 1428*

DEDICATION

To My Parents and My Brother and sisters with Love and Respects

SABA

المستخلص

إن استخدام المؤثرات التفاضلية ذات الرتب الكسرية في النماذج الرياضية ازداد بصورة واسعة في الوقت الحاضر . في هذه الرسالة تم أثبات وجود ووحدانية الحل للمعادلات التفاضلية ذات الرتبة الكسرية العشوائية المتعددة الحد من خلال تحويل هكذا نوع من المعادلات التفاضلية إلى نظام من المعادلات الخطية. ونوقشت استقرارية هكذا نوع من المعادلات التفاضلية ذات الرتب الكسرية من خلال تحويلها إلى معادلات تفاضلية ذات رتب غير كسرية.