Abstract

The aim of this thesis is studying some numerical methods for solving Stochastic Differential Equation. The mathematical preliminary required to understand these numerical methods is proposed. Since many stochastic differential equations do not have explicit solution, Euler-Maruyama and Milstein numerical methods are used. The numerical simulation for different selected examples are implemented. The necessary concluding remarks are provided. The absolute error, the strong convergence error, the weak convergence error and the linear stability for Euler- Maruyama and Milstein's schemes are discussed and supported by numerical test problems. The comparison different type of convergence and error between Euler-Maruyama and Milstein's for some test problems are presented. Some conclusions and comparison in some sense have been presented with discussions. The programs coded in Matlab software are also given with useful discussion.

Acknowledgements

Praise to Allah for providing me the willingness, energy and patience to accomplish this work.

I wish to express sincere gratitude and grateful admiration to my supervisor Asst. Prof. Dr. Radhi Ali Zaboon for his over seeing, guidance, interests and suggestions that were very helpful for the completion of this thesis.

I am deeply indebted to the college of science and to the staff members of the department of mathematics and computer applications, of Al-Nahrain University for allowing me to be one of their students.

Heartfelt thanks goes to my beloved family for their love and patience they gave to me during my work and I'm truly grateful to my wife for her support.

Finally, I'd like to thank all those who hand participated in any way or another during the work.

Adel Sufyan Hussain

October, 2009 ✍

Appendix A

A.1: Discretized Brownian path

```
%Discretized Brownian path
randn('state',100)           % set the state of randn
T = 1; N = 800; dt = T/N;
dw = sqrt(dt)*randn(1,N);    % since dw is N(0,dt)
w = cumsum(dw);              % since w(j) = sum dw(i)for i= 1...j
plot([0:dt:T],[0,w],'r-')    % plot w against t
xlabel('time','Fontsize',12) % Adds x labels
ylabel('w(t)','Fontsize',12,'Rotation',0)   % Adds y labels
```

A.2: Exact solution and Euler scheme

```
%Generator Brownian Motion - Discretized paths
randn('state',100)
xzero = 1; yzero = 0;        % problem parameters
T = 1; N = 2^6; dt = 1/N;
dw = sqrt(dt)*randn(1,N);    % Brownian increments
w = cumsum(dw);              % discretized Brownian path
% Exact solution
R = 1; Dt = R*dt; L = N/R;   % L EM steps of size Dt = R*dt
xam = zeros(1,N);            % preallocate for efficiency
xem = zeros(1,L);            % preallocate for efficiency
ytemp = yzero;
for i = 1:N
    winc = sum(dw(R*(i-1)+1:R*i));
    ytemp = ytemp + (-0.5)*(sin(i*Dt))^2*Dt + sin(i*Dt)*winc;
    xtrue = exp(ytemp);
    xam(i) = xtrue;
end
plot([0:Dt:T],[yzero,xam],'m-'), hold on
% Euler scheme
xtemp = xzero;
for k = 1:L
     winc = sum(dw(R*(k-1)+1:R*k));
     xtemp = xtemp + winc*sin(k*Dt)*xtemp;
     xem(k) = xtemp;
end
plot([0:Dt:T],[xzero,xem],'r--*'), hold off
xlabel('time','Fontsize',10)
ylabel('x(t)','Fontsize',10,'Rotation',0,'HorizontalAlignment','right')
legend('Exact solution','Numerical solution')
% Absolute Error
eme = zeros(1,L);
for d = 1:L
      emerr = abs(xem(d)-xam(d));
      eme(d) = emerr;
end
eme'
plot([0:Dt:T],[0,eme],'b*-'), hold off
xlabel('time(t)','Fontsize',12)
ylabel('error(t)','Fontsize',12,'Rotation',0,'HorizontalAlignment','right')
```

A.3: EM strong convergence

```
% generator Brownian Motion - Discretized paths
randn('state',100)
secma = 1; mu = 2; pzero=1;   % problem parameters
T = 1; N = 2^9; dt = T/N;     %
M = 4000;                     % number of paths sampled
perr = zeros(M,5);            % preallocate array
for s = 1:M,                  % sample over discrete Brownian paths
        dw = sqrt(dt)*randn(1,N);      %   Brownian increments
        w = cumsum(dw);                %   discrete Brownian path.
        ptrue = pzero*exp((secma*w(end))+ (mu - 0.5*secma^2));
for p = 1:5
  R = 2^(p-1); Dt = R*dt; L = N/R;  %L Euler steps of size Dt = R*dt.
  ptemp = pzero;

            for j = 1:L
                Winc = sum(dw(R*(j-1)+1:R*j));
                ptemp = ptemp + Dt*mu*ptemp + secma*ptemp*Winc;
            end

           perr(s,p) = abs(ptemp - ptrue);% store the error at t = 1
        end
end
Dtvals = dt*(2.^([0:4]));
subplot(221)                                        % top LH picture
loglog(Dtvals,mean(perr),'b*-'),hold on
loglog(Dtvals,(Dtvals.^(.5)),'r--'),hold off %reference slope of 1/2.
axis([1e-3 1e-1 1e-4 1])
xlabel('\Delta t'),ylabel('Sample average of | p(T) - p_L |')
%%% Least squares fit of error = c * Dt^q  %%%
A = [ones(5,1),log(Dtvals)']; rhs = log(mean(perr)');
sol = A\rhs;
q = sol(2)
resid = norm(A*sol - rhs)
```

A.4: EM weak convergence

```
% generator Brownian Motion - Discretized paths
randn('state',100)
pzero = 1; T = 1; mu = 2; segma = 1;    % problem parameters
M = 50000;                              % number of paths sampled
pem = zeros(5,1);                       % preallocate arrays
   for p = 1:5                    % take various Euler timesteps
     Dt = 2^(p-10); L = T/Dt;     % L Euler steps of size Dt
     ptemp = pzero*ones(M,1);
        for j = 1:L
        Winc = sqrt(Dt)*randn(M,1);
        %%% Winc = = sqrt(Dt)*sign(randn(M,1)); %% use for weak E-M%%
        ptemp = ptemp + Dt*mu*ptemp + secma*ptemp.*Winc;
        end
              pem(p) = mean(ptemp);
  end
perr = abs(pem - exp(mu));
Dtvals = 2.^([1:5]-10);
subplot(221)                          % top RH picture
loglog(Dtvals,perr,'b*-'),hold on
loglog(Dtvals,Dtvals,'r--'), hold off  %reference slope of 1
axis([1e-3 1e-1 1e-4 1])
```

```
xlabel('\Delta t'),ylabel('| E(x(T)) - Sample average of x_L|')
%%%% Least squares fit of error = c*dt^q %%%%
A = [ones(p,1),log(Dtvals)'];
rhs = log(perr);
sol = A\rhs; q = sol(2)
resid = norm(A*sol - rhs)
```

A.5: Mean – square and asymptotic stability for E-M

```
% generator Brownian Motion - Discretized paths
randn('state',100)
T = 20; M = 50000; Xzero = 1;
ltype = {'b-','r--','m-.'};          % line types for plot
subplot(211) %%%%%%%Mean Square%%%%%%%
 for k = 1:3
     Dt = 2^(1-k);
     N = T/Dt;
     xms = zeros(1,N);
     xtemp = Xzero*ones(M,1);
         for j = 1:N
              winc = sqrt(Dt)*randn(M,1);
              xtemp = xtemp + sin(j*Dt)*xtemp.*winc;
              xms(j) = mean(xtemp.^2);         % mean-square estimate
         end
         semilogy([0:Dt:T],[Xzero,xms],ltype{k},'Linewidth',2), hold on
 end
 axis([0,T,1e-20,1e+20]),hold off
 legend('\Delta t = 1','\Delta t = 1/2','\Delta t = 1/4')
 title('Mean- Square: g(t) = sin(t)','Fontsize',16)
 ylabel('E[x^2]','Fontsize',12)
subplot(212)     %%%% Asymptotic: a single path %%%%%
T =500;
    for k = 1:3
        Dt = 2^(1-k);
        N = T/Dt;
        xemabs = zeros(1,N);
        xtemp = Xzero;
           for j = 1:N
                 winc = sqrt(Dt)*randn;
                  xtemp = xtemp + sin(j*Dt)*xtemp.*winc;
                  xemabs(j) = abs(xtemp);
           end
    semilogy([0:Dt:T],[Xzero,xemabs],ltype{k},'Linewidth',2), hold on
    end
legend('\Delta t = 1','\Delta t = 1/2','\Delta t = 1/4')
title('single path: g(t) = sin(t)','Fontsize',16)
ylabel('|x|','Fontsize',12)
axis([0,T,1e-50,1e+100]),hold off
```

A.6: Exact solution and Milstein scheme

```
% Generator Brownian Motion- Discretized paths
randn('state',100)
xzero = 1; yzero = 0;          % problem parameters
T = 1;  N = 2^7; dt = 1/N;
dw = sqrt(dt)*randn(1,N);      % Brownian increments
w = cumsum(dw);                % discretized Brownian path
% Exact solution
R = 1; Dt = dt; L = N/R;       % L EM steps of size Dt = R*dt.
xam = zeros(1,N);              % preallocate for efficiency
```

```
xem = zeros(1,L);              % preallocate for efficiency
ytemp  = yzero;
    for i = 1:N
         winc = sum(dw(R*(i-1)+1:R*i));
         ytemp = ytemp + (-0.5)*(sin(i*Dt)^2)*Dt + sin(i*Dt)*winc;
         xtrue = exp(ytemp);
         xam(i) = xtrue;
    end
plot([0:Dt:T],[yzero,xam],'m-'),hold on
% Milstein scheme
xtemp = xzero;
for j = 1:L
         winc = sum(dw(R*(j-1)+1:R*j));
         xtemp = xtemp + sin(j*Dt)*xtemp.*winc...
               + 0.5*sin(j*Dt)^2*xtemp.*(winc.^2 - Dt);
     xem(j) = xtemp;
end
plot([0:Dt:T],[xzero,xem],'r--*'), hold off
xlabel('time','Fontsize',10)
ylabel('x(t)','Fontsize',10,'Rotation',0,'HorizontalAlignment','right')
legend('Exact solution','Numerical solution')
% Absolute Error
eme = zeros(1,L);
for d = 1:L
      emerr = abs(xem(d)-xam(d));
      eme(d) = emerr;
end
eme'
plot([0:Dt:T],[0,eme],'b*-'), hold off
xlabel('time(t)','Fontsize',10)
ylabel('error(t)','Fontsize',10,'Rotation',0,'HorizontalAlignment','right
')
```

A.7: strong convergence of Milstein

```
% Generator Brownian Motion - Discretiaed path
randn('state',100)
Xzero = 1;                                  % problem parameters
T = 1; N = 2^(11); dt = T/N;                %
M = 500;                                    % number of paths sampled
R = [1; 16; 32; 64; 128];                   % Milstein stepsizes are R*dt
dw = sqrt(dt)*randn(M,N);                   % Brownian increments
Xmil = zeros(M,5);                          % preallocate array
 for p = 1:5
     Dt = R(p)*dt; L = N/R(p);      % L timesteps of size Dt = R dt
     Xtemp = Xzero*ones(M,1);
      for j = 1:L
              winc = sum(dw(:,R(p)*(j-1)+1:R(p)*j),2);
              Xtemp = Xtemp + sin(j*Dt)*Xtemp.*winc ...
                          + 0.5*(sin(j*Dt))^2*Xtemp.*(winc.^2- Dt);
      end
   Xmil(:,p) = Xtemp;             % store Milstein solution at t = 1
 end
Xref = Xmil(:,1);                                % Reference solution
Xerr = abs(Xmil(:,2:5) - repmat(Xref,1,4));      % Error in each path
mean(Xerr);                                      % Mean pathwise erorrs
Dtvals = dt*R(2:5);                              % Milstein timesteps used
subplot(221)                                     % lower RH picture
loglog(Dtvals,mean(Xerr),'b*-'), hold on
loglog(Dtvals,Dtvals,'r--'), hold off        % reference slope of 1
```

```
axis([1e-3 1e-1 1e-4 1])
xlabel('\Delta t')
ylabel('sample average of |x(T) - x_L|')
%%%% Least squares fit of error = c * Dt^q%%%%
A = [ones(4,1), log(Dtvals)]; rhs = log(mean(Xerr)');
sol = A\rhs; q = sol(2)
resid = norm(A*sol - rhs)
```

A.8: Mean – square and asymptotic stability for Milstein

```
% generator Brownian Motion - Discretized path
randn('state',100)
T = 20; M = 50000; Xzero = 1;
ltype = {'b-','r--','m-.'};          % line types for plot
subplot(211) %%%%%Mean - square%%%%%
lambda = -3; mu = sqrt(3);            % problem parameters
 for k = 1:3
     Dt = 2^(1-k);
     N = T/Dt;
     xms = zeros(1,N);
     xtemp = Xzero*ones(M,1);
        for j = 1:N
            winc = sqrt(Dt)*randn(M,1);
            xtemp = xtemp + lambda*xtemp.*Dt + mu*xtemp.*winc...
                      + 0.5*mu^2*xtemp.*(winc.^2-Dt);
              xms(j) = mean(xtemp.^2);         % mean-square estimate
        end
         semilogy([0:Dt:T],[Xzero,xms],ltype{k},'Linewidth',2), hold on
 end
 axis([0,T,1e-20,1e+20]),hold off
 legend('\Delta t = 1','\Delta t = 1/2','\Delta t = 1/4')
 title('Mean- Square:\lambda = -3, \mu = \surd3','Fontsize',16)
 ylabel('E[x^2]','Fontsize',12)
 subplot(212)%%%% Asymptotic: a single path %%%%%
 T =500;
 lambda = 0.5; mu = sqrt(6);        % problem parameters
    for k = 1:3
        Dt = 2^(1-k);
        N = T/Dt;
        xemabs = zeros(1,N);
        xtemp = Xzero;
           for j = 1:N
                 winc = sqrt(Dt)*randn;
                  xtemp = xtemp + lambda*xtemp.*Dt + mu*xtemp.*winc...
                           + 0.5*mu^2*xtemp.*(winc.^2-Dt);
                  xemabs(j) = abs(xtemp);
           end
           semilogy([0:Dt:T],[Xzero,xemabs],ltype{k},'Linewidth',2), hold
on
    end
legend('\Delta t = 1','\Delta t = 1/2','\Delta t = 1/4')
title('single path:\lambda = -3, \mu = \surd 6','Fontsize',16)
ylabel('|x|','Fontsize',12)
axis([0,T,1e-50,1e+100]),hold off
```

Basic Notations

t	Time.
T	Maturely date.
W_t	Wiener process.
X_t	Stochastic process.
$Var(X)$	Variance of the random variable X.
$E(X)$	The expectation of the random variable X.
$\sim N(\mu, t)$	Normal distribution with expectation μ and variance t.
$\sim$	With distribution.
Δn	Small increment in n.
Ω	Sample space.
$[a, b]$	Closed interval $\{x \in \Re : a \leq x \leq b\}$.
$\Re$	Set of real numbers.
$Cov(X)$	Covariance of the random variable X.
$\|X\|$	The Euclidian norm of a vector X.
$f : \square \longrightarrow \square$	A function f from $\square$ to $\square$.
a.s.	Almost surely.
w.p.1	with probability 1.
P	Probability.
L.i.p.	Limit in probability.

Supervisor Certification

I certify that this thesis was prepared under my supervision at the department of Mathematics, College of Science, Al-Nahrain University, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

Signature:

Name: Asst. Prof. Dr. Radhi Ali Zaboon

Data: / / 2009

In view of the available recommendations, I forward this thesis for debate by the examining committee.

Signature:

Name: Asst. Prof. Dr. Akram M. Al-Abood

Head of the Department of Mathematics and Computer Applications

Data: / / 2009

Examining Committee Certification

We certify that we have read this thesis entitled "*On Numerical Solutions of Some Stochastic Ordinary Differential Equations*" and as examining committee examined the student *(Adel Sufyan Hussain)* in its contents and in what it connected with, and that, in our opinion, it meets the standards of a thesis for the degree of Master of Science in Mathematics.

(Chairman)

Signature:

Name: Dr. Ahlam J. Khaleel

Asst. Prof.

Date: / / 2010

(Member)

Signature:

Name: Dr. Jehad R. Kider

Asst. Prof.

Date: / / 2010

(Member)

Signature:

Name: Dr. Osama H. Mohemmed

Lecturer

Date: / / 2010

(Member and Supervisor)

Signature:

Name: Dr. Radhi A. Zaboon

Asst. Prof.

Date: / / 2010

Approved by the Collage of Science

Signature

Name: Asst. Prof. Dr. Laith Abdul Aziz Al-Ani

Dean of the Collage of Science

Data: / / 2010

Chapter One
Some Stochastic Process Concepts

This chapter presents some stochastic process concepts which is divided into seven sections, the first one describe the set of algebra, the second section deals with the random variable, the third section deals with the stochastic processes, the fourth section deals with the stochastic integral, the fifth section deals with the approximation of functions by step functions, the sixth section deals with the Itô formula, while the seventh section deals with the existence and uniqueness theorem of solution of stochastic differential equations and some of its kinds.

1.1 Algebra of Sets:

The collection of all elementary outcomes of a random experiment is called ***sample space*** and is denoted by Ω. In the terminology, the sample space is termed as the ***universal*** set. Thus, the sample space Ω is a set consisting of mutually exclusive, collectively exhaustive listing of all possible outcomes of a random experiment. That is, $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ denotes the set of all finite outcomes, $\Omega = \{\omega_1, \omega_2, \ldots\}$ denotes the set of all countably infinite outcomes, and , $\Omega = \{0 \leq t \leq T\}$ denotes the set of uncountably infinite outcomes.

Let Ω represent the sample space which is a collection of ω-points as defined earlier. The various set operations are ***complementation***, ***union*** and ***intersection***. Let A and B be two subsets

of the sample space Ω, denoted by $A \subset \Omega$, $B \subset \Omega$. The complement of A, denoted by A^c, represents the set of all ω-points not contained in A, i.e.,

$$A^c = \{\omega: \omega \notin A\} \quad \ldots(1.1)$$

Evidently the complement of Ω is the empty set $\varnothing$. The union of sets A and B, denoted by $A \cup B$ or $A + B$, represents the occurrence of ω-points in either A or B. Similarly, the intersection of sets A and B, denoted by $A \cap B$ or AB, represents the occurrence of ω-points in A and B. Clearly, if there is no commonality of ω-points in A and B, then $A \cap B$ is the empty set $\varnothing$.

Definition (1.1) (Field) (Algebra), [Krishnan, 2006]:

A class of a collection of subsets $A_j \subset \Omega$ denoted by υ is a field, when the following condition are satisfied:

1. If $A_i \in \upsilon$, then $A_i^c \in \upsilon$, $i = 1, 2, \ldots, n$. $\ldots(1.2)$
2. If $\{A_i, i = 1, 2, \ldots, n\} \in \upsilon$, then $\bigcup_{i=1}^{n} A_i \in \upsilon$.

Remark (1.1):

Given the above two conditions, de Morgan's law ensures that finite intersections also belong to the field. Thus a class of subsets is a field if and only if it is closed under all finite set operations like unions, intersection, and complementation.

Definition (1.2) (σ-Field) (σ-Algebra), [Krishnan, 2006]:

A class of a countable infinite collection of subsets $A_j \subset \Omega$ denoted by $\mathcal{F}$ is a σ-field when the following conditions are satisfied:

1. If $A_i \in \mathcal{F}$, then $A_i^c \in \mathcal{F}$.

…(1.3)

2. If $\{A_i, i = 1, 2, \ldots\} \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

In general a σ-field is a field, but a field may not be a σ-field.

Definition (1.3) (Borel σ-Field), [Krishnan, 1984]:

The minimum σ-field generated by the collection of open sets of a topological space Ω is called the Borel σ-field or Borel field. Members of this σ-field are called Borel sets.

Definition (1.4) (Measurable Space), [Stirzaker, 2005]:

A suitable model of the random experiment is therefore a sample space Ω and a σ-field $\mathcal{F}$ of subsets of Ω. The space $(\Omega, \mathcal{F})$ thus created is called a measurable space.

Remarks (1.2), [Krishnan, 1984], [Stirzaker, 2005]:

1. ***Events*** are defined as the subsets of Ω which are elements in the σ-field.
2. In particular, Ω is called the ***certain event.***

3. If two events A and B satisfy $A \cap B = \varnothing$ **,** then they are said to be ***disjoint.***

4. The complement Ω^c is an event called the impossible event, which we denote by $\Omega^c = \varnothing$, the ***empty set.***

5. If $\{A_i, i = 1, 2, \ldots, n\}$ is a class of disjoint sets of Ω, such that $\bigcup_{i=1}^{n} A_i = \Omega$ then the $\{A_i\}$ ***collectively exhaust*** Ω.

Definition (1.5) (Probability Measure), [Krishnan, 2006]:

A ***probability measure*** is a set function defined on a σ-field $\mathcal{F}$ of subsets of a sample space Ω, such that it satisfies the following axioms of Kolmogorov for any $A \in \mathcal{F}$:

1. $P(A) \geq 0$ (nonnegativity).

2. $P(\Omega) = 1$ (normalization). …(1.4)

3. $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$ (σ-additivity), with $A_n \in \mathcal{F}$, and A_i and A_j being pairwise disjoint.

It is also called probability distribution.

Lemma (1.1) (Sequential Monotone Continuity), [Krishnan, 1984]:

Let $\{A_n\}$ be a monotone decreasing sequence in $\mathcal{F}$, such that $A_{n+1} \subset A_n$, and let $\lim_{n\to\infty} A_n = \varnothing$, then:

$$\lim_{n\to\infty} P(A_n) = 0 \qquad \ldots(1.5)$$

The probability measure is said to satisfy the sequential monotone continuity at $\varnothing$.

Proposition (1.1) (Sequential Continuity), [Krishnan, 1984]:

Let $\{A_n\}$ be a convergent sequence of events in $\mathcal{F}$, with $\lim_{n\to\infty} A_n = A$. Then:

$$\lim_{n\to\infty} P(A_n) = P(\lim_{n\to\infty} A_n) = P(A) \qquad \ldots(1.6)$$

The probability measure is sequentially continuous.

1.2 Random Variable, [Krishnan, 1984], [Stirzaker, 2005]:

An important class of functions is the measurable functions which are different from the measure functions, whereas measure functions are set functions, measurable functions are invariably point functions.

Definition (1.6) (Measurable Function), [Krishnan, 1984]:

Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. Let g be a function with domain $E_1 \subset \Omega_1$ and range $E_2 \subset \Omega_2$.

$$g: \Omega_1 \longrightarrow \Omega_2$$

g is called an $\mathcal{F}_1$-measurable function or an $\mathcal{F}_1$-measurable mapping if for every $E_2 \in \mathcal{F}_2$

$$g^{-1}(E_2) = \{\omega: g(\omega) \in E_2\} \in E_1 \qquad \ldots(1.7)$$

is in the σ-field $\mathcal{F}_1$.

Remarks (1.3), [Krishnan, 1984]:

1. If g is measurable with respect to the σ-field $\mathcal{F}$ of sets that are P-measurable, then we might also say that g is P-measurable if there is no confusion.
2. The set given by $g^{-1}(E_2)$ is called the ***inverse image*** or ***inverse mapping of E_2***, and it is measurable set.
3. Inverse mappings preserve all set relations.

Definition (1.7) (Random Variable), [Stirzaker, 2005]:

Measurable space consisting of the real line R and σ-field of Borel sets $\mathcal{R}$. Let the probability measure P be defined on $(\Omega, \mathcal{F})$. The measurable mapping from $(\Omega, \mathcal{F})$ into $(R, \mathcal{R})$ is called a real-valued random variable.

Remarks (1.4), [Krishnan, 1984], [Stirzaker, 2005]:

1. Naturally, the probability measure P induces a probability measure P_X in the space $(R, \mathcal{R})$. If $E_2 \in \mathcal{R}$, then:

$$P_X(E_2) = P(X^{-1}(E_2)) = P(E_1) = P\{\omega: X(\omega) \in E_2\} \quad \ldots(1.8)$$

   Equation (1.8) related the probability measure P_X in $(R, \mathcal{R})$ to the probability measure P in $(\Omega, \mathcal{F})$. Instead of writing $P\{\omega: X(\omega) \in E_2\}$, we shall have the abbreviated notation $P\{X \in E_2\}$.
2. If Ω is a metric topological space, then $\mathcal{F}$ is the σ-field of all Borel sets of Ω. Then a function g mapping $\Omega \longrightarrow$ □ is a Borel function if for every $E_2 \in R$, $g^{-1}(E_2)$ is a Borel set of Ω. Since Borel sets of Ω are measurable by assumption, every Borel function is $\mathcal{F}$-measurable.

1.2.1 Distribution Functions, [Hsu, 1997], [Evans, 2006]:

Let $(\Omega, \mathcal{F}, P)$ be a probability space and suppose $X: \Omega \longrightarrow \mathbb{R}^n$ random variable, in this section some additional concepts about basic statistical definitions and properties of the distribution function are considered.

Definition (1.8) (Distribution Function):

i. The ***distribution function*** of is the function $F_X: \mathbb{R}^n \longrightarrow [0,1]$ defined by:

$$F_X(x) := F(X \leq x) \text{ for all } x \in \mathbb{R}^n \qquad \ldots(1.9)$$

ii. If $X_1, X_2, \ldots, X_m,: \Omega \longrightarrow \mathbb{R}^n$ are random variables, their ***joint distribution function*** is $F_{X_1,X_2,\ldots,X_m}(x_1,x_2,\ldots,x_m): (\mathbb{R}^n)^m \longrightarrow [0, 1]$,

$$F_{X_1,X_2,\ldots,X_m}(x_1,x_2,\ldots,x_m) := P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_m \leq x_m) \qquad \ldots(1.10)$$

for all $x_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$.

Definition (1.9) (Density Function):

Suppose $X: \Omega \longrightarrow \mathbb{R}^n$ is a random variable and $F = F_X$ its distribution function. If there exists a nonnegative, integrable function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$, such that:

$$F(x) = F(x_1, x_2, \ldots, x_n)$$

$$= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \ldots \int_{-\infty}^{x_n} f(y_1, y_2, \ldots, y_n) dy_n \ldots dy_2 dy_1 \qquad \ldots(1.11)$$

Then f is called the ***density function*** for X.

It follows then that:

$$P(X \in B) = \int_B f(x)\,dx, \text{ for all } B \in \mathcal{B} \qquad \ldots(1.12)$$

This formula is important as the expression on the right hand side is an ordinary integral, and can often be explicitly calculated.

Remark (1.5):

If the probability distribution function is differentiable, then we obtain the probability density function f(x)

$$f(x) = \frac{dF(x)}{dx} \qquad \ldots(1.13)$$

1.2.2 Expectation of Random Variables:

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The expectation of a random variable X is usually defined by the Stieltjes integral:

$$EX = \int_{-\infty}^{\infty} X\,dF(x) \qquad \ldots(1.14)$$

Definition (1.10) (Expectation), [Krishnan, 1984]:

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let X be a real random variable. The expectation of X is defined by:

$$EX = \int_{\Omega} X(\omega)\,dP(\omega) \qquad \ldots(1.15)$$

Remarks (1.6):

There are some properties of expectation operation, such as:

1. Linearity: $E(aX + bY) = aEX + bEY$, for all constants a and b.

2. Homogeneity: $E(cX) = cEX$, for constant c.

3. Order preservation $X \geq Y$ implies $EX \geq EY$.

Lemma (1.2), [Evans, 2005]:

Let $X: \Omega \longrightarrow \mathbb{R}^n$ be a random variable, and assume that its distribution function $F = F_X$ which has the density function. Suppose $g: \mathbb{R}^n \longrightarrow \mathbb{R}$, and $Y = g(X)$ is integrable. Then:

$$E(Y) = \int_{\mathbb{R}^n} g(x)f(x)\,dx$$

In particular,

$$E(X) = \int_{\mathbb{R}^n} xf(x)\,dx \qquad \ldots(1.16)$$

and

$$V(X) = \int_{\mathbb{R}^n} |x - E(X)|^2 f(x)\,dx \qquad \ldots(1.17)$$

1.2.3 Convergence of Random Variable:

The convergence of random variable and their kinds are of our interest and then submitted as follows:

Definition (1.11) (Almost Surely Convergence), [Krishnan, 2006]:

A sequence of random variables $\{X_n\}$ converges almost surely (a.s.), or almost certainly, or strongly, to X if for every ω-point not belonging to the null event A,

$$\lim_{n\to\infty} |X_n(\omega) - X(\omega)| = 0 \qquad \ldots(1.18)$$

This type of convergence is known as convergence with probability 1 and is denoted by:

$$X_n(\omega)_{n\to\infty} \xrightarrow{\text{a.s.}} X(\omega)$$

or

$$X(\omega) = \lim_{n\to\infty} X_n(\omega) \quad \text{(a.s.)}$$

Definition (1.12) (Convergence in Probability), [Krishnan, 1984]:

A sequence of random variables $\{X_n\}$ converges in probability to X if for every $\varepsilon > 0$, however small,

$$\lim_{n\to\infty} p(|X_n - X| \geq \varepsilon) = 0, \text{ or}$$

$$\lim_{n\to\infty} p(|X_n - X| < \varepsilon) = 1$$

It is denoted by:

$$X_n(\omega)_{n\to\infty} \xrightarrow{\text{l.i.p.}} X(\omega), \text{ or}$$

$$X(\omega) = \underset{n\to\infty}{\text{l.i.p.}}\, X_n(\omega)$$

(where l.i.p. is standing for limit in probability).

Remarks (1.7), [Krihsnan, 2006]:

The concept of convergence in probability plays an important role in the consistency of estimators and the weak law of large numbers. We give next some results concerning this concept.

i. If a sequence of random variables $\{X_n\}$ converges almost surely to X, then it converges in probability to the same limit. The converse is not true.

ii. If $\{X_n\}$ converges in probability to X, then there exist a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ which converges almost surely to the same limit.

1.3 Stochastic Processes:

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let T be an arbitrary indexed parameter set called the time set. T can be the real line $\mathbb{R}$, the positive real line $\mathbb{R}^+$, the set of positive integers $\mathbb{Z}$, or any semiclosed interval in $\mathbb{R}$ or $\mathbb{R}^+$, unless otherwise specified. We shall assume that T is a semiclosed time interval in $\mathbb{R}^+$. Sometime we will explicitly state that T is in $\mathbb{R}^+$.

Definition (1.13) (Stochastic Process), [Krishnan, 1984]:

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let T be any time set. Let $(\mathbb{R}, \mathcal{R})$ be a measurable space, where $\mathbb{R}$ is the real line and $\mathcal{R}$ is the σ-field of Borel sets on the real line. A stochastic process $\{X_t, t \in T\}$ is a family of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ and taking values in the measurable space $(\mathbb{R}, \mathcal{R})$.

Definition (1.14) (Covariance Matrix), [Raphael, 1972]:

Consider a vector-valued stochastic process W(t). Then we call:

$$m(t) = E\{W(t)\}$$

the mean of the process,

$$C_W(t_1, t_2) = E\{[W(t_1) - m(t_1)][W(t_2) - m(t_2)]^T\} \quad \ldots(1.19)$$

The covariance matrix, and:

$$C_W(t_1, t_2) = E\{W(t_1)W^T(t_2)\} \quad \ldots(1.20)$$

is the second-order joint moment matrix of W(t). $R_W(t, t) = Q(t)$ is termed as the variance matrix, while:

$$C_W(t, t) = Q(t) \quad \ldots(1.21)$$

is the second-order moment matrix of the process.

Remarks (1.8), [Raphael, 1972]:

1. The joint moment matrix written out more explicitly is

$$C_W(t_1, t_2) = E\{W(t_1)W^T(t_2)\}$$

$$= \begin{pmatrix} E\{w_1(t_1)w_1(t_2)\} & E\{w_1(t_1)w_2(t_2)\} & \cdots & E\{w_1(t_1)w_m(t_2)\} \\ E\{w_2(t_1)w_1(t_2)\} & E\{w_2(t_1)w_2(t_2)\} & \cdots & E\{w_2(t_1)w_m(t_2)\} \\ \vdots & \vdots & \ddots & \vdots \\ E\{w_m(t_1)w_1(t_2)\} & E\{w_m(t_1)w_2(t_2)\} & \cdots & E\{w_m(t_1)w_m(t_2)\} \end{pmatrix}$$

$\ldots(1.22)$

2. Each element of $C_W(t_1, t_2)$ is a scalar joint moment function. Similarly, each element of $R_W(t_1, t_2)$ is a scalar covariance function.

1.3.1 Classes of Stochastic Processes:

In this subsection we shall consider several types of stochastic process and discuss their properties.

Definition (1.15) (Stationary Process), [Hsu, 1997]:

Let $\{X_t, t \in T\}$ be a stochastic process with time set T defined on a probability space $(\Omega, \mathcal{F}, P)$ taking values in the state space $(\square, \mathcal{R})$. Let $T = \{t_1, t_2, \ldots, t_n\}$ be any finite set of values belonging to T. Then the process is ***strictly stationary*** or ***stationary*** if for any Δt the joint distribution of the sequence $\{X(t_1), X(t_2), \ldots, X(t_n)\}$is the same as the joint distribution of the sequence $\{X(t_1 + \Delta t), X(t_2 + \Delta t), \ldots, X(t_n + \Delta t)\}$ for any positive integer n.

Definition (1.16) (Wide Sense Stationary), [Krishnan, 1984]:

A real stochastic process $\{X_t, t \in T\}$ is ***wide sense stationary*** or covariance stationary if:

1. $E X_t^2 < \infty$.
2. $\mu_x = EX_t$ a constant.
3. $C_X(t - s) = E\{(X_t - \mu)(X_s - \mu)\}$ depends only on the time difference $t - s$ and not on either t or s.

Remark (1.9):

The strict sense ***stationary*** of definition (1.15) implies ***wide sense stationary*** of definition (1.16), but the converse is not true, [Krishnan, 1984].

