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المستخلص

لهذه الأطروحة هدفين رئيسين يمكن تقسيمهما إلى ما يلي :

الهدف الأول هو لدراسة المجموعات الضبابية (Fuzzy Sets) إضافة إلى بعض الخواص الجبرية لهذه المجموعات وبعض من النتائج النظرية المهمة .

الهدف الثاني هو لدراسة الفضاءات المترية – D (D-Metric Spaces) والفضاءات المترية –M (M-Fuzzy Metric Spaces) وبعضاً من النتائج المهمة في هذين الفضائين كما ويتضمن هدف الاطروحة دراسة كمال الفضاءات المترية الضبابية (Completeness of Fuzzy Metrc Spaces) باستخدام الدوال المترية – M .

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Thesis Entitle: About the Completeness of Fuzzy Metric Spaces

Keywords: fuzzy set, fuzzy topology, fuzzy metric spaces, fuzzy M-metric spaces, completeness of fuzzy metric spaces.

ABSTRACT

The objective of this work may be oriented towards tow aspects.

The first objective is to study fuzzy set theory, as well as some of its basic algebraic properties and theoretical results.

The second objective is to study D-metric space and M-fuzzy metric spaces, and some of their properties. Also, the objective includes the study of complete fuzzy metric space by using M-fuzzy distance function.

المستخلص

لهذه الاطروحة هدفين رئيسيين، وهما:

الهدف الاول هو لدراسة المجموعات الضبابية (Fuzzy Sets) بالاضافة الى بعض الخواص الجبرية لهذه المجموعات وبعض النتائج النظرية المهمة.

الهدف الثاني هو لدراسة الفضاءات المترية−D (D-Metric Spaces) والفضاءات المترية الضبابية−M (M-Fuzzy Metric Spaces) وإعطاء بعضاً من النتائج المهمة في هذين الفضائين. كما ويتضمن هدف الاطروحة دراسة كمال الفضاءات المترية الضبابية (Completeness of Fuzzy Metric Spaces) باستخدام الدوال المترية الضبابية−M.

بسم الله الرحمن الرحيم

قَالُوا سُبْحَٰنَكَ لا عِلمَ لَنا إِلا ما عَلَمْتَنا إِنَّكَ أنتَ العَلِيمُ الحَكِيمُ

صدق الله العظيم

سورة البقرة

الآية (٣٢)

Republic of Iraq
Ministry of Higher Education
and Scientific Research,
Al-Nahrain University,
College of Science, Department of Mathematics
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About the Completeness of Fuzzy Metric Spaces

A Thesis
Submitted to the College of Science of Al-Nahrain University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

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جمهـورية العـراق
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حَول كمال الفضاءات المترية الضبابية

رسالة
مقدمة إلى كلية العلوم – جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
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من قبل
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REFERENCES

[1] Ahmed, H. A., "Some Results About D-Metric Spaces and Fixed Point Theory in D-Metric Spaces", M.Sc. Thesis, College of Education, Ibn Al-Haitham, University of Baghdad, 2007.

[2] Dubois, D. and Prade, H., "Fuzzy Sets and Systems: Theory and Applications", Academic Press, Inc., 1980.

[3] Estabraq, M., "The Numerical Solution of Fuzzy Differential Equation by Using Linear Multistep Methods", M.Sc. Thesis College of Science, Al-Nahrain University, 2005.

[4] Fadhel, F. S., "About Fuzzy Fixed Point Theorem", Ph.D. Thesis, College of Science, Al-Nahrain University, 1998.

[5] George, A. and Veeramani, P., "On Some Result in Fuzzy Metric Space", Fuzzy Sets and System, 64, pp.395-399, 1994.

[6] George, A. and Veeramani, P., "On Some Results of Analysis for Fuzzy Metric Spaces", Fuzzy Sets and System, 90, pp.365-368, 1997.

[7] George, A. and Sapena, A., "On Fixed Point Theorem in Fuzzy Metric Spaces", Fuzzy Sets and System, 125, pp.245-252, 2002.

[8] John, D., "Introduction to Real Analysis", John Wiley and Sons, 1988.

[9] Kelley, J. L., "General Topology", Van Nostrand, 1955.

[10] Mary, M. G., “Further Results About Fuzzy Metric Spaces”, M. Sc. Thesis, College of Education, Al-Mustansiriah University, 2004.

[11] Max, D. L., “Fundamental Concepts of Modern Mathematics”, Addison-Wesley Publishing Company, 1970.

[12] Munir, M. A. K., “On Separation Axioms of Fuzzy Topological Spaces”, Ph.D. Thesis, College of Education, Al-Mustansiriah University, 2006.

[13] Nguyen, H. T., “A Note on the Extension Principle for Fuzzy Sets”, UCB/ERLMEMO M-611, University of California, Berkeley, 1976.

[14] Scheizerb and Skara, “Statistical Metric Spaces”, Pacific J. Math., 10, pp.314-334, 1960.

[15] Sedghi, S. and Shobe, N., “Fixed Point Theorem in M-Fuzzy Metric Spaces with Properties”, ISSN, Vol.1, No.1, pp.55-65, 2006.

[16] Seymour, L., “Theory and Problems of General Topology”, McGraw-Hill Company, Inc., 1965.

[17] Valentin, G. and Salvador, R., “Some Properties of Fuzzy Metric Spaces”, Fuzzy Sets and Systems, 115, pp.485-489, 2002.

[18] Zadeh, L. A., “Fuzzy Sets”, 1965. In Fuzzy Sets and Applications: Selected Papers by L. A. Zadeh, Edited by Yager R. R., Ovchinnikos S., Tong R. M., and Ngnyen W. T., John Wiely and Sons, Inc., 1987.

[19] Zike, D., “Fuzzy Pseudo Metric Space”, J. Math. And Appl., Vol.80, pp.14-95, 1982.

FUTURE WORK

The following problems may be recommended as open problems for future work:

1. Study the completeness of D-metric spaces, in general, and the completeness of fuzzy D-metric spaces, in particular.
2. Studying the compactness of M-fuzzy metric spaces.
3. Study other fixed point theorems of M-fuzzy metric spaces, such as Schuder fixed point theorem, Sadoviski fixed point theorem, etc.
4. Introducing the M-fuzzy topological spaces and its separation axioms.

INTRODUCTION

The concept of fuzzy sets was introduced initially by Zadeh in 1965. Since then, this concept is used in topology and some branches of analysis, many authors have expensively developed the theory of fuzzy sets and application, [2].

We start with the obvious definition of fuzzy sets "A fuzzy set, termed by $\tilde{A}$, in a space of objects X is a class of events with a continuous grade of membership and is characterized by a membership function, termed as $\mu_{\tilde{A}}$, which associates for each $x \in X$ a real number in the interval [0, 1]". The value of $\mu_{\tilde{A}}(x)$ represents of grade of membership of x in $\tilde{A}$, i.e., denotes the degree to which an element or event x may be a member of $\tilde{A}$ or belong to $\tilde{A}$, [18].

The characteristic function $\mu_{\tilde{A}}$, in fact, may be viewed as a weighting function that reflects the ambiguity in a set. As the membership value approaches unity, the grade of membership of an event in $\tilde{A}$ becomes higher. For example, $\mu_{\tilde{A}}(x) = 1$ indicates that the event x is strictly contained in the set $\tilde{A}$, and on the other hand, $\mu_{\tilde{A}}(x) = 0$ indicates that x does not belong to $\tilde{A}$. Any intermediate value would reflects the degree on which x could be a member of $\tilde{A}$, [4].

Classical examples, such as the class of animals, the class of beautiful women or the class of tall men, or large streets, etc. are all good examples that explain the definition of fuzzy sets, [18].

Moreover, every day life, we are used too properties which can not be dealt with satisfactorily on a simple “yes” or “no” basis. Whether these properties perhaps best indicated by a shade of gray, rather than by the black or white. Assigning each individual in a population on a “yes” or “no” value, as is done in ordinary set theory is not an adequate way of dealing with properties of this type, [2].

Historically, the general accepted birth date of the theory of fuzzy sets back to 1965, when the first article entitled “fuzzy sets” by L. A. Zadeh appeared in the journal of information and control. Also, the term “fuzzy” was introduced and coined by Zadeh for the first time in this paper, [4].

Zadeh’s original definition of fuzzy sets is to consider a class of objects with a continuum of grades of membership, such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership value ranging between 0 and 1.

Chang, C. L. in 1968 used the fuzzy set theory for defining and introducing fuzzy topological spaces, while Wong, C. K. in 1973, discussed the covering properties of fuzzy topological spaces, [12].

Ercey, M. A. in 1979, studied fuzzy metric spaces and its connection with statistical metric spaces, [4].

Ming P. P. and Ming L. Y. in 1980, used fuzzy topology to define neighborhood structure of fuzzy point and Moore-Smith convergence, [4].

Zike Deng in 1982, studied the fuzzy point and discussed the fuzzy metric spaces with the metric defined between two fuzzy points, [4].

The main objective of his work is to study and prove the completeness of fuzzy metric spaces using M-fuzzy metric spaces.

This thesis consists of three chapters.

In chapter one, we introduce some of the basic necessary concepts, in which basic definitions and algebraic properties are given with some illustrative examples. An extension principle has been used to generalize crisp mathematical concepts to fuzzy mathematical concepts. As well as the introduction of α-level sets has been considered as an intermediate set between fuzzy sets and ordinary sets. At the end of this chapter, we discusses fuzzy topology, fuzzy metric spaces and its completeness.

In chapter two, a brief introduction to the theory of M-fuzzy metric spaces is given in order to make this thesis, as possible of self contents. This chapter consists of three sections. In section 2.1, we introduce some basic definitions related to this subject, including the definition of T-norms and M-fuzzy metric spaces using functions of three tuples. In section 2.2, we present the concept of D-metric spaces, which has its connectivity with M-fuzzy metric spaces, in which the distance in M-fuzzy metric spaces is defined using the D-function. Also, in this section we give the proof of some important results. In section 2.3, additional results are given with their proofs.

In chapter three, we discuss the completeness of M-fuzzy metric spaces. This chapter consists of two sections. In section 3.1, additional theoretical study to the M-fuzzy metric spaces is given using functions of four tuples where it is noticeable that this section includes some new results to the best of our knowledge. Also, this section consists of some well selected examples, with their solution, which illustrate the M-fuzzy metric spaces. Finally, section 3.2 presents the study of the completeness of M-fuzzy metric spaces.

Dedication

To My Father and Mother

With all Love and Respects

CONTENTS

CHAPTER THREE
COMPLETENESS OF M-FUZZY METRIC SPACES

The purpose of this chapter is to study the completeness of M-fuzzy metric spaces and some basic concepts in this spaces, which are related to the M-convergence of such metric spaces.

For substantially understanding of the idea of M-fuzzy metric spaces, we find it is convenient to present some special examples with details.

George and Veermamani [5], Kramosil and Michalek have introduced the concept of M-fuzzy topological spaces induced by M-fuzzy metric spaces which have very important applications in quantum practical physics particularly in connections which were given and studied by El-Naschie. Many authors [7] have proved the fixed point theorem in M-fuzzy metric spaces and upon such generalization is the generalized metric space. He proved some results on fixed points for a self contractive mappings for M-complete and bounded M-metric spaces, [15].

Here, we obtain the following result, that is the topology generated by any M-fuzzy metric space is metrizable. We also show that if the M-fuzzy metric space is complete, then the generated topology is M-completely metrizable, [17].

3.1 ADDITIONAL RESULTS IN M-FUZZY METRIC SPACES

In this section, some additional important results related to M-fuzzy metric spaces are presented which are necessary for the completeness of M-fuzzy metric spaces.

Now, recall that from chapter two, typical examples of continuous T-norm that may be used in this chapter, which are:

$a*b = ab$

$a*b = \min \{a, b\}$.

We start first with the following generalization of M-fuzzy metric spaces, i.e., generalization of definition (2.3.2).

