Chapter One

Fractional Calculus

3.1 Introduction:

In this chapter we obtain the fractional power series solution of homogeneous linear fractional differential equation of order α (0<α<1) around regular α-singular point x_0 of the equation

$$(x - x_0)^\alpha D^\alpha y(x) + p(x)y(x) = 0 \qquad (3.1)$$

where $p(x)$, due to the regular α-singular character of x_0,is an α-analytic function around x_0.

Definition (3.1),[1]:

A point $x_0 \in [a, b]$ is said to be an α- singular point of the equation (3.1), if the function $p(x)$ is not α-analytic at x_0.

Definition (3.2), [1]:

Let $x_0 \in [a, b]$ be an α-singular point of the equation (3.1), then x_0 is said to be a regular α-singular point of this equation if the functions $(x - x_0)^\alpha p(\alpha)$ is α-analytic in x_0. Otherwise, x_0 is said to be an essential α-singular point.

For example a point $x = x_0 > 1$ is an ordinary point for the following equation:

$$(x - 1)^\alpha D^\alpha y(x) - y(x) = 0 \qquad (3.2)$$

the point $x = 1$ is a regular α-singular point for the equation (3.2).

Theorem (3.1):

Let $x_0 \geq a$ be a regular α-singular point of the differential equation (3.1) of order α, and let

$$p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^{n\alpha} \quad , (\, p_n \in R) \tag{3.3}$$

be the power series expansion of the α-analytic function $p(x)$.Then equation (3.1) is solvable and the function

$$y(x) = (x - x_0)^s \sum_{n=0}^{\infty} a_n (x - x_0)^{n\alpha} \quad , (a_n \in R) \tag{3.4}$$

is the solution for $x \in (x_0, x_0 + \rho)$ where a_0 is a non-zero arbitrary constant, s>-1 is the real solution to the equation

$$\frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} + p_0 = 0$$

and the coefficients $a_n (n \geq 1)$ are given by the following recurrence formula:

$$a_n = \frac{\Gamma(n\alpha + s - \alpha + 1)}{\Gamma(n\alpha + s + 1)} \sum_{i=0}^{n-1} a_i p_{n-i}$$

Moreover, if the series (3.3) converges for all x in the interval $0 < x - x_0 < R$ $(R > 0)$, then the series solution (3.4) of equation (3.1) is also convergent in the same interval.

Proof Theorem (3.1):

Seeking a solution to equation (3.1) in the form (3.4):

$$y(x) = (x - x_0)^s \sum_{n=0}^{\infty} a_n (x - x_0)^{n\alpha}$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n\alpha + s}$$

and taking fractional differentiation of $y(x)$ according to (1.29) we get:

$$D^{\alpha} y(x) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha + s + 1)}{\Gamma(n\alpha + s - \alpha + 1)} (x - x_0)^{n\alpha + s - \alpha} \qquad (3.5)$$

and substituting the result (3.4) ,(3.3) and (3.5) in the general equation (3.1) we have :

$$(x - x_0)^{\alpha} \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha + s + 1)}{\Gamma(n\alpha + s - \alpha + 1)} (x - x_0)^{n\alpha + s - \alpha} - \left(\sum_{n=0}^{\infty} p_n (x - x_0)^{n\alpha}\right)\left(\sum_{n=0}^{\infty} a_n (x - x_0)^{n\alpha + s}\right) = 0 \qquad (3.6)$$

we can write equation (3.6) as:

$$\sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha + s + 1)}{\Gamma(n\alpha + s - \alpha + 1)} (x - x_0)^{n\alpha + s} - \left(\sum_{n=0}^{\infty} p_n (x - x_0)^{n\alpha}\right)\left(\sum_{n=0}^{\infty} a_n (x - x_0)^{n\alpha + s}\right) = 0$$

and we get:

$$\sum_{n=1}^{\infty}\left[a_n\frac{\Gamma(n\alpha+s+1)}{\Gamma(n\alpha+s-\alpha+1)}+\sum_{i=0}^{n}a_ip_{n-i}\right](x-x_0)^{n\alpha+s}$$

