Notations and Abberviations

p.d.f.	Probability Density Function
c.d.f.	Cumulative Distribution Function
m.g.f	Moment Generating Function
r.v.	Random Variable
r.v.'s	Random Variables
r.s.	Random Sample
distn.	distribution
M.L.E.	Maximum Likelihood Estimator
m.ℓ.e	Maximum Likelihood Estimate
$W(\alpha,\theta)$	Weibull Distribution With Parameters α, θ
L.S.M.	Least Square Method
M.M.	Moments Method
M.M.M.	Modified Moments Method
M.L.M.	Maximum Likelihood Method
IT	Inverse Transform
AR	Acceptance-Rejection Method
μ_r	r^{th} moment about the mean
$\mu_r^{/}$	r^{th} moment about the origin
δ^2	Variance
CV.	Coefficient of Variation
$Ext(\delta,\lambda)$	Extreme Value Distribution With Parameters δ, λ

MSE	Mean Square Error
$Exp(\lambda)$	Exponential Distribution With Parameters λ
$N(0,1)$	Standardize Normal Distribution
$C(0,1)$	Cauchy Distribution with parameters $\alpha=0$ and $\beta=1$
MVUE	Minimum Variance Unbiased Estimator
S.S.E.	Sum of Square Error
S.S	Sample Space

Abstract

In this work, we consider the Weibull distribution of two parameters for its importance in statistics and its applications. Mathematical and statistical properties of Weibull distribution are considered, moments and higher moments are illustrated and unified. Four methods of estimation to the distribution parameters namely (Maximum likelihood Method, Moments Method, Modified Moments Method, Least Square Method) are discussed theoretically and assessed practically by utilizing six procedures of Monte-Carlo simulation for generating random variates from the distribution. Efficiency of some procedures are found theoretically and compared practically. Comparisons are made among four methods of estimation by considering the mean square error measurement.

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Appendices

Appendix A
Computer Programs of Estimation Methods

Derivation of Box and Muller Approach.

This procedure due to Box and Muller (1958) [27], where the Weibull variates is generated by utilizing the standard normal distn. .

If U_1 and U_2 is a r.s. of size 2 from $U(0,1)$, then the r.v.'s

$X_1=(-2\ln U_1)^{\frac{1}{2}}\cos(2\pi U_2),\ X_2=(-2\ln U_1)^{\frac{1}{2}}\sin(2\pi U_2)$ represent a r.s. of size 2 from $N(0,1)$.

Since, the joint distn. of U_1 and U_2 are:

$$g(u_1,u_2)=1,\ 0<u_i<1,\ i=1,2\ .$$
$$=0 \qquad ,e.w\ .$$

The function $X_1=(-2\ln U_1)^{\frac{1}{2}}\cos(2\pi U_2),\ X_2=(-2\ln U_1)^{\frac{1}{2}}\sin(2\pi U_2)$ is defined $(1-1)$ transformation that maps $A=\{(u_1,u_2):o<x<1,\ i=1,2\}$ on to the space $B=\{(x_1,x_2):-\infty<x_i<\infty, i=1,2\}$ with inverse transforms

$$x_1^2+x_2^2=(-2\ln u_1)\cos^2(2\pi u_2)+(-2\ln u_1)\sin^2(2\pi u_2)$$
$$=-2\ln u_1\left[\cos^2(2\pi u_2)+\sin^2(2\pi u_2)\right]$$

$$x_1^2+x_2^2=-2\ln u_1 \Rightarrow u_1=e^{-\frac{1}{2}\left(x_1^2+x_2^2\right)}$$

and $\dfrac{x_1}{x_2}=\tan(2\pi u_2)\Rightarrow u_2=\dfrac{1}{2\pi}\tan^{-1}\left(\dfrac{x_1}{x_2}\right)$

With Jacobin transformation

$$J=\frac{\partial(u_1,u_2)}{\partial(x_1,x_2)}=\begin{vmatrix} -x_1 e^{-\frac{1}{2}\left(x_1^2+x_2^2\right)} & -x_2 e^{-\frac{1}{2}\left(x_1^2+x_2^2\right)} \\ \frac{1}{2\pi}\frac{\left(\frac{-x_2}{x_1^2}\right)}{1+\left(\frac{x_1}{x_2}\right)^2} & \frac{1}{2\pi}\frac{\left(\frac{1}{x_1}\right)}{1+\left(\frac{x_2}{x_1}\right)^2} \end{vmatrix}$$

$$=\frac{-1}{2\pi}e^{-\frac{1}{2}\left(x_1^2+x_2^2\right)}$$

Then, the joint distn. of X_1 and X_2 is:

$$f\left(x_1,x_2\right)=g\left(e^{-\frac{1}{2}\left(x_1^2+x_2^2\right)},\frac{1}{2\pi}\tan^{-1}\left(\frac{x_2}{x_1}\right)\right)|J|$$

$$=\frac{1}{2\pi}e^{-\left(x_1^2+x_2^2\right)},-\infty<x<\infty,\ i=1,2\ .$$

$\underset{\sim}{X}=\left(X_1,X_2\right)$ distributed as a r.v. vector of size 2 from $N\left(0,1\right)$.

That is $X_i\sim N\left(0,1\right)$, i=1,2 .

Algorithm:

1- Read where α,θ.

2- Generate U_1 and U_2 from $U\left(0,1\right)$.

3- Set $X_1=\left(-2\ln U_1\right)^{\frac{1}{2}}\cos\left(2\pi U_2\right)$, $X_2=\left(-2\ln U_1\right)^{\frac{1}{2}}\sin\left(2\pi U_2\right)$.

4- Deliver $\underset{\sim}{X}=\left(X_1,X_2\right)$ as a random vector of size 2 generated from N(0,1).

5- Set $Z_1=\sqrt{\theta}X_1$, $Z_2=\sqrt{\theta}X_2$ and $R=\left(\frac{Z_1^2+Z_2^2}{2}\right)^{\frac{1}{\alpha}}$.

6- Deliver R as a r.v. generated from W(α,θ) .

7- Stop.

Program 1:Estimation by Maximum Likelihood Method

Enter your values of α, θ and n

$\alpha :=$ ▮ $\qquad \theta :=$ ▮ $\qquad n :=$ ▮

```
x := | for j ∈ 0.. n − 1
     |   | for i ∈ 0.. n − 1
     |   |   | u1 ← rnd(1)
     |   |   | u2 ← rnd(1)
     |   |   | b1[i,j] ← √(−2·ln(u1))·cos(2π·u2)
     |   |   | b2[i,j] ← √(−2·ln(u1))·sin(2π·u2)
     |   |   | z1 ← √θ·b1[i,j]
     |   |   | z2 ← √θ·b2[i,j]
     |   |   | r[i,j] ← [((z1)^2 + (z2)^2)/2]^(1/α)
     |   | r
     | r
```

$$b1_{i,j} \leftarrow \sqrt{-2\cdot\ln(u1)}\cdot\cos(2\pi\cdot u2) \qquad b2_{i,j} \leftarrow \sqrt{-2\cdot\ln(u1)}\cdot\sin(2\pi\cdot u2)$$

$$z1 \leftarrow \sqrt{\theta}\cdot b1_{i,j} \qquad z2 \leftarrow \sqrt{\theta}\cdot b2_{i,j} \qquad r_{i,j} \leftarrow \left[\frac{(z1)^2 + (z2)^2}{2}\right]^{\frac{1}{\alpha}}$$

Enter your values of α, θ and n

$$p := \left| \begin{array}{l} n \leftarrow \blacksquare \\ \alpha \leftarrow \blacksquare \\ \theta \leftarrow \blacksquare \\ \text{for } i \in 0..n-1 \\ \quad \left| \begin{array}{l} \text{for } j \in 0..n-1 \\ \quad f1 \leftarrow \frac{n}{\alpha} + \sum_{i=0}^{n-1} \ln\left(x_{i,j}\right) - \theta \cdot \sum_{i=0}^{n-1} \left[\left(x_{i,j}\right)^{\alpha} \cdot \ln\left(x_{i,j}\right)\right] \\ f2 \leftarrow \frac{n}{\theta} - \sum_{i=0}^{n-1} \left(x_{i,j}\right)^{\alpha} \\ a \leftarrow \frac{-n}{(\alpha)^2} - \theta \cdot \sum_{i=0}^{n-1} \left[\left(x_{i,j}\right)^{\alpha} \cdot \left(\ln\left(x_{i,j}\right)\right)^2\right] \\ b \leftarrow -\sum_{i=0}^{n-1} \left[\left(x_{i,j}\right)^{\alpha} \cdot \ln\left(x_{i,j}\right)\right] \\ c \leftarrow \frac{-n}{(\theta)^2} \\ h_j \leftarrow \theta - \frac{(-b \cdot f1 + a \cdot f2)}{a \cdot c - (b)^2} \\ z_j \leftarrow \alpha - \frac{(c \cdot f1 - b \cdot f2)}{a \cdot c - (b)^2} \\ \theta \leftarrow \left(h_j\right) \\ \alpha \leftarrow \left(z_j\right) \end{array} \right. \\ \alpha \end{array} \right.$$

$$g := \left| \begin{array}{l} n \leftarrow \blacksquare \\ \alpha \leftarrow \blacksquare \\ \theta \leftarrow \blacksquare \\ \text{for } i \in 0..n-1 \\ \quad \left| \begin{array}{l} \text{for } j \in 0..n-1 \\ \quad f1 \leftarrow \frac{n}{\alpha} + \sum_{i=0}^{n-1} \ln\left(x_{i,j}\right) - \theta \cdot \sum_{i=0}^{n-1} \left[\left(x_{i,j}\right)^{\alpha} \cdot \ln\left(x_{i,j}\right)\right] \\ f2 \leftarrow \frac{n}{\theta} - \sum_{i=0}^{n-1} \left(x_{i,j}\right)^{\alpha} \\ a \leftarrow \frac{-n}{(\alpha)^2} - \theta \cdot \sum_{i=0}^{n-1} \left[\left(x_{i,j}\right)^{\alpha} \cdot \left(\ln\left(x_{i,j}\right)\right)^2\right] \\ b \leftarrow -\sum_{i=0}^{n-1} \left[\left(x_{i,j}\right)^{\alpha} \cdot \ln\left(x_{i,j}\right)\right] \\ c \leftarrow \frac{-n}{(\theta)^2} \\ h_j \leftarrow \theta - \frac{(-b \cdot f1 + a \cdot f2)}{a \cdot c - (b)^2} \\ z_j \leftarrow \alpha - \frac{(c \cdot f1 - b \cdot f2)}{a \cdot c - (b)^2} \\ \alpha \leftarrow \left(z_j\right) \\ \theta \leftarrow \left(h_j\right) \end{array} \right. \\ \theta \end{array} \right.$$

$p = \blacksquare$

$g =$

$$E := \int_{0}^{\infty} p \cdot \alpha \cdot \theta \cdot x^{\alpha - 1} \cdot e^{-\theta \cdot x^{\alpha}} \, dx$$

$$D := \int_{0}^{\infty} g \cdot \alpha \cdot \theta \cdot x^{\alpha - 1} \cdot e^{-\theta \cdot x^{\alpha}} \, dx$$

$E = \blacksquare$

$D = \blacksquare$

$\text{Bias} := E - \theta$

$\text{Bias1} := D - \alpha$

$\text{Bias} = \blacksquare$

$\text{Bias1} = \blacksquare$

$$Q := \int_0^{\infty} (p)^2 \cdot \alpha \cdot \theta \cdot x^{\alpha-1} \cdot e^{-\theta \cdot x^{\alpha}} dx$$

$Q = ▮$

$$Var := Q - (E)^2$$

$Var = ▮$

$$MSE := Var + (Bias)^2$$

$MSE = ▮$

$$W := \int_0^{\infty} (g)^2 \cdot \alpha \cdot \theta \cdot x^{\alpha-1} \cdot e^{-\theta \cdot x^{\alpha}} dx$$

$W = ▮$

$$Variance := W - (D)^2$$

$Variance = ▮$

$$mse := Variance + (Bias1)^2$$

$mse = ▮$

Program 2: Estimation by Moment Method

Enter your values of α, θ and n

$\alpha := ▮$ $\theta := ▮$ $n := ▮$

```
x := | for j ∈ 0.. n − 1
     |   | for i ∈ 0.. n − 1
     |   |   | u1 ← rnd(1)
     |   |   | u2 ← rnd(1)
     |   |   | b1_{i,j} ← √(−2·ln(u1))·cos(2π·u2)
     |   |   | b2_{i,j} ← √(−2·ln(u1))·sin(2π·u2)
     |   |   | z1 ← √θ·b1_{i,j}
     |   |   | z2 ← √θ·b2_{i,j}
     |   |   | r_{i,j} ← [((z1)^2 + (z2)^2)/2]^(1/α)
     |   | r
     | r
```

$$b1_{i,j} \leftarrow \sqrt{-2\cdot\ln(u1)}\cdot\cos(2\pi\cdot u2)$$

$$b2_{i,j} \leftarrow \sqrt{-2\cdot\ln(u1)}\cdot\sin(2\pi\cdot u2)$$

$$r_{i,j} \leftarrow \left[\frac{(z1)^2 + (z2)^2}{2}\right]^{\frac{1}{\alpha}}$$

$j := 0..n-1$

$X_j := \frac{1}{n} \cdot \sum_{i=0}^{n-1} x_{i,j}$

$B_j := \frac{1}{n-1} \cdot \left[\sum_{i=0}^{n-1} \left(x_{i,j}\right)^2 - n \cdot \left(X_j\right)^2 \right]$

$i := 0..n-1$

$\beta_i := 0.5$

$\beta_{i+1} := \beta_i + 0.05$

$\left(cvd_j\right) := \frac{B_j}{\left(X_j\right)^2}$

$\phi_{i,j} := \frac{\sqrt{\Gamma\left(1 + \frac{2}{x_{i,j}}\right) - \left(\Gamma\left(1 + \frac{1}{x_{i,j}}\right)\right)^2}}{\Gamma\left(1 + \frac{1}{x_{i,j}}\right)}$

$p_j :=$ for $i \in 0..n-1$

$\quad \phi_{i,j}$ if $\left(\left|cvd_j - \phi_{i,j}\right|\right) \le 1$

p_i

▮

$w := \frac{1}{n} \cdot \sum_{i=0}^{n-1} p_i$

$w = ▮$

$\lambda_j := \left[\frac{\Gamma\left[1 + \left(p_j\right)\right]}{X_j} \right]^{\frac{1}{\left(p_j\right)}}$

$\lambda = ▮$

$s := \frac{1}{n} \cdot \sum_{i=0}^{n-1} \lambda_i$

$s = ▮$

$E := \int_0^{\infty} s \cdot \alpha \cdot \theta \cdot x^{\alpha-1} \cdot e^{-\theta \cdot x^{\alpha}} \, dx$

$O := \int_0^{\infty} w \cdot \alpha \cdot \theta \cdot x^{\alpha-1} \cdot e^{-\theta \cdot x^{\alpha}} \, dx$

$E = ▮$

$o := ▮$

$Bias := E - \theta$

$Bias = ▮$

$bias := O - \alpha$

$bias := ▮$

$S := \int_0^{\infty} s^2 \cdot \alpha \cdot \theta \cdot x^{\alpha-1} \cdot e^{-\theta \cdot x^{\alpha}} \, dx$

$K := \int_0^{\infty} w^2 \cdot \alpha \cdot \theta \cdot x^{\alpha-1} \cdot e^{-\theta \cdot x^{\alpha}} \, dx$

$S =$ ▮

$K =$

$Variance := K - O^2$

$Variance =$

$Var := S - E^2$

$Var =$ ▮

$mse := Variance + (bias)^2$

$mse :=$ ▮

$MSE := Var + (Bias)^2$

$MSE =$ ▮

Program 3:Estimation by Modified Moment Method

Enter your values of α, θ and n

$\alpha :=$ ▮ $\theta :=$ ▮ $n :=$ ▮

$$
x := \left|
\begin{array}{l}
\text{for } j \in 0..n-1 \\
\quad \left|
\begin{array}{l}
\text{for } i \in 0..n-1 \\
\quad \left|
\begin{array}{l}
u1 \leftarrow rnd(1) \\
u2 \leftarrow rnd(1) \\
b1_{i,j} \leftarrow \sqrt{-2 \cdot \ln(u1)} \cdot \cos(2\pi \cdot u2) \\
b2_{i,j} \leftarrow \sqrt{-2 \cdot \ln(u1)} \cdot \sin(2\pi \cdot u2) \\
z1 \leftarrow \sqrt{\theta} \cdot b1_{i,j} \\
z2 \leftarrow \sqrt{\theta} \cdot b2_{i,j} \\
r_{i,j} \leftarrow \left[\frac{(z1)^2 + (z2)^2}{2} \right]^{\frac{1}{\alpha}}
\end{array}
\right. \\
r
\end{array}
\right. \\
r
\end{array}
\right.
$$

$i := 0..n-1$ $j := 0..n-1$

$y := \min(x)$

$$X_j := \frac{1}{n} \cdot \sum_{i=0}^{n-1} x_{i,j}$$

$$\alpha 1 := \frac{\ln\left(\frac{1}{n}\right)}{\ln\left(\frac{Y_1}{X}\right)}$$

$$a := \frac{1}{n} \cdot \sum_{i=0}^{n-1} \alpha 1_i$$

$$a = ▮$$

$$\theta 1 := \left[\frac{\Gamma\left(1 + \frac{1}{a}\right) \cdot \left[1 - \frac{1}{(n)^{\frac{1}{a}}}\right]}{(X - y)}\right]^a$$

$$k := \frac{1}{n} \cdot \sum_{i=0}^{n-1} \theta 1_i$$

$$k = ▮$$

$$E := \int_0^\infty k \cdot \left[\alpha \cdot \theta \cdot (x)^{\alpha - 1} \cdot e^{-\theta \cdot x^\alpha}\right] dx$$

$$E = ▮$$

$$bais := E - \theta$$

$$bais = ▮$$

$$L := \int_0^\infty (k)^2 \cdot \left[\alpha \cdot \theta \cdot (x)^{\alpha - 1} \cdot e^{-\theta \cdot x^\alpha}\right] dx$$

$$L = ▮$$

$$Var := L - E^2$$

$$Var = ▮$$

$$MSE := Var + (bais)^2$$

$$MSE = ▮$$

$$t := \int_0^\infty a \cdot \left(\alpha \cdot \theta \cdot x^{\alpha - 1} \cdot e^{-\theta \cdot x^\alpha}\right) dx$$

$$t = ▮$$

$$BIAS := t - \alpha$$

$$w := \int_0^\infty a^2 \cdot \left(\alpha \cdot \theta \cdot x^{\alpha - 1} \cdot e^{-\theta \cdot x^\alpha}\right) dx$$

$$w = ▮$$

$$Variance := w - t^2$$

$$Variance = ▮$$

$$mse := Variance + (BIAS)^2$$

$$mse = ▮$$

Program 4: Estimation by Least Square Method

Enter your vales of α, θ and n

$\alpha :=$ ▮ $\theta :=$ ▮ $n :=$ ▮

$$x := \left| \begin{array}{l} \text{for } j \in 0..n-1 \\ \quad \left| \begin{array}{l} \text{for } i \in 0..n-1 \\ \quad \left| \begin{array}{l} u1 \leftarrow rnd(1) \\ u2 \leftarrow rnd(1) \\ b1_{i,j} \leftarrow \sqrt{-2\cdot\ln(u1)}\cdot\cos(2\pi\cdot u2) \\ b2_{i,j} \leftarrow \sqrt{-2\cdot\ln(u1)}\cdot\sin(2\pi\cdot u2) \\ z1 \leftarrow \sqrt{\theta}\cdot b1_{i,j} \\ z2 \leftarrow \sqrt{\theta}\cdot b2_{i,j} \\ r_{i,j} \leftarrow \left[\frac{(z1)^2 + (z2)^2}{2}\right]^{\frac{1}{\alpha}} \end{array} \right. \\ r \end{array} \right. \\ r \end{array} \right.$$