Definition (3.1.1), [15]:

A 4-tuple $(X, M_D, *)$ is called M-fuzzy metric space if X is an arbitrary (nonempty) set, * is M-continuous T-norm and M is a fuzzy subset of $X \times X \times X \times (0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$:

1. $M_D(x, y, z, t) > 0$.
2. $M_D(x, y, z, t) = 1$ if and only if $x = y = z$.
3. $M_D(x, y, z, t) = M_D(p\{x, y, z\}, t)$, where p is a permutation function of x, y and z.
4. $M_D(x, y, a, t)*M_D(a, z, z, s) \le M_D(x, y, z, t + s)$.
5. $M_D(x, y, z, *) : (0, \infty) \longrightarrow [0, 1]$ is a continuous.

<u>Remark (3.1.1), [17]:</u>

Let (X, d) be a metric space and define a*b = ab, for every a, b $\in$ [0, 1]. Let M_d be the function defined on $X \times X \times (0, \infty)$ by:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \ t > 0$$

Then (X, M_d, *) is a M-fuzzy metric space and M_d is called the M-fuzzy metric induced by d.

The next definition may be considered as a generalization to the definitions (2.2.2) and (2.2.3) given in chapter two.

<u>Definition (3.1.2):</u>

Let (X, M_D, *) be a M-fuzzy metric space and let $\tilde{A}$ be a fuzzy subset of X. Let M denote the restriction of M to $\tilde{A} \times \tilde{A} \times \tilde{A}$, then ($\tilde{A}$, M_D, *) is called M-fuzzy metric subspace of (X, M_D, *).

<u>Definition (3.1.3):</u>

An M-fuzzy metric (X, M_D, *) is said to be bounded (M-bounded) if there exists a positive real number k, such that:

$$M_D(x, y, z, t) \leq k, \text{ for all } x, y, z \in X, t > 0$$

and in such a case k is said to be an M-bound for X. Moreover, if $E \subseteq X$, then E is said to be M-bounded subspace if there exists a positive real number n, such that $M_D(x, y, z, t) \leq n$, for all x, y, z $\in$ E.

Definition (3.1.4), [15]:

Let $(X, M_D, *)$ be a M-fuzzy metric space, M is said to be M-continuous function on $X \times X \times X \times (0, \infty)$ if:

$$\lim_{n \to \infty} M_D(x_n, y_n, z_n, t_n) = M_D(x, y, z, t)$$

Whenever a sequence $\{(x_n, y_n, z_n, t_n)\}$ in $X \times X \times X \times (0, \infty)$ is M-converges to a point $(x, y, z, t) \in X \times X \times X \times (0, \infty)$, i.e.,

$$\lim_{n \to \infty} x_n = x, \ \lim_{n \to \infty} y_n = y, \ \lim_{n \to \infty} z_n = z, \quad \text{and}$$

$$\lim_{n \to \infty} M_D(x, y, z, t_n) = M_D(x, y, z, t)$$

The next lemma shows that condition (4) of definition (3.1.1) may be proved in all cases and thus this condition may be violated from the definition.

Lemma (3.1.1):

Let $(X, M_D, *)$ be an M-fuzzy metric space. Define $M_D : X \times X \times X \times (0, \infty) \longrightarrow [0, 1]$, by:

$$M_D(x, y, z, t) = M_d(x, y, t) * M_d(y, z, t) * M_d(z, x, t)$$

Then:

$$M_D(x, y, z, t + s) \geq M_D(x, y, a, t) * M_D(a, z, z, s)$$

for every $t, s > 0$ and $x, y, z \in X$.

Proof:

Since:

$$M_D(x, y, z, t) = M_d(x, y, t)*M_d(y, z, t)*M_d(z, x, t)$$

Then:

$$M_D(x, y, a, t) = M_d(x, y, t)*M_d(y, a, t)*M_d(a, x, t) \quad \text{..................(3.1)}$$

and

$$M_D(a, z, z, s) = M_d(a, z, s)*M_d(z, z, s)*M_d(z, a, s) \quad \text{..................(3.2)}$$

and hence by definition (2.3.2):

$$M_D(x, y, z, t + s) = M_d(x, y, t + s)*M_d(y, z, t + s)*M_d(z, x, t + s)$$

$$\geq M_d(x,y,t)*M_d(y,a,t)*M_d(a,z,s)*M_d(z,a,s)*M_d(a,x,t)$$

$$= M_D(x, y, a, t)*M_d(a, z, s)*M_d(z, a, s) \text{ (using eq.(3.1))}$$

$$= M_D(x, y, a, t)*M_d(a, z, s)*M_d(z, a, s)*1$$

$$= M_D(x, y, a, t)*M_d(a, z, s)*1*M_d(z, a, s)$$

$$= M_D(x, y, a, t)*M_d(a, z, s)*M_d(z, z, s)*M_d(z, a, s)$$

$$= M_D(x, y, a, t)*M_D(a, z, z, s)$$

which follows from definition (3.1.1) and eq.(3.2). ■

Among the main results in this work is the following two lemmas:

Lemma (3.1.2):

Let (X, D) be a D-Metric space, and let:

$$M_D(x, y, z, t) = \frac{t}{t + D(x, y, z)}, \; t > 0$$

where:

$D(x, y, z) = |x - y| + |y - z| + |z - x|$

Then $(X, M_D, *)$ is a fuzzy metric space.

Proof:

To prove that the conditions in definition (3.1.1) are satisfied for all $x, y, z, a \in X$.

1. $M_D(x, y, z, t) > 0$, since $D(x, y, z) > 0$, for all $t > 0$.

2. If $x = y = z$, then $|x - y| = 0$, $|y - z| = 0$, $|z - x| = 0$, and hence $M_D(x, y, z, t)=1$

   If $M_D(x, y, z, t) = 1$, then $|x - y| + |y - z| + |z - x| = 0$, and therefore $x = y = z$.

3. Since from the symmetry of the distance function d,

   $D(x, y, z) = D(y, x, z) = \dots$

   hence:

   $M_D(x, y, z, t) = M_D(y, x, z, t) = \dots$

   i.e., $M_D(x, y, z, t) = M_D(p\{x, y, z\}, t)$

4. From lemma (2.3.1) and lemma (3.1.1), we have:

$$M_D(x, y, z, t + s) = M_d(x, y, t + s)*M_d(y, z, t +s)*M_d(z, x, t + s)$$

$$= \frac{t+s}{t+s+|x-y|} * \frac{t+s}{t+s+|y-z|} * \frac{t+s}{t+s+|z-x|}$$

$$\geq \frac{t}{t+|x-y|} * \frac{t}{t+|y-a|} * \frac{s}{s+|a-z|} * \frac{s}{s+|z-a|} * \frac{t}{t+|a-x|}$$

$$= \frac{t}{t+|x-y|+|y-a|+|a-x|} * \frac{s}{s+|a-z|} * \frac{s}{s+|z-a|} * \frac{s}{s+|z-z|}$$

$$= \frac{t}{t+|x-y|+|y-a|+|a-x|} * \frac{s}{s+|a-z|+|z-z|+|z-a|}$$

$= M_D(x, y, a, t)*M_D(a, z, z, s)$, for every $s > 0$

5. Let $x, y, z \in X$, $t > 0$ and let (x'_n, y'_n, z'_n, t'_n) be a sequence in $X\times X\times X\times(0, \infty)$ that M-converges to (x, y, z, t).

Since (x'_n, y'_n, z'_n, t'_n) is a sequence in $[0, 1]$, then there is a subsequence (x_n, y_n, z_n, t_n) of (x'_n, y'_n, z'_n, t'_n), such that the subsequence (x_n, y_n, z_n, t_n) M-converges to some point of $[0, 1]$.

Fix $\delta > 0$, so that $\delta < \frac{t}{2}$, then there is $n_0 \in \mathbb{N}$, such that:

$|t_n - t| < \delta$, for every $n \geq n_0$

So

$$M_D(x_n, y_n, z_n, t_n) \geq M_D(x_n, y_n, z_n, t - \delta)$$

$$\geq M_D(x_n, y_n, z, t - \frac{4\delta}{3})*M_D(z, z_n, z_n, \frac{\delta}{3})$$

$$\geq M_D(x_n, z, y, t - \frac{5\delta}{3})*M_D(y, y_n, y_n, \frac{\delta}{3})*M_D(z, z_n, z_n, \frac{\delta}{3})$$

$$\geq M_D(z, y, x, t - 2\delta)*M_D(x, x_n, x_n, \frac{\delta}{3})*M_D(y, y_n, y_n, \frac{\delta}{3})* M_D(z, z_n, z_n, \frac{\delta}{3})$$

and

$$M_D(x, y, z, t + 2\delta) \geq M_D(x, y, z, t_n + \delta)$$

$$\geq M_D(x, y, z_n, t_n + \frac{2\delta}{3})*M_D(z_n, z, z, \frac{\delta}{3})$$

$$\geq M_D(x,z_n,y_n,t_n+\frac{\delta}{3})*M_D(y_n, y, y, \frac{\delta}{3})*M_D(z_n, z, z, \frac{\delta}{3})$$

$$\geq M_D(z_n, y_n, x_n, t_n)*M_D(x_n, x, x, \frac{\delta}{3})* M_D(y_n, y, y, \frac{\delta}{3})*$$

$$M_D(z_n, z, z, \frac{\delta}{3})$$

for all $n \geq n_0$. Taking the limits as $n \longrightarrow \infty$, yields:

$$\lim_{n\to\infty} M_D(x_n, y_n, z_n, t_n) \geq M_D(x, y, z, t - 2\delta)*1*1*1$$

$$= M_D(x, y, z, t - 2\delta)$$

and

$$M_D(x, y, z, t + 2\delta) \geq \lim_{n\to\infty} M_D(x_n, y_n, z_n, t_n)*1*1*1$$

$$= \lim_{n\to\infty} M_D(x_n, y_n, z_n, t_n)$$

taking the limit as $\delta \longrightarrow 0$, then

$$\lim_{n\to\infty} M_D(x_n, y_n, z_n, t_n) = M_D(x, y, z, t)$$

Therefore, M_D is continuous on $X\times X\times X\times(0, \infty)$

Hence $(X, M_D, *)$ is M-fuzzy metric space. ■

Lemma (3.1.3):

Let $X = \mathbb{R}$ and let:

$$M_D(x, y, z, t) = \frac{t}{t + D(x, y, z)}, \ t > 0$$

where:

$$D(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}, \ \forall \ x, y, z \in X$$

Then $(X, M_D, *)$ is M-fuzzy metric space.

Proof:

Since, by letting $x, y, z, a \in X$

1. $M_D(x, y, z, t) = \dfrac{t}{t + \max\{|x - y|, |y - z|, |z - x|\}} > 0, \ \forall \ t > 0$ since $|x - y| > 0$, $|y - z| > 0$ and $|z - x| > 0$.

2. If $M_D(x, y, z, t) = \dfrac{t}{t + \max\{|x - y|, |y - z|, |z - x|\}} = 1$ then $\max\{|x - y|, |y - z|, |z - x|\} = 0$ and hence $x = y = z$ and also $\max\{|x - y|, |y - z|, |z - x|\} = 0$ implies that $M_D(x, y, z, t) = 1$.

3. Clear that $M_D(x, y, z, t) = M_D(y, x, z, t) = M_D(z, x, y, t) = \ldots$

4. $M_D(x, y, z, t + s) = \dfrac{t + s}{t + s + \max\{|x - y|, |y - z|, |z - x|\}}$

$$= \frac{t + s}{t + s + \max\{|x - y|\}} * \frac{t + s}{t + s + \max\{|y - z|\}} *$$

$$\frac{t + s}{t + s + \max\{|z - x|\}} \text{ (by lemma (3.1.2))}$$

$$\geq \frac{t}{t+\max\{|x-y|\}} * \frac{t}{t+\max\{|y-a|\}} * \frac{s}{s+\max\{|a-z|\}} *$$

$$\frac{s}{s+\max\{|z-a|\}} * \frac{t}{t+\max\{|a-x|\}} * 1$$

$$= \frac{t}{t+\max\{|x-y|,|y-a|,|a-x|\}} * \frac{s}{s+\max\{|a-z|\}} *$$

$$\frac{s}{s+\max\{|z-a|\}} * \frac{s}{s+\max\{|z-z|\}} \text{ (by lemma (3.1.3))}$$

$$= \frac{t}{t+\max\{|x-y|,|y-a|,|a-x|\}} *$$

$$\frac{s}{s+\max\{|a-z|,|z-z|,|z-a|\}}$$

$= M_D(x, y, a, t)*M_D(a, z, z, s), \forall\ t, s > 0$

5. Let $x, y, z \in X$, $t > 0$ and let $\{(x'_n, y'_n, z'_n, t'_n)\}$ be a sequence in $X\times X\times X\times(0, \infty)$ that M-converges to (x, y, z, t).