Example (1.1), [Krishnan, 1984]:

Let us define the random signal:

$x(t) = \alpha\sin(0.5t + \theta)$

Where α is a positive random variable with mean 0.63 and variance 0.11, θ is uniformly distributed between 0 and 2π, and α and θ are uncorrelated.

where the p.d.f. of uniformly distribution is:

$$f(x) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{e.w.} \end{cases}$$

The mean of this random signal is calculated as:

$$m(t) = E[x(t)]$$

$$= E[\alpha\sin(0.5t + \theta)]$$

$$= E(\alpha)\int_0^{2\pi} \sin(0.5t + \theta)\frac{1}{2\pi}\, d\theta$$

$$= (0.63)\int_0^{2\pi} [\sin(0.5t)\cos(\theta) + \cos(0.5t)\sin(\theta)]\, \frac{1}{2\pi}\, d\theta$$

$$= (0.63)\left[\frac{\sin(0.5t)}{2\pi}\left(-\sin\theta\Big|_0^{2\pi}\right) + \frac{\cos(0.5t)}{2\pi}\left(\cos\theta\Big|_0^{2\pi}\right)\right] = 0.$$

Now, let $t = t_2$, $s = t_1$; $t_2 > t_1$

$$C_X(t - s) = C_X(t_2 - t_1)$$

$$= E\{(X_{t_2} - \mu)(X_{t_1} - \mu)\} = E(X(t_2)X(t_1))$$

$$= \frac{1}{2}E(\alpha^2)\int_0^{2\pi} [\sin\{0.5(t_2 - t_1) + 2\theta\} + \sin\{0.5(t_2 - t_1)\}]\frac{1}{2\pi}\, d\theta$$

$$= \frac{1}{4\pi} E(\alpha^2) \int_0^{2\pi} [\sin(0.5)(t_2 - t_1)\cos(2\theta) + \cos\{0.5(t_2 - t_1)$$

$$\sin(2\theta)\} \sin\{0.5(t_2 - t_1)\}] \, d\theta$$

$$= \frac{1}{4\pi} E(\alpha^2) \left[\frac{\sin(0.5(t_2 - t_1))}{2\pi} \int_0^{2\pi} \cos(2\theta) \, d\theta + \right.$$

$$\frac{\cos(0.5(t_2 - t_1))}{2\pi} \int_0^{2\pi} \sin(2\theta) \, d\theta +$$

$$\left. \frac{\sin(0.5(t_2 - t_1))}{2\pi} \int_0^{2\pi} d\theta \right]$$

$$= 0.03\cos(0.5(t_2 - t_1))$$

The mean is independent of time, and the covariance function depends only on time difference $(t_2 - t_1)$, so this random signal is wide sense stationary. This result is reasonable since there is no preferred time if the phase is uniformly distributed from 0 to 2π.

Definition (1.17) (Independent Increment Process), [Krishnan, 2006]:

A stochastic process $\{X_t, t \in T\}$ defined on the probability space $(\Omega, \mathcal{F}, P)$ is an independent increment process if for any collection $\{t_1, t_2, \ldots, t_n\} \subset T$ satisfying $t_1 < t_2 < \ldots < t_n$ the increment of the process X_t, $X_{t_2} - X_{t_1}$, $X_{t_3} - X_{t_2}$, …, $X_{t_n} - X_{t_{n-1}}$ are a sequence of independent random variables.

1.3.2 White Noise:

The following definitions are needed to complete understanding white noise:

White Noise, [Raphel, 1972]:

One frequently encounters in practice zero-mean scalar stochastic process with the property that $X(t_1)$ and $X(t_2)$ are uncorrelated even for values of $|t_2 - t_1|$ that are quite small, that is:

$$R_X(t_2, t_1) \cong 0, \text{ for } |t_2 - t_1| > \varepsilon \quad \ldots(1.23)$$

where ε is a small number. The covariance function of such stochastic processes can be idealized as follows:

$$C_X(t_2, t_1) = V(t_1)\delta(t_2 - t_1),\ V(t_1) \geq 0 \quad \ldots(1.24)$$

Here $\delta(t_2 - t_1)$ is the delta function and $V(t_1)$ is referred to as the intensity of the process at time t. Such processes are called ***white noise processes***.

We can of course extend the notion of a white noise process to vector-valued process:

Definition (1.18) (White Noise Process) [Raphael, 1972]:

Let X(t) be a zero mean vector-valued stochastic process with covariance matrix:

$$C_X(t_2, t_1) = V(t_1)\delta(t_2 - t_1) \quad \ldots(1.25)$$

where $V(t_1) \geq 0$.

The process X(t) is then said to be a ***white noise stochastic process*** with intensity V(t).

White Noise Differential Equation, [Krishnan, 1984]:

We now investigate the problem of a differential equation driven by white noise. Suppose we are given the differential equation in the following form:

$$\frac{dY_t}{dt} = \alpha(t)Y_t + \beta(t)X_t,\ t \in T,\ Y_a \qquad \ldots(1.26)$$

where Y_a is the initial condition and X_t is a white noise process. Presented in the form (1.26) cannot be interpreted as an ordinary differential equation without making assumptions on differentiability and separability of Y_t and X_t, even if X_t is not white but some other quadratic mean continuous random process. Instead of interpreting this equation as a differential equation, we can interpret it as an integral equation without worrying about these assumptions. We interpret the stochastic process $\{Y_t, t \in [a, b)\}$ with $E|Y_t|^2 < \infty$ as the solution to the differential equation (1.26) if it satisfies the following integral equation:

$$Y_t = Y_a + \int_a^t \alpha(s)Y_s\, ds + \int_a^t \beta(s)\, dZ_s,\ a \le t \le b \qquad \ldots(1.27)$$

where Z_t is the process of orthogonal increment associated with the white noise process X_t, Y_a is the initial condition satisfying $E|Y_a|^2 < \infty$, and $\alpha(t)$ and $\beta(t)$ belong to a class of square integrable functions.

The above integral equation can also be written as:

$$dY_t = \alpha(t)Y_t\, dt + \beta(t)dZ_t,\quad a \le t < b,\ Y_a,\ E|Y_a|^2 < \infty$$

We have more to say about these differential equations when we discuss Itô stochastic differential equations.

1.3.3 Brownian Motion:

Next we define a Brownian motion process assuming that the time set $T = \mathbb{R}^+$ or any interval $[0, a]$, $a > 0$.

Definition (1.19) (Brownian Motion), [Krishnan, 2006]:

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. The stochastic process $\{W_t, t \in T\}$ defined on $(\Omega, \mathcal{F}, P)$ is a Brownian motion process with parameter σ^2 if:

1. $W_0(t) = 0$.
2. $\{W_t\}$ is a stationary independent increment process.
3. For every s and t, $s \leq t$, belonging to the increment $W_t - W_s$ are Gaussian distributed with mean zero and variance $\sigma^2(t - s)$.
4. For almost all $\omega \in \Omega$ the sample functions $t \longrightarrow W_t(\omega)$ are uniformly continuous in the interval T.

With the definition given above we shall now drive the auto covariance function $C_W(t, s)$.

For $t > s$.

$$C_W(t, s) = E(W_t W_s)$$

$$= E(W_t - W_s + W_s)W_s$$

$$= E(W_t - W_s)W_s + E\,W_s^2$$

$$= E\,W_s^2 \quad \text{from 2}$$

$$= \sigma^2 s \qquad \text{from 3}$$

Similarly, for $t < s$, $C_W(t, s) = \sigma^2 t$. Hence $C_W(t, s) = \sigma^2(t \wedge s)$, where $t \wedge s = \min\{t, s\}$.

Remark (1.10), [Stirzaker, 2005]:

If $\sigma^2 = 1$, then W(t) is said to be the ***standard Brownian process (standard Wiener process)***.

1.3.3.1 Computation of Joint Probabilities, [Evans, 2005]:

From the definition if W(.)is a Brownian motion, then for all $t > 0$ and $a \le b$,

$$P(a \le W(t) \le b) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-\frac{x^2}{2t}}\, dx \qquad \ldots(1.28)$$

since W(t) is N(0, t) (for more details see [Evans, 2006]).

Remarks (1.11), [Evans, 2005]:

1. Fix a point $x_0 \in \mathbb{R}^n$ and consider then the ordinary differential equation:

$$\left.\begin{aligned} \dot{X}(t) &= b(X(t)), \quad t > 0 \\ X(0) &= x_0 \end{aligned}\right\} \quad \text{(ODE)} \qquad \ldots(1.29)$$

   where $b: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a given, smooth vector field and the solution is the trajectory $X(.): [0, \infty) \longrightarrow \mathbb{R}^n$.

2. X(t) is the state of the system at time $t \ge 0$,

$$\dot{X}(t) := \frac{d}{dt} X(t) \qquad \ldots(1.30)$$

   In many applications, however, the experimentally measured trajectories of systems modeled by (ODE) do not in fact behave as predicted.

Hence it seems reasonable to modify (ODE's), in such away to include the possibility of random effects disturbing the system. A formal way to do so is to write:

$$\left.\begin{aligned}\dot{X}(t) &= b(X(t)) + B(X(t))\xi(t), \quad t > 0\\ X(0) &= x_0\end{aligned}\right\} \qquad \ldots(1.31)$$

where B: $\mathbb{R}^n \longrightarrow M^{n\times m}$ (= space of n×m matrices) and

$\xi(.)$:= m-dimensional "white noise".

This approach presents us with these ***mathematical problems***:

1. Define the "white noise" ξ (.) as we define.
2. Define what it means for X(.) to solve (1.31).
3. Show (1.31) has a solution, discuss uniqueness, asymptotic behavior, dependence upon x_0, b, B, etc.

Some Heuristics:

Let us first study equation (1.31) in the case m = n, $x_0 = 0$, $b \equiv 0$, and $B \equiv I$. The solution of (1.31) in this setting turns out to be the n-dimensional Wiener process, or Brownian motion, denoted by W(t). Thus we may symbolically write:

$$\dot{W}(t) = \xi(t) \qquad \ldots(1.32)$$

Thereby asserting that "white noise" is the time derivative of the Brownian motion, (if exist).

Now return to the general case of the equation (1.31), write $\frac{d}{dt}$ instead of the dot, yielding:

$$\frac{d}{dt}X(t) = b(X(t)) + B(X(t))\,\frac{dW(t)}{dt} \qquad \ldots(1.33)$$

and finally multiply by "dt":

$$\left.\begin{aligned} dX(t) &= b(X(t))dt + B(X(t))dW(t) \\ X(0) &= x_0 \end{aligned}\right\} \quad \text{(SDE)} \qquad \ldots(1.34)$$

This expression, properly interpreted, is a stochastic differential equation (abbreviated by SDE). We say that X(.) solves the (SDE) provided

$$X(t) = x_0 + \int_0^t b(X(s))\,ds + \int_0^t B(X(s))\,dW, \text{ for all times } t>0 \qquad \ldots(1.35)$$

Now we must:

1. Construct W(t).
2. Define the ***stochastic integral***.
3. Show that equation (1.35) has a solution, etc.

1.4 Stochastic Integral:

It is well-known that stochastic integrals and Itô formula play a central role in modern probability theory and its applications in stochastic differential equation concerned by Brownian motion, etc.

This section concerning the most necessary mathematical principles discussing stochastic integration, Itô formula, Itô SDE, existence of a unique solution of Itô SDEs, as well as some solvable examples.

Now, we shall define the integral:

$$I(T) = \int_0^T f(t)\, dw(t) \qquad \ldots(1.36)$$

where w(t) is a Brownian motion and f(t) is a stochastic function, and to study its basic properties. One may define:

$$I(T) = f(T)w(T) - \int_0^T f'(t)w(t)\, dt$$

If f is absolutely continuous for each w. However, if f is only continuous, or just integrable, this definition does not make sense, [Friedman, 1975].

Remark (1.12), [Friedman, 1975]:

Since w(t) (the Brownian motion) is nowhere differentiable with probability 1, the integral (1.36) cannot be defined in the usual Lebesgue-Stieltjes sense.

Definition (1.20) (Measurability), [Doob, 1953]:

A stochastic process $\{X_t, t \in T\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with a time set T is a measurable process if for all Lebesgue measurable sets B belonging to the σ-field $\mathcal{L}(T)$ generated by Lebesgue measurable sets the mapping $(t, \omega) \longrightarrow X_t(\omega)$ is a measurable on $T \times \Omega$ with respect to the product σ-field $\mathcal{L}(t) \otimes \mathcal{F}$, that is:

$$\{(t, \omega): X_t(\omega) \in B\} \in \mathcal{L}(T) \otimes \mathcal{F} \qquad \ldots(1.37)$$

Theorem (1.1), [Doob, 1953]:

Let $\{X_t, t \in T\}$ be a measurable stochastic process with respect to the product σ-field $\mathcal{L}(t) \otimes \mathcal{F}$. Then:

1. Almost all sample function of this process are Lebesgue measurable function of $t \in T$.
2. If $EX_t(\omega)$ exists for all $t \in T$, then it also defines a Lebesgue measurable function of $t \in T$.
3. If A is a Lebesgue time set in T and if $\int_A E|X_t|\, dt < \infty$, then almost all sample functions $X_t(\omega)$ are Lebesgue integrable on the set A, that is:

$$\int_A |X_t(\omega)|\, dt < \infty, \text{ for almost all w}$$

Since the value of an absolutely convergent integral is independent of the order of integration, we have

$$\int_A EX_t(\omega)\, dt = E\int_A X_t(\omega)\, dt \qquad \ldots(1.38)$$

Definition (1.21) (Increasing σ-Field or Filtration σ-field), [Krishnan, 2006]:

Let$(\Omega, \mathcal{F})$ be a complete measurable space and let $\{\mathcal{F}_t, t \in T, T = \square^+\}$ be a family of sub- σ-field of $\mathcal{F}$ such that for $s \le t$, $\mathcal{F}_s \subset \mathcal{F}_t$. Then $\{\mathcal{F}_t\}$ is called an ***increasing family*** of sub- σ-field on $(\Omega, \mathcal{F})$ or the ***filtration*** σ-field of $(\Omega, \mathcal{F})$.

Remark (1.13):

$\mathcal{F}_t$ is called the σ-field of events prior to t. If $\{X_t, t \in T\}$ is a stochastic process defined on $(\Omega, \mathcal{F}, P)$ then clearly $\mathcal{F}_t$ given by:

$$\mathcal{F}_t = \sigma\{X_s, s \le t, t \in T\} \quad \ldots(1.39)$$

is increasing.

Definition (1.22) (Adaptation of $\{X_t\}$), [Krishnan, 1984]:

Let $\{X_t, t \in T, T = \square^+\}$ be a stochastic process defined on probability space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}_t, t \in T, T = \square^+\}$ be a filtration σ-field. The process $\{X_t\}$ is adapted to the family $\{\mathcal{F}_t\}$, if X_t is $\mathcal{F}_t$-measurable for every $t \in T$, or $E^{\mathcal{F}_t} X_t = X_t$

Remarks (1.14), [Krishnan, 2006]:

1. $E^{\mathcal{F}_t}$ represents the conditional expectation.
2. $\mathcal{F}_t$-adapted random processes are also $\mathcal{F}_t$-measurable and nonanticipative with respect to the σ-field $\mathcal{F}_t$.
3. If $\mathcal{F}_t$ is the σ-field by$\{X_s, s \le t\}$, then clearly the process $\{X_t, t \in T\}$ is adapted to the family $\{\mathcal{F}_t, t \in T\}$, which is called the ***natural family or natural filtration*** of the process $\{X_t\}$.

1.5 Approximation of Functions by Step Functions:

We shall call a stochastic process also a stochastic function.

Let w(t), $t \geq 0$ be Brownian motion on probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_t$ $(t \geq 0)$ be an increasing family of σ-fields, i.e., $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ if $t_1 < t_2$, such that $\mathcal{F}_t \subset \mathcal{F}$, $\mathcal{F}(w(s), 0 \leq s \leq t)$ is in $\mathcal{F}_t$, and $\mathcal{F}(w(\lambda + t) - w(t), \lambda \geq 0)$ is independent of $\mathcal{F}_t$, for all $t \geq 0$. One can take, for instance, $\mathcal{F}_t = \{\mathcal{F}w(s),\ 0 \leq s \leq t\}$. Let $0 \leq \alpha < \beta < \infty$. A stochastic process f(t) defined for $\alpha \leq t < \beta$ is called a nonanticipative function with respect to $\mathcal{F}_t$ if:

(i) f(t) is a separable process; (see definition separable process, [Krishnan, 1984]).

(ii) f(t) is a measurable process, i.e., the function $(t, \omega) \longrightarrow f(t, \omega)$ from $[\alpha, \beta] \times \Omega$ into $\mathbb{R}^1$ is a measurable; (as in definition (1.20)).

(iii) For each $t \in [\alpha, \beta]$, f(t) is $\mathcal{F}_t$ measurable.

Remarks (1.15) [Friedman, 1975]:

1. When (iii) holds we say that f(t) is ***adapted*** to $\mathcal{F}_t$ (see definition (1.22)).
2. Let us define $L^p_\omega[\alpha, \beta]$, $(1 \leq p \leq \infty)$ the class of all nonanticipative functions f(t) satisfying:

$$P\left\{\int_\alpha^\beta |f(t)|^p \ dt < \infty\right\} = 1 \qquad \ldots(1.40)$$

3. We denote by $M_{\omega}^{p}[\alpha, \beta]$ the subset of $L_{\omega}^{p}[\alpha, \beta]$ consisting of all functions f with:

$$E\int_{\alpha}^{\beta} |f(t)|^{p} \, dt < \infty \qquad ...(1.41)$$

Definition (1.23) (Step Function), [Evans, 2005], [Stirzaker, 2005]:

A stochastic process f(t) defined on $[\alpha, \beta]$ is called a step function if there exists a partition $\alpha = t_0 < t_1 < \ldots < t_r = \beta$ of $[\alpha, \beta]$, such that:

$$f(t) = f(t_i) \text{ if } t_i \le t < t_{i+1},\ 0 \le i \le r-1 \qquad \ldots(1.42)$$

Lemma (1.3), [Friedman, 1975]:

Let $f \in L_{\omega}^{2}[\alpha, \beta]$. Then:

(i) There exists a sequence of continuous functions g_n in $L_{\omega}^{2}[\alpha, \beta]$, such that:

$$\lim_{n\to\infty} \int_{\alpha}^{\beta} |f(t) - g_n(t)|^{2} \, dt = 0 \text{ a.s.} \qquad \ldots(1.43)$$

(ii) There exists a sequence of step functions f_n in $L_{\omega}^{2}[\alpha, \beta]$, such that:

$$\lim_{n\to\infty} \int_{\alpha}^{\beta} |f(t) - f_n(t)|^{2} \, dt = 0 \text{ a.s.} \qquad \ldots(1.44)$$

Lemma (1.4), [Friedman, 1975]:

Let $f \in M_{\omega}^{2}[\alpha, \beta]$. Then:

(i) There exists a sequence of continuous functions K_n in $M_\omega^2[\alpha, \beta]$, such that:

$$E\int_\alpha^\beta |f(t)-K_n(t)|^2 \ dt \longrightarrow 0 \qquad \ldots(1.45)$$

if $n \longrightarrow \infty$.

(ii) There exists a sequence of bounded step functions l_n in $M_\omega^2[\alpha, \beta]$, such that:

$$E\int_\alpha^\beta |f(t)-l_n(t)|^2 \ dt \longrightarrow 0 \qquad \ldots(1.46)$$

if $n \longrightarrow \infty$.

Remark (1.16):

The following stochastic integral:

$$\int_0^T W \, dW$$

where W(.) is a 1-dimensional Brownian motion. A reasonable procedure is to construct a Riemann sum approximation, and then–if possible–to pass to limits.

The following definitions are concerning:

Definitions (1.24), [Evans, 2005]:

(i) If [0,T] is an interval, a ***partition K*** of [0,T] is a finite collection of points in [0, T]:

$$K := \{0 = t_0 < t_1 < \ldots < t_m = T\}$$

(ii) Let the ***mesh size*** of K be $|K| := \max_{0 \le k \le m-1} |t_{k+1} - t_k|$

(iii) For fixed $0 \le \lambda \le 1$ and K a given partition of [0, T], set:

$$\tau_k = (1 - \lambda)t_k + \lambda t_{k+1},\ k = 0, 2, \ldots, m - 1$$

For such a partition K and for $0 \le \lambda \le 1$, we define:

$$R = R(K, \lambda) := \sum_{k=0}^{m-1} W(\tau_k)(W(t_{k+1}) - W(t_k))$$

This is the corresponding Riemann sum approximation of $\int_0^T WdW$.

Lemma (1.5) (Quadratic Variation), [Øksendal, 1998]:

Let $[\alpha, \beta]$ be an interval in $[0, \infty)$, and suppose that:

$$P^n := \{\alpha = t_0 < t_1 < \ldots < t_m = \beta\}$$

be a partitions of $[\alpha, \beta]$, with $|P^n| \longrightarrow 0$ as $n \longrightarrow \infty$. Then:

$$\sum_{k=0}^{m_n - 1} (W(t_{k+1}^n) - W(t_k^n)) \longrightarrow \beta - \alpha \qquad \ldots(1.47)$$

In $L_\omega^2[\alpha, \beta]$ as $n \longrightarrow \infty$.

Definition (1.25), [Friedman, 1975]:

Let f(t) be a step function in $L_\omega^2[\alpha, \beta]$, say $f(t) = f_i$ if $t_i \le t < t_{i+1}$, 0

$0 \le i \le r - 1$, where $\{\alpha = t_0 < t_1 < \ldots < t_r = \beta\}$, the random variable:

$$\sum_{k=0}^{r-1} f(t_k)[W(t_{k+1}) - W(t_k)] \qquad \ldots(1.48)$$

where $\max|t_{k+1} - t_k| \longrightarrow 0$, $0 \le k \le r-1$; is denoted by:

$$\int_{\alpha}^{\beta} f(t)\, dw(t) \qquad \ldots(1.49)$$

and is called the ***stochastic integral*** of f with respect to the Brownian motion w; it is also called the ***Itô integral***.

Lemma (1.6), [Evans, 2005]:

We have for all constants $a, b \in \square$ and for all step processes G, $H \in L^2(0, T)$, G in $M_{\omega}^2[\alpha, \beta]$

(i) $\int_0^T (aG + bH)\, dW = a\int_0^T G\, dW + b\int_0^T H\, dW.$

(ii) $E\left(\int_0^T G\, dW\right) = 0.$

(iii) $E\left(\left(\int_0^T G\, dW\right)^2\right) = E\left(\int_0^T G^2\, dW\right)$

Lemma (1.7), [Friedman, 1975]:

If f is a step function in $M_{\omega}^2[\alpha, \beta]$, then:

$$E\int_{\alpha}^{\beta} f(t)\, dw(t) = 0 \qquad \ldots(1.50)$$

$$E\left|\int_{\alpha}^{\beta} f(t)\, dw(t)\right|^2 = E\int_{\alpha}^{\beta} f^2\, dt \qquad \ldots(1.51)$$

Lemma (1.8), [Friedman, 1975]:

Let f, g belong to $L^2_\omega[\alpha, \beta]$, and assume that $f(t) = g(t)$ for all $\alpha \le t \le \beta$, $\omega \in \Omega$. Then:

$$\int_\alpha^\beta f(t)\, dw(t) = \int_\alpha^\beta g(t)\, dw(t), \text{ for a.a. } \omega \in \Omega \qquad \ldots(1.52)$$

Remarks (1.17), [Øksendal, 1998]:

1. Let $f \in L^2_\omega[\alpha, \beta]$ and consider the integral:

$$I(t) = \int_0^t f(s)\, dw(s),\ 0 \le t \le T \qquad \ldots(1.53)$$

2. By definition, $\int_0^0 f(s)\, dw(s) = 0$, and we refer to I(t) as the indefinite integral of f. Notice that I(t) is $\mathcal{F}_t$ measurable.

   If f is a step function, then clearly:

$$\int_\alpha^\beta f(s)\, dw(s) + \int_\beta^\gamma f(s)\, dw(s) = \int_\alpha^\gamma f(s)\, dw(s) \qquad \ldots(1.54)$$

   if $0 \le \alpha < \beta < \gamma \le T$.

   By approximation we find that (1.54) holds for any f in $L^2_\omega[0, T]$.

1.6 Itô Formula:

Definition (1.26), [Evans, 2005]:

Let X(t) ($0 \le t \le T$) be a stochastic process such that for any $0 \le t_1 < t_2 \le T$

$$X(t_2) - X(t_1) = \int_{t_1}^{t_2} a(t)\,dt + \int_{t_1}^{t_2} b(t)\,dw(t)$$

where $a \in L^1_\omega[0, T]$, $b \in L^2_\omega[0, T]$. Then we say that X(t) has stochastic differential dX, on [0, T], given by:

$$dX(t) = a(t)dt + b(t)dw(t)$$

Observe that X(t) is a nonanticipative function. It is also a continuous process. Hence, in particular, it belongs to $L^\infty_\omega[0, T]$.

Definition (1.27), [Friedman, 1975]:

Let X(t) be as in definition (1.26) and let f(t) be a function in $L^\infty_\omega[0, T]$. We define:

$$f(t)dX(t) = f(t)a(t)dt + f(t)b(t)dw(t).$$

Theorem (1.2), [Friedman, 1975], [Øksendal, 1998]:

Let $d\xi(t) = adt + bdw(t)$, and let f(x, t) be a continuous function in $(x, t) \in \mathbb{R}^1 \times [0, \infty)$ together with its partial derivatives f_x, f_{xx}, f_t. Then the process $f(\xi(t), t)$ has a stochastic differential, given by:

$$df(\xi(t), t) = [f_t(\xi(t), t) + f_x(\xi(t), t)a(t) + \frac{1}{2} f_{xx}(\xi(t), t)b^2(t)]dt + f_x(\xi(t), t)b(t)dw(t) \quad \ldots(1.55)$$

This is called the Itô formula. Notice that if w(t) were continuously differentiable in t, then (by the standard calculus formula for total derivatives) the term $\frac{1}{2}f_{xx}b^2dt$ will not appear.

1.7 Existence and Uniqueness Solution of Stochastic Differential Equations, [Øksendal, 1998], [Evans, 2005]:

If $\sigma = (\sigma_{ij})$ is a matrix, we write $|\sigma|^2 = \sum_{i,j} |\sigma_{ij}|^2$.

Let $b(x, t) = (b_1(x, t), b_2(x, t), \ldots, b_n(x, t))$, $\sigma(x, t = (\alpha(x, t))_{i,j=1}^{n}$ and suppose the functions $b_i(x, t)$, $\sigma_{ij}(x, t)$ are measurable in $(x,t)\in \mathbb{R}^n\times[0,T]$. If $\xi(t)$ $(0 \le t \le T)$ is a stochastic process, such that:

$$d\xi(t) = b(\xi(t), t) + \sigma(\xi(t), t)dw(t) \qquad \ldots(1.56)$$

$$\xi(0) = \xi_0 \qquad \ldots(1.57)$$

Then we say that $\xi(t)$ satisfies the system of stochastic differential equations (1.56) and the initial condition (1.57). Note that it is implicitly assumed that $b(\xi(t), t)$ belongs to $L^1_\omega[0, T]$ and $\sigma(\xi(t), t)$ belongs to $L^2_\omega[0, T]$.

Theorem (1.3), [Friedman, 1975], [Øksendal, 1998], [Evans, 2005]:

Suppose $b(x, t)$, $\sigma(x, t)$ are measurable in $(x, t) \in \mathbb{R}^n\times[0, T]$ and

$$|b(x, t) - b(\tilde{x}, t)| \le K_\theta|x - \tilde{x}|, |\sigma(x, t) - \sigma(\tilde{x}, t)| \le K_\theta|x - \tilde{x}|$$

$$|b(x, t)| \le K(1 + |x|), |\sigma(x, t)| \le K(1 + |x|) \qquad \ldots(1.58)$$

where K_θ, K are constants. Let ξ_0 be any n-dimensional random vector independent of $\mathcal{F}(w(t), 0 \le t \le T)$, such that $E|\xi_0|^2 < \infty$. Then there exists a unique solution of (1.56) and (1.57) in $M_\omega^2[0, T]$.

The assertion of uniqueness means that if $\xi_1(t)$, $\xi_2(t)$ are two solutions of (1.56), (1.57) and if they belong to $M_\omega^2[0, T]$, then:

$$P\{\xi_1(t) = \xi_2(t) \text{ for all } 0 \le t \le T\} = 1$$

Theorem (1.4) (Stronger Uniqueness and Existence Theorem):

Suppose $b_i(x, t)$, $\sigma_i(x, t)$ are measurable functions in $(x, t) \in \mathbb{R}^n \times [0, T]$, for i = 1, 2, satisfying:

$$|b_i(x, t) - b_i(\tilde{x}, t)| \le K_\theta|x - \tilde{x}|, \ |\sigma_i(x, t) - \sigma_i(\tilde{x}, t)| \le K_\theta|x - \tilde{x}|$$

$$|b_i(x, t)| \le K(1 + |x|), \ |\sigma_i(x, t)| \le K(1 + |x|)$$

Let D be a domain in $\mathbb{R}^n$ and suppose that:

$$b_1(x, t) = b_2(x, t)$$

$$\sigma_1(x, t) = \sigma_2(x, t) \qquad \ldots(1.59)$$

If $x \in D$, $0 \le t \le T$.

Let $\xi_i(t)$ (i = 1, 2) be the solution of:

$$d\xi(t) = b_i(\xi_i(t), t) + \sigma_i(\xi_i(t), t), \ \xi_i(0) = \xi_{i0}$$

in $M_\omega^2[0, T]$ (with the same family of σ-fields $\mathcal{F}_t$) where $E|\xi_{i0}|^2 < \infty$. Assume finally that $\xi_{10} = \xi_{20}$ for a.a.ω. for which either $\xi_{10} \in D$ or

$\xi_{20} \in$ D. Denote by τ_i the first time $\xi_i(t)$ intersects $\mathbb{R}^n$/D if such time $t \leq$ T exists, and $\tau_i = T$, otherwise. Then:

$$P(\tau_1 = \tau_2) = 1$$

$$P\{\sup_{0 \leq t \leq \tau_1} |\xi_1(s) - \xi_2(s)| = 0\} = 1.$$

Thus if two stochastic equation have the same coefficients in a cylinder Q = D×[0, T] and if the initial condition coincide in D, then the corresponding solution agree until the first time they both leave D; they first leave D at the same time.

Remarks (1.18), [Friedman, 1975]:

1. This is local uniqueness theorem.

2. It remains true for the general domains Q.

Chapter Two
Euler-Maruyama Numerical Method for Solving Stochastic Differential Equations

This chapter deals with the some numerical methods, which have been programmed in Matlab. The strong and weak convergence criteria lead to different discrete time approximation, which are only efficient with respect to one of these two criteria. It is therefore important to clarify the aim of the simulation before choosing an appropriate scheme, deciding on whether a good pathwise approximation of the process is required or whether an approximation of some functional of the Itô process is the real objective.

Initially, Brownian motion is generated. Then Euler-Maruyama numerical scheme, which approximate the solution of the stochastic differential equation, are compared to the exact solution of a linear SDE. This can be easily accomplished since we have a closed solution for the linear SDE. Numerical estimates are provided for the strong convergence schemes of well known estimates for the absolute error using the absolute criterion.

2.1 Vector SDE's, [Evans, 2005]:

We shall interpret a vector as a column vector and its transpose as a row vector and consider an m-dimensional Wiener process $W = \{W_t, t \geq 0\}$, with components W_t^1, W_t^2, …, W_t^m, which are independent

scalar Wiener process. Then, we take a k-dimensional vector values function $a : [t_0, T]\times\mathbb{R}^k \longrightarrow \mathbb{R}^k$, the drift coefficients, and a k×m-matrix valued function $b : [t_0, T]\times\mathbb{R}^k \longrightarrow \mathbb{R}^{k\times m}$, the diffusion coefficient, $t_0 \in [0, T]$, to form a k-dimensional vector stochastic differential equation:

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, dW_t \qquad \ldots(2.1)$$

we interpret this as a stochastic integral equation:

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)\, ds + \int_{t_0}^{t} b(s, X_s)\, dW_s \qquad \ldots(2.2)$$

with initial value $X_{t_0} \in \mathbb{R}^k$, where the Lebsegue and Itô integrals determined component by component, with the component of (2.2) being:

$$X_t^i = X_{t_0}^i + \int_{t_0}^{t} a^i(s, X_s)\, ds + \sum_{j=1}^{m} \int_{t_0}^{t} b^{i,j}(s, X_s)\, dW_s^j$$

If the drift and diffusion coefficients do not depend on the time variable, that is if $a(t, x) \equiv b(x)$, then we say that the stochastic equation is autonomous. We can always write a nonautonomous equation as a vector autonomous equation of one dimension more by setting in the drift component the component of X_t the time variable $W_t^{\ell} = t$.

There is a vector version of the Itô formula. For a sufficiently smooth transformation $f = [t_0, T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^k$ of the solution $X = \{X_t, t_0 \le t \le T\}$ of (2.1), we obtain a k-dimensional process $Y = \{Y_t = f(t, X_t), t_0 \le t \le T\}$ with the vector stochastic differential in component form:

$$dY_t^p = \left(\frac{\partial f^p}{\partial t} + \sum_{i=1}^{d} a^i \frac{\partial f^p}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{\ell=1}^{m} b^{i,\ell} b^{j,\ell} \frac{\partial^2 f^p}{\partial x_i \partial x_j} \right) dt + \sum_{\ell=1}^{m} \sum_{i=1}^{d} b^{i,\ell} \frac{\partial f^p}{\partial x_i} dW_t^{\ell}$$

for $p = 1, 2, \ldots, k$; where the terms are all evaluated at (t, X_t). We can sometimes use this formula to determine the solutions of certain vector stochastic differential equations in terms of known solutions of the other equations, for example linear equations.

2.2 Generating Brownian Motion in Matlab:

The underlying difference between deterministic and probabilistic differential equations is the need to generate the following random increments of the Brownian motion for the SDE:

$$\Delta W_n = W_{t_n} - W_{t_{n-1}} \qquad \ldots(2.3)$$

For computational purpose, it is necessary to describe the Brownian motion, where W_t is specified at discrete t values. Therefore, let $\Delta t = T/N$, for some positive integer N and for T on the interval [0, T]. From the definition of Brownian motion:

$$\Delta W_n = W_{t_n} - W_{t_{n-1}} \sim N(0, t_n - t_{n-1})$$

or equivalently:

$$\Delta W_n = W_{t_n} - W_{t_{n-1}} \sim \sqrt{t_n - t_{n-1}}\, N(0, 1)$$

where N(0, 1) denotes a standard normally distributed random variable with zero mean and variance equal to one. Here $t_n - t_{n-1} = \Delta t$ is the

variance of the Brownian motion random variable. In Matlab, the function **randn (1, N)** will generate N random variables from the standard normal distribution. In order to generate a random variable with variance equal to Δt, random variables from the standard normal distribution are generated using the Matlab function **randn (1, N)** and each of these variables are then multiplied by $\sqrt{\Delta t}$, resulting in the random increments in equation (2.3). From equation (2.3):

$$\Delta W_1 = W_{t_1} - W_{t_0}$$

$$\Delta W_2 = W_{t_2} - W_{t_1}$$

Implying that:

$$\Delta W_1 + \Delta W_2 = W_{t_2} - W_{t_1} + W_{t_1} - W_{t_0}$$

$$= W_{t_2}$$

and since $t_0 = 0$ and $W_0 = 0$, therefore:

$$W_{t_n} = \sum_{j=1}^{n} \Delta W_j$$

For more details, see the computational algorithm for generating Brownian motion supported by Matlab.