$i := 0..n-1$ $j := 0..n-1$

$u := runif(n, 0, 1)$ $t := -\ln(-\ln(u))$

$$Y_j := \frac{1}{n}\cdot\left(\sum_{i=0}^{n-1} \ln(x_{i,j})\right)$$

$$T_j := \frac{1}{n}\cdot\left(\sum_{i=0}^{n-1} t_i\right)$$

$$\alpha1_j := -\left[\frac{T_j\cdot\sum_{i=0}^{n-1} t_i - \sum_{i=0}^{n-1} (t_i)^2}{(T_j)\cdot\sum_{i=0}^{n-1} \left(t_i\cdot\ln(x_{i,j})\right) - (Y_j)\cdot\sum_{i=0}^{n-1} (t_i)^2}\right]$$

$$a := \frac{1}{n} \cdot \sum_{j=0}^{n-1} \alpha 1_j$$

$a = ▮$

$$\theta 1_j := e^{-\left[\frac{(Y_j)\cdot \sum_{i=0}^{n-1} (t_i) - \sum_{i=0}^{n-1} (\ln(x_{i,j})\cdot t_i)}{T_j \cdot \sum_{i=0}^{n-1} t_i - \sum_{i=0}^{n-1} (t_i)^2}\right]\cdot \alpha 1_j}$$

$$b := \frac{1}{n} \cdot \sum_{j=0}^{n-1} \theta 1_j$$

$b = ▮$

$$E := \int_0^{\infty} b\cdot\alpha\cdot\theta\cdot x^{\alpha-1}\cdot e^{-\theta\cdot x^{\alpha}}\, dx$$

$E = ▮$

$Bias := E - \theta$

$$W := \int_0^{\infty} b^2\cdot\alpha\cdot\theta\cdot x^{\alpha-1}\cdot e^{-\theta\cdot x^{\alpha}}\, dx$$

$W = ▮$

$Bias = ▮$

$Var := W - E^2$

$Var = ▮$

$MSE := Var + (Bias)^2$

$MSE = ▮$

$$F := \int_0^{\infty} a\cdot\left[\alpha\cdot\theta\cdot(x)^{\alpha-1}\cdot e^{-\theta\cdot x^{\alpha}}\right] dx$$

$F = ▮$

$Bais := F - \alpha$

$Bais = ▮$

$$O := \int_0^{\infty} (a)^2\cdot\left[\alpha\cdot\theta\cdot(x)^{\alpha-1}\cdot e^{-\theta\cdot x^{\alpha}}\right] dx$$

$O = ▮$

$Variance := O - F^2$

$Variance = ▮$

$mse := Variance + (Bais)^2$

$mse = ▮$

Appendix B
Computer Programs for Generating Random Variates of Weibull Distribution

Program 5: procedure (W-1)

Enter your values of α, θ and n

$\alpha :=$ ▮ $\theta :=$ ▮ $n :=$ ▮

```
x := | for j ∈ 0..n − 1
     |   | for i ∈ 0..n − 1
     |   |   | u ← rnd(1)
     |   |   | b[i,j] ← [ ( (−1/θ)·ln(u) )^(1/α) ]
     |   | b
     | b
```

$$x := \left|\begin{array}{l} \text{for } j \in 0..n-1 \\ \quad \left|\begin{array}{l} \text{for } i \in 0..n-1 \\ \quad \left|\begin{array}{l} u \leftarrow \text{rnd}(1) \\ b_{i,j} \leftarrow \left[\left(\frac{-1}{\theta}\cdot\ln(u)\right)^{\frac{1}{\alpha}}\right] \end{array}\right. \\ b \end{array}\right. \\ b \end{array}\right.$$

$x :=$ ▮

Program 6: Procedure (W-2)

Enter your values of α, θ and n

$$x := \left| \begin{array}{l} \text{for } j \in 0..n-1 \\ \quad \left| \begin{array}{l} \text{for } i \in 0..n-1 \\ \quad \left| \begin{array}{l} u \leftarrow \text{rnd}(1) \\ b_{i,j} \leftarrow -\ln(u) \\ y_{i,j} \leftarrow \left[\frac{1}{\sqrt{\theta}} \cdot e^{-\left(b_{i,j}\right)} \right] \end{array} \right. \\ y \end{array} \right. \\ y \end{array} \right.$$

$x = \blacksquare$

Program 7: Procedure (W-3)

Enter your values of α, θ and n

$n := \blacksquare$ $\theta := \blacksquare$ $\alpha := \blacksquare$

$j := 0..n-1$

$i := 0..n-1$

$$x_{i,j} := \left| \begin{array}{l} u1 \leftarrow \text{rnd}(1) \\ u2 \leftarrow \text{rnd}(1) \\ y \leftarrow \tan\left[\pi\left(u2 - \frac{1}{2}\right)\right] \\ \text{while } u1 > \frac{1}{2}\cdot\left(1 + y^2\right)\cdot e^{\frac{-y^2}{2}} \\ \quad \left| \begin{array}{l} u1 \leftarrow \text{rnd}(1) \\ u2 \leftarrow \text{rnd}(1) \\ y \leftarrow \tan\left[\pi\left(u2 - \frac{1}{2}\right)\right] \end{array} \right. \\ y \end{array} \right.$$

$$R := \left| \begin{array}{l} \text{for } j \in 0..n-1 \\ \quad \text{for } t \in 0..n-1 \\ \quad\quad \left| \begin{array}{l} Z1_{t,j} \leftarrow \sqrt{\theta}\cdot x_{t,j} \\ Z2_{t,j} \leftarrow \sqrt{\theta}\cdot x_{t,j} \\ r_{t,j} \leftarrow \left[\frac{\left(Z1_{t,j}\right)^2 + \left(Z2_{t,j}\right)^2}{2}\right]^{\frac{1}{\alpha}} \\ r \end{array} \right. \\ r \end{array} \right.$$

$x =$ ▮

Program 8: Procedure (W-4)

Enter your values of α, θ and n

$\theta :=$ ▮ $\alpha :=$ ▮ $n :=$ ▮

$j := 0..n-1$

$i := 0..n-1$ $k := \sqrt{\frac{8}{\pi}}$

$$x_{i,j} := \left| \begin{array}{l} u1 \leftarrow rnd(1) \\ u2 \leftarrow rnd(1) \\ y \leftarrow \dfrac{-\ln\left(\tan\left(\frac{\pi}{4}\cdot u2\right)\right)}{k} \\ \text{while } u1 > \dfrac{e^{-\left(\frac{y^2}{2}\right)}\cdot\left(1+e^{-2\cdot k\cdot y}\right)}{2\cdot k\cdot e^{-k}} \\ \quad \left| \begin{array}{l} u1 \leftarrow rnd(1) \\ u2 \leftarrow rnd(1) \\ y \leftarrow \tan\left[\pi\cdot\left(u2-\frac{1}{2}\right)\right] \end{array} \right. \\ u3 \leftarrow rnd(1) \\ -y \ \text{ if } u3 < \frac{1}{2} \\ y \ \text{ if } u3 > \frac{1}{2} \end{array} \right.$$

$$o_{i,j} := \left| \begin{array}{l} u1 \leftarrow rnd(1) \\ u2 \leftarrow rnd(1) \\ y \leftarrow \dfrac{-\ln\left(\tan\left(\frac{\pi}{4}\cdot u2\right)\right)}{k} \\ \text{while } u1 > \dfrac{e^{-\left(\frac{y^2}{2}\right)}\cdot\left(1+e^{-2\cdot k\cdot y}\right)}{2\cdot k\cdot e^{-k}} \\ \quad \left| \begin{array}{l} u1 \leftarrow rnd(1) \\ u2 \leftarrow rnd(1) \\ y \leftarrow \tan\left[\pi\cdot\left(u2-\frac{1}{2}\right)\right] \end{array} \right. \\ u3 \leftarrow rnd(1) \\ -y \ \text{ if } u3 < \frac{1}{2} \\ y \ \text{ if } u3 > \frac{1}{2} \end{array} \right.$$

$$R := \left| \begin{array}{l} \text{for } j \in 0..n-1 \\ \quad \text{for } t \in 0..n-1 \\ \quad\quad \left| \begin{array}{l} Z1_{t,j} \leftarrow \sqrt{\theta}\cdot x_{t,j} \\ Z2_{t,j} \leftarrow \sqrt{\theta}\cdot o_{t,j} \\ r_{t,j} \leftarrow \left[\dfrac{\left(Z1_{t,j}\right)^2+\left(Z2_{t,j}\right)^2}{2}\right]^{\frac{1}{\alpha}} \\ r \end{array} \right. \\ r \end{array} \right.$$

$R =$ ▮

Program 9: Procedure (W-5)

Enter your values of α, θ and n

$n :=$ ▮ $\theta :=$ ▮ $\alpha :=$ ▮

$j := 0..n-1$

$i := 0..n-1$

$$w_{i,j} := \left| \begin{array}{l} u1 \leftarrow \text{rnd}(1) \\ u2 \leftarrow \text{rnd}(1) \\ y \leftarrow -\ln(u2) \\ \text{while } u1 > e^{\frac{-(y-1)^2}{2}} \\ \quad \left| \begin{array}{l} u1 \leftarrow \text{rnd}(1) \\ u2 \leftarrow \text{rnd}(1) \\ y \leftarrow -\ln(u2) \end{array} \right. \\ u3 \leftarrow \text{rnd}(1) \\ -y \text{ if } u3 < \frac{1}{2} \\ y \text{ if } u3 > \frac{1}{2} \end{array} \right.$$

$$o_{i,j} := \left| \begin{array}{l} u1 \leftarrow \text{rnd}(1) \\ u2 \leftarrow \text{rnd}(1) \\ y \leftarrow -\ln(u2) \\ \text{while } u1 > e^{\frac{-(y-1)^2}{2}} \\ \quad \left| \begin{array}{l} u1 \leftarrow \text{rnd}(1) \\ u2 \leftarrow \text{rnd}(1) \\ y \leftarrow -\ln(u2) \end{array} \right. \\ u3 \leftarrow \text{rnd}(1) \\ -y \text{ if } u3 < \frac{1}{2} \\ y \text{ if } u3 > \frac{1}{2} \end{array} \right.$$

$$x := \left| \begin{array}{l} \text{for } j \in 0..n-1 \\ \quad \text{for } t \in 0..n-1 \\ \quad\quad \left| \begin{array}{l} Z1_{t,j} \leftarrow \sqrt{\theta}\cdot w_{t,j} \\ Z2_{t,j} \leftarrow \sqrt{\theta}\cdot o_{t,j} \\ r_{t,j} \leftarrow \left[\frac{\left(Z1_{t,j}\right)^2 + \left(Z2_{t,j}\right)^2}{2}\right]^{\frac{1}{\alpha}} \\ r \end{array} \right. \\ r \end{array} \right.$$

$x =$ ▮

Program 10: Procedure (W-6)

Enter your values of α, θ and n

$\theta :=$ ▮ $\alpha :=$ ▮ $n :=$ ▮ $m :=$ ▮

$i := 0 .. n - 1$

$$
x := \left|
\begin{array}{l}
\text{for } t \in 0 .. n - 1 \\
\quad \left|
\begin{array}{l}
\text{for } i \in 0 .. n - 1 \\
\quad \left|
\begin{array}{l}
\text{for } j \in 0 .. 1 \\
\quad \left|
\begin{array}{l}
U \leftarrow \text{runif}(m, 0, 1) \\
c \leftarrow \frac{1}{m} \cdot \sum_{i=0}^{m-1} U_i \\
b_j \leftarrow \sqrt{12 \cdot m} \cdot \left(c - \frac{1}{2} \right)
\end{array}
\right. \\
b \\
z1 \leftarrow \sqrt{\theta} \cdot b_0 \\
z2 \leftarrow \sqrt{\theta} \cdot b_1 \\
r_{i,t} \leftarrow \left[\frac{(z1)^2 + (z2)^2}{2} \right]^{\frac{1}{\alpha}}
\end{array}
\right. \\
r
\end{array}
\right. \\
r
\end{array}
\right.
$$

$x =$ ▮

Chapter One

1

On Weibull Distribution

1.1 Introduction

The aim of this chapter is to find the estimators of parameters for Weibull distn. by using four methods of estimation and we use Monte Carlo simulation to generate sample from the Weibull distn. by using Box and Muller Method, and comparison between these estimators are made to recognize the best estimator from bias, variance and mean square error and display in tables (1.1) to (1.7).

In this chapter (section 1.2) we introduce some important mathematical and statistical properties of Weibull distribution, in section (1.3) The derivation of the distn. is made by using two different approaches where one approach utilize Normal distn., and the second approach utilize Extreme value distn. .

In section (1.4) Moments properties of the distn. are illustrated and unified, section (1.5) some important definitions, theorems about the estimator are illustrated and four method of estimation, namely, Maximum Likelihood Method, Moments Method, Modified Moments Method and Least Squares Method, are discussed theoretical and assessed practically.

In practice (section 1.6), we use Monte Carlo simulation to generate sample from the Weibull distn. and the estimation of parameters is made by these methods. Statistical properties of the estimators are displayed in tables (1.1) to (1.7).

1.2 Some Mathematical and Statistical Properties of Weibull Distribution

Definition (1.1) [18]:

A continuous r.v. X is said to have a Weibull distn. with parameters α and θ, denoted by $W\left(\alpha,\theta\right)$, if X has the following p.d.f.

$$f\left(x;\alpha,\theta\right)=\alpha\theta x^{\alpha-1}e^{-\theta x^{\alpha}},0<x<\infty \qquad \text{.................(1.1)}$$
$$=0 \qquad ,e.w. \quad ;where\ \alpha,\theta>0.$$

To verify that eq.(1.1) is valid p.d.f., we note that $f\left(x\right)>0,\forall x\in\left(0,\infty\right)$ and the integral $\int_0^{\infty} f\left(x;\alpha,\theta\right)dx$ is unity. Viz

Let $I=\int_0^{\infty} f\left(x;\alpha,\theta\right)dx=\int_0^{\infty}\alpha\theta x^{\alpha-1}e^{-\theta x^{\alpha}}dx$

Set $y=\theta x^{\alpha}$ implies $dy=\alpha\theta x^{\alpha-1}dx$, then $I=\int_0^{\infty} e^{-y}dy=1.$

We note that the Weibull distn. reduces to the Exponential distn. as a special case when $\alpha=1$, and it reduce to Rayleigh distn. when $\alpha=2$, and similar to Normal curve when ($3\leq\alpha\leq4$) [6].

The Weibull distn. depends on two parameters α and θ which are referred to as shape and scale parameters respectively. The variety of p.d.f. shapes can be generated by fixing the values of α once and letting θ vary and fixing the values of θ and letting α vary. The professional MATHCAD, 2005 computer software is used to give a graphical representation of Weibull p.d.f.'s. Figure (1.1) and Figure (1.2) show respectively some Weibull p.d.f.'s for fixed θ with α varying and for fixed α with θ varying as follows:

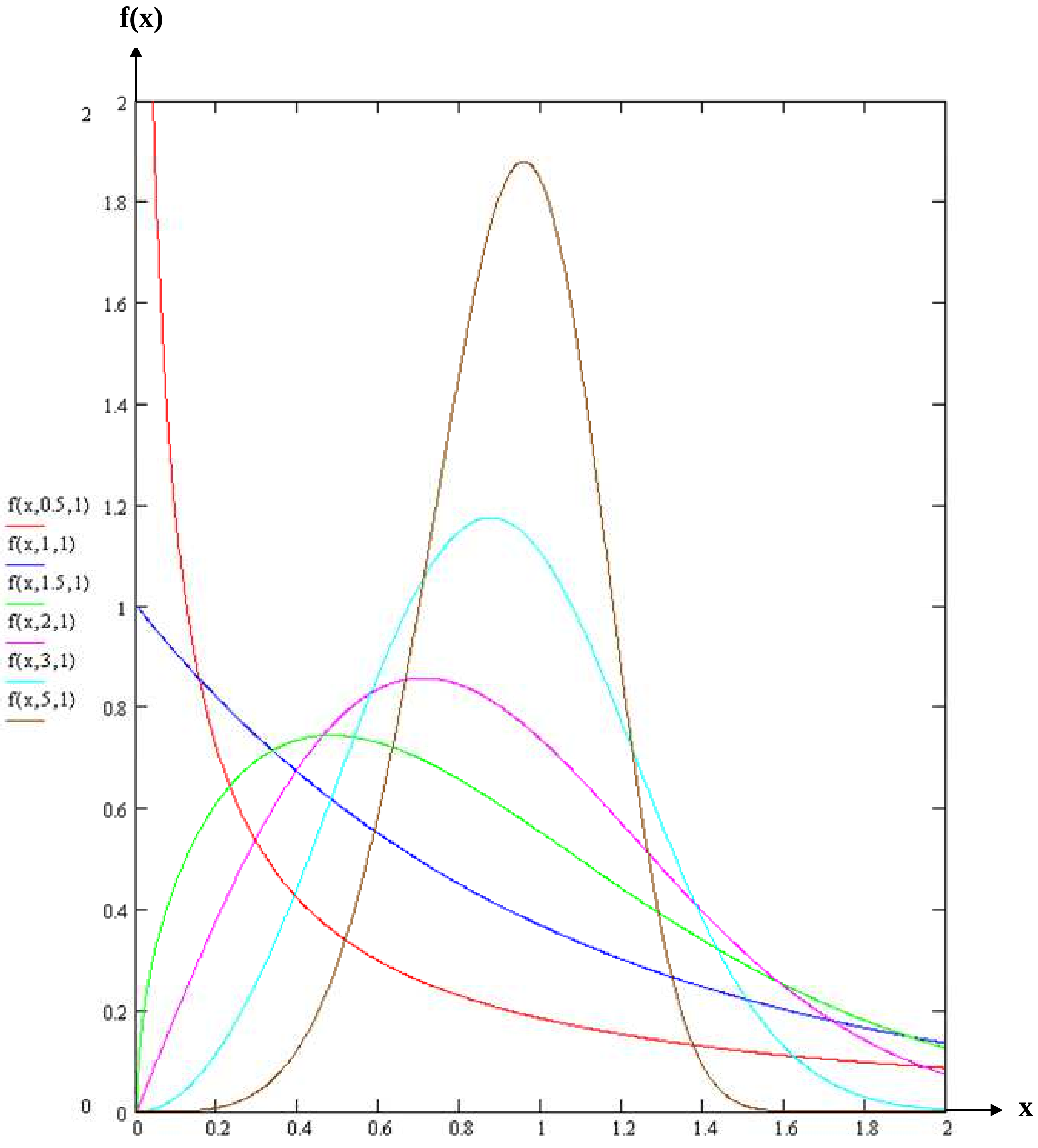

Figure (1.1)
Weibull p.d.f.'s with θ=1 and α=0.5,1,1.5,2,3,5

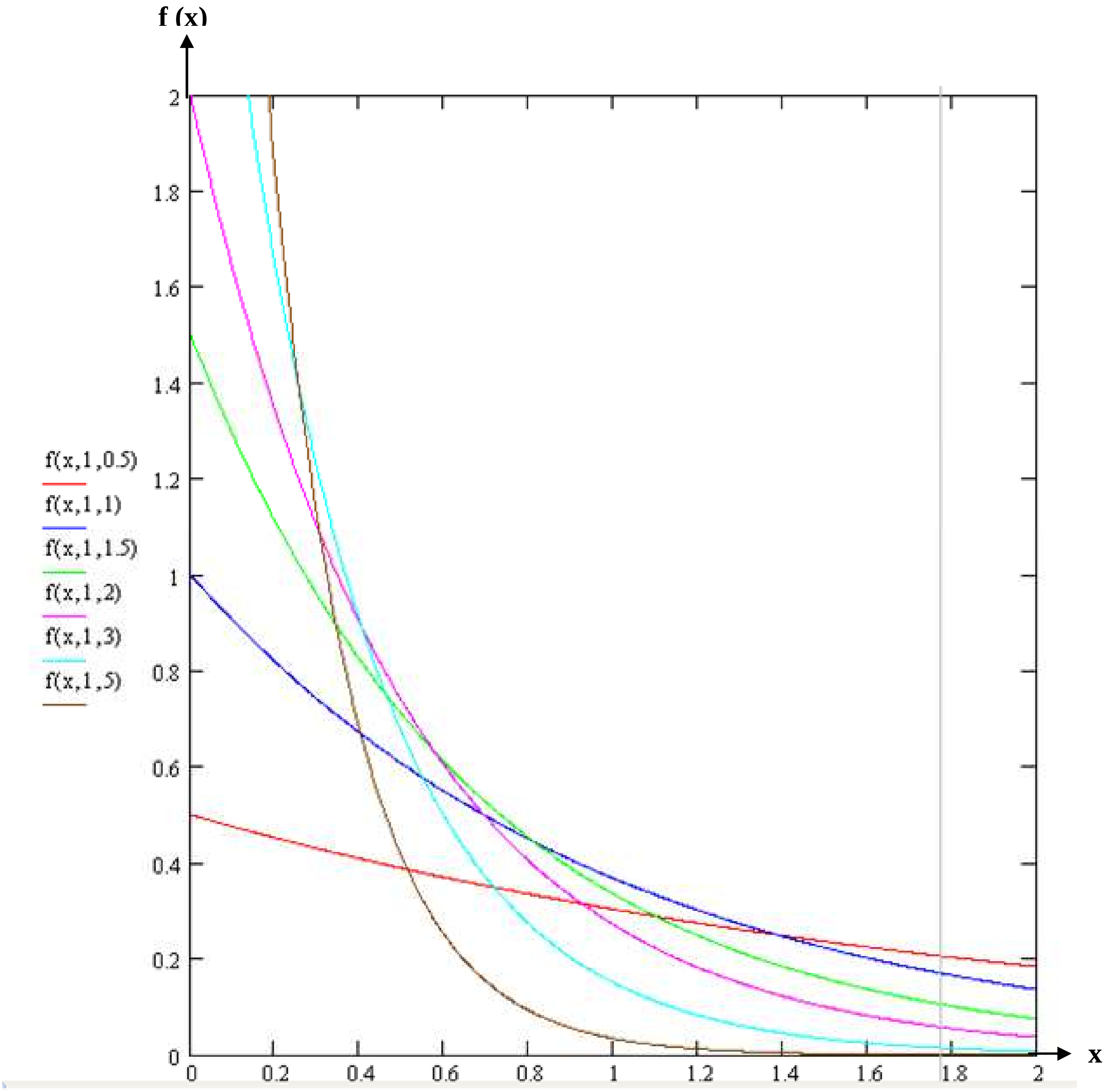

Figure (1.2)
Weibull p.d.f.'s with $\alpha=1$ and $\theta=0.5,1,1.5,2,3,5$

In general we note that the Weibull distn. have the following properties :

1- Have the x-axis as a horizontal asymptote.

