ABSTRACT

The objective of this thesis is to study first the theory of fractional calculus and some of well known methods for evaluating derivatives of fractional orders for certain functions.

The second objective is to study the G-spline interpolation functions and its construction using a new approach in formulating the Heremite-Birkhoff problem using fractional derivatives instead of integer order derivatives.

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CHAPTER ONE

BASIC CONCEPTS OF FRACTIONAL CALCULUS

In the earlier work, the main application of fractional calculus as a technique for solving integral equations and recently fractional derivatives have been used to model physical processes which lead to the formulation of the subject of fractional calculus and hence leading to the formulation of the fractional differential equations which plays an important role in most engineering and mathematical and physics problems. Therefore, in this chapter we give some of the basic concepts related to the subject of fractional calculus, including the basic definitions and properties, as well as some illustrative examples of fractional differentiation.

1.1 BASIC CONCEPTS

In the present section, some fundamental concepts related to the theory of fractional calculus are given in order to avoid vague notions that may arise in this subject.

1.1.1 The Gamma and Beta Functions, [Oldham, 1974]:

Undoubtedly, among the basic functions encountered in fractional calculus which is widely used is the Euler's gamma function $\Gamma(x)$, which generalizes the ordinary definition of factorial function of a positive integer number n and allows n to take also any non-integer positive and even negative numbers or complex values.

As it is known, the gamma function $\Gamma(x)$ is defined by using the following improper integral:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \; x > 0 \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.1)$$

First of all, it is easy to show that the gamma function for any positive integer x can be proved also to be:

(1) $\Gamma(x) = (x-1)!, \; x \in \square$

(2) $\Gamma(x) = (x-1)\,\Gamma(x-1), \; x \in \square^{+}$

(3) $\Gamma(\frac{1}{2} - x) = \dfrac{[-4]^{x} n! \sqrt{\pi}}{(2n)!}$

(4) $\Gamma(\frac{1}{2} + n) = \dfrac{(2n)! \sqrt{\pi}}{4^{n} n!}$

(5) $\Gamma(-x) = \dfrac{-\pi \csc(\pi x)}{\Gamma(x+1)}$

which enable us to calculate for any positive real x the gamma function in terms of the fractional part of x.

For positive integer q and j with $q > j$, the expression $\dfrac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}$ may be regarded as the binomial coefficient, as follows:

$$\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} = \frac{(j-q-1)(j-q-2)...(-q+1)(-q)}{j!}$$

$$= (-1)^j \binom{q}{j} \quad \text{...}(1.2)$$

where; $\binom{q}{j} = \frac{q!}{j!(q-j)!}$, $j = 0,1,2,...,q$.

A second function of important is the beta function $B(m,n)$ which is defined by:

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx\,,\ n, m > 0 \quad \text{……………...…………...}(1.3)$$

An important relationship connecting between gamma and beta functions is given by:

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\ n, m > 0 \quad \text{………………………...………}(1.4)$$

1.1.2 Riemann – Liouville Formula of Fractional Derivatives, [Oldham, 1974]:

Riemann and Liouville in (1832) introduced a differential operates of fractional order q>0, which takes the form:

$$D^q{}_t y(t) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_{t_0}^{t} \frac{y(u)}{(t-u)^{q-m+1}} du \quad \text{………………………} \quad (1.5)$$

where m is a positive integer number defined by $m - 1 < q \leq m$.

Equation (1.5) is a Volterra integral equation with singular kernel. Differential equations involving these fractional derivatives have been proved to be valuable tolls in modeling many physical and engineering phenomenas.

1.2 FRACTIONAL DIFFERENTIATION AND INTEGRATION

Fractional differentiation and integration may be defined and evaluated using several approaches depending on the used definition of differentiation or integration. Therefore, this section presents some of such types of differentiation and integration which is termed for simplicity as differintegration.

1.2.1 Fractional Derivatives:

The usual formulation of the fractional derivative, given in standard references such as [Samko, 1993] and [Oldham, 1974] is by using the Riemann-Liouville formula that requires initial values expressed as fractional derivatives. This is very inconvenient, since it is usually not clear what the physical meaning of these fractional order initial value would be and they are therefore hard to derive from a physical system. In applications, it is often more convenient to use the formulation of the fractional derivative suggested by Grünwald, Osler, Caputo, etc. (see [Caputo, 1971]).

The Grünwald derivatives which requires the same starting conditions as in ordinary differential equations of the next higher order. The Grünwald definition of fractional derivatives is given by:

$$\frac{d^q f(t)}{dt^q} = \lim_{N\to\infty} \frac{\left(\frac{t}{N}\right)^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(t - j\left(\frac{t}{N}\right)\right) \quad \text{............................} (1.6)$$

where $q < 0$ indicates fractional integration and $q>0$ indicates fractional differentiation.

The Bertram Ross definition of fractional derivative as follows:

$$\frac{d^q}{dt^q} y(t) = \frac{\Gamma(q+1)}{2\pi i} \int_C \frac{y(u)}{(u-t)^{q+1}} du \quad \text{...................................} (1.7)$$

where he made a branch cut from t to infinity through the origin and the integral curve C is an open contour which encloses t in the positive sense and $u \notin C$ (i.e., C is an integral curve along that cut).

The equivalent between these formulas may be proved, but it have more computations and therefore are omitted.

It is remarkable to notice that additional definitions of fractional derivatives may be found in liteatmes in addition to Riemann – Liouville definition given by eq. (1.5).

1.2.2 Fractional Integration:

The common formulation for the fractional integration can be

derived directly from a traditional expression of the repeated integration of a function. This approach is commonly referred to as Riemann – Liouville approach.

The Riemann-Liouville definition of fractional integration is given by:

$$f_v^+(a,x) = \frac{1}{\Gamma(q)}\int_a^x (x-t)^{q-1} f(t)dt \quad \text{(right hand integration)}$$

$$f_q^-(x,b) = \frac{1}{\Gamma(q)}\int_x^b (t-x)^{q-1} f(t)dt \quad \text{(left hand integration)}$$

where q<0.

Combining the least two eqs. gives:

$$I^q f(x) = \frac{1}{\Gamma(q)}\int_{x_0}^x \frac{f(u)}{(x-u)^{1-q}} du$$

The Weyle definition of the right and left hand fractional integrals are given respectively by:

$$f_v^+(-\infty,x) = \frac{1}{\Gamma(q)}\int_{-\infty}^x (x-t)^{q-1} f(t)dt \text{ ....(1.8)}$$

$$f_q^-(x,\infty) = \frac{1}{\Gamma(q)}\int_x^\infty (t-x)^{q-1} f(t)dt \text{ ....(1.9)}$$

where f(t) is a periodic function and its mean value for one period is zero. But eqs. (1.8) and (1.9) are used as the definition of the integration without any condition at the present time.

Because we will concern ourselves to fractional differentiations only, then we will focus our attention on fractional derivatives only and give some illustrative examples for such type of differentiations in the next section.

1.3 FRACTIONAL DERIVATIVES OF SOME WELL KNOWN FUNCTIONS

In this section, some fractional derivatives using the above definitions of derivatives will be evaluated as an illustrative examples to fractional differentiations. Other functions derivatives may be derived, such as sinh($\sqrt{x}$), sin($\sqrt{x}$), etc., (see [Oldham, 1974]).

1. The Unit Functions, [Oldham, 1974]:

Consider first the differintegral to the order q of the function f = 1, for which it is found that it is convenient to reserve the special notation which function will be referred to as the unit function.

Using Riemann-Liouville fractional derivatives given by eq.(1.5) with $m_0 = 0, t_0 = 0$ to give

$$\frac{d^q}{dx^q} f(x) = \frac{1}{\Gamma(-q)} \int_0^x \frac{f(u)}{(x-u)^{q+1}} du, \ -1 < q \leq 0$$