2.3 Stochastic Taylor Expansion, [Kloeden & Platen, 1992]:

We consider the equation $X = \{X_t, t \in [t_0, T]\}$ of one-dimensional stochastic ordinary differential equation:

$$\frac{d}{dt} X_t = a(X_t)$$

with initial value X_{t_0}, for $t \in [t_0, T]$, where $0 \leq t_0 < T$, which we can write in the equivalent integral equation form as:

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)\, ds \qquad \ldots(2.4)$$

To justify the following constructions we require that the function a satisfies appropriate properties, for instance to be sufficiently smooth with a linear growth bound. Let $f : \square \longrightarrow \square$ be continuously differentiable function. Then by the chain rule, we have:

$$\frac{d}{dt} f(X_t) = a(X_t) f(X_t) \qquad \ldots(2.5)$$

Which using the operator:

$$Lf = af$$

we can express (2.5) as the integral relation:

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} Lf(X_s)\, ds \qquad \ldots(2.6)$$

for all $t \in [t_0, T]$. When $f(x) = x$, we have $Lf = a$, $L^2 f = La$, …, and (2.6) reduces to:

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)\, ds \qquad \ldots(2.7)$$

that is, to equation (2.4). If we apply the relation (2.6) to the function $f = a$ in the integral in (2.7), we obtain:

$$X_t = X_{t_0} + \int_{t_0}^{t} \left(a(X_{t_0}) + \int_{t_0}^{s} La(X_z)\,dz \right) ds$$

$$= X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + \int_{t_0}^{t} \int_{t_0}^{s} La(X_z)\,dz\,ds \qquad \ldots(2.8)$$

Which is the simplest nontrivial Taylor expansion for X_t. We can apply (2.6) again to the function f = La in the double integral of (2.8) to derive:

$$X_t = X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + La(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dz\,ds + R_3$$

where:

$$R_3 = \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^2 a(X_u)\,du\,dz\,ds$$

for $t \in [t_0, T]$. For a general r + 1 times continuously differentiable function $f : \square \longrightarrow \square$, this method gives the classical Taylor formula in integral form:

$$f(X_t) = f(X_0) + \sum_{\ell=1}^{r} \frac{(t-t_0)^{\ell}}{\ell!} L^{\ell} f(X_{t_0}) + \int_{t_0}^{t} \ldots \int_{t_0}^{s_2} L^{r+1} f(X_{s_1})\,ds_1 \qquad \ldots(2.9)$$

for $t \in [t_0, T]$ and r = 1, 2, ..; since:

$$\int_{t_0}^{t} \int_{t_0}^{s_1} \ldots \int_{t_0}^{s_{\ell-1}} ds_1 \ldots ds_{\ell} = \frac{1}{\ell!}(t-t_0)^{\ell}$$

for $\ell = 1, 2, \ldots$. The Taylor formula (2.9) has proven to be a very useful tool in both theoretical and practical investigations, particularly in numerical analysis. It follows the approximation of a sufficiently smooth

function in a neighborhood of a given point to any desired order of accuracy. The expansion depends on the values of the function and some of its higher derivatives at the expansion point, weighted by corresponding multiple time integrals. In addition, there is a remainder term which contains the next following multiple time integral, but with a time dependent integrand.

A stochastic counterpart of the deterministic Taylor formula for the expansion of smooth functions of an Itô process about a given value has many potential.

As with the deterministic Taylor expansion, the stochastic Taylor formula for the expansion of smooth functions of an Itô process is used to construct numerical methods for stochastic differential equations. The stochastic Taylor formula, which is called the Itô-Taylor expansion is derived by repeatedly applying the Itô formula (1.55). For any twice continuously differentiable function f : □ ⟶ □ , apply the Itô formula to obtain:

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} \left(a(X_1)f'(X_1) + \frac{1}{2}b^2(X_1)f''(X_1) \right) ds +$$

$$\int_{t_0}^{t} b(X_1)f'(X_1)\,dX_1$$

Introduce the following operators:

$$L^0 f = af' + \frac{1}{2}b^2 f''$$

$$L^1 f = bf' \qquad \ldots(2.10)$$

To obtain:

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_1)\, ds + \int_{t_0}^{t} L^1 f(X_1)\, dW_1 \qquad \ldots(2.11)$$

for any $t \in [t_0, T]$. If $f(x) = x$, then $L^0 f = a$ and $L^1 f = b$. Thus, the above is just the original Itô equation for X_1:

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)\, ds + \int_{t_0}^{t} b(X_s)\, dW_s \qquad \ldots(2.12)$$

If formula (2.11) is again applied to the functions f = a and b in equation (2.12), the following is obtained:

$$X_t = X_{t_0} + \int_{t_0}^{t} \left(a(X_{t_0}) + \int_{t_0}^{s} L^0 a(X_z)\, dz + \int_{t_0}^{s} L^0 a(X_z)\, dW_z \right) ds +$$

$$\int_{t_0}^{t} \left(b(X_{t_0}) + \int_{t_0}^{s} L^0 b(X_z)\, dz + \int_{t_0}^{s} L^0 b(X_z)\, dW_z \right) dW_z$$

$$= X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_z + R \qquad \ldots(2.13)$$

Where:

$$R = \int_{t_0}^{t} \int_{t_0}^{z} L^0 a(X_z)\, dz ds + \int_{t_0}^{t} \int_{t_0}^{z} L^1 a(X_z)\, dW_z ds +$$

$$\int_{t_0}^{t} \int_{t_0}^{z} L^0 b(X_z)\, dz dW_s + \int_{t_0}^{t} \int_{t_0}^{z} L^0 b(X_z)\, dW_z dW_s$$

Repeat this procedure by applying the formula (2.11) to $f = L^1 f$ in equation (2.13) to obtain:

$$X_t = X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_z +$$

$$L^1 b(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{z} dW_z dW_s + \bar{R} \qquad \ldots(2.14)$$

Where:

$$\bar{R} = \int_{t_0}^{t} \int_{t_0}^{s} L^0 a(X_z)\, dz ds + \int_{t_0}^{t} \int_{t_0}^{s} L^1 a(X_z)\, dW_z ds +$$

$$\int_{t_0}^{t} \int_{t_0}^{s} L^0 b(X_z)\, dz dW_s + \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^0 L^1 b(X_u)\, du dW_z dW_s +$$

$$\int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^1 L^1 b(X_u)\, dW_u dW_z dW_s$$

The Itô-Taylor expansion can thus be considered as a generalization of both the Itô formula and the deterministic Taylor formula.

2.4 Euler-Maruyama Method, [Evans, 2005]:

The Euler-Maruyama method applied to an SDE is similar to the Euler method used to solve an ordinary differential equation. Consider the following scalar SDE:

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

or in integral form:

$$X_t(\omega) = X_{t_0}(\omega) + \int_{t_0}^{t} a(s, X_s(\omega))\, ds + \int_{t_0}^{t} b(s, X_s(\omega))\, dW_s(\omega) \qquad \ldots(2.15)$$

where $X = \{X_t : t_0 \le t \le T\}$ is an Itô process with initial value $X_{t_0} = X_0$. Subdivide the interval $[t_0, T]$ into N-subintervals according to the following discretisation:

$$t_0 = \tau_0 < \tau_1 < \ldots < \tau_n < \ldots < \tau_N = T$$

The Euler approximation is defined as a continuous time stochastic process $Y = \{Y(T), t_0 \le t \le T\}$ satisfying the iterative scheme:

$$Y_{n+1} = Y(\tau_n) + a(\tau_n, Y(\tau_n))(\tau_{n+1} - \tau_n) + b(\tau_n, Y(\tau_n))(X_{t_{n+1}} - X_{t_n}) \quad \ldots(2.16)$$

for $n = 0, 1, \ldots, N - 1$; with initial value $Y_0 = X_0$. The Euler scheme is obtained by considering the first three terms of the Itô-Taylor expansion (2.14):

$$X_t = X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_z + L^1 b(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{z} dW_z dW_s + \bar{R} \quad \ldots(2.17)$$

where $\bar{R}$ is the remainder and is defined in equation (2.14). Equation (2.17) is the Itô-Taylor expansion of $X_t(\omega)$ in equation (2.15). The Itô-Taylor expansion is useful in approximating a sufficiently smooth function in a neighborhood of a given point to a desired order of accuracy. Thus, considering the first three terms of equation (2.17) provides the Euler scheme in (2.16) where each term in the right hand side of equation (2.16) approximates the corresponding term on the right hand side of equation (2.15). For brevity, equation (2.16) is written as:

$$Y_{n+1} = Y_n + a\Delta n + b\Delta W_n$$

where:

$$\Delta n = \tau_{n+1} - \tau_n = \int_{\tau_n}^{\tau_{n+1}} ds$$

$$\Delta W_n = X_{t_{n+1}} - X_{t_n} = \int_{\tau_n}^{\tau_{n+1}} dW_s$$

$$a = a(\tau_n, Y(\tau_n))$$

$$b = b(\tau_n, Y(\tau_n))$$

$$Y_n = Y(\tau_n)$$

The Euler scheme for a deterministic ordinary differential equations is obtained if b = 0 in equation (2.16). Thus, the main difference between the Euler scheme for deterministic ordinary differential equations and the Euler-Maruyama scheme for SDE's is the following random increments need to be generated for the SDE:

$$\Delta W_n = X_{t_{n+1}} - X_{t_n}$$

for n = 0, 1, …, N – 1; of the Wiener process $W = \{W_t, t \geq 0\}$, as defined in, [Evans, 2005].

The Euler scheme determines values of the approximating process at the discretisation times only. The values at the intermediate instances can be calculated by using either the piecewise constant interpolation method or the linear interpolation method. An overview method is provided in [Kloeden & Platen, 1994].

The Euler scheme is an example of a time discrete approximation (or difference method) in which the continuous time differential

equation is replaced by a discrete time difference equation generating values Y_1, Y_2, …, Y_n to approximate X_{t_1}, X_{t_2}, …, X_{t_n} at given discretisation times $t_0 < t_1 < \ldots < t_n$. The Euler scheme is the simplest strong Taylor approximation and attains an order of convergence $\gamma = 0.5$. The proof is given in [Kloeden & Platen, 1992].

2.5 Convergence Criteria:

There are five commonly used concepts for the convergence of random sequences, [Kloeden & Platen, 1992]. These are:

(i) Convergence with probability one:

$$P\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \left|X_n(\omega) - X(\omega)\right| = 0\right\}\right) = 1$$

(ii) Mean-square convergence:

$E(X_n^2) < \infty$, for $n = 1, 2, \ldots$

$E(X^2) < \infty$, and

$$\lim_{n\to\infty} E(|X_n - X|^2) = 0$$

(iii) Convergence in probability:

$$\lim_{n\to\infty} P(\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\}) = 0, \text{ for all } \varepsilon > 0$$

(iv) Convergence in distribution:

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

for all continuity points of F_X.

(v) Weak convergence:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\, dF_{X_n}(x) = \int_{-\infty}^{\infty} f(x)\, dF_X(x)$$

for all test functions $f : \square \longrightarrow \square$.

In these definitions, all random variables are defined on a common probability space (Ω, F, P).

Convergence of random sequences is classified into two classes, namely, strong convergence and weak convergence. Convergence with probability one, mean square convergence and convergence in probability are the most commonly used convergence in the strong class while convergence in distribution and weak convergence are classified from the weak class. For the weak class, only the distribution function is required and not the actual random variables of the underlying probability space.

Since many SDE's cannot be solved explicitly, numerical schemes are employed. There are various numerical schemes (see [Kloeden & Platen, 1992]) and in order to access their usefulness and practicality, certain criteria are required in which to access the various schemes. The convergence criterion is just one of many other criteria, like mean square stability, asymptotic stability and cost of computation, which can be used when assessing the usefulness of different numerical schemes.

This work uses the strong and weak convergence criteria defined below.

2.5.1 Strong Convergence Criterion:

In many practical areas, like direct simulations, filtering or testing statistical estimators, a good pathwise approximation is usually required and for these instances, the absolute error criterion is appropriate. The criterion gives a measure of pathwise closeness at the end of the time interval [0, T], [Kloeden & Platen, 1992].

Consider a practical sample path of the Wiener process, i.e., W_T is given (and hence known) therefore there is no randomness in the SDE and hence no randomness in X_T [Cao & Pope, 2003]. The increments in the given Wiener process are then used to obtain the numerical approximation Y(T). The absolute error criterion is defined as:

$$\varepsilon = E(|X_T - Y(T)|)$$

Here, the Euclidean norm is use, X_T is the Itô process at time T while Y(T) is the approximation obtained by approximately integrating the SDE in a sequence of time steps, i.e., from the numerical scheme. Therefore, the error is the expectation of the absolute value of the difference between the approximation Y(T) and the Itô process X_T at time T.

The numerical scheme is consistent if the approximation Y(T) converges to X_T at Δt tends to zero. Therefore, a discrete time approximation Y(T) with maximum time step size δ converges strongly to X at time T if [Kloeden & Platen, 1992]:

$$\lim_{\delta \to 0} E(|X_T - Y(T)|) = 0 \qquad \ldots(2.18)$$

There are various discrete time approximations that can be derived from the Itô-Taylor expansion and in order to compare different discrete time approximations, the order of convergence of the numerical scheme is used.

A discrete time approximation Y^p converges strongly with order $p > 0$ at time T if there exists a positive constant C, which does not depend on δ, and a $\delta_0 > 0$, such that:

$$\varepsilon(\delta) = E(|X_T - Y(T)|) \leq C\Delta^p$$

for each $\delta \in (0, \delta_0)$. Thus, the numerical scheme is strong p^{th} order accurate if the error is of order Δt^p, [Cao & Pope, 2003].

2.5.2 Weak Convergence Criterion:

In practical problems, approximating some functional of the Itô process is of interest, such as the probability distribution, its mean and variance. Thus, the weak convergence criterion is used since the requirements for their simulation are not as demanding as for pathwise approximations, [Kloeden & Platen, 1992]. Here the sample path W_T is not known but is drawn from the distribution of Wiener processes.

Since W_T is a random variable, X_T is a random variable. The numerical approximation Y(T) is also a random variable because Y(T) is obtained using samples of Wiener-process increments. The convergence in distribution is analyzed in terms of means g(X(T)) of test functions g(x), [Cao & Pope, 2003].

The test functions g(x) are bounded, infinitely differentiable and the means exist as |x| tends to infinity. The numerical scheme is weak p^{th} order accurate if the error:

$$\varepsilon = |E(g(X(T))) - E(g(Y(T)))|$$

is of order Δt^p. Thus:

$$|E(g(X(T))) - E(g(Y(T)))| \leq C\Delta t^p$$

A general discrete time approximation Y with maximum time step size δ converges weakly to X at time T as $\delta \longrightarrow 0$ with respect to a class C of test functions $g : \mathbb{R}^d \longrightarrow \mathbb{R}$, if we have:

$$\lim_{\delta \to 0} |E(g(X_T)) - E(g(T)))| = 0, \text{ for } g \in C \qquad \ldots(2.19)$$

A time discrete approximation Y converges weakly with order $\beta > 0$ to X at time T as $\delta \longrightarrow 0$, if for each polynomial g, there exists a positive constant C, which does not depend on δ, and a finite number δ_0, such that:

$$|E(g(X_T)) - E(g(T)))| \leq C\Delta t^{\beta}, \text{ for each } \delta \in (0, \delta_0)$$

Whereas, the strong convergence criterion gives the measure of the closeness of the positive approximation to the Itô process, the weak convergence criterion gives an approximation of the probability distribution of X_T. [Carletti, 2006], states that:

(The strange of convergence measures the rate at which the "mean of the error" decays as $\Delta t \longrightarrow 0$. The weak order of convergence measures the rate of decay of the "error of the means", as $\Delta t \longrightarrow 0$).

Theorem (2.1), [Evans, 2005]:

Suppose that n = 1, but m ≥ 1 is arbitrary, then the solution of:

$$dx = (c(t) + d(t)x)dt + \sum_{\ell=1}^{m}(e^{\ell}(t) + f^{\ell}(t)x)dw^{\ell}\ ,\ x(0) = 0$$

is:

$$x(t) = \Phi(t)\left(x_0 + \int_0^t \Phi^{-1}(s)\left(c(s) - \sum_{\ell=1}^{m} e^{\ell}(s)f^{\ell}(s)\right)ds \right) + \sum_{\ell=1}^{m} \Phi^{-1}(s)e^{\ell}(s)dw^{\ell}$$

where:

$$\Phi(t) = \exp\left(\int_0^t d - \sum_{\ell=1}^{m} \frac{(f^{\ell})^2}{2}ds + \int_0^t \sum_{\ell=1}^{m} f^{\ell}dw^{\ell} \right)$$

2.6 Examples of Linear Stochastic Differential Equations:

The following examples with discussions are needed later on:

Example (2.1), [Evans, 2005]:

Let m = n = 1 and suppose g is a continuous function (not a random variable). Then the solution of:

$$\left.\begin{aligned} dX &= gXdW \\ X(0) &= 1 \end{aligned}\right\} \qquad \ldots(2.20)$$

is:

$$X(t) = e^{-\frac{1}{2}\int_0^t g^2\,ds + \int_0^t g\,dW}$$

for $0 \le t \le T$. To verify this, note that:

$$Y(t) = -\frac{1}{2}\int_0^t g^2\,ds + \int_0^t g\,dW$$

satisfies:

$$dY = -\frac{1}{2}\, g^2\, dt + g\, dW$$

Thus, Itô's lemma for $u(x) = e^x$, gives:

$$dX = \frac{\partial u}{\partial x} dY + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} g^2 dt$$

$$= e^Y\left(-\frac{1}{2}g^2 dt + g dW + \frac{1}{2}g^2 dt\right) = gX\, dW, \text{ as claimed}$$

We will prove uniqueness later.

Example (2.2), [Evans, 2005]:

Similarly, the unique solution of:

$$\left.\begin{aligned} dX &= fX dt + gX dW \\ X(0) &= 1 \end{aligned}\right\} \qquad \ldots(2.21)$$

is:

$$X(t) = e^{\int_0^t f - \frac{1}{2}g^2 dt + \int_0^t g dW}, \text{ for } 0 \le t \le T.$$

Example (2.3) (Stock Prices), [Evans, 2005]:

Let P(t) denote the price of a stock at time t. We can model the evolution of P(t) in time by supposing that $\frac{dP}{P}$, the relative change of price, evolves according to the SDE:

$$\frac{dP}{P} = \mu dt + \sigma dW$$

for certain constants $\mu > 0$ and σ, called the drift and the volatility of the stock. Hence:

$$dP = \mu P dt + \sigma P dW \qquad \ldots(2.22)$$

and so:

$$d(\log(P)) = \frac{dP}{P} - \frac{1}{2}\frac{\sigma^2 P^2 dt}{P^2}, \text{ by Itô formula}$$

$$= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW$$

Consequently:

$$P(t) = p_0 e^{\sigma W(t) + \left(\mu - \frac{\sigma^2}{2}\right)t}$$

similarly to example (2.2). Observe that the price is always positive, assuming the initial price p_0 is positive.

Since (2.22) implies:

$$P(t) = p_0 + \int_0^t \mu P ds + \int_0^t \sigma P dW$$

and $E\left(\int_0^t \sigma P\, dW\right) = 0$, we see that:

$$E(P(t)) = p_0 + \int_0^t \mu E(P(s))\, ds$$

Hence:

$$E(P(t)) = p_0 e^{\mu t}, \text{ for } t \geq 0$$

The expected value of the stock price consequently agrees with the deterministic solution of (2.22) corresponding to $\sigma = 0$.

Example (2.4) (Langevin's Equation) [Evans, 2005]:

A possible improvement of our mathematical model of the motion of the Brownian particle models frictional forces as follows for the one-dimensional case:

$$\frac{dX}{dt} = -bX + \sigma\xi$$

where $\xi(.)$ is "white noise", $b > 0$ is a coefficient of friction, and σ is a diffusion coefficient. In this interpretation $X(.)$ is the velocity of the Brownian particle. We interpret this to mean:

$$\left.\begin{aligned} dX &= -bXdt + \sigma dW \\ X(0) &= X_0 \end{aligned}\right\} \qquad \ldots(2.23)$$

for some initial distribution X_0, independent of the Brownian motion. This is the Langevin equation. The solution is:

$$X(t) = e^{-bt}X_0 + \sigma\int_0^t e^{-b(t-s)}\, dW,\ t \geq 0$$

as is straightforward to verify. Observe that:

$$E(X(t)) = e^{-bt}E(X_0)$$

and

$$E(X^2(t)) = E\left(e^{-2bt}X_0^2 + 2\sigma e^{-bt}X_0\int_0^t e^{-b(t-s)}dW + \sigma^2\left(\int_0^t e^{-b(t-s)}dW\right)^2\right)$$

$$= e^{-2bt}E(X_0^2) + 2\sigma e^{-bt}E(X_0)E\left(\int_0^t e^{-b(t-s)}dW\right) +$$

$$\sigma^2\int_0^t e^{-2b(t-s)}dW$$

$$= e^{-2bt}E(X_0^2) + \frac{\sigma^2}{2b}(1 - e^{-2bt})$$

Thus, the variance:

$$V(X(t)) = E(X^2(t)) - E(X(t))^2$$

is given by:

$$V(X(t)) = e^{-2bt}V(X_0) + \frac{\sigma^2}{2b}(1 - e^{-2bt})$$

assuming that, $V(X_0) < \infty$. For any such initial condition X_0, we therefore have:

$$\left.\begin{array}{l} E(X(t)) \longrightarrow 0 \\ V(X(t)) \longrightarrow \dfrac{\sigma^2}{2b} \end{array}\right\}, \text{ as } t \longrightarrow \infty$$

From the explicit form of the solution we see that the distribution of X(t) approaches $N(0, \frac{\sigma^2}{2b})$ as $t \longrightarrow \infty$. We interpret this to mean that irrespective of the initial distribution, the solution of the SDE for large time "settles down" into a Gaussian distribution whose variance $\frac{\sigma^2}{2b}$ represents a balance between the random disturbing force $\sigma\xi(.)$ and the frictional damping force $-bX(.)$.

Example (2.5), [Evans, 2005]:

Consider next the general equation:

$$\left.\begin{aligned} dX &= (c(t) + d(t)X)dt + (e(t) + f(t)X)dW \\ X(0) &= X_0 \end{aligned}\right\} \quad \ldots(2.24)$$

again for m = n = 1. As above, we try for a solution of the form:

$$X(t) = X_1(t)X_2(t)$$

where now:

$$\left.\begin{aligned} dX_1 &= d(t)X_1dt + f(t)X_1dW \\ X_1(0) &= 1 \end{aligned}\right\} \quad \ldots(2.25)$$

and

$$\left.\begin{aligned} dX_2 &= A(t)dt + B(t)dW \\ X_2(0) &= X_0 \end{aligned}\right\} \quad \ldots(2.26)$$

the functions A, B to be chosen. Then:

$$dX = X_2dX_1 + X_1dX_2 + f(t)X_1B(t)dt$$

$$= d(t)Xdt + f(t)XdW + X_1(A(t)dt + B(t)dW) + f(t)X_1B(t)dt$$

We now require:

$$X_1(A(t)dt + B(t)dW) + f(t)X_1B(t)dt = c(t)dt + e(t)dW;$$

and this identity will hold if we take:

$$\left.\begin{aligned} A(t) &:= [c(t) - f(t)e(t)](X_1(t))^{-1} \\ B(t) &:= e(t)(X_1(t))^{-1} \end{aligned}\right\}$$

Observe that since:

$$X_1(t) = e^{\int_0^t f dW + \int_0^t d - \frac{1}{2} f^2 ds}$$

We have $X_1(t) > 0$ almost surely. Consequently:

$$X_2(t) = X_0 + \int_0^t [c(s) - f(s)e(s)](X_1(s))^{-1}\, ds + \int_0^t e(s)(X_1(s))^{-1}\, dW$$

Employing this and the expression above for X_1, we arrive at the formula:

$$X(t) = X_1(t)X_2(t)$$

$$= \exp\left(\int_0^t d(s) - \frac{1}{2} f^2(s) ds + \int_0^t f(s) dW\right)\left(X_0 + \int_0^t \exp\left(-\int_0^r d(r) - \right.\right.$$

$$\left.\frac{1}{2} f^2(r) dr - \int_0^s f(r) dW\right)(c(s) - e(s)f(s)) ds +$$

$$\left.\int_0^t \exp\left(-\int_0^s d(R) - \frac{1}{2} f^2(r) dr - \int_0^s f(r) dW\right) e(s) dW\right)$$

Example (2.6), [Evans, 2005]:

Consider the linear stochastic differential equation:

$$\left.\begin{aligned} dX &= d(t)X dt + f(t)X dW \\ X(0) &= X_0 \end{aligned}\right\} \qquad (2.27)$$

for m = n = 1. We will try to find a solution having the product form:

$$X(t) = X_1(t)X_2(t)$$

where:

$$\left.\begin{aligned} dX_1 &= f(t)X_1 dW \\ X_1(0) &= X_0 \end{aligned}\right\} \quad \ldots(2.28)$$

and:

$$\left.\begin{aligned} dX_2 &= A(t)dt + B(t)dW \\ X_2(0) &= 1 \end{aligned}\right\} \quad \ldots(2.29)$$

where the functions A and B are to be considered. Then:

$$\begin{aligned} dX &= d(X_1X_2) \\ &= X_1dX_2 + X_2dX_1 + f(t)X_1B(t)dt \\ &= f(t)XdW + (X_1dX_2 + f(t)X_1B(t)dt) \end{aligned}$$

according to (2.28). Now, we try to choose A, B, so that:

$$dX_2 + f(t)B(t)\,dt = d(t)X_2dt$$

for this, $B \equiv 0$ and $A(t) = d(t)X_2(t)$ will work. Thus (2.29) reads:

$$\left.\begin{aligned} dX_2 &= d(t)X_2dt \\ X_2(0) &= 1 \end{aligned}\right\}$$

This is non-random differential equation, which have the solution:

$$X_2(t) = e^{\int_0^t d(s)ds}$$

Since the solution of (2.28) is:

$$X_1(t) = X_0 e^{\int_0^t f(s)dW - \frac{1}{2}\int_0^t f^2(s)ds}$$

We conclude that:

$$\begin{aligned} X(t) &= X_1(t)X_2(t) \\ &= X_0 e^{\int_0^t f(s)dW + \int_0^t \left(d(s) - \frac{1}{2}f^2(s)\right)ds} \end{aligned}$$

a formula noted earlier.

Example (2.7), [Evans, 2005]:

Let m = n = 1 and suppose λ, μ are constants. Then:

$$\left.\begin{aligned} dX &= \lambda X dt + \mu X dW \\ X(0) &= 1 \end{aligned}\right\} \qquad \ldots(2.30)$$

and so:

$$d(\text{Log}X(t)) = \frac{dX(t)}{X(t)} - \frac{1}{2}\frac{[dX(t)]^2}{X^2(t)} \quad \text{(by Itô formula)}$$

$$= (\lambda dt + \mu dW) - \frac{1}{2}\mu^2 dt$$

$$= \left(\lambda - \frac{\mu^2}{2}\right)dt + dW$$

integrate both sides to obtain:

$$\text{Log}X(t) - \text{Log}X(0) = \int_0^t \left(\lambda - \frac{\mu^2}{2}\right) ds + \int_0^t \mu \, dW_s$$

$$\Rightarrow \quad \text{Log}[X(t)/X(0)] = \left(\lambda - \frac{\mu^2}{2}\right)t + \mu W(t)$$

$$\Rightarrow \quad X(t)/X(0) = \exp\{(\lambda - \frac{\mu^2}{2})t + \mu W(t)\}$$

$$\Rightarrow \quad X(t) = X(0)\exp\{(\lambda - \frac{\mu^2}{2})t + \mu W(t)\}$$

Therefore, the solution for the above linear SDE equation is:

$$X_t = X_0\exp\{(\lambda - \frac{\mu^2}{2})t + \mu W(t)\},\ 0 \le t \le T \qquad \ldots(2.31)$$

The following are very useful steps for solving SDE numerically as follows:

Concluding Remarks (2.1) (Numerical Steps for Explicit Euler-Maruyama Method), [Desmond J. Higham, 2000]:

1. The SDE can be written in integral form as:

$$X(t) = X(0) + \int_0^t f(X(s))\, ds + \int_0^t g(X(s))\, dW(s),\ 0 \le t \le T \quad \ldots(2.32)$$

Here f and g are scalar functions and the initial condition X_0 is a random variable. The second integral on the right hand side of eq.(2.32) is to be taken with respect to Brownian motion as discussed in the previous section.

2. It is usual to rewrite (2.32) in differential forms:

$$dX(t) = f(X(t))\, dt + g(X(t))\, dW(t),\ X(0) = X_0,\ 0 \le t \le T \quad \ldots(2.33)$$

This is nothing more than a compact way of saying that X(t) solves (2.32). To keep with convection, we will emphasize the SDE from (2.33) rather than the integral form (2.32).

3. Note that we are not allowed to write dW(t)/dt, since Brownian motion is nowhere differentiable with probability 1.

4. If g = 0 and X_0 is constant, then the problem becomes deterministic and (2.33) reduces to the ordinary differential equation:

$$\frac{dX(t)}{dt} = f(X(t)),\ \text{with } X(0) = X_0$$

5. To apply a numerical method to (2.33) over [0, T], we first discretize the interval. Let $\Delta t = T/L$, for some positive integer L and $\tau_j = j\Delta t$, for $j = 1, 2, \ldots, L$. Our numerical approximation to $X(\tau_j)$ will be denoted by X_j.

   The Euler-method takes the form:

$$X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{j-1})(W(\tau_j) - W(\tau_{j-1})) \quad \ldots(2.34)$$

   for all $j = 1, 2, \ldots, L$

6. To understand where (2.34) comes from, notice from the integral (2.32) that:

$$X(\tau_j) = X(\tau_{j-1}) + \int_{\tau_{j-1}}^{\tau_j} f(X(s))\, ds + \int_{\tau_{j-1}}^{\tau_j} g(X(s))\, dW(s) \quad \ldots(2.35)$$

   Each of the three terms on the right hand side of (2.34) approximates the corresponding term on the right-hand side of (2.35).

7. We also note that in the deterministic case ($g = 0$ and X_0 constant) (2.34) reduces to Euler's method.

8. For computational purpose it is useful to consider discretized Brownian motion, where W(t) is specified at discrete t values. We thus set $\delta t = T/N$ for some positive integer N and let W_j denote $W(t_j)$ with $t_j = j\delta t$. Condition 1 says $W_0 = 0$ with probability 1, tell us that:

$$W_j = W_{j-1} + dW_j, \; j = 1, 2, \ldots, N \quad \ldots(2.36)$$

   where each dW_j is an independent random vector of the form $\sqrt{\delta t}\, N(0, 1)$.

9. For convenience, we always choose the step size Δt for the numerical method to be an integral multiple $R \geq 1$ of the δt for the Brownian path.

10. This ensures that the set of points $\{t_j\}$ on which the discretized Brownian path is based contains the points $\{J_j\}$ at which the EM solution is computed.

Concluding Remarks (2.2) (Numerical Steps of Strong Convergence for EM), {Desmond J. Higham, 2000]:

1. Keeping in mind that $X(T_n)$ and X_n are random variables, in order to make the notion of convergence precise, we must decide how to measure their difference.

   1.1 Using $E|X_n - X(T_n)|$ where E(.) denotes the expected value leads to the concept of strong convergence.

   1.2 The method is said to have strong order of convergence equal to η if there exists a constant C, such that:

$$E|X_n - X(T)| \leq C\Delta t^{\eta} \qquad \ldots(2.37)$$

   for any fixed $T = n\Delta t \in [0, T]$ and Δt sufficiently small.

2. If SDE functions satisfy appropriate conditions, it can be shown that Euler-Maruyama has strong order has strong order of convergence $\eta = \frac{1}{2}$.

3. This marks a departure from the deterministic setting if $g \equiv 0$ and X_0 is constant, then the expected value can be deleted from the left hand side of (2.37) and the inequality is true with $\eta = 1$.

4. In our numerical tests, we will focus on the error at the end point t = T, so that:

$$e_{\Delta t}^{strong} := E|X_t - X(T)|, \text{ where } L\Delta t = T \qquad \ldots(2.38)$$

denote the EM end point error in this strong sense. If the EM bound (2.37) holds with $\eta = \frac{1}{2}$ at any fixed point in [0, T], then certainly holds at the end point, so we have:

$$e_{\Delta t}^{strong} \leq C\Delta t^{1/2} \qquad \ldots(2.39)$$

for sufficiently small Δt.

5. To study the numerical solution of SDE's, there is a need to determine the type of convergence. The strong convergence is then adapted and for the following example on dyonding of the result of section (2.5.1) the main steps the strong error for different examples using different values and functions are simulated in the following tables and figures.

6. The least square's error is:

$$e_{\Delta t}^{strong} \leq C\Delta t^{q}$$

$$Log\, e_{\Delta t}^{strong} = LogC + qLog\Delta t_i + e_i \qquad \ldots(2.40)$$

$$e_i = Log\, e_{\Delta t}^{strong} - LogC - qLog\Delta t_i$$

$$\Omega = \sum_{i=1}^{N} e_i^2$$

$$= \sum_{i=1}^{N} \left(Log\, e_{\Delta t}^{strong} - LogC - qLog\Delta t_i \right)^2 \longrightarrow 0$$

On setting $\frac{\partial \Omega}{\partial \text{LogC}} = 0$ and $\frac{\partial \Omega}{\partial q} = 0$, we obtain:

$$-2\sum_{i=1}^{N}\left(\text{Log}\, e_{\Delta t}^{strong} - \text{LogC} - q\text{Log}\Delta t_i\right) = 0$$

$$-2\sum_{i=1}^{N}\left(\text{Log}\, e_{\Delta t}^{strong} - \text{LogC} - q\text{Log}\Delta t_i\right)\text{Log}\Delta t_i = 0$$

$$\sum_{i=1}^{N}\text{Log}\, e_{\Delta t}^{strong} - N\text{LogC} - q\sum_{i=1}^{N}\text{Log}\Delta t_i = 0$$

$$\sum_{i=1}^{N}\text{Log}\, e_{\Delta t}^{strong}\,\text{Log}\Delta t_i - \text{LogC}\sum_{i=1}^{N}\text{Log}\Delta t_i - q\sum_{i=1}^{N}\left(\text{Log}\Delta t_i\right)^2 = 0$$

$$\begin{pmatrix} N & \sum_{i=1}^{N}\text{Log}\Delta t_i \\ \sum_{i=1}^{N}\text{Log}\Delta t_i & \sum_{i=1}^{N}\left(\text{Log}\Delta t_i\right)^2 \end{pmatrix}\begin{pmatrix} \text{LogC} \\ q \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{N}\text{Log}\, e_{\Delta t}^{strong} \\ \sum_{i=1}^{N}\text{Log}\, e_{\Delta t}^{strong}\text{Log}\Delta t_i \end{pmatrix}$$

By Gramer's rule:

$$\begin{vmatrix} N & \sum_{i=1}^{N}\text{Log}\Delta t_i \\ \sum_{i=1}^{N}\text{Log}\Delta t_i & \sum_{i=1}^{N}\left(\text{Log}\Delta t_i\right)^2 \end{vmatrix} = N\sum_{i=1}^{N}\left(\text{Log}\Delta t_i\right)^2 - \left(\sum_{i=1}^{N}\text{Log}\Delta t_i\right)^2 \neq 0$$

Hence:

$$\text{LogC} = \frac{\sum_{i=1}^{N}\text{Log}\, e_{\Delta t}^{strong}\left(\sum_{i=1}^{N}\text{Log}\Delta t_i\right)^2 - \sum_{i=1}^{N}\text{Log}\Delta t_i\sum_{i=1}^{N}\text{Log}\, e_{\Delta t}^{strong}\text{Log}\Delta t_i}{N\sum_{i=1}^{N}\left(\text{Log}\Delta t_i\right)^2 - \left(\sum_{i=1}^{N}\text{Log}\Delta t_i\right)^2}$$

and:

$$q = \frac{N\sum_{i=1}^{N} \text{Log}\, e_{\Delta t}^{strong} \text{Log}\Delta t_i - \sum_{i=1}^{N} \text{Log}\, e_{\Delta t}^{strong} \sum_{i=1}^{N} \text{Log}\Delta t_i}{N\sum_{i=1}^{N} (\text{Log}\Delta t_i)^2 - \left(\sum_{i=1}^{N} \text{Log}\Delta t_i\right)^2}$$

Then, we have that:

$$y = ax + b$$

where:

$$a = \text{Log}\Delta t_i;\ b = \text{Log}\ c \text{ and } y = \text{Log}\, e_{\Delta t}^{strong}$$

on comparison with eq.(2.40), one can see that q; the slope of the line of fitting is the ordered of strong convergence which need to be 0.5 as discussed in section (2.5.1).