$$+a_0\left(\frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)}+p_0\right)(x-x_0)^s=0 \qquad (3.7)$$

now if we put $f_0(s)=\left(\frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)}+p_0\right)$, then from (3.7) we obtain

$$a_0f_0(s)=0 \qquad (3.8)$$

and

$$a_n=\frac{-\sum_{i=0}^{n-1}a_ip_{n-i}}{f_0(n\alpha+s)} \qquad (3.9)$$

suppose that $a_0\neq 0$, then $f_0(s)=0$. Thus, if s is the only real root of the equation$f_0(s)=0$, then the expression (3.9) provides, by recurrence ,the coefficients a_n of (3.4) in terms of a_0.

Now we prove the convergence of the series. Let $0<r<R$. Since the series (3.3) is convergent, there exists a constant $M>0$, such that

$$|p_{n-i}|\leq\frac{Mr^{i\alpha}}{r^{n\alpha}} \qquad n\in N \qquad (3.10)$$

and therefore:

$$|a_n|\leq\frac{M}{|f_0(n\alpha+s)|}\sum_{k=1}^{n}\frac{|a_{n-k}|}{r^{k\alpha}}$$

now we define $c_0=|a_0|$ and $c_n=\frac{M}{|f_0(n\alpha+s)|}\sum_{k=1}^{n}\frac{|a_{n-k}|}{r^{k\alpha}}$ for $n\geq 1$. Then

$$\left|\frac{c_{n+1}}{c_n}\right|=\left[\frac{M}{|f_0((n+1)\alpha+s)|}\Big/\frac{f_0(n\alpha+s)}{f_0((n+1)\alpha+s}\right]\frac{1}{r^{\alpha}}$$

and we get

$$\left|\frac{c_{n+1}(x-x_0)^{(n+1)\alpha}}{c_n(x-x_0)^{n\alpha}}\right| \to \left(\frac{|x-x_0|}{r}\right)^{\alpha}$$

when $n \to \infty$. Therefore the series $\sum_{n=0}^{\infty} c_n(x-x_0)^{n\alpha}$ converges for all x such that $0 < |x-x_0| < r$.

From this we conclude that (3.4) converges for $0 < |x-x_0| < R$. And the proof is complete.

Now, to demonstrate the application of the above theorem, we present the following example:

Example:

Consider the fractional differential equation:

$$(x-1)^{\alpha} D^{\alpha} y(x) - y(x) = 0 \tag{3.11}$$

where$D^{\alpha} y(x)$ represent the Riemann-Liouville fractional derivative.

Solution:

Since the point $x = 1$ is a regular α-singular point of (3.11), we shall seek a solution to this equation around the point $x = 1$of the form:

$$y(x) = (x-1)^s \sum_{n=0}^{\infty} a_n (x-1)^{n\alpha}$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^{n\alpha+s} \tag{3.12}$$

and the fractional derivative of $y(x)$ is given by:

$$D^{\alpha} y(x) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+s+1)}{\Gamma(n\alpha+s-\alpha+1)} (x-1)^{n\alpha+s-\alpha} \tag{3.13}$$

substitute (3.12) and (3.13) in the general (3.11) we get:

$$\sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+s+1)}{\Gamma(n\alpha+s-\alpha+1)}(x-1)^{n\alpha+s-\alpha+\alpha} - \sum_{n=0}^{\infty} a_n (x-1)^{n\alpha+s} = 0$$

and we obtain

$$\sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+s+1)}{\Gamma(n\alpha+s-\alpha+1)}(x-1)^{n\alpha+s} - \sum_{n=0}^{\infty} a_n (x-1)^{n\alpha+s} = 0$$

$$\left[\sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+s+1)}{\Gamma(n\alpha+s-\alpha+1)} - \sum_{n=0}^{\infty} a_n\right](x-1)^{n\alpha+s} = 0$$

$$a_0 \frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} - a_0 = 0$$

$$a_0\left(\frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} - 1\right) = 0$$

suppose that $a_0 \neq 0$ then $\frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} - 1 = 0$

$$\frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} = 1 \qquad (s > -1) \tag{3.14}$$

suppose that the solution is $s = \beta, \ (-1 < \beta < 0)$.