Then with the unite function $f = 1,$ we have

$$\frac{d^q}{dx^q} f(x) = \frac{1}{\Gamma(-q)} \int_0^x \frac{du}{(x-u)^{q+1}}$$

$$= \frac{1}{\Gamma(-q)} \frac{(x-u)^{-q}}{q} \Bigg|_0^x$$

$$= \frac{1}{q\Gamma(-q)} \left(0 - x^{-q}\right)$$

$$= \frac{x^{-q}}{q\Gamma(-q)}$$

$$= \frac{x^{-q}}{\Gamma(1-q)}.$$

2. The Constant Function,[Oldham, 1998]:

For a function f = c, where c is any constant including zero, then the fractional differential of order q>0 may be indicated using the differential the fractional derivative of the unit function, then

$$\frac{d^q}{d(x-a)^q}(c) = c\frac{d^q}{d(x-a)^q}(1)$$

$$= c\frac{(x-a)^{-q}}{\Gamma(1-q)}$$

and hence,

$$\frac{d^q}{d(x-a)^q}(0) = 0,\ \forall q \quad \text{.......}(1.10)$$

3. The Function $(x-a)^p$, [Oldham, 1998]:

The function of fractional degree that may be considerd in this case is an important function given by $f(x) = (x-a)^p$, where p is initially arbitrary, and one can see that p must exceed −1 for differintegration to have the properties that the researchers demand of the operator. For integer n of either sign, one can show that:

$$\frac{d^n x^p}{dx^n} = p(p-1)\ldots(p-n+1)x^{p-n},\ n = 0, 1, \ldots$$

and from classical calculus, the first encounter with non-integer q, will be restricted to negative q so that we may exploit the Riemann-Liouville definition. Thus:

$$\frac{d^q (x-a)^p}{(d(x-a))^q} = \frac{1}{\Gamma(-q)} \int_a^x \frac{(y-a)^p}{(x-y)^{q+1}} dy$$

$$= \frac{1}{\Gamma(-q)} \int_0^{x-a} \frac{v^p}{(x-a-v)^{q+1}} dv\,,\ q<0$$

where v has replaced by y – a. By further replacement of v by (x – a)u, then the integral may be cast into the structure of the beta function as follows:

$$\frac{d^q (x-a)^p}{(d(x-a))^q} = \frac{(x-a)^{p-q}}{\Gamma(-q)} \int_0^1 u^p (1-u)^{-q-1} du, q<0$$

$$= \frac{\Gamma(p+1)(x-a)^{p-q}}{\Gamma(p-q+1)}, p<1, q>-1 \text{ ....}(1.11)$$

4. The Exponential Function exp(r – cx):

With r and c are an arbitrary constants, then the power-series expansion of $\exp(r-cx)$ is given by:

$$\exp(r-cx) = \exp(r-ca) \sum_{j=0}^{\infty} \frac{(-c(x-a))^j}{\Gamma(j+1)}$$

which is valid for all x – a.

Differintegration term-by-term with respect to c(x – a), yields:

$$\frac{d^q \exp(r - cx)}{\left(d(cx - ca)\right)^q} = \left\{c(x-a)\right\}^{-q} \exp(r - ca) \sum_{j=0}^{\infty} \frac{\left\{-c(x-a)\right\}^j}{\Gamma(j - q + 1)}$$

The summation may be expressed as an incomplete gamma function of argument –c(x – a) and parameter –q, then the final result appears to be as:

$$\frac{d^q \exp(r - cx)}{\left(d(x-a)\right)^q} = \frac{\exp(r - cx)}{\left(x - a\right)^q} \gamma^*\left(-q, -c(x-a)\right)$$

where $\gamma^*(-n, y) = y^n$ for non negative integer n. The above result seems to be reduced to the well-known formula for multiple differentiation of an exponential function, reduction to the simple formula:

$$\frac{d^q \exp(\mp x)}{dx^q} = \frac{\exp(\mp x)}{x^q} \gamma^*(-q, \mp x)$$

occurs on substituting k = a = 0 and c = ±1 into the general result.

5. The Functions $\frac{x^q}{1 - x}$ and $\frac{x^p}{1 - x}$:

By using the Macluarion expansion of $(1 - x)^{-1}$ and the technique of term-by-term differentegration which is from the (linearity of differentiation), one can arrive at:

$$\frac{d^q}{dx^q}\left(\frac{x^q}{1-x}\right) = \sum_{j=0}^{\infty} \frac{d^q}{dx^q} x^{j+q}$$

Identification of the summation as a Macluarion expansion produces:

$$\frac{d^q}{dx^q}\left[\frac{x^q}{1-x}\right]=\sum_{j=0}^{\infty}\frac{\Gamma(j+q+1)}{\Gamma(j+1)}x^j$$

$$=\sum_{j=o}^{\infty}\frac{(j+q)!}{j!}x^j$$

$$=q!\sum_{j=o}^{\infty}\frac{(j+q)!}{j!q!}x^j$$

$$=\Gamma(q+1)\sum_{j=0}^{\infty}\binom{j+1}{j}x^j$$

$$=\Gamma(q+1)\sum_{j=0}^{\infty}\binom{-q-1}{j}(-x)^j$$

and hence:

$$\frac{d^q}{dx^q}\left(\frac{x^q}{1-x}\right)=\frac{\Gamma(q+1)}{(1-x)^{q+1}}$$

$$=\Gamma(q+1)(1-x)^{-(q+1)}$$

The technique for differintegrating $\frac{x^p}{1-x}$ follows in such a similar result above, that is, it will suffice to cite one intermediate and the final result of differentiation:

$$\frac{d^q}{dx^q}\left(\frac{x^p}{1-x}\right)=\frac{\Gamma(p+1)\beta(p-q,q+1)}{\Gamma(p-q)(1-x)^{q+1}}$$

$$=\frac{\Gamma(q+1)}{(1-x)^{q+1}}$$

where $B(p, q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ together with the restriction, namely, -1 < x < 1 and p > −1, which is assumed during the derivation.

1.4 SOME PROPERTIES OF FRACTIONAL ORDER DIFFERENTIAL OPERATOR

Some properties of fractional differentiation operators are of great importance in applications and in approximating functions using g-spline, therefore some of such properties of fractional order differential operator D will be stated.

(1) The operator D_x^{α} is linear, i.e.,

$$D_x^{\alpha}\left[c_1 f(x) + c_2 g(x)\right] = c_1 D_x^{\alpha} f(x) + c_2 D_x^{\alpha} g(x)$$

(2) The operator D_x^{α} f gives the same results as the usual differentiation of order $n \in N$, where $\alpha = n$ is positive integer, and the same effect of an n-fold integration if $\alpha = -n$, is negative integer.

(3) The operator of order $\alpha = 0$ is the identity operator, i.e., $D_x^0 f = f$.

(4) For $\alpha > 0$ and $\beta > 0$ (Re$\alpha > 0$, Re$\beta > 0$) the additive index hold, i.e.,

$$D_x^{-\alpha} D_x^{-\beta} f(x) = D_x^{-(\alpha+\beta)} f(x).$$

The proofs are obtained directly from the definitions. Other properties may be found in [Bertram, 1974], [Igor, 2001], [Lixia, 1998].

1.5 ILLUSTRATIVE EXAMPLES

Following are some examples which illustrate the applicability of fractional derivatives for some well known functions.

Example (1.1):

Consider $f(x) = x$ then the following may be carried using Riemann – Liouville formula of functional derivatives:

(1) $$D_x^{-0.5}\, x = \frac{1}{\Gamma(0.5)}\int_0^x (x-t)^{-1/2}\, t\, dt$$

$$= \frac{4\sqrt{x^3}}{3\sqrt{\pi}}$$

(2) $$D_x^{0.5}\, x = \frac{d}{dx}\frac{1}{\Gamma(1-0.5)}\int_0^x (x-t)^{-1/2}\, t\, dt$$

$$= 2\sqrt{\frac{x}{\pi}}$$

(3) $$D_x^{1.5}\, x = \frac{d^2}{dx^2}\frac{1}{\Gamma(1-\frac{3}{2})}\int_0^x (x-t)^{-1/2}\, t\, dt$$

$$= \frac{1}{\sqrt{\pi x}}$$

In generality, to derive the fractional derivative of x^m when $\alpha = 1/2$, we get an important relation:

$$\frac{d^{1/2}}{dx^{1/2}}x^m = \frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}x^{m-\frac{1}{2}}$$

$$= \frac{m!}{\dfrac{1.3.5...(2m-1)\Gamma(\frac{1}{2})}{2^m}}\frac{x^m}{\sqrt{x}}$$

$$= \frac{2^m m! 2.4. \ldots .2m}{\sqrt{\pi x}1.2.3. \ldots .(2m-1).2m}x^m$$

$$= \frac{2^m m!\, 2^m m!}{\Gamma(\pi x)(2m)!} x^m$$

$$\frac{(m!)^2 (4x)^m}{(2m)!\Gamma(\pi x)} \quad m = 0, 1, \ldots, n \ldots\ldots\ldots\ldots\ldots\ldots (1.12)$$

Example (1.2):

Consider the function f(x) = sinx, hence the fractional derivatives of order $\alpha = 1/2$, may be evaluated as follows:

$${}_0D_x^{0.5} \sin x = \frac{d}{dx}\frac{1}{\sqrt{\pi}}\left(\int_0^x (x-t)^{-0.5} \sin t \; dt \right)$$

then letting $x - t = u^2$, which implise $dt = -2u\, du$ and so:

$${}_0D_x^{0.5} \sin x = \frac{1}{\sqrt{\pi}}\frac{d}{dx}\left(\int_0^{\sqrt{x}} \frac{\sin(x-u^2)}{2} 2u\, du \right)$$

$$= \frac{2}{\sqrt{\pi}}\frac{d}{dx}\left(\int_0^{\sqrt{x}} (\sin x \cos u^2 - \cos x \sin u^2)\, du \right)$$

$$= \frac{2}{\sqrt{\pi}}\frac{d}{dx}\left(\sin x \int_0^{\sqrt{x}} \cos u^2 \; du - \cos x \int_0^{\sqrt{x}} \sin u^2 \; du \right)$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\sin x \cos x}{2\sqrt{x}} + \cos x \int_0^{\sqrt{x}} \cos u^2\, du - \right.$$

$$\left. \frac{\cos x \sin x}{2\sqrt{x}} + \sin x \int_0^{\sqrt{x}} \sin u^2\, du \right]$$

and using Fresnel Integrals given by [Kai, 1999]:

(i) Fresenl sine integral is defined as:

$$s(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin u^2 \, du$$

such that $s(-x) = -s(x)$, $s(0) = 0$, $s(\infty) = 1/2$.