2- Increasing for $0 < x < \left(\frac{\alpha - 1}{\alpha\theta}\right)^{\frac{1}{\alpha}}$ and decreasing for $\left(\frac{\alpha - 1}{\alpha\theta}\right)^{\frac{1}{\alpha}} < x < \infty$.

3- Have a maximum point at $x = \left(\frac{\alpha - 1}{\alpha\theta}\right)^{\frac{1}{\alpha}}$.

4- Have two inflection points at $x = \left(\frac{3\alpha + \sqrt{5\alpha^2 - 6\alpha + 1} - 3}{2\alpha\theta}\right)^{\frac{1}{\alpha}}$ and

$$x = \left(\frac{3\alpha - \sqrt{5\alpha^2 - 6\alpha + 1} - 3}{2\alpha\theta}\right)^{\frac{1}{\alpha}}.$$

5- Concave up for $0 < x < \left(\frac{3\alpha - \sqrt{5\alpha^2 - 6\alpha + 1} - 3}{2\alpha\theta}\right)^{\frac{1}{\alpha}}$ and

$\left(\frac{3\alpha + \sqrt{5\alpha^2 - 6\alpha + 1} - 3}{2\alpha\theta}\right)^{\frac{1}{\alpha}} < x < \infty$ and Concave down for

$$\left(\frac{3\alpha - \sqrt{5\alpha^2 - 6\alpha + 1} - 3}{2\alpha\theta}\right)^{\frac{1}{\alpha}} < x < \left(\frac{3\alpha + \sqrt{5\alpha^2 - 6\alpha + 1} - 3}{2\alpha\theta}\right)^{\frac{1}{\alpha}}.$$

6- The distn. is limited to the left and unlimited to the right, it is never symmetric, but may appear symmetric for certain values of α.

7- The p.d.f is a bell shape for $\alpha > 1$ and a J shape for $0 < \alpha \le 1$.

1.2.1 The Cumulative Distribution Function

The Weibull c.d.f. is defined as

$$F(x) = \int_{-\infty}^{x} f\,(t;\alpha,\theta)\,dt = \int_{0}^{x} \alpha\theta t^{\alpha-1} e^{-\theta t^{\alpha}}\,dt \text{ implies}$$

$$F(x) = \Pr(X \le x) = \begin{cases} 0, x < 0 \\ 1-e^{-\theta x^{\alpha}}, 0 < x < \infty \\ 1, x \to \infty \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1.2)$$

1.3 Derivation of the Weibull Distribution

There are many ways in which one expected the Weibull distn. can arise to give a useful description of observed variation.

1.3.1 Genesis Derivation by Utilizing Normal Distribution

A new derivation is made by extending the original idea as stated in the following context. Suppose we are trying to locate an object in plane and we determine its distance from the origin by measuring the distance along the x and y axes and applying the Phythagoras formula $r^2 = x^2 + y^2$. Suppose the measurements are subjecting to random errors with X and Y representing the errors in the measurement. The errors are assumed to be independent and normally distributed with constant variance [8]. We can develop the derivation of Rayleigh distn. [20] to obtaine the r.v. $R = \left(\frac{X^2+Y^2}{2}\right)^{\frac{1}{\alpha}}$ has Weibull distn. .

Theorem (1.1):

The idea of this derivation is developed from Rayleigh distn. [20]. If X and Y are two independent r.v.'s $\sim N(0,\frac{1}{\sqrt{\theta}})$. Then the r.v.

$$R=(\frac{X^2+Y^2}{2})^{\frac{1}{\alpha}}\sim W(\alpha,\theta) \quad ;\alpha>0,\theta>0.$$

Proof:

The joint p.d.f. of r.v.'s of X and Y is:

$$f(x,y)=\frac{\theta}{2\pi}e^{-\frac{\theta}{2}(x^2+y^2)},\ -\infty<x<\infty,-\infty<y<\infty$$

With transformation $R=(\frac{X^2+Y^2}{2})^{\frac{1}{\alpha}}$, set $W=\frac{Y}{X}$.

The functions $r=(\frac{x^2+y^2}{2})^{\frac{1}{\alpha}}$, set $w=\frac{y}{x}$ does not define $(1-1)$ transformation that maps the space

$A=\{(x,y):-\infty<x<\infty,-\infty<y<\infty\}$ onto the space

$B=\{(r,w):0<r<\infty,-\infty<w<\infty\}$. We write the space A as a union of two disjoint subset say, $A_1=\{(x,y):-\infty<x<0,-\infty<y<0\}$,

$A_2=\{(x,y):0<x<\infty,0<y<\infty\}$ where $A=A_1\cup A_2$.

Now, the functions $r=(\frac{x^2+y^2}{2})^{\frac{1}{\alpha}}$, $w=\frac{y}{x}$ defined $(1-1)$ transformation that maps each of A_1 and A_2 onto B the with inverse transforms:

$$x=\pm\frac{\sqrt{2r^{\frac{\alpha}{2}}}}{\sqrt{1+w^2}},\ y=\pm\frac{\sqrt{2r^{\frac{\alpha}{2}}w}}{\sqrt{1+w^2}}$$

In A_1

$$x = -\frac{\sqrt{2}r^{\frac{\alpha}{2}}}{\sqrt{1+w^2}}, \; y = -\frac{\sqrt{2}r^{\frac{\alpha}{2}}w}{\sqrt{1+w^2}}$$

$$J_1 = \frac{\partial(x,y)}{\partial(r,w)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} -\frac{\alpha r^{\frac{\alpha}{2}-1}}{\sqrt{2}\sqrt{1+w^2}} & \frac{\sqrt{2}r^{\frac{\alpha}{2}}w}{(1+w^2)^{\frac{3}{2}}} \\ -\frac{\alpha r^{\frac{\alpha}{2}-1}w}{\sqrt{2}\sqrt{1+w^2}} & -\frac{\sqrt{2}r^{\frac{\alpha}{2}}}{(1+w^2)^{\frac{3}{2}}} \end{vmatrix}$$

$$= \frac{\alpha r^{\alpha-1}}{(1+w^2)^2} + \frac{\alpha r^{\alpha-1}w^2}{(1+w^2)^2} = \frac{\alpha r^{\alpha-1}}{(1+w^2)^2}(1+w^2) = \frac{\alpha r^{\alpha-1}}{(1+w^2)}$$

In A_2

$$x = +\frac{\sqrt{2}r^{\frac{\alpha}{2}}}{\sqrt{1+w^2}}, \; y = +\frac{\sqrt{2}r^{\frac{\alpha}{2}}w}{\sqrt{1+w^2}}$$

Then, the joint p.d.f. of R and W is:

$$J_2 = \frac{\partial(x,y)}{\partial(r,w)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{\alpha r^{\frac{\alpha}{2}-1}}{\sqrt{2}\sqrt{1+w^2}} & -\frac{\sqrt{2}r^{\frac{\alpha}{2}}w}{(1+w^2)^{\frac{3}{2}}} \\ \frac{\alpha r^{\frac{\alpha}{2}-1}w}{\sqrt{2}\sqrt{1+w^2}} & \frac{\sqrt{2}r^{\frac{\alpha}{2}}}{(1+w^2)^{\frac{3}{2}}} \end{vmatrix}$$

$$= \frac{\alpha r^{\alpha-1}}{(1+w^2)^2} + \frac{\alpha r^{\alpha-1}w^2}{(1+w^2)^2} = \frac{\alpha r^{\alpha-1}}{(1+w^2)^2}(1+w^2) = \frac{\alpha r^{\alpha-1}}{(1+w^2)}$$

$$g(r,w)=f\left(\frac{-\sqrt{2}r^{\frac{\alpha}{2}}}{\sqrt{1+w^2}},\frac{-\sqrt{2}r^{\frac{\alpha}{2}}w}{\sqrt{1+w^2}}\right)|J_1|+f\left(\frac{\sqrt{2}r^{\frac{\alpha}{2}}}{\sqrt{1+w^2}},\frac{\sqrt{2}r^{\frac{\alpha}{2}}w}{\sqrt{1+w^2}}\right)|J_2|$$

$$=\frac{\theta}{2\pi}e^{-\frac{\theta}{2}2r^{\alpha}}\frac{\alpha r^{\alpha-1}}{1+w^2}+\frac{\theta}{2\pi}e^{-\frac{\theta}{2}2r^{\alpha}}\frac{\alpha r^{\alpha-1}}{1+w^2}$$

$$=\frac{\alpha\theta}{\pi(1+w^2)}r^{\alpha-1}e^{-\theta r^{\alpha}} \quad ;0<r<\infty,-\infty<w<\infty$$

The marginal p.d.f. of R is:

$$g_1(r)=\int_w g(r,w)dw=\alpha\theta r^{\alpha-1}e^{-\theta r^{\alpha}}\int_{-\infty}^{\infty}\frac{dw}{\pi(1+w^2)}$$

Since, $\int_{-\infty}^{\infty}\frac{dw}{\pi(1+w^2)}=1$, which is C(0,1).

$$\Rightarrow g_1(r)=\alpha\theta r^{\alpha-1}e^{-\alpha x^{\theta}} \quad ,0<r<\infty \quad ,\alpha>0,\theta>0$$
$$=0 \quad ,e.w.$$

Which is the p.d.f. of Weibull distn. as given by eq.(1.1) .

Q.D.E.

1.3.2 Derivation by Utilizing Extreme Value Distribution

This derivation to Weibull distn. can arise by using the Extreme Value distn. . The details is given by theorem (1.2).

Theorem (1.2) [23]:

Let $X \sim Ext(\delta,\lambda)$ then the p.d.f. of X is:

$f(x)=\frac{1}{\lambda}\exp\left[-\left(\frac{x-\delta}{\lambda}\right)-e^{-\left(\frac{x-\delta}{\lambda}\right)}\right],-\infty<x<\infty$. If we rewrite *f(x)*

$f(x)=\alpha\theta e^{-\alpha x}e^{-\theta e^{-\alpha x}}, -\infty < x < \infty$. Where $\alpha=\frac{1}{\lambda}$ and $\theta=e^{\frac{\delta}{\lambda}}$, then the r.v.

$Y=e^{-X} \sim W(\alpha,\theta)$.

Proof:

The function $y=e^{-x}$ define a $(1-1)$ transformation that maps the space $A=\{x:-\infty<x<\infty\}$ onto the space $B=\{y:0<y<\infty\}$ with inverse transform $x=-\ln(y)$ and the Jacobin transform is $|J|=\left|\frac{dx}{dy}\right|=\left|\frac{-1}{y}\right|=\frac{1}{y}$

Thus, the p.d.f. of Y is:

$$g(y)=f(-\ln y)|J|$$

implies $g(y)=\alpha\theta y^{\alpha}e^{-\theta y^{\alpha}}\frac{1}{y}$

We have

$$g(y)=\alpha\theta y^{\alpha-1}e^{-\theta y^{\alpha}},\ 0<y<\infty$$
$$=0 \qquad ,e.w.$$

Which is the p.d.f. of Weibull distn. as given by eq.(1.1) .

Q.D.E.

1.4 Moments and Higher Moments Properties of Weibull Distribution [18]

Moments are set of constants used for measuring a distn. properties and under certain circumstance they specify the distn. . The moments of r.v. X (or distn.) are defined in terms of the mathematical expectation of certain power of X when they exist. For instance $\mu_r'=E(X^r)$ is called the

r^{th} moment of X about the origin and $\mu_r = E\left[(X-\mu)^r\right]$ is called the r^{th} central moments of X . That is

$$\mu'_r = E(X^r) = \begin{cases} \sum_x x^r f(x) & ,x \text{ is discrete } r.v. \\ \int_x x^r f(x)dx & ,x \text{ is continuous } r.v. \end{cases}$$

and

$$\mu_r = E\left[(X-\mu)^r\right] = \begin{cases} \sum_x (x-\mu)^r f(x) & ,x \text{ is discrete } r.v. \\ \int_x (x-\mu)^r f(x)dx & ,x \text{ is continuous } r.v. \end{cases}$$

Sometimes they are defining the distn.'s, and also have a particular usefulness in connection with sums of independent r.v.'s .

The moment generating function of Weibull distn. does not have an implicit form, so it is more convenient to find the moments of Weibull distn. by using direct expectation approach.

The r^{th} moment $\mu'_r = E(X^r)$ of the distn. about the origin is

$$\mu'_r = E(X^r) = \int_0^\infty x^r f(x;\alpha,\theta)dx$$

$$= \int_0^\infty x^r \alpha\theta x^{\alpha-1} e^{-\theta x^\alpha} dx$$

Set $u = \theta x^\alpha \Rightarrow x = \left(\frac{u}{\theta}\right)^{\frac{1}{\alpha}} \Rightarrow dx = \frac{1}{\alpha\theta}\left(\frac{u}{\theta}\right)^{\frac{1}{\alpha}-1} du$ implies

$$\mu'_r = \frac{1}{\theta^{\frac{r}{\alpha}}}\int_0^\infty u^{(1+\frac{r}{\alpha})-1} e^{-u} du \qquad \text{……………………………..…………………(1.4)}$$

But,

$\int_{0}^{\infty} u^{(1+\frac{r}{\alpha})-1} e^{-u} du$ is known as incomplete gamma function and equal to $\Gamma(1+\frac{r}{\alpha})$.

Therefore;

$$\mu_r' = E(X^r) = \theta^{\frac{-r}{\alpha}} \Gamma(1+\frac{r}{\alpha}) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.5)$$

1.4.1 Mean and Variance:

The mean and variance are respectively obtained from eq.(1.5) by setting $r = 1, 2$.

(i) Mean:

$E(X) = \mu = \mu_1'$ is called the mean of r.v. X (or distn.). It is a measure of central tendency.

$$\mu = \frac{1}{\theta^{\frac{1}{\alpha}}} \Gamma(1+\frac{1}{\alpha}) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.6)$$

(ii) Variance:

$Var(X) = \delta^2 = E\left[(X-\mu)^2\right] = E\left(X^2\right) - \mu^2$ is called the variance of r.v. X (or distn.). It is a measure of dispersion, where

$$\mu_2' = E(X^2) = \frac{1}{\theta^{\frac{2}{\alpha}}} \Gamma(1+\frac{2}{\alpha})$$

Hence,

$$Var(X) = \delta^2 = \frac{1}{\theta^{\frac{2}{\alpha}}}\Gamma(1+\frac{2}{\alpha}) - \left\{\frac{1}{\theta^{\frac{1}{\alpha}}}\Gamma(1+\frac{1}{\alpha})\right\}^2 \quad \text{.........................(1.7)}$$

1.4.2 Other Moments:

(i) Mode:

A mode of a distn. is the value x of r.v. X that maximize the p.d.f. $f(x)$.

For continuous distn.'s the mode x is a solution of $\frac{df(x)}{dx} = 0$ and $\frac{d^2f(x)}{dx^2} < 0$.

The mode is measure of location. Also we note that the mode may not exist or we may have more than one mode.

For Weibull case with p.d.f.

$$f(x) = \alpha\theta x^{\alpha-1} e^{-\theta x^{\alpha}}$$

$$\Rightarrow \quad \frac{df(x)}{dx} = e^{-\theta x^{\alpha}}\left[-\left(\alpha\theta x^{\alpha-1}\right)^2 + \alpha\theta(\alpha-1)x^{\alpha-2}\right] \text{..................(1.8)}$$

Equating eq.(1.8) to zero, and solving for x, we have $-\alpha\theta x^{\alpha-1} + \alpha - 1 = 0$ which implies the critical point is

$$x = (\frac{\alpha-1}{\alpha\theta})^{\frac{1}{\alpha}} \text{...(1.9)}$$

This critical point satisfy that x is the distn. mode where condition

$\frac{d^2 f(x)}{dx^2} < 0$ at $x = (\frac{\alpha - 1}{\alpha\theta})^{\frac{1}{\alpha}}$ is hold .

(ii) Median:

A median of a distn. is defined to the value x of r.v. X such that $F(x) = \Pr(X \le x) = \frac{1}{2}$. The median is measure of location.

For Weibull case,

We equate the c.d.f. given by eq.(1.2) to $\frac{1}{2}$, that is

$$\frac{1}{2} = 1 - e^{-\theta x^{\alpha}} \qquad \text{……(1.10)}$$

Solving for x in eq.(1.10) lead to the median

$$x = \left(\frac{\ln 2}{\theta}\right)^{\frac{1}{\alpha}} \qquad \text{……(1.11)}$$

(iii) Coefficient of Variation:

The variational coefficient of r.v. X (or distn.) is defined by the ratio $\frac{\delta}{\mu}$. It is a measure of dispersion. It is independent of scale measurement and denoted by CV.

For Weibull case:

$$\text{CV} = \frac{\delta}{\mu} = \left\{ \frac{\Gamma\left(\frac{2}{\alpha} + 1\right)}{\left\{\Gamma\left(\frac{1}{\alpha} + 1\right)\right\}^2} - 1 \right\}^{\frac{1}{2}} \qquad \text{……(1.12)}$$

(iv) Coefficient of Skewness:

$\gamma_1 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}}$ is called the coefficient of Skewness. It is a measure of the departure of the frequency curve from symmetry. If $\gamma_1 = 0$, the curve is not skewed, $\gamma_1 > 0$, the curve is positively skewed, and $\gamma_1 < 0$, the curve is negatively skewed.