It is directly verified that $a_n = 0 \ (n \in N)$. Then the general solution to equation (3.11) has the following form:

$$y(x) = a_0(x-1)^{\beta} \qquad a_0 \neq 0. \tag{3.15}$$

Naturally, the result would be the same if we directly seek a solution as:

$$y(x) = C(x-1)^{s} \tag{3.16}$$

2.1 Introduction:

The use of fractional orders differential and integral operators in mathematical models has become increasingly widespread in recent year [3] [16]. Several forms of fractional differential equations have been proposed in standard models and there has been significant interest in developing their schemes solution [6], [11].

In this chapter and next chapter we shall discuss the existence solution of fractional order differential equations with variable coefficients In this chapter, we present the implementation of Laplace transform method to construct the required solution and the power series method, to obtain the ordinary of fractional order differential equation.

2.2 Power Series Method:

In this section we are going to use the power series method for obtaining the solution around an ordinary points of the fractional differential equation of order α.

The concept of α-analyticly function which generalized the concepts of an analytic function is presented, and constructing the solution for the equation:

$$D^{\alpha}y(x) + p(x)y(x) = o \qquad (2.1)$$

around an α-ordinary point $x_0 \in [a, b]$ with $p(x)$ defined in the interval $[a, b]$ and $\alpha \in (0,1)$. We consider $D^{\alpha}y$ to represent the Riemann-Liouiville fractional derivatives of order α of the function $y(x)$ since x_0 is an α-ordinary point , $p(x)$ can be expressed in power series expansion as follows:

$$p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^{n\alpha} \qquad (p_n \in R) \qquad (2.2)$$

this series being convergent for $x \in [x_0, x_0 + \rho]$ with ρ>0.

Definition (2.1), [1]:

Let $\alpha \in (0,1]$, $f(x)$ be a function defined on the interval $[a, b]$ and $x_0 \in [a, b]$. Then $f(x)$ is said to be α-analytic at x_0 if there exists an interval $N(x_0)$ such that for all $x \in N(x_0), f(x)$ can be expressed as a series of natural power of $(x - x_0)^\alpha$. That is $f(x)$ can be expressed as $\sum_{n=0}^{\infty} c_n (x -$ *x0)nα* with *0<α<1 (cn∈ R)* this series being absolutely convergent for $|x - x_0| < \rho$. The radius of convergence of the series is ρ.

Definition (2.2), [1]:

A point $x_0 \in [a, b]$ is said to be an α-ordinary point of equation (2.1), if the function $p(x)$ is α-analytic at x_0.

For example a point $x = x_0 > 0$ is an ordinary point for the following equation:

$$D^\alpha y(x) - x^\alpha y(x) = 0$$

Theorem (2.1)

Let $\alpha \in (0,1]$ and $a_0 \in R$ and let $x_0 \in [a, b]$ be an α -ordinary point for the equation (2.1) is solvable and the function

$$y(x) = (x - x_0)^{\alpha-1} \sum_{n=0}^{\infty} a_n (x - x_0)^{n\alpha} \quad , (a_n \in R) \tag{2.3}$$

is its solution , for $x \in (x_0, x_0 + \rho)$ and a_0 is the initial condition.