(ii) Fresenl cosine integral which is defined as:

$$c(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos u^2 \, du$$

with $c(-x) = -c(x)$ and $c(0) = 0$, $s(\infty) = 1/2$.

yields:

$$ {}_0D_x^{0.5} \sin x = \frac{2}{\sqrt{\pi}} \left[\cos x \left(\frac{\sqrt{\pi}}{\sqrt{2}} c(\sqrt{x}) \right) + \sin x \left(\frac{\sqrt{\pi}}{\sqrt{2}} s(\sqrt{x}) \right) \right] $$

$$ = \sqrt{2} \cos x \; c(\sqrt{x}) + \sqrt{2} \sin x \; s(\sqrt{x}) $$

and following the same approach, one may arrive at:

$$ {}_0D_x^{0.5} \cos x = \frac{1}{\sqrt{\pi}} - \sqrt{2}\, c(\sqrt{x}) \sin x + \sqrt{2}\; s(x) \cos x. $$

CHAPTER THREE

G-SPLINE INTERPOLATION USING HERMITE-BIRKHOFF PROBLEM WITH FRACTIONAL DERIVATIVES

Just over 138 years ago, since Lagrange in 1870 has constructed the polynomial of minimal degree such that the polynomial assumed prescribed values at a given knots and the derivatives of certain orders of the polynomial also assumed prescribed values at the knots.

In 1968, Schoenberg extended the idea of Hermite for splines to specify that the orders of the derivatives specified may vary from knot to knot. Schoenberg used the term "G-spline" instead of generalized splines, because the usual spline term "generalized splines" already described as an extension in a different direction, as it is mentioned in section three of chapter two, [Schoenberg, 1968].

G-spline functions are used to interpolate the Heremite-Birkhoff data (problem), which is abbreviated by HB-problem, the data in this problem are the values of the function and its derivatives but without Hermite's condition that only consecutive derivatives be used at each knot.

Again, Schoenberge has defined G-splines as a smooth picewise polynomials, where the smoothness is governed by the incidence matrix E, and then he proved that the G-splines satisfies what is called the "minimum norm property", which is used for the optimality of the G-spline functions, which is given mathematically by the following inequality:

$$\int_I \left[f^{(m)}(x)\right]^2 dx > \int_I \left[S^{(m)}(x)\right]^2 dx$$

where the function S is called a G-spline function and it is a polynomial spline of degree $2m - 1$ over the interval I. If the only polynomial that solve the homogeneous HB-interpolation problem is identically the zero polynomial, then the problem is said to be m-poised. We will see later the consideration of the HB problem that the m-poised problem will play an important role for the uniqueness of the solution of the HB-problem (that is if the HB-problem is m-poised, then there is a unique G-spline function of degree $2m - 1$ that solves the HB-problem).

3.1 THE HERMITE-BIRKHOFF PROBLEM WITH FRACTIONAL DERIVATIVES

As it is mentioned earlier, that the G-spline interpolation functions are calculated using the HB-problem. In this section a modified approach is used to define and give the HB-problem which is by using the fractional derivatives instead of positive integer order derivatives.

First of all, a tractable formal definition of the natural G-spline interpolation is given.

Let us consider the knots:

$x_1 < x_2 < \ldots < x_k$

to be distinct and real and let α to be the maximum of the orders of the derivatives given by:

$$r-1<\alpha\leq r$$

where α is fractional number and r is a natural number.

Define an incidence matrix E, by:

$E = [a_{ij}]$, $i = 1, 2, \ldots, k$; $j = 0, 1, \ldots, r$

where:

$$a_{ij} = \begin{cases} 1, & \text{if} \quad (i, j) \in e \\ 0, & \text{if} \quad (i, j) \notin e \end{cases}$$

Here $e = \{(i, j)\}$ has been chosen in such a way that i takes the values 1, 2, …, k; one or more times, while $j \in \{0, 1, \ldots, r\}$ and $j = r$ is attained in at least one element (i, j) of e. Assume also that each row of the incidence matrix E and the last column of E should contain some element equals 1.

Let $y_i^{(\alpha_i)}$ for each $r_i-1<\alpha\leq r_i$ be a prescribed real numbers for each $(i, j) \in e$ and let $\alpha=\max\limits_{i}\alpha_i$, with $r-1<\alpha\leq r$, then the HB-problem using fractional order derivatives is to find $f(x) \in C^{r+1}[x_i, x_k]$, which satisfies the interpolatory condition:

$$\left.\begin{array}{l} f^{(\alpha_i)}(x_i) = y_i^{(\alpha_i)} \\ \text{for all } r_i - 1 < \alpha_i \le r_i,\ i = 1,2,\ldots,k,\ j = 0,1,\ldots,r_i \text{ for } (i,j) \in e \end{array}\right\} \ldots\ldots\ldots (3.1)$$

Therefore, in practical applications if $r_i - 1 < \alpha_i \le r_i$, we operate both sides of $f^{(\alpha_i)}(x_i) = y_i^{(\alpha_i)}$ with $D^{r_i - \alpha_i}$ operator to the both sides to get an equivalent problem with:

$$\begin{aligned} f^{(r_i)}(x_i) &= D^{r_i - \alpha_i} y_i^{(\alpha_i)} \\ &= y_i^{(\alpha_i)} D^{r_i - \alpha_i}(1) \\ &= y_i^{(\alpha_i)} \frac{(x-a)^{\alpha_i - r_i}}{\Gamma(\alpha_i - r_i + 1)}. \end{aligned}$$

The matrix E will likewise describes the set of equations (3.1) if the set e is defined by:

$$e = \{(i, j) \mid a_{ij} = 1\}$$

then the integer:

$$n = \sum_{i,j} a_{ij}$$

is really the number of interpolatory conditions required to constitute the system (3.1).

Therefore, at each knot x_i of the system (3.1), we prescribes the value of $f(x_i)$ for all $i = 1,2,\ldots,k$ and may be also a certain number of consecutive derivatives $f^{(\alpha_i)}(x_i)$ for j = 1, 2, …, r_i; where r_i satisfy $r_i - 1 < \alpha \le r_i$, for each i.

As an illustration we consider first the following HB-problem with positive integer derivative.

Example (3.1):

Consider the HB-problem:

$f(x_1) = y_1,\ f'(x_2) = y'_2,\ f(x_3) = y_3.$

Then the incidence matrix E takes the form:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the set e is given by:

$e = \{(1, 0), (2, 1), (3, 0)\}$

The next example discusses the HB-problem with fractional derivatives with its transformation to an integer order HB-problem, but with different function derivatives

Example (3.2):

Consider the HB-problem with fractional derivatives:

$$f(x_0) = y_0, f(x_1) = y_1, f^{(\frac{1}{2})}(x_1) = y_1^{(\frac{1}{2})}, f(x_2) = y_2, f^{(\frac{1}{3})}(x_2) = y_2^{(\frac{1}{3})}$$

In this problem we have $n = \sum_{i.j} a_{i,j} = 5$ and $r_1 = r_2 = 1$ and hence $r = \max\{r_1, r_2\} = 1$, then it is 3-poised problem. The transformed interpolatory conditions are:

$$f(x_0) = y_0, f(x_1) = y_1, f'(x_1) = y'_1, f(x_2) = y_2, f'(x_2) = y'_2$$

The incidence matrix is given by:

$$E = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Since we have $i=0,1,2$ and $j=0,1$ and the HB-set e will take the form:

$$e = \{(0,0),(1,0),(1,1),(2,0),(2,1)\}$$

Definition (3.1), [Schoenberg, 1968]:

The HB-problem (3.1) is said to be normal provided that (3.1) can always be solved uniquely by an $f(x) \in \Pi_{n-1}$.

Remark (3.1):

The condition that the HB-problem (3.1) is normal may be equivalently expressed by the following requirement.

If:

$p(x) \in \Pi_{n-1}$..(3.2)

$p^{(j)}(x_i) = 0$, if $(i, j) \in e$, ...(3.3)

then:

$p(x) = 0$

A closely related concept of the normal HB-problem is presented in the following definition:

Definition (3.2), [Schoenberg, 1968]:

Let m be a natural number, then the HB-problem (3.1) is said to be m-poised provided that:

$p(x) \in \Pi_{m-1}$

$p^{(j)}(x_i) = 0$ if $(i, j) \in e$

then:

$p(x) = 0.$

Remarks (3.2):

1- The HB-problem (3.1) is normal if and only if it is n-poised.