7. Since $\text{Log}\, e_{\Delta t}^{strong}$ is taken on both sides. Then one can looking for the slops of the figure to be equal to that one of curve fitting.
8. The best results of the numerical simulation of strong curves is obtained on the line simulation has a slop 0.5 as one can concluded this fact from (6).

Concluding Remarks (2.3) (Numerical Steps of Weak Convergence for EM), [Desmond J. Higham, 2000]:

1. The strong order of convergence (2.36) measures the rate at which the "mean of the error" decays as $\Delta t \longrightarrow 0$ of a less demanding alternative is to measure the rate of decay of the "error of means".

2. This leads to the concept of weak convergence. A method is said to be of weak order of convergence equal to η if there exists a constant C, such that for all functions p in some class:

$$|Ep(X_n) - Ep(X(T))| \leq C\Delta t^{\eta} \quad \ldots(2.41)$$

at any fixed $T = n\Delta t \in [0, T]$ and Δt sufficiently small. Typically, the function p allowed in (2.41) must satisfy smoothness and polynomial growth conditions.

3. For appropriate function of SDE, it can be shown that EM has weak order of convergence $\eta = 1$. Mimicking our strong convergence tests, we let:

$$e_{\Delta t}^{weak} := |EX_L - EX(T)| \quad \ldots(2.42)$$

where $L\Delta t = T$ denote the weak end point error in EM. So (2.41) for $p(x) = x$ with $\eta = 1$ immediately implies that:

$$e_{\Delta t}^{weak} \leq C\Delta t \quad \ldots(2.43)$$

for sufficiently small Δt.

4. This improves the execution time at the expense of extra strong requirements. To compensate, we have used different paths for each Δt so that only the current increments, rather than the complete paths, need to be stored. Further, we choose the path increment $\delta t = \Delta t$ for extra efficiency. The sample average approximation to EX_L it follows from (2.1) that $EX(T) = e^{\lambda T}$, for the true solution and X_{error} stores the corresponding weak endpoint error for each Δt.

Concluding Remarks (2.4) (Numerical Steps of Liner Stability for EM), [Desmond J. Higham, 2000]:

1. The concepts of strong and weak convergence concern the accuracy of numerical methods over a finite interval [0, T], for small step sizes Δt. However, in many applications the long term, $t \longrightarrow \infty$, behaviour of an SDE is of interest.

2. Convergence bounds of the form:

$$E|X_n - X(T)| \leq C\Delta t^{\gamma} \text{ or } |Ep(X_n) - Ep(X(T))| \leq C\Delta t^{\gamma}$$

are not relevant in this context, since generally, the constant C grows unboundedly with T.

3. For deterministic ODE methods, a large body of stability theory has been developed that gives insight into the behavior of numerical methods in the Δt fixed, $t_j \longrightarrow \infty$.

4. Typically, a numerical method is applied to a class of problems with some qualitative feature, and the ability of the method to reproduce this feature is analyzed. Although, a wide variety of problem classes have been analyzed, the simplest, and perhaps the most revealing, is the linear test equation $dX/dt = \lambda X$, where $\lambda \in \square$ is a constant parameter. For SDE's it is possible to develop an analogous linear stability theory, as we now indicate:

5. We return to the linear SDE:

$$dX(t) = \lambda(X(t))\, dt + \mu(X(t))\, dW(t),\ X(0) = X_0 \qquad \ldots(2.44)$$

where the function of SDE allowed to be complex in the case where $\mu = 0$ and X_0 is constant, (2.44) reduces to the deterministic linear test

equation, if we use the term stable to mean that $\lim_{t\to\infty} X(t) = 0$, for any X_0. Then we see that stability is characterized by $R\{\lambda\} < 0$.

6. We will consider the two most common measures of stability; Mean-Square and Asymptotic, [D. J. Higham, 2000]. Assuming that $X_0 \neq 0$ with probability 1, solutions of SDE is:

$$dX(t) = \lambda X(t)\,dt + \mu X(t)\,dW(t),\ X(0) = X_0$$

Satisfying:

$$\lim_{t\to\infty} EX^2(t) = 0 \Leftrightarrow R\{\lambda\} + \frac{1}{2}|\mu|^2 < 0 \qquad \ldots(2.45)$$

$$\lim_{t\to\infty} |X(t)| = 0 \text{ with probability } 1 \Leftrightarrow R\{\lambda - \frac{1}{2}\mu^2\} < 0 \qquad \ldots(2.46)$$

The left-hand side of (2.45) defines what is meant by mean-square stability. The right-hand side of (2.45) completely characterizes this property in terms of the function SDE. Similarly (2.46) defines and characterizes asymptotic stability.

7. Setting the characterization collapse to the same condition $R\{f\} < 0$, which of course, a rose for deterministic stability. It follows immediately from (2.45) and (2.46) that if the SDE:

$$dX(t) = \lambda X(t)\,dt + \mu X(t)\,dW(t),\ X(0) = X_0$$

is mean-square stable, then it is automatically asymptotic stable, but not vice versa.

8. Now, suppose that the functions f and g are chosen so that the SDE:

$$dX(t) = \lambda X(t)\,dt + \mu X(t)\,dW(t),\ X(0) = X_0$$

is stable in the mean-square or asymptotic sense. A simple properties of the expected value show that:

$$\lim_{j\to\infty} E\,X_j^2 = 0 \Leftrightarrow |1 + \Delta t\lambda|^2 + \Delta t|\mu|^2 < 1 \qquad \ldots(2.47)$$

for EM applied to equation:

$$dX(t) = \lambda X(t)\,dt + \mu X(t)\,dW(t),\ X(0) = X_0$$

The asymptotic version of the question can be studied via the strong low of large numbers and the low of the iterated logarithm, leading to:

$$\lim_{j\to\infty} |X_j|=0, \text{ with probability } 1 \Leftrightarrow E\text{Log}|1 + \Delta t\lambda + \sqrt{\Delta t}\,\mu N(0,1)|<0$$

2.7 Algorithm and Illustration:

Algorithm (2.1) (Euler-Maruyama Method):

Input: The dynamic stochastic differential equation in problem formulation:

$$\left.\begin{aligned} dX(t) &= f(X_t, t)dt + g(X_t, t)\,dW(t) \\ X(0) &= X_0 \end{aligned}\right\} \qquad \ldots(2.48)$$

Output: Numerical (sample path) solution of stochastic process.

Step 1: Consider problem formulation (2.48).

Step 2: Generating a Brownian motion as follows (see, concluding remark (2.1)):

Step 2.1: Generate a random number.

Step 2.2: Consider $T = t_0$; $N = n_0$; $dt = T/N$.

Step 2.3: $W(t) \sim N(0, 1)$.

Step 2.4: $W_0 = 0$ with probability 1.

Step 2.5: $W_j = W_{j-1} + dW_j$, $j = 1, 2, \ldots, N$.

Step 2.6: $dW_j \sim \sqrt{\delta t}\, N(0, 1)$.

Step 3: Set $j = 1 \longrightarrow L$

$$W(T_j) - W(T_{j-1}) = W(jR\delta t) - W((j-1)R\delta t)$$

$$= \sum_{k=jR-R+1}^{jR} dW_k$$

$$X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{j-1})(W(T_j) - W(T_{j-1})),$$

If $j \geq L$ stop.

Step 4: Computation of error, depending on the type of error for example, the following is absolute error

Step 4.1: If $\Delta t = \delta t$

Set $i = 1 \longrightarrow L$

$error = abs(X_n(i) - X_T(i))$

Step 4.2: If $\Delta t \neq \delta t$

$error = abs(X_n(final) - X_T(final))$.

Figure (2.1) Shows a signal simulation of discretized Brownian motion of the interval [0, 1] and N = 500, so that $\Delta t = 1/500$ (see program A.1 in Appendix).

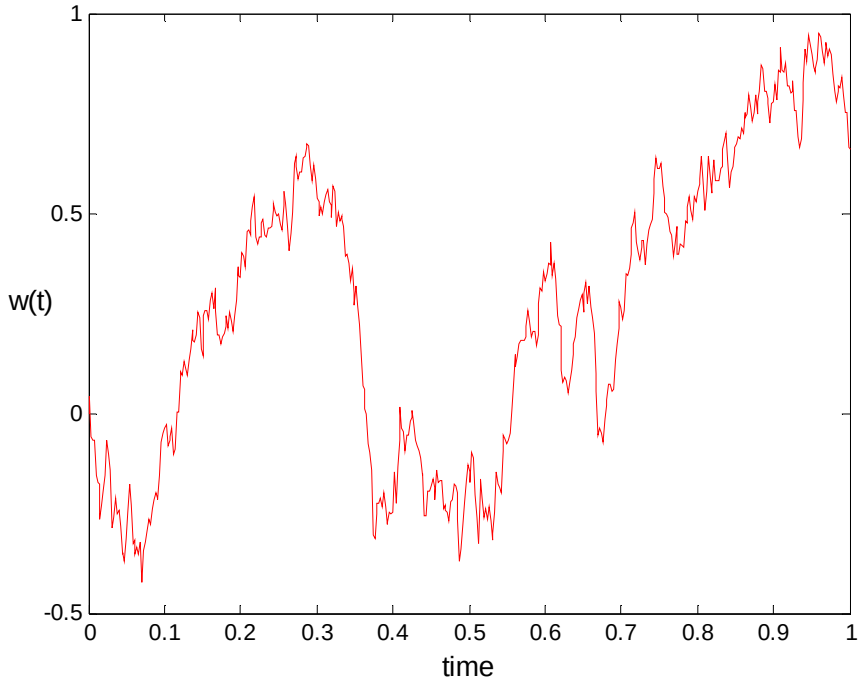


Figure (2.1) Discretized Brownian paths.

Figure (2.2) Shows a signal simulation of discretized Brownian motion of the interval [0, 1] and N = 600, where $\Delta t = 1/600$.

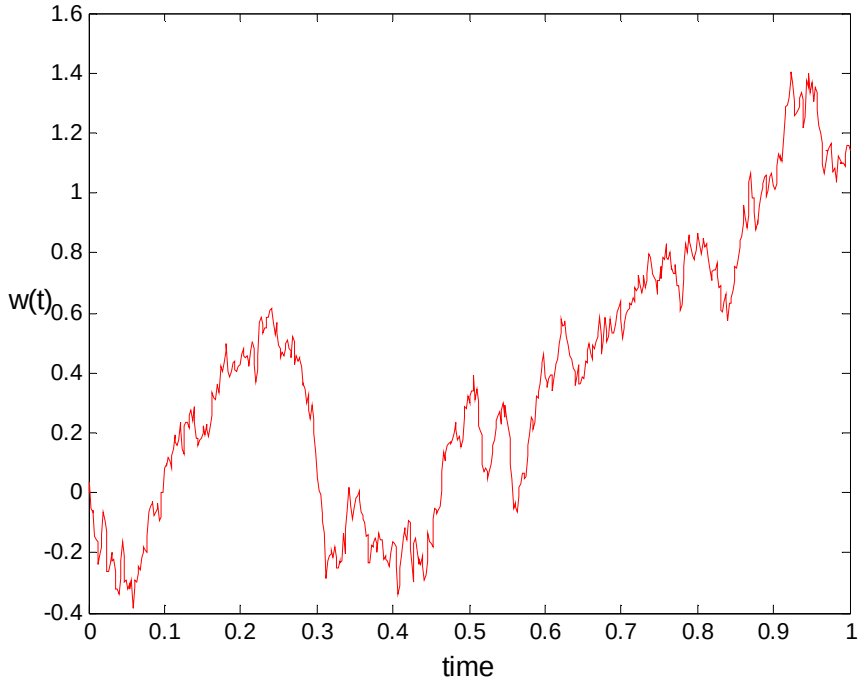


Figure (2.2) Discretized Brownian paths.

While Figure (2.3) Shows a signal simulation of discretized Brownian motion of the interval [0, 1] and N = 800, with $\Delta t = 1/800$.

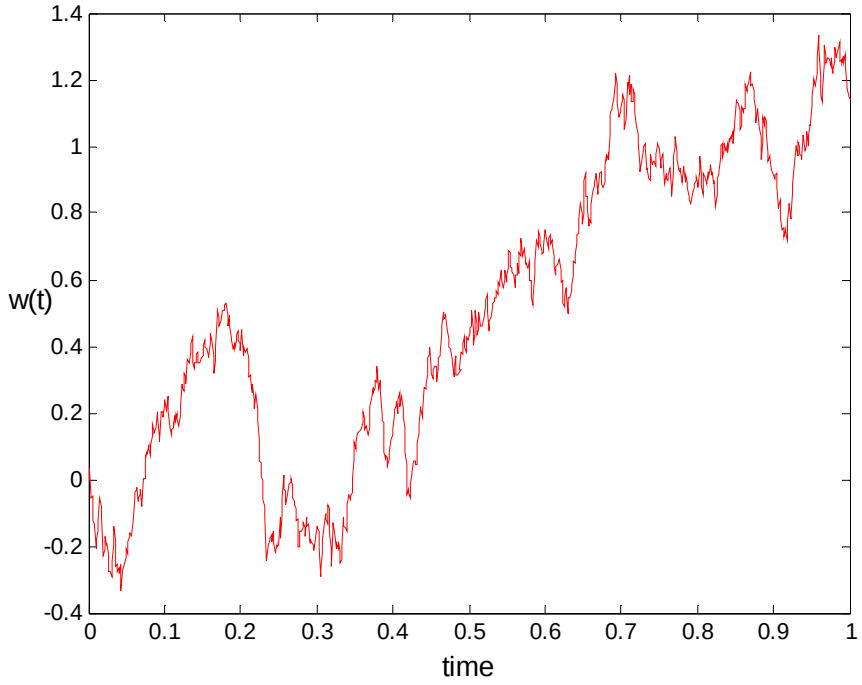


Figure (2.3) Discretized Brownian paths.

Illustration (2.1) (With Absolute Error Test and Comparisons):

Consider the example (2.1) of section (2.6) which is:

$dX = gXdW$

$X(0) = 1$

Where:

$g(t) = \sin t$; $X_0 = 1$; $y_0 = 0$

The absolute error at the final time interval for different sample space numbers, where $\Delta t = \delta t$; R = 1; the step time for discritization of Brownian motion equals to the step time of Euler scheme, are shown in

the following (table (2.1) and Figure (2.4.1)). As one can see, increasing the number of sample (N) leads to improving the absolute error at the different time steps, where $\Delta t = \delta t$.

The figure (2.4) shows the very good agreement between the exact solution and the corresponding numerical solution on simulating different selected function and their nature leads to well understand the behaviour of numerical method. The absolute error at final time interval is depending not only on the number of sample N, but also on the nature of the selection function of SDE and on the selection of R not equal to 1, as one can see this fact from the following figure (2.4) and table (2.1) is:

Table (2.1) Error generated by the Euler scheme.

R	***N***	***Error at final time***
1	2^5	0.0409
	2^6	0.0058

On using R = 1, N = 2^6, the following numerical solution is obtained and presented in the following figure (2.4.1) (see program A.2 in Appendix A).

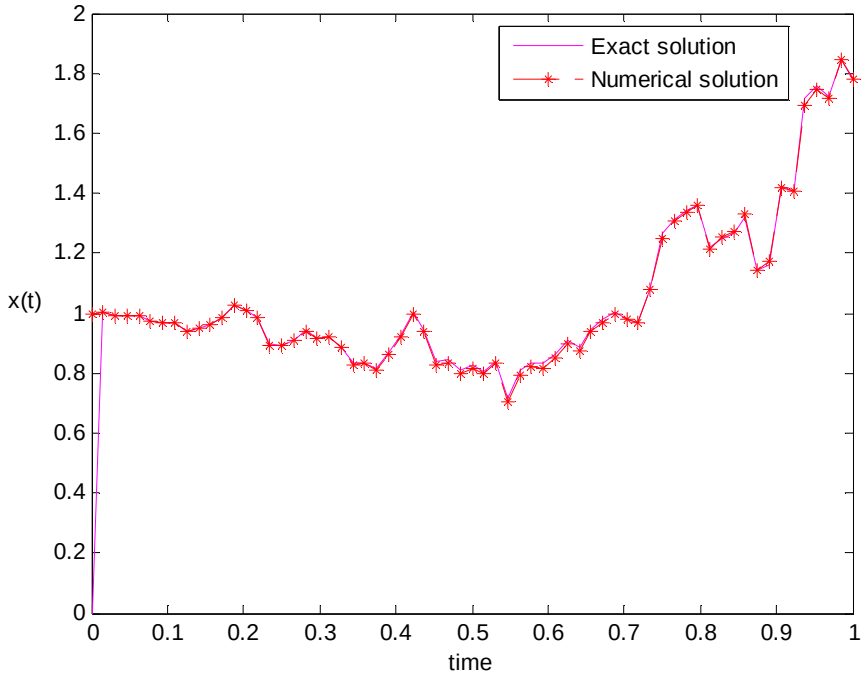


Figure (2.4) Exact solution and the numerical solution by Euler scheme with $N = 2^6$; $R = 1$.

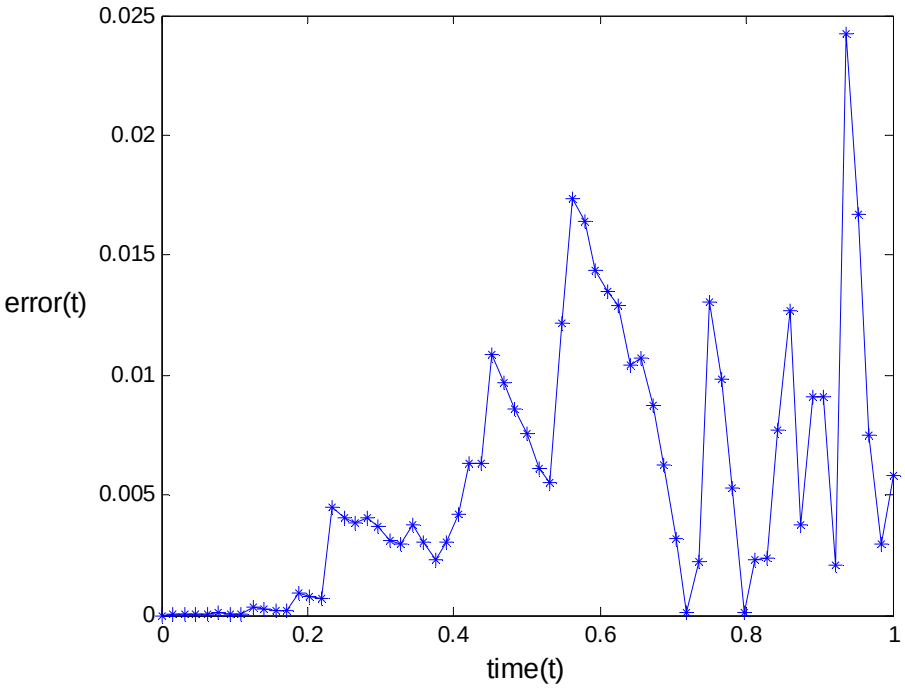


Figure (2.4.1) Absolute error between the Euler scheme and exact solution with $N = 2^6$; $R = 1$.

Illustration (2.2) (With Absolute Error Test and Comparisons):

Consider the example (2.2) of section (2.6) which is:

$dX = fX\,dt + gX\,dW$

$X(0) = 1$

Where:

$f(t) = \cos t$; $g(t) = \sin t$; $X_0 = 1$; $Y_0 = 0$

As discussed previously in illustration (2.1), the following table (2.2) is needed for error analysis and as follows:

Table (2.2) Error generated by the Euler scheme.

R	***N***	***Error at final time***
1	2^5	0.0821
	2^6	0.0250
	2^8	0.0123
	2^9	0.0526
	2^{10}	0.0230
	2^{11}	0.0088

On using R = 1, N = 2^{11}, the following numerical solution is obtained and presented in the following figure (2.5.1).

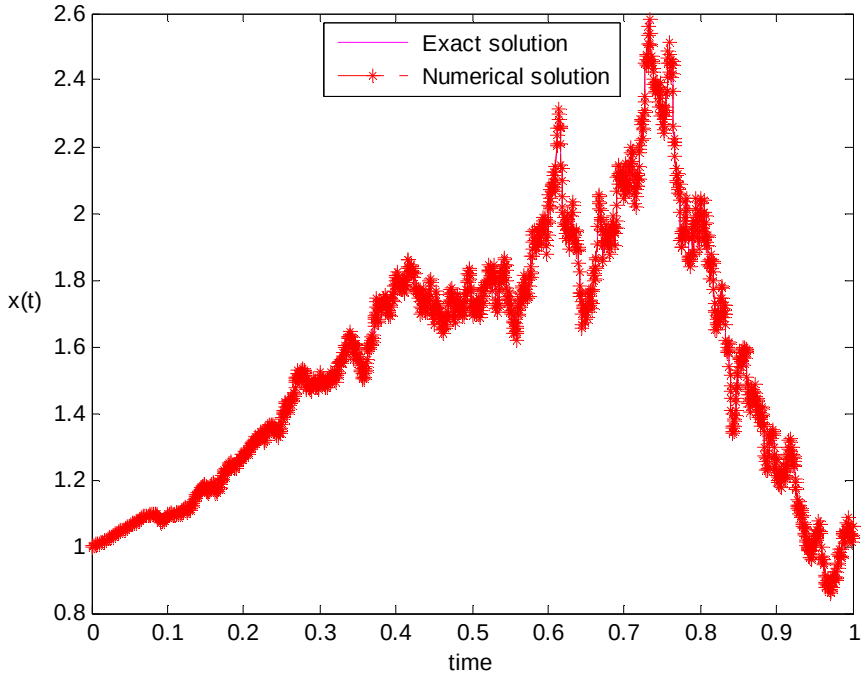


Figure (2.5) Exact solution and the numerical solution by Euler scheme with $N = 2^{11}$; $R = 1$.

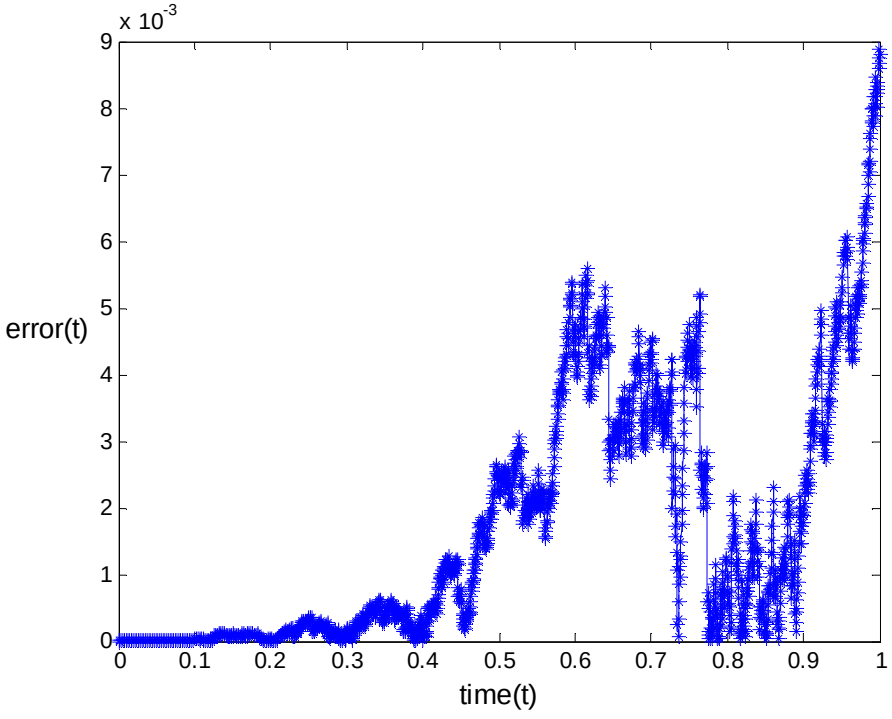


Figure (2.5.1) Absolute error between the Euler scheme and exact solution with $N = 2^{11}$; $R = 1$.

Illustration (2.3) (With Absolute Error Test and Comparisons):

Consider the example (2.3) of section (2.6) which is:

$dP = \mu P\, dt + \sigma P\, dW$

$P(0) = P_0$

Where:

$P_0 = 1;\ \mu = 1;\ \sigma = 2$

The error at final time interval for R = 1 and different number of sample N is discussed in the following table (2.3)

Table (2.3) Error generated by the Euler scheme.

R	***N***	***Error at final time***
1	2^5	0.0284
	2^8	0.0266
	2^{11}	0.0128

One can select R = 1, N = 2^{11} for accuracy, the following numerical solution is then obtained and presented in the following figure (2.6.1).

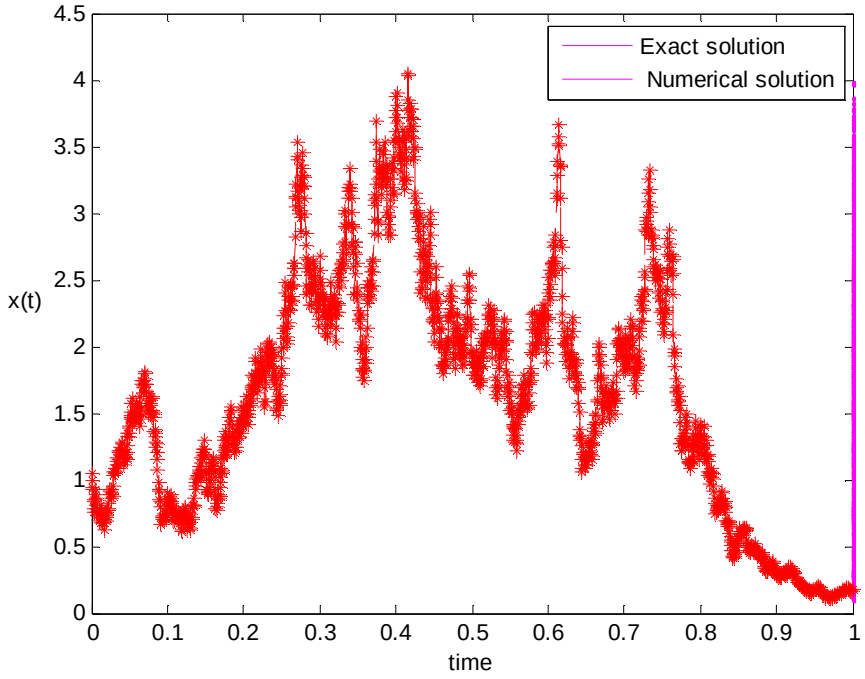


Figure (2.6) Exact solution and the numerical solution by Euler scheme with $N = 2^{11}$; $R = 1$.

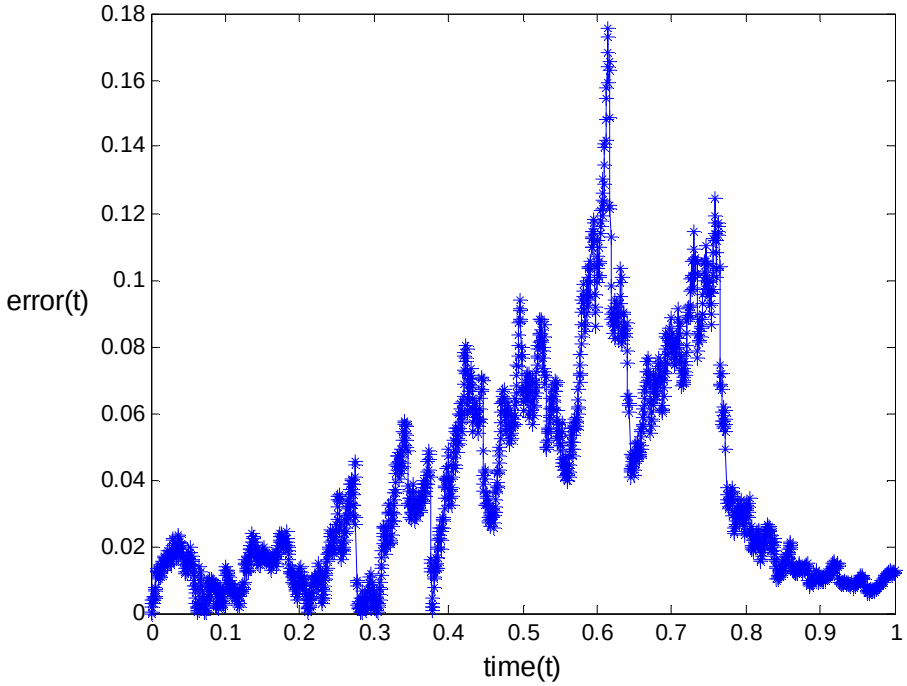


Figure (2.6.1) Absolute error between the Euler scheme and exact solution with $N = 2^{11}$; $R = 1$.

Illustration (2.4) (With Absolute Error Test and Comparisons):

Consider the example (2.4) of section (2.6) which is:

$$dX = -bX\,dt + \sigma\,dW$$

$$X(0) = X_0$$

Where:

$$X_0 = 1;\ Y_0 = 0;\ b = 2;\ \sigma = 1$$

Where $R = 1$ and $N = 2^8$ is adapted for the numerical solution as discussed in the following table (2.4) and Figures (2.7.1).

Table (2.4) Error generated by the Euler scheme.

R	***N***	***Error at final time***
1	2^5	0.8634
	2^6	0.3335
	2^7	0.3793
	2^8	0.0686

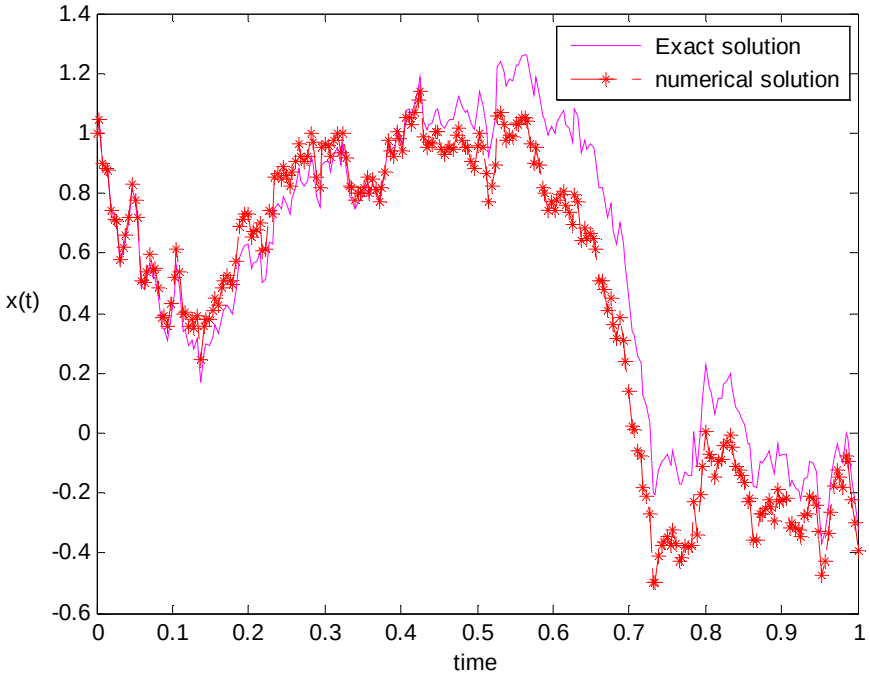


Figure (2.7) Exact solution and the numerical solution by Euler scheme with $N = 2^8$; $R = 1$.

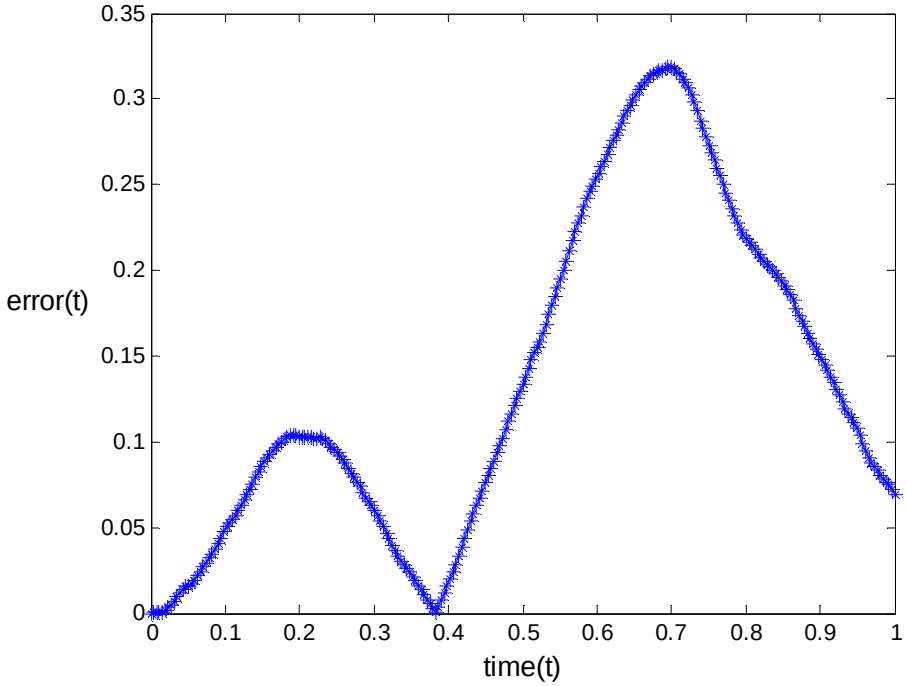


Figure (2.7.1) Absolute error between the Euler scheme and exact solution with $N = 2^8$; $R = 1$.

Illustration (2.5) (With Absolute Error Test and Comparisons):

Consider the example (2.7) of section (2.6) which is:

$$dX = \lambda X\, dt + \mu X\, dW$$

$$X(0) = 1$$

Where:

$$\lambda = 2;\ \mu = 1;\ X_0 = 1$$

As discussed previously in illustration (2.1), the following table (2.5) is needed for error analysis and as follows.