Since $\{(x'_n, y'_n, z'_n, t'_n)\}$ is a sequence in $[0, 1]$, then there is a subsequence $\{(x_n, y_n, z_n, t_n)\}$ of $\{(x'_n, y'_n, z'_n, t'_n)\}$, such that the subsequence $\{(x_n, y_n, z_n, t_n)\}$ M-converges to some point of $[0, 1]$.

Fix $\delta > 0$, so that $\delta = \frac{t}{2}$, then there is $n_0 \in \mathbb{N}$, such that:

$|t_n - t| < \delta$, for every $n \geq n_0$

So

$M_D(x_n, y_n, z_n, t_n) \geq M_D(x_n, y_n, z_n, t - \delta)$

$$= \frac{t-\delta}{t-\delta+\max\{|x_n-y_n|,|y_n-z_n|,|z_n-x_n|\}}$$

$$= \frac{t-\frac{4}{3}\delta+\frac{\delta}{3}}{t-\frac{4}{3}\delta+\frac{\delta}{3}+\max\{|x_n-y_n|,|y_n-z_n|,|z_n-x_n|\}}$$

$$\geq \frac{t-\frac{4\delta}{3}}{t-\frac{4\delta}{3}+\max\{|x_n-y_n|,|y_n-z|,|z-x_n|\}} * \frac{\frac{\delta}{3}}{\frac{\delta}{3}+\max\{|z-z_n|,|z_n-z_n|,|z_n-z|\}}$$

$$\geq \frac{t-\frac{4\delta}{3}}{t-\frac{4\delta}{3}+\max\{|x_n-y_n|,|y_n-z|,|z-x_n|\}} * \frac{\frac{\delta}{3}}{\frac{\delta}{3}+\max\{|z-z_n|,0,|z_n-z|\}}$$

$$\geq \frac{t-\frac{4\delta}{3}}{t-\frac{4\delta}{3}+\max\{|x_n-y_n|,|y_n-z|,|z-x_n|\}} * \frac{\frac{\delta}{3}}{\frac{\delta}{3}+\max\{|z-z_n|,0,|z-z_n|\}}$$

$$= \frac{t-\frac{4\delta}{3}}{t-\frac{4\delta}{3}+\max\{|x_n-y_n|,|y_n-z|,|z-x_n|\}} * \frac{\frac{\delta}{3}}{\frac{\delta}{3}+|z-z_n|}$$

$$\geq \frac{t-\frac{5\delta}{3}}{t-\frac{5\delta}{3}+\max\{|x_n-z|,|z-y|,|y-x_n|\}} * \frac{\frac{\delta}{3}}{\frac{\delta}{3}+\max\{|y-y_n|,|y_n-y_n|,|y_n-y|\}} * \frac{\frac{\delta}{3}}{\frac{\delta}{3}+\max\{|z-z_n|,|z_n-z_n|,|z_n-z|\}}$$

$$\geq \frac{t-\frac{5\delta}{3}}{t-\frac{5\delta}{3}+\max\{|x_n-z|,|z-y|,|y-x_n|\}} * \frac{\frac{\delta}{3}}{\frac{\delta}{3}+|y-y_n|} * \frac{\frac{\delta}{3}}{\frac{\delta}{3}+|z-z_n|}$$

$$\geq \frac{t-2\delta}{t-2\delta+\max\{|z-y|,|y-x|,|x-z|\}} * \frac{\frac{\delta}{3}}{\frac{\delta}{3}+\max\{|x-x_n|,|x_n-x_n|,|x_n-x|\}} * \frac{\frac{\delta}{3}}{\frac{\delta}{3}+\max\{|y-y_n|,|y_n-y_n|,|y_n-y|\}} * \frac{\frac{\delta}{3}}{\frac{\delta}{3}+\max\{|z-z_n|,|z_n-z_n|,|z_n-z|\}}$$

and with $\delta = \frac{t}{2}$

$$\geq \frac{\frac{t}{6}}{\frac{t}{6}+\max\{|z-y|,|y-x|,|x-z|\}} * \frac{\frac{t}{6}}{\frac{t}{6}+|x-x_n|} *$$

$$\frac{\frac{t}{6}}{\frac{t}{6}+|y-y_n|} * \frac{\frac{t}{6}}{\frac{t}{6}+|z-z_n|}$$

and

$$M(x, y, z, t + 2\delta) \geq M(x, y, z, t_n + \delta)$$

$$\geq \frac{t_n+\delta}{t_n+\delta+\max\{|x-y|,|y-z|,|z-x|\}}$$

$$\geq \frac{t_n+\frac{2\delta}{3}}{t_n+\frac{2\delta}{3}+\max\{|x-y|,|y-z_n|,|z_n-x|\}} *$$

$$\frac{\frac{\delta}{3}}{\frac{\delta}{3}+\max\{|z_n-z|,|z-z|,|z-z_n|\}}$$

$$= \frac{t_n+\frac{2\delta}{3}}{t_n+\frac{2\delta}{3}+\max\{|x-y|,|y-z_n|,|z_n-x|\}} * \frac{\frac{\delta}{3}}{\frac{\delta}{3}+|z_n-z|}$$

$$\geq \frac{t_n + \frac{\delta}{3}}{t_n + \frac{\delta}{3} + \max\{|x - z_n|, |z_n - y_n|, |y_n - x|\}} *$$

$$\frac{\frac{\delta}{3}}{\frac{\delta}{3} + \max\{|y_n - y|, |y - y|, |y - y_n|\}} *$$

$$\frac{\frac{\delta}{3}}{\frac{\delta}{3} + \max\{|z_n - z|, |z - z|, |z - z_n|\}}$$

$$= \frac{t_n + \frac{\delta}{3}}{t_n + \frac{\delta}{3} + \max\{|x - z_n|, |z_n - y_n|, |y_n - x|\}} * \frac{\frac{\delta}{3}}{\frac{\delta}{3} + |y_n - y|} *$$

$$\frac{\frac{\delta}{3}}{\frac{\delta}{3} + |z_n - z|}$$

$$\geq \frac{t_n}{t_n + \max\{|z_n - y_n|, |y_n - x_n|, |x_n - z_n|\}} *$$

$$\frac{\frac{\delta}{3}}{\frac{\delta}{3} + \max\{|x_n - x|, |x - x|, |x - x_n|\}} *$$

$$\frac{\frac{\delta}{3}}{\frac{\delta}{3} + \max\{|y_n - y|, |y - y|, |y - y_n|\}} *$$

$$\frac{\frac{\delta}{3}}{\frac{\delta}{3} + \max\{|z_n - z|, |z - z|, |z - z_n|\}}$$

$$\geq \frac{t_n}{t_n + \max\{|z_n - y_n|, |y_n - x_n|, |x_n - z_n|\}} * \frac{\frac{\delta}{3}}{\frac{\delta}{3} + |x_n - x|} *$$

$$\frac{\frac{\delta}{3}}{\frac{\delta}{3} + |y_n - y|} * \frac{\frac{\delta}{3}}{\frac{\delta}{3} + |z_n - z|}$$

$$\geq \frac{t_n}{t_n + \max\{|z_n - y_n|, |y_n - x_n|, |x_n - z_n|\}}$$

for all $n \geq n_0$ and by taking the limit as $n \longrightarrow \infty$, yields:

$$\lim_{n\to\infty} M_D(x_n, y_n, z_n, t_n) \geq M_D(x, y, z, t - 2\delta)*1*1*1$$

$$\geq M_D(x, y, z, t - 2\delta)$$

and

$$M_D(x, y, z, t + 2\delta) \geq \lim_{n\to\infty} M_D(x_n, y_n, z_n, t_n)*1*1*1$$

$$= \lim_{n\to\infty} M_D(x_n, y_n, z_n, t_n)$$

taking the limit as $\delta \longrightarrow 0$, one can immediately deduce that:

$$\lim_{n\to\infty} M_D(x_n, y_n, z_n, t_n) = M_D(x, y, z, t)$$

Therefore, M_D is continuous on $X\times X\times X\times(0, \infty)$

Then $(X, M_D, *)$ is M-fuzzy metric space. ■

Remark (3.1.2), [15]:

Let $(X, M_D, *)$ be M-fuzzy metric space, then one can prove that for every $t > 0$:

$$M_D(x, x, y, t) = M_D(x, y, y, t)$$

Which follows directly from lemma (2.2.1).

Let $(X, M_D, *)$ be an M-fuzzy metric space, for $t > 0$ the M-open ball $B_M(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by:

$$B_M(x, r, t) = \{y \in X : M_D(x, y, y, t) > 1 - r\}$$

A subset A of X is called M-open set if for each $x \in A$, there exists $t > 0$ and $0 < r < 1$, such that $B_M(x, r, t) \subseteq A$.

A sequence $\{x_n\}$ in X is M-converges to $x \in X$ if and only if $M_D(x, x, x_n, t) \longrightarrow 1$ as $n \longrightarrow \infty$, for each $t > 0$

A sequence $\{x_n\}$ is called M-Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$, such that:

$$M_D(x_n, x_n, x_m, t) > 1 - \varepsilon, \forall\, n, m \geq n_0$$

The M-fuzzy metric space $(X, M_D, *)$ is said to be M-complete if every M-Cauchy sequence is M-converges.

The following result is given in [15] without details. Here, we give the details of the proof.

Lemma (3.1.4), [15]:

Let $(X, M_D, *)$ be an M-fuzzy metric space. Then $M_D(x, y, z, t)$ is nondecreasing with respect to t, for all $x, y, z \in X$, where:

$$M_D(x, y, z, t) = M_d(x, y, t)*M_d(y, z, t)*M_d(z, x, t)$$

Proof:

By definition (3.1.1)(4) , for each $x, y, z, a \in X$ and $t, s > 0$, then from eq.(3.2):

$$M_D(a, z, z, s) = M_d(a, z, s)*M_d(z, a, s)*M_d(z, z, s)$$

hence

$$M_D(x, y, a, t)*M_D(a, z, z, s) \leq M_D(x, y, z, t + s)$$

Setting $a = z$, and then using lemma (2.3.1) and lemma (3.1.1):

$$M_D(x, y, z, t)*M_D(z, z, z, s) \leq M_D(x, y, z, t + s)$$

That is:

$$M_D(x, y, z, t + s) \geq M_D(x, y, z, t)$$

Hence M_D is nondecreasing. ■

3.2 FUNDAMENTAL RESULTS

Completeness of M-fuzzy metric spaces play an important role in the analysis of the subject.

In the next lemma, we use the set U_n, which is defined by:

$$U_n = \{(x, y, z) \in X\times X\times X \mid M_D(x, y, z, \frac{1}{n}) > 1 - \frac{1}{n}\}$$

Lemma (3.2.1):

Let $\{U_n, n \in W\}$, be a sequence of subsets of X×X×X, such that U_0 = X×X×X, where W is any index set and U_n contains the diagonal (the identity relation is called the diagonal), and $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subset U_n$, $\forall$ n, where $\circ$ denotes the composition of three uniformly M-continuous functions is a given uniformly M-continuous. Then there is a non-negative real valued function d on X×X, such that:

a. $d(x, z) \leq d(x, y) + d(y, z)$.
b. $U_n \subset \{(x, y) \mid d(x, y) < 2^{-n} = f(x, y)\} \subseteq U_{n-1}$, for each n ∈ □ and if each U_n is symmetric (i.e., $U = U^{-1}$).

Then there is a pseudo-metric d satisfying condition (b).

Proof:

See [9]. ■

Definition (3.2.1), [9]:

A family $\mathcal{A}$ is σ-locally finite (σ-discrete) if it is the union of a countable number of locally finite (respectively, discrete) subfamilies of $\mathcal{A}$.

The statement of the metrization theorem may be decomposed into the following two lemmas:

Lemma (3.2.2), [9]:

If X is a uniform space which has a countable base, then X is pseudo-metrizable.