2- If the HB-problem (3.1) is m-poised, then the inequality $m \leq n$ must be hold.

3.2 INTERPOLATION BY G-SPLINE FUNCTIONS

In this section, we shall assume that the HB-problem (3.1) is m-poised and $r < m \leq n$, where $r-1 < \alpha \leq r$, α is the highest fractional derivatives that appears in interpolation problem.

The definition of the G-spline function is facilitated by defining a matrix E* which is obtained from the incidence matrix E by adding

$m - r + 1$ columns of zeros to the matrix E, i.e., $E^* = [a^*_{ij}]$, where $(i = 1, 2, \ldots, k;\ j = 0, 1, \ldots, m)$, and:

$$a^*_{ij} = \begin{cases} a_{ij}, \text{if } j \le r \\ 0, \text{if } j = r+1, r+3, \ldots, m \end{cases}$$

If $j = r+1$, then $E^* = E$.

Definition (3.3), [Schoenberge, 1968]:

A function S(x) is called natural G-spline for the knots $x_1, x_2, \ldots, x_k$ and the matrix E^* of order m provided that it satisfies the following conditions:

(1) $S(x) \in \Pi_{2m-1}$ in (x_i, x_{i+1}), $i = 1, 2, \ldots, k - 1$.

(2) $S(x) \in \Pi_{m-1}$ in $(-\infty, x_1)$ and in (x_k, ∞).

(3) $S(x) \in C^{m-1}(-\infty, \infty)$.

(4) If $a^*_{ij} = 0$, then $S^{(2m-j-1)}(x)$ is continuous at $x = x_i$; that is, $S^{(2m-j-1)}(x_i - 0) = S^{(2m-j-1)}(x_i + 0)$, where $x_i + 0$ and $x_i - 0$ refers to the right and left hand limits of the function $S^{(2m-j-1)}$.

We denote the set of all natural G-splines interpolation polynomials of a given function with knots $x_1, x_2, \ldots, x_k$, by:

$$\mathcal{S}_m = S\,(E^*; x_1, x_2, \ldots, x_k)$$

$\mathcal{S}_m$ is a non empty set and this is shown by the inclusion relation:

$\Pi_{m-1} \subset \mathcal{S}_m$

Indeed, if $S(x) \in \Pi_{m-1}$, then $S(x)$ satisfies the conditions from (1) to (4).

A special case when (3.1) is given by:

$$f(x_i) = y_i, f^{(\alpha_1)}(x_i) = y_i^{(\alpha_1)}, \ldots, f^{(\alpha_{i-1})}(x_i) = y_i^{(\alpha_{i-1})},$$

for all $i = 1, 2, \ldots, k$ and $r_i - 1 \leq \alpha_i \leq r_i$

Then the HB-problem is equivalent to the Hermite problem in approximation theory with integer order derivatives.

It is clearly that $\alpha = \max_i \alpha_i$, and $r - 1 < \alpha_i \leq r$ where $r \leq m \leq n$; $S^{(2m-j-1)}(x)$ is continuous at $x = x_i$, for $j = r, \ldots, m - 1$. In other words, $S^{(v)}(x)$ is continuous at $x = x_i$, for $v = m, m + 1, \ldots, 2m - r$ together with condition (3), of definition (3.3) we conclude that:

$$S(x) \in C^{2m-r}, \text{ near } x = x_i, i = 1, 2, \ldots, k \qquad (3.4)$$

Conditions (1), (2) of definition (3.3) and eq. (3.4) shows that S_m is identical with the natural spline function of degree $2m - 1$ having x_i, $i = 1, 2, \ldots, k$; a multiple knot of multiplicity r_i, where $r_i \leq m$.

Another special case is the Lagrange problem which occurs if we assume that $n = k + 1$ and $e = \{(i, 0), i = 0, 1, \ldots, k\}$.

In this case, $m = k$ and we can show that S_m is identical with the

class of natural spline functions of degree $2m - 1$ having knots $x_0, x_1, \ldots, x_k$, [Schoenberg, 1968].

The existence of a unique G-spline function may be generated from the following theorem.

Theorem (3.1):

If the HB-problem (3.1) is m-poised, then there exists a unique G-spline function

$$S(x) \in S_m(E^*; x_1, ..., x_k)$$

such that

$$S^{(\alpha_j)}(x_i) = y^{(\alpha_i)}(x_i), r_j - 1 < \alpha_j < r_j \quad \text{.............................} \quad (3.5)$$

In order to prove this theorem, first transform the HB-conditions (3.5) using the fractional operator $D^{1-\alpha_j}$ to the both sides, yields:

$$S(x_i) = y^{(\alpha_i)}(x_i) D^{1-\alpha_j}(1)$$

$$= y^{(\alpha_j)}(x_i) \frac{(x-a)^{1-\alpha_j}}{\Gamma(\alpha_j)}$$

and hence the proof may proceed similarly as in the usual case (see [Schoenberg, 1968]).

Now, the most difficulty in the study of G-spline interpolation functions is the constructing of the G-spline functions itself, because most of literatures give no details about these functions, therefore we will illustrate in details the method of construction of such functions.

Next, the construction of the G-spline interpolation formula in a more efficient approach leading to a system of only m + n equations is given as follows:

From conditions (1), (2) and (3) of definition (3.3) it is clear that the most suitable form of the G-spline function S(x) must take the form:

$$S(x) = P_{m-1}(x) + \sum_{i=1}^{k}\sum_{j=0}^{m-1} c_{ij}\frac{(x-x_i)_+^{2m-j-1}}{(2m-j-1)!} \quad \text{.................................(3.6)}$$

where $P_{m-1}(x) \in \Pi_{m-1}$, while the c_{ij} are constants to be determined. Any function of the form (3.6) satisfies the conditions (1), (2) and (3) except:

$$S(x) \in \Pi_{m-1} \text{ if } x_k < x \quad \text{.................................(3.7)}$$

and according to the definition of the truncated power basis, from (3.6).

One can see that $S^{(2m-j-1)}(x)$ is continuous at $x = x_i$ if and only if $c_{ij} = 0$, while condition (4) of definition (3.3) requires that $S^{(2m-j-1)}(x)$ is continuous if and only if $a_{ij}^* = 0$. Leaving out all such terms, yields:

$$S(x) = p_{m-1}(x) + \sum_{(i,j)\in e} c_{ij}\frac{(x-x_i)_+^{2m-j-1}}{(2m-j-1)!} \quad \text{.................................(3.8)}$$

In order to satisfy (3.8), expand all binomial terms and equating to zero those coefficients of x^m, x^{m+1}, …, x^{2m-1}, the following equations are obtained:

$$\sum_{\substack{(i,j)\in e \\ j\le v}} \frac{c_{ij}}{(2m-j-1)!}\binom{2m-j-1}{2m-v-1}(-x_i)^{v-j} = 0,\ v=0,1,\ldots,m-1 \quad \text{...(3.9)}$$

and also have the equations:

$S^{(j)}(x_i) = y^{(j)}(x_i)$, $(i, j) \in e$..(3.10)

Therefore, we get n + m algebraic equations from (3.9) and (3.10) and writing the solution of the unique G-spline so as to exhibit the $f^{(j)}(x_i)$,), to get:

$$S(x) = \sum_{(i,j)\in e} y_i^{(j)} L_{ij}(x)$$

which is the final form of G-spline approximation function. It is clear that the final form of the G-spline function depends on the fundamental G-spline functions $L_{ij}(x)$, $(i, j) \in e$.

3.3 ILLUSTRATIVE EXAMPLES

In this section, some illustrative examples are considered. The first example is for an ordinary case of the HB-problem without fractional derivatives.

Example (3.3):

Consider the following HB-problem:

$f(-1) = y_1$, $f'(0) = y'_2$, $f(1) = y_3$

and to find the G-spline function which interpolate (3.2). In this problem we have r = 1, n = 3.