Here, the numerical solution are discussed for different values of R, i.e., $\Delta t = R\delta t$. The absolute error can be evaluated at all the values in the selected interval, but only on the final time interval. As one can see the effect of selection of R on the error.

Table (2.5) Error generated by the Euler scheme.

R	*N*	*Error at final time*
1	2^5	0.3430
	2^6	1.0219
	2^7	0.3420
	2^8	0.0821
	2^9	0.1269
	2^{10}	0.1581
	2^{11}	0.0603
2	2^5	0.3030
	2^6	0.9630
	2^7	0.1084
	2^8	0.1595
	2^9	0.1100
	2^{10}	0.0735
	2^{11}	0.0035
3	2^5	0.6355
	2^6	2.1141
	2^7	3.1173
	2^8	0.1075
	2^9	0.8522
	2^{10}	0.1285
	2^{11}	0.2298
4	2^5	0.6320
	2^6	2.4777
	2^7	1.8936
	2^8	0.6907
	2^9	1.3507
	2^{10}	0.7935
	2^{11}	0.1515

Different solution for illustration (2.5) have been obtained by using different values of R and N. The comparison behaviour between the given exact solution and numerical one have also been given as one can see this from the following figures:

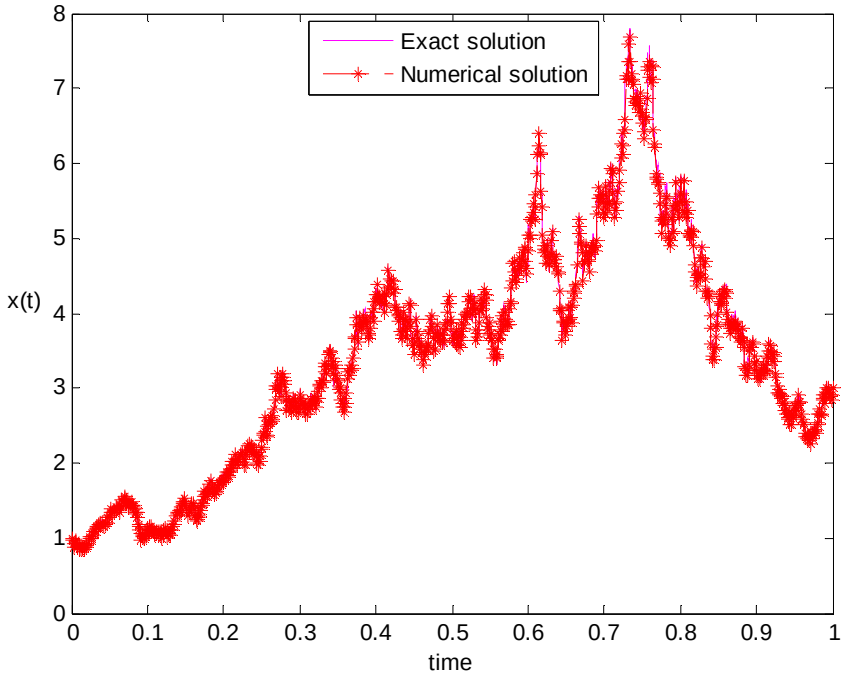


Figure (2.8) Exact solution and the numerical solution by Euler scheme with $N = 2^{11}$; $R = 2$.

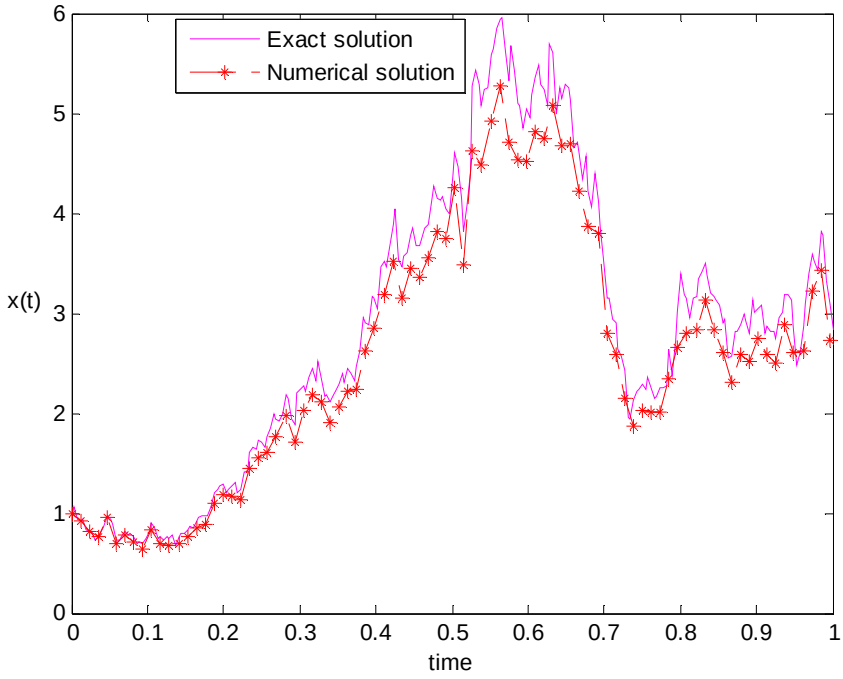


Figure (2.9) Exact solution and the numerical solution by Euler scheme with $N = 2^8$; $R = 3$.

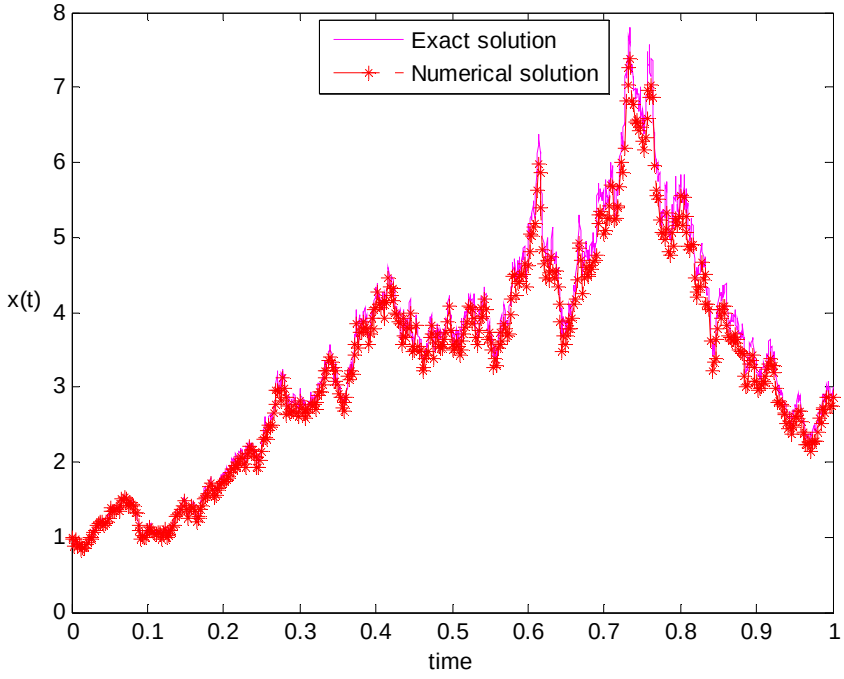


Figure (2.10) Exact solution and the numerical solution by Euler scheme with $N = 2^{11}$; $R = 4$.

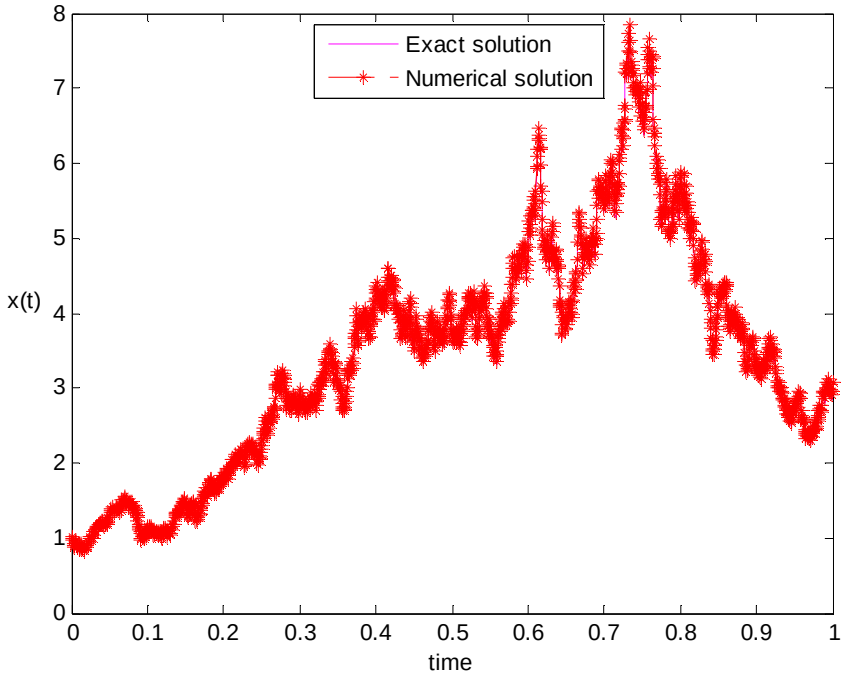


Figure (2.11) Exact solution and the numerical solution by Euler scheme with $N = 2^{11}$; $R = 1$.

Illustration (2.6) (With Strong Convergence Test):

Consider the example (2.3) of section (2.6) which is:

$$dP = \mu P\, dt + \sigma P\, dW$$

$$P(0) = P_0$$

Where:

$P_0 = 1$; $\sigma = 1$ and $\mu = 2$

$N = 2^9$; $T = 1$ and sample path $\mu = 4000$

Depending on the concluding remark (2.2), the strong convergence for the numerical solution using Euler-Maruyama method is obtained. This produces the blue asterisks connected with solid lines in the plot of Figure (2.12). For reference, a dashed red line of slope one-

half is added. The least-squares power law fit gives q = 0.5194 and resid = 0.0355 (see program A.3 in Appendix A).

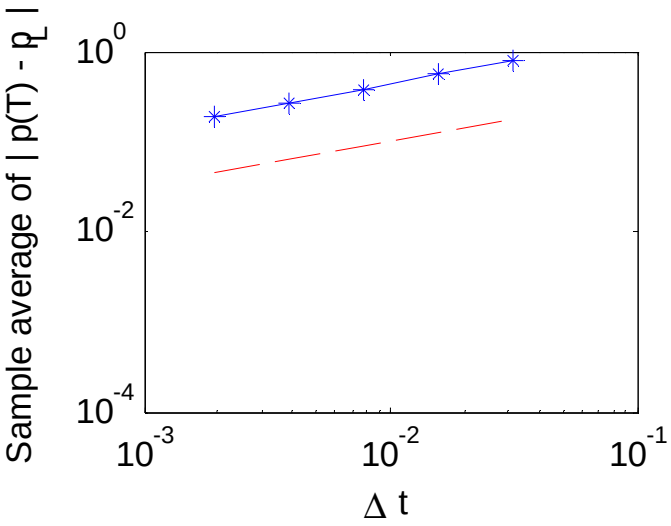


Figure (2.12) Euler-Maruyama strong convergence.

Illustration (2.7) (With Strong Convergence Test):

Consider the example (2.7) of section (2.6) which is:

$dX = \lambda X\, dt + \mu X\, dW$

$X(0) = 1$

Where:

$\lambda = 2$; $\mu = 1$; $X_0 = 1$; $N = 2^9$, $T = 1$ and sample path $\mu = 4000$

The strong convergence for the numerical solution using Euler-Maruyama method of this example is obtained, where the least-squares power law fit gives q = 0.5490 and resid = 0.0940.

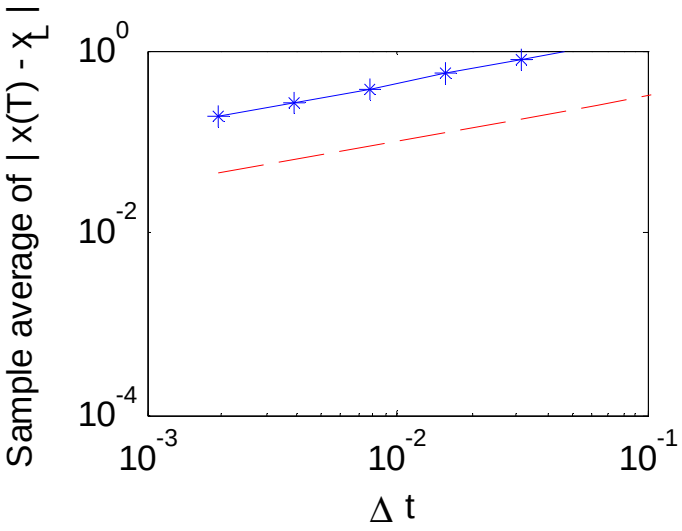


Figure (2.13) Euler-Maruyama strong convergence.

Illustration (2.8) (With Strong Convergence Test):

Consider the example (2.3) of section (2.6) which is:

$dP = \mu P\, dt + \sigma P\, dW$

$P(0) = P_0$

Where:

$P_0 = 1$; $T = 1$; $\mu = 2$; $\sigma = 1$ and sample path $\mu = 50000$

The Figure (2.14) shows that weak error varies with Δt on a Log-Log scale. A dashed red reference line of slope one is added. The least-squares power law fit gives q = 1.2617 and resid = 0.7709 (see program A.4 in Appendix a).

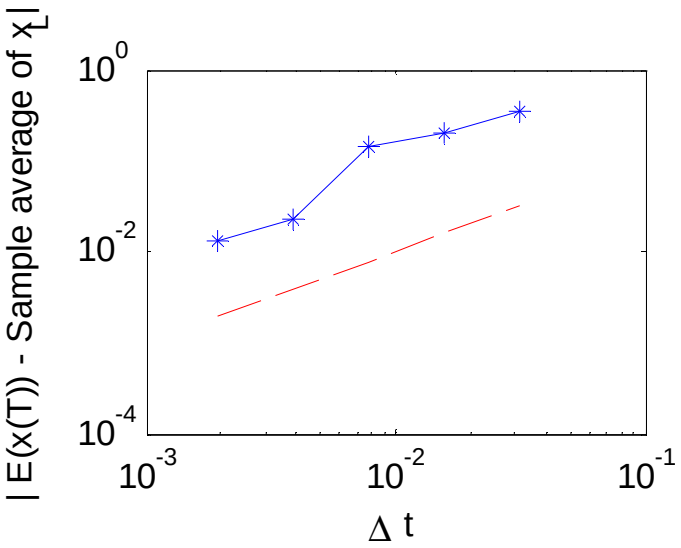


Figure (2.14) Euler-Maruyama strong convergence.

Illustration (2.9) (With Weak Convergence Test):

Consider the example (2.7) of section (2.6) which is:

$dX = \lambda X\, dt + \mu X\, dW$

$X(0) = 1$

Where:

$X_0 = 1$; $\lambda = 2$; $\mu = 0.1$; and sample path $\mu = 50000$

For this example, the weak convergence for the numerical solution using Euler-Maruyama method is obtained, where Figure (2.15) shows the weak error varies with Δt on a Log-Log scale. A dashed red reference line of slope one is added, where the least-squares power law fit gives q = 0.9858 and resid = 0.0508.

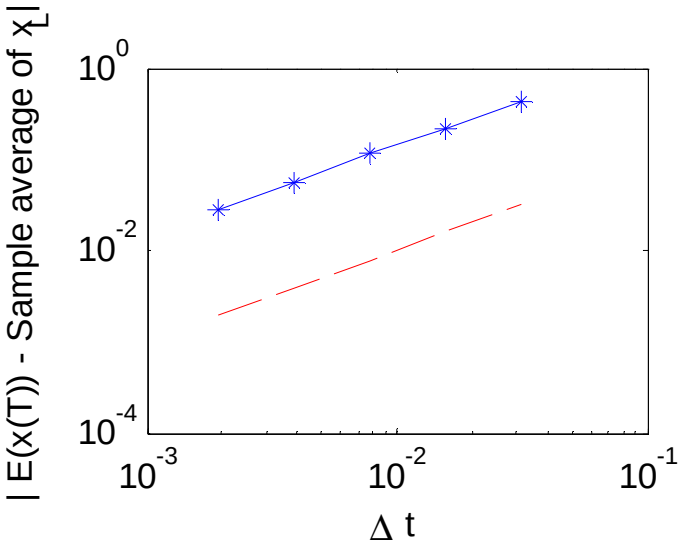


Figure (2.15) Euler-Maruyama strong convergence.

Illustration (2.10) (With Linear Stability Test):

Consider the example (2.1) of section (2.6) which is:

$dX = gX\,dW$

$X(0) = 1$

(1) Mean-Square stability:

Where $g(t) = \sin t$; $T = 20$; $\mu = 50000$; $X_0 = 1$; $\Delta t = 1, 2, 1/4$;

and $N = T/\Delta t$

(2) Asymptotic stability:

Where $g(t) = \sin t$; $T = 500$; $\Delta t = 1, 2, 1/4$; and $N = T/\Delta t$

The Figure (2.16) plots the sample average of $E(X^2)$ against t in this picture the $\Delta t = 1$ and $\Delta t = 1/2$ curves increase with t, while the $\Delta t = 1/4$ curve decays toward zero (see program A.5 in Appendix A).

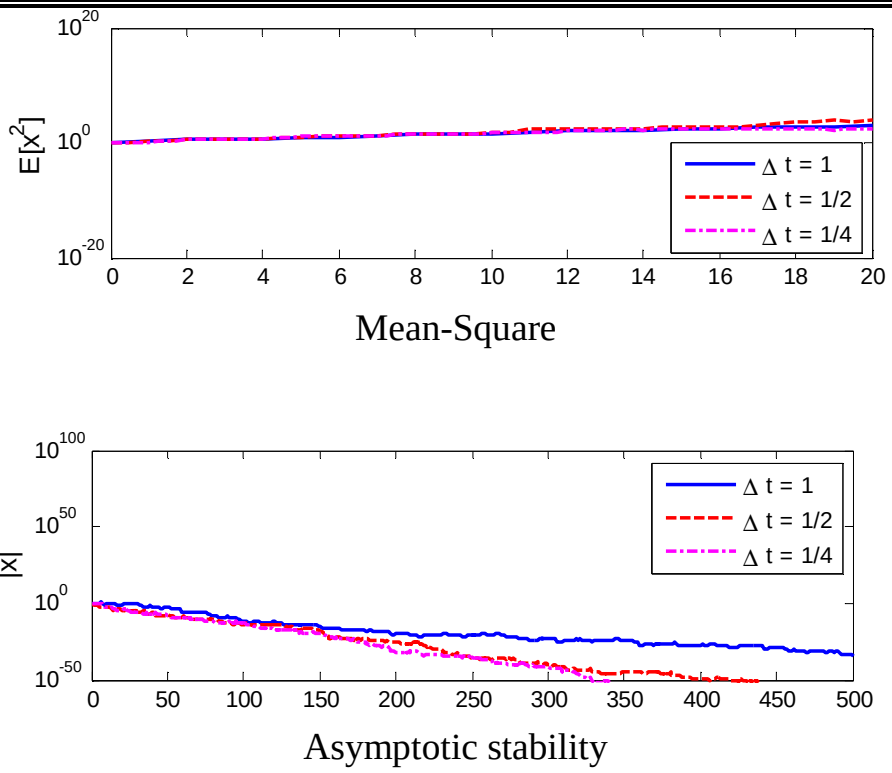


Figure (2.16) Mean-Square and Asymptotic stability.

Illustration (2.11) (With Linear Stability Test):

Consider the example (2.2) of section (2.6) which is:

$dX = fX\,dt + gX\,dW$

$X(0) = 1$

(1) Mean-Square stability:

Where f(t) = cost; g(t) = sint; sample path $\mu = 50000$; T = 2;

$X_0 = 1$ and using step size $\Delta t = 1, 1/2, 1/4$; and $N = T/\Delta t$.

(2) Asymptotic stability:

Where f(t) = cost; g(t) = sint; T = 500; and using step size $\Delta t = 1, 1/2, 1/4$; and $N = T/\Delta t$.

The Figure (2.17) plots the sample average of $E(X^2)$ against t in this picture the $\Delta t = 1$ and $\Delta t = 1/2$ curves increase with t, while the $\Delta t = 1/4$ curve decays toward zero.

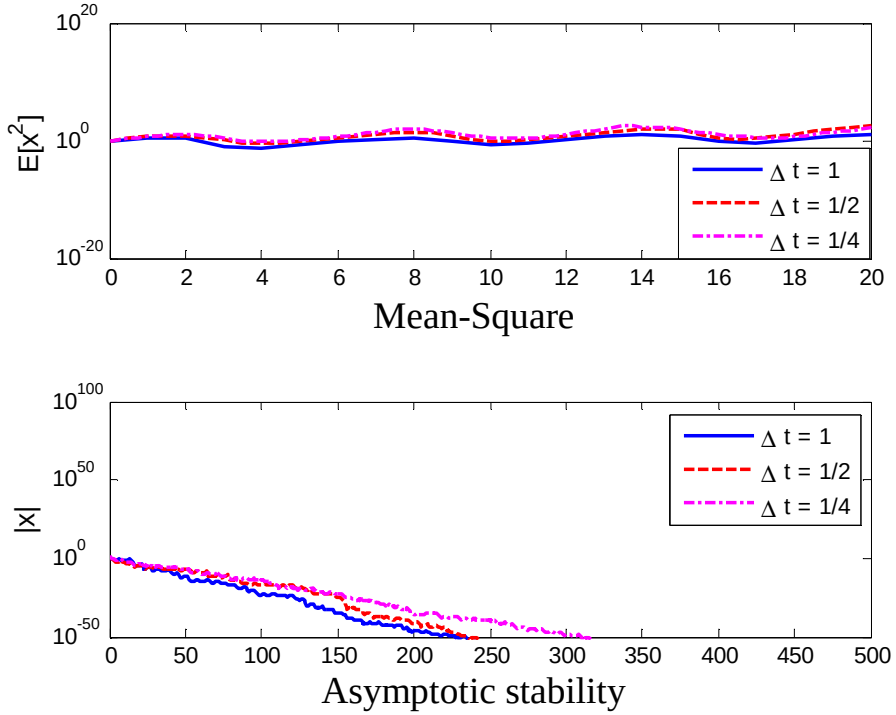


Figure (2.17) Mean-Square and Asymptotic stability for EM.

Illustration (2.12) (With Linear Stability Test):

Consider the example (2.3) is (Stock prices):

$dP = \mu P\, dt + \sigma P\, dW$

$P_0 = 1$

(1) Mean-Square stability:

Where $T = 20$; $\mu = 50000$; $P_0 = 1$ and $\Delta t = 1, 1/2, 1/4$; $N = T/\Delta t$.

$\mu = -3$; $\sigma = 3$

(2) Asymptotic stability:

Where T = 500; $\mu = 1/2$; $\sigma = 3$ and $\Delta t = 1, 1/2, 1/4$; $N = T/\Delta t$.

Show in the following Figure (2.18) plots the sample average for ($\mu = -3$, $\sigma = 3$) and ($\mu = 1/2$, $\sigma = 3$, respectively.

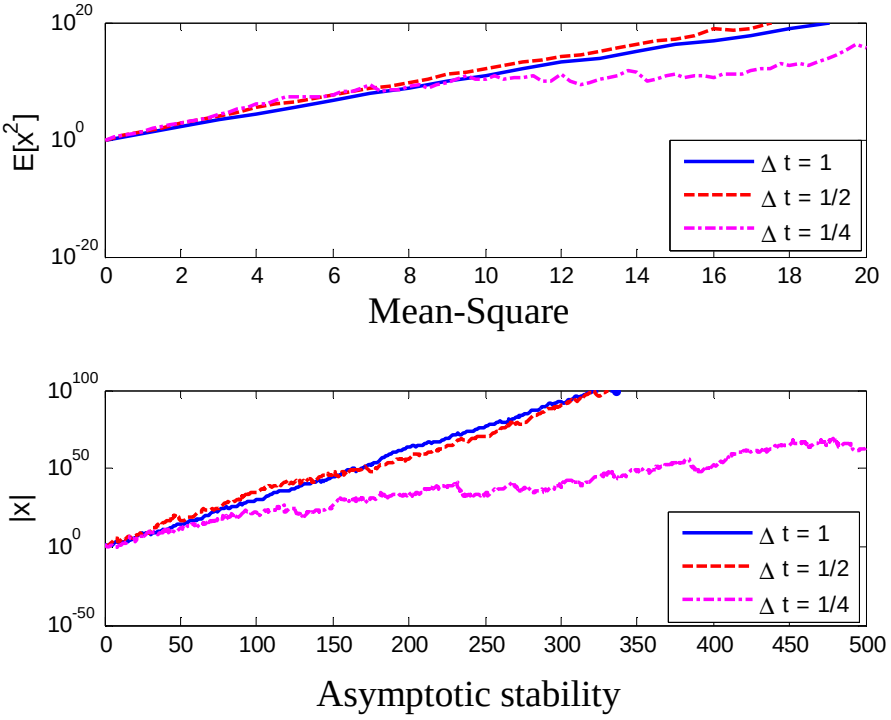


Figure (2.18) Mean-Square and Asymptotic stability for EM.

Illustration (2.13) (With Linear Stability Test):

Consider the example (2.4) of section (2.6) which is:

$dX = -bXdt + \sigma\, dW$

$X(0) = 1$

(1) Mean-Square stability:

Where $\mu = 50000$; $T = 20$; $b = 0$; $\sigma = 1$ and using step size

$\Delta t = 1, 1/2, 1/4$; $N = T/\Delta t$.

(2) Asymptotic stability:

Where T = 500; b = 1; and σ = 2.

The first part of Figure (2.19) represent the mean-square for b = 0 and σ = 1, while the second part is standing for asymptotic stability which plots the asymptotic behaviour of this example.

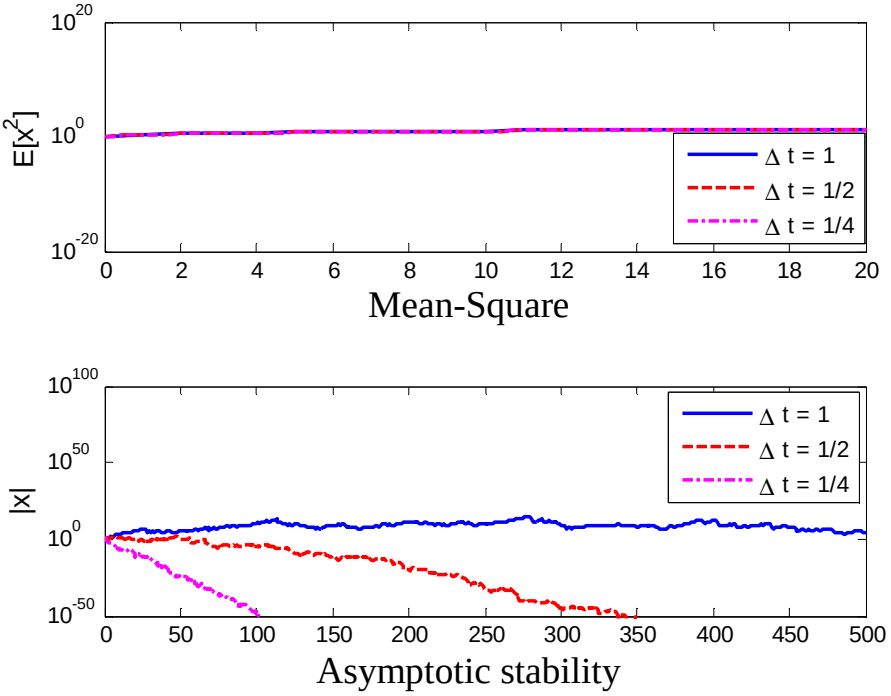


Figure (2.19) Mean-Square and Asymptotic stability for EM.

Illustration (2.14) (With Linear Stability Test):

Consider the example (2.7) of section (2.6) which is:

$dX = \lambda X dt + \mu X\, dW$

$X_0 = 1$

(1) Mean-Square stability:

$\lambda = -3$; $\mu = \sqrt{3}$; T = 20; sample path μ = 50000 and step size

$\Delta t = 1, 1/2, 1/4$; $N = T/\Delta t$.

(2) Asymptotic stability:

$\lambda = -3$; $\mu = \sqrt{6}$; T = 500; and step size $\Delta t = 1, 1/2, 1/4$; $N = T/\Delta t$.

The information of mean-square error and asymptotic linear stability may be found in the following Figure (2.20).

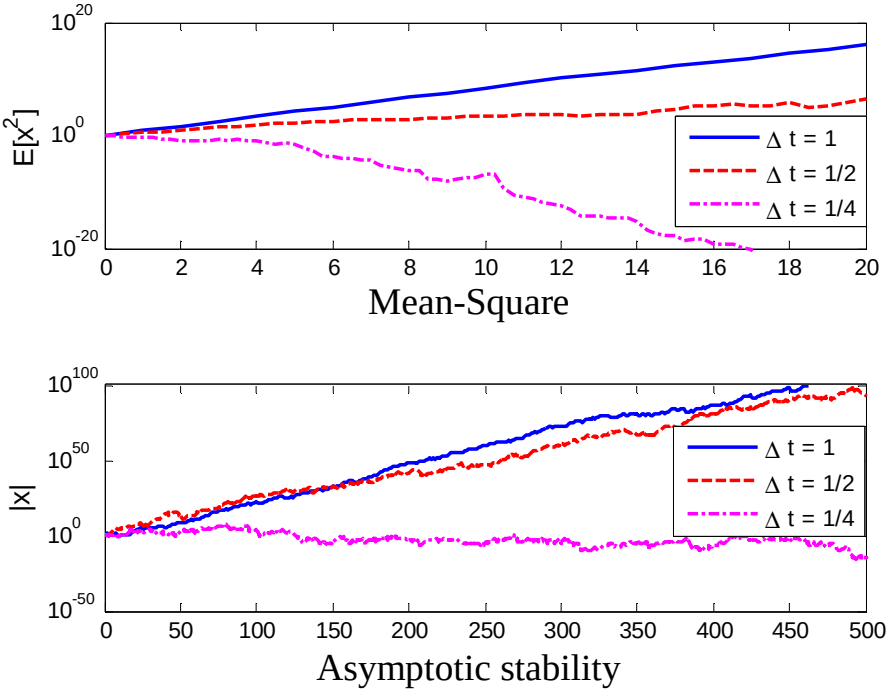


Figure (2.20) Mean-Square and Asymptotic stability for EM.

Chapter Three
Milstein's Numerical Method for Solving Stochastic Differential Equations

Early attempts are made in the area of numerical methods for stochastic differential equations using Euler-Maruyama method. [Milstein, 1974] provides an early account for constructing a numerical method for solving stochastic differential equations. This method is known as the Milstein method. [Hovanessian & Chang, 1977] proved an application of the central difference and predictor methods for finding a solution of differential equations with stochastic inputs.

Numerical methods for SDE's constructed by translating a deterministic numerical method (like the Euler method or Runge-Kutta method, etc). and applying it to a stochastic ordinary differential equation. However, merely translating a deterministic numerical method and applying it to an SDE will generally not provide accurate methods, [Burrage & Burrage, 1996]. Suitably appropriate numerical methods for SDE's should take into account a detailed analysis of the order of convergence as well as stability of the numerical scheme and the behaviour of the errors. The Euler-Maruyama (see chapter two) method for SDE's is the simplest method which is a direct translation of, the deterministic Euler method, but according to [Burrage & Burrage, 1996] this method is not very accurate. However, this method is useful in that it provides a starting point for more advanced numerical methods for SDE's.

A very concise publication by [Kloeden & Platen, 1992] provides a comprehensive and systematic presentation of numerical methods available for SDE's. The book focuses on time discretisation methods for initial value problems of SDE's with its diffusion as their solutions. Numerical methods for both the strong and weak order of convergence are presented.

The following is then focuses into understanding the Milstein numerical method for solving SDE's and their stability, error as well as its line stadety.

3.1 Milstein Scheme:

The Milstein scheme is obtained by considering the first four terms of Taylor expansion of section (2.3) in chapter two. It is given by:

$$X_t = X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s +$$

$$L^1 b(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dW_z dW_s$$

Use formula:

$$\int_0^t W_s(\omega)\, dW_s(\omega) = \frac{1}{2} W_t^2(\omega) - \frac{1}{2} t$$

to obtain:

$$\int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s dW_t = \frac{1}{2}\{(\Delta W_n)^2 - \Delta n\}$$

From equation (2.10), $L^1b = bb^1$, thus:

$$L^1b(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dW_z dW_s = \frac{1}{2} bb^t\{(\Delta W_n)^2 - \Delta n\}$$

Therefore, the Milstein scheme is defined as:

$$Y_{n+1} = Y(\tau_n) + a(\tau_n, Y(\tau_n))(\tau_{n+1} - \tau_n) + b(\tau_n, Y(\tau_n))(W_{\tau_{n+1}} - W_{\tau_n}) + \frac{1}{2} b(\tau_n, Y(\tau_n))b'(\tau_n, Y(\tau_n))\{(W_{\tau_{n+1}} - W_{\tau_n})^2 - (\tau_{n+1} - \tau_n)\}$$

For brevity, this is written as:

$$Y_{n+1} = Y_n + a\Delta n + b\Delta W_n + \frac{1}{2} bb'\{(\Delta W_n)^2 - (\Delta n)\}$$

The term b' is the partial derivative of b with respect to x., i.e., $b' = \frac{\partial b}{\partial x}$.

Concluding Remarks (3.1) (Explicit and Implicit Numerical Method):

1. The Euler scheme is an example of a time discrete approximation (or difference method) in which the continuous time differential equation is replaced by a discrete time difference equation generating values Y_1, Y_2, …, Y_n to approximate X_{i_1}, X_{i_2}, …, X_{i_n} at given discretisation times $t_0 < t_l < ... < t_n$. The Euler scheme is the simplest strong Taylor approximation and attains an order of convergence $y = 0.5$. The proof is given in [Koleden & Platen, 1992].
2. The Euler-Maruyama scheme has order $\gamma = 0.5$, the Milistein scheme has and order $\gamma = 1$, [Kloden & Platen, 1992].
3. The orders of strong and weak convergence of the stochastic Euler and Milistein schemes are low. In order to improve the order of convergence, multiple stochastic integrals of W_t are included in the

numerical scheme. This is because the simple increments ΔW_t do not provide enough information about the sample paths of Wiener process W_t inside the discretization subinterval $[\tau_{n+1}, \tau_n]$ to ensure higher order of approximation.

4. Generally, the numerical scheme implicitly uses a linear interpolation in the subinterval. Multiple stochastic integrals of W_t that occur in the stochastic Taylor expansion provide additional information about the sample paths of the driving Wiener process within the discretization interval.

5. A more accurate order 1.5 strong Taylor scheme can be obtained by including further multiple stochastic integrals from the stochastic Taylor expansion in the scheme. The order $\gamma = 1.5$ strong Taylor scheme is derived by adding more terms from the Itô-Taylor expansion to the Milstein scheme.

