Abstract

In this thesis, we introduce a modified approach for solving fractional order boundary value problems. This approach is given by applying the Riesz-Feller operator to obtain a modified finite difference equation, which is symmetric to the equation of fractional boundary value problems.

Also, the main objective of this work is to study the existence and uniqueness theorem of solutions of the fractional boundary value problems, and to present their proof depending on Schauder fixed point theorem for fractional order integral operator.

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Chapter One
Fundamental Theory of Fractional Differential Equations

In this chapter, some general concepts are presented including, fractional calculus and fractional differential equations of initial value problems.

Fractional calculus is that field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order.

This chapter consists of two sections. In section 1.1 a brief and general introduction to fractional calculus is given which is necessary for understanding fractional differential equations. In section 1.2, fractional differential equations, as well as, some of its analytical and numerical methods are given and illustrated with well solved examples.

1.1 Fractional Calculus

In this section, we introduce some of the basic and fundamental concepts and definitions related to the subject of fractional calculus for completeness.

1.1.1 Basic Notations:

1.1.1.1 The Gamma and Beta Functions, [Oldham, 1974]:

Gamma and beta functions are two of the most important notations in fractional calculus, since they play an important role in fractional differentiation and integration.

First, the gamma function Γ(x) of a positive real x, is defined by:

$$\Gamma(x)=\int_0^\infty y^{x-1}e^{-y}dy,\ \ x>0 \quad\ldots\ldots(1.1)$$

Following are some of the most important properties of the gamma function:

1. $\Gamma(1)=1$.

2. $\Gamma(x+1)=x\Gamma(x)$.

3. $\Gamma(x+1)=x!$.

4. $\Gamma\left(\frac{1}{2}-n\right)=\frac{(-4)^n n!\sqrt{\pi}}{(2n)!}$.

5. $\Gamma\left(\frac{1}{2}+n\right)=\frac{(2n)!\sqrt{\pi}}{4^n n!}$.

6. $\Gamma(-x)=\frac{-\pi\csc(\pi x)}{\Gamma(x+1)}$.

7. $\Gamma(nx)=\sqrt{\frac{2\pi}{n}}\left[\frac{n^x}{\sqrt{2\pi}}\right]\prod_{k=0}^{n-1}\left(n+\frac{k}{n}\right),\ \forall\ n\in\mathbb{N}$.

The second function is the beta function with positive parameters p and q is defined by:

$$B(p,q)=\int_0^1 y^{p-1}(1-y)^{q-1}\,dy \quad \text{...(1.2)}$$

If either p or q is non-positive, the integral diverges.

The incomplete beta function can be defined in terms of gamma function by the following relationship:

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \ \forall \ p \text{ and } q$$

The beta function of argument t is defined by the integral:

$$B_t(p,q)=\int_0^t y^{p-1}(1-y)^{q-1}\,dy \quad \text{...(1.3)}$$

1.1.1.2 Fractional Derivatives:

Many literatures discussed and presented fractional derivatives of certain functions, therefore this subsection some definitions of fractional derivatives are presented:

1. Riemann-Liouville Formula of Fractional Differentiation and Integration, [Oldham, 1974], [Nishimoto, 1983]:

Among the most important formulae used in fractional calculus is the Riemann-Liouville formula. For a given function f(x), ∀ x ∈ [a, b], the left and right hand Riemann-Liouville fractional derivatives of order q > 0 and m is a natural number, are given by:

$${}_xD^q_{a+}f(x)=\frac{1}{\Gamma(m-q)}\frac{d^m}{dx^m}\int_a^x \frac{f(t)}{(x-t)^{q-m+1}}\,dt \quad \text{..........................(1.4)}$$

$$ {}_{x}D_{b-}^{q} f(x) = \frac{(-1)^m}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_{x}^{b} \frac{f(t)}{(x-t)^{q-m+1}} dt \quad \text{..........................(1.5)} $$

where $m - 1 < q \leq m$, $m \in$ □ . These equations are usually named as the Riemann-Liouville fractional derivatives.

2. Caputo Fractional Derivatives, [El-Sayed, 2006]:

Another type of equation used in fractional calculus is the Caputo fractional derivatives and according to the left and right hand Riemann-Liouville derivatives, the left and right Caputo derivatives of order $q > 0$ of a given function $f(x)$, $x \in [a, b]$, can be defined as:

$$ {}_{x}D_{a+}^{q} f(x) = \frac{1}{\Gamma(m-q)} \int_{a}^{x} \frac{f^{(m)}(t)}{(x-t)^{q-m+1}} dt \quad \text{...................................(1.6)} $$

$$ {}_{x}D_{b-}^{q} f(x) = \frac{(-1)^m}{\Gamma(m-q)} \int_{x}^{b} \frac{f^{(m)}(t)}{(x-t)^{q-m+1}} dt \quad \text{...................................(1.7)} $$

where $m \in$ □ , $m - 1 < q < m$. It is remarkable that, the Caputo derivatives will be used in the derivation of the finite difference equation related to the boundary fractional ordinary differential equations (see chapter three).

The Relationship Between Riemann-Liouville and Caputo Fractional Derivatives, [El-Sayed, 2006]:

When $q \in (0, 1)$ the following relationships between the operator ${}_{x}D_{a+}^{q}$ and ${}_{x}\mathsf{D}_{a+}^{q}$, ${}_{x}D_{b-}^{q}$ and ${}_{x}\mathsf{D}_{b-}^{q}$ have been introduced.

By integrating by parts of eqs.(1.6) and (1.7) will leads to:

$$ {}_x\mathsf{D}^{q}_{a+} f(x) = \frac{1}{\Gamma(1-q)}\left[\frac{f(a)}{(a-x)^{q}} + {}_x D^{q}_{a+} f(x)\right] $$

and similarly:

$$ {}_x\mathsf{D}^{q}_{b-} f(x) = \frac{1}{\Gamma(1-q)}\left[\frac{f(b)}{(b-x)^{q}} - {}_x D^{q}_{b-} f(x)\right] $$

Such relations can be extended easily to the case that $q \in (m-1, m)$ as follows:

$$ {}_x\mathsf{D}^{q}_{a+} f(x) = {}_x D^{q}_{a+} f(x) + \sum_{k=0}^{m-1} \frac{(a-x)^{k-q}}{\Gamma(k-q+1)} \frac{d^k}{dx^k} f(x)\Big|_{x=a} \quad \text{............(1.8)} $$

$$ {}_x\mathsf{D}^{q}_{b-} f(x) = {}_x D^{q}_{b-} f(x) + \sum_{k=0}^{m-1} \frac{(-1)^{m-k}(x-b)^{k-q}}{\Gamma(k-q+1)} \frac{d^k}{dx^k} f(x)\Big|_{x=b} \quad (1.9) $$

and based on the assumption that $\lim\limits_{a+\to-\infty} \frac{d^k}{dx^k} f(x)\Big|_{x=a} < \infty$ and as well $\lim\limits_{b-\to\infty} \frac{d^k}{dx^k} f(x)\Big|_{x=b} < \infty$, for $k = 0, 1, \ldots, m-1$. The terms accruing on the left and right sides of eqs.(1.8) and (1.9) tend to zero, thus when the lower and upper limit of integration approaches to $+\infty$ and $-\infty$, yields:

$$ {}_{-\infty}D^{q}_{a+} f(x) = \mathsf{D}^{q}_{a+} f(x) \quad \text{and} \quad {}_{b-}D^{q}_{\infty} f(x) = {}_{b-}\mathsf{D}^{q}_{\infty} f(x) $$

3. Riesz-Feller Fractional Derivative, [Mainardi, 2001]:

There is also another kind of equations of fractional derivatives which is not less important than Riemann-Lioville and Caputo fractional derivatives which is called Riesz-Feller derivative. For a given function the Riesz-Feller fractional derivative of order $0 < q < 2$, $q \neq 1$ is defined by:

$$_{|x|}\mathbf{D}_0^q f(x) = \frac{d^q}{d|x|_0^q} f(x)$$

$$= \frac{\Gamma(1+q)}{\pi} \sin\left(\frac{q\pi}{2}\right) \int_0^{\infty} \frac{f(x+t) - 2f(x) + f(x-t)}{t^{1+q}} dt \quad ..(1.10)$$

and for $q = 1$, the Riesz derivative related to Hilbert transform by Feller in 1952 [Hahn, 1996] can be defined as:

$$_{|x|}\mathbf{D}_0^1 f(x) = -\frac{1}{\pi} \frac{d}{dx} \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt$$

and for $0 < q < 2$, $q \neq 1$ and $|\theta| \leq \min\{q, 2-q\}$ the Riesz-Feller derivative can be written as:

$$_{|x|}\mathbf{D}_\theta^q f(x) = \frac{\Gamma(1+q)}{\pi} \left\{ \sin\left(\frac{(q+\theta)\pi}{2}\right) \int_0^{\infty} \frac{f(x+t) - f(x)}{t^{1+q}} dt + \right.$$

$$\left. \sin\left(\frac{(q-\theta)\pi}{2}\right) \int_0^{\infty} \frac{f(x-t) - f(x)}{t^{1+q}} dt \right\} \quad ....(1.11)$$

for $q = 1$, we obtain the composite formula:

$$_{|x|}\mathbf{D}_\theta^1 f(x) = \left[\cos\left(\frac{\theta\pi}{2}\right) {}_{|x|}\mathbf{D}_0^1 + \sin\left(\frac{\theta\pi}{2}\right) {}_{|x|}\mathbf{D}_0^1 \right] f(x)$$

4. Gruünwald Fractional Derivatives, [Oldham, 1974], [Odibat, 2006]:

The Gruünwald derivatives of any integer order to any fractional order derivative which takes the form:

$$\frac{d^q}{dx^q} f(x) = \lim_{N \to \infty} \left\{ \frac{\left(\frac{x}{N}\right)^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(x - j\left(\frac{x}{N}\right)\right) \right\} \text{...........(1.12)}$$

The following are some examples for fractional differentiations:

1. $\frac{d^q}{dx^q}(1) = \frac{x^{-q}}{\Gamma(1-q)}$, $x > 0$.

2. $\frac{d^q}{dx^q}(c) = c\frac{d^q}{dx^q}(1) = c\frac{x^{-q}}{\Gamma(1-q)}$.

3. $\frac{d^q}{dx^q} x^p = \frac{\Gamma(p+1)x^{p-q}}{\Gamma(p-q+1)}$.

4. $\frac{d^q}{dx^q}\left(\frac{x^q}{1-x}\right) = \frac{\Gamma(p+1)}{(1-x)^{q+1}}$.

5. $\frac{d^q}{dx^q}\left(\frac{x^p}{1-x}\right) = x^{p-q}\sum_{j=0}^{\infty}\frac{\Gamma(j+p+1)}{\Gamma(j+p-q+1)}$

$$= \frac{\Gamma\left(p+1\right)B_x(p-q,q+1)}{\Gamma(p-q)(1-x)^{q+1}}$$

6. $\frac{d^q}{d(x-a)^q}[\exp(L - cx)] = \frac{\exp(L-cx)}{(x-a)^q}\gamma^*(-q, -c(x-a))$, where $0<x<1$, $p > -1$, L an arbitrary constant, and $\gamma^*(c, x)$ is the incomplete gamma function defined by:

$$\gamma^*(c, x) = \frac{c^{-x}}{\Gamma(x)}\int_0^c y^{x-1}e^{-y}\,dy$$

$$= \exp(-x)\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+c+1)}$$

7. $\frac{d^q}{dx^q}\sin(ax) = x^q\sum_{k=0}^{\infty}\frac{(-1)^k(ax)^{2k+1}}{\Gamma(q+2k+2)}$.

8. $\frac{d^q}{dx^q}\cos(ax) = x^q\sum_{k=0}^{\infty}\frac{(-1)^k(ax)^{2k}}{\Gamma(q+2k+1)}$.

1.1.1.3 Fractional Integration:

As in fractional ordinary derivatives, there are many literatures introduces different definitions of fractional integration, such as:

1. Riemann-Liouville Fractional Integral, [Oldham, 1974]:

The generalization to non-integer q of Riemann-Liouville integral can be written for a suitable function f(x) ($x \in \mathbb{R}$) as:

$$\frac{d^q}{dx^q}f(x) = \frac{1}{\Gamma(-q)}\int_0^x (x-y)^{-q-1}f(y)\,dy,\ q < 0 \qquad (1.13)$$

2. Weyl Fractional Integral, [Oldham, 1974]:

The left handed fractional order integral of order q > 0 of a given function f is defined as:

$$ {}_{-\infty}D_x^q f(x) = \frac{1}{\Gamma(q)} \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{1-q}}\, dy, \; x > -\infty \text{.........................} (1.14)$$

and the right handed fractional order integral of order q > 0:

$$ {}_{\infty}D_x^q f(x) = \frac{1}{\Gamma(q)} \int_{x}^{\infty} \frac{f(y)}{(y-x)^{1-q}}\, dy, \; x < \infty \text{..............................} (1.15)$$

1.1.2 Some Properties of the Fractional Differential Operator $\boldsymbol{D_x^q}$:

The fractional differential operator D_x^q has some important properties that can be described as follows:

1. The operator D_x^q is linear, i.e.,

$$D_x^q(c_1 f_1(x) + c_2 f_2(x)) = c_1 D_x^q(f_1(x)) + c_2 D_x^q(f_2(x))$$

2. For real numbers p and q, the equality between $D_x^p D_x^q f(x)$ and $D_x^{p+q} f(x)$ is always valid if p ≤ 0, but is not so if p > 0, even when p is an integer, except when q is positive integer which implies that:

$$D_x^p D_x^q f(x) = D_x^{p+q} f(x)$$

1.2 Fractional Differential Equations

An important type of differential equations is the so called ordinary fractional differential equations which is an equation containing fractional order derivatives in the independent variable with suitable initial or boundary conditions. Such type of equations may be considered to have the form:

$$D^{q}y(x) = f(x, y(x))$$

with initial conditions:

$$D^{q-k}y(x_0) = y_0^{k},\ k = 1, 2, \ldots, m + 1;\ m - 1 < q \leq m$$

where n is a positive integer, $q \in$ □ .

In this respect, two kinds of conditions have been introduced associated with the fractional differential equations, initial and boundary conditions, and in this chapter we will concern with analytical and numerical methods for solving initial ordinary fractional differential equations.

1.2.1 Analytic Methods for Solving Ordinary Fractional Differential Equations, [Oldham, 1974]:

Several analytical methods are available to solve fractional differential equations and many of such methods are the following:

1. The inverse operator method:

Consider the simplest type of all fractional differential equations:

$$\frac{d^q f}{dx^q} = F \quad \text{.......................................}(1.16)$$

where f is an unknown function and q is an arbitrary real number, F is a given function of x. Hence taking the inverse operator $\frac{d^{-q}}{dx^{-q}}$ to the both sides of eq.(1.16), yields:

$$f = \frac{d^{-q}}{dx^{-q}} F$$

where it is clear it is not always the case that they are equal, but this is not the most general solution:

$$f - \frac{d^{-q}}{dx^{-q}} \frac{d^q}{dx^q} f = 0 \quad \text{.......................................}(1.17)$$

The differentiation $f - \frac{d^{-q}}{dx^{-q}} \frac{d^q}{dx^q} f$, will not, in general, vanish but will consist of these portions of the differentiable series units $\{f_u\}$ in f that are sent to zero under the action $\frac{d^q}{dx^q}$.

We decompose f into differentiable units $f_{u,i}$; where:

$$f_{u,i} = x^{p_i} \sum_{j=0}^{\infty} a_{ij} x^j, \; p_i > -1, \; a_{i0} \neq 0, \; i = 1, 2, \ldots$$

The condition on $f_{u,i}$ required to give:

$$f - \frac{d^{-q}}{dx^{-q}} \frac{d^q}{dx^q} f \neq 0$$

This condition is obvious if and only if for some i in the range $1 \le i \le n$, $\Gamma(p_i - q + 1)$ is infinite, this condition can occur, however, only when $p_i - q + 1 = 0, -1, -2, \ldots$; and hence $p_i = q - 1, q - 2, \ldots$; then:

$$f - \frac{d^{-q}}{dx^{-q}}\frac{d^{q}}{dx^{q}}f = c_1x^{q-1} + c_2x^{q-2} + \ldots + c_mx^{q-m}$$

where $c_1, c_2, \ldots, c_m$ are arbitrary constants, and $m - 1 < q \le m$, thus:

$$f - c_1x^{q-1} - c_2x^{q-2} - \ldots - c_mx^{q-m} = \frac{d^{-q}}{dx^{-q}}\frac{d^{q}}{dx^{q}}f$$

$$= \frac{d^{-q}}{dx^{-q}}F$$

Finally, the general solution of eq.(1.16) can be written as:

$$f = \frac{d^{-q}}{dx^{-q}}F + c_1x^{q-1} + c_2x^{q-2} + \ldots + c_mx^{q-m} \qquad \text{.............................(1.18)}$$

The next example illustrate the above method of solution.