Proof:

If X has a uniformity $\mathcal{U}$ with countable base $\{U_n\}$, then by the principle of mathematical induction, we can construct a subsequence $\{U_n\}$, such that:

1. Each U_n is symmetric.
2. $U_n o U_n o U_n \subseteq U_{n-1}$.
3. $U_n \subseteq V_n$, $\forall\, n \in \mathbb{N}$.

Hence $\{U_n\}$ form a base for $\mathcal{U}$ and hence by the metrization lemma (3.2.1), we have a uniform space $(X, \mathcal{U})$ is a pseudo-metrizable. ■

Lemma (3.2.3), [9]:

A regular T_1-space whose topology has a σ-locally finite base is metrizable.

Proof:

It will be shown that there is a countable family F of pseudo metrics on the space X, such that each member of F is M-continuous on X×X and such that for each closed subset A of X and each point x of X − A, there is member d of F such that d-distance from x to A is positive.

This will be prove metrizability, for the map of X into each of the pseudo metric spaces (X, d) will then be M-continuous, and since from the embedding lemma [9], and since if we let $\{(X_n, d_n), n \in W\}$ be a sequence of pseudo metric spaces, each of diameter at most one, and define d by:

$$d(x, y) = \sum_{n\in \square} 2^{-n} d_n(x_n, y_n)$$

Then d is a pseudo-metric for the Cartesian product and the pseudo-metric topology is the product topology, [9].

The problem is then to construct the family F.

Let B be a σ-locally finite base for the topology of X, and suppose that:

$$B = \cup \{B_n : n \in W\}$$

where each B_n is locally finite.

For every ordered pair of integers m and n and for each member U of B_m, let U' be the union of all members of B_n whose closures are contained in U.

Because B_n is locally finite the closure of U' is a subset of U and there is a M-continuous function f_U on X to the unit interval which is one on U' and zero on $X - U$.

Hence letting (Since a regular space whose topology has a σ-locally finite base is normal), [9]:

$$d(x, y) = \sum_{U\in B_m} \left| f_U(x) - f_U(y) \right|$$

The continuity of d on X×X is straight forward consequence of the local finiteness of B_m.

Finally, let F be the family of pseudo-metrics so obtained, since pseudo-metric was constructed for each ordered pair of integers, F is countable, if A is a closed subset of X and $x \in X - A$, then for some m and some U in B_m it is true that $x \in U \subset X - A$, and for some n and some V in B_n it is true that $x \in V$ and $V' \subset U$.

For the pseudo-metric d constructed for this pair it is clear that the d-distance from x to A is at least one. ■

From the theory of topological spaces, the definition of Hausdorff topological space that; a topological space (X, τ) is called Hausdorff space if given distinct points x, y ∈ X, there exists an open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

The result may be generalized to fuzzy set theory (M-fuzzy topological spaces) using the properties of α-level sets depending on the relation between α-level sets (crisp sets) and fuzzy sets, where:

$$\tilde{A} = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha$$

which says that, if the α-level set A_α, $\forall \alpha \in [0, 1]$ satisfy or have certain property, then the fuzzy set $\tilde{A}$ has also that property.

Now, if A_α, $\forall \alpha \in [0, 1]$ is a Hausdorff space, then $\tilde{A}$ is a fuzzy Hausdorff space.

Lemma (3.2.4), [17]:

Let $(X, M_D, *)$ be a M-fuzzy metric space. Then τ_M is a Hausdorff topological space and for each $x \in X$, $\{B(x, 1/n, 1/n) \mid n \in \mathbb{N}\}$ is a neighborhood base at x for the topology τ_M.

From the above lemma one can note that every fuzzy metric space indeed is a fuzzy Hausdorff space.

Theorem (3.2.1):

Let $(X, M_D, *)$ be an M-fuzzy metric space, and let:

$$M_D(x, y, z, t) = \frac{t}{t + D(x,y,z)},\ t > 0$$

where:

$$D(x, y, z) = |x - y| + |y - z| + |z - x|$$

then (X, τ_M) is metrizable fuzzy topological space.

Proof:

For each $n \in \mathbb{N}$, define:

$$U_n = \{(x, y, z) \in X \times X \times X \mid M_D(x, y, z, 1/n) > 1 - \frac{1}{n}\}$$

where:

$$M_D(x, y, z, t) = \frac{t}{t + D(x,y,z)}$$

$$= \frac{\frac{1}{n}}{\frac{1}{n} + |x - y| + |y - z| + |z - x|}$$

It is sufficient to prove that the sequence $\{U_n\}$ is a base for a uniformity $\mathcal{U}$ on X, whose induced topology coincides with τ_M.

First, for each $n \in \mathbb{N}$, to prove that:

$$\{(x, x, x) \mid x \in X\} \subseteq U_n,\ U_{n+1} \subseteq U_n \text{ and } U_n = U_n^{-1}$$

Since:

$$M_D(x, x, x, t) = \frac{t}{t+|x-x|+|x-x|+|x-x|}$$

$$= \frac{t}{t} = 1$$

Hence $M_D \geq 1$ and therefore $\{(x, x, x) \mid x \in X\} \subseteq U_n$, i.e., the diagonal is contained in U_n.

Now, to prove that $U_{n+1} \subseteq U_n$, $\forall\ n \in \mathbb{N}$, and since $n + 1 > n$, hence $\frac{1}{n+1} < \frac{1}{n}$ and so:

$$1 - \frac{1}{n+1} > 1 - \frac{1}{n}$$

Therefore $U_{n+1} \subseteq U_n$ and $U_n = U_n^{-1}$

If $U = U^{-1}$, then U is called symmetric.

On the other hand, for each $n \in \mathbb{N}$, there is, by the M-continuity of *, $m \in \mathbb{N}$ such that $m > 3n$. Hence:

$$\frac{1}{m} < \frac{1}{3n}$$

and with * to be the usual product, gives:

$$\left[1-\frac{1}{m}\right]*\left[1-\frac{1}{m}\right]*\left[1-\frac{1}{m}\right] = \left[1-\frac{1}{m}\right]\left[1-\frac{1}{m}\right]\left[1-\frac{1}{m}\right]$$

$$< \left[1-\frac{1}{3n}\right]\left[1-\frac{1}{3n}\right]\left[1-\frac{1}{3n}\right]$$

$$< \left[1-\frac{1}{n}\right]\left[1-\frac{1}{n}\right]\left[1-\frac{1}{n}\right]$$

$$< 1-\frac{1}{n}, \forall\, n \in \mathbb{N}$$

Therefore $U_m o U_m o U_m \subseteq U_n$ (by lemma (3.2.2))

Indeed, let $(x, y) \in U_m$, $(y, y) \in U_m$ and $(y, a) \in U_m$

Since $M_D(x, y, z, *)$ is non decreasing (by lemma (3.1.4))

Then $M_d(x, a, 1/n) \geq M_d(x, a, 3/m)$, and so:

$$M_d(x, a, 1/n) \geq M_d(x, y, 1/m)* M_d(y, y, 1/m)*M_d(y, a, 1/m)$$

$$\geq \frac{\frac{1}{m}}{\frac{1}{m}+|x-y|}*\frac{\frac{1}{m}}{\frac{1}{m}+|y-y|}*\frac{\frac{1}{m}}{\frac{1}{m}+|y-a|}$$

$$\geq \frac{\frac{1}{m}}{\frac{1}{m}+|x-y|}*\frac{\frac{1}{m}}{\frac{1}{m}+|y-a|}$$

$$\geq \left[1-\frac{1}{m}\right]*\left[1-\frac{1}{m}\right]$$

$$\geq 1-\frac{1}{n}$$

Therefore, $(x, a) \in U_n$ and thus $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity $\mathcal{U}$ on X.

Since for each $x \in X$ and each $n \in \mathbb{N}$

$$U_n(x) = \{y \in X : M_D(x, y, y, 1/n) > 1 - \frac{1}{n}\}$$

$$= B(x, 1/n, 1/n)$$

Hence from lemma (3.2.4), that the fuzzy topology induced by $\mathcal{U}$ coincide with τ_M.

By lemma (3.2.3), (X, τ_M) is a metrizable fuzzy topological space. ■

Definition (3.2.2), [6]:

An M-fuzzy metric space is said to be completely M-fuzzy metrizable if every M-fuzzy Cauchy sequence is M-fuzzy convergent.

Theorem (3.2.2), [17]:

Let $(X, M_D, *)$ be a M-complete fuzzy metric space, and let:

$$M_D(x, y, z, t) = \frac{t}{t + D(x, y, z)}, \ t > 0$$

where:

$$D(x, y, z) = |x - y| + |y - z| + |z - x|$$

Then (X, τ_M) is M-completely fuzzy metrizable.

Proof:

It follows from the proof of theorem (3.2.1) that $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity $\mathcal{U}$ on X compatible with τ_M, where:

$$U_n = \{(x, y, y) \in X\times X\times X \mid M_D(x, y, y, 1/n) > 1 - \frac{1}{n}, \forall\, n \in \mathbb{N}\}$$

Then there is a metric d on X whose induced uniformity coincides with $\mathcal{U}$.

To show that the metric (X, d) is M-complete fuzzy metric spase, indeed given a fuzzy M-Cauchy sequence $\{x_n\}$ in (X, d) and to show that $\{x_n\}$ is a fuzzy M-Cauchy sequence in $(X, M_D, *)$

To do this, fix r, t with $0 < r < 1$ and $t > 0$ and choose $k \in \mathbb{N}$, such that $\frac{1}{k} \le \min\{t, r\}$

Then, there is $n_0 \in \mathbb{N}$, such that $(x_n, x_n, x_m) \in U_k$, for every $n, m \ge n_0$

Consequently, for each $n, m \ge n_0$

$$M_D(x_n, x_n, x_m, t) \ge M_D(x_n, x_n, x_m, 1/k)$$

$$> 1 - \frac{1}{k} \ge 1 - r$$

Hence $\{x_n\}$ is a fuzzy M-Cauchy sequence in the M-complete fuzzy metric space $(X, M_D, *)$ (by assumption)

So it is M-convergent with respect to τ_M

Hence, (X, d) is a M-complete metric space on X

Therefore, (X, τ_M) is M-completely fuzzy metrizable (by definition (3.2.3). ■

Corollary (3.2.1), [17]:

A topological space (X, τ_M) is M-completely metrizable if and only if it is a compatible complete fuzzy metric space, where:

$$M_D(x, y, z, t) = \frac{t}{t + D(x, y, z)}, \ t > 0$$

and

$$D(x, y, z) = |x - y| + |y - z| + |z - x|$$

Proof:

Suppose that (X, τ) is a M-completely metrizable space and let (X, d) be a M-complete metric space such that d is compatible with τ since one can prove that the induced fuzzy metric space $(X, M_D, *)$ is M-complete if and only if the metric space (X, d) is M-complete, where:

$$M_D(x, y, z, t) = \frac{t}{t + D(x, y, z)}, \ \forall\ x, y, z, \in X, t \in (0, \infty)$$

and it is compatible with t

The converse follows immediately from theorem (3.2.2). ■

Now, we are in a place to give the main result related to this work which is the M-completeness of fuzzy metric spaces.

Definition (3.2.3), [17]:

An M-fuzzy metric space $(X, M_D, *)$ is called precompact if for each r, with $0 < r < 1$ and each $t > 0$, there is a finite subset A of X, such that:

$$X = \bigcup_{a \in A} B(a, r, t)$$

In this case, M is called a precompact M-fuzzy metric space on X.

Theorem (3.2.3), [17]:

Compact M-fuzzy metric space is M-complete.

Proof:

Suppose that $(X, M_D, *)$ is a compact fuzzy metric space, for each r, with $0 < r < 1$ and each $t > 0$ the open cover $\{B(x, r, t) : x \in X\}$ of X, has a finite subcover by definition (1.4.5)

Hence $(X, M_D, *)$ is precompact (by definition (3.2.3))

On the other hand, every M-Cauchy sequence $\{x_n\}$ in $(X, M_D, *)$ has a limit point $y \in X$

Let $\{x_n\}$ be a fuzzy M-Cauchy sequence in $(X, M_D, *)$ having a limit point $x \in X$, then there is a subsequence $\{x_{kn}\}$ of $\{x_n\}$ that M-converges to y with respect to τ_M.