   The order 1.5 strong Taylor scheme is given as (see [Kloeden & Platen, 1992]):

$$X_{n+1} = X_n + a\Delta n + b\Delta W_n + \frac{1}{2}bb'\{(\Delta W_n)^2 - \Delta n\} + b\frac{\partial a}{\partial x}\Delta Z_n +$$
$$\left\{\frac{\partial b}{\partial t} + \frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\right\}(\Delta W_n \Delta n - \Delta Z_n\} +$$
$$\frac{1}{2}\left\{\frac{\partial a}{\partial t} + a\frac{\partial a}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 a}{\partial x^2}\right\}(\Delta n)^2 +$$
$$\frac{1}{2}\left\{b\frac{\partial^2 b}{\partial x^2} + \left(\frac{\partial b}{\partial x}\right)^2\right\}\left\{\frac{1}{3}(\Delta W_n)^2 - \Delta n\right\}\Delta W_n$$

6. Using strong Taylor approximations involves determining and evaluating the derivatives of the various orders of the drift and

diffusion coefficient as well as the coefficients themselves for each step. Implementing this procedure can be time consuming. Therefore, to avoid the use of derivatives, the derivatives in the strong schemes are replaced by their corresponding finite differences.

Using the Milstein scheme, one can derive the explicit order 1 scheme by replacing the derivatives by the corresponding difference ratios. However, these differences require the use of supporting values of the coefficients at additional points.

Use the following Milstein scheme:

$$Y_{n+1} = Y_n + a\Delta n + b\Delta W_n + \frac{1}{2}bb'\{(\Delta W_n)^2 - (\Delta n)\}$$

and replace the derivative b' with finite differences, to obtain the explicit order 1 strong scheme which has the following form [KIoeden & Platen, 1992]

$$Y_{n+1} = Y_n + a\Delta n + b\Delta W_n + \frac{1}{2\sqrt{\Delta n}}\{b(\tau_n, \bar{Y}_n) - b(\tau_n, Y_n)\}\{(\Delta Y_n)^2 - \Delta n\}$$

Where:

$$\bar{Y}_n = Y_n + a\Delta n + b\sqrt{\Delta n}$$

An explicit order 1.5 strong scheme can also be derived by replacing the derivatives in the order 1.5 strong Taylor scheme by corresponding finite differences.

7. In principle, the derivatives of the strong Taylor schemes can be replaced to obtain corresponding explicit schemes. This procedure may work well for low order explicit schemes but as the order is increased the formulae become more complicated. Sometimes, the

special structure of the equation under consideration can be used to derive relatively simple higher order explicit schemes which do not involve the derivatives of the drift and diffusion coefficients. [Kloeden & Platen, 1992] provide an explicit order 2 strong scheme for additive noise, using the Stratonovich notation, to simplify the notation. This is possible, since an SDE which is written using Ito integrals can be equivalently written using Stratonovich integrals (see [Kloeden & Platen, 1992]).

8. The implicit Milstein scheme is derived analogously:

$$Y_{n+1} = Y_n + a(\tau_{n+1}, Y_{n+1})\Delta n + b\Delta W + \frac{1}{2}bb'\{(\Delta W)^2 - (\Delta n)\} \ldots(3.1)$$

where again only the drift term contains the unknown y_{n+1}.

The family of implicit schemes:

$$Y_{n+1} = Y_n + \{\alpha a(\tau_{n+1}, Y_{n+1}) + (1-\alpha)a\}\Delta n + b\Delta W + \frac{1}{2}bb'\{(\Delta W)^2 - (\Delta n)\}$$

where again, $\alpha \in [0, l]$ characterizes the degree of implicitness. When $\alpha = 0$, the explicit Milstein scheme is obtained; the implicit scheme when $\alpha = 1$ and for $\alpha = 0.5$ the generalization of the deterministic trapezoidal method is obtained.

Implicit schemes for the order 1.5 and order 2 strong Taylor schemes can also be obtained, as well as derivative free schemes and multi-step schemes. [Kloeden & Platen, 1992] provide the implicit versions of these schemes for the 1-dimensional as well as multi-dimensional case. In addition, they also consider the special cases when there is additive and commutative noise, sometimes using the equivalent Stratonovich representation to ease the notation.

Concluding Remarks (3.1) (Numerical Steps for Explicit Milstein's Methods and Linear Stability):

1. SDE can be written in integral form as:

$$X(t) = X_0 + \int_0^t \lambda X(s)\, ds + \int_0^t \mu X(s)\, dW(s) + \int_0^t \int_0^s \mu X(z)\mu' X(s)\, dW(z)dW(s),\ 0 \le t \le T \qquad \ldots(3.2)$$

   Here f and g are scalar functions and the time condition X_0 is the random variable. The second integral on the right-hand side of (3.2) is to be taken with respect to Brownian motion as discussed in the previous section.

2. To rewrite (3.2) in differential equation form as:

$$dX(t) = \lambda X(t)\, dt + \mu X(t)\, dW(t) + \frac{1}{2}\mu X(t)\mu' X(t)((dW(t))^2 - dt),$$

$$X(0) = X_0,\ 0 \le t \le T \qquad \ldots(3.3)$$

   this is nothing more than a compact way of using that X(t) solves (3.2). To keep with connection, we will emphasize the SDE from (3.3) rather than the integral from (3.2).

3. Note that, we are not allowed to write dW(t)/dt. Since motion is nowhere differentiable with probability 1.

4. If $g = 0$ and x_0 is constant, then the problem becomes deterministic and (3.3) reduces to the ordinary differential equation $dx(t)/dt = \lambda x(t)$, with $x(0) = x_0$.

5. To apply a numerical method to (3.3) over [0, T] we first discretize the interval. Let $\Delta t = T/L$ for some positive integer L, and $T_j = j\Delta t$ for

$j = 1, 2, \ldots, L$. Our numerical approximation to $X(T_j)$ will be denoted X_j. The Milstein's method takes the form:

$$X_j = X_{j-1} + \lambda X_{j-1}\Delta t + \mu X_{j-1}(W(T_j) - W(T_{j-1})) + \frac{1}{2}\mu X_{j-1}\mu' X_{j-1}((W(T_j)-W(T_{j-1}))^2-\Delta t),\ j=1,2,\ldots,L \qquad \ldots(3.4)$$

6. To understand where (3.4) comes from, notice from the integral form (3.2) that:

$$X(T_j) = X(T_{j-1}) + \int_{T_{j-1}}^{T_j} \lambda X(s)\, ds + \int_{T_{j-1}}^{T_j} \mu X(s)\, dW(s) + \frac{1}{2}\int_{T_{j-1}}^{0}\int_{0}^{T_j} \mu X(z)\mu'(s))\, dW_z dW_s \qquad \ldots(3.5)$$

Each of the three terms on the right-hand side of (3.4) approximates the corresponding term on the right-hand side of (3.5).

7. We also note that in the deterministic case ($g = 0$ and X_0 constant) (3.4) reduce to Milstein's method.

8. For computational purposes, it is useful to consider discrtized Brownian motion, where $W(t)$ is specified at discrete t values, we thus set $\delta t = T/N$ for some positive integer N and let W_j denote $W(t_j)$ with $t_j - j\delta t$ condition 1 says $W_0 = 0$ with probability 1 us that:

$$W_j = W_{j-1} + dW_j,\ j = 1, 2, \ldots, N \qquad \ldots(3.6)$$

Where each dW_j is an independent random variable of the form $\sqrt{\delta t}\ N(0, 1)$.

9. In this chapter, we will compute our own discretized Brownian path and use them to generate the increments $W(T_j) - W(T_{j-1})$ needed in (3.4). For convenience, we always choose the step size Δt for the numerical method to be an integer multiple $R \geq 1$ of the increment δt for the Brownian path.
10. This ensures that the set of points $\{t_j\}$ on which the discretized Brownian path is based contains the points $\{T_j\}$ at which the Milstein's method solution is computed.
11. Linear stability: For stability, one can see concluding remark (2.4) in chapter two, the same is true for Milestien's method excepted changing:

$$dX(t) = \lambda X(t)\,dt + \mu X(t)\,dW(t),\ X(0) = X_0$$

by:

$$dX(t) = \lambda X(t)\,dt + \mu X(t)\,dW(t) + \frac{1}{2}\mu X(t)\mu' X(t)((dW(t))^2 - dt),$$

$$X(0) = X_0$$

Concluding Remarks (3.2) (Numerical Steps of Strong Converge for Milstein's):

1. We saw in the previous section that EM has strong order of converges $\gamma = 1/2$ in $E|X_n - X(T)| \leq C\Delta t^{\gamma}$, the underlying deterministic Milestien's method converges with classical order 1. It is possible to raise the strong order of EM 1 by adding a correction to the stochastic increment gives Milstein's method.

2. The correction arises because the traditional Taylor expansion must be modified in the case of Itô calculus. A so called Itô Taylor can be formed by applying Itô's result, which is a fundamental tool of stochastic calculus truncating the Itô-Taylor expansion at an appropriate point produces Milstein's method for the SDE (2.33).

$$X_j = X_{j-1} + \Delta t f(X_{j-1}) + g(X_{j-1})(W(T_j) - W(T_{j-1})) + \frac{1}{2} g(X_{j-1}) g'(X_{j-1})((W(T_j) - W(T_{j-1}))^2 - \Delta t,\ j = 1, 2, \ldots, L \quad \ldots(3.7)$$

3.2 Algorithm and Illustrations:

Algorithm (2.1) (The Explicit Milstein's Method):

Input: The dynamic stochastic differential equation in problem formulation (3.3).

Output: Numerical (sample path) solution of stochastic process.

Step 1: Consider problem formulation (3.3).

Step 2: Generating a Brownian motion as follows (see concluding remark (3.1)):

Step 2.1: Generate a normal random number.

Step 2.2: Consider $T = t_0$; $N = n_0$; step size $dt = T/N$;

Step 2.3: $W(t) \sim N(0, 1)$.

Step 2.4: Set $W_0 = 0$ with probability 1.

Step 3.5: $W_j = W_{j-1} + dW_j$; $j = 1, 2, \ldots, N$.

Step 2.6: $dW_j \sim \sqrt{\delta t}\, N(0, 1)$.

Step 3: Set $j = 1 \longrightarrow L$

$$W(T_j) - W(T_{j-1}) = W(jR\delta t) - W((j-1)R\delta t)$$

$$= \sum_{k=jR-R+1}^{jR} dW_k$$

$$X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{j-1})(W(T_j) - W(T_{j-1})) +$$

$$\frac{1}{2} g(X_{j-1})g'(X_{j-1})((W(T_j)-W(T_{j-1}))^2-\Delta t),$$

If $j \geq L$ stop.

Step 4: Computation of error, depending on the type of error for example, the following is absolute error

Step 4.1: If $\Delta t = \delta t$

Set $i = 1 \longrightarrow L$

$error = abs(X_n(i) - X_T(i))$

Step 4.2: If $\Delta t \neq \delta t$

$error = abs(X_n(final) - X_T(final))$.

Step 4.3: For strong convergence (see concluding remark (3.2)).

Illustration (3.1):

Consider the example (2.1) of section (2.6) in chapter two, which is:

$$dX = gX\, dW$$

$$X(0) = 1$$

Where, $g(t) = \sin t$; $X_0 = 1$; $Y_0 = 0$

The absolute error at the final time interval for different sample space numbers, where $\Delta t = \delta t$; R = 1; the step time for discritization of Brownian motion equals to the step time discretization of explicit Euler scheme are shown in the following (table (3.1)) and Figure (3.1.1). As one can see, increasing the sample space generated randomly (N) leads to improving the absolute error at the different time steps, where $\Delta t = \delta t$. The Figure 3.1) show the very good agreement between the exact solution and the corresponding numerical solution (see program A.6 in Appendix A).

Table (3.1) Error generated by the explicit Milstein scheme.

R	***N***	***Error at final time***
1	2^5	0.0023
	2^6	0.0012
	2^7	0.0009

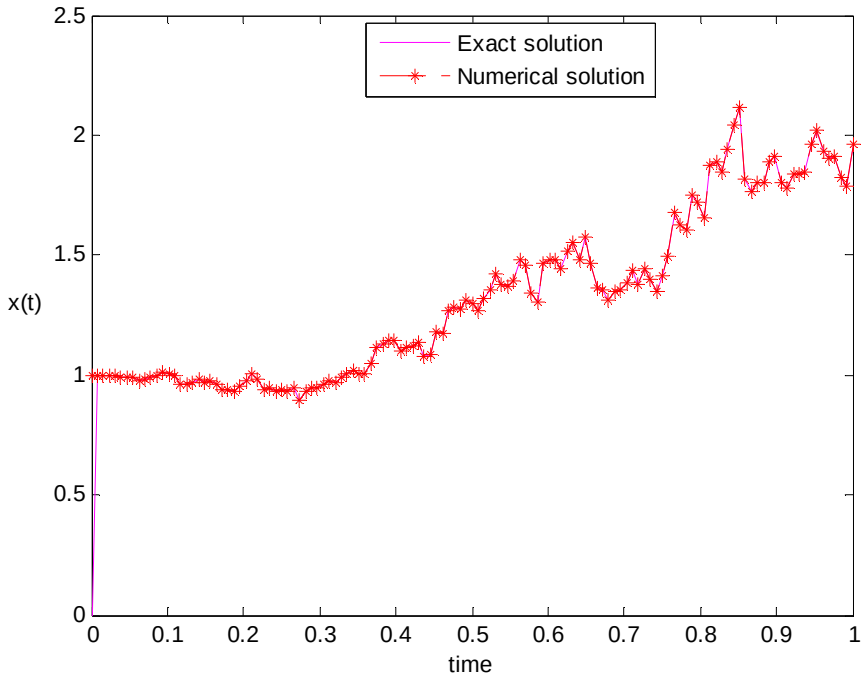


Figure (3.1) Exact solution and the numerical solution by Milstein's scheme with $N = 2^7$; $R = 1$.

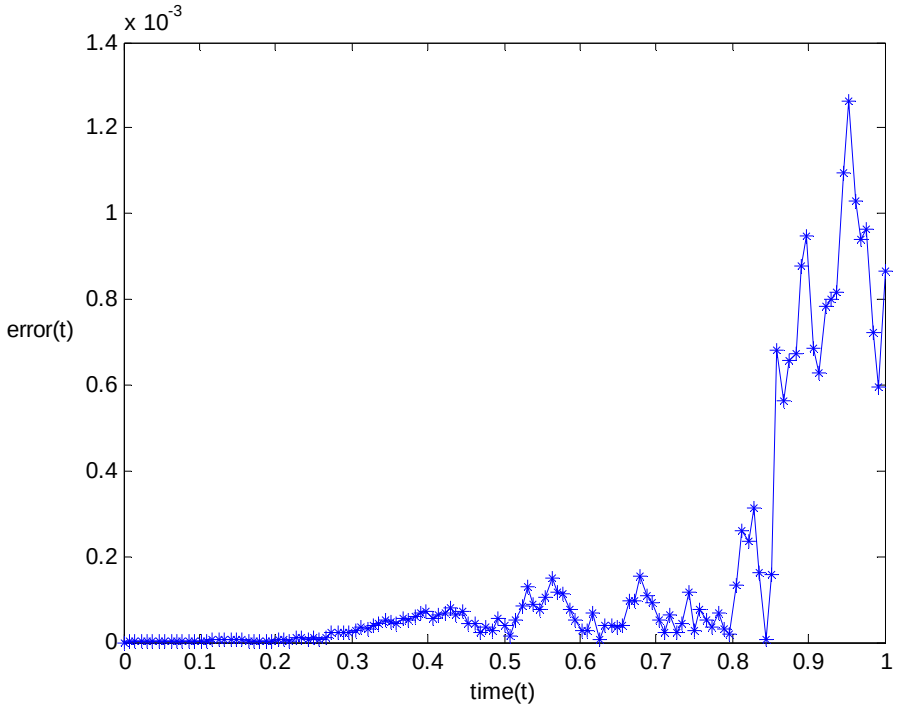


Figure (3.1.1) Absolute error between the Milstein's scheme and exact solution with $N = 2^7$; $R = 1$.

Illustration (3.2):

Consider the example (2.2) of section (2.6) in chapter two, which is:

$$dX = fX\,dt + gX\,dW$$

$$X(0) = 1$$

Where, $f(t) = \cos t$; $g(t) = \sin t$; $X_0 = 1$; $Y_0 = 0$

The values absolute error at the final time interval $T = 1$; for $R = 1$, can be shown in table (3.2). As one can see, generating final time error is obtained for sample space $N = 2^8$.

Table (3.2) Error generated by the explicit Milstein scheme.

R	*N*	*Error at final time*
1	2^5	0.0152
	2^6	0.0356
	2^7	0.0270
	2^8	0.0011

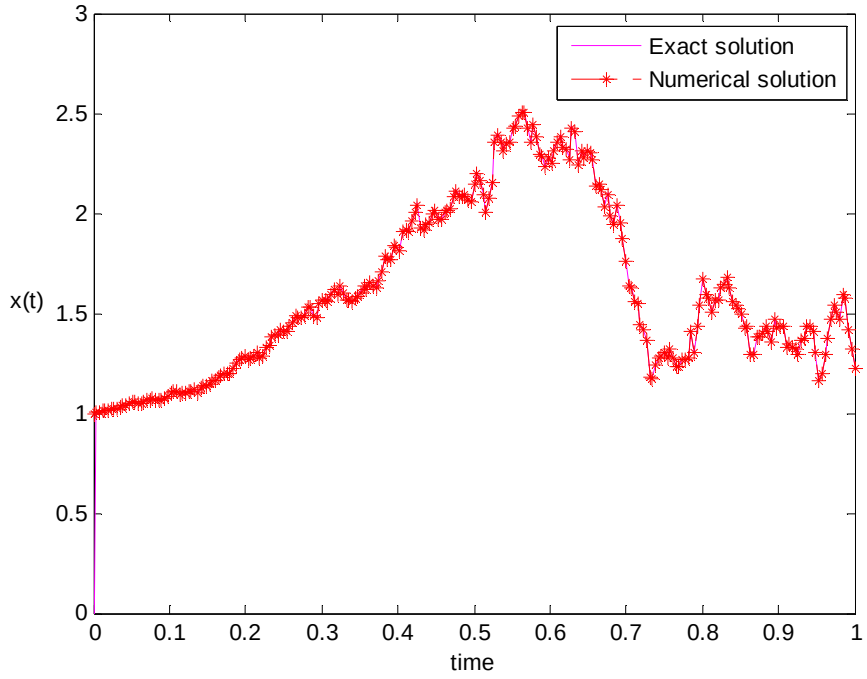


Figure (3.2) Exact solution and the numerical solution by Milstein's scheme with $N = 2^8$; $R = 1$.

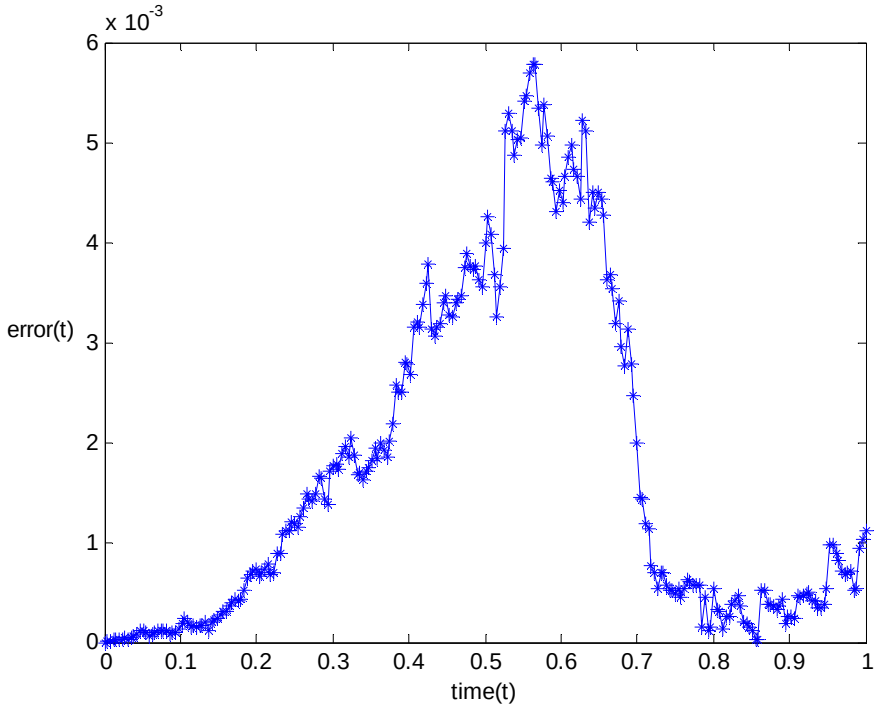


Figure (3.2.1) Absolute error between the Milstein's scheme and exact solution with $N = 2^8$; $R = 1$.

Illustration (3.3):

Consider the example (2.3) of section (2.6) in chapter two, which is:

$$dP = \mu P\, dt + \sigma P\, dW$$

$$P(0) = P_0$$

Where, $P_0 = 1$; $\mu = 1$; $\sigma = 2$.

Same error estimations is represented in the following table (3.3).

Table (3.3) Error generated by the explicit Milstein scheme.

R	***N***	***Error at final time***
1	2^5	0.0304
	2^6	0.1997
	2^7	0.2294
	2^8	0.0071

Figure (3.3) on using the R = 1; N = 2^8, the following plot of numerical solution of Milstein method is showing with the exact solution.

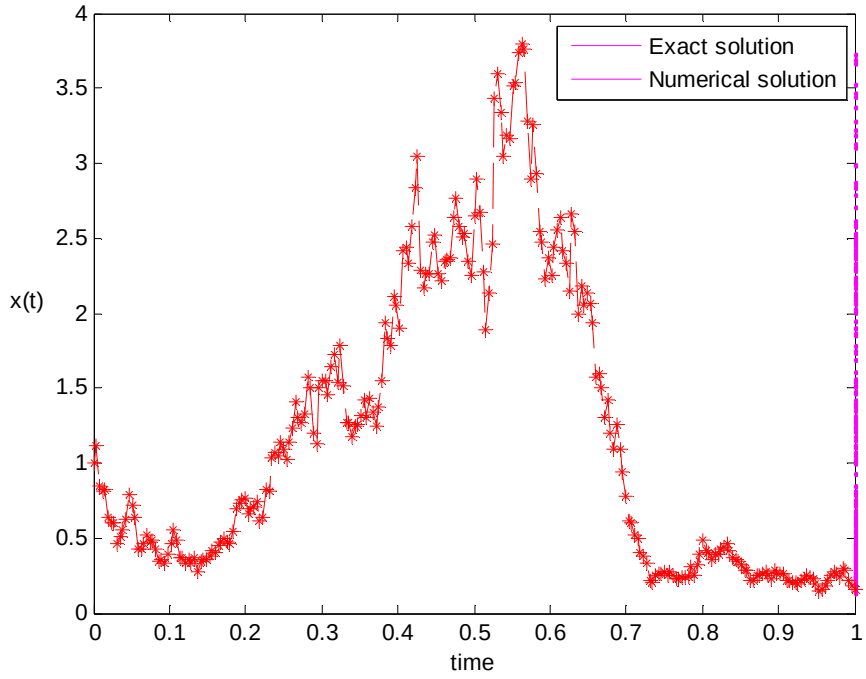


Figure (3.3) Exact solution and the numerical solution by Milstein's scheme with $N = 2^8$; $R = 1$.

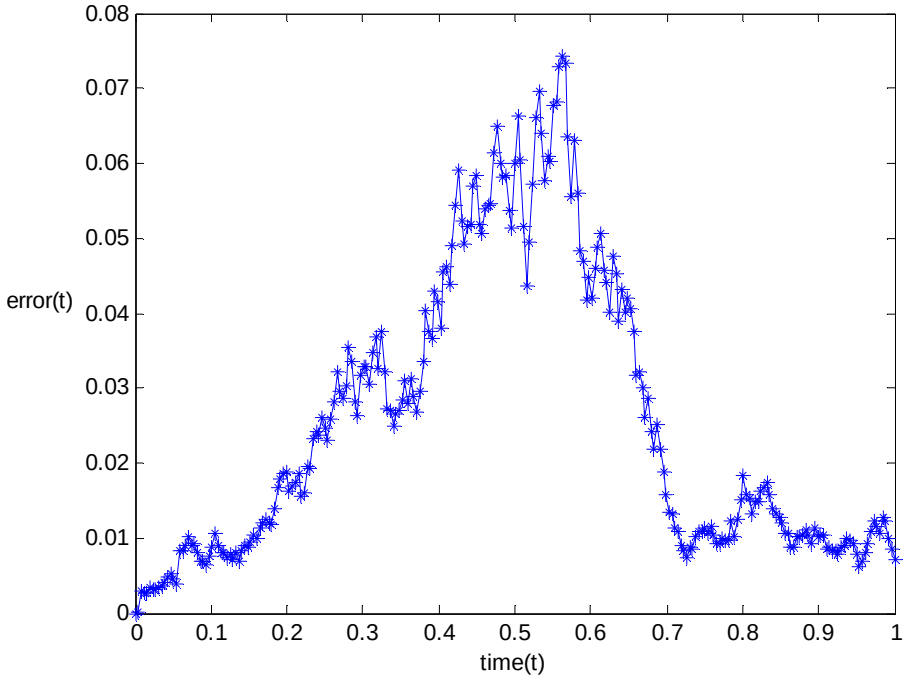


Figure (3.3.1) Absolute error between the Milstein's scheme and exact solution with $N = 2^8$; $R = 1$.

Illustration (3.4):

Consider the example (2.4) of section (2.6) in chapter two, which is:

$$dX = -bX\,dt + \sigma\,dW$$

$$X(0) = X_0$$

Where, $X_0 = 1$; $Y_0 = 0$; $b = 2$; $\sigma = 1$

As discussed previously in illustration (3.1), the following table (3.4) is needed for error analysis and as follows:

Table (3.4) Error generated by the explicit Milstein scheme.

R	***N***	***Error at final time***
1	2^5	0.9633
	2^6	0.3539
	2^7	0.4183
	2^8	0.0560

Figure (3.4) on using the R = 1; N = 2^8, the following plot of numerical solution of Milstein method is showing with the exact solution.

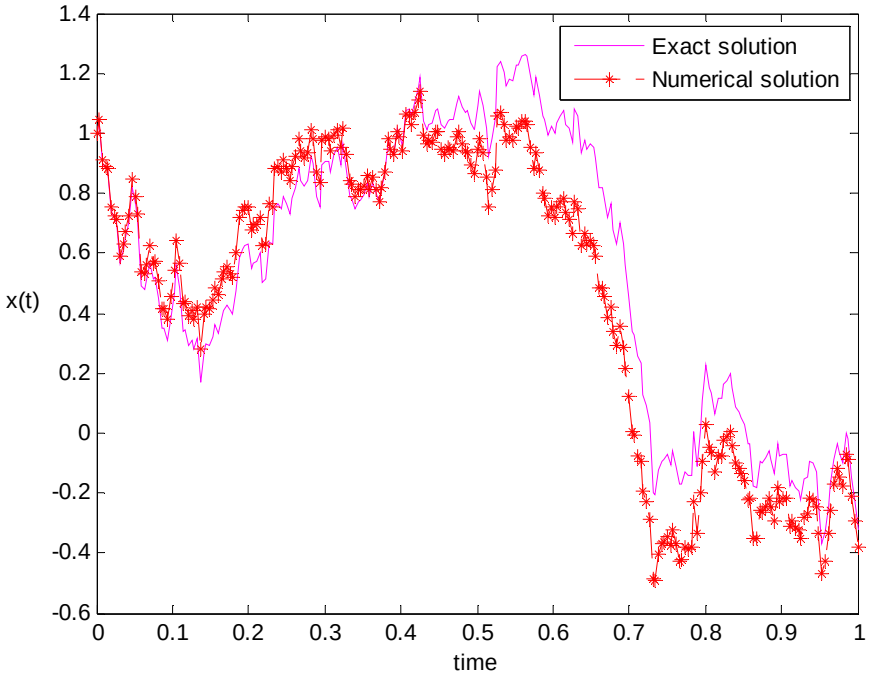


Figure (3.4) Exact solution and the numerical solution by Milstein's scheme with $N = 2^8$; $R = 1$.

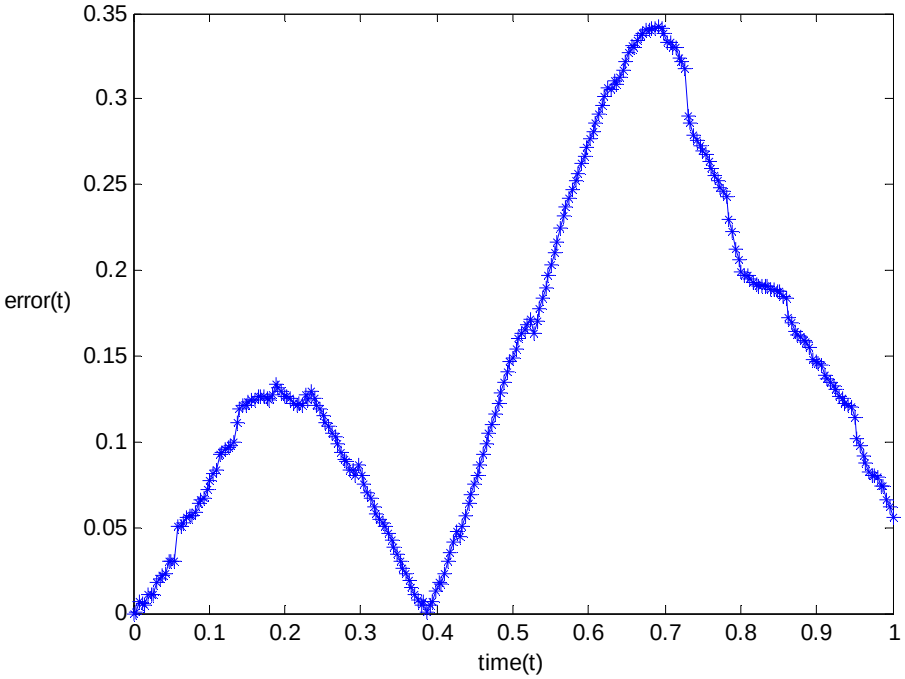


Figure (3.4.1) Absolute error between the Milstein's scheme and exact solution with $N = 2^8$; $R = 1$.

Illustration (3.5):

Consider the example (2.7) of section (2.6) in chapter two, which is:

$$dX = \lambda X\, dt + \mu X\, dW$$

$$X(0) = 1$$

Where, $\lambda = 2$; $\mu = 1$; $X_0 = 1$.

As one can see, the error is improved for this case. The following numerical solution is adopted for R =1; $N = 2^{11}$; and R = 2; $N = 2^{11}$ and R = 3; $N = 2^{11}$; R = 4 and $N = 2^{10}$ as one can see in figures (3.5)-(3.7).

Table (3.5) Error generated by the Milstein scheme.

R	*N*	*Error at final time*
1	2^5	0.1826
	2^6	0.0834
	2^7	0.1083
	2^8	0.0196
	2^9	0.1083
	2^{10}	0.0118
	2^{11}	0.0015
2	2^5	0.2469
	2^6	0.0743
	2^7	0.3855
	2^8	0.0291
	2^9	0.0558
	2^{10}	0.0256
	2^{11}	0.0032
3	2^5	0.4462
	2^6	0.1569
	2^7	1.7240
	2^8	0.2899
	2^9	0.4118
	2^{10}	0.2884
	2^{11}	0.1519
4	2^5	0.7802
	2^6	0.2335
	2^7	0.5157
	2^8	0.1018
	2^9	0.0345
	2^{10}	0.0198

Here, the numerical solution are discussed for different values of R, i.e., $\Delta t = R\delta t$, there are different below the time steps of Brownian motion and the numerical one. The absolute error can be evaluated at all the values in the selected interval, but only on the final time interval. As one can see the effect of selection of R on the absolute error.

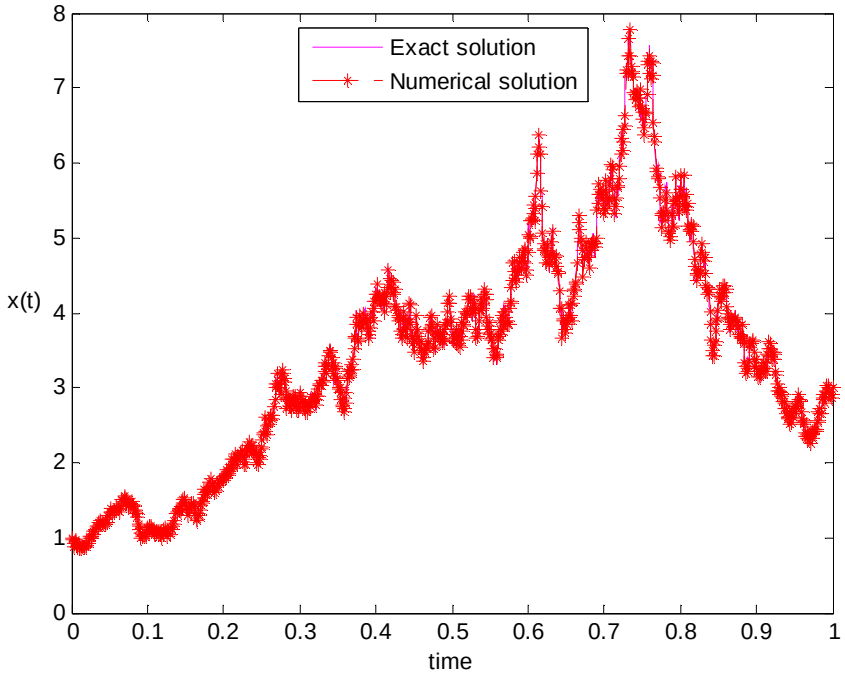


Figure (3.5) Exact solution and the numerical solution by Milstein's scheme with $N = 2^{11}$; $R = 2$.

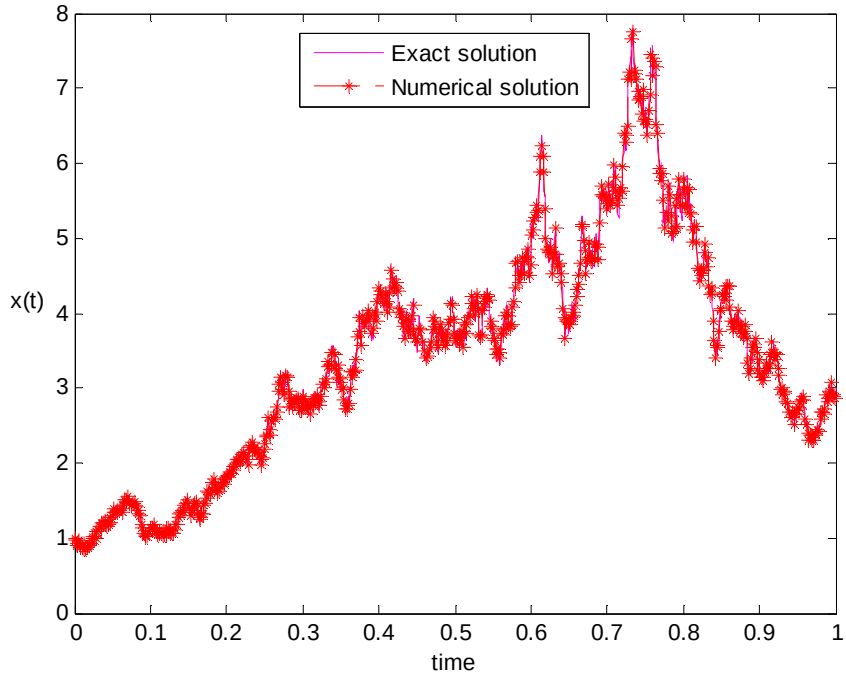


Figure (3.6) Exact solution and the numerical solution by Milstein's scheme with $N = 2^{11}$; $R = 3$.