For Weibull case:

$E\left(X^3\right) = \frac{1}{\theta^{\frac{3}{\alpha}}}\Gamma(1+\frac{3}{\alpha})$ implies

$$\gamma_1 = \frac{\Gamma\left(1+\frac{3}{\alpha}\right) - 3\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(1+\frac{2}{\alpha}\right) + 2\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^3}{\left[\Gamma\left(1+\frac{2}{\alpha}\right) - \left\{\Gamma\left(1+\frac{1}{\alpha}\right)\right\}^2\right]^{\frac{3}{2}}} \qquad \text{…………...(1.13)}$$

(v) Coefficient of Kurtosis:

$\gamma_2 = \frac{\mu_4}{\mu_2^2} - 3$ is called the coefficient of Kurtosis . It is a measure of the departure of the degree of flatting of the frequency curve. If $\gamma_2 = 0$, the curve is not mesokurtic, $\gamma_2 > 0$, the curve is leptokurtic, and $\gamma_2 < 0$, the curve is phtykuritic .

For Weibull case:

$E\left(X^4\right) = \frac{1}{\theta^{\frac{4}{\alpha}}}\Gamma\left(1+\frac{4}{\alpha}\right)$ implies

$$\gamma_2 = \frac{\Gamma\left(1+\frac{4}{\alpha}\right) - 4\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(1+\frac{3}{\alpha}\right) + 6\Gamma\left(1+\frac{2}{\alpha}\right)\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2 + 3\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^4}{\left[\Gamma\left(1+\frac{2}{\alpha}\right) - \left\{\Gamma\left(1+\frac{1}{\alpha}\right)\right\}^2\right]^2}$$

…………(1.14)

1.5 Estimation of Parameters for Weibull Distribution:

We shall introduce in this section some definitions, methods, and theorems that are needed for parameters estimation.

1.5.1 Point Estimation [27]:

Point estimation is concerned with inference about the unknown parameters of a distn. from a sample. It provides a single value for each unknown parameter.

Point estimation admits two problems:

1^{st} developing methods of obtaining statistics whose values could be used to estimate the unknown parameters of the distn., such statistics are called point estimators .

2^{nd} selecting criteria and technique that obtain a best estimator among possible estimators.

1.5.2 Some Basic Definitions:

Definition (1.2) (Statistic) [14]:

A statistic is a function of one or more r.v.'s which is not depends upon any unknown parameters.

Definition (1.3) (Estimator) [14]:

Any statistic whose value is used to estimate the unknown parameter θ or some function of θ say $\tau(\theta)$ is called point estimator.

Definition (1.4) (Unbiased Estimator) [18]:

An estimator $\hat{\theta}=u\left(X_1,X_2,...,X_n\right)$ is defined to be an unbiased estimator of θ iff $E\left(\hat{\theta}\right)=\theta$ for all $\theta\in\Omega$, where Ω is a parameter space. The term $E\left(\hat{\theta}\right)-\theta$ is called the bias of the estimator $\hat{\theta}$.

Definition (1.5) (Asymptotically Unbiased Estimator) [18]:

An estimator $\hat{\theta}=u\left(X_1,X_2,...,X_n\right)$ is defined to be asymptotically unbiased estimator for θ if $\lim_{n\to\infty} E(\hat{\theta})=\theta$.

Definition (1.6) (Consistence Estimator) [18]:

Let the statistic $\hat{\theta}=u\left(X_1,X_2,...,X_n\right)$ be an estimator of the unknown parameter θ is said to be consistent estimator if $\lim_{n\to\infty} \Pr\left(\left|\hat{\theta}-\theta\right|>\varepsilon\right)=0$; for each $\theta\in\Omega$.

Definition (1.7) (Minimum Variance Unbiased Estimator) [18]:

Let X_1, X_2,…, X_n be a r.s. of size n whose p.d.f. $f\,(\underset{\sim}{x},\theta)$. An estimator $\hat{\theta}=u\left(X_1,X_2,...,X_n\right)$ of θ is defined to be a minimum variance unbiased estimator of θ iff

(i) $E(\hat{\theta})=\theta$, that is , $\hat{\theta}$ is unbiased.

(ii) The variance of $\hat{\theta}$ is less than or equal to the variance of every other unbiased estimators of θ.

1.5.3 Methods of Finding Estimators [2]

Many technique have been proposed in the literatures of finding estimators for the distn. parameters such as Moments, Maximum Likelihood, Minimum Chi-square, Minimum Distance, Least Square, and Bayesian method. These methods provides a single value for each unknown parameter of the distn. .

For Weibull case we shall consider four methods for finding the estimator of distn. parameters.

(i) Maximum Likelihood Method .

(ii) Moments Method.

(iii) Modified Moments Method.

(iv) Least square method.

Definition (1.8) (Likelihood Function) [2]:

The likelihood function of r.s. $X_1, X_2,\ldots, X_n$ of size n from a distn. having p.d.f. $f(\underset{\sim}{x};\underset{\sim}{\theta})$, where $\underset{\sim}{\theta}=(\theta_1,\theta_2,\ldots,\theta_k)$ is a vector of unknown parameters, is defined to be the joint p.d.f. of the r.v.'s $X_1, X_2,\ldots, X_n$ which is considered as a function of $\underset{\sim}{\theta}$ and denoted by $L(\underset{\sim}{\theta},\underset{\sim}{x})$ that

is $L=L(\underset{\sim}{\theta},\underset{\sim}{x})=f(\underset{\sim}{x};\underset{\sim}{\theta})=\prod_{i=1}^{n} f(x_i;\underset{\sim}{\theta})$.

1.5.3.1 Estimation of Parameters by Maximum Likelihood Method [2]:

Let $L(\underset{\sim}{\theta},\underset{\sim}{x})$ be the likelihood function of a r.s. $X_1, X_2, \ldots, X_n$ of size n from a distn. whose p.d.f. $f(\underset{\sim}{x};\underset{\sim}{\theta})$, $\underset{\sim}{\theta}=(\theta_1,\theta_2,\ldots,\theta_k)$ is a vector of unknown parameters. Let $\hat{\underset{\sim}{\theta}}=\underset{\sim}{u}(\underset{\sim}{x})=(u_1(\underset{\sim}{x}),u_2(\underset{\sim}{x}),\ldots,u_k(\underset{\sim}{x}))$ be a vector of unknown statistics of observations $\underset{\sim}{x}=(x_1,x_2,\ldots,x_n)$. If $\hat{\underset{\sim}{\theta}}$ have the value of $\underset{\sim}{\theta}$ which maximize $L(\hat{\underset{\sim}{\theta}},\underset{\sim}{x})$, then $\hat{\underset{\sim}{\theta}}$ is the m.ℓ.e. of $\underset{\sim}{\theta}$ and the corresponding statistic $\hat{\underset{\sim}{\Theta}}=\underset{\sim}{u}(\underset{\sim}{X})$ is the M.L.E of $\underset{\sim}{\theta}$. We note that

(i) Many likelihood function satisfy the condition that the m.ℓ.e is a solution of the likelihood eq.'s.

$$\frac{\partial L(\underset{\sim}{\theta},\underset{\sim}{x})}{\partial\theta_r}=0, \text{ at } \underset{\sim}{\theta}=\hat{\underset{\sim}{\theta}} \quad r=1,2,\ldots,k.$$

(ii) Since L($\underset{\sim}{\theta}$, $\underset{\sim}{x}$) and ln L($\underset{\sim}{\theta}$, $\underset{\sim}{x}$) have their maximum at the same value of $\underset{\sim}{\theta}$ so sometimes it is easier to find the maximum of the logarithm of the likelihood.

In such case, the m.ℓ.e $\hat{\underset{\sim}{\theta}}$ of $\underset{\sim}{\theta}$ which maximizes L($\underset{\sim}{\theta}$, $\underset{\sim}{x}$) may be given the solution of the likelihood eq.'s.

$$\frac{\partial \ln L(\underset{\sim}{\theta},\underset{\sim}{x})}{\partial\theta_r}=0, \text{ at } \underset{\sim}{\theta}=\hat{\underset{\sim}{\theta}}, \ r=1,2,\ldots,k.$$

For Weibull distn. case:

Let X_1, X_2, …, X_n be a r.s. of size n from $W\left(\alpha,\theta\right)$ where the distn. p.d.f. is given by eq.(1.1). The likelihood function is

$$L(\alpha,\theta,\underset{\sim}{x}) = f(\underset{\sim}{x},\alpha,\theta) = \prod_{i=1}^{n} f(x_i,\alpha,\theta)$$

$$= \prod_{i=1}^{n} \alpha\theta x_i^{\alpha-1} e^{-\theta x_i^{\alpha}}$$

$$= \left(\alpha\theta\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\theta\sum_{i=1}^{n} x_i^{\alpha}}$$

$$\ln L = n\ln\theta + n\ln\alpha + \left(\alpha-1\right)\sum_{i=1}^{n}\ln x_i - \theta\sum_{i=1}^{n} x_i^{\alpha}$$

$$\frac{\partial \ln L}{\partial\alpha} = \frac{-n}{\alpha} + \sum_{i=1}^{n}\ln x_i - \theta\sum_{i=1}^{n} x_i^{\alpha}\ln x_i \quad \text{.......................................} \quad (1.15)$$

And

$$\frac{\partial \ln L}{\partial\theta} = \frac{n}{\alpha} - \sum_{i=1}^{n} x_i^{\alpha} \quad \text{...} \quad (1.16)$$

Set $\frac{\partial \ln L}{\partial\alpha} = 0$ and $\frac{\partial \ln L}{\partial\theta} = 0$ at $\alpha = \hat{\alpha}, \theta = \hat{\theta}$.

We have:

$$\frac{-n}{\alpha} + \sum_{i=1}^{n}\ln x_i - \theta\sum_{i=1}^{n} x_i^{\alpha}\ln x_i = 0 \quad \text{...} \quad (1.17)$$

And

$$\frac{n}{\alpha} - \sum_{i=1}^{n} x_i^{\alpha} = 0 \quad \text{……………………………………………………} (1.18)$$

Solution for $\hat{\alpha}$ and $\hat{\theta}$ can not be found analytically from the non-linear eq.'s (1.17) and (1.18).

An approximate solution for $\hat{\alpha}$ and $\hat{\theta}$ from eq.'s (1.17) and (1.18) can be made iteratively by using Newton-Raphson method for solving a non-linear eq.'s as follows:

Let $f_1 = f_1(\hat{\alpha},\hat{\theta}) = \dfrac{n}{\hat{\alpha}} + \sum_{i=1}^{n} \ln x_i - \hat{\theta}\sum_{i=1}^{n} x_i^{\hat{\alpha}} \ln x_i$

and

$$f_2 = f_2(\hat{\alpha},\hat{\theta}) = \frac{n}{\hat{\alpha}} - \sum_{i=1}^{n} x_i^{\hat{\alpha}}$$

Suppose that $(\hat{\alpha}^{(s)},\hat{\theta}^{(s)})$ represent the approximate solution of $(\hat{\alpha},\hat{\theta})$ at stage (s). Then the approximate solution at stage (s+1) for $(\hat{\alpha}^{(s)},\hat{\theta}^{(s)})$ is

$$\hat{\alpha}^{(s+1)} = \hat{\alpha}^{(s)} + \delta_1 \quad \ldots\ldots(1.19)$$

$$\hat{\theta}^{(s+1)} = \hat{\theta}^{(s)} + \delta_2 \quad \ldots\ldots(1.20)$$

Where

$$\underset{\sim}{\delta} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = -\begin{pmatrix} \dfrac{\partial f_1}{\partial \hat{\alpha}^{(s)}} & \dfrac{\partial f_1}{\partial \hat{\theta}^{(s)}} \\ \dfrac{\partial f_2}{\partial \hat{\alpha}^{(s)}} & \dfrac{\partial f_2}{\partial \hat{\theta}^{(s)}} \end{pmatrix}^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \text{ Set}$$

$$a = \frac{\partial f_1}{\partial \hat{\alpha}} = \frac{-n}{\hat{\alpha}^2} - \hat{\theta}\sum_{i=1}^{n} x_i^{\hat{\alpha}} \left(\ln x_i\right)^2$$

$$b = \frac{\partial f_1}{\partial \hat{\theta}} = \frac{\partial f_2}{\partial \hat{\alpha}} = -\sum_{i=1}^{n} x_i^{\hat{\alpha}} \left(\ln x_i\right)$$

$$c = \frac{\partial f_2}{\partial \hat{\theta}} = \frac{-n}{\hat{\theta}^2}$$

We have

$$\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix} = -\begin{pmatrix}a & b\\ b & c\end{pmatrix}^{-1}\begin{pmatrix}f_1\\ f_2\end{pmatrix} = \frac{-1}{ac-b^2}\begin{pmatrix}c & -b\\ -b & a\end{pmatrix}\begin{pmatrix}f_1\\ f_2\end{pmatrix}$$

Then,

$$\delta_1 = -\frac{1}{ac-b^2}(cf_1 - bf_2)$$

$$\delta_2 = -\frac{1}{ac-b^2}(-bf_1 + af_2)$$

and according to the eq.'s (1.19) and (1.20), we have

$$\hat{\alpha}^{(s+1)} = \hat{\alpha}^{(s)} - \frac{1}{ac-b^2}(cf_1 - bf_2) \quad \ldots\ldots\ldots(1.21)$$

$$\hat{\theta}^{(s+1)} = \hat{\theta}^{(s)} - \frac{1}{ac-b^2}(-bf_1 + af_2) \quad \ldots\ldots\ldots(1.22)$$

1.5.3.2 Estimation of Parameters by Moments Method [2]:

Let $X_1, X_2, \ldots, X_n$ be a r.s of size n from a distn. whose p.d.f $f(x;\underset{\sim}{\theta})$, $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ is a vector of unknown parameters, let

$\mu'_r = E(X^r)$ be the r^th^ moment of the distn. about origin and $M_r = \frac{1}{n}\sum_{i=1}^{n} X_i^r$

be the r^th^ moment of the sample about origin. The method of moments can be described as follows:

Since, we have k unknown parameters, equate μ'_r to M_r at $\underset{\sim}{\theta} = \underset{\sim}{\hat{\theta}}$.

That is

$$\mu'_r = M_r \quad \text{at} \quad \underset{\sim}{\theta} = \underset{\sim}{\hat{\theta}},\ r = 1, 2, \ldots, k .$$

For these k equations, we find a unique solution for $\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_k$ and we say that $\hat{\theta}_r (r = 1, 2, ..., k)$ is an estimate of θ_r obtained by method of Moments and the corresponding statistic $\hat{\Theta}_r$ is an estimator of θ_r.

Now, to estimate α and θ by method of moments we let $X_1, X_2, \ldots, X_n$ be a r.s. of size n from $W\ (\alpha, \theta)$ is taken.

Since, $W\ (\alpha, \theta)$ distn. involve two unknown parameters,

We set $\mu'_r = M_r$ at $\alpha = \hat{\alpha},\ \theta = \hat{\theta}$, r = 1,2 .

r = 1 implies $\mu'_1 = E(X) = (\frac{1}{\theta})^{\frac{1}{\alpha}} \Gamma(1 + \frac{1}{\alpha})$ and $M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$.

r = 2 implies $\mu'_2 = E(X^2) = (\frac{1}{\theta})^{\frac{2}{\alpha}} \Gamma(1 + \frac{2}{\alpha})$

And

$$M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{n-1}{n} S^2 + \bar{X}^2 .$$

We set

$\mu'_1 = M_1$ and $\mu'_2 = M_2$ at $\alpha = \hat{\alpha},\ \theta = \hat{\theta}$ we have

$$(\frac{1}{\hat{\theta}})^{\frac{1}{\hat{\alpha}}} \Gamma(1 + \frac{1}{\hat{\alpha}}) = \bar{X} \quad \ldots\ldots\ldots\ldots(1.23)$$

$$(\frac{1}{\hat{\theta}})^{\frac{2}{\hat{\alpha}}} \Gamma(1 + \frac{2}{\hat{\alpha}}) = \frac{1}{n-1} S^2 + \bar{X}^2 \quad \ldots\ldots\ldots\ldots(1.24)$$

For finding the estimators of α and θ, we follow the approach made by [7] as follows:

$$(CV)_d = \frac{\delta^2}{\mu_1^2} \quad \ldots\ldots(1.25)$$

Since, the unbiased estimator of δ^2 is S^2 and the estimator of μ is $\bar{X}$, then

$$(\hat{CV})_d = \frac{S^2}{(\bar{X})^2} \quad \ldots\ldots(1.26)$$

Which can be calculated from the given set of observation but, the coefficient of variation is:

$$CV = \frac{\delta}{\mu} = \frac{\sqrt{\Gamma(1+\frac{2}{\hat{\alpha}}) - \Gamma^2(1+\frac{1}{\hat{\alpha}})}}{\Gamma(1+\frac{1}{\hat{\alpha}})} \quad \ldots\ldots(1.27)$$

By taking different configuration values of $\hat{\alpha}$ in eq. (1.27), randomly. The value $\hat{\alpha}$ is adopted when CV is very close to of $(\hat{CV})_d$. The scale parameter(θ) can then be estimated using eq. (1.23) as

$$\hat{\theta} = \left(\frac{\Gamma(1+\frac{1}{\hat{\alpha}})}{\bar{X}} \right)^{\hat{\alpha}} \quad \ldots\ldots(1.28)$$

1.5.3.3 Estimation of Parameters by Modified Moments Method [20] :

Let $X_1, X_2, \ldots, X_n$ be a r.s of size n from a distn. p.d.f. $f(x, \underset{\sim}{\theta})$ where $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ is a vector of k unknown parameters.

Let $Y_1, Y_2, \ldots, Y_n$ represent the arrangement of the sample set $\{X_i\}$ in a ascending order of magnitude.

Let $\mu_r' = E(X^r)$ be the r^{th} sample moment about the origin, r=1,2, … .
In this method we equate $\mu_r' = E(X^r)$ with $r = 1$ and ranking
$E(Y_i) = Y_i$ beginning with $i = 1$ until $i = k$ this process will gives k eq.'s to provide a unique solution for $\theta_i, i = 1,2,...,k$ say $\hat{\theta}_i, i = 1,2,...,k$ and the obtained $\hat{\theta}_i$, this method is called modified moment estimator.

For Weibull case:

we have two unknown parameters α and θ and if we take a r.s. of size n from $W(\alpha,\theta)$, we let Y_1 represent the first order statistic of the sample.
From the order statistic theory the p.d.f. of Y_1 is

$$g_1(y_1) = n\left(1 - F(y_1)\right)^{n-1} f(y_1)$$

$$\Rightarrow g_1(y_1) = n\alpha\theta y_1^{\alpha-1} e^{-n\theta y_1^{\alpha}}, \quad 0 < y_1 < \infty$$
$$= 0 \quad , e.w. \qquad ; \alpha,\theta > 0$$

This shown that $Y_1 \sim W(\alpha, n\theta)$.

Accordingly, $E(Y_1) = \dfrac{1}{(n\theta)^{\frac{1}{\alpha}}}\Gamma(1+\dfrac{1}{\alpha})$.

Now, we apply the Modified Moment Method by setting $\mu_1' = \bar{X}$ and
$E(Y_1) = Y_1$ at $\alpha = \hat{\alpha}$, $\theta = \hat{\theta}$ which leads to

$$\left(\frac{1}{\hat{\theta}}\right)^{\frac{1}{\hat{\alpha}}} \Gamma\left(1 + \frac{1}{\hat{\alpha}}\right) = \bar{X} \quad \text{……………………………………………….(1.29)}$$

$$\left(\frac{1}{n\hat{\theta}}\right)^{\frac{1}{\hat{\alpha}}} \Gamma\left(1 + \frac{1}{\hat{\alpha}}\right) = Y_1 \quad \text{……………………………………………..(1.30)}$$

From eq.'s (1.29) and (1.30), the estimators of α and θ are respectively

$$\hat{\alpha} = \frac{\ln\left(\frac{1}{n}\right)}{\ln\left(\frac{Y_1}{\bar{X}}\right)} \qquad \ldots\ldots(1.31)$$

$$\hat{\theta} = \left(\frac{\left(1 - \frac{1}{n^{\frac{1}{\hat{\alpha}}}}\right)\Gamma\left(1+\frac{1}{\hat{\alpha}}\right)}{\bar{X} - Y_1} \right)^{\hat{\alpha}} \qquad \ldots\ldots(1.32)$$

1.5.3.4 Estimation of Parameters by Least Squares Method [20]:

The least squares method is general technique for estimating parameters in fitting a set of points to generate a curve whose trend might be linear, quadratic, or of higher order. In order to utilize this method, the error terms due to experiment must satisfy the following conditions:

(i) They have zero mean.