Thus, given r, with $0 < r < 1$ and $t > 0$, there is $n_0 \in \mathbb{N}$, such that for each $n \geq n_0$

$$M_D(x, x, x_{kn}, \frac{t}{3}) > 1 - s, \text{ where } s > 0$$

Which satisfies $(1-s)*(1-s) > 1-r$

Also, there exists $n_1 \geq k(n_0)$, such that for each n, $m \geq n_1$

$$M_D(x_n, x_n, x_m, \frac{t}{3}) > 1 - s$$

Therefore, for each $n \geq n_1$

$$M_D(x, x, x_n, t) \geq M_D(x, x, x_{kn}, \frac{t}{3})*M_D(x_{kn}, x_n, x_n, \frac{t}{3})$$

$$\geq (1-s)*(1-s)$$

$$> 1 - r$$

Hence the fuzzy M-Cauchy sequence $\{x_n\}$ M-converges to x.

Thus $(X, M_D, *)$ is an M-complete fuzzy metric space. ■

CHAPTER TWO
M-FUZZY METRIC SPACES

Fuzzy metric spaces are of great importance in studying fuzzy dynamical systems. Fuzzy metric spaces have been introduced by several mathematicians using different approaches, either by using α-level sets, or by using fuzzy numbers, or by using the cooperation of fuzzy topological spaces, etc.

This chapter consists of studying an important type of fuzzy metric spaces, which have not been studied commonly by other researchers, which is the M-fuzzy metric spaces and its relationship with fuzzy topological spaces.

2.1 BASIC DEFINITIONS

Following are some definitions and basic concepts in fuzzy metric spaces which are given by several literatures and further illustrated by Mary in 2004, [10].

Definition (2.1.1), [10]:

A fuzzy set $\tilde{A}$ is bounded if there exists a real number $h > 0$, such that:

$$d^*(x, y) < h, \forall\ x, y \in \tilde{A}$$

Now, some additional concepts are given for completeness in ordinary set theory which are necessary in this chapter to define M-fuzzy metric spaces.

Definition (2.1.2), [11]:

Let A be a non empty set. A binary operation * on A is a correspondence which associates to each ordered pair (a, b) of elements of $A \times A$ a unique element a*b of A.

An alternative definition of binary operations is as follows:

It is a mapping from the Cartesian product $A \times A$ to A, where the image of (a, b) is denoted by a*b.

Three words in the above definition are given, which merit extra emphasis:

(1) "each". If * is to be a binary operation on a set A, it must define a*b for every pair of elements a, b of A, i.e., there is no elements of A that can not be combined.

(2) "unique". For each pair of elements a, b of A, there must be only one "answer" a*b when combining a and b.

In particular, this says that if a_1, a_2, b_1 and $b_2 \in A$, $a_1 = a_2$ and $b_1 = b_2$, then $a_1 * b_1 = a_2 * b_2$. When the operation in the question has this "uniqueness" property, then the operation is said to be well defined.

(3) "of". For * to be a binary operation on A, a*b must be an element of A. Hence for every a, b $\in$ A, a*b $\in$ A. This property is usually called the closure property.

Now, it is easy for the reader to recall some additional basic definitions in ordinary set theory, such as commutative, associative, identity element of an operation, etc. For symbolic definitions, let A be any nonempty set, then we let K(A) to denote the set of all one to one mappings of A onto itself.

Some properties of K(A) are listed below which are needed later in chapter three.

Properties of K(A) (2.1.1), [11]:

Consider three mappings $\alpha : A \longrightarrow A$, $\beta : A \longrightarrow A$ and $\gamma : A \longrightarrow A$ and let o be the usual composition of mappings, then:

1. $\alpha o \beta$ is an element of K(A), where o is the usual mapping composition.
2. $\alpha o(\beta o \gamma) = (\alpha o \beta) o \gamma$.
3. There exists an element L (the identity mapping on A) in K(A), such that:

   $\alpha o L = L o \alpha = \alpha$

4. There exists an element α^{-1} for each α in K(A), such that:

   $\alpha o \alpha^{-1} = \alpha^{-1} o \alpha = L$

In the present section, we will set also the basic fundamental definitions which are needed later on to study the completeness of fuzzy metric spaces.

We start first with some classical definitions and notions in fuzzy set theory, in general, and in fuzzy metric spaces, in particular.

Definition (2.1.3), [14]:

A two place function T : $[0,1]\times[0,1] \longrightarrow [0.1]$ is said to be T-norm if it satisfy:

(a) $0 \leq T(a, b) \leq 1$.

(b) $T(c, d) \geq T(a, b)$, for $c \geq a$, $d \geq b$.

(c) $T(a, b) = T(b, a)$.

(d) $T(1, 1) = 1$

(e) $T(a, 1) > 0$, for all $a > 0$.

As an example of some of the most well known T-norms, are the following:

For all $a, b \in [0, 1]$

T_1: $T(a, b) = \max\{a + b - 1, 0\}$, i.e., $T = \max\{\text{sum} - 1, 0\}$.

T_2: $T(a, b) = ab$, i.e., T = product.

T_3: $T(a, b) = \min\{a, b\}$, i.e., T = min.

T_4: $T(a, b) = \max\{a, b\}$, i.e., T = max.

T_5: $T(a, b) = a + b - ab$, i.e., T =sum −product.

T_6: $T(a, b) = \min\{a + b, 1\}$, i.e., $T = \min\{\text{sum}, 1\}$

Definition (2.1.4), [17]:

A binary operation $* : [0, 1]\times[0, 1] \longrightarrow [0, 1]$ is continuous T-norm if it satisfies the following conditions:

1. $*$ is associative and commutative.
2. $*$ is continuous mapping.
3. $a*1 = a$, for all $a \in [0, 1]$.
4. $a*b \leq c*d$, whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0, 1]$.

As an example of continuous T-norm which will be used next in chapter three is the usual product, which is given in [17] without details, here we give its details.

Example (2.1.1):

Consider the binary operation $* : [0, 1]\times[0, 1] \longrightarrow [0, 1]$ defined by $a*b = ab$, then it is clear that $*$ is commutative and associative.

Also, consider $f(a, b) = a*b = ab, \forall\ a, b \in [0, 1]$

Now, let $A = [a, b] \subseteq [0, 1]$

Hence $f^{-1}(A) = [c, d] \subseteq [0, 1]$

Then $a*b$ is continuous

Also, it clear that $a*1 = a, \forall\ a \in [0, 1]$

Finally:

$$
\begin{aligned}
a*b &= ab, \text{ if } a \leq c, b \leq d \\
&\leq cd \\
&= c*d, (a, b, c, d \in [0, 1])
\end{aligned}
$$

Therefore, $a*b = ab$ is continuous T-norm.

2.2 D-METRIC SPACES

It is important to mention that the definition of M-fuzzy metric spaces depends implicitly on another type of metric spaces, which is the so called D-metric spaces (as it will be seen next in chapter three), and hence because of this strong relationship between M-fuzzy metric spaces and D-metric spaces, we discuss in this section (for completion purpose) D-metric spaces, as well as, some of its important properties.

We start with the following basic definition of D-metric spaces:

Definition (2.2.1), [15]:

Let X be a nonempty set. A generalized metric (or D-metric) on X is a function $D : X \times X \times X \longrightarrow \mathbb{R}^+$, that satisfies the following conditions for each x, y, z, a $\in$ X:

(1) D(x, y, z) $\geq$ 0.

(2) D(x, y, z) = 0 if and only if x = y = z.

(3) D(x, y, z) =D(p{x, y, z}), where p is the permutation function.

(4) D(x, y, z) $\leq$ D(x, y, a) + D(a, z, z).

The pair (X, D) is called the generalized metric or D-metric space.

Immediate examples of such a function which are of great importance, are:

(a) D(x, y, z)= max {d(x, y), d(y, z), d(z, x)}, where d is the ordinary metric on X.

(b) D(x, y, z) = d(x, y) + d(y, z) + d(z, x).

(c) If $X = \mathbb{R}^n$, then for every $p \in \mathbb{R}^+$

$$D(x, y, z) = \left(\| x - y \|^p + \| y - z \|^p + \| z - x \|^p \right)^{1/p}$$

(d) If $X = \mathbb{R}^+$, then:

$$D(x, y, z) = \begin{cases} 0, & \text{if } x = y = z \\ \max\{x, y, z\}, & \text{otherwise} \end{cases}$$

Definition (2.2.2), [1]:

Let (X, D) be a D-metric space and let $A \subseteq X$. Let r denote the restriction of D to $A \times A \times A$, then (A, r) is called a D-metric subspace of (X, D).

Definition (2.2.3), [1]:

A D-metric space (X, D) is said to be D-bounded if there exists a positive real number N, such that $D(x, y, z) \leq N$, for all $x, y, z \in X$.

In such a case N is said to be the D-bound for X. Moreover, if $E \subseteq X$, then E is said to be D-bounded subspace of X if there exists a positive real number M, such that $D(x, y, z) \leq M$, for all $x, y, z \in E$.

Now, some illustrative examples are considered for completeness purpose.

Example (2.2.1), [15], [1]:

Let $X = \mathbb{R}$ and $D(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in X$. Then (X, D) is unbounded D-metric space.

Since if we let $x, y, z, a \in X$, then:

i- $D(x, y, z) = |x - y| + |y - z| + |z - x| > 0$ if and only if x, y, z are distinct and if $x = y = z$ then $|x - y| + |y - z| + |z - x| = 0$ and hence $D(x, y, z) = 0$

if $D(x, y, z) = 0$, then $|x - y| + |y - z| + |z - x| = 0$, which is true only if $|x - y| = 0$, $|y - z| = 0$, $|z - x| = 0$ and therefore $x = y = z$.

ii- $$\begin{aligned} D(x, y, z) &= |x - y| + |y - z| + |z - x| \\ &= |x - z| + |z - y| + |y - x| \\ &= D(x, z, y) \end{aligned}$$

Similarly, $D(x, y, z) = D(x, z, y) = D(z, x, y) = \dots$

i.e., $D(x, y, z) = D(p\{x, y, z\})$, where p is the permutation function.

iii- $$\begin{aligned} D(x, y, z) &= |x - y| + |y - z| + |z - x| \\ &\le |x - y| + |y - a| + |a - x| + |a - z| + |z - z| + |z - a| \\ &= D(x, y, a) + D(a, z, z) \end{aligned}$$

Therefore (X, D) is a D-metric space.

But if there is no positive real number N, such that $D(x, y, z) \le N$, for all $x, y, z \in X$, then (X, D) is unbounded D-metric space.

Also, consider the following example which may be solved similarly as example (2.2.1).

Example (2.2.2), [15], [1]:

Let $X = \mathbb{R}$ and $D(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$, for all x, y, z $\in$ X. Then (X, D) is unbounded D-metric space.

The next lemma is given in [15] without details, we give here the details of the proof.

Lemma (2.2.1):

Let (X, D) be a D-metric space, then $D(x, x, y) = D(x, y, y)$.

Proof:

(i) $D(x, y, z) \leq D(x, y, a) + D(a, z, z)$ (by definition)

By substituting x instead of y and y instead of z, gives:

$D(x, x, y) \leq D(x, x, a) + D(a, y, y)$, and if a = x

$\leq D(x, x, x) + D(x, y, y)$

and since $D(x, y, z) = 0$ if and only if $x = y = z$, then $D(x, x, x) = 0$

Therefore, $D(x, x, y) \leq D(x, y, y)$

(ii) Similarly, as in (i)

$D(x, y, y) = D(y, y, x)$

$\leq D(y, y, a)+D(a, x, x)$

$\leq D(y, y, y) + D(y, x, x)$

$= D(y, y, y) + D(y, x, x)$

$= D(x, x, y)$

Therefore, $D(x, y, y) \leq D(x, x, y)$

Then from (i) and (ii):

$D(x, x, y) = D(x, y, y)$. ■

Definition (2.2.4), [15]:

Let (X, D) be a D-metric space, for $r > 0$, define:

$B_D(x, r) = \{y \in X : D(x, y, y) < r\}$

If for every $x \in A$, there exists $r > 0$, such that $B_D(x, r) \subset A$, then the subset A of X is called D-open ball.

Example (2.2.3), [15]:

Let $X = \mathbb{R}$, and define $D(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in \mathbb{R}$, then:

$B_D(1, 2) = \{y \in \mathbb{R} : D(1, y, y) < 2\}$

$= \{y \in \mathbb{R} : |y - 1| + |y - 1| < 2\}$

$= \{y \in \mathbb{R} : |y - 1| < 1\}$

$= (0, 2)$

Definitions (2.2.5), [15]:

Let (X, D) be a D-metric space and $A \subset X$.