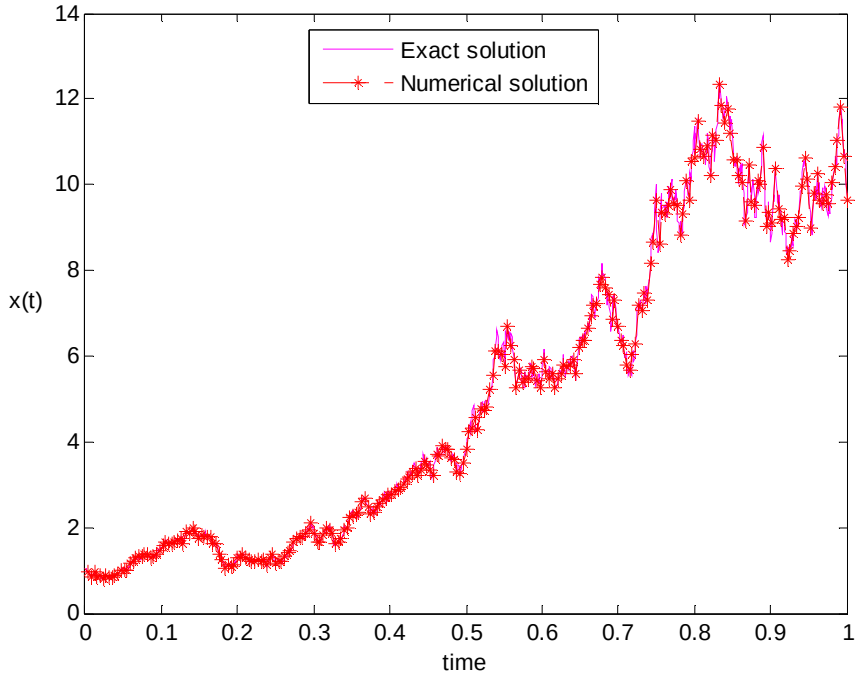


Figure (3.7) Exact solution and the numerical solution by Milstein's scheme with $N = 2^{10}$; $R = 4$.

The strong convergence for some test examples are shown and discussed in the following:

Illustration (3.6):

Consider the example (2.1) of section (2.6) in chapter two, which is:

$$dX = gX\,dW$$

$$X(0) = 1$$

Where:

$g(t) = \sin t$; $X_0 = 1$; $N = 2^{11}$; $T = 1$; sample path $\mu = 500$

Depending on the concluding remark (3.2), the strong convergence for the numerical solution using Euler-Maruyama method is obtained. This produces the blue asterisks connected with solid lines in the plot of Figure (3.8). For reference, a dashed red line of slope one-half is added. The least-squares power law fit gives q = 1.0132 and resid = 0.0166 (see program A.7 in Appendix A).

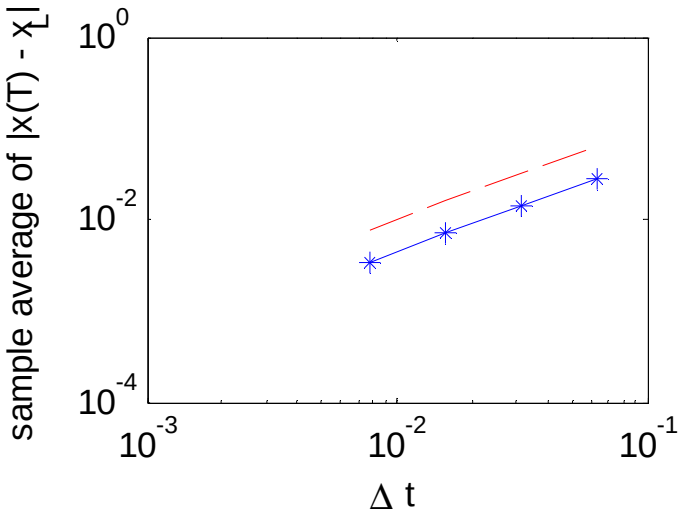


Figure (3.8) Milstein strong convergence.

Illustration (3.7):

Consider the example (2.2) of section (2.6) in chapter two, which is:

$$dX = fX\,dt + gX\,dW$$

$$X(0) = 1$$

Where:

$F(t) = \cos t$; $g(t) = \sin t$; $X_0 = 1$; $N = 2^{11}$; $T = 1$;

sample path $\mu = 500$

The strong convergence plot is as follows:

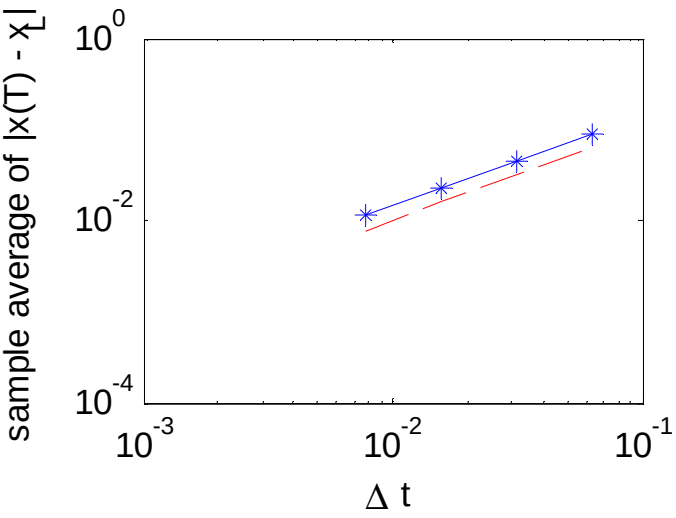


Figure (3.9) Milstein strong convergence.

Illustration (3.8):

Consider the example (2.7) of section (2.6) in chapter two, which is:

$$dX = \lambda X\,dt + \mu X\,dW$$

$$X(0) = 1$$

(1) Mean-Square stability:

Where $\lambda = -3$, $\mu = \sqrt{3}$; and $T = 20$; $\mu = 50000$; $X_0 = 1$;

$\Delta t = 1, 1/2, 1/4$.

(2) Asymptotic stability:

Where $\lambda = 0.5$, $\mu = \sqrt{6}$; and $T = 500$; $\Delta t = 1, 1/2, 1/4$.

The following Figure (3.11) plots the sample average of $E(X^2)$ against t in this picture the $\Delta t = 1$ and $\Delta t = 1/2$ curves increase with t, while the $\Delta t = 1/4$ curve decays toward zero (see program A.8 in Appendix A).

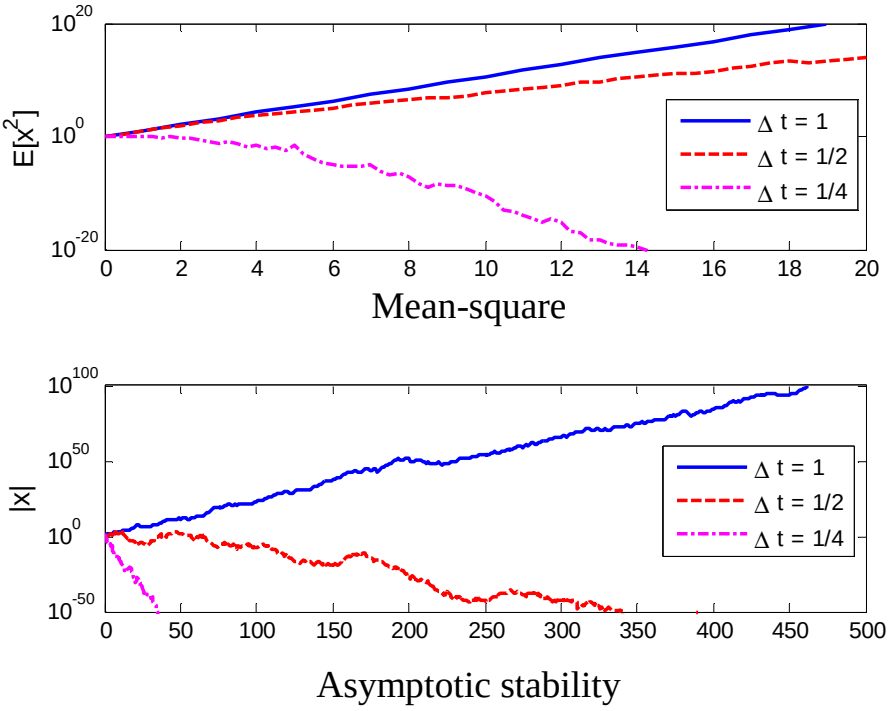


Figure (3.10) Mean-Square and Asymptotic stability by Milstein scheme.

Illustration (3.9):

Consider the example (2.1) of section (2.6) in chapter two, which is:

$$dX = gX\,dW$$

$$X_0 = 1$$

(1) Mean-Square stability:

Where g(t) = sint; T = 20; $X_0 = 1$; $\Delta t = 1, 1/2, 1/4$;

and sample path $\mu = 50000$

(2) Asymptotic stability:

Where g(t) = sint; T = 500; $\Delta t = 1, 1/2, 1/4$;

The following figures are then obtained:

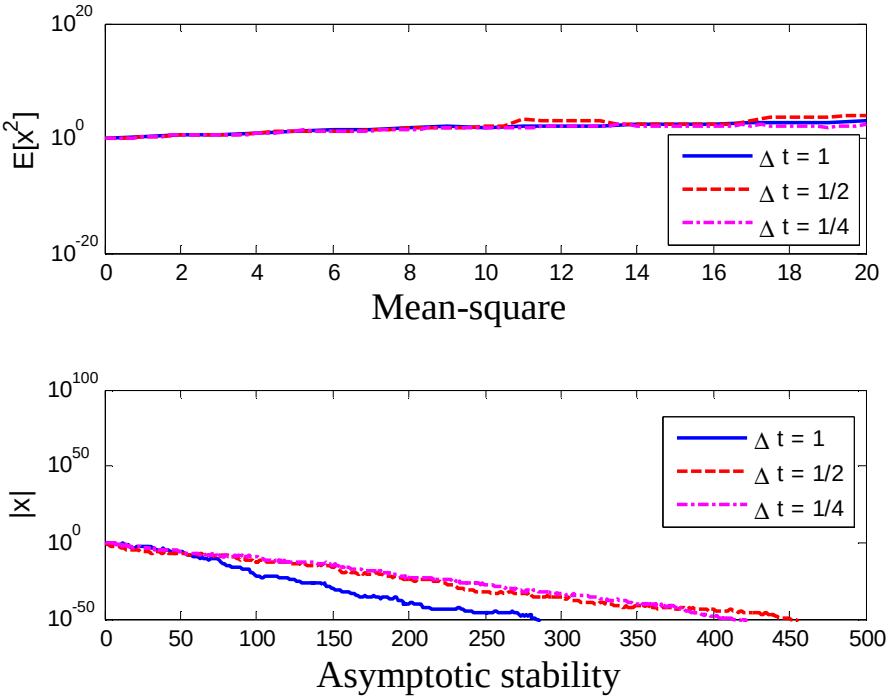


Figure (3.11) Mean-Square and Asymptotic stability by Milstein scheme.

Illustration (3.10):

Consider the example (2.2) of section (2.6) in chapter two, which is:

$$dX = fX\,dt + gX\,dW$$

$$X_0 = 1$$

(1) Mean-Square stability:

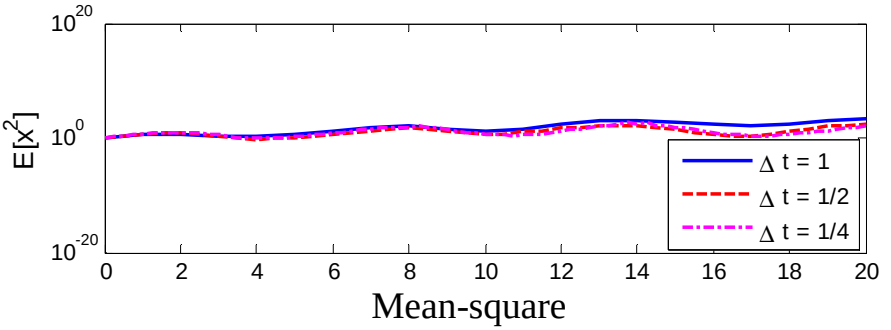
Where f(t) = cost; g(t) = sint; T = 20; $X_0 = 1$; $\Delta t = 1, 1/2, 1/4$;

and sample path $\mu = 50000$

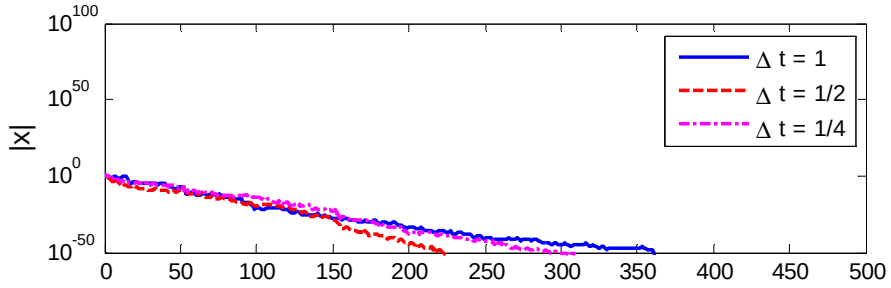
(2) Asymptotic stability:

Where g((t) = cost; g(t) = sint; T = 500; $\Delta t = 1, 1/2, 1/4$;

The numerical solution can be shown in the following figures:



Mean-square



Asymptotic stability

Figure (3.12) Mean-Square and Asymptotic stability by Milstein scheme.

Chapter Four
Comparison and Conclusions

4.1 Comparisons (Summary of Numerical Results):

4.1.1 Comparison (of Example (2.1) of Section (2.6) in Chapter Two):

This section focuses on some comparisons and conclusions of the presented work based on the test problems and methods, like, Euler scheme and Milstein scheme. The following section step-by-step comparisons are obtained for (example (2.1) of section (2.6) in chapter two) and parameters to study the proposed numerical method, so that one can conclude his final decision easily and as follows.

Table (4.1) provides a summary of the numerical schemes and the associated error at final time for the SDE with EM and Milstein as follows:

Table (4.1) Comparison of errors: SDE with EM and Milstein.

Scheme	***R***	$N = 2^5$	$N = 2^6$	$N = 2^7$	***Absolute error at final time***
Euler scheme	1	0.0409	0.0058	0.0742	
Milstein scheme	1	0.0023	0.0012	0.0009	

The error of the Euler scheme decreases when the number of discritization points increases from $N = 2^5$ to $N = 2^7$. The Milstein scheme performs better than the Euler scheme. Also, the error analysis

for N = 26 and R = 1 for t ∈ [0, 1] using example (2.1) have been plotted in the following figures for comparison point of view.

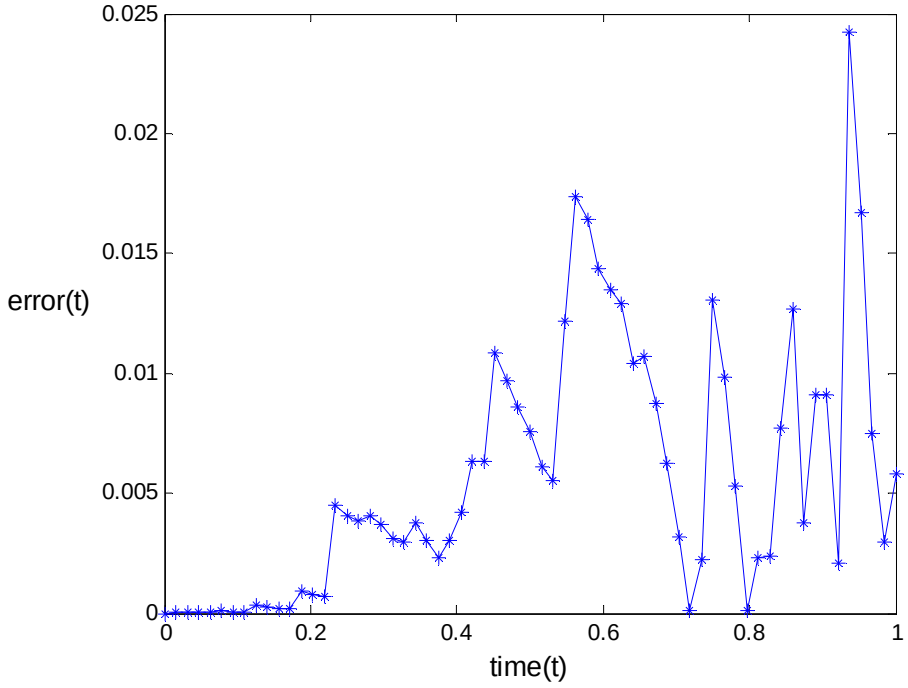


Figure (2.4.1) Absolute error between the Euler scheme and exact solution with $N = 2^6$; $R = 1$.

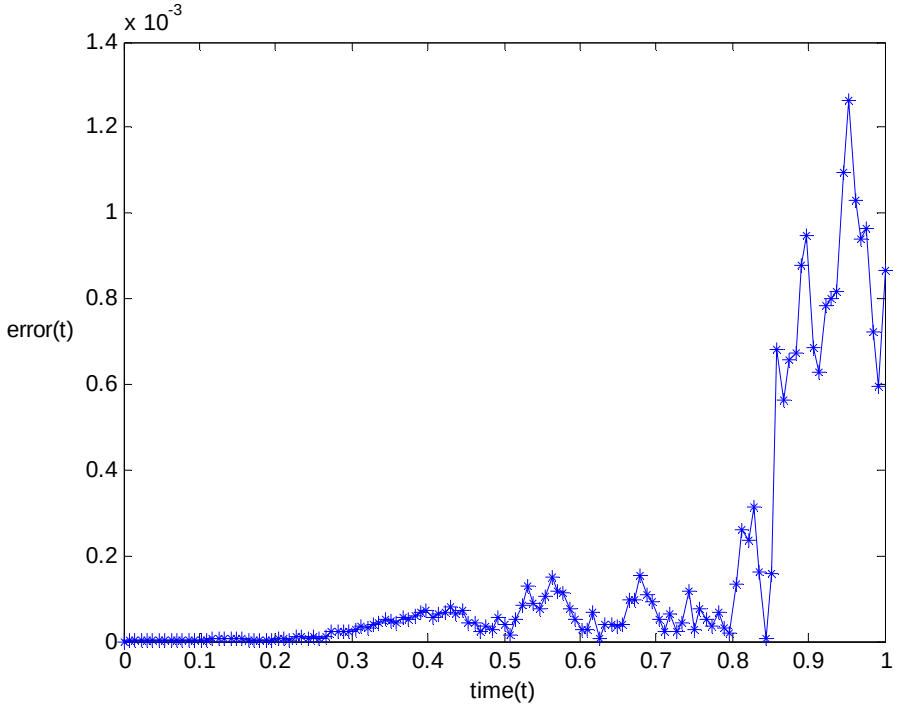


Figure (3.1.1) Absolute error between the Milstein's scheme and exact solution with $N = 2^7$; $R = 1$.

4.1.2 Comparison (of Example (2.2) of Section (2.6) in Chapter Two):

Table (4.2) provides a summary of the numerical schemes and the associated error at final time for the SDE with EM and Milstein for example (2.2) of section (2.6) in chapter two, as follows:

Table (4.2) Comparison of errors: SDE with EM and Milstein.

Scheme	***R***	$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$	$N=2^{11}$	***Absolute error at final time***
Euler scheme	1	0.0821	0.0250	0.1409	0.0123	0.0526	0.0230	0.0088	
Milstein scheme	1	0.0125	0.0356	0.0270	0.0011	0.0031	0.0012	0.0005	

The error of the Euler scheme decreases when the number of discritization points increases from $N = 2^7$ to $N = 2^{11}$. The Milstein scheme performs better than the Euler scheme as one can see this from the above table, even for lesser number of point laws than Euler method.

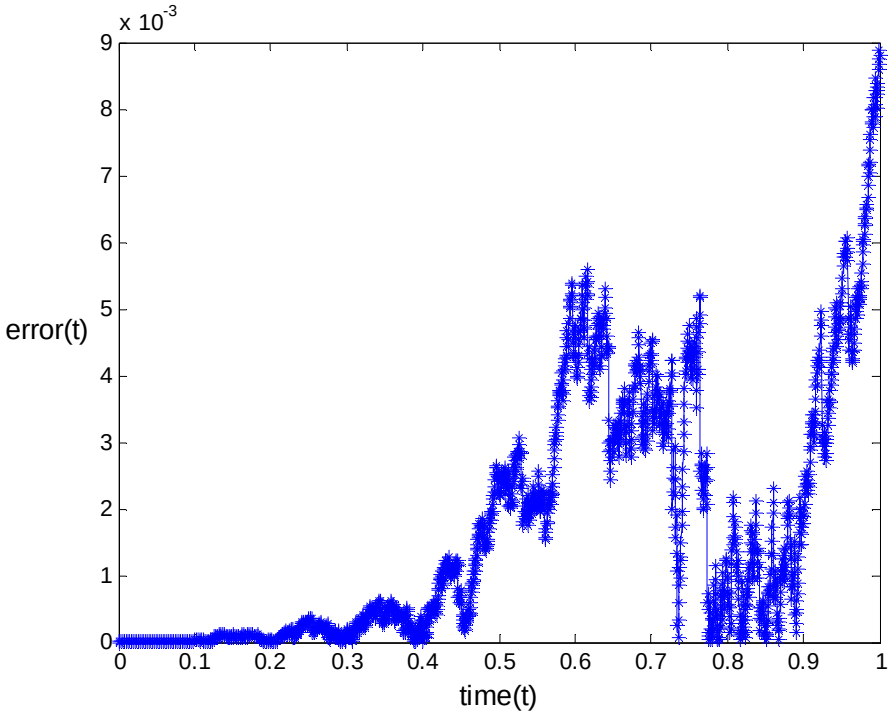


Figure (2.5.1) Absolute error between the Euler scheme and exact solution with $N = 2^{11}$; $R = 1$.

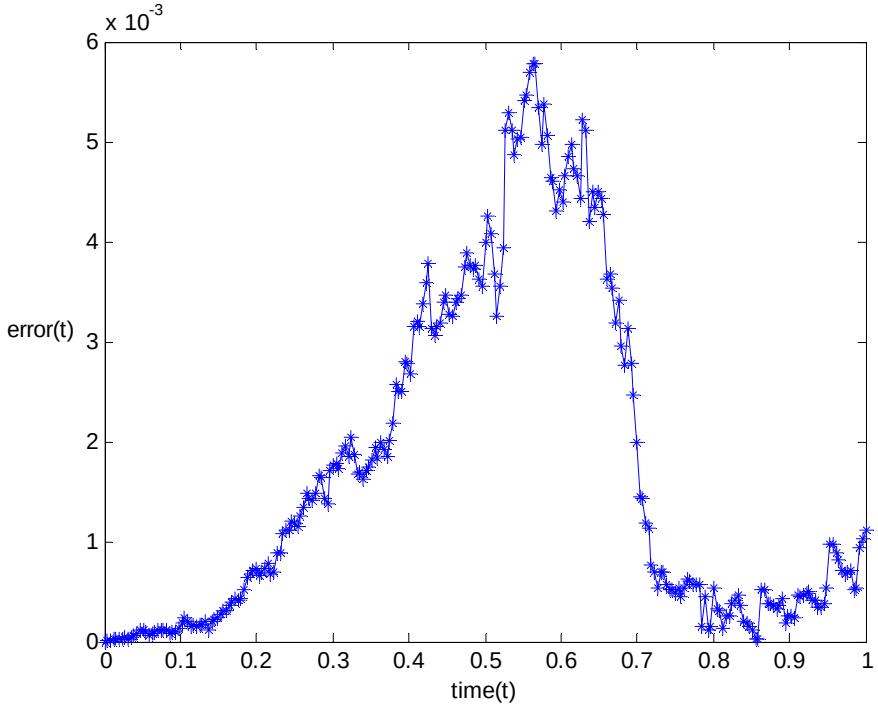


Figure (3.2.1) Absolute error between the Milstein's scheme and exact solution with $N = 2^{8}$; $R = 1$.

4.1.3 Comparison (of Example (2.3) of Section (2.6) in Chapter Two):

As one discussed earlier, the following comparisons are obtained:

Table (4.3) Comparison of errors: SDE with EM and Milstein.

Scheme	*R*	$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$	$N=2^{11}$	*Absolute error at final time*
Euler scheme	1	0.0248	0.9980	0.4442	0.0266	0.0841	0.1149	0.0128	
Milstein scheme	1	0.0304	0.1997	0.2294	0.0071	0.0309	0.0074	0.0005	

The Milstein scheme performs better than the Euler scheme as one can see for $N = 2^{11}$. The following graphs are also plotted:

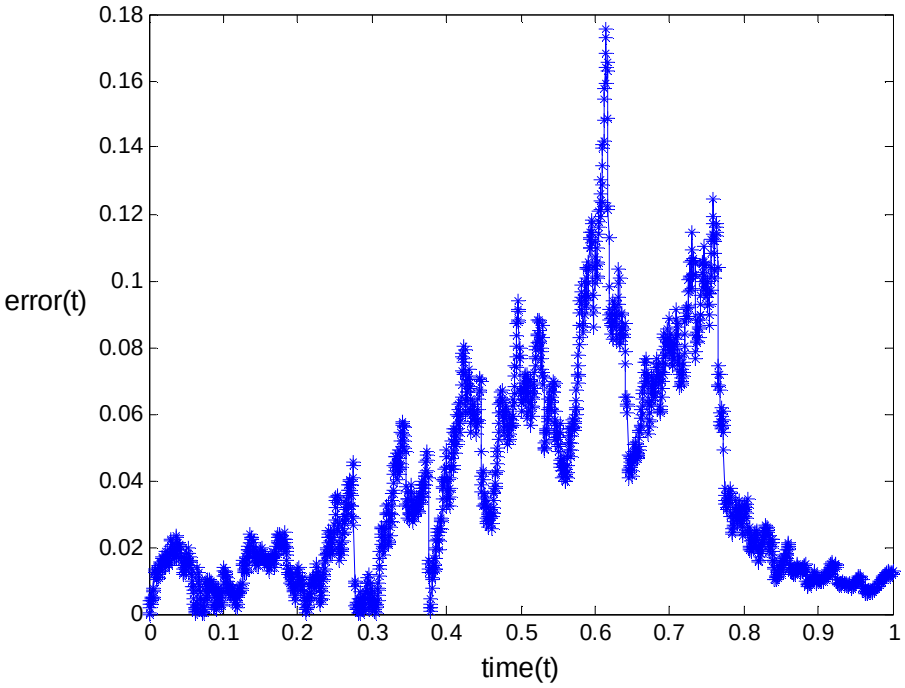


Figure (2.6.1) Absolute error between the Euler scheme and exact solution with $N = 2^{11}$; $R = 1$.

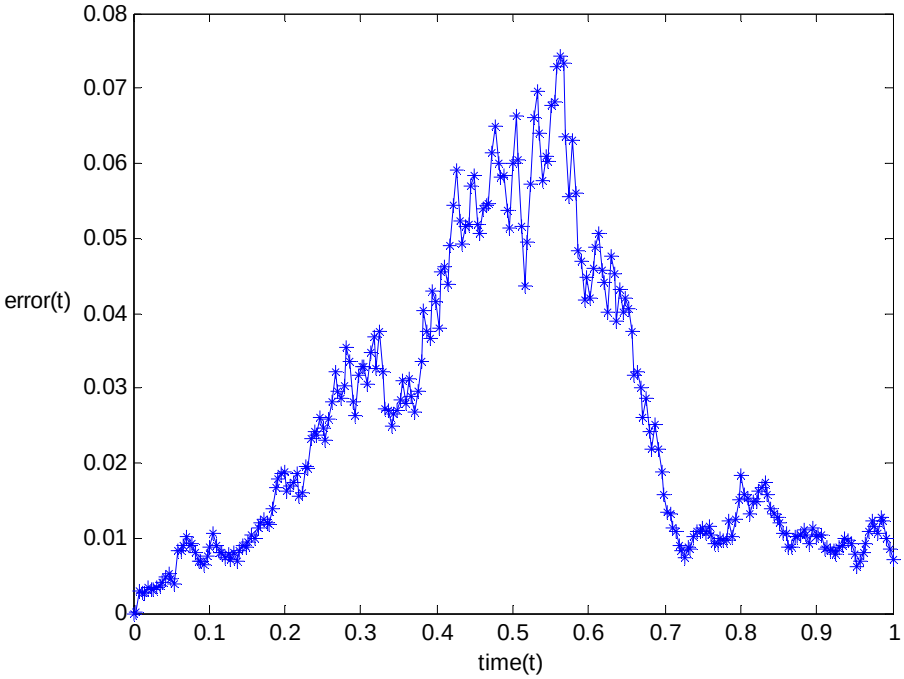


Figure (3.3.1) Absolute error between the Milstein's scheme and exact solution with $N = 2^8$; $R = 1$.

4.1.4 Comparison (Absolute and Strong Converge):

For example 92.4) of section (2.6) in chapter two, the following is given:

Table (4.4) Comparison of errors: SDE with EM and Milstein.

Scheme	***R***	$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	***Absolute error at final time***
Euler scheme	1	0.8634	0.3335	0.3793	0.0686	
Milstein scheme	1	0.9633	0.3539	0.4183	0.0560	

and

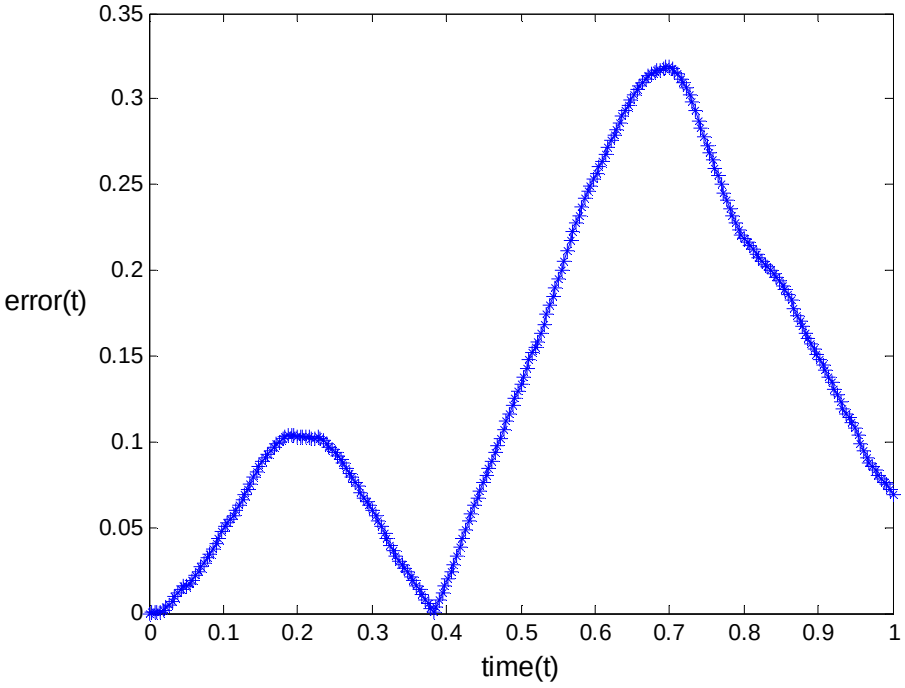


Figure (2.7.1) Aabsolute error between the Euler scheme and exact solution with $N = 2^8$; $R = 1$.

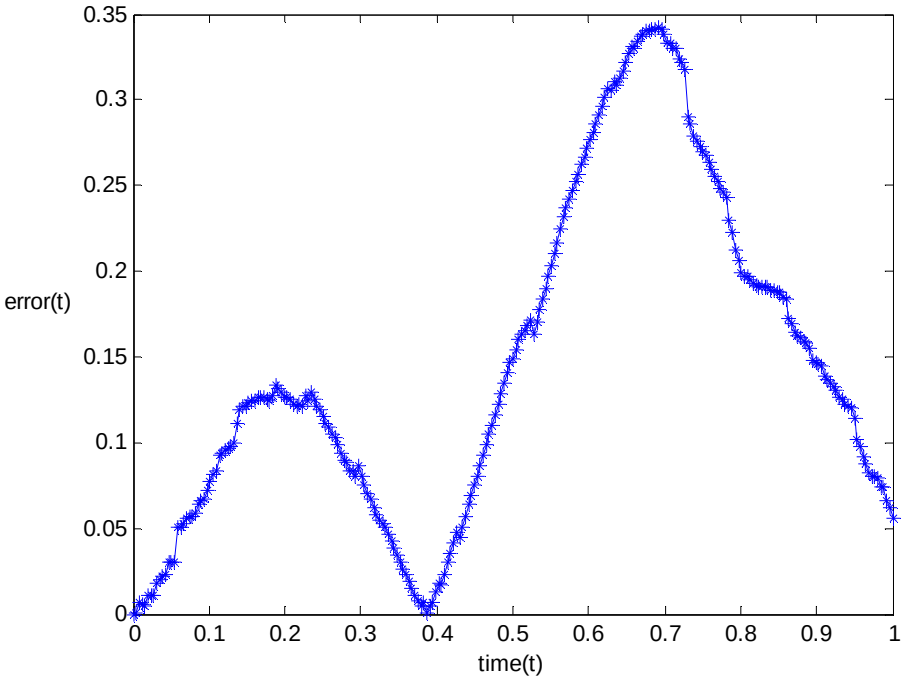


Figure (3.4.1) Absolute error between the Milstein's scheme and exact solution with $N = 2^8$; $R = 1$.

One can also see the following order of strong convergence of Euler-Maruyama method is 0.5, while the order of strong convergence of Milstein method is 1, depending on:

$$E|X_n - X(T)| \leq C\Delta t^{\eta}$$

Which present the Milstein method to be the best than Euler method.

The linear stability between Euler-Maruyama method and Milstein method shows that Milstein method is better than Euler-Maruyama method, as one can see this final time in figures (2.16)-(2.20) and (3.10)-(3.12) for example (2.1), (2.2), (2.3), (2.4) and (2.7), respectively.