Proof Theorem (2.1):

We shall seek for a solution of equation (2.1) as follows:

$$y(x) = (x - x_0)^{\alpha-1} \sum_{n=0}^{\infty} a_n (x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+1)\alpha-1} \qquad (2.4)$$

and the different ion of the equation (2.4) we get

$$D^{\alpha} y(x) = \sum_{n=0}^{\infty} \frac{\Gamma[(n+1)\alpha]}{\Gamma(n\alpha)} a_n (x - x_0)^{n\alpha-1} \qquad (2.5)$$

in order to obtain the recurrence formula, we put (2.2),(2.4) and (2.5) in (2.1) we get:

$$\sum_{n=0}^{\infty} \frac{\Gamma[(n+1)\alpha]}{\Gamma(n\alpha)} a_n \,(x - x_0)^{n\alpha-1} + \left(\sum_{n=0}^{\infty} p_n (x - x_0)^{n\alpha}\right)\left(\sum_{n=0}^{\infty} a_n \,(x - x_0)^{(n+1)\alpha-1}\right) = 0 \qquad (2.6)$$

We can write equation (2.6) as:

$$\sum_{n=0}^{\infty} \frac{\Gamma[(n+1)\alpha]}{\Gamma(n\alpha)} a_n (x - x_0)^{n\alpha-1} = -\left(\sum_{n=0}^{\infty} p_n (x - x_0)^{n\alpha}\right)\left(\sum_{n=0}^{\infty} a_n \,(x - x_0)^{(n+1)\alpha-1}\right) \qquad (2.7)$$

from equation (2.7) we can write as:

$$\frac{\Gamma(\alpha)}{\Gamma(0)} a_0 (x - x_0)^{-1} + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} a_1 (x - x_0)^{\alpha-1} + \frac{\Gamma(3\alpha)}{\Gamma(2\alpha)} a_2 (x - x_0)^{2\alpha-1} + \cdots + \frac{\Gamma[(n+1)\alpha]}{\Gamma(n\alpha)} a_n (x - x_0)^{n\alpha-1} + \cdots$$

$$= -[(p_0 + p_1 (x - x_0)^{\alpha} + p_2 (x - x_0)^{2\alpha} + \cdots p_n (x - x_0)^{n\alpha} \ldots)$$

$$(a_0(x-x_0)^{\alpha-1} + a_1(x-x_0)^{2\alpha-1} + a_2(x-x_0)^{3\alpha-1} + \cdots$$

$$\dots + a_n\big(x-x_0\big)^{(n+1)\alpha-1} + \cdots\big)]$$

i.e

$$\frac{\Gamma(\alpha)}{\Gamma(0)}a_0(x-x_0)^{-1} + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}a_1(x-x_0)^{\alpha-1} + \frac{\Gamma(3\alpha)}{\Gamma(2\alpha)}a_2(x-x_0)^{2\alpha-1} + \cdots$$

$$\dots + \frac{\Gamma[(n+1)\alpha]}{\Gamma(n\alpha)}a_n(x-x_0)^{n\alpha-1} + \cdots$$

$$= -[p_0a_0(x-x_0)^{\alpha-1} + p_0a_1(x-x_0)^{2\alpha-1} + p_0a_2(x-x_0)^{3\alpha-1} + \cdots$$

$$\dots + p_0a_n(x-x_0)^{(n+1)\alpha-1} + \cdots$$

$$+p_1a_0(x-x_0)^{2\alpha-1} + p_1a_1(x-x_0)^{3\alpha-1} + p_1a_2(x-x_0)^{4\alpha-1} + \cdots$$

$$..+p_1a_n(x-x_0)^{(n+2)\alpha-1} + \cdots$$

$$+p_2a_0(x-x_0)^{3\alpha-1}+p_2a_1(x-x_0)^{4\alpha-1} + p_2a_2(x-x_0)^{5\alpha-1} + \cdots$$

$$\dots + p_2a_n(x-x_0)^{(n+3)\alpha-1} + \cdots$$

$$\vdots$$

$$+p_na_0(x-x_0)^{(n+1)\alpha-1}+p_na_1(x-x_0)^{(n+2)\alpha-1}+p_na_2(x-x_0)^{(n+3)\alpha-1} + \cdots]$$