Example (1.1):

Consider the ordinary fractional differential equation:

$$\frac{d^{1/2}}{dx^{1/2}}y = x^5,\ y^{(-1/2)}(0) = 0.1$$

Now, since $q = 1/2$, $F = x^5$ hence $0 < q < 1$, so using eq.(1.18) yields:

$$y = \frac{d^{-1/2}}{dx^{-1/2}}x^5 + c_1x^{1/2}$$

$$= \frac{\Gamma(6)}{\Gamma(13/2)}x^{11/2} + c_1x^{-1/2}$$

and applying the first initial conditions, gives:

$$\frac{d^{-1/2}y}{dx^{1/2}} = \frac{\Gamma(6)}{\Gamma(13/2)}\frac{d^{-1/2}}{dx^{-1/2}}x^{11/2} + c_1\frac{d^{-1/2}}{dx^{-1/2}}x^{-1/2}$$

and hence $c_1 = \frac{0.1}{\Gamma(1/2)}$. Therefore:

$$y = \frac{\Gamma(6)}{\Gamma(13/2)}x^{11/2} + \frac{0.1}{\Gamma(1/2)}x^{-1/2}$$

2. Laplace Transformation Method:

Another type of analytic methods for solving fractional differential equations which will be discussed in this section by using Laplace transformation method.

The Laplace transformation of $\frac{d^q}{dx^q}$, $q \in \mathbb{R}^+$ is given by:

$$L\left\{\frac{d^q f}{dx^q}\right\} = \int_0^\infty \exp(-sx)\frac{d^q f}{dx^q}dx$$

But first let us recall the well-known transforms of integer order derivatives:

$$L\left\{\frac{d^n f}{dx^n}\right\} = s^n L\{f\} - \sum_{k=0}^{n-1} s^{n-1-k}\frac{d^k f(0)}{dx^k}, n = 1, 2, \ldots$$

and

$$L\left\{\frac{d^n f}{dx^n}\right\} = s^n L\{f\}, n = 0, -1, -2, \ldots \quad \ldots\ldots\ldots\ldots(1.19)$$

note that both formulas are embraced by:

$$\mathcal{L}\left\{\frac{d^q f}{dx^q}\right\} = s^q \mathcal{L}\{f\} - \sum_{k=0}^{q-1} s^k \frac{d^{q-1-k} f(0)}{dx^{q-1-k}}, q = 0, \mp 1, \ldots \quad \text{............(1.20)}$$

Notice that eq.(1.20) may be generalized to include non integer q by the simple extension:

$$\mathcal{L}\left\{\frac{d^q f}{dx^q}\right\} = s^q \mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k} f(0)}{dx^{q-1-k}}, \ \forall \ q \in \square \quad \text{............(1.21)}$$

where n is integer such than $n - 1 < q \leq n$.

The sum vanishes when $q \leq 0$. To prove (1.21) first consider $q < 0$, so that the Riemann-Liouville definition gives:

$$\frac{d^q f}{dx^q} = \frac{1}{\Gamma(-q)} \int_0^x \frac{f(y)}{(x-y)^{q+1}} dy, q < 0 \quad \text{..(1.22)}$$

may be adopted and upon direct application of the convolution theorem [Churchill, 1948]:

$$\mathcal{L}\left\{\int_0^x f_1(x-y) f_2(y) dy\right\} = \mathcal{L}\{f_1\} \mathcal{L}\{f_2\}$$

Then gives:

$$\mathcal{L}\left\{\frac{d^q f}{dx^q}\right\} = \frac{1}{\Gamma(-q)} \mathcal{L}\left\{x^{1-q}\right\} \mathcal{L}\{f\}$$

$$= s^q \mathcal{L}\{f\}, q < 0$$

so that eq. (1.19) unchanged may be generalized for negative q.

For positive non integer q, the following result is used:

$$\frac{d^q f}{dx^q} = \frac{d^n}{dx^n}\frac{d^{q-n} f}{dx^{q-n}}$$

where n is an integer number such that n – 1 < q < n. Now, on application of eq. (1.20), one can find that:

$$L\left\{\frac{d^q f}{dx^q}\right\} = L\left\{\frac{d^n}{dx^n}\left[\frac{d^{q-n} f}{dx^{q-n}}\right]\right\}$$

$$= s^n L\left\{\frac{d^{q-n} f}{dx^{q-n}}\right\} - \sum_{k=0}^{n-1} s^k \frac{d^{n-1-k}}{dx^{n-1-k}}\left[\frac{d^{q-n} f}{dx^{q-n}}\right](0)$$

The difference q – n being negative, the first right-hard term may be evaluated by use of equation (1.22), since q – n < 0, the composition rule may be applied to the terms within the summation which is:

$$L\left\{\frac{d^q f}{dx^q}\right\} = s^q L\{f\} - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k} f(0)}{dx^{q-1-k}},\ q \in \mathbb{R}^+$$

For the linear fractional ordinary differential equations with constant coefficients, consider the equation:

$$\sum_{i=0}^{n} c_i \frac{d^{q_i} f(x)}{dx^{q_i}} = g(x),\ \text{where} -1 < q_i < n_i$$

and taking the Laplace transformation to the both sides of the above equation and using the homogeneous property, yields:

$$L\left\{\sum_{i=0}^{n} c_i \frac{d^{q_i} f(x)}{dx^{q_i}}\right\} = L\{g(x)\}$$

or equivalently:

$$\sum_{i=0}^{n} c_i L\left\{\frac{d^{q_i} f(x)}{dx^{q_i}}\right\} = L\{g(x)\}$$

And by using eq.(1.21) with f(x), for all $x \in (0,\infty)$, one can find $L\{f(x)\} = G(x)$ which is the solution of the linear equation.

It is remarkable that, in this method the following initial conditions can be used:

$$\frac{d^{q-1-k} f(0)}{dx^{q-1-k}} = 0,\ k = 0,1,\ldots,n-1$$

The next example illustrate the above method of solution:

Example (1.2):

Consider the fractional differential equation:

$$\frac{d^{1/2} f(x)}{dx^{1/2}} + \frac{d^{-1/2} f(x)}{dx^{-1/2}} + 2f(x) = \frac{2}{\sqrt{\pi x}} + 6\sqrt{\frac{x}{\pi}} + 4\frac{x^{3/2}}{3\sqrt{\pi}} + 2x + 4$$

...............................(1.23)

$$\frac{d^{q-1-k} f(0)}{dx^{q-1-k}} = 0,\ k = 0,1$$

To solve this equation using Laplace transformation method, first take the Laplace transformation to the both sides of eq.(1.23)

$$L\left\{\frac{d^{1/2} f(x)}{dx^{1/2}}\right\} + L\left\{\frac{d^{-1/2} f(x)}{dx^{-1/2}}\right\} + 2\ell\{f(x)\} = \frac{2}{\sqrt{\pi}} L\left\{\frac{1}{\sqrt{x}}\right\} +$$

$$\frac{6}{\sqrt{\pi}} L\left\{\sqrt{x}\right\} + \frac{4}{3\sqrt{\pi}} L\left\{x^{3/2}\right\} + 2\ell\{x\} + \ell\{4\}$$

which is equivalent to:

$$\mathcal{L}\{f\} = \{s^3 + s^2 + 2s^{5/2}\} = 2s^2 + 3s + 1 + 2s^{1/2} + 4s\sqrt{s}$$

and hence:

$$\mathcal{L}\{f\} = \frac{2s^2 + 3s + 1 + 2\sqrt{s} + 4s\sqrt{s}}{s^2(s + 1 + 2\sqrt{s})}$$

$$= \frac{(2s^2 + 3s + 1) + 2\sqrt{s}(1 + 2s)}{s^2(s + 1 + 2\sqrt{s})}$$

$$= \frac{(2s + 1)(s + 1) + 2\sqrt{s}(1 + 2s)}{s^2(s + 1 + 2\sqrt{s})}$$

$$= \frac{(2s + 1)(s + 1 + 2\sqrt{s})}{s^2(s + 1 + 2\sqrt{s})} = \frac{2s}{s^2} + \frac{1}{s^2}$$

Therefore:

$$\mathcal{L}\{f\} = \frac{2}{s} + \frac{1}{s^2}$$

Then using the inverse Laplace transformation, gives the solution:

$$f(x) = 2 + x$$

1.2.2 Numerical and Approximate Methods for Solving Ordinary Fractional Differential Equations:

The choice of approximate method for approximating the solution to problems is influenced significantly by changes in calculator and computer technology since 50 years ago, and since the mathematical problem ordinarily does not solve the physical problem exactly in any case, it is often more appropriate to find an approximate solution to more complicated mathematical model of physical problem, [Burden, 1985].

In this subsection several numerical and approximate methods will be discussed which can be used to solve “fractional differential equations”.

1. The least-square method, [Burden, 1985]:

Among the most important methods used to approximate the solution of fractional differential equations which is called the least-square method and has the general idea of minimizing the square of residual error. To illustrate this method, consider the fractional differential equation:

$$D^q y = f(x)$$

where $f \in C[0,1]$, $q > 0$ and approximate the solution by:

$$y(x_n) = \sum_{j=0}^{n} c_j x^j \text{ ., } n \in \mathbb{N}$$

where c_j, $\forall$ j = 0, 1, …, n are constants to be determined. Hence, substituting in the differential equation and minimizing the sequence of the residual error, i.e.,

$$E(c_0, c_1, \ldots, c_n) = \int_a^b \left\{ f(x) - D_x^q y_n(x) \right\}^2 dx, \quad q > 0 \text{ ....(1.24)}$$

For this residual error, we have upon using the linear property:

$$E(c_0, c_1, \ldots, c_n) = \int_a^b \left\{ f(x) - D_x^q \sum_{i=0}^{n} c_i x^i \right\}^2 dx$$

$$= \int_a^b \left\{ f(x) - \sum_{i=0}^{n} c_i D_x^q x^i \right\}^2 dx \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.25)$$

Hence, the problem now is reduced to find the coefficients c_j, $j = 0, 1, \ldots, n$. A necessary condition for the coefficient c_j, $j = 0, 1, \ldots, n$; which minimizes E is that:

$$\frac{\partial E}{\partial c_j} = 0, \text{ for each } j = 0, 1, \ldots, n$$

hence:

$$\frac{\partial E}{\partial c_j} = -2\int_a^b f(x) D_x^q x^j \, dx + 2\sum_{i=0}^{n} c_i \int_a^b D_x^q x^{j+i} \, dx, \ \forall\, j = 0, 1, \ldots, n$$

Therefore, in order to find y_n, we have the following n + 1 linear system:

$$\sum_{i=0}^{n} c_i \int_a^b D_x^q x^{j+i} \, dx = \int_a^b f(x) D_x^q x^j \, dx, \ j = 0, 1, \ldots, n \quad \ldots\ldots\ldots\ldots(1.26)$$

which must be solved for n + 1 unknowns c_j, $j = 0, 1, \ldots, n$.

2. The collocation method, [Al-Hussainy, 2006]:

The collocation method is one of the approximate methods which is used “in general” to solve differential equations and to solve “in particular” fractional differential equations. The method has another application in solving other equations, such as integral equations, partial differential equations, integrodifferential equations, etc. This method has its basis on approximating the solution of the fractional differential equation by a complete sequence of functions $\{\phi_i\}$, where ϕ_i, $\forall$ i satisfy the homogeneous conditions and certain function ψ which satisfy the non-homogenous initial and boundary conditions, i.e.:

$$y(x) = \psi(x) + \sum_{j=1}^{m} c_j \phi_j(x) \quad \text{...} (1.27)$$

where c_j's, $\forall$ j = 1, 2, …, m; are an arbitrary constants to be evaluated. Therefore, to solve the last equation, we must evaluate the coefficients c_j's, j = 1, 2, …, m; which will produce a linear system of algebraic equations.

After substituting y(x) in the different equation and evaluating the resulting equation of m-distinct points in the domain of solution.

3. Adam's method, [Diethelm. 1999]:

Consider the fractional differential equation:

$$D^q y = f(x, y(x)),\ y(x_0) = y_0,\ m - 1 < q \le m,\ m \in \mathbb{N} \quad \text{.............}(1.28)$$

In order to solve this equation, we must first converting the problem into the following equivalent equation:

$$y(x) = y(x_0) + \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x - v)^{q-1} f(v, y(v))\, dv \quad \text{.......................} (1.29)$$

which is a Volterra singular integral equation of the second kind and also called Riemann-Liouville integration formula.

Second, use any quadrature formula with nodes x_j, j=0,1,…,n+1, taken with the weighted function $(x_{n+1} - v)^{q-1}$ and use the approximation:

$$\int_{x_0}^{x_{n+1}} (x_{n+1} - v)^{q-1} g(v)\, dv \approx \int_{x_0}^{x_{n+1}} (x_{n+1} - v)^{q-1} g_{n+1}(v)\, dv \quad \text{.......} (1.30)$$

where g_{n+1} is the piecewise linear interpolation for g whose nodes are chosen at the x_j, j = 0, 1, …, n + 1. Then use Legendre quadrature integration method. Then the right hand side of eq.(1.30) may be written as:

$$\int_{x_0}^{x_{n+1}} (x_{n+1} - v)^{q-1} g_{n+1}(v) dv = \sum_{j=0}^{n+1} a_{j,n+1}\, g(x_j) \quad \text{.............................}(1.31)$$

where:

$$a_{j,n+1} = \int_{x_0}^{x_{n+1}} (x_{n+1} - v)^{q-1} \xi_{j,n+1}(v)\, dv \quad \text{.......................................}(1.32)$$

In the case of equispaced nodes $t_j = t_0 + jh$ with some fixed h, the relationship of eq.(1.32) reduced to:

$$\xi_{j,n+1}(v) = \begin{cases} (v - x_{j-1})/(x_j - x_{j-1}), & \text{if}, x_{j-1} < v < x_j \\ (x_{j+1} - v)/(x_{j+1} - x_j), & \text{if}, x_j < v < x_{j+1} \\ 0, & \text{otherwise} \end{cases} \quad \text{..................}(1.33)$$

and:

$$a_{j,n+1} = \begin{cases} \dfrac{h^q}{q(q+1)}\left[n^{q+1} - (n-q)(n+1)^q\right], & \text{if} \quad j = 0 \\ \dfrac{h^q}{q(q+1)}, & \text{if} \quad j = n+1 \end{cases} \quad \text{.........}(1.34)$$

where $1 \le j \le n$, then eq.(1.32) becomes:

$$a_{j,n+1} = \frac{h^q}{q(q+1)} \left[(n-j+2)-2(n-j+1)^{q+1}+(n-j)^{q+1}\right] \text{......(1.35)}$$

where $a_{j,n+1}$ are termed as the coefficients of the method and $\xi_{j,n+1}(u)$ as the linear basis functions.

Now, substituting eq.(1.31) into eq.(1.29) and using the fractional variant of one step Adam-Moulton method, yields:

$$y^c_{n+1} = y_0 + \frac{1}{\Gamma(q)}\left[\sum_{j=0}^{n} a_{j,n+1} f(x_j, y_j) + a_{n+1,n+1} f(x_{n+1}, y^p_{n+1})\right] \quad (1.36)$$

Now, the problem is the determination of the predictor formula to calculate the value y^p_{n+1}. By using the one-step Adams-Bashforth method which is described above, we replace the integral on the right-hand side of eq. (1.29), by any quadrature rule, i.e.,

$$\int_{x_0}^{x_{n+1}} (x_{n+1}-v)^{q-1} g(v)\, dv = \sum_{j=0}^{n} c_{j,n+1}\, g(x_j) \text{............................(1.37)}$$

where:

$$c_{j,n+1} = \int_{x_j}^{x_{j+1}} (x_{n+1}-v)^{q-1}\, dv = \frac{1}{q}\left[(x_{n+1}-x_j)^q - (x_{n+1}-x_{j+1})^q\right]$$

$$\text{................................(1.38)}$$

Thus, for equispaced case, one has:

$$c_{j,n+1} = \frac{h^q}{q}\left[(n+1-j)^q - (n-j)^q\right] \text{.. (1.39)}$$

Hence y^p_{n+1}, is given by:

$$y_{n+1}^{p} = y_0 + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} c_{j,n+1} f(x_j, y_j) \quad \text{...} (1.40)$$

The next example is an illustrative example which is given in [Ford, 2003] and solved here using the discussed numerical and approximate methods.