(ii) They have same variance.

(iii) Must be uncorrelated.

For good result of fitting curve to the data set, the error must be minimized as small as possible .

Let us assume that we have a set of n data points (x_i, t_i) through which we desire to pass a straight line. This line is representing the best fit in the least square sense.

Suppose that the best fitting straight line to the data (x_i, t_i) is $x = \beta_0 + \beta_1 t$. Where β_0 and β_1 are two unknown parameters representing respectively the vertical intercept and the slop.

To assist in visualizing the process, assume the fitted line shown in figure (1.3) which depicts the data points as well as the line to be fitted, unless the data fall in a straight line, usually the general curve will not pass through all of the data points. For convenience let us consider the ith point where ordinate of the point is given as x_i .

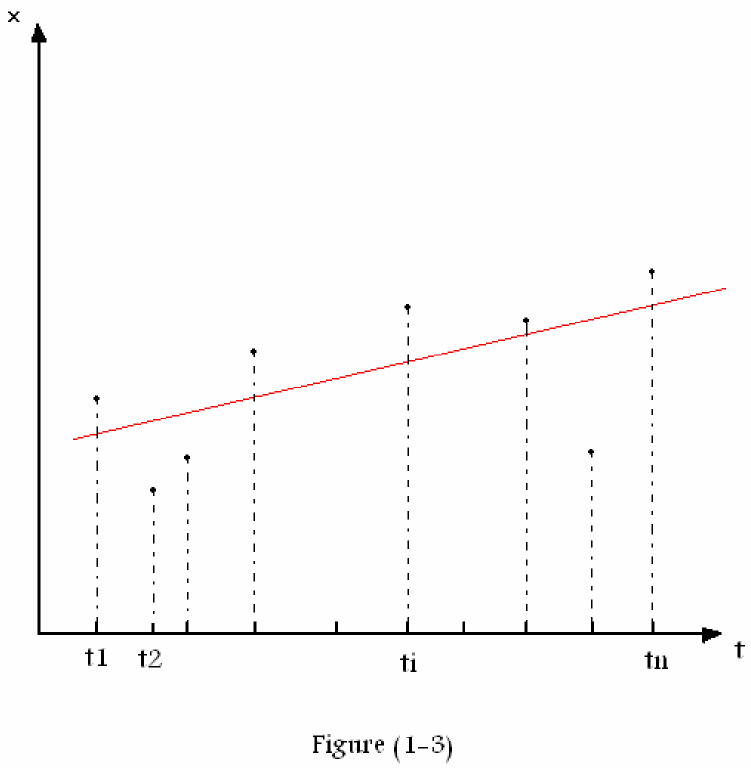

Figure (1.3) show the best fitted line to the data (x_i, t_i)

The ordinate x_i as given by the general line is $\beta_0 + \beta_1 t_i$. The difference between these two values is the error of fit at the ith point $\varepsilon_i = x_i - (\beta_0 + \beta_1 t_i)$ .Let the sum squares of all errors at the data points be

$SSE=\sum_{i=1}^{n}\varepsilon_i^2=\sum_{i=1}^{n}\left(x_i-\beta_0-\beta_1 t_i\right)^2$. For minimum, we set $\frac{\partial SSE}{\partial\beta_0}=0$ and $\frac{\partial SSE}{\partial\beta_1}=0$, at $\beta_0=\hat{\beta}_0,\beta_1=\hat{\beta}_1$.

$$\left.\frac{\partial SSE}{\partial\hat{\beta}_0}\right|_{\substack{\beta_0=\hat{\beta}_0\\ \beta_1=\hat{\beta}_1}}=-2\sum_{i=1}^{n}\left(x_i-\hat{\beta}_0-\hat{\beta}_1 t_i\right)=0 \quad \text{.....(1.33)}$$

$$\left.\frac{\partial SSE}{\partial\hat{\beta}_1}\right|_{\substack{\beta_0=\hat{\beta}_0\\ \beta_1=\hat{\beta}_1}}=-2\sum_{i=1}^{n}\left(x_i-\hat{\beta}_0-\hat{\beta}_1 t_i\right)t_i=0 \quad \text{.....(1.34)}$$

From (1.33) and (1.34) we can get two eq.'s as

$$n\hat{\beta}_0+\hat{\beta}_1\sum_{i=1}^{n}t_i=\sum_{i=1}^{n}x_i \quad \text{.....(1.35)}$$

$$\hat{\beta}_0\sum_{i=1}^{n}t_i+\hat{\beta}_1\sum_{i=1}^{n}t_i^2=\sum_{i=1}^{n}t_i\,x_i \quad \text{.....(1.36)}$$

Solving eq.'s (1.35) and (1.36), lead to

$$\hat{\beta}_0=\frac{\left(\sum_{i=1}^{n}x_i\,t_i\right)\left(\sum_{i=1}^{n}t_i\right)-\left(\sum_{i=1}^{n}t_i^2\right)\left(\sum_{i=1}^{n}x_i\right)}{\left(\sum_{i=1}^{n}t_i\right)^2-n\left(\sum_{i=1}^{n}t_i^2\right)} \quad \text{.....(1.37)}$$

and $\hat{\beta}_1=\frac{\bar{x}-\hat{\beta}_0}{\bar{t}}$ .....(1.38)

Since, $\bar{x}=\hat{\beta}_0+\hat{\beta}_1\bar{t}$.

For Weibull case:

Suppose that X_1, X_2, …, X_n be a r.s of size n from a distn. having cumulative function $F(x)=\Pr(X \le x)=\begin{cases} 0, & x<0 \\ 1-e^{-\theta x^{\alpha}} & ,0<x<\infty \\ 1, & x\to\infty \end{cases}$

Then $u_i = 1-e^{-\theta x_i^{\alpha}} \Rightarrow e^{-\theta x_i^{\alpha}} = 1-u_i = u_i$ since, $u_i \sim U(0,1)$, then $1-u_i \sim U(0,1)$ which implies

$$ln(x_i)=\frac{1}{\alpha}ln(-Ln\, u_i)-\frac{1}{\alpha}ln(\theta),\ i=1,2,....,n. \quad ………………….(1.39)$$

Set $y_i = ln(x_i)$, $t_i = ln(-ln\, u_i)$ $i=1,2,....,n$ and $\beta_0=-\frac{1}{\alpha}ln(\theta)$, $\beta_1=\frac{1}{\alpha}$

then , eq.(1.40) becomes $y_i=\hat{\beta}_0+\hat{\beta}_1 t_i$,$i=1,2,3,...,n$.

Utilizing eq.(1.39) for obtaining the estimators $\hat{\beta}_0$ and $\hat{\beta}_1$. Therefore; the least squares estimators $\hat{\alpha}$ and $\hat{\theta}$ can be obtained from the eq.(1.39) $\hat{\alpha}=\frac{1}{\hat{\beta}_1}$ and $\hat{\theta}=\exp\left(-\frac{\hat{\beta}_0}{\hat{\beta}_1}\right)$.

1.5.4 Some Important Concepts (Definitions and Theorems)

Definition (1.9) (Sufficient Statistic) [18]:

Let X_1, X_2,…, X_n be a r.s. of size n whose joint p.d.f. $f(\underset{\sim}{x},\underset{\sim}{\theta})$, where $\underset{\sim}{\theta}=(\theta_1,\theta_2,...,\theta_m)$ is a vector of unknown parameters and let $Y_i=u_i(X_1,X_2,...,X_n)$, $i=1,2,...,m$ be k statistics whose joint p.d.f.

$g(\underset{\sim}{y},\underset{\sim}{\theta})$.Then the k statistics are called jointly sufficient statistics for $\underset{\sim}{\theta}$ iff

$\frac{f(\underset{\sim}{x},\underset{\sim}{\theta})}{g(\underset{\sim}{y},\underset{\sim}{\theta})}=H(\underset{\sim}{x})$ .Where $H(\underset{\sim}{x})$ does not depends on $\underset{\sim}{\theta}$ for all fixed values of

$y_i=u_i(x_1,x_2,...,x_n),\ i=1,2,...,m$.

Theorem (1.3) (Neymann Factorization Theorem) [18]:

Let X_1, X_2,…, X_n be a r.s. of size n whose p.d.f. $f(\underset{\sim}{x},\theta)$, where, $\underset{\sim}{\theta}=(\theta_1,\theta_2,...,\theta_m)$ be a vector of unknown parameters. A set of statistics $Y_i=u_i(X_1,X_2,...,X_n), i=1,2,...,m$ are jointly sufficient statistics for $\underset{\sim}{\theta}$ iff, we can find two non-negative functions K_1 and K_2 such that

$$\begin{aligned} f(\underset{\sim}{x},\underset{\sim}{\theta}) &= f(x_1,x_2,...,x_n;\theta_1,\theta_2,...,\theta_m) \\ &= K_1(u_1(\underset{\sim}{x}),u_2(\underset{\sim}{x}),...,u_m(\underset{\sim}{x});\theta_1,\theta_2,...,\theta_m)\ K_2(\underset{\sim}{x}) \end{aligned}$$

Where $K_2(\underset{\sim}{x})$ is independent of $\underset{\sim}{\theta}$.

In general we note that every functions of a sufficient statistics is also sufficient statistics.

Definition (1.10) (completeness) [18]:

Let $X_1,X_2,...,X_n$ be a r.s. of size n from a distn. (continuous or discrete) whose p.d.f. belongs to the family $\left\{f(\underset{\sim}{x},\underset{\sim}{\theta}),\underset{\sim}{\theta}=(\theta_1,\theta_2,...,\theta_m)\in\Omega^m\right\}$ of p.d.f.'s , and let $u(\underset{\sim}{x})$ be a continuous function of $\underset{\sim}{x}=(x_1,x_2,...,x_n)$. If $E\left[u(\underset{\sim}{X})\right]=0$, implies $u(\underset{\sim}{x})=0,\forall\underset{\sim}{x}$, then the family $\left\{f(x,\underset{\sim}{\theta}),\theta\in\Omega\right\}$ is called a complete family of p.d.f.'s .

Remark [18]:

If $Y = u(\underset{\sim}{X})$ is a sufficient statistic for θ whose p.d.f. belong to the complete family, then Y is a complete sufficient statistic for θ. We note that

Theorem (1.4) (Lehman-scheffe'-1st Theorem) [18]:

Let $X_1, X_2, \ldots, X_n$ be a r.s. of size n whose p.d.f. $f(\underset{\sim}{x}, \theta)$, $\theta \in \Omega$.

Let $Y = u(\underset{\sim}{X})$ be a sufficient statistic for θ whose p.d.f. belong to the complete family $\{g(y, \theta), \theta \in \Omega\}$. If $\Phi(Y)$ is a function of Y which is an unbiased estimator for θ, then $\Phi(Y)$ is a unique MVUE for θ.

Definition (Exponential Family of p.d.f.'s) (1.11) [18]:

Consider the family $\left\{f(x; \underset{\sim}{\theta}), \underset{\sim}{\theta} \in \Omega^m\right\}$ of p.d.f.'s which can be expressed as

$$f(x; \underset{\sim}{\theta}) = \exp\left[\sum_{j=1}^{m} p_j(\underset{\sim}{\theta}) k_j(x) + q(\underset{\sim}{\theta}) + s(x)\right], a < x < b \quad \ldots\ldots\ldots\ldots(1.40)$$

$$= 0 \qquad , e.w.$$

Such p.d.f. is said to be a member of exponential class of p.d.f.'s and satisfying the following conditions:

(i) Neither a nor b depends on $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_m)$.

(ii) $p_j(\underset{\sim}{\theta})$ is nontrivial, continuous functions of θ_j, $j = 1, 2, \ldots, m$.

(iii) $k'_j(x) \neq 0$ and $s(x)$ is continuous function of x for $a < x < b$.

(iv) $q(\underset{\sim}{\theta})$ is a continuous function of $\underset{\sim}{\theta}$, $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_m)$.

Now, if a r.s. $X_1, X_2, ..., X_n$ is taken from a distn. whose p.d.f. $f(x;\underset{\sim}{\theta})$, then the joint p.d.f. of the sample set $\{X_i\}$ is

$$f(\underset{\sim}{x};\underset{\sim}{\theta}) = \prod_{i=1}^{n} f(x_i;\underset{\sim}{\theta}) = \prod_{i=1}^{n} \exp\left[\sum_{j=1}^{m} p_j(\underset{\sim}{\theta}) k_j(x_i) + q(\underset{\sim}{\theta}) + s(x_i)\right], a < x < b$$

$$= \exp\left[\sum_{j=1}^{m} p_j(\underset{\sim}{\theta}) \sum_{i=1}^{n} k_j(x_i) + n q(\underset{\sim}{\theta}) + \sum_{i=1}^{n} s(x_i)\right]$$

$$= \exp\left[\sum_{j=1}^{m} p_j(\underset{\sim}{\theta}) \sum_{i=1}^{n} k_j(x_i) + n q(\underset{\sim}{\theta})\right] \exp\left[\sum_{i=1}^{n} s(x_i)\right]$$

Then, according to factorization theorem (1.3), the statistics $Y_1 = \sum_{i=1}^{n} k_1(X_i), Y_2 = \sum_{i=1}^{n} k_2(X_i), ..., Y_m = \sum_{i=1}^{n} k_m(X_i)$ are jointly sufficient statistics for m parameters $\theta_1, \theta_2, ..., \theta_m$.

For Weibull $W(\alpha, \theta)$ with p.d.f.

$$f(x;\alpha,\theta) = \alpha\theta x^{\alpha-1} e^{-\theta x^{\alpha}}, 0 < x < \infty$$
$$= 0 \qquad ,e.w. \qquad ; where\ \alpha, \theta > 0.$$

Which can be written as

$$\Rightarrow \quad f(x;\alpha,\theta) = e^{\ln(\alpha\theta) + (\alpha-1)\ln x - \theta x^{\alpha}}$$

We note that there are three cases:

(***i***) $f(x;\alpha,\theta)$ is not exponential when α and θ are unknown.

(***ii***) $f(x,\alpha)$ is not exponential when α is unknown and θ is known.

(***iii***) $f(x,\theta)$ is exponential when α is known and θ is unknown.

In case (***iii***) we write $f\left(x,\theta\right)$ as

$$f\left(x;\alpha,\theta\right)=e^{-\theta x^{\alpha}+\ln\theta+\ln\alpha+(\alpha-1)\ln x}$$

Where $p(\theta)=-\theta$, $k(x)=x^{\alpha}$, $q(\theta)=\ln\theta$, $s(x)=\ln\alpha+(\alpha-1)\ln x$

If a sample $X_1, X_2, \ldots, X_n$ is taken from $W\left(\alpha,\theta\right)$, (α known), then according to factorization theorem (1.3), the statistic $Y=\sum_{i=1}^{n}k\left(x_i\right)=\sum_{i=1}^{n}X_i^{\alpha}$ …..(1.41)

Is a sufficient statistic for θ.

Theorem (1.5) (Lehman-scheffe'-2nd Theorem) [18]:

Let $X_1, X_2, \ldots, X_n$ be a r.s. of size n whose p.d.f. $f\left(\underset{\sim}{x},\underset{\sim}{\theta}\right)$ where $\underset{\sim}{\theta}=\left(\theta_1,\theta_2,\ldots,\theta_m\right)$, belong to the exponential family and let $Y_1, Y_2, \ldots, Y_m$ be jointly sufficient statistics for $\theta_1,\theta_2,\ldots,\theta_m$, then the family of p.d.f.'s $\left\{g\left(\underset{\sim}{y},\underset{\sim}{\theta}\right),\theta\in\Omega^m\right\}$ is complete and the statistics $Y_1, Y_2, \ldots, Y_m$ are jointly complete sufficient statistics for $\theta_1,\theta_2,\ldots,\theta_m$.

Now, according to Lehman- Scheffe' 2nd theorem, the statistic given by eq.(1.41) $Y=\sum_{i=1}^{n}X_i^{\alpha}$ is complete sufficient statistic for θ. To find the MVUE for θ, consider the transformation $Z=X^{\alpha}$.

The function $z=x^{\alpha}$ define one-to-one transformation that maps the space $A=\left\{x:0<x<\infty\right\}$ onto the space $B=\left\{z:0<z<\infty\right\}$ with inverse $x=z^{\frac{1}{\alpha}}$

and the Jacobin of transformation $J = \frac{dx}{dz} = \frac{1}{\alpha} z^{\frac{1}{\alpha}-1}$. Then the p.d.f. of Y is

$$g(z) = f(z^{\frac{1}{\alpha}})|J| = \alpha\theta z^{1-\frac{1}{\alpha}} e^{-\theta y} \frac{1}{\alpha} z^{\frac{1}{\alpha}-1}$$

$$= \theta e^{-\theta z}, \quad 0 < z < \infty$$

$$= \qquad , e.w.$$

That is $Z \sim Exp(\frac{1}{\theta})$.

Since, $Y = \sum_{i=1}^{n} X_i^{\alpha} = \sum_{i=1}^{n} Z_i$, then according to the additive property of

exponential distn. lead to $Y \sim G(n, \frac{1}{\theta})$ with p.d.f.

$$h(y) = \frac{\theta^n}{\Gamma(n)} y^{n-1} e^{-\theta y}, \ 0 < y < \infty$$

$$= 0 \qquad , e.w.$$

Now, consider the expectation

$$E(\frac{1}{Y}) = \int_y \frac{1}{y} h(y) dy = \int_{y=0}^{\infty} \frac{1}{y} \frac{\theta^n}{\Gamma(n)} y^{n-1} e^{-\theta y} dy$$

$$= \frac{\theta^n \Gamma(n-1)}{\Gamma(n)\theta^{n-1}} \int_0^{\infty} \frac{\theta^{n-1}}{\Gamma(n-1)} y^{(n-1)} e^{-\theta y} dy = \frac{\theta}{n-1}$$

Accordingly, the MVUE for θ is $\frac{n-1}{Y} = \frac{n-1}{\sum_{i=1}^{n} X_i^{\alpha}}$.

1.6 Monte Carlo Results:

To access the results obtained by the four methods of estimation practically, we generate the Normal variates according to Box and Muller Method [27], [See Appendix A] and then these normal variates are trans-feered to Weibull variate as shown in section (1.3). In practice sample from $W\left(\frac{1}{2}, 1\right)$ are generated with size $n = 5,10(5)50$ and the run size used is 100. The estimates of the four methods are shown in table (1.1).

Table (1.1)

Parameters Estimation

Sample Size	Estimation of $(\hat{\alpha},\hat{\theta})$			
	M.L.M	M.M	M.M.M	L.S.M
5	$(0.464,1.151)$	$(0.310,0.858)$	$(0.430,1.387)$	$(0.444,0.993)$
10	$(0.478,1.110)$	$(0.407,1.030)$	$(0.397,1.306)$	$(0.479,0.988)$
20	$(0.477,1.092)$	$(0.413,0.952)$	$(0.457,1.216)$	$(0.522,0.997)$
30	$(0.480,1.017)$	$(0.535,1.123)$	$(0.460,0.923)$	$(0.522,0.984)$
40	$(0.502,1.136)$	$(0.461,1.104)$	$(0.486,0.945)$	$(0.504,0.984)$
50	$(0.498,0.939)$	$(0.495,0.962)$	$(0.491,0.962)$	$(0.501,1.019)$

This table show the bias of estimator $\hat{\alpha}$ which can be obtained by :

$Bais\,(\hat{\alpha}) = E\,(\hat{\alpha}) - \alpha$.