1. A sequence $\{x_n\}$ in X is said to be D-converge to x if and only if $D(x_n, x_n, x) \longrightarrow 0$, as $n \longrightarrow \infty$.

   That is, for each $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$, such that:

   $$D(x_n, x, x) < \varepsilon, \forall\, n \geq n_0$$

   One can easily prove for a D-convergent sequence

   $$D(x_n, x_n, x) = D(x, x, x_n)$$

2. A sequence $\{x_n\}$ in X is called a D-Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $D(x_n, x_n, x_m) < \varepsilon$, for each $n, m \geq n_0$.

3. The D-metric space (X, D) is said to be D-complete if every D-Cauchy sequence is D-converge.

Now, let τ be the set of all $A \subset X$ with $x \in A$ if there exists $r > 0$, such that $B_D(x, r) \subset A$, then τ is called a topology on X induced by the D-metric D.

<u>Lemma (2.2.2), [15]:</u>

Let (X, D) be a D-metric space. If $r > 0$, then the ball $B_D(x, r)$ with center $x \in X$ and radius r is an open set.

Proof:

Let $z \in B_D(x, r)$, hence $D(x, z, z) < r$

Letting $D(x, z, z) = \delta$ and let $r' = r - \delta$, and since $r' < r$, then $B_D(x, r') \subset B_D(x, r)$

Let $y \in B_D(z, r')$, hence by the triangular inequality:

$$\begin{aligned} D(x, y, y) &= D(y, y, x) \\ &\leq D(y, y, z) + D(z, x, x) \\ &< r' + \delta = r \end{aligned}$$

Therefore, $B_D(z, r') \subseteq B_D(x, r)$

That is the ball $B_D(x, r)$ is an open set. ■

Lemma (2.2.3), [15]:

Let (X, D) be a D-metric space. If a sequence $\{x_n\}$ in X is D-converges to x, then x is unique.

Proof:

Suppose that $\{x_n\}$ has two D-limit points x and y, such that $x \neq y$

Since $\{x_n\}$ converge to x and y, hence for each $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$, such that for every $n \geq n_1$, $D(x, x, x_n) < \frac{\varepsilon}{2}$ and $n_2 \in \mathbb{N}$, such that for every $n \geq n_2$, $D(y, y, x_n) < \frac{\varepsilon}{2}$

Setting $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ and by the triangular inequality:

$$\begin{aligned} D(x, x, y) &\leq D(x, x, x_n) + D(x_n, y, y) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Hence $D(x, x, y) = 0$ which is a contradiction, so $x = y$. ■

Lemma (2.2.4), [15]:

Let (X, D) be a D-metric space. If a sequence $\{x_n\}$ in X is D-converges to $x \in X$, then the sequence $\{x_n\}$ is a D-Cauchy sequence.

Proof:

Since $\{x_n\}$ is D-converges to x, hence for every $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$, such that for every $n \geq n_1$, $D(x_n, x_n, x) < \frac{\varepsilon}{2}$ and $n_2 \in \mathbb{N}$, such that for every $m > n_2$, $D(x, x_m, x_m) < \frac{\varepsilon}{2}$

Setting $n_0 = \max\{n_1, n_2\}$, then for every $n, m \geq n_0$ and by the triangular inequality:

$$D(x_n, x_n, x_m) \leq D(x_n, x_n, x) + D(x, x_m, x_m)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence the sequence $\{x_n\}$ is a D-Cauchy sequence. ■

2.3 ELEMENTARY CONSEPTS IN M-FUZZY METRIC SPACES

In this section fundamental concepts are recalled in M-fuzzy metric spaces.

Definition (2.3.1), [17]:

A 3-tuple $(X, M_d, *)$ is said to be M-fuzzy metric space if X is an arbitrary set, * is a continuous T-norm and M is a fuzzy subset of

$X \times X \times (0, \infty)$ satisfying the following conditions for all x, y, z $\in$ X and t, s > 0:

(1) M(x, y, t) > 0.

(2) M(x, y, t) = 1 if and only if x = y.

(3) M(x, y, t) = M(y, x, t).

(4) M(x, y, t)*M(y, z, s) $\leq$ M(x, z, t + s).

(5) M(x, y, *) : $(0, \infty) \longrightarrow [0, 1]$ is continuous.

Let (X, M_d, *) be a M-fuzzy metric space, then among the important notions in M-fuzzy metric spaces is the family:

$\{B(x, r, t) \mid x \in X, 0 < r < 1, t > 0\}$

which is the neighborhood system for a Hausdorff topology on X, that is called the topology induced by the M-fuzzy metric M, with topology τ defined by:

$\tau = \{ A \subset X : x \in A$ if and only if there exists t > 0, 0 < r < 1, such that $B(x, r, t) \subset A\}$

Definition (2.3.2), [16]:

Let (X, τ) be a topological space, then (X, τ) is called T2-space (Hausdorff space) if for all a, b $\in$ X, there exists an open sets G and H, such that:

A $\in$ G, b $\in$ H and $G \cap H = \varnothing$

Now, we are going to recall some basic concepts in ordinary metric spaces and topological spaces with some fundamental concepts.

Definition (2.3.3), [16]:

A topological space (X, τ) is a first countable space if there exists a countable local base $\mathcal{B}$ at every point $p \in X$.

Definition (2.3.4), [8], [16]:

A topological space (X, τ) is called a second countable space if there exists a countable base $\mathcal{B}$ for the topology τ.

Definition (2.3.5), [17]:

A topological space (X, τ) admits a compatible fuzzy metric if there is a M-fuzzy metric M on X, such that $\tau = \tau_M$.

Lemma (2.3.1):

Let (X, d) be a metric space and consider the M-fuzzy metric space (X, M_d, *) and define M_d by:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $M_d(x, y, t + s) \geq M_d(x, y, t)$, $\forall$ s, t, > 0.

Proof:

Suppose that t, s > 0, then:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

$$= \frac{t + d(x, y) - d(x, y)}{t + d(x, y)}$$

$$= \frac{t + d(x, y)}{t + d(x, y)} - \frac{d(x, y)}{t + d(x, y)}$$

$$= 1 - \frac{d(x, y)}{t + d(x, y)}$$

$$\leq 1 - \frac{d(x, y)}{t + s + d(x, y)}$$

$$= \frac{t + s + d(x, y) - d(x, y)}{t + s + d(x, y)}$$

$$= \frac{t + s}{t + s + d(x, y)}$$

$$= M_d(x, y, t + s).$$

and hence the inequality follows. ■

The proof that (X, τ_M) is a Hausdorff first countable topological space, is given in the next two theorems which are given in [16] without details, here we give the details of the proof.

<u>Theorem (2.3.1):</u>

Let X be a Hausdorff space, then every M-convergent sequence in X has a unique limit point.

Proof:

Suppose $\{a_n\}_{n=1}^{\infty}$ is a M-convergent sequence in X to a and b, and suppose that $a \neq b$. Since X is Hausdorff, then there exists two open sets G and H, such that:

$a \in G$, $b \in H$ and $G \cap H = \varnothing$

By hypothesis, the sequence $\{a_n\}$ is M-converges to a; hence there exists an $n_0 \in \mathbb{N}$, such that for all $n > n_0$, implies that $a_n \in G$, i.e., G contains all except a finite number of the terms of the sequence

But G and H are disjoints, hence H can only contain those terms of the sequence which do not belong to G and there are only a finite number of such terms.

Accordingly, $\{a_n\}$ can not M-converge to b

But this violates the hypothesis that the sequence M-converge to b also, which is a contradiction

Hence $a = b$. ■

<u>Theorem (2.3.2):</u>

Let X be a first countable space, then the following are equivalent:

(1) X is Hausdorff space.

(2) Every M-convergent sequence has a unique limit.

Proof:

(1) $\Rightarrow$ (2). Clear from theorem (2.3.1).

(2) $\Rightarrow$ (1). Suppose that X is not a Hausdorff space, then there exists a, b $\in$ X, a $\neq$ b, with the property that every open set containing a has a nonempty intersection with every open set containing b.

Now, let $\{G_n\}$ and $\{H_n\}$ be a nested local bases at a and b, respectively.

Then, $G_n \cap H_n \neq \emptyset$, for every $n \in \mathbb{N}$, and so there exist a sequence $\{a_i\}_{i=1}^{\infty}$ such that:

$$a_1 \in G_1 \cap H_1,\ a_2 \in G_2 \cap H_2,\ \ldots$$

Accordingly, $\{a_i\}$ is M-converges to both a and b and hence the limit point is not unique, which is a contradiction

Hence, X is a Hausdorff space. ■

Definition (2.3.6), [8], [9]:

A pseudo-metric space is a pair (X, d) such that d is a pseudo-metric on X.

An equivalent definition of pseudo-metrics is the following which has its connectivity with the topological spaces:

Definition (2.3.7), [19]:

A pseudo-metric space is a metric space if and only if the topology is T_1 space, i.e., each singleton set {x} is closed, $\forall$ x $\in$ X.

Definition (2.3.8), [9]:

A topological space (X, τ) is pseudo-metrizable if and only if there is a pseudo-metric, such that the topology is the pseudo-metric topology.

Definition (2.3.9), [9]:

A topological space (X, τ) is metrizable if and only if it is T_1 and pseudo-metrizable.

CHAPTER ONE
FUNDAMENTAL CONCEPTS OF FUZZY SET THEORY

In this chapter, some of the most important concepts related to fuzzy set theory, will be presented. These concepts include a brief introduction to the theory of fuzzy sets, fuzzy metric spaces and fuzzy topological spaces. These concepts are of great importance for this work.

1.1 BASIC CONCEPTS OF FUZZY SETS [3]

Fuzzy set theory is a generalization of abstract set theory; it has a widr scope of applicability than abstract set theory in solving problems that involve to some degree subjective evaluation or vague notions.

Let X be any non-empty set of elements. A ***fuzzy set*** $\tilde{A}$ in X is the set of all $x \in X$, which are characterized by a ***membership function*** $\mu_{\tilde{A}}(x) : X \longrightarrow I$, where I is the closed unit interval [0, 1]. The grades 0 and 1 represent respectively non-membership and full membership in a fuzzy set $\tilde{A}$. A fuzzy set $\tilde{A}$ may be written mathematically as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X, 0 \le \mu_{\tilde{A}}(x) \le 1\}$$

The following concepts may be defined in fuzzy sets:

1. The ***support*** of $\tilde{A}$ is the crisp set of all $x \in X$, such that $\mu_{\tilde{A}}(x) > 0$ and is denoted by $S(\tilde{A})$.

2. The ***core (uncleus)*** of a fuzzy set $\tilde{A}$ is the set of all points x ∈ X, such that $\mu_{\tilde{A}}(x) = 1$.

3. The ***height*** of a fuzzy set $\tilde{A}$ (denoted by hgt $(\tilde{A})$) is the supremum value of $\mu_{\tilde{A}}(x)$ over all x ∈ X. If hgt $(\tilde{A}) = 1$, then $\tilde{A}$ is ***normal,*** otherwise it is ***subnormal***, and a fuzzy set may be always ***normalized*** by defining the scaled membership function:

$$\mu^{*}_{\tilde{A}}(x) = \frac{\mu_{\tilde{A}}(x)}{\underset{x\in X}{\mathrm{Sup}}\,\mu_{\tilde{A}}(x)}, \ \forall\, x \in X$$

4. The ***crossover point*** of a fuzzy set $\tilde{A}$ is that point in X, whose grade of membership in $\tilde{A}$ is 0.5.

5. ***Fuzzy singleton*** is a fuzzy set whose support is a single point x ∈ X, with $\mu_{\tilde{A}}(x) = \alpha$, $\alpha \in (0, 1]$.

<u>Remarks (1.1.1), [3]:</u>

Some important concepts related to fuzzy subset of a universal set X may be listed below. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy subsets of the universal set X with membership functions $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$, respectively, then:

1. $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x)$, ∀ x ∈ X.

2. $\tilde{A} = \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$, ∀ x ∈ X.

3. $\tilde{A}^{c}$ is the complement of $\tilde{A}$ with membership function $\mu_{\tilde{A}^{c}}(x) = 1 - \mu_{\tilde{A}}(x)$, ∀ x ∈ X.