Since a_0 is arbitrary:

$$\frac{\Gamma(2\alpha)}{\Gamma(\alpha)}a_1 = -p_0a_0$$

$$\frac{\Gamma(3\alpha)}{\Gamma(2\alpha)}a_2 = -(p_1a_0+p_0a_1)$$

$$\frac{\Gamma(4\alpha)}{\Gamma(3\alpha)}a_3 = -(p_2a_0+p_1a_1 + p_0a_2)$$

$$\vdots$$

$$\frac{\Gamma[(n+2)\alpha]}{\Gamma[(n+1)\alpha]}a_{n+1} = -(p_na_0 + p_{n-1}a_1 + p_{n-2}a_2+\cdots p_0a_n).$$

from this result we obtain the following recurrence formula which allows us to express a_n $(n > 0)$ in terms of a_0 :

$$\frac{\Gamma[(n+2)\alpha]}{\Gamma[(n+1)\alpha]} a_{n+1} = -\sum_{k=0}^{n} p_{n-k} a_k \tag{2.8}$$

where $k = (0,1,2,3,\dots)$

we show that for $x \in (x_0, x_0 + \rho)$ the series in (2.4) converges.

let $r < \rho$ since (2.2) converges. There exists a constant $M > 0$ such that:

$$|p_{n-k}| \leq \frac{M r^{k\alpha}}{r^{n\alpha}}$$

and therefore,

$$\frac{\Gamma[(n+2)\alpha]}{\Gamma[(n+1)\alpha]} |a_{n+1}| \leq \frac{M}{r^{n\alpha}} \sum_{k=0}^{n} a_k r^{k\alpha}$$

denoting $b_0 = |a_0|$ by recurrence we define $b_n (n \in N)$as follows:

$$\frac{\Gamma[(n+2)\alpha]}{\Gamma[(n+1)\alpha]} b_{n+1} = \frac{M}{r^{n\alpha}} \sum_{k=0}^{n} b_k \, r^{k\alpha}$$

where $(n \in N_0)$. It is clear that $0 \leq |a_n| \leq b_n$ for $n \in N_0$the series:

$$g_r(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^{n\alpha + \alpha - 1} = (x - x_0)^{(\alpha - 1)} \sum_{n=0}^{\infty} b_n (x - x_0)^{n\alpha}$$

is convergent for $|x - x_0| < r$ .And the following estimate is holds

$$\lim_{n \to \infty} \left| \frac{b_{n+1} (x - x_0)^{(n+1)\alpha}}{b_n (x - x_0)^{n\alpha}} \right| = \left(\frac{|x - x_0|}{r} \right)^{\alpha} < 1$$

this proves the convergence of (2.4) for $x \in (x_0, x_0 + \rho)$

Example:

Consider the following fractional differential equation of order $\alpha(0<\alpha<1)$:

$$D^{\alpha}y(x)+(x+1)^{\alpha}y(x)=0 \tag{2.9}$$

Solution :

The point $x_0=-1$ is an α -ordinary point for this equation.

Let us find the general solution to (2.9) around the point $x_0=-1$. We seek solutions of the form:

$$y(x)=(x+1)^{\alpha-1}\sum_{n=0}^{\infty}a_n(x+1)^{n\alpha}=\sum_{n=0}^{\infty}a_n(x+1)^{(n+1)\alpha-1} \tag{2.10}$$

to find a_n substiute (2.10) into the original equation (2.9):

$$\left(D^{\alpha}\sum_{n=0}^{\infty}a_n\,(x+1)^{(n+1)\alpha-1}\right)+(x+1)^{\alpha}\left(\sum_{n=0}^{\infty}a_n\,(x+1)^{(n+2)\alpha-1}\right)=0$$

$$\sum_{n=0}^{\infty}\frac{\Gamma[(n+1)\alpha]}{\Gamma(n\alpha)}a_n(x+1)^{n\alpha-1}=-\sum_{n=0}^{\infty}a_n(x+1)^{(n+2)\alpha-1}$$

we have $a_{2k+1}=0 \quad (k\in N)$ (2.11)

and $a_{2k}=(-1)^kQ(k)a_0$ (2.12)

with $Q(k)=\prod_{i=1}^{k}\frac{\Gamma(2i\alpha)}{\Gamma[(2i+1)\alpha]}$ (2.13)

therefore the general solution to (2.9) has the form:

$$y(x)=a_0(x+1)^{\alpha-1}+\sum_{k=1}^{\infty}(-1)^kQ(k)a_0(x+1)^{(2k+1)\alpha-1} \tag{2.14}$$

$$y(x) = a_0(x+1)^{\alpha-1}\left[1+\sum_{k=1}^{\infty}(-1)^k\,Q(k)(x+1)^{2k\alpha}\right]. \qquad (2.15)$$