Hence, the incidence matrix E is then given by:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the HB-set e will take the form:

$$e = \{(1, 0), (2, 1), (3, 0)\}.$$

Therefore, the G-spline interpolation function will be:

$$S(x) = a_0 + a_1 x + \frac{1}{6} c_{10} (x+1)_+^3 + \frac{1}{2} c_{21} x_+^2 + \frac{1}{6} c_{30} (x-1)_+^3$$

Now, we must solve the following linear system of algebraic equations obtained from eq. (3.9) and (3.10):

$$\frac{1}{6} c_{10} + \frac{1}{6} c_{30} = 0$$

$$\frac{1}{2} c_{10} + \frac{1}{2} c_{21} - \frac{1}{2} c_{30} = 0$$

$$a_0 - a_1 = y_1$$

$$a_1 + \frac{1}{2} c_{10} = y'_2$$

$$a_0 + a_1 + \frac{8}{6} c_{10} + \frac{1}{2} c_{21} = y_3$$

Hence, we get:

$$c_{10} = \frac{3}{2} y_1 + 3y'_2 - y_3$$

$$c_{21} = -3y_1 - 6y'_2 + 3y_3$$

$$c_{30} = -\frac{3}{2} y_1 - 3y'_2 + \frac{3}{2} y_3$$

$$a_0 = \frac{1}{4}y_1 - \frac{1}{2}y'_2 + \frac{3}{4}y_3$$

$$a_1 = -\frac{3}{4}y_1 - \frac{1}{2}y'_2 + \frac{3}{4}y_3$$

where y_1, y'_2 and y_3 are given in applications. Therefore, for $y_1 = -1, y'_2 = 0$ and $y_3 = 1$ we get the following G-spline function:

$$S(x) = y_1L_{10}(x) + y'_2L_{21}(x) + y_3L_{30}(x)$$

where:

$$L_{10}(x) = \frac{1}{4}(1 - 3x) + \frac{1}{4}(x+1)^3_+ - \frac{3}{2}x^2_+ - \frac{1}{4}(x-1)^3_+$$

$$L_{21}(x) = -\frac{1}{2}(1 + x) + \frac{1}{2}(x+1)^3_+ - 3x^2_+ - \frac{1}{2}(x-1)^3_+$$

$$L_{30}(x) = \frac{3}{4}(1 + x) - \frac{1}{4}(x+1)^3_+ + \frac{3}{2}x^2_+ + \frac{1}{4}(x-1)^3_+$$

The G-spline function and its comparison with the exact approximated function $f(x) = x^3, -1 \le x \le 1$ is given in fig. (3.1).

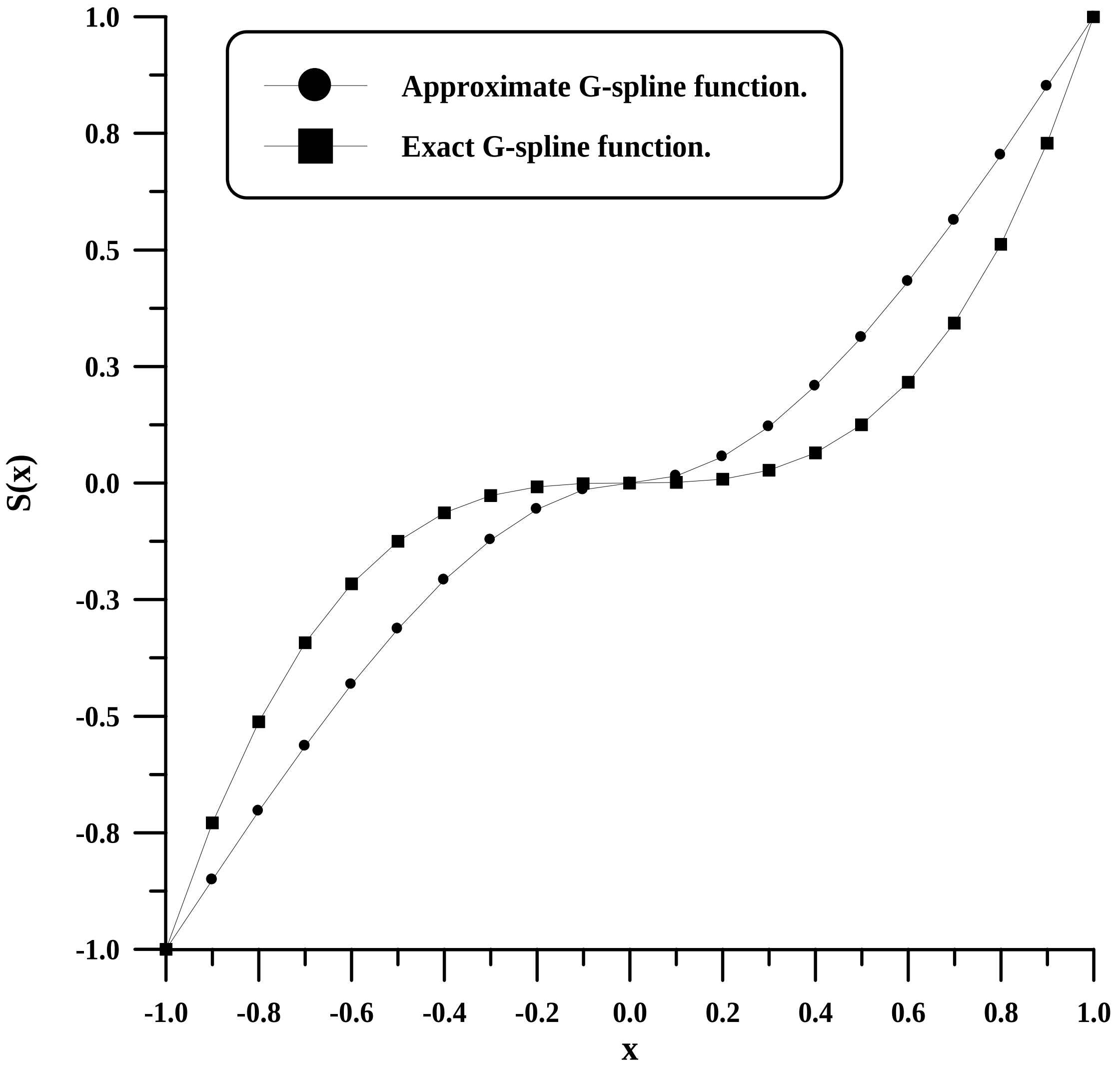

Figure (3.1) Approximate Normal G-spline function for $f(x) = x^3$.

From the graph of the results of the last example, one can notice the error between the G-spline approximation for $f(x) = x^3$ and f(x). This is because no usage to the derivative has been made at the knot points.

<u>Example (3.4):</u>

Consider the HB-problem

$$f(x_0) = y_0, f'(x_0) = y_0, f(x_1) = y_1, f'(x_1) = y_1', f(x_2) = y_2, f'(x_2) = y_2'$$

where $x_0 = -1, x_1 = 0, x_2 = 1.$

and to find the G-spline function which interpolate (3.2). In this problem we have $n=\sum_{i.j} a_{i,j}=6$ and $r=1$, then it is m-poised (5 poised).

The incidence matrix is given by:

$$E=\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \end{bmatrix}$$

Hence,

$$E^*=\begin{bmatrix} a_{00} & a_{01} & 0 \\ a_{10} & a_{11} & 0 \\ a_{20} & a_{21} & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Since we have i=0,1,2 and j=0,1 and the HB-set e will take the form:

$$e=\{(1,0),(1,1),(2,0),(2,1),(3,0),(3,1)\}$$

Form eq. (3.8) we get

$$S(x)=a_0+a_1x+a_2x^2+a_3x^3+c_{10}\frac{(x+1)_+^9}{9!}+c_{11}\frac{(x+1)_+^8}{8!}+c_{20}\frac{(x)_+^9}{9!}$$

$$+c_{21}\frac{(x)_+^8}{8!}+c_{30}\frac{(x-1)_+^9}{9!}+c_{31}\frac{(x-1)_+^8}{8!}$$

and form (3.9) we get

$$c_{10}+c_{20}+c_{30}=0$$

$$c_{10}+c_{11}+c_{20}+c_{21}+c_{30}+c_{31}=0$$

$$\frac{1}{7!2!}c_{10}+\frac{1}{7!}c_{11}+\frac{1}{8!}c_{30}+\frac{1}{8!}c_{31}=0$$

$$\frac{1}{6!3!}c_{10}+\frac{1}{6!2!}c_{11}+\frac{1}{6!3!}c_{30}+\frac{1}{6!2!}c_{31}=0$$

Now from eq. (3.10)

$$a_0+\frac{1}{9!}c_{10}+\frac{1}{8!}c_{11}=0$$

$$c_{10}+c_{11}=0$$

$$a_0-a_1+a_2-a_3=-1$$

$$a_1-2a_2+3a_3=3$$

$$a_0+a_1+a_2+a_3+\frac{2^9}{9!}c_{10}+\frac{2^8}{8!}c_{11}+\frac{2^9}{9!}c_{20}+\frac{2^8}{8!}c_{21}=1$$

$$a_1+2a_2+3a_3+\frac{2^8}{8!}c_{10}+\frac{2^7}{7!}c_{11}+\frac{1}{8!}c_{20}+\frac{1}{7!}c_{21}=3$$