Table (1.2)

Bias of Estimator $(\hat{\alpha})$

Sample Size	Bias of Estimation $(\hat{\alpha})$			
	M.L.M	M.M	M.M.M	L.S.M
5	-0.036	-0.19	-0.070	-0.056
10	-0.022	-0.093	-0.103	-0.021
20	-0.023	-0.087	-0.043	-0.022
30	-0.020	0.035	-0.040	0.022
40	0.002	-0.039	-0.014	0.004
50	-0.002	-0.005	-0.009	0.001

This table show the bias of estimator $\hat{\theta}$ which can be obtained by :

$Bais\left(\hat{\theta}\right)=E\left(\hat{\theta}\right)-\theta$.

Table (1.3)

Bias of Estimator $\left(\hat{\theta}\right)$

Sample Size	Bias of Estimation $\left(\hat{\theta}\right)$			
	M.L.M	M.M	M.M.M	L.S.M
5	0.151	-0.142	0.387	-0.007
10	0.11	0.03	0.306	-0.012
20	0.092	-0.048	0.216	-0.003
30	0.017	0.123	-0.077	-0.016
40	0.136	0.104	-0.055	-0.016
50	-0.061	-0.038	-0.038	0.019

This table show the variance of estimator $\hat{\alpha}$ which can be obtained by :

$Var(\hat{\alpha}) = E(\hat{\alpha}^2) - [E(\hat{\alpha})]^2$.

Table (1.4)

Variance of Estimator $(\hat{\alpha})$

Sample Size	Variance of Estimation $(\hat{\alpha})$			
	M.L.M	M.M	M.M.M	L.S.M
5	4.121×10^{-7}	3.284×10^{-6}	4.361×10^{-6}	3.411×10^{-6}
10	3.961×10^{-7}	3.245×10^{-6}	3.217×10^{-6}	3.697×10^{-6}
20	3.629×10^{-7}	3.559×10^{-6}	3.988×10^{-6}	3.371×10^{-6}
30	4.578×10^{-7}	3.187×10^{-6}	3.640×10^{-6}	2.342×10^{-6}
40	4.359×10^{-7}	3.081×10^{-6}	4.44×10^{-6}	2.101×10^{-6}
50	5.710×10^{-7}	3.946×10^{-6}	3.756×10^{-6}	2.322×10^{-6}

This table show the variance of estimator $\hat{\theta}$ which can be obtained by :

$Var\left(\hat{\theta}\right)=E\left(\hat{\theta}^2\right)-\left[E\left(\hat{\theta}\right)\right]^2$.

Table (1.5)

Variance of Estimator $\left(\hat{\theta}\right)$

Sample Size	Variance of Estimation $\left(\hat{\theta}\right)$			
	M.L.M	**M.M**	**M.M.M**	**L.S.M**
5	4.614×10^{-6}	3.041×10^{-7}	3.046×10^{-7}	5.241×10^{-7}
10	3.572×10^{-6}	5.118×10^{-7}	2.602×10^{-7}	5.761×10^{-7}
20	2.863×10^{-6}	3.698×10^{-7}	5.112×10^{-7}	5.321×10^{-7}
30	2.346×10^{-6}	2.545×10^{-7}	7.81×10^{-7}	4.404×10^{-7}
40	2.710×10^{-6}	2.311×10^{-7}	4.78×10^{-7}	4.513×10^{-7}
50	2.224×10^{-6}	4.102×10^{-7}	4.102×10^{-7}	5.846×10^{-7}

This table show the mean square error of estimator $\hat{\alpha}$ which can be obtained by :

$$MSE(\hat{\alpha}) = Var(\hat{\alpha}) + (Bais(\hat{\alpha}))^2 .$$

Table (1.6)

Mean Square Error of Estimator $(\hat{\alpha})$

Sample Size	Mean Square Error of Estimator $(\hat{\alpha})$			
	M.L.M	M.M	M.M.M	L.S.M
5	0.00129	0.036	0.0049	0.0031
10	0.00048	0.008	0.010	0.00044
20	0.00052	0.007	0.001	0.00048
30	0.0004	0.001	0.001	0.00048
40	4×10^{-6}	0.001	0.0002	18×10^{-6}
50	4×10^{-6}	0.0002	8×10^{-5}	3×10^{-6}

This table show the mean square error of estimator $\hat{\alpha}$ which can be obtained by :

$$MSE\left(\hat{\alpha}\right)=Var\left(\hat{\alpha}\right)+\left(Bais\left(\hat{\theta}\right)\right)^{2}.$$

Table (1.7)

Mean Square Error of Estimator $\left(\hat{\theta}\right)$

Sample Size	Mean square Error of Estimation $\left(\hat{\theta}\right)$			
	M.L.M	M.M	M.M.M	L.S.M
5	0.022	0.002	0.151	49×10^{-6}
10	0.012	0.0009	0.094	14×10^{-6}
20	0.008	0.002	0.046	9×10^{-6}
30	0.0002	0.015	0.005	25×10^{-5}
40	0.018	0.011	0.003	25×10^{-5}
50	0.003	0.001	0.001	36×10^{-5}

Chapter Three

3

Monte Carlo Application

3.1 Introduction

In this chapter we shall utilize the procedures given in section (2.5) of chapter two for generating random varaites from Weibull $W(\frac{1}{2},1)$ distribution. Efficiency of some procedures was made theoritically and assessed practicaly. The simulated Weibull samples are observed by the six procedures mensioned in section (2.5.1), (2.5.2), (2.5.3), (2.5.4), (2.5.5) and (2.5.6) of chapter two and used to estimate the distribution parameters by the four methods given by sections (1.4.3), (1.4.4), (1.4.5), and (1.4.6) of chapter one.

3.2 Application of Procedure $(W-1)$

A computer program for procedure $(W-1)$ of section (2.5.1) which utilize the Inverse Transform Method to generate the $W(\frac{1}{2},1)$ varaites is shown in program (5) of Appendix (B). Sample size $n=5,10(10)50$ are taken. For high accuracy the procedure repeats itself 100 times. The result is displayed in table (3.1).

Table (3.1)

Parameters estimation using procedure $(W-1)$

Sample Size	Estimation of $(\hat{\alpha},\hat{\theta})$			
	M.L.M	M.M	M.M.M	L.S.M
5	$(0.401,1.053)$	$(0.547,0.858)$	$(0.625,1.145)$	$(0.571,1.316)$
10	$(0.522,1.502)$	$(0.410,1.030)$	$(0.591,1.172)$	$(0.411,1.352)$
20	$(0.440,1.157)$	$(0.514,0.952)$	$(0.522,1.152)$	$(0.540,1.024)$
30	$(0.458,1.299)$	$(0.431,1.123)$	$(0.476,0.900)$	$(0.436,1.188)$
40	$(0.476,0.971)$	$(0.453,1.104)$	$(0.460,1.014)$	$(0.510,1.053)$
50	$(0.494,0.902)$	$(0.470,0.962)$	$(0.501,0.950)$	$(0.501,1.131)$

3.3 Application of Procedure $(W-2)$

A computer program for procedure $(W-2)$ of section (2.5.2) which utilize theorem (1.2) to generate the $W(\frac{1}{2},1)$ varaites is shown in program (6) of Appendix (B). Sample size $n=5,10(10)50$ are taken. For high accuracy the procedure repeats itself 100 times. The Result is displayed in table (3.3).

Table (3.3)

Parameters estimation using procedure $(W-2)$

Sample size	Estimation of $(\hat{\alpha},\hat{\theta})$			
	M.L.M	M.M	M.M.M	L.S.M
5	$(0,541,1.246)$	$(0.562,1.155)$	$(0.510,1.282)$	$(0.412,0.721)$
10	$(0.405,1.085)$	$(0.533,0.905)$	$(0.511,1.082)$	$(0.442,1.332)$
20	$(0.463,0.923)$	$(0.453,1.040)$	$(0.455,0.988)$	$(0.480,1.169)$
30	$(0.484,0.940)$	$(0.460,1.033)$	$(0.495,1.085)$	$(0.490,0.912)$
40	$(0.472,1.060)$	$(0.481,1.191)$	$(0.484,0.999)$	$(0.497,1.096)$
50	$(0.498,0.905)$	$(0.493,1.014)$	$(0.499,0.950)$	$(0.498,1.092)$

3.4 Application of Procedure $(W-3)$

A computer program for procedure $(W-3)$ of section (2.5.3) which utilize the Acceptance-Rejection method generate the $W(\frac{1}{2},1)$ varaites is shown in program (7) of Appendix (B). Sample size $n=5(10)50$ are taken. For high accuracy the procedure repeats itself 100 times to calculates the procedure efficiency and the run size 100 were made for the efficiency average. The result is displayed in table (3.4).

Table (3.4)

Efficiency of Procedure $(W-3)$

Theory Efficiency	Simulation Efficiency	**Error=Theory-Simulation**
0.40	0.3999	0.0001

Procedure $(W-3)$ is used to each one of the four methods of estimation with sample size $n=5(10)50$ and the repetition is 100 was made. The result are displayed in table (3.5).

Table (3.5)

Parameters estimation using procedure $(W-3)$

Sample Size	Estimation of $(\hat{\alpha},\hat{\theta})$			
	M.L.M	M.M	M.M.M	L.S.M
5	$(0.593,0.990)$	$(0.562,1.126)$	$(0.583,1.368)$	$(0.588,0.771)$
10	$(0.557,1.263)$	$(0.543,1.003)$	$(0.550,0.937)$	$(0.400,1.189)$
20	$(0.535,1.069)$	$(0.437,1.149)$	$(0.415,1.069)$	$(0.411,1.127)$
30	$(0.521,1.037)$	$(0.433,0.916)$	$(0.529,1.056)$	$(0.538,1.066)$
40	$(0.454,0.933)$	$(0.453,1.143)$	$(0.460,0.930)$	$(0.444,1.122)$
50	$(0.477,1.026)$	$(0.483,0.917)$	$(0.470,0.995)$	$(0.495,0.107)$

3.5 Application of Procedure $(W-4)$

A computer program for procedure $(W-4)$ of section (2.5.4) which utilize the Acceptance-Rejection method generate the $W(\frac{1}{2},1)$ varaites is shown in program (8) of Appendix (B). Sample size $n=5,10(10)50$ are taken. For high accuracy the procedure repeats itself 100 times to calculates the procedure efficiency and the run size 100 were made for the efficiency average. The result is displayed in table (3.6).

Table (3.6)

Efficiency of Procedure $(W-4)$

Theory Efficiency	Simulation Efficiency	**Error=Theory-Simulation**
0.266	0.263	0.003

Procedure $(W-4)$ is used to each one of the four methods of estimation with sample size $n=5,10(10)50$ and the repetition is 100 was made. The result are displayed in table (3.7).

Table (3.7)

Parameters estimation using procedure $(W-4)$

Sample Size	Estimation of $(\hat{\alpha},\hat{\theta})$			
	M.L.M	M.M	M.M.M	L.S.M
5	$(0.409,1.116)$	$(0.562,1.151)$	$(0.522,0.969)$	$(0.533,1.107)$
10	$(0.547,1.154)$	$(0.523,1.360)$	$(0.511,0.890)$	$(0.520,0.982)$
20	$(0.532,1.393)$	$(0.419,0.871)$	$(0.437,1.161)$	$(0.431,1.105)$
30	$(0.503,1.348)$	$(0.441,1.262)$	$(0.510,1.124)$	$(0.444,1.103)$
40	$(0.481,1.011)$	$(0.480,1.220)$	$(0.477,1.001)$	$(0.490,1.022)$
50	$(0.493,1.186)$	$(0.499,1.099)$	$(0.482,0.952)$	$(0.496,1.058)$

3.6 Application of Procedure $(W-5)$

A computer program for procedure $(W-5)$ of section (2.5.5) which utilize the Acceptance-Rejection method generate the $W(\frac{1}{2},1)$ varaites is shown in program (9) of Appendix (B). Sample size $n=5,10(10)50$ are taken. For high accuracy the procedure repeats itself 100 times to calculates the procedure efficiency and the run size 100 were made for the efficiency average. The result is displayed in table (3.8).

Table (3.8)

Efficiency of Procedure $(W-5)$

Theory Efficiency	Simulation Efficiency	**Error=Theory-Simulation**
0.76	0.759	0.001

Procedure $(W-6)$ is used to each one of the four methods of estimation with sample size $n=5,10(10)50$ and the repetition is 100 was made. The result are displayed in table (3.9).

Table (3.9)

Parameters estimation using procedure $(W-5)$

Sample size	Estimation of $(\hat{\alpha},\hat{\theta})$			
	M.L.M	M.M	M.M.M	L.S.M
5	$(0.522,1.007)$	$(0.535,1.110)$	$(0.531,0.983)$	$(0.415,1.373)$
10	$(0.512,1.046)$	$(0.526,0.824)$	$(0.511,1.213)$	$(0.433,1.023)$
20	$(0.434,0.919)$	$(0.401,1.021)$	$(0.460,1.213)$	$(0.452,1.064)$
30	$(0.439,0.981)$	$(0.537,1.150)$	$(0.464,1.058)$	$(0.483,1.099)$
40	$(0.483,0.951)$	$(0.440,1.248)$	$(0.494,1.023)$	$(0.496,1.056)$
50	$(0.498,0.928)$	$(0.491,1.044)$	$(0.499,0.988)$	$(0.500,0.972)$

3.7 Application of Procedure $(W-6)$

A computer program for procedure $(W-6)$ of section (2.5.6) which utilize the Central Limit Theorem to generate the $W(\frac{1}{2},1)$ varaites is shown in program (10) of Appendix (B). Sample size $n=5,10(10)50$ are taken. For high accuracy the procedure repeats itself 100 times. The result is displayed in table (3.10).

Table (3.10)

Parameters estimation using procedure $(W-6)$

Sample Size	Estimation of $(\hat{\alpha},\hat{\theta})$			
	M.L.M	M.M	M.M.M	L.S.M
5	(0.512,1.056)	(0.411,1.245)	(0.418,0.936	(0.549,0.811)
10	(0.510,1.099)	(0.531,1.084)	(0.527,0.922)	(0.530,0.806)
20	(0.478,0.901)	(0.476,0.776)	(0.423,1.022)	(0.409,1.083)
30	(0.487,0.881)	(0.509,1.232)	(0.453,0.926)	(0.452,1.020)
40	(0.491,1.083)	(0.485,0.928)	(0.466,1.047)	(0.513,1.003)
50	(0.498,1.020)	(0.501,1.074)	(0.496,1.037)	(0.495,1.199)

Chapter Two

Various Techniques of Sampling Weibull Variates by Monte Carlo Simulation

2

2.1 Introduction

The goal of this chapter is to generate random variates from Weibull distn. by using Inverse Transform Method, Theorem (1.2), Acceptance Rejection Method and Central Limit Theorem. In section (2.2) we introduced the genesis of Monte Carlo simulation and the uses of Monte Carlo methods; in section (2.3) we observed the random number generation and the algorithm which will be used to generate random variates. Section (2.4) show the random variates generation from continuous distn.'s which consist of two methods namely, Inverse Transform method (IT) and Acceptance Rejection method (AR). Section (2.5) we shall consider six procedures for generating random variates from Weibull distn. by Inverse Transform method, Theorem (1.2), Acceptance Rejection method and Central Limit Theorem.

After constructing a mathematical model for the problem under consideration, the next step is to derive a solution. There are analytic and numerical solution methods. The analytic solution is usually obtained directly from its mathematical representation in the form of formula, while the numerical solution is generally an approximate solution obtained as a result of substitution of numerical values for the variables and parameters of the model [27]. Many numerical methods are iterative, that is, each successive step in the solution uses the result from the previous step such as Newton's method for approximating the root of non-linear eq. Two special types of numerical methods simulation and the Monte Carlo are designed for a solution of deterministic and stochastic problem.

Simulation in a wide sense is defined as a numerical technique for conducting experiments on a digital computer which involve certain types of mathematical and logical models that describe the behavior of system over extended periods of real time, for example, simulating a football game, supersonic jet flight, a telephone communication system, wind tunnel [27], a large scale military battle (to evaluate defensive or offensive weapon system), or a mainterinance operation (to determine the optimal size of repair crews) and a live applications of real equipment in mock combat scenarios or firing range, these allow pilots, tank derivers and others soldiers to practice the physical activates of a war with their real equipment [26], etc. .

Whereas simulation in a narrow sense (also called stochastic simulation) is defined as experimenting with the model over time, it includes sampling stochastic variates from probability distn. Often simulation is viewed as a “Method of Last Resort” to be used when every things else has failed [27]. Software building and technical development have made simulation one of the most widely used and accepted tools for designers in the system analysis and operation research.

In this chapter, we shall introduce two methods to generate random variates from continuous probability distn., namely

1- Inverse Transform Method.

2- Acceptance-Rejection Method.

These methods specifically applied on six procedures for generating random variates from Weibull distn. .

2.2 Monte Carlo Simulation

Stochastic simulation is sometimes called Monte Carlo simulation, because sampling from a particular distn. involve the use of random numbers [27]. Historically, the Monte Carlo method was considered as a technique using random or pseudorandom numbers for a solution of a model. These random numbers are essentially independent random variables uniformly distributed over the unit interval [0,1].

Actually there are arithmetic codes available at computer center for generating sequence of pseudorandom number digits where each digits (0 through 9) occurs with approximately equal probability (imagine flip of a fair ten-side die). Such codes are called random number generators.

In the beginning of the 20th century the Monte Carlo method was used to examine the Boltzmann eq. [27].

In (1908) the famous statistician W.S.Gosset (student) used the Monte Carlo Method (experimental sampling) for estimating the correlation coefficient in his t-distn. [27]. One of earliest problem connected with Monte Carlo method is the famous Buffon's needle problem, who found the probability of a needle of length L thrown randomly onto a floor composed of parallel planks of width D>L is $p = \frac{2L}{\pi D}$ which can be estimated as the ratio of the number of throws hitting the crack to the total number of throws.

The term Monte Carlo was introduce by Von Numann and Ulam during world war *II* as a code word for the secret work at Los Alamos; it was suggested by the gambling casinos at the city of Monte Carlo in Monaco [30]. The Monte Carlo method was then applied to problems related to the a atomic bomb where the work involved direct simulation of behavior concerned with neutron random diffusion in fissionable material.

Shortly thereafter Monte Carlo methods used to evaluate complex multidimensional integrals, stochastic problems, and deterministic problems if they have the same formal expressions as some stochastic process. Also Monte Carlo method is used for solution of certain integrals and differential equations, sampling of random variates from probability distn.'s, and for analyzing complex problem (such as radiation transport to rivers). Useful references related to Monte Carlo simulation by Rubinstein (1981) [27] and Norman (1988) [22].

2.3 Random Number Generation [27]

Many techniques for generating random numbers on digital computer by Monte Carlo method and simulation have been suggested, tested, and used in recent years. Some of these methods are based on random phenomena, others on deterministic recurrence procedures.

Initially manual methods were used to generate, a sequence of numbers such as coin flipping, dice rolling, card shuffling, and roulette wheels, but these methods were too slow for general use and moreover the generated sequence of such methods could not reproduced .

With the computer aid it becomes possible to obtained random numbers. In (1951) Von Neumann suggested the mid-square method using the arithmetic operations of a computer. His idea was to take the square of the preceding random number and extract the middle digits. For instance, suppose we wish to generate 4-digits numbers.

1 – Choose any 4-digits to generate 4-digits numbers, say 3201.

2 – Square it, to have 10246401.

3 – The next 4-digits numbers is the middle 4-digit in step (2), that is 2464.

4 – Repeat the process.

This method proved slow and not suitable for statistical analysis, furthermore the sequence tends to cyclicity, and once a zero is encountered the sequence terminates [27].