Example (1.3):

Consider the fractional differential equation:

$$y^{(1/2)} = -y + x^2 + \frac{\Gamma(3)}{\Gamma(5/2)} x^{3/2}, \ y(0) = 0$$

where the exact solution is given by $y(x) = x^2$.

The numerical results obtained using the above three approaches are given in table (1.1) as well as the exact solution.

Table (1.1)

The numerical results of example (1.3).

n	*x*	*Collocation*	*Least square*	*Adam-Moulton*	*Exact Solution*
0	0	0	0	0	0
1	0.1	0.01	0.01	0.14	0.01
2	0.2	0.04	0.04	0.47	0.04
3	0.3	0.09	0.09	0.99	0.09
4	0.4	0.16	0.16	0.162	0.16

From the above obtained results, one can see that the accuracy of the results, where the approximate solution of the fractional differential equation using the least square and collocation methods are more accurate than the solution obtained by using the linear approximation of Adam’s method.

Chapter Two
Existence and Uniqueness Theorems of Fractional Boundary Value Problems

In this chapter we shall state and prove an important theorems concerning the existence and uniqueness theorem of solution of fractional order boundary value problems (FBVP's).

Also, some necessary definitions and results which are important to state and prove those theorems.

This chapter consists of two sections. Section one consists of some preliminary concepts of fractional order boundary value problems. While in section two, we state and prove the existence and uniqueness theorems of fractional boundary value problem, using Schuader fixed point theorem.

2.1 Preliminaries

Consider the fractional boundary value problem of the α^{th} order described as:

$$x^{(\alpha)} = f(t, x(t)),\ t \in [a, b],\ \alpha \in (0, 1) \quad \text{.......................................(2.1)}$$

subject to the boundary conditions:

$$Mx(a) + R\,x(b) = \beta,\ a, b \in \mathbb{R} \quad \text{...(2.2)}$$

where $f:[a, b]\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous, non-linear function, $x^{(\alpha)}$ is the fractional derivative of x, M and R are given constants in $\mathbb{R}$, β is a number in $\mathbb{R}^n$.

Equations (2.1) and (2.2) are known as a fractional boundary value problem (FBVP).

In this section, we shall introduce an important definitions, lemmas and theorems which are necessary to prove the existence and uniqueness theorem.

But first, some of the most important and necessary results for the existence and uniqueness of solution for fractional differential equations are given. The proofs of these results are given in details which seem to be necessary here.

Lemma (2.1), [Tisdell, 2005]:

Suppose M + R ≠ 0 holds, and if $x(t) \in C([a, b], \mathbb{R}^n)$ satisfies eqs. (2.1) and (2.2), then:

$$x(t) = \psi + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\, f(s, x(s))\, ds, \quad t \in [a, b]$$

where:

$$\psi = \frac{1}{(M+R)}\left[\beta - \frac{R}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\, f(s,x(s))\, ds\right]$$

where $\alpha \in (0, 1)$.

Proof:

Using Riemann-Liouville integration formula given by eq.(1.29) from a to t, yields:

$$x(t) = x(a) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s, x(s))\, ds \quad \text{.........(2.3)}$$

and hence substituting t = b, yields

$$x(b) = x(a) + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s, x(s))\, ds \quad \text{.........(2.4)}$$

Now, substitute eq.(2.4) in eq.(2.2), gives:

$$M\, x(a) + R\left(x(a) + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s, x(s))\, ds \right) = \beta$$

$$M\, x(a) + +Rx(a) + \frac{R}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s, x(s))\, ds = \beta \quad \text{.........(2.5)}$$

rearranging eq.(2.5), give:

$$x(a) = \frac{1}{(M+R)}\left[\beta - \frac{R}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s, x(s))\, ds \right] \quad \text{.........(2.6)}$$

Now, substituting eq.(2.6) in eq.(2.3) gives for $t \in [a, b]$

$$x(t) = \frac{1}{(M+R)}\left[\beta - \frac{R}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s, x(s))\, ds \right] +$$

$$\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s, x(s))\, ds \quad \text{.........(2.7)}$$

■

Now, the fractional operator related to the fractional boundary value problem given by eq. (2.7) is:

$$T = I. + \frac{R}{\Gamma(\alpha)(M+R)} \int_a^b (b-s)^{\alpha-1} f(s,.)\, ds -$$

$$\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s,.)\, ds \quad \text{...(2.8)}$$

and if f is taken to be linear, then eq.(2.8), become of the form:

$$T = I. + \frac{R}{\Gamma(\alpha)(M+R)} \int_a^b (b-s)^{\alpha-1} K(t,s).\, ds -$$

$$\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} K(t,s).\, ds$$

where K(t, s) is an function of s and t, I is the identity operator.

Lemma (2.2):

The fractional operator T is linear.

Proof:

To prove T is linear, i.e., is to prove that:

$T(c_1x_1 + c_2x_2) = c_1T(x_1) + c_2T(x_2)$

Now:

$$T(c_1x_1 + c_2x_2) = (c_1x_1 + c_2x_2) + \frac{R}{(M+R)\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} K(s,t)$$

$$(c_1x_1 + c_2x_2)\, ds - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} K(s,t)(c_1x_1 + c_2x_2)\, ds$$

$$= c_1 x_1 + c_2 x_2 + c_1 \frac{R}{(M+R)\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} K(s,t) x_1 \, ds +$$

$$\frac{c_2 R}{(M+R)\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} K(s,t) x_2 \, ds -$$

$$\frac{c_1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} K(s,t) x_1 \, ds -$$

$$\frac{c_2}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} K(s,t) x_2 \, ds$$

$$= c_1 \left[x_1 + \frac{R}{(M+R)\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} K(s,t) x_1 \, ds - \right.$$

$$\left. \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} K(s,t) x_1 \, ds \right] +$$

$$c_2 \left[x_2 + \frac{R}{(M+R)\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} K(s,t) x_2 \, ds - \right.$$

$$\left. \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} K(s,t) x_2 \, ds \right]$$

$$= c_1 \left[I. + \frac{R}{(M+R)\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} K(s,t). \, ds - \right.$$

$$\left. \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} K(s,t). \, ds \right] x_1 +$$

$$c_2 \left[I. + \frac{R}{(M+R)\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} K(s,t). \, ds - \right.$$

$$\left. \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} K(s,t). \, ds \right] x_2$$

$$= c_1 T(x_1) + c_2 T(x_2)$$

Hence T is linear. ■

Definition (2.1), [Erwin, 1978]:

Given a point $x_0 \in \mathbb{R}^n$ and a number r > 0, an open ball B of radius r and center x_0, is defined by:

$$B(x_0, r) = \left\{ x \in \mathbb{R}^n : \|x - x_0\| < r \right\}$$

Definition (2.2), [Erwin, 1978]:

Let X and Y be two normed spaces and $T : X \longrightarrow Y$ a linear operator and let $B(x_0, r)$ be a ball, then the fractional operator T is said to be ***bounded*** if there is a real number c, such that:

$\|Tx\| \leq c\|x\|$, for all $x \in B_{L+1}$.

The next theorem is of great importance which will be used in the proof of the existence and uniqueness theorem of (FBVP) that is called "Schauder fixed point theorem".

Theorem (2.1), (Schauder Fixed Point Theorem), [Rao, 1980]:

Let X be a nonempty, closed, bounded and convex subset of a Banach space B, and suppose that $T : X \longrightarrow X$ is a compact operator, then T has a fixed point.

The next theorem has a tremendous importance in the proof of the existence and uniqueness theorem:

Theorem (2.2), (Finite Dimensional Rang), [Erwin, 1978]:

Let X and Y be two normed spaces and T : X $\longrightarrow$ Y, a Linear operator, then:

(a) If T is bounded and dim T(x) < ∞, the operator T is compact.

(b) If dim X < ∞, then operator T is compact.

2.2 The Existence and Uniqueness Theorem

Because of the importance of the existence and uniqueness theorem in the theory of fractional boundary value problems, in this section we shall state and prove this theorem by using Schauder fixed point:

Theorem (2.3):

Suppose M + R ≠ 0 hold and $f \in C([a, b]\times\mathbb{R}^n, \mathbb{R}^n)$ and if there exist a function $p \in C([a, b];\mathbb{R}^+)$, such that:

$$\|f(t, q)\| \leq p(t)\|q\|, \text{ for all } t \in [a, b], q \in \mathbb{R}^n \quad \text{............................(2.9)}$$

and if:

$$\frac{1}{\Gamma(\alpha)}\left[1+\left|\frac{1}{(M+R)}R\right|\right]\int_a^b (b-s)^{\alpha-1}p(s)\,ds \leq 1 \quad \text{......................(2.10)}$$

Then the boundary value problem (2.1) and (2.2) has at least one solution in $C([a, b], \mathbb{R}^n)$.

Proof:

The existence of at least one solution to the fractional boundary value problem (2.1) and (2.2) is equivalent to the proof that the fractional integral equation given by eq.(2.7) has a fixed point, by using the Schauder fixed point theorem.

Consider the mapping $T : C([a, b], \mathbb{R}^n) \longrightarrow C([a, b], \mathbb{R}^n)$, is defined by:

$$T(x(t)) = \frac{1}{(M+R)}\left[\beta - \frac{R}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s, x(s))\, ds\right] + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s, x(s))\, ds,\ \forall\, t \in [a, b] \quad \text{.............. (2.11)}$$

Thus the problem is to prove the existence of at least one x, such that:

$$Tx = x \quad \text{.. (2.12)}$$

In order to prove that x = Tx, one must consider first an associated problem, namely:

$$x = \lambda Tx,\ \lambda \in [0, 1]$$

and hence one can prove that all possible solutions of $x = \lambda Tx$ (for the proof see [Tisdell, 2005]) with $\lambda = 1$ is a solution for x = Tx, therefore:

$$\|Tx\| = \|x\|$$

and hence:

$$\|x\| \leq \|Tx\|$$

$$= \left\| \frac{1}{(M+R)}\left[\beta - \frac{R}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s,x(s))\,ds\right] + \right.$$

$$\left. \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,x(s))\,ds \right\|$$

Also, since $t \in [a, b]$, i.e., $t \leq b$, hence:

$$\int_a^t (t-s)^{\alpha-1} f(s,x(s))\,ds \leq \int_a^b (b-s)^{\alpha-1} f(s,x(s))\,ds$$

and therefore:

$$\|x\| \leq \frac{1}{\Gamma(\alpha)}\left(1+\left|\frac{1}{M+R}\right|R\right)\int_a^b \|(b-s)^{\alpha-1} f(s,x(s))ds\| + \left|\frac{1}{M+R}\right|\|\beta\|$$

$$\leq \frac{1}{\Gamma(\alpha)}\left(1+\left|\frac{1}{M+R}\right|R\right)\int_a^b (b-s)^{\alpha-1} p(s)\|x(s)\|\,ds\| + \left|\frac{1}{M+R}\right|\|\beta\|$$

Therefore:

$$\|x\| \leq \frac{1}{\Gamma(\alpha)}\left(1+\left|\frac{1}{M+R}\right|R\right)\left\{\sup_{t\in[a,b]}|x(t)|\int_a^b (b-s)^{\alpha-1} p(s)ds\right\} + \left|\frac{1}{M+R}\right|\|\beta\|$$

$$\sup_{t\in[a,b]}|x(t)|\left\{1-\frac{1}{\Gamma(\alpha)}\left(1+\left|\frac{1}{M+R}\right|R\right)\int_a^b (b-s)^{\alpha-1} p(s)\,ds\right\} \leq \left|\frac{1}{M+R}\right|\|\beta\|$$

$$\sup_{t\in[a,b]}|x(t)| \leq \frac{\left|\frac{1}{M+R}\right|\|\beta\|}{1-\frac{1}{\Gamma(\alpha)}\left(1+\left|\frac{1}{M+R}\right|R\right)\int_a^b (b-s)^{\alpha-1} p(s)\,ds} = L \quad .(2.13)$$

Now, define the open ball with center 0 and radius L + 1, by:

$$B_{L+1} = \left\{ x \in C([a, b], \mathbb{R}^n) : \|x(t)\| < L+1, \forall\, t \in [a, b] \right\}$$

From eq.(2.13) it is clear that $\|x(t)\| \le L + 1$, $\forall$ $t \in [a, b]$

Now, we define a fractional operator T, as follow:

$$T : B_{L+1} \subset C([a, b], \mathbb{R}^n) \longrightarrow C([a, b], \mathbb{R}^n)$$

To prove, B_{L+1} is closed, bounded and convex fractional subset of $C([a, b], \mathbb{R}^n)$. First, from the construction of B_{L+1} it is clear that B_{L+1} is closed and bounded set.

Now to Prove, B_{L+1} is convex set of fractional solutions.

Let $x_1(t), x_2(t) \in B_{L+1}$, hence we have:

$$x_1(t) \in C([a, b], \mathbb{R}^n), \text{ where } \|x_1(t)\| \le L + 1, \forall\, t \in [a, b]$$

and

$$x_2(t) \in C([a, b], \mathbb{R}^n), \text{ where } \|x_2(t)\| \le L + 1, \forall\, t \in [a, b]$$

To prove:

$$x(t) = \lambda x_1(t) + (1 - \lambda)x_2(t) \in B_{L+1}$$

i.e., to prove that $x(t) \in C([a, b], \mathbb{R}^n)$ and $\|x(t)\| \le L + 1$, $\forall$ $t \in [a, b]$.

Also:

$$\begin{aligned}
\|x\| &= \|\lambda x_1(t) + (1-\lambda) x_2(t)\| \\
&\le \|\lambda x_1(t)\| + \|(1-\lambda) x_2(t)\| \\
&= |\lambda| \, \|x_1(t)\| + |1-\lambda| \, \|x_2(t)\| \\
&\le \lambda(L + 1) + (1 - \lambda)(L + 1) \\
&= L + 1
\end{aligned}$$

Hence, $\|x\| \le L + 1$, i.e., $x(t) \in B_{L+1}$

Hence, B_{L+1} is convex set

Now, to prove that T is bounded, i.e., to prove $\|Tx\| \le M\|x\|$, for any $x(t) \in C([a, b], \mathbb{R}^n)$, we have:

$$T(x(t)) = x(t) + \frac{R}{(M+R)\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s, x(s))\, ds - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s, x(s))\, ds$$

therefore

$$\| T(x(t)) \| = \left\| x(t) + \frac{R}{(M+R)\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s, x(s))\, ds - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s, x(s))\, ds \right\|$$

$$\le \| x(t) \| + \frac{1}{\Gamma(\alpha)}\left(1 + \left|\frac{R}{M+R}\right|\right)\int_a^b (b-s)^{\alpha-1} \| f(s, x(s)) \|\, ds$$

$$\le \| x(t) \| + \left[\frac{1}{\Gamma(\alpha)}\left(1 + \left|\frac{R}{M+R}\right|\right)\right]\int_a^b (b-s)^{\alpha-1} p(s) \| x(s) \|\, ds$$

$$\le \sup_{t\in[a,b]} |x(t)| + \frac{1}{\Gamma(\alpha)}\left(1 + \left|\frac{R}{M+R}\right|\right)\sup_{t\in[a,b]} | x(t) | \int_a^b (b-s)^{\alpha-1} p(s)\, ds$$

$$\le \left\{1 + \left[\frac{1}{\Gamma(\alpha)}\left(1 + \left|\frac{R}{M+R}\right|\right)\right]\int_a^b (b-s)^{\alpha-1} p(s)\, ds\right\} \sup_{t\in[a,b]} | x(t) |$$

$$= \left\{1 + \left[\frac{1}{\Gamma(\alpha)}\left(1 + \left|\frac{R}{M+R}\right|\right)\right]\int_a^b (b-s)^{\alpha-1} p(s)\, ds\right\} \| x \| \le M\|x\|$$

Hence the fractional operator is bounded, and since T(x(t)) has a finite dimension, then by using theorem (2.2) T is compact fractional operator.