The particular solution to (2.9) which satisfies $y(0) = 1$ is given by (2.15), with a_0 given by:

$$a_0 = \frac{1}{1+\sum_{k=1}^{\infty}(-1)^k\,Q(k)} \qquad (2.16)$$

Conclusions and Future work

Conclusions and Future Work

The main conclusions of this work are that we can obtain the solution of fractional order differential equation with variable coefficients by using the power series solution in the same manner as in ordinary differential equation.

In future work, we suggested the following cases to study:

1. Differential equations with variable coefficients having multi-fractional order α such that $\alpha > 1$

2. Differential equations with variable coefficients having fractional order α such that $\alpha > 1$

3. Non-homogenous fractional order differential equations with variable coefficients.

4. Other definitions of fractional order differential equations (such as Caputo, Hadmard,…etc) to be considered to investigate the solutions of such fractional order differential equations

5. The solutions of the system of fractional order differential equations with variable coefficients.

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References

Abstract

In this work the solutions of ordinary homogenous fractional order (with values between zero and one) differential equations with variable coefficients are investigated. Also the existence of the solution is by presenting theorems, using the method of Power Series for ordinary and singular type of fractional order differential equations with variable coefficients. Example has been presented for each case.

Introduction

Introduction

Fractional Calculus is a field of mathematical study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. According to our primary knowledge exponents provide a short notation for what is essentially a repeated multiplication of numerical value.

This concept in itself is easy to grasp and straight forward. However, this physical interpretation can clearly become confused when considering exponents of non integer value. While almost anyone can verify that $x^3 = x \cdot x \cdot x$, how might one describe the physical meaning of $x^{3.4}$ or moreover the transcendental exponent x^{π}. One cannot conceive what it might be like to multiply a number or quantity by itself 3.4 times or π times and yet these expressions have a definite value for any value x, verifiable by infinite series expansion, or more practically by calculator.[10]

Most authors on this topic will cite a particular date as the birthday of so called 'fractional calculus'. In a letter dated September 30th ,1695 L'Hopital wrote to Leibniz asking him about a particular notation he had used in his publications for the nth-derivative of the linear function $f(x) = x$. L'Hopital's posed the question to Leibniz, what would the result be if $n = 1/2$ Leibniz's response:"An apparent paradox, from which one day useful consequences will be drawn". In these words fractional calculus was born.

Following L'Hopital's and Liebniz's first quisition, fractional calculus was primarily a study reserved for the best minds in mathematics. Fourier, Euler, Laplace are among the many that dabbled with fractional calculus and the mathematical consequences [7].

Many found, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative. The most famous of these definitions that have been popularized in the world of fractional calculus are the Riemann-Liouville and Groünwaled-Letnikov definition.

Most of the mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the 20th century. However it is in the past 100 years that the most intriguing leaps in engineering and scientific application have been found. The mathematics has in some cases had to change to meet the requirements of physical reality. Caputo reformulated the more 'classic' definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions to solve his fractional order differential equations [16].

As recently in 1996, Kolowankar reformulated again, the Riemann-Liouville fractional derivative in order to differentiate no-where differentiable fractal functions [8].Leibniz's response, based on studies over the intervening 300 years, has proven at least half right. It is clear that within the 20th century especially numerous applications and physical manifestations of fractional calculus have been found. However, these applications and the mathematical background surrounding fractional calculus are far from paradoxical. While the physical meaning is difficult (arguably impossible) to grasp, the definitions themselves are no more rigorous than those of their integer order counterparts [10].

In this work we are studying and investigating the existence solutions of fractional order differential equations with variable coefficients.