Where $y_1, y_1', y_1, y_2', y_3$ and y_3' are given in applications. Therefore, we get the following G-spline function:

$$S(x)=-0.262x+0.511x^2+1.253x^3+205.948\frac{(x+1)_+^9}{9!}$$

$$+205.948\frac{(x+1)_+^8}{8!}-1.648.10^3\frac{(x)_+^9}{9!}-411.896\frac{(x)_+^8}{8!}$$

$$+1.442\cdot 10^3\frac{(x-1)_+^9}{9!}+617.843\frac{(x-1)_+^8}{8!}$$

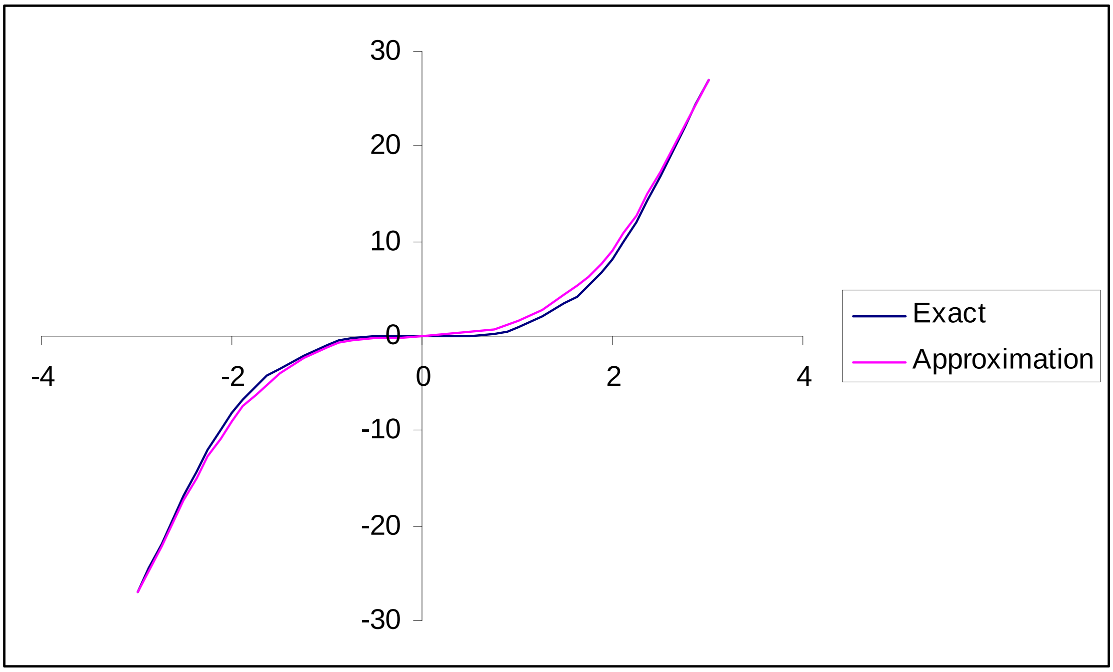

Figure (3.2) Approximation G-spline function

Example (3.4):

Consider the HB-problem

$$f(x_0)=y_0, f(x_1)=y_1, f^{(\frac{1}{2})}(x_1)=y_1^{(\frac{1}{2})}, f(x_2)=y_2, f^{(\frac{1}{3})}(x_2)=y_2^{(\frac{1}{3})}$$

where $x_0=0, x_1=1, x_2=2$.

and to find the G-spline function which interpolate (3.2). In this problem we have $n=\sum_{i.j} a_{i,j}=5$ and $r=1$, then it is m-poised (3 poised).

The incidence matrix is given by:

$$E=\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{22} \end{bmatrix}$$

Hence,

$$E^* = \begin{bmatrix} a_{00} & a_{01} & 0 \\ a_{10} & a_{11} & 0 \\ a_{20} & a_{12} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Since we have $i=0,1,2$ and $j=0,1$ and the HB-set e will take the form:

$$e = \{(0,0),(1,0),(1,1),(2,0),(2,1)\}$$

From (3.8) we have,

$$S(x) = a_0 + a_1 x + a_2 x^2 + c_{00}\frac{(x-0)_+^5}{5!} + c_{10}\frac{(x-1)_+^5}{5!} + c_{11}\frac{(x-1)_+^2}{4!} + c_{20}\frac{(x-2)_+^5}{5!} + c_{21}\frac{(x-2)_+^4}{4!}$$

And from (3.9) we get

$$\frac{1}{5!}c_{00} + \frac{1}{5!}c_{10} + \frac{1}{5!}c_{20} = 0$$

$$-\frac{1}{4!}c_{10} + \frac{1}{4!}c_{11} + \frac{1}{4!}c_{20} + \frac{1}{4!}c_{21} = 0$$

$$\frac{1}{12}c_{10} + \frac{1}{6}c_{11} + \frac{1}{3}c_{20} - \frac{1}{3}c_{21} = 0$$

Now from eq. (3.10)

$$a_0 = 0$$

$$a_0 + a_1 + a_2 + \frac{1}{5!}c_{00} = 1$$

$$a_1 + 2a_2 + \frac{1}{4!}c_{00} = 3$$

$$a_0 + 2a_1 + 4a_2 + \frac{2^5}{5!}c_{00} + \frac{1}{5!}c_{10} + \frac{1}{4!}c_{11} = 8$$

$$a_1 + 4a_2 + 0.667c_{00} + \frac{1}{4!}c_{10} + \frac{1}{3!}c_{11} = 12$$

where y_1, y_2' and y_3 are given in applications. Therefore, we get the following G-spline function:

$$S(x) = -0.618x + 1.491x^2 + 15.275\frac{(x)_+^5}{5!} - 37.393\frac{(x-1)_+^5}{5!} - 11.853\frac{(x-1)_+^4}{4!}$$

$$+22.118\frac{(x-2)_+^5}{5!} + 18.697\frac{(x-2)_+^4}{4!}$$

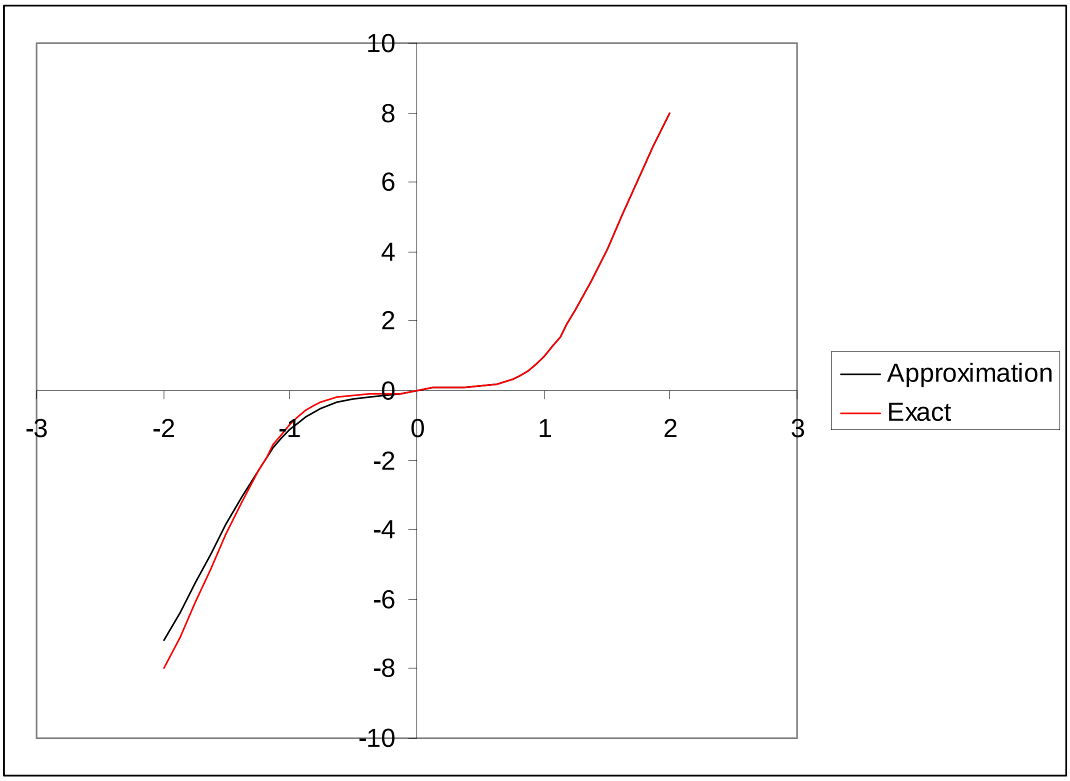

Figure (3.2) Approximate G-spline function

Example (3.5):

Consider the HB-problem

$$f(x_0)=y_0, f(x_1)=y_1, f^{(\frac{1}{2})}(x_1)=y_1^{(\frac{1}{2})}, f(x_2)=y_2, f^{(\frac{3}{2})}(x_2)=y_2^{(\frac{3}{2})}$$

where $x_0=-1, x_1=0, x_2=1$.

and to find the G-spline function which interpolate (3.2). In this problem we have $n=\sum_{i.j} a_{i,j}=5$ and $r=2$, then it is m-poised (3 poised).