Finally, by using Schauder fixed point theorem, then T has a fixed point which shows that the existence of at least one solution in B_{L+1} and hence to (2.1) and (2.2). ■

Theorem (2.4):

Suppose $M + R \neq 0$ hold and $f \in C([a, b]\times\mathbb{R}^n, \mathbb{R}^n)$ and if there exists a function $p \in C([a, b];\mathbb{R}^+)$, such that:

$$\|f(t, u) - f(t, v)\| \leq (b - s)^{\alpha-1}p(t)\|u - v\|, \forall\, t \in [a, b], u, v \in \mathbb{R}^n \quad \text{...............................(2.14)}$$

and eq.(2.10) holds, then the fractional boundary value problem (2.1) and (2.2) has a unique solution in $C([a, b], \mathbb{R}^n)$.

Proof:

Suppose that there exist two solutions u_1 and u_2 for the FBVP, and let $z = u_1 - u_2$. Now, consider the fractional boundary value problem:

$$z^{(\alpha)} = u_1^{(\alpha)} - u_2^{(\alpha)}$$

$$= f(t, u_1) - f(t, u_2),\ t \in [a, b] \quad \text{...(2.15)}$$

subject to:

$$Mz(a) + Rz(b) = 0 \quad \text{..(2.16)}$$

As in the proofs of lemma (2.1) and theorem (2.3) for $t \in [a, b]$, integrating the boundary value problem given by eq. (2.15) from a to t, yields:

$$z(t) = z(a) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left[f(s,u_1) - f(s,u_2)\right] ds, t \in [a, b] \text{ .(2.17)}$$

Putting t = b, we have:

$$z(b) = z(a) + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}[f(s, u_1) - f(s, u_2)]\, ds \text{ ....(2.18)}$$

Now, substituting eq. (2.18) in eq. (2.16) gives:

$$M z(a) + R\left[z(a) + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}[f(s,u_1) - f(s,u_2)]\, ds\right] = 0 \text{ .(2.19)}$$

and hence:

$$M z(a) + R z(a) = -\frac{R}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}[f(s,u_1) - f(s,u_2)]\, ds$$

Therefore:

$$z(a) = \frac{1}{(M+R)}\left[\frac{-R}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}[f(s,u_1) - f(s,u_2)]\, ds\right] \text{ .....(2.20)}$$

So substituting eq. (2.20) in eq. (2.17), to get:

$$z(t) = \frac{1}{(M+R)}\left[\frac{-R}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}[f(s,u_1) - f(s,u_2)]\, ds\right] + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}[f(s,u_1) - f(s,u_2)]\, ds$$

taking the norm on both sides of the last equation, yields:

$$\| z(t) \| = \left\| \frac{1}{(M+R)} \left[\frac{-R}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} [\, f(s,u_1) - f(s,u_2) \,] \, ds \right] + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} [\, f(s,u_1) - f(s,u_2) \,] ds \right\|$$

$$\leq \left\{ \frac{1}{\Gamma(\alpha)} \left(1 + \left| \frac{R}{M+R} \right| \right) \right\} \int_a^b (b-s)^{\alpha-1} \| f(s,u_1) - f(s,u_2) \| \, ds$$

$$\leq \left\{ \frac{1}{\Gamma(\alpha)} \left(1 + \left| \frac{R}{M+R} \right| \right) \right\} \sup_{t \in [a,b]} | z(t) | \int_a^b (b-s)^{\alpha-1} p(s) \, ds$$

and rearranging the last inequality, we obtain:

$$\sup_{t \in [a,\, b]} | z(t) | \left\{ 1 - \frac{1}{\Gamma(\alpha)} \left(1 + \left| \frac{R}{M+R} \right| \right) \int_a^b (b-s)^{\alpha-1} p(s) \, ds \right\} \leq 0$$

So we have $\|z(t)\| = 0$, $\forall\ t \in [a, b]$ and from the properties of the norm, we have $z(t) = 0$, i.e., $u_1(t) = u_2(t)$, $\forall\ t \in [a, b]$, which shows the uniqueness of the solution. ■

Chapter Three

Solution of Fractional Boundary Value Problems

Some physical problems that are position depend rather than time-dependent are often described in terms of differential equation with conditions imposed at more than one point. Because of this reason and more the boundary conditions are required to solve some problems, [Burden, 1985].

Three types of boundary conditions are possible, [Hoffman, 2001]:

1. The function y(x) may be specified “called Dirichlet boundary conditions”.
2. The derivative ordinary $y'(x)$ or $y^{(\alpha)}(x)$, $\alpha \in \mathbb{R}$ may be specified “called Neumann boundary conditions”.
3. A combination of y(x) and $y'(x)$ or $y^{(\alpha)}(x)$ may be specified “called mixed boundary conditions”.

This chapter deals with ordinary fractional boundary value problems and the methods for solving such kind of problems. This chapter consists of two sections. In section (3.1), the analytic solution of ordinary fractional boundary value problems is introduced and using the Green’s function to solve such problems.

In section (3.2), the numerical solution of boundary value problems of fractional order differential equations have been introduced using the Riesz-Feller fractional operator.

3.1 Analytic Solution of Ordinary Fractional Boundary Value Problems

The only analytical method proposed by a number of researchers to solve fractional differential equations with boundary conditions. In this section, we will discuss this method in details.

3.1.1 Fractional Green's Function Method for Solving Fractional Boundary Value Problems, [Zhang, 2006]:

A Green function is one of the most important functions which can be used to solve differential equations. In this subsection, we first derive the corresponding Green's function, named as fractional Green's function with boundary conditions and then use this function it to solve these kind of fractional differential equations.

Let $h \in C[0, 1]$ be a given function, then the fractional boundary value problem is defined by:

$$\left.\begin{array}{l} D_{0^+}^{\alpha} u(t) = h(t),\ 0 < t < 1,\ 1 < \alpha \le 2 \\ u(0) + u'(0) = 0,\ u(1) + u'(1) = 0 \end{array}\right\} \quad \text{...(3.1)}$$

Then a unique solution, given by:

$$u(t) = \int_0^1 G(t, s)h(s)\, ds \quad \text{...(3.2)}$$

where:

$$G(t, s)=\begin{cases} \dfrac{(1-s)^{\alpha-1}(1-t)+(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}, & s \le t \\ \dfrac{(1-s)^{\alpha-1}(1-t)}{\Gamma(\alpha)}+\dfrac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}, & t \le s \end{cases} \quad .(3.3)$$

and G(t, s) is called the Green's function of the boundary value problem (3.1) and $D_{0^+}^{\alpha}$ is the Caputo fractional derivative which is described by eq.(1.6).

Equation (3.2) may be proved as follows, since the equivalent integral equation is given by:

$$u(t) = I_{0^+}^{\alpha} h(t) - c_1 - c_2 t$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds - c_1 - c_2 t \quad \text{..................................(3.4)}$$

For some $c_1, c_2 \in \mathbb{R}$.

Now, using the relation $D^{\alpha} I^{\alpha} u(t) = u(t)$ and $I_0^{\alpha} I_0^{\beta} u(t) = I_0^{\alpha+\beta} u(t)$, for some $\alpha, \beta > 0$, one can have:

$$u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} h(s)\, ds - c_2 \quad \text{.....................................(3.5)}$$

Equation (3.4) and the boundary conditions, yields:

$$u(0) + u'(0) = 0$$

which implies that $c_2 = -c_1$, and also:

$$u(1) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1 - s)^{\alpha-1}h(s)\,ds - c_1 - c_2$$

and

$$u'(1) = \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1 - s)^{\alpha-2}h(s)\,ds - c_2$$

Therefore:

$$u(1) + u'(1) = 0$$

implies that:

$$\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)ds - c_1 - c_2 + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}h(s)ds - c_2 = 0$$

and hence

$$c_1 + 2c_2 = \frac{1}{\Gamma(\alpha)}\int_0^1 (1 - s)^{\alpha-1}h(s)\,ds + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1 - s)^{\alpha-2}h(s)\,ds \qquad \text{....................................(3.6)}$$

and substituting $c_2 = -c_1$ in eq.(3.6), gives:

$$c_1 - 2c_1 = \frac{1}{\Gamma(\alpha)}\int_0^1 (1 - s)^{\alpha-1}h(s)\,ds + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1 - s)^{\alpha-2}h(s)\,ds$$

hence:

$$c_1 = -\frac{1}{\Gamma(\alpha)}\int_0^1 (1 - s)^{\alpha-1}h(s)\,ds - \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1 - s)^{\alpha-2}h(s)\,ds \quad \text{....(3.7)}$$

and

$$c_2 = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)\,ds + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}h(s)\,ds \text{.......}(3.8)$$

therefore, the unique solution of eq.(3.1) is:

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)\,ds +$$

$$\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}h(s)\,ds - \frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)\,ds -$$

$$\frac{t}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}h(s)\,ds$$

Finally, to find the Green function, divide the interval (0, 1) into two subintervals (0, t) and (t, 1), yields:

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (1-s)^{\alpha-1}h(s)\,ds +$$

$$\frac{1}{\Gamma(\alpha)}\int_t^1 (1-s)^{\alpha-1}h(s)\,ds + \frac{1}{\Gamma(\alpha-1)}\int_0^t (1-s)^{\alpha-2}h(s)\,ds +$$

$$\frac{1}{\Gamma(\alpha-1)}\int_t^1 (1-s)^{\alpha-2}h(s)\,ds - \frac{t}{\Gamma(\alpha)}\int_0^t (1-s)^{\alpha-1}h(s)\,ds -$$

$$\frac{t}{\Gamma(\alpha)}\int_t^1 (1-s)^{\alpha-1}h(s)\,ds - \frac{t}{\Gamma(\alpha-1)}\int_0^t (1-s)^{\alpha-2}h(s)\,ds -$$

$$\frac{t}{\Gamma(\alpha-1)}\int_t^1 (1-s)^{\alpha-2}h(s)\,ds$$

implies that:

$$u(t) = \int_0^t \left[\frac{(1-s)^{\alpha-1}(1-t)+(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)} \right] h(s)\,ds \; +$$

$$\int_t^1 \left[\frac{(1-s)^{\alpha-1}(1-t)}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)} \right] h(s)\,ds$$

$$= \int_0^1 G(t,s)\,h(s)\,ds$$

which completes the proof of the relation.

Now, the function G(t, s) has some properties that can be described as:

1. $G \in C([0, 1]\times[0, 1])$ and $G(t, s) > 0$, for $t, s \in (0, 1)$.
2. There exists a positive function $\gamma \in C(0, 1)$, such that:

$$\min_{\frac{1}{4}\le t\le \frac{3}{4}} G(t, s) \ge \gamma(s)M(s), \; s \in (0, 1)$$

$$\max_{0\le t\le 1} G(t, s) \le M(s)$$

where:

$$M(s) = \frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}, \; s \in [0, 1)$$

for more details, see [Zhang, 2006].

The next example illustrate the above method of solution:

Example (3.1):

Consider the fractional boundary value problem:

$D_0^{3/2} u(t) = t,\ 0 \le t \le 1$

$u(0) + u'(0) = 0,\ u(1) + u'(1) = 0$

Hence by substituting h(t) = t and α = 3/2 in G(t, s) of eq. (3.3), yields:

$$G(t, s)=\begin{cases} \dfrac{(1-s)^{1/2}(1-t)+(t-s)^{1/2}}{\Gamma(3/2)}+\dfrac{(1-s)^{-1/2}(1-t)}{\Gamma(1/2)}, & s \le t \\ \dfrac{(1-s)^{1/2}(1-t)}{\Gamma(3/2)}+\dfrac{(1-s)^{-1/2}(1-t)}{\Gamma(1/2)}, & t \le s \end{cases}$$

Then, put G(t, s) in eq.(3.2) one can have:

$$U(t) = \int_0^1 G(t,s)\,h(s)\,ds = \int_0^t \left[\frac{(1-s)^{1/2}(1-t)+(t-s)^{1/2}}{\Gamma(3/2)} + \frac{(1-s)^{-1/2}(1-t)}{\Gamma(1/2)}\right] s\,ds + \int_t^1 \left[\frac{(1-s)^{1/2}(1-t)}{\Gamma(3/2)} + \frac{(1-s)^{-1/2}(1-t)}{\Gamma(1/2)}\right] s\,ds$$

$$= 1.053(1 - t) + 0.3t^{5/2}$$

3.2 Numerical Solution of Fractional Boundary Value Problems

Analytic solution of fractional boundary value problems is so difficult and limited and it does not work in all cases and functions, because of these reasons this section deals with the numerical solution to an ordinary differential equations of fractional order, which is based on the finite difference method (FDM), [Smith, 1978], [Hoffman, 2001], but we start with the Riesz-Feller fractional operator and then describe an approximate method for solving fractional boundary value problems.

3.2.1 Riesz-Feller Operator, [Ciesielski, 2006]:

Consider an ordinary differential equation of fractional order of the following form:

$$\frac{d^{\alpha}}{d|x|_{\theta}^{\alpha}} T(x) = 0, \; x \in \mathbb{R}, \; 0 < \alpha \leq 2 \quad \text{...(3.9)}$$

where T(x) is a variable depending on x and $\frac{d^{\alpha}}{d|x|_{\theta}^{\alpha}} T(x)$ is the Riesz-Feller fractional operator, α is the real order of this operator and θ is a parameter.

The Riesz-Feller fractional operator is defined as [Georenflo, 1998]:

$$\frac{d^{\alpha}}{d|x|_{\theta}^{\alpha}} T(x) = {}_{x}D_{\theta}^{\alpha} T(x)$$

$$= -[C_L(\alpha, \theta)\, {}_{-\infty}D_{x}^{\alpha} T(x) + C_R(\alpha, \theta)\, {}_{x}D_{+\infty}^{\alpha} T(x)] \quad \text{.........(3.10)}$$

for $0 < \alpha \leq 2$, $\alpha \neq 1$, where:

$${}_{-\infty}D_{x}^{\alpha} T(x) = \left(\frac{d}{dx}\right)^{m} \left[{}_{-\infty}I_{x}^{m-\alpha} T(x)\right]$$

$$\text{.......................(3.11)}$$

$${}_{x}D_{+\infty}^{\alpha} T(x) = (-1)^{m} \left(\frac{d}{dx}\right)^{m} \left[{}_{x}I_{+\infty}^{m-\alpha} T(x)\right]$$

for $m \in \mathbb{N}$, $m - 1 < \alpha \le m$, and the coefficients $C_L(\alpha, \theta)$, $C_R(\alpha, \theta)$ (for $0 < \alpha \le 2$, $\alpha \ne 1$, $|\theta| \le \min\{\alpha, 2 - \alpha\}$) are defined as:

$$\left.\begin{aligned} C_L(\alpha,\theta) &= \frac{\sin\left(\frac{(\alpha-\theta)\pi}{2}\right)}{\sin(\alpha\pi)} \\ C_R(\alpha,\theta) &= \frac{\sin\left(\frac{(\alpha+\theta)\pi}{2}\right)}{\sin(\alpha\pi)} \end{aligned}\right\} \qquad \ldots(3.12)$$

and the fractional integral operator of order α : ${}_{-\infty}I_x^{\alpha}T(x)$ and ${}_xI_{+\infty}^{\alpha}T(x)$ are defined as the left and right hand of Weyl fractional integration and when $x \in [a, b]$, i.e., the integration is proper, then it is called Caputo's fractional derivatives as in eqs.(1.6) and (1.7).

In this section, consider eq.(3.9) for $1 < \alpha \le 2$ in one-dimensional domain $\Omega : L \le x \le R$, with boundary conditions of the first kinds (Dirichlet conditions) as:

$$\left.\begin{aligned} T(L) &= g_L \\ T(R) &= g_R \end{aligned}\right\} \qquad \ldots(3.13)$$

3.2.2 Approximation of Riesz-Feller Operator:

In order to develop a discrete form of the operator in eq.(3.10), consider a homogeneous grid $-\infty < \ldots < x_{i-2} < x_{i-1} < x_i < x_{i+1} < x_{i+2} < \ldots < \infty$, with uniform step size $h = x_k - x_{k-1}$.and denoting the value of the function $T(x)$ at the point x_k as $T_k = T(x_k)$, for $k \in \mathbb{Z}$ taking into account only the function of one variable in order to simplify notations and denote $C_L = C_L(\alpha, \theta)$ and $C_R = C_R(\alpha, \theta)$.

In accordance with changes parameter α in eq.(3.9) the following two cases will be described a discrete approximation of Riesz-Feller derivative depending on the value of the fractional derivative.