One method of generating random numbers on digital computer was published by RAND Corporation (1955); consist of preparing a table of million random digits stored in the computer memory [25]. The advantage of this method is reproducibility and its disadvantage, was its slow and the risk of exhausting the table.

We say that, the random numbers generated by any method is a "good" one if the random numbers are uniformly distributed, statistically independent and reproducible; moreover the method is necessarily fast and requires minimum capacity in the computer memory.

The Congruential methods for generating pseudorandom numbers are designed specifically to satisfy as many of these requirements as possible.

These methods produce a nonrandom sequence of numbers according to some recursive formula based on calculating the residues module of some integer m of a linear transformation. Knuth [27], show that numbers generated by such sequence appear to be uniformly distributed and statistically independent [4].

The Congruential methods are based on a fundamental congruence relationship, which may be formulated as:

$$X_{i+1} = (aX_i + c)(\bmod m),\ i = 1,2,...,m . \qquad \text{.................... (2.1)}$$

where a is the multiplier, c is the increment, and m is the modulus (a, c, m are non-negative integers), (mod m) mean that eq.(3.1) can be written as:

$$X_{i+1} = aX_i + c - m[z] \qquad \text{.................................(2.2)}$$

where $[z] = \left[\frac{aX_i + c}{m}\right]$ is the largest integer in z.

Given an initial starting value X_1 with fixed values of a, c and m, then eq. (2.2) yields congruence relationship (modulo m) for any values i of the sequence {X_i}. The seq. $\{X_i\}$ will repeat itself in at most m steps and will be therefore periodic.

For example:

Let $a = c = X_1 = 4$, and $m = 9$, then the sequence obtained from the recursive formula

$X_{i+1} = (4X_i + 4)(\text{mod } 9)$ is $X_i = 4,2,3,7,5,6,1,0,4,\ldots\ ; i = 1,2,3,\ldots$.

The random number on the unit interval [0,1] can be obtained by:

$$U_i = \frac{X_i}{m},\ i = 1,2,\ldots,m\ . \qquad \ldots\ldots(2.3)$$

It follows from eq.(2.3) that $X_i \le m$, $\forall i$, this inequality mean that the period of the generator cannot exceed m, that is, the sequence $\{X_i\}$ contains at most m distinct numbers. So we should choose m as large as possible to ensure, a sufficiently large sequence of distinct numbers in the cycle.

It is noted in the literature, [16] that good statistical result can be achieved from computers by choosing $a = 2^{7+1}$, $c = 1$, and $m = 2^{35}$.

2.4 Random Variates Generation From Continuous Distribution

Many methods and procedures are proposed in the literature for generating random variates from different distribution. We shall utilize most well known methods namely, Inverse Transform Method (IT), and Acceptance-Rejection Method (AR).

2.4.1 Inverse Transform Method [27]:

Let X be a r.v. with c.d.f. *F(x),* since *F(x)* is non-decreasing function. The inverse function $F^{-1}(y)$ may be defined for any value of y between 0 and 1 as $F^{-1}(y)$ is the smallest x satisfying $F(x) \geq y$, that is,

$F^{-1}(y) = \inf\{x : F(x) \geq y\}$. ……….…………………………….(2.4)

It is important to prove the following theorem.

Theorem (2.1) [27]:

The random variable $U = F(X) \sim U(0,1)$ if and only if, the random variable $X = F^{-1}(U)$ has c.d.f. $\Pr(X \leq x) = F(X)$.

Proof :

Let the random variable $U = F(X) \sim U(0,1)$ then U has c.d.f.

$$G(u) = \Pr(U \leq u) = \begin{cases} 0, u \leq 0 \\ u, 0 < u < 1 \\ 1, u \geq 1 \end{cases}$$

Now,

$$\Pr(X \leq x) = \Pr[F^{-1}(U) \leq x] = \Pr[U \leq F(x)] = F(x).$$

Conversely, Let the random variable has X c.d.f. $\Pr(X \leq x) = F(x)$ and let G(u) be the c.d.f. of random variable U, then

$$G(u) = \Pr(U \leq u) = \Pr\left[F(X) \leq u\right] = \Pr\left[X \leq F^{-1}(u)\right] = F\left[F^{-1}(u)\right] = u$$

Q.D.E.

The algorithm of generating random variates by inverse transform method can be described by the steps of IT-Algorithm.

IT-Algorithm:

1- Generate U from $U(0,1)$.

2- Set $X = F^{-1}(U)$.

3- Deliver X as a random variable generated from the p.d.f. $f\left(x\right)$.

4- Stop.

As an application of IT- Algorithm, we shall consider the following two examples.

2.4.2 Examples:

Example (2.1) [27]:

Generate a r.v. X from $C\left(0,1\right)$ where the distn. p.d.f.

$$f(x)=\frac{1}{\pi(1+x^2)}\ ,\ -\infty<x<\infty$$

Solution:

The c.d.f. of this p.d.f. is

$$F(x)=\Pr\left(X\le x\right)=\int_{-\infty}^{x} f\left(t\right)dt=\frac{1}{\pi}\int_{-\infty}^{x}\frac{1}{1+t^2}dt$$

$F\left(x\right)=\dfrac{\tan^{-1}x}{\pi}+\dfrac{1}{2}$ set $u=F(x)$ implies

$$x=\tan\left[\pi(u-\frac{1}{2})\right] .$$

Apply IT-Algorithm:

1- Generate U from $U\left(0,1\right)$.

2- Set $X=\tan\left[\pi(U-\frac{1}{2})\right]$.

3- Deliver X as a r.v. generated from $f(x)=\dfrac{1}{\pi(1+x^2)}$.

4- Stop.

Example (2.2) [27]: Generate a random variable X from the distribution whose distn. p.d.f.

$$f(x)=\frac{1}{2}e^{-|x|}=\begin{cases}\frac{1}{2}e^{x}, & -\infty<x<0\\ \frac{1}{2}e^{-x}, & 0\le x<\infty\end{cases}$$

Solution:

The c.d.f. of this p.d.f. is

$$F(x)=\Pr(X\le x)=\begin{cases}0, & x\to-\infty\\ \int_{-\infty}^{x}f(t)dt, & -\infty<x<0\\ \int_{-\infty}^{0}f(t)dt+\int_{0}^{x}f(t)dt, & 0\le x<\infty\\ 1, & x\to\infty\end{cases}$$

So,

$$F(x)=\begin{cases}0, & x\to-\infty\\ \frac{1}{2}e^{x}, & -\infty<x\le 0\\ 1-\frac{1}{2}e^{-x}, & 0\le x<\infty\\ 1, & x\to\infty\end{cases}$$

For $-\infty<x<0$, set $u=F(x)\Rightarrow u=\frac{1}{2}e^{x}$ implies $x=\ln(2u)$, for $0<u<\frac{1}{2}$.

For $0\le x<\infty$, set $u=F(x)\Rightarrow u=1-\frac{1}{2}e^{-x}$ implies $x=-\ln(2u)$, for $\frac{1}{2}\le u<1$.

Apply IT-Algorithm:

1- Generate U from $U(0,1)$.

2- If $0<U<\frac{1}{2}$ set $X=\ln(2U)$; go to step (4).

3- Else, set $X=-\ln(2U)$.

4- Deliver X as a random variable generated from.

$$f(x)=\frac{1}{2}e^{-|x|},-\infty<x<\infty.$$

5- Stop.

We note that:

To apply Inverse Transform Method the c.d.f. $F(x)$ must exist in a form for which the corresponding inverse transform can found analytically. Some probability distn., it's either impossible or difficult to find the inverse transform, that is, to solve $u=F(x)=\int_{-\infty}^{x} f(t)dt$

For example :

1- $X\sim Exp(\lambda)$ where $f(x)=\frac{1}{\lambda}e^{-\frac{x}{\lambda}}$, $0<x<\infty$ (possible) .

2- $X\sim G(2,1)$ where $f(x)=x\,e^{-x}$, $0<x<\infty$ (difficult) .

3- $X\sim N(0,1)$ where $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, $-\infty<x<\infty$ (impossible) .

2.4.3 Acceptance Rejection Method [16]

This method is due to Von Neumann. This method can be applied to generate variable from an appropriate distn. and subjecting it to a test to determine whether or not it will be acceptable for use.

To carry out the method, we represent the p.d.f. $f(x)$ of the generated random variable X as $f(x)=c\,h(x)g(x)$ where $c\geq 1$, *h(x)* is also a p.d.f. and $0\leq g(x)\leq 1$. Then we generate two random variables U and Y from $U(0,1)$ and h(y), respectively, and test to see whether or not the inequality $U\leq g(Y)$ holds.

1- If the inequality holds, then accept *Y*=*X* as a variate generated from $f(x)$.

2- If the inequality is violated, reject the pair U, Y and try again.

Theorem (2.2) [27]:

Let X be a random variable distributed with the p.d.f. $f(x)$, $x\in I$, which is represented as $f(x)=c\,h(x)g(x)$ where $c\geq 1$, h(x) is also a p.d.f. and $0\leq g(x)\leq 1$.

Let U and Y be a distributed $U(0,1)$ and h(y), respectively, then $\Pr\left[Y=x\,\middle|\,U\leq g(Y)\right]=f(x)$.

Proof:

$$\Pr\left[Y=x\,\middle|\,U\leq g(Y)\right]=\frac{\Pr\left[Y=x,U\leq g(Y)\right]}{\Pr\left[U\leq g(Y)\right]}$$

$$=\frac{\Pr\left[Y=x,U\leq g(Y)\right]}{\int_x \Pr\left[Y=x,U\leq g(Y)\right]dx}$$

Using Bayes theorem [18], we have:

$$\Pr\left[Y=x\,\middle|\,U\leq g(Y)\right]=\frac{\Pr\left(U\leq g(Y)\,\middle|\,Y=x\right)\Pr\left(Y=x\right)}{\int_x \Pr\left[U\leq g(Y)\,\middle|\,Y=x\right]\Pr\left(Y=x\right)dx}$$

Since,

$\Pr\left[U \le g(Y) \middle| Y = x\right] = \Pr\left[U \le g(x)\right] = g(x)$ and $\Pr(Y = x) = h(x)$

Therefore ;

$$\Pr\left[Y = x \middle| U \le g(Y)\right] = \frac{g(x)h(x)}{\int_x g(x)h(x)dx} = \frac{g(x)h(x)}{\int_x \frac{f(x)}{c}dx} = \frac{g(x)h(x)}{\frac{1}{c}}$$

$$= c\,h(x)g(x).$$

The efficiency of Acceptance-Rejection Method is to determined by the inequality $U \le g(Y)$ where efficiency is: $\Pr\left[U \le g(Y)\right] = \frac{1}{c} = p$.

Because the trails are independent, the probability of success in each trials is $p = \frac{1}{c}$. The number of trials N before a successful pair (U,Y) has geometric distn. with p.d.f.

$$\Pr(N = n) = p(1-p)^{n-1},\ n = 1,2,3,...$$
$$= 0 \qquad ,e.w.$$

With the expected number of trails $E(N) = \frac{1}{p} = c$.

The AR-Algorithm describes the necessary steps of generating a random variable by Acceptance-Rejection Method.

AR-Algorithm:

1- Generate U from $U(0,1)$.
2- Generate Y from h(y) .
3- If $U \le g(Y)$, deliver (we accept) $Y=X$ as a random variable generated from the p.d.f. $f(x)$. Go to step (5).
4- Else go to step (1).
5- Stop.

We note that, for the Acceptance-Rejection method to be of practical interest, the following criteria must be used.

(i) It should be easy to generate from *h(x).*

(ii) The efficiency (probability) of the procedure $\frac{1}{c}$ should be large, that is, c closed to one .

As an application of AR-Algorithm, we shall consider the following two example.

To illustrate the method, we choose *c≥1* such that $f(x) \le c\,h(x) = \varphi(x), \forall x \in I$. Then the problem is to find the function $\varphi(x)$ and the function $h(x) = \frac{\varphi(x)}{c}$ from which the r.v. can be generated.

2.4.4 Examples

Example(2.3) [27]:

Solution:

Since, $\sqrt{R^2 - x^2} \le R, \quad \forall x \in [-R, R]$

Then, $f(x) = \frac{2}{\pi R^2}\sqrt{R^2 - x^2} \le \frac{2R}{\pi R^2} = \frac{2}{\pi R} = \varphi(x)$

But, $c\,h(x) = \varphi(x)$ implies $c = \int_{-R}^{R} \varphi(x)dx = \int_{-R}^{R} \frac{2}{\pi R}dx = \frac{4}{\pi}$

So $h(x) = \frac{\varphi(x)}{c} = \frac{\frac{2}{\pi R}}{\frac{4}{\pi}} = \frac{1}{2R}, -R \le x \le R$ and

$$g(x) = \frac{f(x)}{c\,h(x)} = \frac{\sqrt{R^2 - x^2}}{R} \ .$$

Now, the c.d.f. of the p.d.f. *h(x)* is:

$$H(x) = \int_{-\infty}^{x} h(t)dt = \int_{-R}^{x} \frac{1}{2R}dt = \frac{x+R}{2R}.$$

Set

$u_2 = H(y) = \dfrac{y+R}{2R}$ implies $y = (2u_2 - 1)R$.

Apply AR-Algorithm:

1- Read R.

2- Generate U_1 and U_2 from $U(0,1)$.

3- Set $Y = (2u_2 - 1)R$.

4- If $U_1 \le g(Y) = \dfrac{1}{R}\sqrt{R^2 - Y^2}$, deliver (we accept) Y=X as a r.v. generated from *f(x)*.Go to step (6).

5- Else Go to step (2).

6- Stop.

The expected number of trials $c = \dfrac{4}{\pi} = 1.273$ and the efficiency is:

$\dfrac{1}{c} = \dfrac{\pi}{4} = 0.785$.

Example(2.4) [27]:

Generate a r.v. from the distn. p.d.f.

$$f(x) = 6x(1-x), 0 < x < 1$$
$$= 0 \qquad , e.w.$$

Solution:

Since, $x(1-x) \le x \Rightarrow f(x) = 6x(1-x) \le 6x = \varphi(x)$

$c\,h(x) = \varphi(x) \Rightarrow \int_0^1 c\,h(x)dx = \int_0^1 \varphi(x)dx \Rightarrow c = \int_0^1 6x\,dx$ implies $c = 3$.

Then,

$$h(x)=\begin{cases}2x, 0<x<1\\ 0, e.w.\end{cases}$$

$$g(x)=\frac{f(x)}{c\,h(x)}=\frac{6x(1-x)}{3\times 2x}=\frac{6x(1-x)}{6x}\Rightarrow g(x)=1-x, 0<g(x)<1.$$

Now,

$$H(x)=\Pr(X\le x)=\begin{cases}0, x\le 0\\ x^2, 0<x<1\\ 1, x>1\end{cases} \qquad \text{setting } u_2=H(y)=y^2$$

implies $y=\sqrt{u_2}$.

Apply AR-Algorithm:

1- Generate U_1 and U_2 from $U(0,1)$.

2- Set $Y=(U_2)^{\frac{1}{2}}$.

3- If $U_1\le 1-Y$, deliver Y=X as a r.v. generated from p.d.f.

$f(x)=6x(1-x)$.

4- Else go to step (1).

5- Stop

The expectation number of trials $c=3$, and the efficiency is:

$$\frac{1}{c}=\frac{1}{3}=0.333$$

2.5 Procedures for Generating Random Variates for Weibull Distribution

In this section we shall consider six procedures for generating random variates from Weibull distn. by using Inverse Transform method, Theorem (1.2), Acceptance-Rejection method and Central Limit Theorem.

2.5.1 Procedure (W-1):

This procedure is based on Inverse Transform Method.

$$f(x) = \alpha\theta x^{\alpha-1} e^{-\theta x^{\alpha}}, 0 < x < \infty$$
$$= 0 \qquad , e.w. \ ; \alpha > 0, \theta > 0$$

The c.d.f. of this p.d.f. is

$$F(x) = \Pr(X \le x) = \int_{-\infty}^{x} f(t)dt = \alpha\theta \int_{0}^{x} t^{\alpha-1} e^{-\theta t^{\alpha}} dt \text{ implies}$$

$F(x) = 1 - e^{-\theta x^{\alpha}}$, setting $u = F(x)$ implies $u = 1 - e^{-\theta x^{\alpha}}$ implies

$$x = \left(\frac{-1}{\theta}\ln(u)\right)^{\frac{1}{\alpha}}.$$

Algorithm (W-1):

1- Read α, θ.

2- Generated U from $U(0,1)$.

3- Set $X = \left[\frac{-1}{\theta}\ln(u)\right]^{\frac{1}{\alpha}}$.

4- Deliver X as a r.v. generated from $f(x) = \alpha\theta x^{\alpha-1} e^{-\theta x^{\alpha}}$.

5- Stop.

2.5.2 Procedure (W-2):

This procedure is based on Theorem (1.2) as follows:

Algorithm (W-2):

1- Read α, θ.

2- Generate U from $U(0,1)$.

3- Set $X = -\ln(U)$.

4- Set $Y = e^{-X}$.

5- Deliver Y as a r.v. generated from W(α,θ).

6- Stop.

2.5.3 Procedure (W-3):

The procedure is based on Acceptance-Rejection method, where the Weibull variate is generated by utilizing the standard normal distn. as follows:

The p.d.f. of r.v. $X \sim N(0,1)$ is

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty$ where we make use the inequality

$e^{\frac{-x^2}{2}} \le \frac{2}{1+x^2}, \forall x\,(-\infty,\infty)$.

To apply the Acceptance-Rejection method, we need to write the p.d.f. as $f(x) = c\,h(x)\,g(x)$ as shown in section (2.4.3).

Now, we consider the inequality

$e^{\frac{-x^2}{2}} \le \frac{2}{1+x^2} \Rightarrow \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \le \frac{1}{\sqrt{2\pi}} \frac{2}{1+x^2}$, then

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \le \frac{1}{\sqrt{2\pi}} \frac{2}{1+x^2} = \varphi(x) \ .$$

$$c\,h(x) = \varphi(x) \Rightarrow \int_{-\infty}^{\infty} \varphi(x)\,dx = \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}\left(1+x^2\right)} dx \Rightarrow c = \sqrt{2\pi}$$

$$h(x)=\frac{\varphi(x)}{c}=\frac{\frac{\sqrt{2}}{\sqrt{\pi}(1+x^2)}}{\sqrt{2}\sqrt{\pi}}=\frac{1}{\pi(1+x^2)}.$$

$$H(x)=\int_{-\infty}^{x}\frac{dx}{\pi(1+x^2)}=\frac{1}{\pi}\tan^{-1}t\Big|_{-\infty}^{x}=\frac{1}{\pi}(\tan^{-1}x+\frac{\pi}{2})=\frac{1}{\pi}\tan^{-1}x+\frac{1}{2}$$

$$g(x)=\frac{f(x)}{\varphi(x)}=\frac{1}{2}(1+x^2)e^{-\frac{x^2}{2}}\text{, where } 0\le g(x)\le 1.$$

Set $u_2=H(y)=\frac{1}{\pi}\tan^{-1}y+\frac{1}{2}\Rightarrow \pi(u_2-\frac{1}{2})=\tan^{-1}y$ implies

$$y=\tan\left(\pi(u_2-\frac{1}{2})\right).$$

The number of trials equal to $c=\sqrt{2\pi}\approx 2.51$and the efficiency (probability) of the method is equal to $\frac{1}{c}=\frac{1}{\sqrt{2\pi}}\approx 0.40$

Algorithm (W-3):

1- Read α,θ.

2- For *i=1 to 2.*

3- Generate U_1 and U_2 from $U(0,1)$.

4- Set $Y=\tan\left(\pi(u_2-\frac{1}{2})\right)$.

5- If $U_1>g(Y)$ go to step (3).

6- Else set $X_i=Y$ as a r.v. generated from $N(0,1)$.

7- Next *i.*

8- Set $Z_1=\sqrt{\theta}X_1,\ Z_2=\sqrt{\theta}X_2$ and $R=\left(\frac{Z_1^2+Z_2^2}{2}\right)^{\frac{1}{\alpha}}$.