4. The empty fuzzy set $\tilde{\varnothing}$ and the universal set X, when for all $x \in X$ $\mu_{\tilde{A}}(x) = 0$ and $\mu_{\tilde{A}}(x) = 1$, respectively.

5. $\tilde{C} = \tilde{A} \cap \tilde{B}$ is a fuzzy set with membership function:

$$\mu_{\tilde{C}}(x) = \text{Min}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \ \forall\ x \in X$$

More generally, for any index set J, then $\bigcap_{j \in J} \tilde{A}_j$ is also a fuzzy set of X with membership function:

$$\mu_{\bigcap_{j \in J} \tilde{A}_j}(x) = \inf_{i \in J} \mu_{\tilde{A}_j}(x), \ \forall\ x \in X$$

6. $\tilde{D} = \tilde{A} \cup \tilde{B}$ is a fuzzy set with membership function

$$\mu_{\tilde{D}}(x) = \text{Max}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \ \forall\ x \in X$$

More generally, for any index set J, then $\bigcup_{j \in J} \tilde{A}_j$ is also a fuzzy set of X with membership function:

$$\mu_{\bigcup_{j \in J} \tilde{A}_j}(x) = \sup_{i \in J} \mu_{\tilde{A}_j}(x), \ \forall\ x \in X$$

7. If $\mu_{\tilde{A} \cap \tilde{B}}(x) = 0$, $\forall\ x \in X$, then $\tilde{A}$ and $\tilde{B}$ are said to be ***disjoint***.

More additional concepts may be found in any text reference, such as [8], [11].

Remark (1.1.2), [3]:

It is notable that the only law is $A \cup A^c = X$ and the law $A \cap A^c = \emptyset$ are broken for the fuzzy sets, since $\tilde{A} \cup \tilde{A}^c \neq X$ and $\tilde{A} \cap \tilde{A}^c \neq \tilde{\emptyset}$. Indeed, for all $x \in X$, if $\mu_{\tilde{A}}(x) = \alpha$, $0 < \alpha < 1$, then:

$$\mu_{\tilde{A} \cup \tilde{A}^c}(x) = \max \{\alpha, 1 - \alpha\} \neq 1$$

$$\mu_{\tilde{A} \cap \tilde{A}^c}(x) = \min \{\alpha, 1 - \alpha\} \neq 0$$

1.2 THE EXTENSION PRINCIPLE

The extension principle of fuzzy set theory may be used to generalize crisp mathematical concepts to fuzzy mathematical concepts, which is used also to define fuzzy functions, [1].

Definition (1.2.1), [2]:

Let X be the Cartesion product of universes $X_1, X_2, \ldots, X_r$ and $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_r$ be r- fuzzy sets in $X_1, X_2, \ldots, X_r$, respectively, f is a mapping from X to a universe Y ($y = f(x_1, x_2, \ldots, x_r)$). Then the fuzzy set $\tilde{B}$ in Y is defined, by:

$$\tilde{B} = f(\tilde{A}) = \left\{ \left(y, \mu_{\tilde{B}}(y)\right) \middle|\ y = f(x_1, x_2, \ldots, x_r), (x_1, x_2, \ldots, x_r) \in X \right\}$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{(x_1, x_2, \ldots, x_r) \in f^{-1}(y)} \min\left\{\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_r}(x_r)\right\}, & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{, Otherwise} \end{cases}$$

where f^{-1} is the inverse image of f.

For r = 1, the extension principle, of course, reduces to:

$$\tilde{B} = f(\tilde{A}) = \left\{ \left(y, \mu_{\tilde{B}}(y) \right) \middle| \; y = f(x), \, x \in X \right\}$$

where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x), & \text{if } f^{-1}(y) \neq \varnothing \\ 0 & , \text{ Otherwise} \end{cases}$$

which is the definition of a fuzzy function.

1.3 α-LEVEL SETS [4]

Among the basic concepts in fuzzy set theory is the concept of α-level (α-cut) sets of a fuzzy set $\tilde{A}$, which is used as an intermediate set that connect between fuzzy and non fuzzy sets.

Given a fuzzy set $\tilde{A}$ defined on a universal X and any number $\alpha \in (0, 1]$ the α-level, A_α is the crisp set that contains all elements of the universal set X, whose membership grades in $\tilde{A}$ are greater than or equal to a pre specified value of α, i.e.,

$$A_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha, \; \forall \, x \in X\}$$

Also, the strong α-level set is defined by:

$$A_{\alpha+} = \{x : \mu_{\tilde{A}}(x) > \alpha, \; \forall \, x \in X\}$$

The following properties are satisfied for all $\alpha \in [0, 1]$, which may be proved easily for all $\alpha, \beta \in (0, 1]$:

1. $\tilde{A} = \tilde{B}$ if and only if $A_\alpha = B_\alpha$.
2. If $\tilde{A} \subseteq \tilde{B}$ then $A_\alpha \subseteq B_\alpha$.
3. $(\tilde{A} \cup \tilde{B})_\alpha = A_\alpha \cup B_\alpha$.
4. $(\tilde{A} \cap \tilde{B})_\alpha = A_\alpha \cap B_\alpha$.
5. If $\alpha \leq \beta$, then $A_\alpha \supseteq A_\beta$.
6. $A_\alpha \cap A_\beta = A_\beta$ and $A_\alpha \cup A_\beta = A_\alpha$, if $\alpha \leq \beta$.
7. If $\tilde{A}$ is a fuzzy set, $\{A_\alpha\}$, $\forall\, \alpha \in (0, 1]$ is a family of subsets of the universal set X, then:

$$\tilde{A} = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha$$

Now, because of its importance in fuzzy sets, the following property which is given in some literatures without proof, and appeared in [13] without any details, here we give the details of the proof.

8. Let X and Y be two universal sets, and $f : X \times X \longrightarrow Y$ be an ordinary function, and $\tilde{A}$, $\tilde{B}$ be any two fuzzy subset of X, then:

$$f(\tilde{A}, \tilde{B}) = \bigcup_{\alpha \in [0,1]} \alpha f(A_\alpha, B_\alpha)$$

One can prove this property as follows:

Since $\tilde{A} = \bigcup_{\alpha \in (0,1]} \alpha A_\alpha$, $\tilde{B} = \bigcup_{\alpha \in (0,1]} \alpha B_\alpha$

For the left hand side and using the extension principle:

$$\mu_{f(\tilde{A},\tilde{B})}(z) = \sup_{(x,y)\in f^{-1}(z)} \min\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\right\}$$

$$= \sup_{(x,y)\in f^{-1}(z)} \min\left\{\sup_{\alpha\in[0,1]} \alpha 1_{A_\alpha}(x), \sup_{\alpha\in[0,1]} \alpha 1_{B_\alpha}(y)\right\}$$

where $1_{A_\alpha}(x)$, $1_{B_\alpha}(y)$ refers to the characteristic functions of the crisp sets A_α and B_α, respectively.

Now, from the uniqueness of the supremum then one can write:

$$\mu_{f(\tilde{A},\tilde{B})}(z) = \sup_{\substack{(x,y)\in f^{-1}(z)\\ \alpha\in[0,1]}} \min\left\{\alpha 1_{A_\alpha}(x), \alpha 1_{B_\alpha}(y)\right\} \text{..................}(1.1)$$

Also, for the right hand side and by letting:

$$\tilde{C} = \bigcup_{\alpha\in[0,1]} \alpha f(A_\alpha, B_\alpha)$$

Hence:

$$\mu_{\tilde{C}}(z) = \sup_{\alpha\in[0,1]} \alpha 1_{f(A_\alpha, B_\alpha)}(z)$$

$$= \sup_{\alpha\in[0,1]} \left\{ \sup_{(x,y)\in f^{-1}(z)} \min\left\{\alpha 1_{A_\alpha}(x), \alpha 1_{B_\alpha}(y)\right\}\right\}$$

$$= \sup_{\substack{(x,y)\in f^{-1}(z)\\ \alpha\in[0,1]}} \min\left\{\alpha 1_{A_\alpha}(x), \alpha 1_{B_\alpha}(y)\right\} \text{.........................}(1.2)$$

Hence, from eqs. (1.1) and (1.2):

$$f(\tilde{A}, \tilde{B}) = \bigcup_{\alpha\in[0,1]} \alpha f(A_\alpha, B_\alpha). \quad \blacksquare$$

Also, the following property is of great importance, which appears in [13] without proof. Here, we give for completeness the details of the proof.

9. If $f : X \longrightarrow Y$, and $\tilde{A}$ and $\tilde{B}$ a fuzzy subset of X, and:

$$\sup_{(x,y)\in f^{-1}(z)} \min\left\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(y)\right\} = \alpha,\ \forall\ \alpha \in [0, 1]$$

then $[f(\tilde{A}, \tilde{B})]_\alpha = f(A_\alpha, B_\alpha)$.

Proof:

From (7) and (8), we have:

$f(A_\alpha, B_\alpha) \subseteq [f(\tilde{A}, \tilde{B})]_\alpha,\ \forall\ \alpha \in [0, 1]$......................................(1.3)

Now, let $z \in [f(\tilde{A}, \tilde{B})]_\alpha$, i.e.,

$$\mu_{f(\tilde{A},\tilde{B})}(z) = \sup_{(x,y)\in f^{-1}(z)} \min\left\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(y)\right\} \geq \alpha$$

Then, there exists $(\hat{x}, \hat{y}) \in f^{-1}(z)$, such that:

$$\alpha < \min\left\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(y)\right\}$$

Then $\mu_{\tilde{A}}(x) \geq \alpha$, $\mu_{\tilde{B}}(y) \geq \alpha$

Therefore, $\hat{x} \in A_\alpha$, $\hat{y} \in B_\alpha$

Then $z = f(\hat{x}, \hat{y}) = \alpha$, and hence by hypothesis, there exists $(x', y') \in f^{-1}(z)$, such that:

$$\min\left\{\mu_{\tilde{A}}(x'),\mu_{\tilde{B}}(y')\right\} = \sup_{\substack{(x',y')\in f^{-1}(z)\\ \alpha\in[0,1]}} \min\left\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(y)\right\} = \alpha$$

hence $x' \in A_\alpha$, $y' \in B_\alpha$

therefore, $z = f(x', y') \in f(A_\alpha, B_\alpha)$, and hence:

$$[f(\tilde{A}, \tilde{B})]_\alpha \subseteq f(A_\alpha, B_\alpha) \qquad \text{......(1.4)}$$

From (1.3) and (1.4), we have:

$$[f(\tilde{A}, \tilde{B})]_\alpha = f(A_\alpha, B_\alpha), \ \forall\ \alpha \in [0, 1]. \quad \blacksquare$$

<u>Remarks (1.3.1), [3]:</u>

1. The support of $\tilde{A}$ is exactly the same as the strong α-level of $\tilde{A}$ for $\alpha = 0$, $A_{0^+} = S(\tilde{A})$.
2. The core of $\tilde{A}$ is exactly the same as the α-level set of $\tilde{A}$ with $\alpha = 1$, i.e. $A_1 = \text{core}(\tilde{A})$.
3. The height of $\tilde{A}$ may also be viewed as the supremum of α-level set for which $A_{\alpha+} \neq \emptyset$.

1.4 FUZZY TOPOLOGY AND FUZZY METRIC SPACES, [8], [10], [12]

Chang, C. L. in 1968 introduced the notion of fuzzy topological spaces, which is a non-empty set X together with a family of fuzzy sets in X which is closed under arbitrary union and finite intersection. Erceg, M. A. in 1979, studied fuzzy metric spaces and its connection with statistical metric spaces. Ming P. P. and Ming L. T. in 1980 used fuzzy topology to define the neighborhood structure of fuzzy point. Zike Deng in 1982, studied the fuzzy point and discussed the fuzzy metric spaces with certain metric defined between two fuzzy points.

We start with the following definition of fuzzy topological spaces:

Definition (1.4.1):

A family τ^* of fuzzy sets of X is called a fuzzy topology for X if and only if:

a. $\tilde{\emptyset}$, X $\in \tau^*$.

b. $\tilde{A} \cap \tilde{B} \in \tau^*$ whenever $\tilde{A}, \tilde{B} \in \tau^*$.

c. $\cup\{\tilde{A}_i \mid i \in J\} \in \tau^*$ whenever each $\tilde{A}_i \in \tau^*$ $(i \in J)$, where J is any index set.