This thesis consists of three chapters:

In chapter one, we study the fundamental concepts and definitions related to fractional calculus, while the main objective of this chapter is to give an overview about fractional differentiation and integration.

In chapter two we discussed the existence of the solution of the fractional differential equations with variable coefficients around ordinary point by using Power Series Method.

In chapter three we discussed the existence of the solution of fractional differential equations with variable coefficients around singular point of fractional differential equations of order $0 < \alpha < 1$.

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May,2008

Supervisor Certification

I certify that this thesis was prepared under my supervision at the Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

Signature:
Name: Assist. Prof. Dr. Alauldin N.Ahmed
Date: /5/2008

In view of the available recommendations; I forward this thesis for debate by examination committee.

Signature:
Name: Assist. Prof. Dr. Akram M. Al-Abood
Head of the Department of Mathematics and Computer Applications
Date: /5/2008

Examining Committee's Certification

We certify that we read this thesis entitled ***"Solutions of Ordinary Homogenous Fractional Order Differential Equations with Variable Coefficients"*** and as examining committee examined the student, ***Dhmyaa Salim Mooter*** in its contents and in what it connected with, and that is in our opinion it meet the standard of thesis for the degree of Master of Science in Mathematics.

(Chairman)

Signature:

Name: *Dr. Ahlam J.Khaleel*

Date: / /2008

(Member)

signature:

Name: *Dr. Saad Naji Ali*

Date: / /2008

(Member)

Signature:

Name: *Dr. Fadhel S.Fadhel*

Date: / /2008

(Member and Supervisor)

signature:

Name: *Dr. Alauldin N. Ahmed*

Date: / /2008

Approved by the Dean of the College of Science of Al-Nahrain University

Signature:

Name: *Assist. Prof. Dr. Laith Abdul-Aziz Al-Ani*

Dean of College of Science of Al-Nahrain University

Date: / /2008

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المشرف على الاطروحة: د. علاء الدين نوري احمد

اسم الطالبة: ضمياء سالم مطر

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رقم الهاتف: ٠٧٨٠٥١٥٤٦٢٤

البريد الالكتروني: Dhmyaa_Salim@yahoo.com

Republic of Iraq
Ministry of Higher Education
and Scientific Research
Al-Nahrain University
College of Science

Solutions of Ordinary Homogenous Fractional Order Differential Equations with Variable Coefficients

A Thesis

Submitted to the College of Science, Al-Nahrain University, as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