The incidence matrix is given by:

$$E=\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}$$

Hence,

$$E=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Since we have i=0,1,2 and j=0,1,2 and the HB-set e will take the form:

$$e=\{(0,0),(1,0),(1,1),(2,0),(2,2)\}$$

From (3.8) we have,

$$S(x)=a_0+a_1x+a_2x^2+c_{00}\frac{(x+1)_+^3}{3!}+c_{10}\frac{(x)_+^3}{3!}+c_{11}\frac{(x)_+^2}{2!}$$
$$+c_{20}\frac{(x-1)_+^5}{5!}+c_{22}(x-1)_+^1$$

And from eq. (3.9) we get

$$\frac{1}{5!}c_{00}+\frac{1}{5!}c_{20}=0$$

$$\frac{1}{4!}c_{00}+\frac{1}{4!}c_{10}-\frac{1}{4!}c_{20}=0$$

$$\frac{1}{12}c_{00}+\frac{1}{12}c_{20}+\frac{1}{3!}c_{22}=0$$

Now from eq. (3.10)

$$a_0-a_1+a_2=-1$$

$$a_0+\frac{1}{5!}c_{10}=0$$

$$a_1+\frac{1}{2!}c_{00}+c_{22}=0$$

$$a_0+a_1+a_2+\frac{1}{3!}c_{10}+\frac{1}{2!}c_{11}=1$$

$$2a_2+c_{10}+c_{11}=6$$

where y_1, y_2' and y_3 are given in applications. Therefore, we get the following G-spline function:

$$S(x)=-0.182+5.455x+4.636x^2-10.909\frac{(x+1)_+^3}{3!}-21.818\frac{(x)_+^3}{3!}$$

$$-25.091\frac{(x)_+^2}{2!}+10.909\frac{(x-1)_+^3}{3!}$$

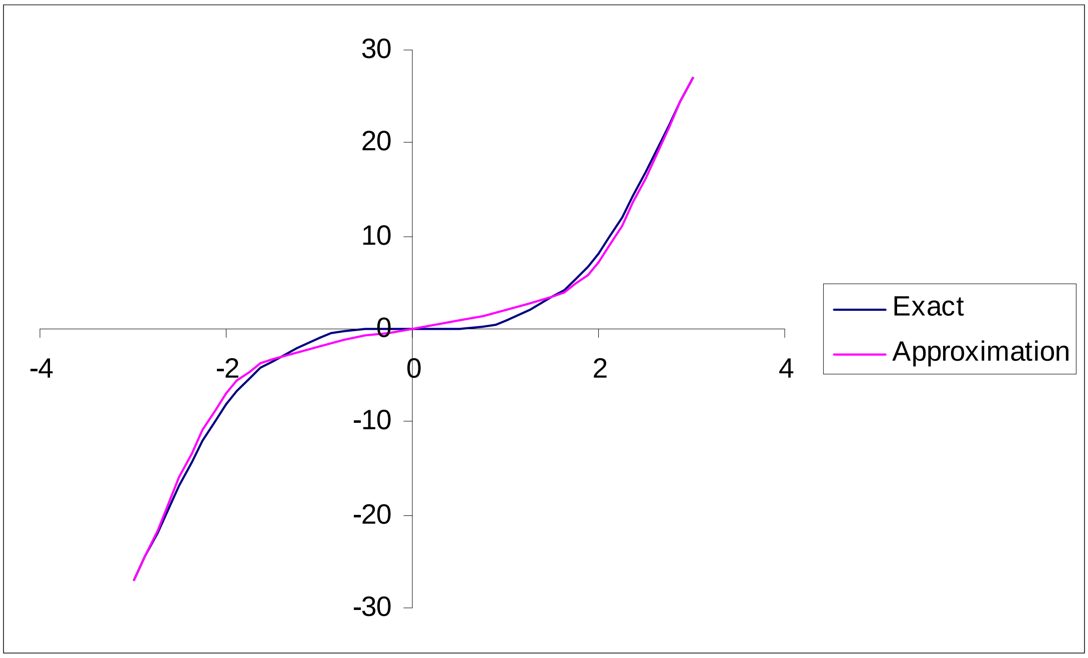

Figure (3.3) Approximate G-spline function

CHAPTER TWO

APPROXIMATION BY SPLINE FUNCTIONS

The name "spline function" comes from the fact that a third degree spline function approximates the behaviour of mechanical spline, a device used by draughtsmen to draw a smooth curve, that consists of a flexible strip to which weights are attached at certain points in order to force a fit to the given data points, [Ahlberg, 1967].

In order to avoid the oscillatory in approximation by high degree polynomials, it is important to remark that the spline function is a piecewise polynomial function drawn in such a way that its derivatives up to and including the order one less the degree of polynomials used are continuous everywhere in the domain of definition.

For the purposes of interpolation, the use of spline function offers substational advantages such as by employing polynomials of relatively low degree, and then one can often avoid the marked undulatory behaviour that commonly arises from fitting a single polynomial exactly to a large number of empirical observations, [Ahlberg, 1967].

A spline function obviously provides continuity of the greatest possible number of derivatives for the interpolatory function which must be consistent with the use of polynomials of lower degree than would be required to fit all data points by using polynomial, [Schoenberg, 1963].

2.1 POLYNOMIAL INTERPOLATION

One cause of using polynomials for approximation and interpolation of a given function or data points is that they may be evaluated, differentiated and integrated easily and in finitely many steps using just the basic mathematical operations of addition, subtraction and multiplication.

Following are the most elementary types or methods of interpolation.

2.1.1 The Lagrange Interpolation Polynomial:

Suppose that an approximation of a function $f \in C[a,b]$ is evaluated by a polynomial of degree n (or order n+1) as:

$$p(x) = \sum_{i=0}^{n} c_i x^i \text{ , } a \le x \le b \text{..(2.1)}$$

where $c_i, \forall i = 0,1,2,...,n$; are the polynomial coefficients must be evaluated and hence, the problem here is reduced to the evaluation of the coefficients c_i's, $i = 0, 1, \ldots, n$. The most straightforward method for evaluating p(x) is to calculate the value of f at (n + 1) distinct points x_i's, $i = 0, 1, \ldots, n$ of [a, b] and to satisfy the equations:

$$p(x_i) = f(x_i),\ i = 0, 1, \ldots, n \text{..(2.2)}$$

which will give a linear system of algebraic equations in $c_0, c_1, \ldots, c_n$.

The following theorem shows that one can determine the polynomial $p \in \Pi_n$ uniquely, where Π_n is the set of all polynomials of degree at most $n \in \mathbb{N}$.

In addition the next result is given [Burden, 1985], but we give here more details about the proof for completeness.

Theorem (2.1.1):

Let x_i, $i = 0, 1, \ldots, n$, be any set of $(n + 1)$ distinct points in $[a, b]$, and let $f \in C[a, b]$. Then there is exactly one polynomial $p \in \Pi_n$ that satisfy the equations of interpolatory conditions:

$$p(x_i) = f(x_i),\ i = 0, 1, \ldots, n$$

Proof:

To prove the existence, define the function $\ell_k(x)$ as the basis functions, by:

$$\ell_k(x) = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)},\ a \le x \le b,\ k = 0, 1, \ldots, n$$

then $\ell_k(x) \in \Pi_n$, has the values:

$$\ell_k(x_i) = \delta_{ki},\ i,k = 0, 1, \ldots, n$$

such that:

$$\delta_{ki} = \begin{cases} 1, & k = i \\ 0, & k \neq i \end{cases}$$

It follows that the function:

$$p(x) = \sum_{k=0}^{n} f(x_k)\ell_k(x)$$

is in Π_n and

$$p(x_i) = \sum_{k=0}^{n} f(x_k)\ell_k(x_i)$$
$$= \sum_{k=0}^{n} f(x_k)\delta_{ki}$$
$$= f(x_i), \ \forall i = 0,1,\dots,n.$$

and therefore p(x) satisfy the condition (2.2).

To prove the uniqueness, suppose that there exist another polynomial $g(x) \in \Pi_n$, such that $g(x_i) = f(x_i)$, for all i = 0, 1, …, n, i.e., g(x) satisfy also condition (2.12); and define a function $H(x) = p(x) - g(x)$. Now,

$$H(x) = p(x) - g(x)$$

$$= \sum_{k=0}^{n} f(x_k)\ell_k(x) - \sum_{k=0}^{n} g(x_k)\ell_k(x)$$

$$= \sum_{k=0}^{n} \big(f(x_k) - g(x_k)\big)\ell_k(x)$$

$$= \sum_{k=1}^{n} H(x_k)\ell_k(x).$$

hence, $H(x) \in \Pi_n$, and also:

$$H(x_i) = p(x_i) - g(x_i)$$
$$= f(x_i) - f(x_i) = 0$$

therefore $H(x_i) = 0$, $i = 0, 1, \ldots, n$

Hence from the fundamental theorem of algebra, H(x) has (n+1) roots.

Therfor H(x) must be identically zero for all $x \in [a,b]$ (since H is a polynomial of degree n that has n+1 roots).

therefore p(x)=g(x), $\forall x \in [a,b]$

Hence, p(x) is unique ■

If e denotes the error function encountered in the approximation, by Lagrange method, i.e.,

$$e(x) = f(x) - p(x), \quad a \le x \le b \qquad (2.3)$$

where $p(x) \in \Pi_n$ and satisfies the interpolatory conditions (2.2). It should be clear that, if f is changed by adding element of Π_n, then the interpolation process automatically adds the same element to p, which leaves e unchanged. Expressions for the error should show this property. It is therefore appropriate, when $f \in C^{n+1}[a, b]$ to state e in terms of the derivative $f^{(n+1)}$, which is given in the next theorem:

The next theorem is given also in [Burden, 1985] and we give here more details of the proof.