Case (1):

The first case includes changes in the parameter α in the range $0 < \alpha < 1$, by rewriting the Riesz-Feller operator in eq.(3.10) using Caputo definition in eqs.(1.6) and (1.7) as:

$$ {}_{x_i}D_\theta^\alpha T(x_i) = -\left[c_L \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x_i} \frac{T'(\xi)}{(x_i-\xi)^\alpha} d\xi - c_R \frac{1}{\Gamma(1-\alpha)} \int_{x_i}^{\infty} \frac{T'(\xi)}{(\xi-x_i)^\alpha} d\xi \right] $$

after using numerical integration schemes, replace the above integration by the sum of discrete integrals as:

$$ {}_{x_i}D_\theta^\alpha T(x_i) = \frac{-1}{\Gamma(1-\alpha)} \left[c_L \sum_{k=0}^{\infty} \int_{x_{i-k-1}}^{x_{i-k}} \frac{T'(\xi)}{(x_i-\xi)^\alpha} d\xi - c_R \sum_{k=0}^{\infty} \int_{x_{i+k}}^{x_{i+k+1}} \frac{T'(\xi)}{(\xi-x_i)^\alpha} d\xi \right] $$

$$ \approx \frac{-1}{\Gamma(1-\alpha)} \left[c_L \sum_{k=0}^{\infty} \tilde{T}'_{i-k} \int_{x_{i-k-1}}^{x_{i-k}} \frac{1}{(x_i-\xi)^\alpha} d\xi - c_R \sum_{k=0}^{\infty} \tilde{\tilde{T}}'_{i+k} \int_{x_{i+k}}^{x_{i+k+1}} \frac{1}{(\xi-x_i)^\alpha} d\xi \right] \quad \text{.............(3.14)} $$

where $\tilde{T}'_j$ and $\tilde{\tilde{T}}'_j$ are the difference schemes which approximate the first derivative of integer order in the interval $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$, respectively. The following weighted forms of these schemes are obtained:

$$\tilde{T}'_j = \frac{1}{2}\left[\frac{T_j - T_{j-1}}{h} + \frac{(1-\lambda_1)(T_j - T_{j-1}) + \lambda_1(T_{j+1} - T_j)}{h}\right]$$

$$= \frac{1}{2h}\left[\lambda_1 T_{j+1} + 2(1-\lambda_1)T_j + (\lambda_1 - 2)T_{j-1}\right] \text{.........................(3.15)}$$

$$\tilde{\tilde{T}}'_j = \frac{1}{2}\left[\frac{T_{j+1} - T_j}{h} + \frac{(1-\lambda_1)(T_{j+1} - T_j) + \lambda_1(T_j - T_{j-1})}{h}\right]$$

$$= \frac{1}{2h}\left[(2-\lambda_1)T_{j+1} + 2(\lambda_1 - 1)T_j + (-\lambda_1)T_{j-1}\right] \text{....................(3.16)}$$

where $\lambda_1 = \lambda_1(\alpha, \theta) = \alpha - |\theta|$, $\lambda_1 \in [0, 1]$.

The above formulae have been introduced in order to obtain various transitions between the difference schemes. For example, if substituting $\lambda_1 = 1$ in eqs.(3.15) and (3.16) will give the central-difference approximation of first derivative, and after putting $\lambda_1 = 0$ get the backward difference equation in eq.(3.15) and forward difference equation in eq.(3.16).

Denoting by:

$$v_k^\alpha = \frac{1}{\Gamma(1-\alpha)} \int_{x_{i-k-1}}^{x_{i-k}} \frac{1}{(x_i - \xi)^\alpha} d\xi$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_{x_{i+k}}^{x_{i+k+1}} \frac{1}{(\xi - x_i)^{\alpha}} d\xi$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{(\xi - x_i)^{-\alpha+1}}{-\alpha+1} \Bigg|_{x_{i+k}}^{x_{i+k+1}}$$

and since $\Gamma(2 - \alpha) = (1 - \alpha)\Gamma(1 - \alpha)$, then:

$$\frac{1}{\Gamma(2-\alpha)} [(x_{i+k+1} - x_i)^{1-\alpha} - (x_{i+k} - x_i)^{1-\alpha}]$$

$$= \frac{1}{\Gamma(2-\alpha)} [(x_{i+k} + h - x_i)^{1-\alpha} - (x_i + kh - x_i)^{1-\alpha}]$$

$$= \frac{1}{\Gamma(2-\alpha)} [(x_i + kh + h - x_i)^{1-\alpha} - (x_i + kh - x_i)^{1-\alpha}]$$

$$= \frac{[(k+1)h]^{1-\alpha} - k^{1-\alpha}}{\Gamma(2-\alpha)}$$

Hence:

$$v_k^{\alpha} = h^{1-\alpha} \frac{(k+1)^{1-\alpha} - k^{1-\alpha}}{\Gamma(2-\alpha)} \quad \text{...(3.17)}$$

Now, substituting eqs.(3.15), (3.16) into eq.(3.14), yields:

$$ {}_{x_i}D_{\theta}^{\alpha} T(x_i) \approx -\Bigg[c_L \sum_{k=0}^{\infty} \frac{1}{2h} [\lambda_1 T_{i-k+1} + 2(1-\lambda_1) T_{i-k} + (\lambda_1 - 2) T_{i-k-1}] v_k^{\alpha} - c_R \sum_{k=0}^{\infty} \frac{1}{2h} [(2-\lambda_1) T_{i+k+1} + 2(\lambda_1 - 1) T_{i+k} + (-\lambda_1) T_{i+k-1}] v_k^{\alpha} \Bigg] \quad \text{........................(3.18)}$$

Finally, from eqs. (3.17), (3.18) the discrete form of Riesz-Feller operator in eq.(3.10) for $0 < \alpha < 1$ can be written as:

$$ {}_{x_i}D_{\theta}^{\alpha} T(x_i) \approx \frac{1}{h^{\alpha}} \sum_{k=-\infty}^{\infty} T_{i+k} w_k^{(\alpha,\theta)} \quad \text{..(3.19)} $$

where the coefficient $w_k = w_k(\alpha, \theta)$ have the following form:

$$ w_k^{(\alpha,\theta)} = \frac{-1}{2\Gamma(2-\alpha)} \begin{cases} \Big[(|k|+2)^{1-\alpha}\lambda_1 + (|k|+1)^{1-\alpha}(2-3\lambda_1) + \\ |k|^{1-\alpha}(3\lambda_1-4) + (|k|-1)^{1-\alpha}(2-\lambda_1)\Big]c_L, & k \le -2 \\ \Big[3^{1-\alpha}\lambda_1 + 2^{1-\alpha}(2-3\lambda_1) + 3\lambda_1 - 4\Big]c_L + \lambda_1 c_R, & k = -1 \\ \Big[2^{1-\alpha}\lambda_1 - 3\lambda_1 + 2\Big](c_L + c_R), & k = 0 \\ \Big[3^{1-\alpha}\lambda_1 + 2^{1-\alpha}(2-3\lambda_1) + 3\lambda_1 - 4\Big]c_R + \lambda_1 c_L, & k = 1 \\ \Big[(k+2)^{1-\alpha}\lambda_1 + (k+1)^{1-\alpha}(2-3\lambda_1) + \\ k^{1-\alpha}(3\lambda_1-4) + (k-1)^{1-\alpha}(2-\lambda_1)\Big]c_R, & k \ge 2 \end{cases} $$

.................................(3.20)

Case (2):

The second case involves changes in the parameter α for the range $1 < \alpha \le 2$ by rewriting the Riesz-Feller operator in eq.(3.10) using Caputo definition in eqs.(1.6) and (1.7) as:

$$ {}_{x_i}D_{\theta}^{\alpha} T(x_i) = -\left[c_L \frac{1}{\Gamma(2-\alpha)} \int_{-\infty}^{x_i} \frac{T''(\xi)}{(x_i-\xi)^{\alpha-1}} d\xi + c_R \frac{1}{\Gamma(2-\alpha)} \int_{x_i}^{\infty} \frac{T''(\xi)}{(\xi-x_i)^{\alpha-1}} d\xi \right] $$

Similarly, as in case (1) after using numerical integration scheme, replace the above integral by the sum of discrete integrals as:

$$ {}_{x_i}D_\theta^\alpha T(x_i) \approx \frac{-1}{\Gamma(2-\alpha)}\left[c_L \sum_{k=0}^{\infty} \tilde{T}''_{i-k} \int_{x_{i-k-1}}^{x_{i-k}} \frac{1}{(x_i-\xi)^{\alpha-1}} d\xi + \right. $$

$$ \left. c_R \sum_{k=0}^{\infty} \tilde{\tilde{T}}''_{i+k} \int_{x_{i+k}}^{x_{i+k+1}} \frac{1}{(\xi-x_i)^{\alpha-1}} d\xi \right] \quad \text{.......................(3.21)} $$

where $\tilde{T}''_j$ and $\tilde{\tilde{T}}''_j$ are difference schemes of the second derivative of integer order, which approximated by the following formulae:

$$ \tilde{T}''_j = \frac{1}{2}\left[\frac{T_{j+1}-2T_j+T_{j-1}}{h^2} + \frac{(1-\lambda_2)(T_{j+1}-2T_j+T_{j-1})+\lambda_2(T_j-2T_{j-1}+T_{j-2})}{h^2} \right] $$

$$ = \frac{1}{2h^2}[(2-\lambda_2)T_{j+1} + (3\lambda_2-4)T_j + (2-3\lambda_2)T_{j-1} + \lambda_2 T_{j-2}] \quad (3.22) $$

$$ \tilde{\tilde{T}}''_j = \frac{1}{2}\left[\frac{T_{j+1}-2T_j+T_{j-1}}{h^2} + \frac{(1-\lambda_2)(T_{j+1}-2T_j+T_{j-1})+\lambda_2(T_{j+2}-2T_{j+1}+T_j)}{h^2} \right] $$

$$ = \frac{1}{2h^2}[\lambda_2 T_{j+2} + (2-3\lambda_2)T_{j+1} + (3\lambda_2-4)T_j + (2-\lambda_2)T_{j-1}] \quad \text{.....(3.23)} $$

where $\lambda_2 = \lambda_2(\alpha, \theta) = 2 - (\alpha + |\theta|)$, $\lambda_2 \in [0, 1]$.

By letting $\lambda_2 = 0$ into eqs.(3.22) and (3.23), the classical central difference schemes are obtained, and for $\lambda_2 = 1$ in eq. (3.22) the backward four-point of the second derivative of integer order is obtained and in eq.(3.23) the forward four-point of the second derivative of integer is also obtained.

Denoting by:

$$u_k^\alpha = \frac{1}{\Gamma(2-\alpha)} \int_{x_{i-k-1}}^{x_{i-k}} \frac{1}{(x_i-\xi)^{\alpha-1}} d\xi$$

$$= \frac{1}{\Gamma(2-\alpha)} \int_{x_{i+k}}^{x_{i+k+1}} \frac{1}{(\xi-x_i)^{\alpha-1}} d\xi$$

$$= \frac{1}{\Gamma(2-\alpha)} \frac{(\xi-x_i)^{-\alpha+2}}{2-\alpha}\Bigg|_{x_{i+k}}^{x_{i+k+1}}$$

and since $\Gamma(3-\alpha) = (2-\alpha)\Gamma(2-\alpha)$, then:

$$\frac{1}{\Gamma(3-\alpha)}[(x_{i+k+1}-x_i)^{2-\alpha} - (x_{i+k}-x_i)^{2-\alpha}]$$

$$= \frac{1}{\Gamma(3-\alpha)}[(x_i+kh+h-x_i)^{2-\alpha} - (x_i+h-x_i)^{2-\alpha}]$$

$$= \frac{1}{\Gamma(3-\alpha)}[(h(k+1))^{2-\alpha} - (kh)^{2-\alpha}]$$

Hence:

$$u_k^\alpha = h^{2-\alpha}\frac{(k+1)^{2-\alpha}-k^{2-\alpha}}{\Gamma(3-\alpha)} \quad \text{...(3.24)}$$

Now, substituting eqs.(3.22), (3.23) into eq.(3.21), yields:

$$ {}_{x_i}D_\theta^\alpha T(x_i) \approx -\Bigg[c_L\sum_{k=0}^{\infty}\frac{1}{2h^2}[(2-\lambda_2)T_{i-k+1}+(3\lambda_2-4)T_{i-k}+$$

$$(2-3\lambda_2)T_{i-k-1}+\lambda_2T_{i-k-2}]u_k^\alpha + c_R\sum_{k=0}^{\infty}\frac{1}{2h^2}[\lambda_2T_{i+k+2}+$$

$$(2-3\lambda_2)T_{i+k+1}+(3\lambda_2-4)T_{i+k}+(2-\lambda_2)T_{i+k-1}]u_k^\alpha\Bigg] \quad \text{..(3.25)}$$

Finally, from eqs.(3.24) and (3.25) the discrete form of the Riesz-Feller operator in eq.(3.10) for $1 < \alpha \leq 2$ can be written as:

$$ {}_{x_i}D_\theta^\alpha T(x_i) \approx \frac{1}{h^\alpha} \sum_{k=-\infty}^{\infty} T_{i+k} w_k^{(\alpha,\theta)} \quad \text{.......................................(3.26)} $$

Where the coefficient $w_k = w_k(\alpha, \theta)$ have the following form:

$$ w_k^{(\alpha,\theta)} = \frac{-1}{2\Gamma(3-\alpha)} \begin{cases} \Big[(|k|+2)^{2-\alpha}(2-\lambda_2) + (|k|+1)^{2-\alpha}(4\lambda_2 - 6) + |k|^{2-\alpha} \\ (6-6\lambda_2) + (|k|-1)^{2-\alpha}(4\lambda_2 - 2) + (|k|-2)^{2-\alpha}(-\lambda_2)\Big] c_L, & k \leq -2 \\ \Big[3^{2-\alpha}(2-\lambda_2) + 2^{2-\alpha}(4\lambda_2 - 6) - 6\lambda_2 + 6\Big] c_L + (2-\lambda_2) c_R, & k = -1 \\ \Big[2^{2-\alpha}(2-\lambda_2) + 4\lambda_2 - 6\Big](c_L + c_R), & k = 0 \\ \Big[3^{2-\alpha}(2-\lambda_2) + 2^{2-\alpha}(4\lambda_2 - 6) - 6\lambda_2 + 6\Big] c_R + (2-\lambda_2) c_L, & k = 1 \\ \Big[(k+2)^{2-\alpha}(2-\lambda_2) + (k+1)^{2-\alpha}(4\lambda_2 - 6) + k^{2-\alpha}(6 - \\ 6\lambda_2) + (k-1)^{2-\alpha}(4\lambda_2 - 2) + (k-2)^{2-\alpha}(-\lambda_2)\Big] c_R, & k \geq 2 \end{cases} $$

...............................(3.27)

In both cases, when $\alpha = 1$, the Riesz-Feller operator is singular, and hence the method failed to be applied.

3.2.3 Finite Difference Method for Fractional Differential Equations:

In this subsection, $\frac{d^\alpha}{d|x|_\theta^\alpha} T(x)$ can be described using the finite difference by restricting the numerical solution in comparison with eq.(3.9) where the discritization of the fractional derivative can be approximated using the central difference method of the second order. The difference appears in the setting of boundary conditions.