Dhmyaa Salim

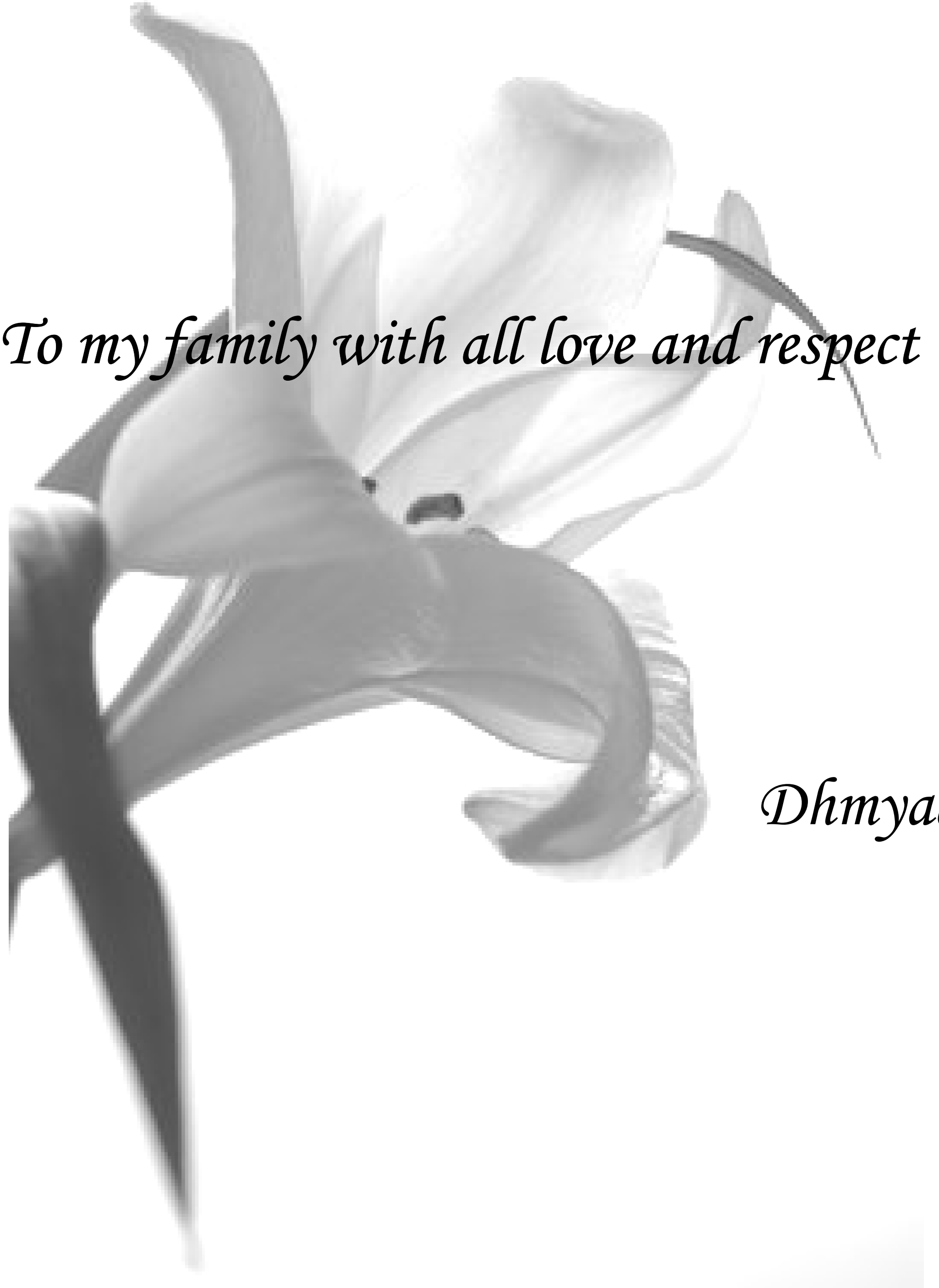
(B.Sc. 2004)

Supervised by

Dr. Alauldin N. Ahmed

May 2008

Dedication



To my family with all love and respect

Dhmyaa

بِسْمِ اللهِ الرَّحْمنِ الرَّحِيمِ

(وقُلْ رَبِّ أدْخِلْني مُدْخَلَ صِدْقٍ وَأخْرِجْني مُخرَجَ صِدْقٍ واجْعَلْ لي مِنْ لدُنْكَ سُلْطاناً نَصيراً)

صدق الله العلي العظيم

(سورة الاسراء/ ٨٠)

المستخلص

في هذا العمل تم دراسة حلول المعادلات التفاضلية الاعتيادية المتجانسة من الرتب الكسرية (والتي قيمها مابين الصفر والواحد) ذات المعاملات المتغيرة.

لقد تم استنباط وجود هذه الحلول من خلال عرض نظرية استخدم فيها طريقة Power Series للحالات الاعتيادية (ordinary point) و المفردة (singular point) من المعادلات التفاضلية الاعتيادية من الرتب الكسرية ذات المعاملات المتغيرة و قد تم عرض مثال لكل نوع.

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعــة النهرين
كلية العلوم

حلول المعادلات التفاضلية الاعتيادية المتجانسة من الرتب الكسرية ذات المعاملات المتغيرة

رسالة
مقدمه إلى كلية العلوم/جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير
علوم في الرياضيات

من قبل
ضمياء سالم

(بكالوريوس علوم ٢٠٠٤)

بإشراف
د. علاء الدين نوري احمد

آيار ٢٠٠٨