The pair (X, τ^*) is called fuzzy topological space.

Definition (1.4.2):

Let (X, τ^*) be a fuzzy topological space. A subfamily β of τ^* is called a base for τ^* if and only if for each $\tilde{A} \in \tau^*$, there exists $B_{\tilde{A}} \subseteq \beta$ such that $\tilde{A} = \cup B_{\tilde{A}}$, and a subfamily σ of τ^* is called a subbase for τ^* if and only if the family $\beta = \{\cap F \mid F$ is a finite subset of $\sigma\}$ is a base τ^*.

There is a strong relationship between fuzzy metric spaces and fuzzy topological spaces, as the next definition shows:

Definition (1.4.3):

Let (X*, d*) be a fuzzy metric space. The fuzzy topology $\tilde{\tau}$ on X generated by the class of open fuzzy balls in X* is called the fuzzy topological space (or, the fuzzy topology induced by the fuzzy metric d*), where:

$$d^*(\tilde{A},\tilde{B})=\sup_{\alpha\in I}\left\{\operatorname{Max}\left\{\sup_{x\in A_\alpha}\inf_{y\in B_\alpha} d(x,y), \sup_{y\in B_\alpha}\inf_{x\in A_\alpha} d(y,x)\right\}\right\}$$

Definition (1.4.4):

Let (X, $\tilde{\tau}$) be a fuzzy topological space, a family $\mathcal{A}$ of fuzzy sets is a cover of a fuzzy set $\tilde{B}$ if and only if $\tilde{B} \subseteq \cup\{\tilde{A} \mid \tilde{A} \in \mathcal{A}\}$, it is an open cover if and only if each member of $\mathcal{A}$ is an open fuzzy set. A subcover of $\mathcal{A}$ is a subfamily, which is also cover.

Definition (1.4.5):

A fuzzy topological space is compact if and only if each open cover of the space has a finite subcover.

Theorem (1.4.1):

The fuzzy topological space (X, τ^*) is compact if and only if every family of closed subsets of (X, τ^*) satisfies the finite intersection property and it's intersection is non-empty.

Remark (1.4.1):

A fuzzy metric space is a fuzzy topological space in which the topology is induced by a fuzzy metric. Accordingly, all concepts defined for fuzzy topological spaces are also defined for fuzzy metric spaces.

For example, by theorem (1.4.1) and definition (1.4.3), we obtain (X*, d*) which is compact fuzzy metric space.

Definition (1.4.6):

Let (X*, d*) be a fuzzy metric space. A neighborhood of a fuzzy point p is a fuzzy set $N_\varepsilon(p)$ consisting of all fuzzy points q such that $d^*(p, q) < \varepsilon$, the number ε is called the radius of $N_\varepsilon(p)$, i.e.,

$$N_\varepsilon(p) = \cup \{q \tilde{\in} X^* \mid d^*(p, q) < \varepsilon, \varepsilon > 0\}$$

Definition (1.4.7):

Let (X*, d*) be a fuzzy metric space. A fuzzy point p is said to be fuzzy limit point of the fuzzy set $\tilde{E}$ if every neighborhood of p contains fuzzy points $q \neq p$ such that $q \tilde{\in} \tilde{E}$.

Theorem (1.4.2):

If (X*, d*) is a compact fuzzy metric space, then every infinite fuzzy subset of X* has at least one fuzzy limit point in X*.

Definition (1.4.8):

Let $\tilde{A}$ be a fuzzy subset of a fuzzy metric space X* and let $\varepsilon > 0$. A finite fuzzy set of fuzzy points $\tilde{W} = \{(e_1, \alpha_1), (e_2, \alpha_2), \ldots, (e_m, \alpha_m)\}$ is called an ε-fuzzy net for $\tilde{A}$ if for every fuzzy point $p \tilde{\in} \tilde{A}$. There exists an $e_{i0} \tilde{\in} \tilde{W}$ with $d^*(p, e_{i0}) < \varepsilon$.

Definition (1.4.9):

A fuzzy subset $\tilde{A}$ of a fuzzy metric space X* is totally bounded if $\tilde{A}$ possesses an ε-fuzzy net for every ε > 0.

Definition (1.4.10):

A fuzzy subset $\tilde{E}$ of a fuzzy topological space (X, τ*) is said to be countably compact if and only if there exist a fuzzy limit point for every infinite fuzzy subset of $\tilde{E}$.

It is said that a fuzzy set $\tilde{E}$ in a fuzzy topological space is countably compact if and only if there exists for each infinite fuzzy set of $\tilde{E}$ has limit point.

Remark (1.4.2):

By theorem (1.4.2) and definition (1.4.10), we obtain that (X*, d*) is countably compact fuzzy metric space.

Theorem (1.4.3):

If a fuzzy metric space is countably compact, then it is also totally bounded.

1.5 COMPLETENESS OF FUZZY METRIC SPACES

The completeness of metric spaces is one of the fundamental aspects in real analysis, in general, and of fuzzy metric spaces in particular, [4].

Therefore, several approaches are proposed to study this subject.

Hence, in this section, we will give one of such approaches as a theorem without proof. Also, we will stand and present some of the basic ideas for the construction and the proof of the completeness of fuzzy metric spaces, which are given in [4]. Where the following abbreviation is used, X* is the set of all closed and bounded fuzzy subsets of X.

We start with the following definitions:

<u>Definition (1.5.1), [4]:</u>

A fuzzy set $\tilde{A}$ is closed and bounded fuzzy subset of X if and only if for all $\alpha \in$ I, the α-level sets A_α are closed and bounded ordinary subset of X.

The distance function between two fuzzy sets is given as in the following definition:

<u>Definition (1.5.2), [4]:</u>

Let (X, d) be a closed and bounded metric space and let $\tilde{A}$, $\tilde{B} \in$ X* be any two closed and bounded fuzzy subsets of X, i.e., A_α, B_α are closed and bounded subsets of X for each $\alpha \in$ I, then a function $d^* : X^* \times X^* \longrightarrow \mathbb{R}^+$, defined by:

$$d^*(\tilde{A},\tilde{B}) = \sup_{\alpha \in I}\left\{ \mathrm{Max}\left\{ \sup_{x \in A_\alpha} \inf_{y \in B_\alpha} d(x,y), \sup_{y \in B_\alpha} \inf_{x \in A_\alpha} d(y,x) \right\}\right\} \quad ..(1.5)$$

is said to be a distance function between fuzzy sets.

The last function given by (1.5) can be proved to be a distance function (see [4]) and therefore (X*, d*) is a fuzzy metric space.

In [4], the author is interested in the completeness of the fuzzy metric space (X*, d*) whenever the original metric space (X, d) is complete, i.e., the completeness of the space (X*, d*) is an inheritable property that may be accomplished from the completeness of the original crisp metric space (X, d).

<u>Definition (1.5.3), [4]:</u>

Suppose that (X*, d*) is a fuzzy metric space and $\tilde{S}$ be a fuzzy subset in X* with membership function $\mu_{\tilde{S}} : X \longrightarrow I$, and let $\delta > 0$ be any given real number. The δ-neighborhood $\tilde{S} + \delta$ of $\tilde{S}$ in X* is defined by using the α-level sets as:

$$(S + \delta)_\alpha = \{y \in X : \exists\, x \in S_\alpha \text{ such that } d(x, y) \leq \delta, \forall\, \alpha \in I \text{ and } \mu_{\tilde{S}}(y) + \delta \in [0, 1]\}$$

where $(S + \delta)_\alpha = S_\alpha + \delta$, $\forall\, \alpha \in I$.

It is clear that $\tilde{S} + \delta$ is also a fuzzy subset of X*.

The next theorem generalizes the ideas of the neighborhood of a point to a neighborhood of a fuzzy set.

Theorem (1.5.1), [4]:

Suppose that $\tilde{A}$ is a closed and bounded fuzzy subset of X*. The δ-neighborhood of $\tilde{A}$ (denoted by $\tilde{A}$ + δ) is also closed and bounded fuzzy subset of X*, i.e., if A_α is closed and bounded, then $A_\alpha + \delta$ is also closed and bounded for each $\alpha \in I$.

The following lemma is of great importance, since it gives an equivalence definition to the distance between two fuzzy sets.

Lemma (1.5.1), [4]:

Let $\tilde{A}$, $\tilde{B} \in X^*$ and $\varepsilon > 0$ be any given real number. Then $d^*(\tilde{A}, \tilde{B}) \leq \varepsilon$ if and only if $A_\alpha \subset B_\alpha + \varepsilon$ and $B_\alpha \subset A_\alpha + \varepsilon$, $\forall\ \alpha \in I$.

The above lemma may be considered as one of the most important properties of α-level sets.

Definition (1.5.4), [10]:

A sequence $\{\tilde{A}_n\}_{n=1}^{\infty}$ of fuzzy subsets of X* is said to be convergent to a fuzzy set $\tilde{A}$, if for any real number $\varepsilon > 0$, there exists a natural number $k \in \mathbb{N}$, such that:

$$A_{n_\alpha} \subset A_\alpha + \varepsilon \quad \text{and} \quad A_\alpha \subset A_{n_\alpha} + \varepsilon,\ \forall\ n > k.$$

i.e., $d^*(\tilde{A}_n, \tilde{A}) \leq \varepsilon$.

Definition (1.5.5), [10]:

A sequence $\{\tilde{A}_n\}_{n=1}^{\infty}$ of fuzzy subsets of X* is said to be fuzzy Cauchy sequence if for every real number $\varepsilon > 0$, there is a natural number N such that for all n, m ≥ N implies that $d^*(\tilde{A}_n, \tilde{A}_m) \leq \varepsilon$ which is equivalent to $A_{n_\alpha} \subset A_{m_\alpha} + \varepsilon$ and $A_{m_\alpha} \subset A_{n_\alpha} + \varepsilon$.

This Cauchy sequence of fuzzy sets has the following property. There is a Cauchy sequence $\{x_n\}_{n=1}^{\infty} \in X$ with the condition that $x_n \in \tilde{A}_n$, ∀ n; and there is a natural number N, such that:

$d(x_n, x_m) < \varepsilon$, ∀ n, m > N

As it is mentioned previously, many results and properties may be proved and generalized for fuzzy metric spaces. But Fadhel in 1998 [4] interested in the completeness of the space (X*, d*) whenever the ordinary metric space (X, d) is complete, i.e., the completeness of the space (X*, d*) is an inheritable property that can be concluded from the completeness of the space (X, d).

Theorem (1.5.2):

Let (X, d) be a complete metric space, then (X*, d*) is also a complete metric space, i.e., if $\{\tilde{A}_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X*. Then $\lim_{n\to\infty} \tilde{A}_n = \tilde{A} \in X^*$, which can be characterized if we define:

$\tilde{A} = \{x \in X$:There is a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in (\tilde{A}_n)_\alpha$, ∀ n and $\{x_n\}$ converge to x, where $\mu_{\tilde{A}}(x) \in [0, 1]\}$

In this theorem, one must prove first that the limit fuzzy set $\tilde{A}$ is a nonempty, closed and bounded fuzzy subset of X* (see [4]).

Theorem (1.5.3):

A fuzzy metric space (X*, d*) is complete if there exist a fuzzy limit point for every totally bounded and infinite fuzzy subset of X*.

SUPERVISOR CERTIFICATION

I certify that this thesis was prepared under my supervision at Al-Nahrain University, College of Science, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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EXAMINING COMMITTEE CERTIFICATION

We certify that we have read this thesis entitled *"About the Completeness of Fuzzy Metric Spaces"* and as examining committee examined the student *(Amani Eltifat Kadhem)* in its contents and in what it connected with, and that is in our opinion it meets the standards of a thesis for the degree of Master of Science in Mathematics.

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Amani Eltifat Kadhem

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ABSTRACT

The objective of this work may be oriented toward two objectives.

The first objective is to study fuzzy set theory, as well as some of its basic algebraic properties and theoretical results.

The second objective is to study D-metric spaces and M-fuzzy metric spaces, and some of their properties. Also, this objective includes the study of complete fuzzy metric spaces using M-fuzzy distance function. In addition, some additional results are presented and proved in this work.