Here, replace eq.(3.9) by eqs.(3.19) and (3.26) depending on α as:

$$\frac{1}{h^{\alpha}}\sum_{k=-\infty}^{\infty} T_{i+k} w_k^{(\alpha,\theta)} = 0 \qquad (3.28)$$

But eq.(3.28) includes unbounded domain $-\infty < x < \infty$ and this unbounded domain has no practical implementation in computer simulation. So, to solve this problem in the finite domain $\Omega : L \le x < R$ with boundary conditions (3.13), one can follow the following procedure:

Divide the domain Ω into N subdomains with step length $h = \frac{R-L}{N}$, $N \in \mathbb{N}$, and in order to introduce the Dirichlet boundary conditions, propose a numerical treatment which assumes that the values of the function T in outside points are identical as the values in the boundary nodes x_0 or x_N, i.e.,

$$T(x_k) = \begin{cases} T(x_0) = g_L, & \text{for } x < 0 \\ T(x_N) = g_R, & \text{for } x > N \end{cases} \qquad (3.29)$$

On the basis of the above considerations, modify eqs.(3.19) and (3.26) for the discritization of the Riesz-Feller derivative, as:

$$ {}_{x_i}D_{\theta}^{\alpha} T(x_i) \approx \frac{1}{h^{\alpha}}\sum_{k=-\infty}^{\infty} T_{i+k} w_k^{(\alpha,\theta)}$$

$$= \frac{1}{h^{\alpha}}\left[\sum_{k=-\infty}^{-i-1} T_{i+k} w_k^{(\alpha,\theta)} + \sum_{k=-i}^{N-i} T_{i+k} w_k^{(\alpha,\theta)} + \sum_{k=N-i+1}^{\infty} T_{i+k} w_k^{(\alpha,\theta)}\right]$$

$$= \frac{1}{h^{\alpha}}\left[g_L \sum_{k=-\infty}^{-i-1} w_k^{(\alpha,\theta)} + \sum_{k=-i}^{N-i} T_{i+k} w_k^{(\alpha,\theta)} + g_R \sum_{k=N-i+1}^{\infty} w_k^{(\alpha,\theta)}\right]$$

Hence:

$$ {}_{x_i}D_\theta^\alpha T(x_i) \approx \frac{1}{h^\alpha}\left[\sum_{k=-i}^{N-i} T_{i+k} w_k^{(\alpha,\theta)} + g_L \delta L_i^{(\alpha,\theta)} + g_R \delta R_{N-i}^{(\alpha,\theta)}\right] \quad ..(3.30) $$

for i = 1, 2, …, N − 1, where:

$$ \delta L_j^{(\alpha,\theta)} = \sum_{k=-\infty}^{-j-1} w_k^{(\alpha,\theta)} $$

$$ = \begin{cases} c_L(\alpha,\theta) r_j, & \text{for } 0<\alpha<1 \\ c_L(\alpha,\theta)\ell_j, & \text{for } 1<\alpha\le 2 \end{cases} \quad ....(3.31) $$

$$ \delta R_j^{(\alpha,\theta)} = \sum_{k=j+1}^{\infty} w_k^{(\alpha,\theta)} $$

$$ = \begin{cases} c_R(\alpha,\theta) r_j, & \text{for } 0<\alpha<1 \\ c_R(\alpha,\theta)\ell_j, & \text{for } 1<\alpha\le 2 \end{cases} \quad ....(3.32) $$

and

$$ r_j = \frac{(j+2)^{1-\alpha}\lambda_1 + (j+1)^{1-\alpha}(2-2\lambda_1) + j^{1-\alpha}(\lambda_1 - 2)}{2\Gamma(2-\alpha)} \quad ....(3.33) $$

$$ \ell_j = \frac{(j+2)^{2-\alpha}(2-\lambda_2) + (j+1)^{2-\alpha}(3\lambda_2 - 4) + j^{2-\alpha}(2-3\lambda_2)}{2\Gamma(3-\alpha)} + \frac{(j-1)^{2-\alpha}\lambda_2}{2\Gamma(3-\alpha)} $$

Substituting eq.(3.30) into eq.(3.28), the following finite difference scheme have been obtained:

$$\sum_{k=-i}^{N-i} T_{i+k} w_k^{(\alpha,\theta)} + g_L \delta L_i^{(\alpha,\theta)} + g_R \delta R_{N-i}^{(\alpha,\theta)} = 0 \quad \text{..........................(3.34)}$$

for i = 1, 2, …, N – 1, with boundary conditions:

$T_0 = g_L$, $T_N = g_R$

Finally, eq.(3.34) may be written as a linear system in matrix form as:

$AT = B$ ....(3.35)

where:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_{-1} & a_0 & a_1 & a_2 & \cdots & a_{N-3} & a_{N-2} & a_{N-1} \\ a_{-2} & a_{-1} & a_0 & a_1 & \cdots & a_{N-4} & a_{N-3} & a_{N-2} \\ a_{-3} & a_{-2} & a_{-1} & a_0 & \cdots & a_{N-5} & a_{N-4} & a_{N-3} \\ a_{-4} & a_{-3} & a_{-2} & a_{-1} & \cdots & a_{N-6} & a_{N-5} & a_{N-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{-N+2} & a_{-N+3} & a_{-N+4} & a_{-N+5} & \cdots & a_0 & a_1 & a_2 \\ a_{-N+1} & a_{-N+2} & a_{-N+3} & a_{-N+4} & \cdots & a_{-1} & a_0 & a_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

................................(3.36)

and

$B = [g_L, b_1, b_2, \ldots, b_{N-1}, g_R]^t$

with

$a_j = w_k^{(\alpha,\theta)}$, for j = –N+1, –N+2, …, N–1

$b_j = g_L \delta L_j^{(\alpha,\theta)} + g_R \delta R_j^{(\alpha,\theta)}$, for j = 1, 2, …, N–1

and

$$T = [T_0, T_1, \ldots, T_N]^t$$

is the vector of unknown values of the function T, and t is the matrix transportation.

The next example illustrate the above method of numerical solution:

Example (3.2):

Consider the fractional boundary value problem:

$$_{x_i}D_{0.1}^{1.01}T(x) = 0$$

$$T(x_k) = \begin{cases} T(x_0) = 2, & \text{for } x < 0 \\ T(x_N) = 1, & \text{for } x > 10 \end{cases}$$

where $\alpha = 1.01$ and $\theta = 0.1$, $0 \le x \le 1$, and to solve this problem using Riesz-Feller operator.

Let the number of node points be equal to 9, i.e., N = 10 and hence h = 0.1. To solve this problem, the resulting finite difference equation from eq.(3.34):

$$\sum_{k=-i}^{10-i} T_{i+k} w_k^{(1.01,0.1)} + g_L \delta L_i^{(1.01,0.1)} + g_R \delta R_{10-i}^{(1.01,0.1)} = 0,\ i = 1,2,\ldots,9$$

First, from eq.(3.12), $c_L = -31.519$ and $c_R = -31.362$.

Second, after carrying out some calculations from eq.(3.27), w_j with $\lambda_2 = 0.89$, j = 1, 2, …, 9, yields:

$w_{-9} = 3.847\times10^{-3}$, $w_{-8} = 4.892\times10^{-3}$, $w_{-7} = 6.434\times10^{-3}$,

$w_{-6} = 8.851\times10^{-3}$, $w_{-5} = 0.013$, $w_{-4} = 0.21$, $w_{-3} = 0.041$,

$w_{-2} = 0.154$, $w_{-1} = 3.352$, $w_0 = -7.43$, $w_1 = 3.509$, $w_2 = 0.153$,

$w_3 = 0.041$, $w_4 = 0.021$, $w_5 = 0.013$, $w_6 = 8.807\times10^{-3}$,

$w_7 = 6.402\times10^{-3}$, $w_8 = 4.868\times10^{-3}$, $w_9 = 3.828\times10^{-3}$

Then, from eq.(3.31) and after carrying some calculations, δL_j, j = 1, 2, …, 9, becomes:

$\delta L_1 = 0.285$, $\delta L_2 = 0.131$, $\delta L_3 = 0.09$, $\delta L_4 = 0.069$, $\delta L_5 = 0.056$, $\delta L_6 = 0.047$, $\delta L_7 = 0.041$, $\delta L_8 = 0.036$, $\delta L_9 = 0.032$

and δR_j, j =1, 2, …, 9, from eq.(3.32), yields:

$\delta R_1 = 0.284$, $\delta R_2 = 0.13$, $\delta R_3 = 0.09$, $\delta R_4 = 0.069$, $\delta R_5 = 0.056$, $\delta R_6 = 0.047$, $\delta R_7 = 0.041$, $\delta R_8 = 0.036$, $\delta R_9 = 0.032$

Finally, from the results of the finite difference equation (3.34) the linear system may be written as in the following matrix form:

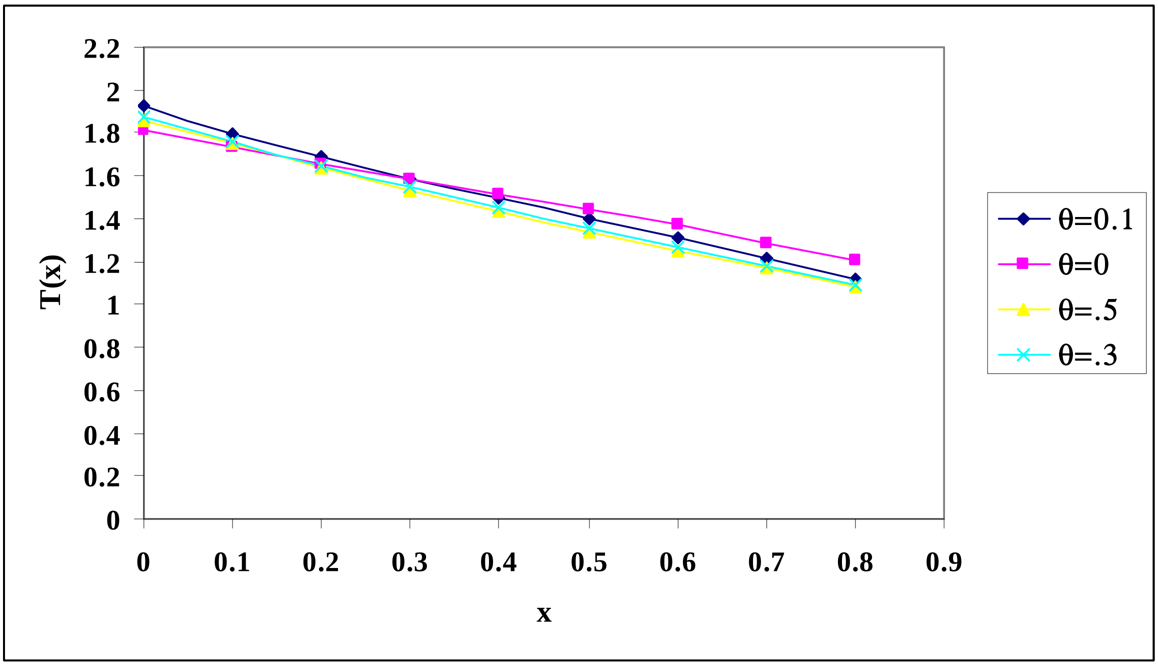
$$AT = b$$

where:

$$A=\begin{bmatrix} -7.43 & 3.509 & 0.153 & 0.041 & 0.021 & 0.013 & 8.807\times10^{-3} & 6.402\times10^{-3} & 4.868\times10^{-3} \\ 3.352 & -7.43 & 3.509 & 0.153 & 0.041 & 0.021 & 0.013 & 8.807\times10^{-3} & 6.402\times10^{-3} \\ 0.154 & 3.352 & -7.43 & 3.509 & 0.153 & 0.041 & 0.021 & 0.013 & 8.807\times10^{-3} \\ 0.041 & 0.154 & 3.352 & -7.43 & 3.509 & 0.153 & 0.041 & 0.021 & 0.013 \\ 0.021 & 0.041 & 0.154 & 3.352 & -7.43 & 3.509 & 0.153 & 0.041 & 0.021 \\ 0.013 & 0.021 & 0.041 & 0.154 & 3.352 & -7.43 & 3.509 & 0.153 & 0.041 \\ 8.851\times10^{-3} & 0.013 & 0.021 & 0.041 & 0.154 & 3.352 & -7.43 & 3.509 & 0.153 \\ 6.434\times10^{-3} & 8.851\times10^{-3} & 0.013 & 0.021 & 0.041 & 0.154 & 3.352 & -7.43 & 3.509 \\ 4.892\times10^{-3} & 6.434\times10^{-3} & 8.851\times10^{-3} & 0.013 & 0.021 & 0.041 & 0.154 & 3.352 & -7.43 \end{bmatrix}$$

$$T=\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{bmatrix} \quad \text{and} \quad b=\begin{bmatrix} -7.314 \\ -0.611 \\ -0.309 \\ -0.236 \\ -0.207 \\ -0.202 \\ -0.226 \\ -0.365 \\ -3.685 \end{bmatrix}$$

Solving this system, one can get the following numerical results presented in table (3.1). Also, the results for $\alpha = 1.01$ and $\theta = 0, 0.1, 0.3, 0.5$ are also illustrated in figure (3.1).



T(x)
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
x
θ=0.1
θ=0
θ=.5
θ=.3

3.2.4 Other Types of Boundary Conditions (Neumann Condition) for Solving Fractional Differential Equations by Riesz-Feller Operator:

In this subsection, two kinds of new boundary conditions that may be encountered in boundary value problem are considered, which are:

I- The First type of boundary conditions:

This kind can describe the boundary conditions in the shape of the ordinary differential equation which may be converted into a Dirichlet condition by the forward and backwards difference method, then solving the problem using Riesz-Feller operator as in subsection (3.2.3).

So, the boundary conditions may be described as follows:

$$T(x_k) = \begin{cases} T'(x_0) = g'_L, & \text{for } k < 0 \\ T'(x_N) = g'_R, & \text{for } k > N \end{cases} \quad \text{..(3.37)}$$

The following example illustrates the above conditions:

Example (3.3):

Consider the boundary value problem:

$${}_xD_0^{0.1}T(x) = 0$$

$$T(x_k) = \begin{cases} T'(x_0) = 2, & \text{for } k < 0 \\ T'(x_9) = 1, & \text{for } k > 9 \end{cases}$$

for $\alpha = 0.1$ and $\theta = 0$, $0 \le x \le 1$, and let the number of node points be equal to 10, i.e., $N = 9$ and hence $h = 1/9$.

Hence, using the forward and backward difference methods to convert the conditions (3.37) to Dirichlet conditions as; for $i = 0$, by forward difference method

$$-T_0 + T_1 = 0.222$$

and for $i = 9$, by backward difference method

$$-T_8 + T_9 = 0.111$$

and for $i = 1, 2, \ldots, 8$ from eq.(3.34) the finite difference equation to solve this problem is:

$$\sum_{k=-i}^{9-i} T_{i+k} w_k^{(0.1,0)} + g_L \delta L_i^{(0.1,0)} + g_R \delta R_{9-i}^{(0.1,0)} = 0$$

from eq.(3.12) and since $\theta = 0$, then

$$c_L(\alpha, \theta) = c_R(\alpha, \theta) = 0.506$$

and after carrying out some calculations from eq.(3.20), w_j with $\lambda_1 = \alpha - \theta = 0.1$, $j = 0, 1, \ldots, 8$ and since $\theta = 0$, therefore:

$$w_0 = -0.993,\ w_{\pm1} = 0.042,\ w_{\pm2} = 0.023,\ w_{\pm3} = 0.014,$$

$$w_{\pm4} = 0.01,\ w_{\pm5} = 8.054\times10^{-3},\ w_{\pm6} = 6.584\times10^{-3},\ w_{\pm7} = 5.554,$$

$$w_{\pm8} = 4.795\times10^{-3}$$

Then, from eq.(3.31) and after some calculations for δL_j and δR_j, $j = 1, 2, \ldots, 8$, yields:

$$\delta L_1 = \delta R_1 = 0.454,\ \delta L_2 = \delta R_2 = 0.432,\ \delta L_3 = \delta R_3 = 0.417,$$

$\delta L_4 = \delta R_4 = 0.407$, $\delta L_5 = \delta R_5 = 0.399$, $\delta L_6 = \delta R_6 = 0.393$,

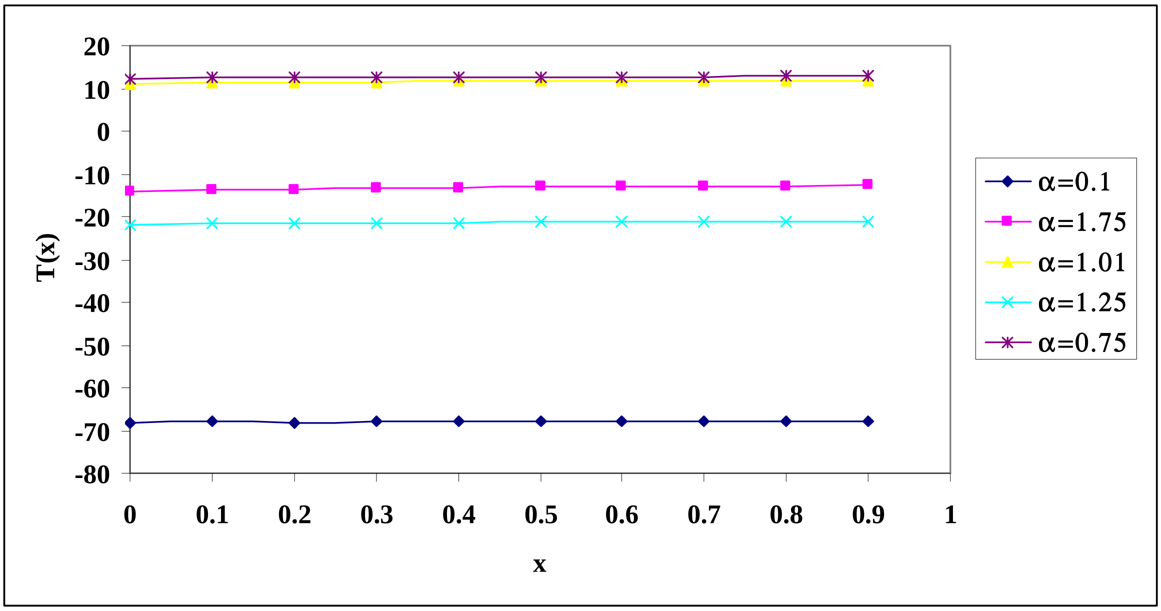
$\delta L_7 = \delta R_7 = 0.387$, $\delta L_8 = \delta R_8 = 0.382$