9- Deliver R as a r.v. generated from W(α,θ).

10- Stop.

2.5.4 Procedure (W-4):

This procedure is based on Acceptance-Rejection method, where the Weibull variate is generated by utilizing the Normal distn. as follows:

Since, the standard Normal distn. is symmetric about origin, then the p.d.f. of r.v. $X \sim N^{+}(0,1)$ can be written as:

$$f(x)=\sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}},\ 0<x<\infty$$
$$=0 \quad ,e.w.$$

We can make use of the inequality $e^{-\frac{x^2}{2}} \le \frac{6ke^{-kx}}{1+e^{-2kx}}$, where $k=\sqrt{\frac{8}{\pi}}$

[27] .

To apply the Acceptance-Rejection method, we need to write the p.d.f. as $f(x)=c\,h(x)g(x)$ as shown in section (2.4.3).

Now, we consider the inequality $e^{-\frac{x^2}{2}} \le \frac{6ke^{-kx}}{1+e^{-2kx}},\ 0<x<\infty$.

$$f(x)=\sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}} \le \frac{3k^2e^{-kx}}{1+e^{-2kx}}=\varphi(x)$$

$$c\,h(x)=\varphi(x) \Rightarrow c=\int_0^\infty \varphi(x)dx=\int_0^\infty \frac{3k^2e^{-kx}}{1+e^{-2kx}}dx \text{ implies } c=3\sqrt{\frac{\pi}{2}}$$

$$h(x)=\frac{\varphi(x)}{c}=\sqrt{\frac{2}{\pi}}\frac{k^2e^{-kx}}{1+e^{-2kx}} \quad ,0<x<\infty$$
$$=0,e.w.$$

$$H(x)=\Pr(X\le x)=\int_{-\infty}^{x}h(t)dt=\begin{cases}0, & x\le 0\\ 1-\frac{4}{\pi}\tan^{-1}\left(e^{-kx}\right), & 0<x<\infty\\ 1, & x\to\infty\end{cases}$$

$$g(x)=\frac{f(x)}{\varphi(x)}=\frac{\sqrt{\frac{2}{\pi}}\left(1+e^{-2kx}\right)e^{-\frac{x^2}{2}}}{\sqrt{\frac{8}{\pi}}3ke^{-kx}}=\frac{e^{-\frac{x^2}{2}}(1+e^{-2kx})}{6ke^{-kx}}\ \text{where}$$

Set $u_2=H(y)\Rightarrow u_2=1-\frac{4}{\pi}\tan^{-1}\left(e^{-ky}\right)$ implies

$$y=-\frac{1}{k}\ln\left(\tan\left(\frac{\pi}{4}(u_2)\right)\right).$$

The number of trials equal to $c=3\sqrt{\frac{\pi}{2}}\approx 3.76$ and the efficiency (probability) of the method is equal to $\frac{1}{c}=\frac{1}{3}\sqrt{\frac{2}{\pi}}\approx 0.266$

Algorithm (W-4):

1- Read where α, θ, k.

2- For *i=1 to 2*.

3- Generate U_1 and U_2 from $U(0,1)$.

4- Set $Y=-\frac{1}{k}\ln\left[\tan\left(\frac{\pi}{4}(u_2)\right)\right]$.

5- If $U_1>g(Y)$ go to step(3).

6- Generate U_3 from $U(0,1)$.

7- If $U_3<\frac{1}{2}$ set $X_i=-Y$ as a r.v. generated from $N^{-}(0,1)$.

8- Else set $X_i=Y$ as a r.v. generated from $N^{+}(0,1)$.

9- Next i.

10- Set $Z_1 = \sqrt{\theta} X_1, Z_2 = \sqrt{\theta} X_2$ and $R = \left(\frac{Z_1^2 + Z_2^2}{2} \right)^{\frac{1}{\alpha}}$.

11- Deliver R as a r.v. generated from W(α,θ) .

12- Stop.

2.5.5 Procedure (W-5):

This procedure is based on Acceptance-Rejection method, where the Weibull variate is generated by utilizing the standard normal distn. as follows:

Since the standard normal distn. is symmetric about origin, then the p.d.f. of r.v. $X \sim N^+(0,1)$ can be written as:

$$f(x) = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}, 0 < x < \infty$$
$$= 0 \quad , e.w.$$

Where we use of inequality $(x-1)^2 \geq 0$ [27].

To apply the Acceptance-Rejection method, we need to write the p.d.f. as $f(x) = c\,h(x)g(x)$ as shown in section (2.4.3).

Now, we consider the inequality $(x-1)^2 \geq 0 \Rightarrow x^2 - 2x + 1 \geq 0$.

$$\Rightarrow \frac{-x^2}{2} \leq \frac{1}{2} - x \Rightarrow e^{\frac{-x^2}{2}} \leq e^{\frac{1}{2}-x}$$, then

$$\sqrt{\frac{2}{\pi}} e^{\frac{-x^2}{2}} \leq \sqrt{\frac{2}{\pi}} e^{\frac{1}{2}-x} = \sqrt{\frac{2e}{\pi}} e^{-x} = \varphi(x).$$

$$c\,h(x)=\varphi(x)\Rightarrow c=\int_0^\infty \varphi(x)\,dx=\sqrt{\frac{2e}{\pi}}\int_0^\infty e^{-x}dx \text{ implies } c=\sqrt{\frac{2e}{\pi}}$$

$$h(x)=\frac{\varphi(x)}{c}=e^{-x}\ ,0<x<\infty$$
$$=0\quad ,e.w.$$

$$H(x)=\Pr(X\le x)=\int_0^\infty h(t)dt=\begin{cases}0,x\le 0\\ 1-e^{-x}\ ,\ 0<x<\infty\\ 1,x\to\infty\end{cases}$$

$$g(x)=\frac{f(x)}{\varphi(x)}=e^{-\frac{(x-1)^2}{2}} \text{ where } 0<g(x)<1.$$

Set $u_2=H(y)\Rightarrow u_2=1-e^{-y}\Rightarrow y=-\ln(u_2)$.

The number of trails equal to $c=\sqrt{\frac{2e}{\pi}}\approx 1.32$ and the efficiency (probability) of the method is equal to $\frac{1}{c}=\sqrt{\frac{\pi}{2e}}\approx 0.76$

Algorithm (W-5):

1- Read where α,θ.

2- For *i=1 to 2.*

3- Generate U_1 and U_2 from $U(0,1)$.

4- Set $Y=-\ln(u_2)$.

5- If $U_1>g(Y)$ go to step (3) .

6- Generate U_3 from $U(0,1)$.

7- If $U_3<\frac{1}{2}$ set $X_i=-Y$ as a r.v. generated from $N^-(0,1)$.

8- Else set $X=Y$ as a r.v. generated from $N^+(0,1)$.

9- Next i.

10- Set $Z_1 = \sqrt{\theta} X_1, Z_2 = \sqrt{\theta} X_2$ and $R = \left(\frac{Z^2_1 + Z^2_2}{2} \right)^{\frac{1}{\alpha}}$.

11- Deliver R as a r.v. generated from W(α,θ).

12- Stop.

2.5.6 Procedure (W-6):

This procedure is based on a Central Limit Theorem [27], viz. For large n, Let $X_1, X_2, \ldots, X_n$ be a r.s of size n from any distn. . (discrete. and continuous) having mean μ and variance δ^2 with existence of *M(t)*. Then the r.v. $X = \frac{\sqrt{n}\left(\overline{X} - \mu\right)}{\delta} \sim app\, N\,(0,1)$.

To apply this procedure, we consider a r.s. $U_1, U_2, \ldots, U_n$ of size n from $U\,(0,1)$ where p.d.f.

$$g(u) = 1 \quad ,0 < u < 1$$
$$= 0 \quad ,e.w.$$

Since, $\mu = E(U) = \frac{1}{2}$ and $\delta^2 = Var(U) = \frac{1}{12}$ implies

$$X = \frac{\sqrt{n}(\overline{U} - \frac{1}{2})}{\sqrt{\frac{1}{12}}} = \sqrt{12n}(\overline{U} - \frac{1}{2}).$$

Algorithm (W−6):

1- Read where α, θ.

2- For *i=1 to 2*.

3- Generate $U_1, U_2, \ldots, U_n$ from $U\,(0,1)$.

4- Set $\overline{U} = \frac{1}{n}\sum_{i=1}^{n} U_i$, $X_i = \sqrt{12n}(\overline{U} - \frac{1}{2})$.

5- Set $Z_1 = \sqrt{\theta} X_1, Z_2 = \sqrt{\theta} X_2$ and $R = \left(\frac{Z^2_1 + Z^2_2}{2} \right)^{\frac{1}{\alpha}}$.

6- Deliver R as a r.v. generated from *W(α,θ)*.

7- Stop.

Conclusions

From the present study, we can conclude the following:

1- *Inverse Transform procedure has less time consuming in comparison with the other procedures of generation.*

2- *The theory and practice show that the efficiency of procedure (W-5) is superior than procedures (W-3) and (W-4).*

3- *For all sample sizes, the M.L.M and L.S.M give estimates* $\hat{\alpha}$ *and* $\hat{\theta}$ *which is close to the exact values of* α *and* θ.

4- *For moderate sample sizes, the M.M.M gives estimate close to the exact values of* α *and* θ.

5- *The M.M.M. and M.M. gives small bias for estimating* α*, while M.M.M and M.M. gives small bias for estimating* θ *in comparison with other methods of estimation.*

6- *M.L.M. gives small variance of* $\hat{\alpha}$ *in comparison with other methods.*

7- *L.S.M. gives small variance of* $\hat{\theta}$ *in comparison with other methods.*

8- *M.L.M. and L.S.M. gives small MSE of* $\hat{\alpha}$ *in comparison with other methods.*

9- *L.S.M. gives small MSE of* $\hat{\theta}$ *in comparison with other methods.*

10- *The disadvantage of Monte-Carlo methods depends on generating pseudorandom variates and that might carry dirty data.*

Future Work and Recommendation

1- This work can be use for generalized Weibull distribution of three parameters and other life distribution.

2- Another methods can be used to estimate the distribution parameters α *and* θ *like Minimum Chi-square, Minimum Distance, Bayesian Method, … etc.*

3- It can be generate r.v.'s from Weibull distribution by other new procedures which can be compare their efficiency with our used procedures.

4- The bias of estimation is a r.v. of unknown distribution which can be investigated approximately by using well-known statistical tests such as Chi-Square Goodness-of-Fit Test, Kolmogorov-Smirnov Goodness-of-Fit Test, Serial Test, …etc.

Contents

Chapter Two: Various Techniques of Sampling Weibull Variates by Monte Carlo Simulation

Chapter Three: Monte Carlo Application

Examining Committee's Certification

We certify that we read this thesis entitled "*Estimation of Parameters for Weibull Distribution with Application by Using Monte Carlo Simulation*" and as examining committee examined the student, *Salam Adel Ahmed* in its contents and in what it connected with, and that is in our opinion it meet the standard of thesis for the degree of Master of Science, in Mathematics.

(Chairman)

Signature:

Name:

Date:

(Member)

Signature:

Name:

Date:

(Member)

Signature:

Name:

Date:

(Member and Supervisor)

Signature:

Name: Dr. Akram M.AL-Abood

Date:

Approved for the Dean of the College of Science

Signature:

Name: Assis. Prof. Dr. LAITH ABDUL -AZIZ

Dean of College of Science of Al-Nahrain University

Date: / /2008

Introduction

The Weibull distn. has been widely used as a model in many areas of applications, specifically in the studies of failure components and as a model for product life. It has also been used as the distn. of strength of certain materials. It is named after the Swedish scientist Weibull [31] who first proposed the distn. in connection with his studies on strength of materials [31]. One reason for its popularity is that it has a great variety of shapes, which make it extremely flexible in fitting many kinds of empirical data. Weibull [32] showed that the distn. is also useful in describing the "wear-out" or fatigue failures. The distn. has been showed to provide a useful probabilistic model for time to failure of system, which consist of a large number of components where system failure occurs as soon as one component fails. Kao [19] used it as a model for vacuum tube failure [15], Mann gave a variety of situations in which the distn. is used for other types of failure data, Whitmore and Altschalerf used it in studies on the time interval to the occurrence of tumors in human population. Cohen [5] derived the maximum likelihood eq.'s to estimate the distn. parameters from ***(i)*** complete sample ***(ii)*** singly censored samples and ***(iii)*** progressively (multiple) censored samples. Bain and Antle [17] used a Maximum Likelihood method to obtain two simple estimators of parameters for Weibull distn. . These estimators are similar to the estimators given by Gumbel, Miller, Freund, and Menon. Some useful properties of these estimators are developed to make it practical use in Monte Carlo methods to determine the variance and biases of the estimators for various sample sizes. Comparison between the estimators are made and unbiasing factors calculated in some cases. The variance of the estimators were also compared with the Cramer-Rao lower bounds for regular unbiasing estimators. Darrel, et al [10]. stated the problems of estimation and testing hypothesis regarding the parameters of the Weibull distn., and they obtained the following result ***(i)*** Exact confidence intervals for the parameters based upon maximum likelihood estimators. ***(ii)*** A table of unbiasing factors (developing upon sample size) for the maximum likelihood estimator of the shape parameters. ***(iii)*** Tests of hypotheses regarding the parameters and the power of the test regarding the shape parameter are developed. Nancy [21] assumed the Weibull distn. model of two parameters to obtain the exact confidence bounds for the shape parameter and for reliable life and shown that the analytically derived bounds for a few ordered observations be highly efficient w.r.t. those derivable made by Monte Carlo procedures using all order observation. Pandey and Upadhyay [24] studied shrinkage method to estimate the two parameters of Weibull distn. when the shape parameter known and when the scale parameter, shape parameter unknown. Al-Badhani [1] developed a new parameterization and general form for the distn. with three parameters. This formulation avoids many problems that appear in estimation and applications where his studied concern the strength of ceramic materials.

Ishioka and Nonaka [28] presented a stable technique for obtaining the maximum Likelihood estimate of Weibull parameters of the life distn.,s of two components that form a series system. This technique requires much more computation than a previously published procedure. The simulation results, however, showed the standard deviation of the estimated values of the Weibull parameters greatly reduced. This technique does not require the concomitarite indicator, and can be applied not only for complete data but for randomly censored data. Seki and Yokoyama [29] were proposed robust estimation methods for the Weibull parameters, and applies bootstrap estimators of order statistics to the parametric estimation procedure. Estimates of the Weibull parameters are equivalent to the estimates using the extreme value distn. .They examined the bootstrap estimators of order statistics for the parameters of the extreme value distn. . Accuracy and robustness for outliers are examined by Monte Carlo experiments which indicate adequate efficiency of the proposed estimators for data with some outliers. Al-Ali [3] studied some estimators of parameters and reliability function for Weibull distn. and suggested four methods to estimate the shape parameter when the scale parameter is known. Al-Fawzan [7] presented two categories methods, ***(i)*** graphical method and ***(ii)*** analytic method, for estimating the shape and scale parameters of Weibull distn., and he reported the computational experiments on the present methods. Dongfung and Guanzhong [11] used Monte Carlo simulations to search for the optimal probability estimator for estimating Weibull parameters with the linear regression method. Compared with commonly used probability estimators, the optimal one obtained gives a more accurate estimation of the Weibull modulus and the same estimation precision of the scale parameter. They will also concluded that the maximum likelihood method results in the highest precision, however, less conservative than the linear regression method**.** Abed [12] compared the parameters and reliability function of Weibull distn. with three parameters expressed as a failure model, using some classical methods of estimation (Maximum Likelihood Method, Moment Method), and Bayesian Methods (Bayes Method, shrinkage Method), and he used the Monte Carlo simulation to compare these methods. Montanari, et al. [13], applied unbiasing procedures for the maximum likelihood method to parameters of Weibull function are dealt with. The performance of unbiasing methods applied to expected values and point estimates of the Weibull parameters, as well they discussed the Monte Carlo method for the estimation of the expected values, and they shown that the accuracy of the unbiasing methods can be significantly affected by several factors, such as the value of the shape parameter and the estimation of the expected value, and that some methods can be successfully applied to the point estimates of the Weibull parameters.

The aim of thesis is to estimates the parameters of Weibull distn. by using four methods of estimation and generating a procedures of random variates from Weibull distn. by using Monte Carlo simulation. The professional MATHCAD, 2005 computer software is used in make the programs of thesis.

This thesis includes three chapters. In chapter one, we present some important mathematical and statistical properties of Weibull distn. .

Genesis of the distn. is derived by extending the idea of obtaining the Rayleigh distn. which utilize some specific transformation related to Normal distn. . Also, we show that the Weibull distn. can arise by two different approaches. Moment properties of the distn. are illustrated and unified. Four methods of estimation for the distn. parameters are discussed theoretically and assessed practically. Monte Carlo simulation is made by four methods of estimation.

In chapter two, we introduce some concepts of the history of stochastic simulation. Procedures for generating random numbers and random variates from different distn. is discussed theoretically and supported by various examples.

Six procedures for generating random variates from Weibull distn. are considered, and then some of these are discussed with efficiency and with out efficiency and number of trails are illustrated.

In chapter three, we utilize practically the procedures of generating variates from Weibull distn. as discussed theoretically in chapter two. These procedures are applied from parameters estimation with efficiency of some procedures.

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Supervisor Certification

I certify that this thesis was prepared under my supervision at the Al-Nahrain University, College of Science, in partial fulfillment of as of the requirements for the degree of Master of Science in Mathematics.

Signature:
Name: Assist. Prof. Dr. Akram M. Al-Abood
Date: /   /2008

In view of the available recommendations, I forward this thesis for debate by the examination committee.

Signature:
Name: Assist. Prof. Dr. Akram M. Al-Abood
Chairman of Mathematics Department
Date: /   /2008

بِسْمِ اللّٰهِ الرَّحْمَنِ الرَّحِيْمِ

وانزل الله عليك الكتاب والحكمة وعلمك ما لم تكن تعلم وكان فضل الله عليك عظيما ﴿١١٣﴾

صدق الله العلي العظيم

سورة النساء الآية (١١٣)

المُستَخلَص

تطرقنا في هذه الرسالة إلى توزيع ويبل ذو المعلمتين لأهميته في مجالات الإحصاء وتطبيقاته من حيث استعراض لخواص التوزيع الرياضية والإحصائية والعزوم والعزوم العليا. ثم تطرقنا إلى التخمين وخواصه ومناقشة أربعة طرق لتخمين معالم التوزيع وهي:

طريقة الترجيح الأعظم، طريقة العزوم، طريقة العزوم المعدلة وطريقة المربعات الصغرى.

نوقشت هذه الطرق نظريا و طبقت عمليا باستخدام ستة أساليب من محاكاة مونت كارلو لتوليد المتغيرات العشوائية من توزيع ويبل. أوجدت كفاءة بعض هذه الأساليب نظريا و قورنت عمليا. تمت المقارنة بين الطرائق الأربعة التخمينية باستخدام مقياس معدل مربعات الخطأ.

Republic of Iraq
Ministry of Higher Education
and Scientific Research
Al-Nahrain University
College of Science
Department of Mathematics
and Computer Applications

Estimation of Parameters for Weibull Distribution with Application by Using Monte Carlo Simulation

A Thesis

Submitted to the College of Science Al-Nahrain University as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By
Salam Adel Ahmed
(B.Sc., Al-Nahrain University, 2005)

Supervised by

Asst. Prof. Dr. Akram M. Al-Abood

February 2008 | Muharam 1429

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة النهرين

كلية العلوم

قسم الرياضيات و تطبيقات الحاسوب

تخمين معلمات توزيع ويبل مع تطبيق باستخدام محاكاة مونت كارلو

رسالة

مقدمة إلى كلية العلوم – جامعة النهرين وهي جزء من متطلبات نيل درجة ماجستير في

علوم الرياضيات

من قبل

سلام عادل احمد

(بكالوريوس علوم, جامعة النهرين, ٢٠٠٥)

بأشراف

د. أكرم محمد العبود

شباط ٢٠٠٨

محرم ١٤٢٩