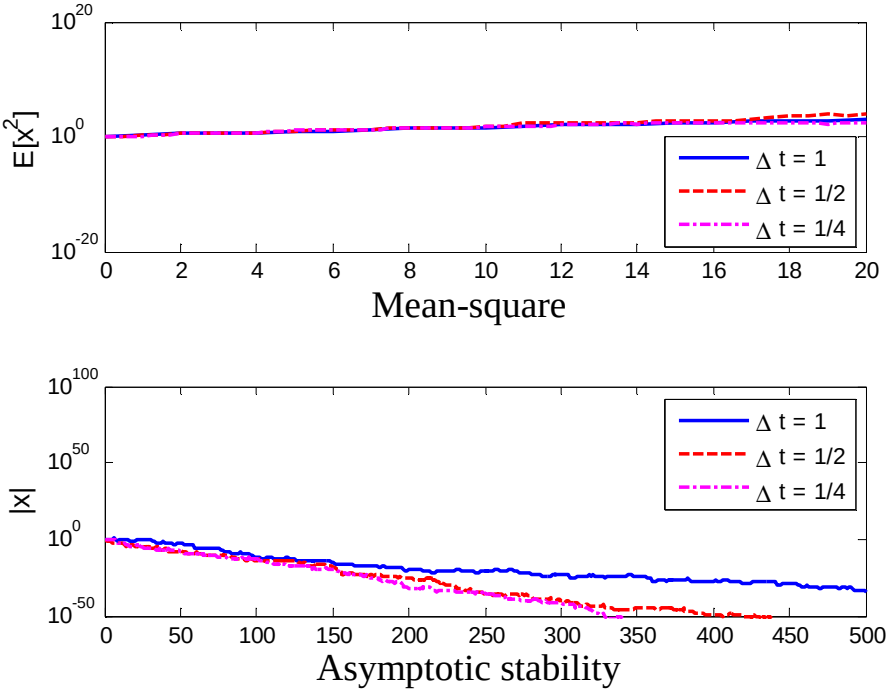


Figure (2.16) Mean-Square and Asymptotic stability.

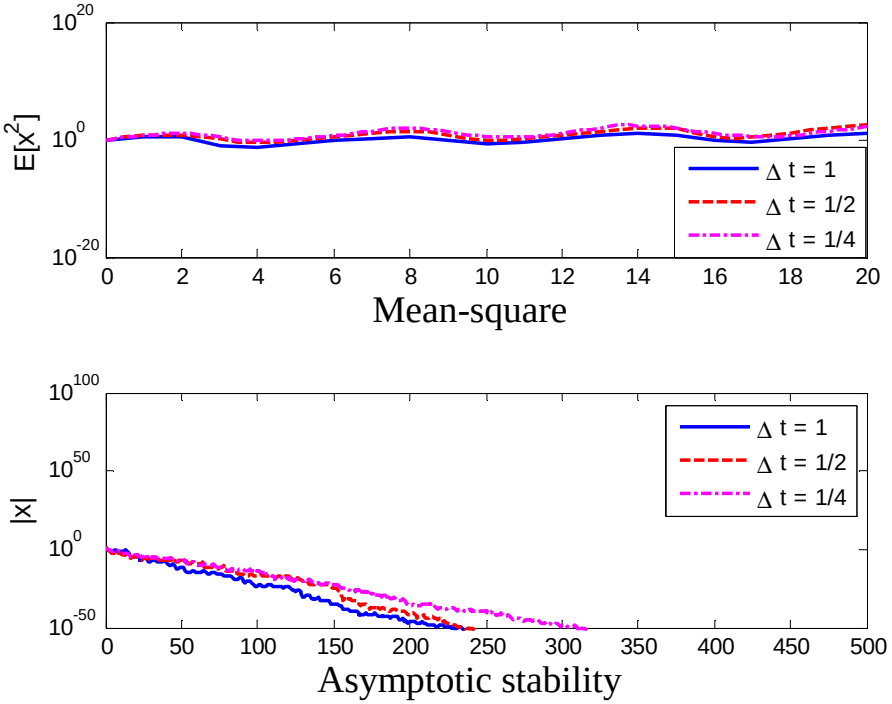


Figure (2.17) Mean-Square and Asymptotic stability for EM.

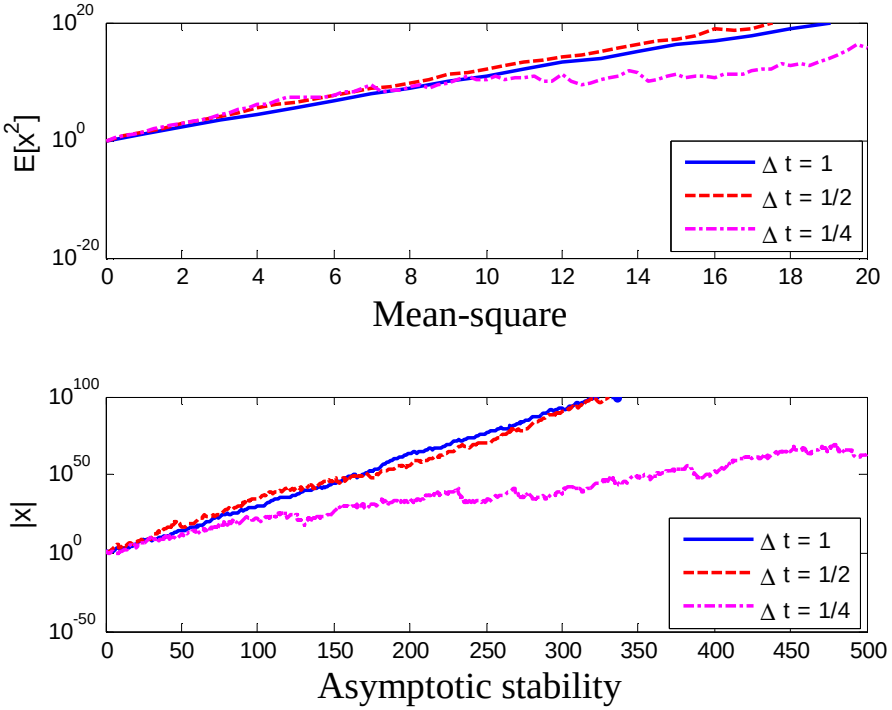


Figure (2.18) Mean-Square and Asymptotic stability for EM.

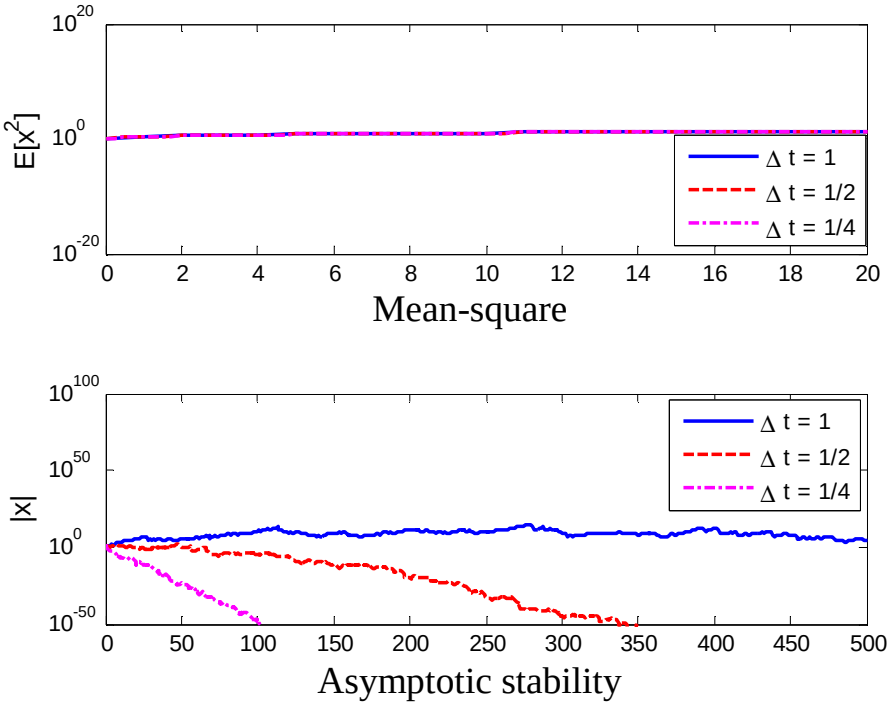


Figure (2.19) Mean-Square and Asymptotic stability for EM.

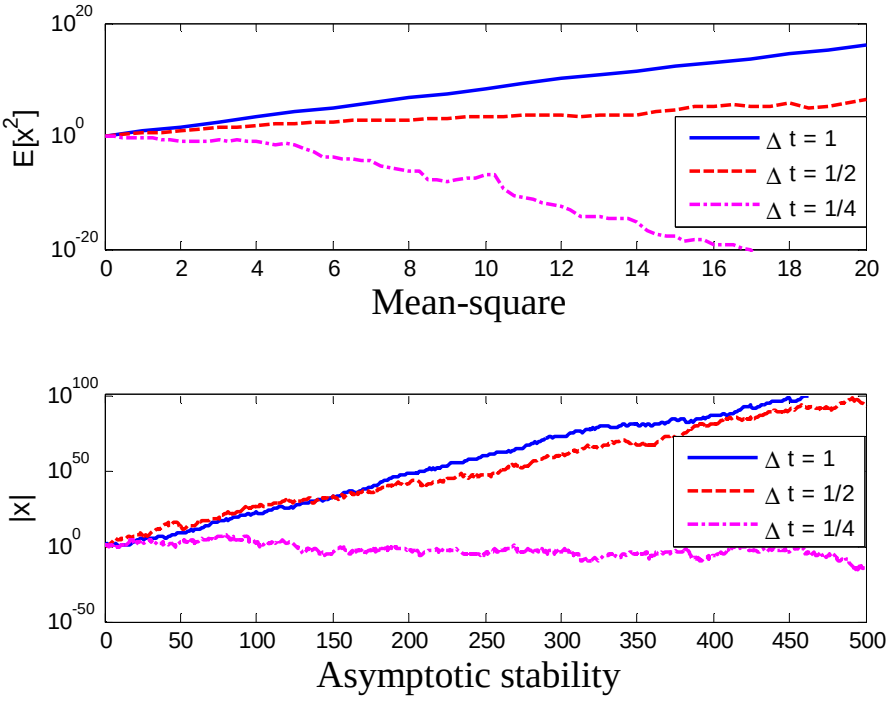


Figure (2.20) Mean-Square and Asymptotic stability for EM.

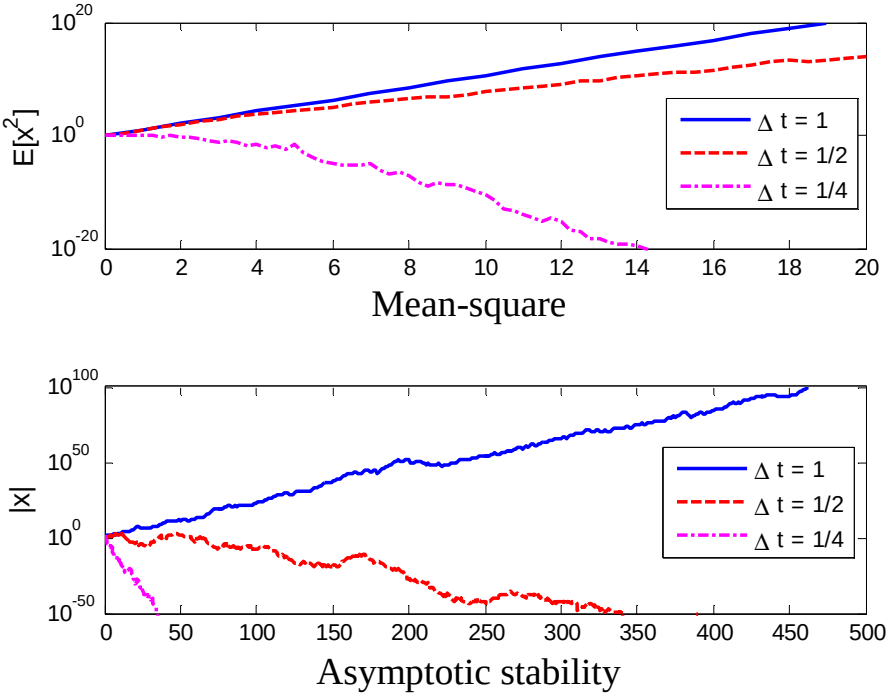


Figure (3.10) Mean-Square and Asymptotic stability by Milstein scheme.

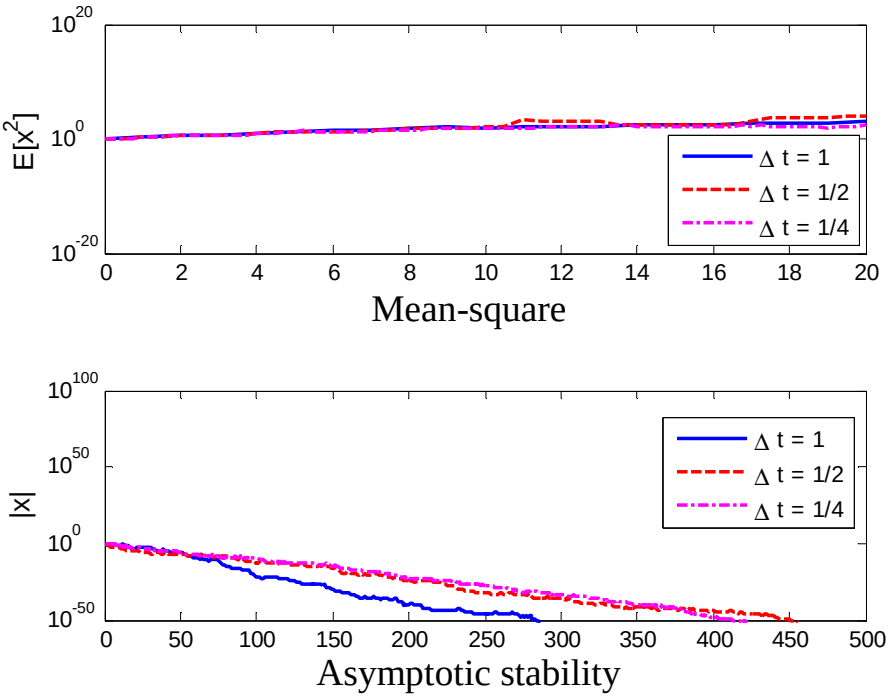


Figure (3.11) Mean-Square and Asymptotic stability by Milstein scheme.

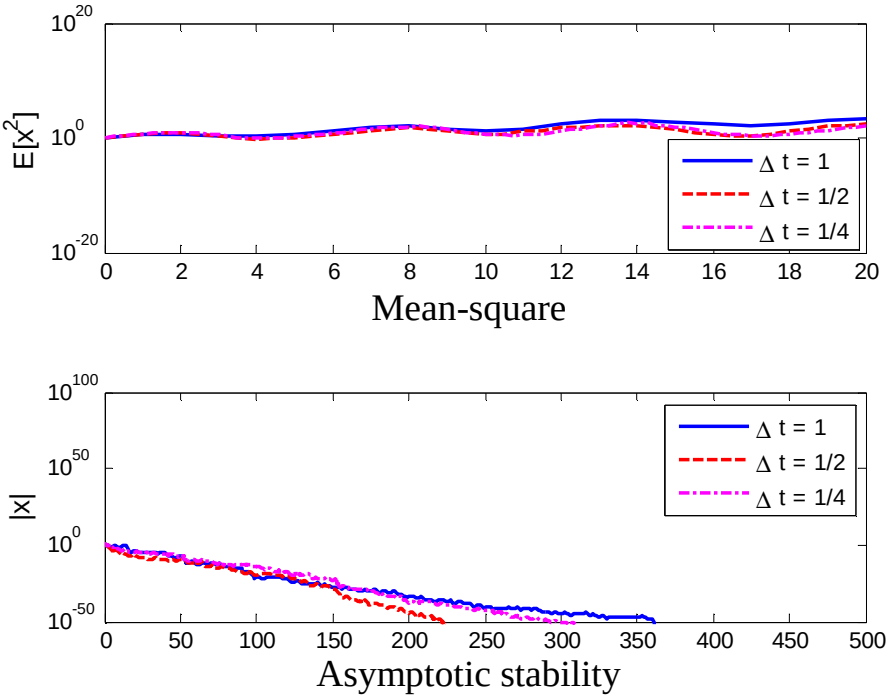


Figure (3.12) Mean-Square and Asymptotic stability by Milstein scheme.

The final decision, the numerical result for Milstein method is better than Euler-Maruyama method, even for small number of sample size and R (Brownian motion step size) as one can see from previous tables (4.1)-(4.4), for different type of comparison.

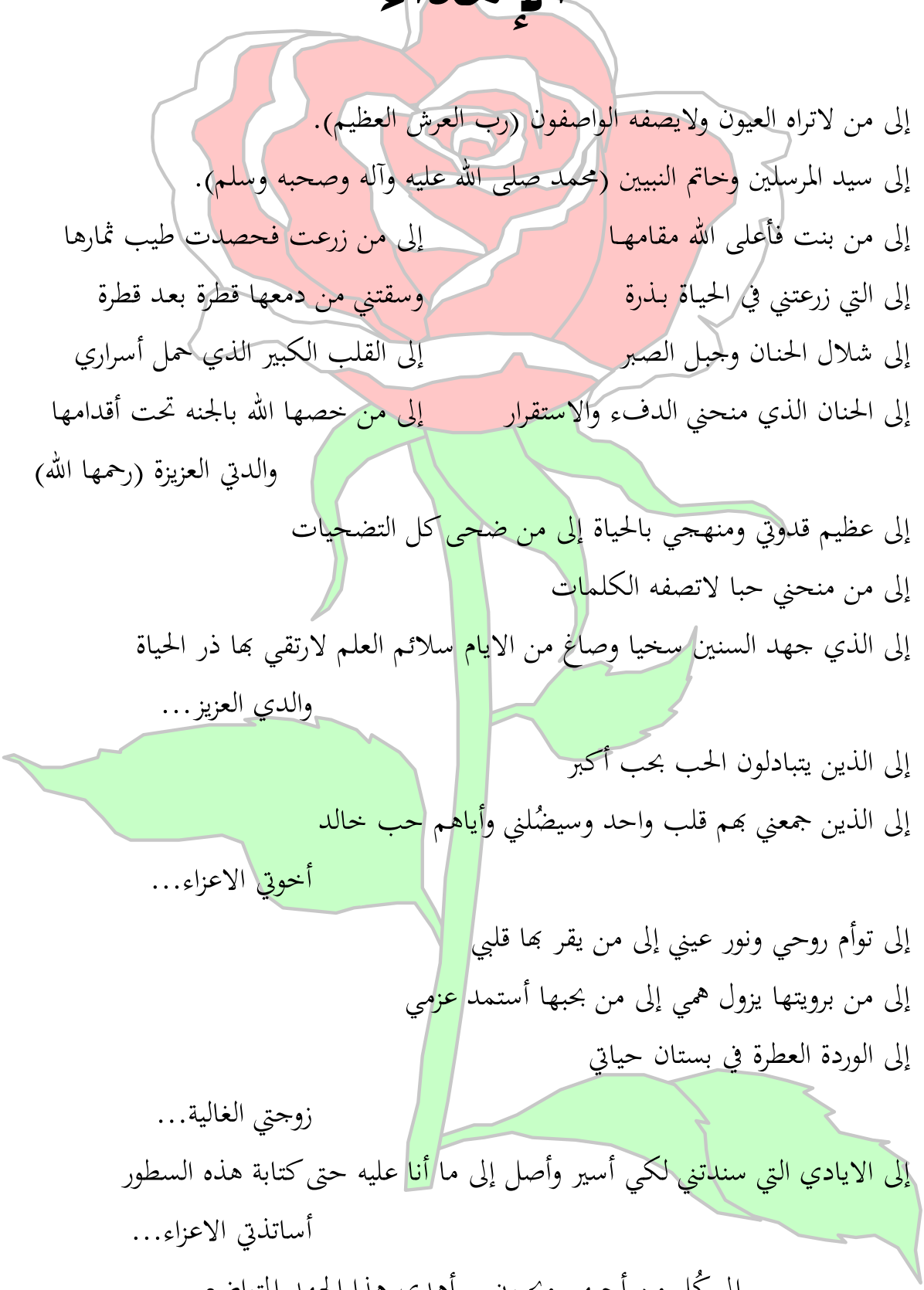
4.2 Future Work:

1. On using the present numerical methods, one can go further to study the numerical solution of stochastic partial differential equations.
2. One can focus on study the implicit numerical solutions for stochastic differential equations to overcome the problem of stability and convergence.
3. Systems had nonlinearity with differential kinds of randomness (not necessary Brownian motion) may be adopted and study the numerical solution.
4. Another numerical method, like Runge-Kutta method, etc., may be studied for solving some SDE's.

Contents

Contents

الإهداء



إلى من لاتراه العيون ولايصفه الواصفون (رب العرش العظيم).

إلى سيد المرسلين وخاتم النبيين (محمد صلى الله عليه وآله وصحبه وسلم).

إلى من بنت فأعلى الله مقامها إلى من زرعت فحصدت طيب ثمارها

إلى التي زرعتني في الحياة بذرة وسقتني من دمعها قطرة بعد قطرة

إلى شلال الحنان وجبل الصبر إلى القلب الكبير الذي حمل أسراري

إلى الحنان الذي منحني الدفء والاستقرار إلى من خصها الله بالجنه تحت أقدامها

والدتي العزيزة (رحمها الله)

إلى عظيم قدوتي ومنهجي بالحياة إلى من ضحى كل التضحيات

إلى من منحني حبا لاتصفه الكلمات

إلى الذي جهد السنين سخيا وصاغ من الايام سلائم العلم لارتقي بها ذر الحياة

والدي العزيز...

إلى الذين يتبادلون الحب بحب أكبر

إلى الذين جمعني بهم قلب واحد وسيضُلني وأياهم حب خالد

أخوتي الاعزاء...

إلى توأم روحي ونور عيني إلى من يقر بها قلبي

إلى من برويتها يزول همي إلى من بحبها أستمد عزمي

إلى الوردة العطرة في بستان حياتي

زوجتي الغالية...

إلى الايادي التي سندتني لكي أسير وأصل إلى ما أنا عليه حتى كتابة هذه السطور

أساتذتي الاعزاء...

إلى كُل من أحبهم ويحبوني...أهدي هذا الجهد المتواضع

عادل

Introduction

Stochastic differential equations (SDE's) constitute an ideal mathematical model for a multitude of phenomena and processes encountered in areas such as differential equations, stochastic control, signal processes and mathematical finance, most notably in option pricing (see for example [Kloeden & Platen, 1992] and [ϕksendal, 1998]). Unlike their deterministic counterparts, SDE's do not have explicit solutions, a part from in a few exceptional cases; hence the necessity for a sound theory of their numerical approximation is important.

It is well-known that stochastic integrals and Itô formula play a central role in modern probability theory and its applications to stochastic differential equations concerned by Brownian motion.

The theory of Itô stochastic differential equations is one of the most beautiful and most useful areas of the theory of stochastic processes. However, until recently the range of investigations in this theory have been in our view, justifiably restricted only equations were studied which can in analogy with the deterministic case, be called ordinary stochastic equations. The situation had begun to change in the last 12-18 years. The necessity of considering equations combining the features of partial differential equations and Itô equations has appeared both in the theory of stochastic processes and in related areas, [Krylov & Rozovskii, 2007].

During the past twenty years, there has been an accelerating interest in developing numerical methods for stochastic differential equations, especially in engineering and physical sciences, [Burrage & Burrage, 1996]. This has been supported by continuous improvements in computing capability and the equivalent decrease in costs of personal computers.

In the light of the volume of interest rate of related derivatives trade worldwide, there is a need to highlight and to understand the available numerical methods that could be used to solve the stochastic differential equations, thus providing a more accurate and efficient way for the pricing and hedging of derivatives products. Further, these numerical methods aide in bridging the gap between the well advanced theory of SDE's and its application to specific examples.

There are various methods that have been proposed to solve SDE's numerically, Monte Carlo methods can be used to simulate the behaviour of the system. Under this method, the physical process is simulated directly using a sequence of random numbers and there is no need to specify the differential equation that describes the behaviour of the system. The physical system is described by probability density functions and then the Monte Carlo simulation can begin by random sampling from the probability density function [Casella & Robert, 2005]. Many simulations are performed and the desired result is taken as an average over the number of observations.

However [Kloeden & Platen, 1992] claim that this method is to some extent inefficient because it does not use the special structure of the drift and diffusion coefficients. Another method to solve SDE's is to

make use of the discretisation of both time and space variables, so that the solution is approximated as finite state Markov chains. This method is plausible for simple problems, but for high dimension problems, this method can involve a considerable amount of computing time because the transition matrices contain a lot of unnecessary information which must be repeatedly reprocessed during computations.

Another method involves the finite discretisation of the time interval [0,T] only and not the state variable. This time discrete approximation can be used to generate approximate values of the sample paths at each step of the discretisation times. The simulated sample paths can then be analysed using statistical methods to determine how good the approximation is to the exact solution. This method is efficient and can be easily implemented on a digital computer. Consequently, it has been used widely and preferred to other methods because it has lower computational costs, [Fridman, 1975].

Having realised the importance and the recent increased use of SDE's, the main aim of this thesis is to present a brief analysis of the two numerical methods that have been developed for solving SDE's, focusing, on strong and weak schemes. In addition, this thesis shows how these numerical schemes can be derived from Taylor expansions of the stochastic differential equations, thus providing opportunities for the derivation of more advanced numerical schemes and the application of existing schemes by other researchers that do not have a solid background in modern probability theory, [Stirzaker, 2005].

Analogous to deterministic ordinary differential equations where the Taylor expansion is used to derive various numerical methods, the

ltô-Taylor expansion for stochastic differential equations is used to derive various numerical methods. The Euler and Milstein schemes provide a good starting point to introduce numerical methods for SDE's. This is then extended to higher order Taylor schemes. These schemes involve simulating the derivatives of the coefficients of the drift and diffusion terms. Further schemes are presented which replace the derivatives with finite differences, implicit schemes are then presented, which takes into account previous simulated values and thus involves less computing. These schemes are presented for both the strong and weak convergence criteria, [Platen, 1999].

The numerical methods are based on time discrete approximations. Time discrete approximations for both the strong and weak convergence criteria will be presented. Whereas time discrete approximations which satisfy the strong convergence criterion involves the simulation of sample paths at each step of the discretisation time, approximations that satisfy the weak convergence criterion involve the approximation of some function of the ltô process such as the first and second moments at a given final time T. Also, the linear stability of the considered numerical schemes is also presented in this thesis. Further, the thesis also contrasts the different numerical schemes by providing some analytical results of the scheme and comparing it with the known solution. This is done by using Matlab software. The effect of varying the step size is considered.

Some examples of SDE have been taken and the numerical solution are compared with the derived exact solution (if any) depending on some types of error like absolute, strong, weak, etc.

This thesis consists of four chapters. The first chapter deals with the, Some Stochastic Process Concepts.

In chapter two, Euler-Maruyama numerical method for solving SDE is discussed and some illustration have been implemented the absolute error, strong error as well as weak convergence error and linear stability are also been discussed supported by some illustration. Some concluding remarks have also been proposed.

In chapter three, Milstein's numerical method for solving SDE is proposed. Some illustrations have also been implemented. The absolute error, strong convergence error and linear stability are also been discussed and supported by some illustration. Some concluding remarks have also been presented.

In chapter four, comparisons and conclusions, future work, are presented. Then, appendixes with result programs and references are given at the end of this thesis.

List of Figures

List of Tables

References

1. **[Abdukhaled, 2004]:** Abdukhaled M., "Mean Square Stability of Second-Order Weak Numerical Methods for Stochastic Differential Equations", Applied Numerical Mathematics, Vo.48, 127-134, 2004.

2. **[Arnold, 1974]:** Arnold J., "Stochastic Differential Equations; Theory and Applications", John Wiley and Sons, New York, 1974.

3. **[Bernard & Fleury, 2001]:** Bernard P. and Fleury G., "Convergence of Schemes for Stochastic Differential Equations; Monte Carlo Methods", Applied, Vol.7(1), 35-53, 2001.

4. **[Burrage & Burrage, 1996]:** Burrage K. and Burrage P. M., "High Strong Order Explicit Runge-Kutta Methods for Stochastic Ordinary Differential Equations", Applied Numerical Mathematics, Vol.22, 81-101, 1996.

5. **[Cao & Pope, 2003]:** Cao R. ad Pope S. B., "Numerical Integration of Stochastic Differential Equations: Weak Second-Order Mid-Point Scheme for Application in the Composition PDF Method", Journal of Computational Physics, Vol.185(1), 194-212, 2003.

6. **[Carletti, 2006]:** Carletti M., "Numerical Solution of Stochastic Differential Problems in the Bioscience", Journal of Computational and Applied Mathematics, Vol.185, 422-450, 2006.

7. **[D. J. Highan, 2000]:** Highan D. J., "Mean-Square and Asymptotic Stability of the Stochastic Theta Method", SIAM J. Numer. Anal., 38 (2000), pp.753-769.

8. **[Doob, 1953]:** Doob, J. L., "Stochastic Processes", Wiley, New York, 1953.

9. **[Evans, 2005]:** Lawrence C. Evance, "An Introduction to Stochastic Differential Equations", Version 1.2, Lecture Notes, Short Course at SIAM Meeting, July, 2005.

10. **[Fleury, 2006]:** Fleury G., "Convergence of Schemes for Stochastic Differential Equations", Probabilistic Engineering Mechanics, Vol.21, 35-43, 2006.

11. **[Fridman, 1975]:** Avner Fridman, "Stochastic Differential Equations and Applications", Volume 1, Academic Press, Inc., 1975.

12. **[Gard, 1988]:** T. G. Gard, "Introduction to Stochastic Differential Equations", Marcel Dekker, New York, 1988.

13. **[Higman, 2000]:** D. J. Higman, "Mean-Square and Asymptotic Stability of the Stochastic Theta Method", SIAM J. Numer. Anal., 38, 2000, pp.753-769.

14. **[Hovanessian & Chang, 1977]:** Hovanessian S. A. and Chang C. C., "Finite Difference Solution of Differential Equations with Stochastic Inputs", Computers and Electrical Engineering, Vol.4, March, 25-35, 1977.

15. **[Hsu, 1997]:** Hwei P. Hsu, "Theory and Problems of Probability Random Variables and Random Processes", McGraw-Hill Companies, Inc., 1997.

16. **[Karatzas & Shereve, 1988]:** Karatzas I. and Shereve S. E., "Brownian Motion and Stochastic Calculus", Springer Verlag, New York, 1988.

17. **[Karatzas & Shreve, 1991]:** I. Karatzas and S. E. Shreve, "Brownian Motion and Stochastic Calcuus", 2nd Edition, Springer-Verlag, Berlin, 1991.

18. **[Kloeden & Platen, 1992]:** P. E. Kloeden and E. Platen, "Numerical Solution of Stochastic Differential Equations", V.23, Applications of Mathematics, New York, Springer-Verlag, Berlin, 1992.

19. **[Kloeden & Platen, 1999]:** P. E. Kloeden and Platen, "Numerical Solution of Stochastic Differential Equations", Springer-Verlag, Berlin, 1999.

20. **[Krishnan, 2006]:** Ven Katarama Krishnan, "Probability and Random Processes", John Wiley & Sons, Inc., 2006.

21. **[Krishnan, 1984]:** Krishnan V., "Nonlinear Filtering and Smoothing", John Wiley and Sons, Inc., 1984.

22. **[Mao, 1997]:** X. Mao, "Stochastic Differential Equations and Applications", Horwood, Chichester, 1997.

23. **[Milstein, 1974]:** Milstein G. N., "Approximating Integration of Stochastic Differential Equations", Theory and Probability and its Application, Vol.19, 557-562, 1974.

24. **[Platen, 1999]:** P. Platen "An Introduction to Numerical Methods for Stochastic Differential Equations", Acta Numer., 8, 1999, pp.197-246.

25. **[Raphael, 1972]:** Sivan Raphael and Kwakernaak, H., "Linear Optimal Control Systems", John Wiley and Sons, Inc., 1972.

26. **[Saito & Mitsui, 1996]:** Y. Saito and T. Mitsui, "Stability Analysis of Numerical Schemes for Stochastic Differential Equations", SIAM J. Numer. Anal., 33, 1996, pp.2254-2267.

27. **[Strizaker , 2005]:** Strizaker D., "Stochastic Processes and Models", Oxford University Press, Inc., New York, 2005.

28. **[ϕksendal, 1998]:** B. ϕksendal, "Stochastic Differential Equations", University Text, Springer-Verlag, Berlin, 5th Edition, An Introduction with Applications, 1998.

Republic of Iraq
Ministry of Higher Education
and Scientific Research,
Al-Nahrain University,
College of Science,
Department of Mathematics and
Computer Applications

On Numerical Solutions of Some Stochastic Ordinary Differential Equations

A Thesis
Submitted to the College of Science of Al-Nahrain University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By
Adel Sufyan Hussain
(B.Sc., Al-Nahrain University, 2006)

Supervised by
Asst. Prof. Dr. Radhi Ali Zaboon

October 2009 *Shawaal 1430*

جمهـورية العـراق
وزارة التعليم العالي والبحث العلمي
جـــامعة النهـــريـن
كليــة العلوم
قسم الريـاضيات وتطبيقات الحاسوب

حول الحلول العددية لبعض المعادلات التفاضلية الاعتيادية الصدفية

رسالة
مقدمة إلى كلية العلوم – جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل
عادل سفيان حسين
(بكالوريوس علوم، جامعة النهرين، ٢٠٠٦)

إشراف
أ.م.د. راضي علي زبون

تشرين الاول ٢٠٠٩ شوال ١٤٣٠

بسم الله الرحمن الرحيم

فَأَمَّا الزَّبَدُ فَيَذْهَبُ جُفَاءً وَأَمَّا مَا يَنْفَعُ النَّاسَ فَيَمْكُثُ فِي الأَرْضِ

كَذَلِكَ يَضْرِبُ اللَّهُ الأَمْثَالَ

صدق الله العظيم

سورة الرعد

الآية (١٧)

المستخلص

الهدف الرئيس لهذه الرسالة هو دراسة بعض الطرق العددية لحل المعادلات التفاضلية الصدفية (Stochastic Differential Equations) حلاً عددياً. لقد تم عرض المفاهيم الاساسية لفهم ودراسة الطرق العددية المقترحة.

بسبب صعوبة إيجاد الحلول التحليلية لكثير من المعادلات التفاضلية الصدفية، تم إستخدام طريقتي أويلر ميرما و ميلستين العدديتين. ولقد تم تنفيذ بعض المحاكات العددية لعدد من الامثلة الاختيارية. وقدمت الملاحظات الاستنتاجية الضرورية لذلك.

لقد تم كذلك دراسة وتقديم الخطأ المطلق، خطأ التقارب القوي، خطأ التقارب الضعيف بالاضافة الى الاستقرارية الخطية لطريقتي أويلر ميرما وميلستين مدعمة بأختبارات عددية.

عُرضت مع المناقشة المقارنة لانواع مختلفة من التقاربات والخطأ العددية للطريقتين (أويلر–ميرما وميلستين) ولبعض الامثلة الاختيارية وأخيراً لقد قدمت ونوقشت بعض الاستنتاجات والمقارنات لانواع معينة من الدراسة. مع عرض البرامج الحاسوبية مبرمجة ضمن لغة Matlab Software مع الشروحات الكافية لفهما.