Finally, from the result of the finite difference equation as in eq.(3.34) and the forward, backward difference methods, the linear system AT = b, can be written as the following matrix form:

$$A=\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.496 & -0.993 & 0.042 & 0.023 & 0.014 & 0.01 & 8.054\times10^{-3} & 6.584\times10^{-3} & 5.554\times10^{-3} & 0.387 \\ 0.455 & 0.042 & -0.993 & 0.042 & 0.023 & 0.014 & 0.01 & 8.054\times10^{-3} & 6.584\times10^{-3} & 0.393 \\ 0.431 & 0.023 & 0.042 & -0.993 & 0.042 & 0.023 & 0.014 & 0.01 & 8.054\times10^{-3} & 0.4 \\ 0.417 & 0.014 & 0.023 & 0.042 & -0.993 & 0.042 & 0.023 & 0.014 & 0.01 & 0.407 \\ 0.407 & 0.01 & 0.014 & 0.023 & 0.042 & -0.993 & 0.042 & 0.023 & 0.014 & 0.417 \\ 0.4 & 8.054\times10^{-3} & 0.01 & 0.014 & 0.023 & 0.042 & -0.993 & 0.042 & 0.023 & 0.431 \\ 0.393 & 6.584\times10^{-3} & 8.054\times10^{-3} & 0.01 & 0.014 & 0.023 & 0.042 & -0.993 & 0.042 & 0.455 \\ 0.387 & 5.554\times10^{-3} & 6.584\times10^{-3} & 8.054\times10^{-3} & 0.01 & 0.014 & 0.023 & 0.042 & -0.993 & 0.496 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$T=\begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{bmatrix} \quad \text{and} \quad b=\begin{bmatrix} 0.222 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.111 \end{bmatrix}$$

Solving this system using any numerical method, one can get the following numerical results presented in table (3.2). Also, the results for $\alpha = 0.1, 0.75, 1.01, 1.25, 1.75$ and $\theta = 0$, are also illustrated in figure (3.2).



20
10
0
-10
-20
-30
-40
-50
-60
-70
-80
T(x)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
α=0.1
α=1.75
α=1.01
α=1.25
α=0.75

II- The Second type of boundary conditions:

This kind can be described as the boundary conditions in the shape of the fractional differential equation, which can be described as follows:

$$T(x_k) = \begin{cases} T^{(\alpha)}(x_0) = g^{(\alpha)}{}_L, & \text{for } k < 0 \\ T^{(\alpha)}(x_N) = g^{(\alpha)}{}_R, & \text{for } k > N \end{cases} \quad \text{.................................(3.38)}$$

and may be converted into Drichlit condition by the following finite difference method. To do this, let us write the boundary conditions as:

$$T^{(\alpha)}(x_i) = f(x_i, T(x_i)) \quad \text{...(3.39)}$$

where $x_i = L + ih$, $i = 0, 1, \ldots, N$, $h = \frac{R-L}{N}$, where $N \in$ ￼ is the number of subintervals of the interval [L, R].

Now, recall the left-hand fractional derivatives of Grünwald definition:

$$\frac{d^\alpha}{dx^\alpha}T(x) = \lim_{N\to\infty} \frac{1}{h^\alpha}\sum_{j=0}^{N} g_j T(x - jh) \quad \text{.....................................(3.40)}$$

where $g_0 = 1$, and:

$$g_j = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-j+1)}{j!}, \text{ for } j = 1, 2, \ldots$$

Next, to obtain a good approximation, define the left hand shifted Grünwald estimate to left hand derivative.

$$\frac{d^\alpha}{dx^\alpha}T(x) = \frac{1}{h^\alpha}\sum_{j=0}^{N} g_j T(x - (j-1)h)$$

Therefore:

$$T^{(\alpha)}(x_i) = \frac{1}{h^{\alpha}}\sum_{j=0}^{N} g_j T(x-(j-1)h)$$

$$= \frac{1}{h^{\alpha}}\sum_{j=0}^{N} g_j T_{i-j+1} \qquad (3.41)$$

and by substituting eq.(3.41) in eq.(3.39), one can have:

$$\frac{1}{h^{\alpha}}\sum_{j=0}^{i+1} g_j T_{i-j+1} = f(x_i, T_i),\ i = 0, 1, \ldots, n-1 \qquad (3.42)$$

To illustrate this method, consider the following design example:

Example (3.4):

Consider the boundary value problem:

$${}_xD_0^{0.5}T(x) = 0,\ 0 \le x \le 1$$

$$T(x_k) = \begin{cases} T^{(\alpha)}(x_0) = 1, & \text{for } k < 0 \\ T^{(\alpha)}(x_9) = 0, & \text{for } k > 9 \end{cases}$$

and if we let N = 9, then h = 0.111

The first step is to convert the two boundary conditions to Dirichlet conditions using eq.(3.42) for i = 0 and i = 9, respectively as follows:

$$\frac{1}{h^{\alpha}}\sum_{j=0}^{1} g_j T_{i-j+1} = 1$$

and

$$\frac{1}{h^{\alpha}}\sum_{j=0}^{9} g_j T_{i-j+1} = 0$$

Now, for i = 1, 2, …, 8, from eq.(3.34) the finite difference equation to solve this problem is:

$$\sum_{k=-i}^{9-i} T_{i+k} w_k^{(0.5,0)} + g_L \delta L_i^{(0.5,0)} + g_R \delta R_{N-i}^{(0.5,0)} = 0$$

from eq.(3.12) and since $\theta = 0$, then:

$$c_L(\alpha, \theta) = c_R(\alpha, \theta) = 0.707$$

and after carrying some calculations from eq.(3.20), w_j with $\lambda_1 = \alpha - |\theta| = 0.5$, j = 0, 1, …, 8, and since $\theta = 0$, therefore:

$w_0 = -0.963$, $w_{\pm1} = 0.17$, $w_{\pm2} = 0.068$, $w_{\pm3} = 0.036$,

$w_{\pm4} = 0.024$, $w_{\pm5} = 0.017$, $w_{\pm6} = 0.013$, $w_{\pm7} = 0.01$,

$w_{\pm8} = 8.498\times10^{-3}$

Then, from eq.(3.31)and after some calculations δL_j, δR_j, j = 1, 2, …, 8, yields:

$\delta L_1 = \delta R_1 = 0.311$, $\delta L_2 = \delta R_2 = 0.244$, $\delta L_3 = \delta R_3 = 0.208$,

$\delta L_4 = \delta R_4 = 0.184$, $\delta L_5 = \delta R_5 = 0.167$, $\delta L_6 = \delta R_6 = 0.154$,

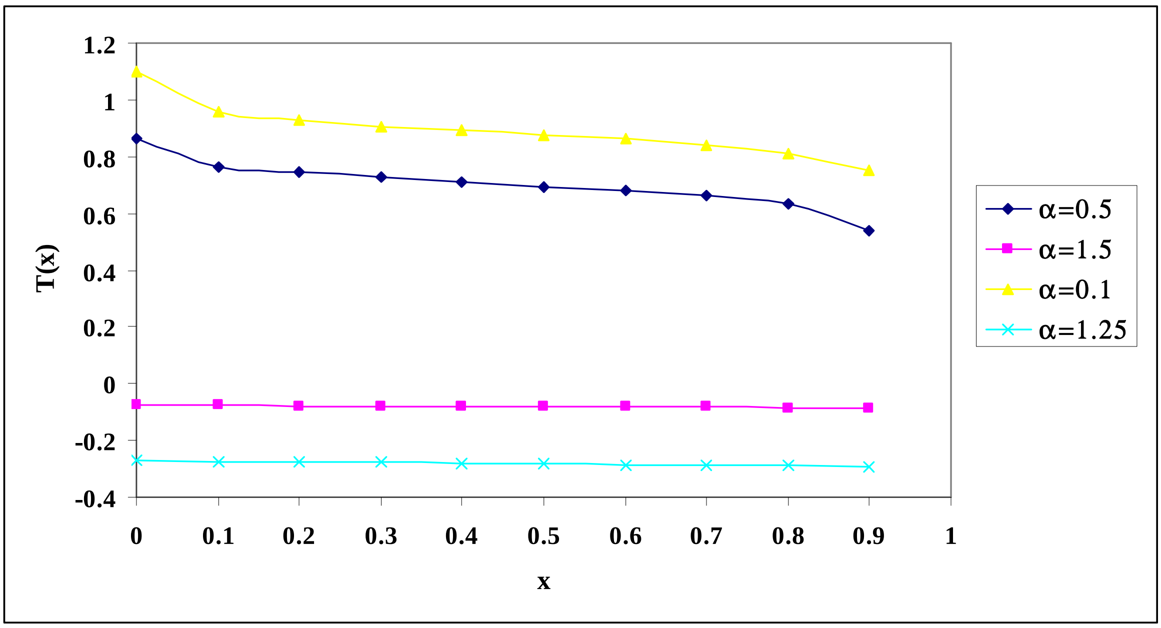
$\delta L_7 = \delta R_7 = 0.144$, $\delta L_8 = \delta R_8 = 0.135$

Finally, from the result of the finite difference equation as in eq.(3.34) and by eqs.(3.42), then the system AT = b, which can be written as the following matrix form:

$$A=\begin{bmatrix} -0.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.481 & -0.963 & 0.17 & 0.068 & 0.036 & 0.024 & 0.017 & 0.013 & 0.01 & 0.143 \\ 0.312 & 0.17 & -0.963 & 0.17 & 0.068 & 0.036 & 0.024 & 0.017 & 0.013 & 0.154 \\ 0.244 & 0.068 & 0.17 & -0.963 & 0.17 & 0.068 & 0.036 & 0.024 & 0.017 & 0.167 \\ 0.208 & 0.036 & 0.068 & 0.17 & -0.963 & 0.17 & 0.068 & 0.036 & 0.024 & 0.184 \\ 0.184 & 0.024 & 0.036 & 0.068 & 0.17 & -0.963 & 0.17 & 0.068 & 0.036 & 0.208 \\ 0.167 & 0.017 & 0.024 & 0.036 & 0.068 & 0.17 & -0.963 & 0.17 & 0.068 & 0.244 \\ 0.154 & 0.013 & 0.017 & 0.024 & 0.036 & 0.068 & 0.17 & -0.963 & 0.17 & 0.312 \\ 0.143 & 0.01 & 0.013 & 0.017 & 0.024 & 0.036 & 0.068 & 0.17 & -0.963 & 0.481 \\ -0.011 & -0.013 & -0.016 & -0.021 & -0.027 & -0.039 & -0.063 & -0.123 & -0.5 & 1 \end{bmatrix}$$

$$T=\begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{bmatrix} \quad \text{and} \quad b=\begin{bmatrix} 0.333 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system using any numerical method, one can get the following numerical results presented in table (3.3). Also, the results for $\alpha = 0.1, 0.5, 1.25, 1.5$ and $\theta = 0$, are also illustrated in figure (3.3).



1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
T(x)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
α=0.5
α=1.5
α=0.1
α=1.25

Conclusions and Recommendations

The following conclusions may be drown from the present study:

1. The finite difference method used in the numerical solution of fractional differential equations depends on the Grüunwald fractional derivative approximation of the fractional order derivative, which is the only applicable method for solving FBV’s.
2. The accuracy of the results may be improved when considering fractional differential equations with Drichlit boundary conditions, of integer or fractional order (see examples (3.3) and (3.4)).
3. From the illustrative figures of examples (3.2), (3.3) and (3.4), one can see that the behavior of the solutions is unchanged for different values of α and θ.

Also, the following problems may be recommended for future work as an open problems:

1. Solving fractional boundary value problems using Riesz-Feller fractional derivative with fractional order $\alpha > 2$.
2. Using the Green’s function method to solve fractional boundary value problems with boundary conditions of fractional order.
3. Proposing a modified approach for solving partial differential equations with fractional order derivatives using Riesz-Feller fractional derivative.

4. Modifying the present approach for solving homogeneous fractional boundary value problems to solve non-homogeneous fractional boundary value problems.
5. Solving nonlinear fractional boundary value problems using Riesz-Feller fractional derivative.
6. Using other methods for solving fractional boundary value problems, such as the shooting method, the collocation method, etc.

Contents

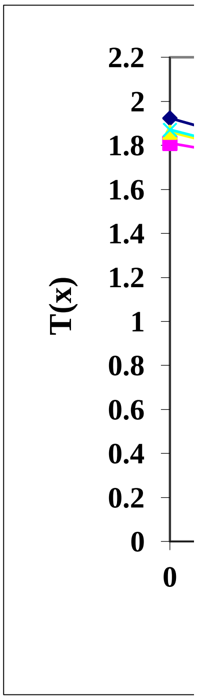
Dedication

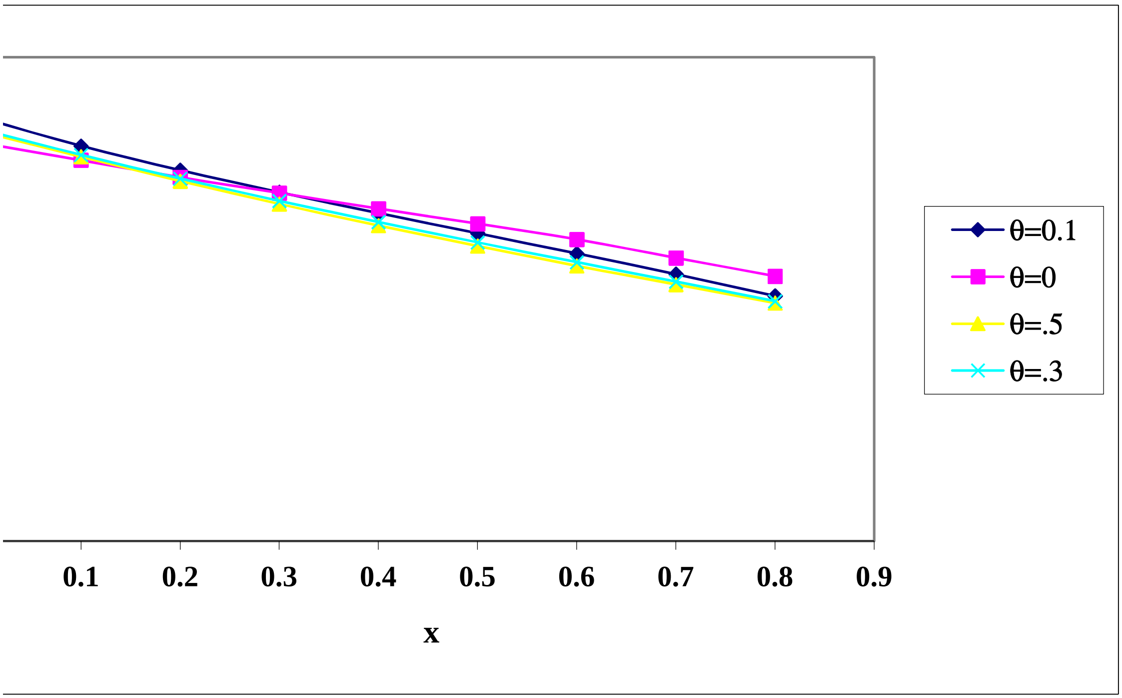
To …

My Father, Mother and the Spirit of My Grandfather Mohammed

All Persons who Encouraged and Supported Me

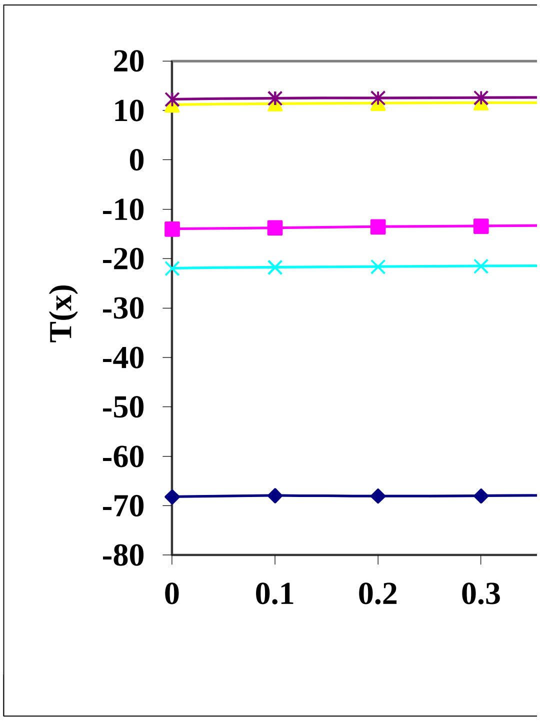
0	1.924	1.81	1.861	1.871
0.1	1.797	1.732	1.748	1.756
0.2	1.687	1.654	1.637	1.648
0.3	1.587	1.583	1.533	1.547
0.4	1.492	1.512	1.435	1.451
0.5	1.4	1.443	1.342	1.358
0.6	1.308	1.372	1.252	1.268
0.7	1.214	1.288	1.167	1.18
0.8	1.115	1.205	1.083	1.091

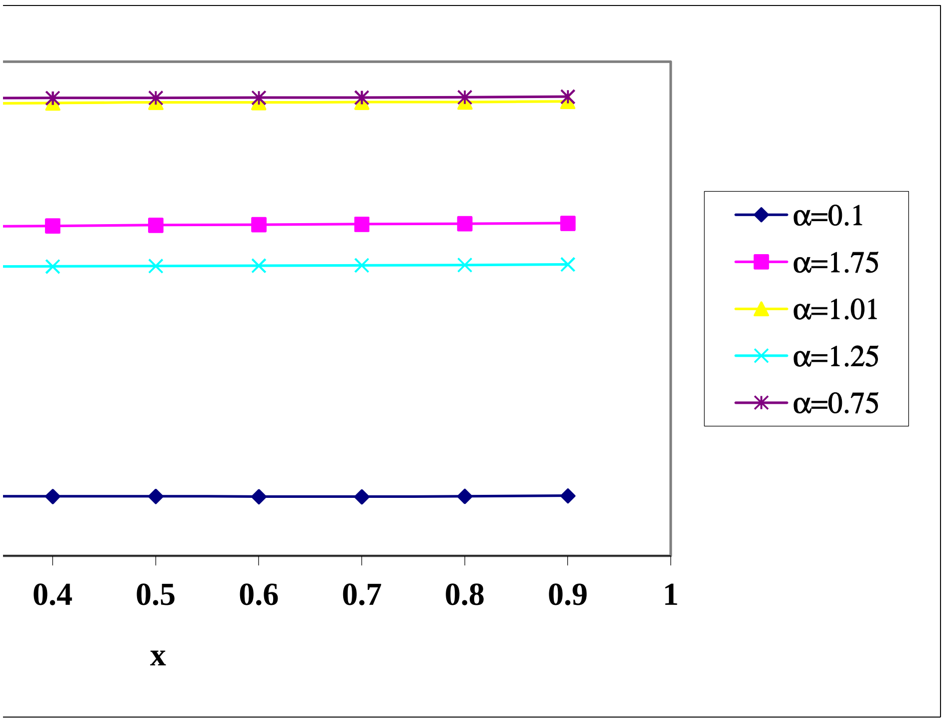




0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
x
θ=0.1
θ=0
θ=.5
θ=.3

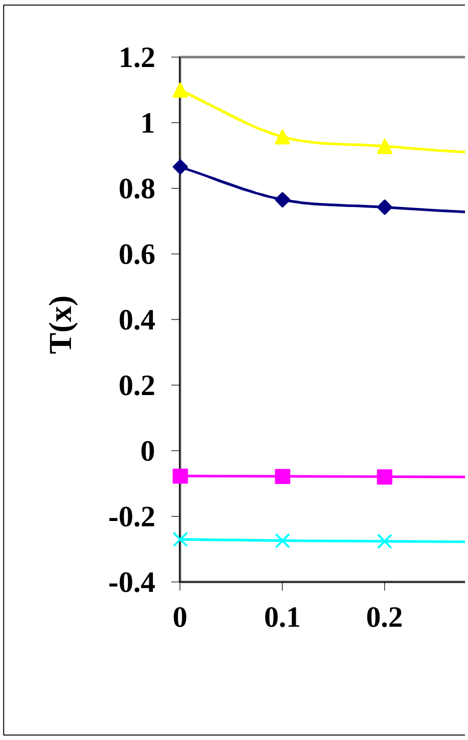
0	-68.184	-13.99	11.159	-21.966	12.252
0.1	-67.962	-13.768	11.381	-21.744	12.474
0.2	-68.048	-13.561	11.469	-21.624	12.53
0.3	-68.002	-13.394	11.55	-21.519	12.581
0.4	-67.927	-13.24	11.617	-21.426	12.636
0.5	-67.923	-13.104	11.767	-21.343	12.675
0.6	-67.989	-12.984	11.735		
0.7	-68.023	-12.873	11.793		
0.8	-67.92	-12.792	11.842		
0.9	-67.809	-12.681	11.953		

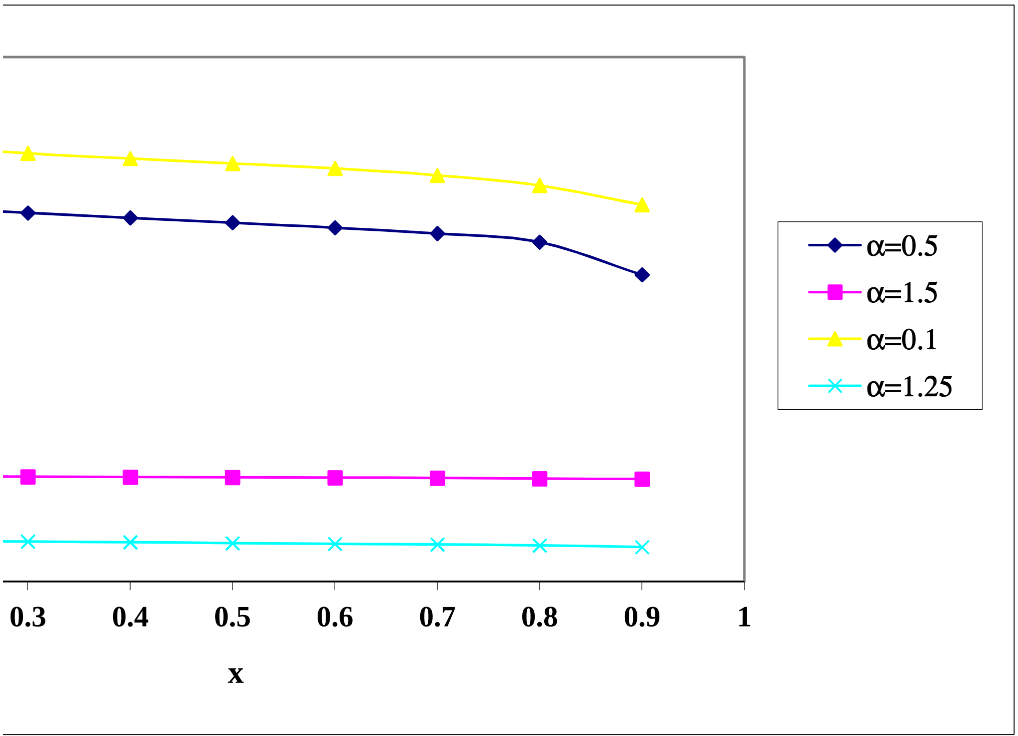




0.4
0.5
0.6
0.7
0.8
0.9
1
x
α=0.1
α=1.75
α=1.01
α=1.25
α=0.75

0	0.866	-0.077	1.1	-0.27
0.1	0.766	-0.078	0.958	-0.274
0.2	0.743	-0.079	0.929	-0.276
0.3	0.725	-0.08	0.907	-0.278
0.4	0.71	-0.081	0.891	-0.28
0.5	0.695	-0.082	0.876	-0.283
0.6	0.68	-0.083	0.861	-0.285
0.7	0.662	-0.084	0.84	-0.287
0.8	0.636	-0.086	0.809	-0.29
0.9	0.536	-0.087	0.75	-0.295





α=0.5
α=1.5
α=0.1
α=1.25
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x

Introduction

From the 16th century until now, the fractional calculus have an important place in many fields, because it deals with the investigation and applications of integrals and derivatives of arbitrary order. Moreover, it has played a significant role in engineering, science, economy and more particularly in transport of chemical contaminant through water around rocks, diffusion process involving cells, signals theory such as radar and sonar applications, control theory and many more, [El-Sayed, 2006].

The subject of fractional calculus may be considered as an old and yet novel topic. It is an old since, starting from some speculations of G. W. Leibniz (1695, 1697), L. Euler (1730) who suggested to use this relationship for negative or non-integer (rational) values of n.

Historically, S. Locroix (1819) first mentioned derivatives of arbitrary order in a text published in (1819) and it has been developed up to nowadays by J. B. Fourier (1820-1822) who put the first steps to the generalization of the notion of differential equations of arbitrary function. Also, the first application of fractional derivatives was given in (1823) by Abel who applied the fractional calculus in the solution of an integral equations. Liouville (1832) who attempted to give logical definitions of fractional derivatives.

Moreover, one can state that the whole theory of fractional derivative and calculus was established on the bounds of many scientists in the 2nd half of the 19th century (for more detailed overview concerning the history of fractional derivative and calculus, see [Ross, 1975]).

Many books and papers on fractional calculus, fractional differential equations have appeared recently, such as [Samko, 1993], [Diethelm, 1997], [Podlubny, 1997], [Gorenflo, 1998], [Odibat, 2006].

Many scientists who works in fractional calculus starting from G. W. Leibniz (1695-1697), P. S. Laplace (1812), J. B. Fourier (1822), N. H. Abel (1823-1828), J. Liouville (1832-1873), B. Riemann (1847), H. Holmgren (1865-1867), A. K. Grünwald (1867-1872) until the 19th century like S. Pincherle (1902), G. H. Hordy and J. E. Littlewood (1917-1928), E. R. Love (1938-1996), A. Erdelyi (1939-1965), H. Kober (1940), D. V. Widder (1941), M. Riesz, (1949), etc., [Oldham, 1974].

Finally, in recent years, the interesting of fractional calculus have been stimulated by using the subject in many applications like the subject of finding the numerical solution of differential equations and in sciences, such as physics and engineering, etc., [Al-Hussieny, 2006].

Nowadays, many researchers works in fractional differential equations with initial conditions and the methods of solution it such as [Al-Shather, 2003] who presented some approximate solutions for fractional delay integro-differential equations, [Al-Azawi, 2004] who gave some results in fractional calculus, [Al-Authab, 2005] gave some numerical methods for solving fractional differential equations, [Khalil, 2006] used linear multistep method to approximate some fractional order differential equations, [Aziz, 2006] use some approximated methods for solving fractional partial differential equations, [Al-Husseiny, 2006] who gave some type of solution in fuzzy fractional differential equations, [Ghareeb, 2007] used the finite difference methods for solving fractional differential equations and [Farjo, 2007] used the Laplace transformation method to solve ordinary fractional differential equations with constant coefficients. But a little of mathematicians or papers deals with fractional differential equation with boundary conditions like [Zhang, 2006], [Ciesielski, 2006] and [Zhanbing, 2005], thus this thesis is oriented towards introducing fractional boundary value problems and the numerical methods for solving such type of equations.

This thesis consists of three chapters, the first chapter devoted to introduce the general concepts of fractional calculus and fractional differential equations, while the main objective of this chapter is to give an overview about fractional differential equations with initial conditions, these problems has the form:

$$D^{q}y(x) = f(x, y(x)),\ D^{q-k}y(x_0) = y_0^k,\ q > k$$

where $k = 1, 2, \ldots, m + 1$; $m - 1 < q \le m$, and m is a positive integer. The methods used to solve the above problems are discussed with some illustrative examples, which are necessary for understanding these methods of solution.

In order to ensure the existence of a unique solution, chapter two is devoted to discusses the existence and uniqueness theorem of solutions of fractional differential equations with boundary conditions using the Schauder's fixed point theorem.

Finally, in chapter three we present the only well known fundamental analytic method for solving fractional differential equations with boundary conditions. Since the numerical methods may be sometimes the most applicable methods for solving differential equations, in general, and fractional differential equations, in particular, therefore in this chapter we derived one of the most successful methods for solving fractional boundary value problems and present some illustrative examples are given.

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Department of Mathematics and
Computer Applications

Solutions of Fractional Boundary Value Problems

A Thesis
Submitted to the College of Science of Al-Nahrain University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By
Sema'a Abdul Sattar Mohammad Al-Fayadh
(B.Sc., Al-Nahrain University, 2005)

Supervised by
Asit. Prof. Dr. Fadhel Subhi Fadhel

April 2008 *Rabia'a Al-Thani 1429*

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جـــامعـة النهـــريـن
كلية العلوم
قسم الرياضيات وتطبيقات الحاسوب

حلول المعادلات التفاضلية الكسرية الحدودية

رسالة
مقدمة إلى كلية العلوم ـ جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل
سيماء عبد الستار محمد الفياض
(بكالوريوس علوم، جامعة النهرين، ٢٠٠٥)

إشراف
أ.م.د. فاضل صبحي فاضل

نيسان ٢٠٠٨ ربيع الثاني ١٤٢٩

بسم الله الرحمن الرحيم

نَرفَعُ دَرَجتٍ مَّن نّشَاءُ وفَوقَ كُلِّ ذِي عِلمٍ عَليمٌ

صدق الله العظيم

سورة يوسف

الآية (٧٦)

المستخلص

في هذه الرسالة، قمنا بتقديم اسلوب مطور لحل المعادلات التفاضلية الحدودية ذات الرتب الكسرية (Fractional order boundary value problems). حيث اعتمدنا في هذا الاسلوب على تطبيق مؤثر رايسز–فيلر (Riesz-Feller operator) والحصول على الصيغة المطورة لمعادلة الفروقات المنتهية المناظرة للمعادلة التفاضلية الحدودية الكسرية.

كما وان من أهداف هذا العمل هو دراسة مبرهنة وجود ووحدانية حلول المعادلات التفاضلية الحدودية الكسرية، وتقديم برهان لهتين المبرهنتين بالاعتماد على مبرهنة شاودر للنقطة الصامدة (Schauder fixed point theorem) للمؤثر التكاملي الكسري (Fractional integral operator).

جامعة النهرين

كلية العلوم

قسم الرياضيات وتطبيقات الحاسوب

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التخصص : الرياضيات (معادلات تفاضلية)

تاريخ المناقشة : ٢٩ / ٦ / ٢٠٠٨

اسم المشرف : د. فاضل صبحي فاضل

عنوان الأطروحة : حلول المعادلات التفاضلية الكسرية الحدودية

مفاتيح الكلمات: المعادلات التفاضلية الكسرية، المعادلات الحدودية، الحلول العددية، دوال كرين.

المستخلص

في هذه الرسالة، قمنا بتقديم اسلوب مطور لحل المعادلات التفاضلية الحدودية ذات الرتب الكسرية (Fractional order boundary value problems). حيث اعتمدنا في هذا الاسلوب على تطبيق مؤثر رايسز–فيلر (Riesz-Feller operator) والحصول على الصيغة المطورة لمعادلة الفروقات المنتهية المناظرة للمعادلة التفاضلية الحدودية الكسرية.

كما وان من أهداف هذا العمل هو دراسة مبرهنة وجود ووحدانية حلول المعادلات التفاضلية الحدودية الكسرية، وتقديم برهان لهتين المبرهنتين بالاعتماد على مبرهنة شاودر للنقطة الصامدة (Schauder fixed point theorem) للمؤثر التكاملي الكسري (Fractional integral operator).

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ABSTRACT

In this thesis, we introduce a modified approach for solving fractional order boundary value problems. This approach is given by applying the Riesz-Feller operator to obtain a modified finite difference equation, which is symmetric to the equation of fractional boundary value problems.

Also, the main objective of this work is to study the existence and uniqueness theorem of solutions of the fractional boundary value problems, and to present their proof depending on Schauder fixed point theorem for fractional order integral operator.