Theorem (2.1.2):

For any set of distinct interpolation points $\{x_i, i = 0, 1, \ldots, n\}$ in [a, b] and for any $f \in C^{(n+1)}[a, b]$, let $p \in \Pi_n$ that satisfies (2.3). Then, for any $x \in [a, b]$, the error of Lagrange interpolation polynomial is:

$$e(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i) \quad \text{..(2.4)}$$

where $\xi \in [a, b]$ that depends on x.

Proof:

For certain x_k, if $x = x_k$, for $k = 0, 1, \ldots, n$; then $f(x_k) = p(x_k)$ and choosing $\xi(x_k)$ arbitrarily in (a, b) to satisfy (2.4).

If $x \neq x_k$, for any $k = 0, 1, \ldots, n$; define a function g for $t \in [a, b]$ by:

$$g(t) = f(t) - p(t) - [f(x) - p(x)] \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)}$$

since $f \in C^{n+1}[a, b]$, $p \in C^{\infty}[a, b]$, and $x \neq x_k$, for any k, it follows that $g \in C^{n+1}[a, b]$. For $t = x_k$, $g(x_k) = 0$. Moreover:

$$g(x_k) = f(x_k) - p(x_k) - \big(f(x) - g(x)\big) \prod_{i=0}^{n} \frac{(x_k - x_i)}{(x - x_i)}$$

$$= 0 - \big(f(x) - g(x)\big) * 0 = 0$$

Also

$$g(x) = f(x) - p(x) - \big(f(x) - p(x)\big) \prod_{i=0}^{n} \frac{(x - x_i)}{(x - x_i)}$$

$$= f(x) - p(x) - f(x) + p(x)$$

$$= 0, \ \forall x \in [a, b]$$

Thus, g vanishes at the n + 2 distinct numbers $x, x_0, x_1, \ldots, x_n$.

By the generalized Roll's theorem, there exists $\xi = \xi(x)$ in (a, b) for which $g^{(n+1)}(\xi) = 0$.

Evaluating $g^{(n+1)}$ at ξ gives:

$$0 = g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - p^{(n+1)}(\xi) - [f(x) - p(x)] \frac{(n+1)!}{\prod_{i=0}^{n}(x - x_i)}$$

Since p is a polynomial of degree at most n, then $p^{(n+1)}(\xi)$ must be identically zero. Hence:

$$e(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n}(x - x_i) . \quad \blacksquare$$

2.1.2 Hermite Interpolation Polynomial:

In certain cases, it happens that in addition to the function values $f(x_i)$, i = 0, 1, …, n, some additional values of the derivative of f are required also. The general Hermite interpolation problem is to calculate $p \in \Pi_n$ that satisfies the interpolatory conditions:

$$p^{(j)}(x_i) = f^{(j)}(x_i),\ j = 0, 1, \dots, \alpha_i,\ i = 0, 1, \dots, n \quad \text{.........................(2.5)}$$

where the number of coefficients of p equals to the number of data points, and α_i it is the highest derivative of f in x_i, for all $i = 0,1,\dots,n$ which implies that n is defined by the following equation:

$$n + 1 = \sum_{i=0}^{n}(\alpha_i + 1) \quad \text{..(2.6)}$$

The polynomial p may be obtained from an interesting extension of Newton's interpolation method (or Newton's divided difference formula). The data on the right hand side of (2.5) define the required interpolation polynomial uniquely, as shown in the following theorem:

Theorem (2.1.3), [Powell, 1981]:

Let $\{x_i, i = 0, 1, \ldots, m\}$ be a set of distinct points from [a, b], and let the real numbers $\{f^{(j)}(x_i), j = 0, 1, \ldots, \alpha_i, i = 0, 1, \ldots, n\}$ be given. Then there is exactly one polynomial $p \in \Pi_n$ that satisfies eq. (2.5), where the value of n is defined by eq. (2.6).

Proof:

The first part of the proof is a highly useful general method for demonstrating the uniqueness of approximation from the linear space of all polynomial functions.

The approximating functions are parameterized by choosing a basis for the linear space and in the present case every member of Π_n can be expressed in the form

$$p(x) = \sum_{i=0}^{n} c_i x^i, \qquad a \le x \le b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.7)$$

because the number of conditions on p equals to the number of parameters, the required coefficients $\{c_i : i = 0,1,\ldots,n\}$ satisfy a square system of linear algebraic equations.

It is therefore sufficient to prove that the matrix of the system obtained is non-singular.

An equivalent condition is that, setting the right-hand sides of the equations to zero, then they are satisfied only if all the parameters equals to zero. Hence it suffices to prove that, if all the coefficients c_i 's, $\forall i$ are zero, then p is identically zero.

Hence, when the data are zero, then p is a multiple of the following polynomial

$$\prod_{i=0}^{n}(x - x_i)^{l_i+1}, \quad a \le x \le b.$$

Because this polynomial includes the term x^{n+1}, the multiplying factor must be zero.

Hence p is identically zero. ■

Remark (2.1.1):

The Hermite interpolation polynomial may be generalized for any node point in terms of x_{i-1} and x_{i+1} which is the so called Hermite-Berkhoff problem.

2.2 SPLINE FUNCTIONS

Suppose one want to interpolate n-given data points (x_i, y_i), for all i = 1, 2, …, n; and $a = x_1 < x_2 < \ldots < x_n = b$, by means of a function S, which has continuous derivatives of order 1, 2, …, k; where k is an integer number and $1 \le k \le n$. Furthermore, to find an approximation function S, which minimizes:

$$\sigma = \int_a^b [g^{(k)}(x)]^2 dx$$

overall approximation functions g.

The problem just described does not have a unique solution if $k > n$, because there is an infinite number of polynomials of degree $k - 1$ that interpolate with the data points exactly and all of these polynomials lead to $\sigma = 0$.

For $k < n$, there is a unique solution which is a picewise function given in any interval $[x_i, x_{i+1}]$, $i=1,2,\ldots,n-1$ by a polynomial of degree $2k - 1$, which is, moreover, by a different polynomial in each such interval.

Furthermore, the polynomial functions that make the graph of the function S "join smoothly" in the sense that, for each two polynomials that represent g on the subintervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$, we have to the left and to the right of x_i the same ordinate and the same values of the derivatives of order 1, 2, …, $2k - 2$ for $x = x_i$.

The function S also satisfies the following property; in each of the intervals $(-\infty, x_1)$ and (x_n, ∞) it is reduced to a polynomial of degree $k - 1$. This function, which we described just now belongs to a class of functions known as ***spline functions***.

The next definition gives the alternate mathematical definition of the spline functions.

<u>Definition (2.2.1), [Greville, 1967]:</u>

A spline function $S(x)$ of degree $m \in \mathbb{N}$ with knots or "nodes" $a= x_1 < x_2 < \ldots < x_n = b$ is a function which satisfies following two conditions:

(i) In each of the intervals $(-\infty, x_1)$, $[x_1, x_2)$, …, $[x_n, \infty)$, the spline function S(x) is a polynomial of degree m.

(ii) $S(x) \in C^{m-1}(-\infty, \infty)$.

It is important to mention that, there is an essential difference between spline of even and odd degrees. One finds, for example, that, polynomial splines of even degree interpolating a prescribed function at certain mesh points need not to be exist, and for more details see [Ahlberg, 1967], while for an odd degree this problem is violated.

<u>Definition (2.2.2), [Greville, 1967]:</u>

A spline function S(x) of odd degree $2m - 1, m \in \mathbb{N}$ with knots $a = x_0 < x_1 < \ldots < x_n = b$ is called a natural spline function if it satisfies the following conditions:

(i) $S(x) \in \Pi_{2m-1}$ in $[x_1, x_2), [x_2, x_3), \ldots, [x_{n-1}, x_n]$.

(ii) $S(x) \in C^{2m-2}(-\infty, \infty)$.

(iii) $S(x) \in \Pi_{m-1}$ in $(-\infty, x_0)$ and (x_n, ∞).

where the symbol Π_{m-1} is used to denote the set of all polynomials of degree less than or equal to $m - 1$.

It is clear that from the above definitions that any sum or difference of spline functions of a given degree with given knots is also a spline function of the same degree with the same knots. Perhaps, the simplest spline function is the truncated power function x_+^m, defined by [Schoenberg, 1946]:

$$x_+^m = \begin{cases} x^m, & x > 0 \\ 0, & x \le 0 \end{cases}$$

where m is a positive integer. The differentiation rule for this function is similar to that for ordinary powers:

$$\frac{d}{dx} x_+^m = m\, x_+^{m-1}$$

The m-th derivative of x_+^m is $m!\, x_+^0$, where x_+^0 is taken to be the Heaviside function, defined as:

$$x_+^0 = \begin{cases} 1, & x > 0 \\ 0, & x \le 0 \end{cases}$$

for the rest of this work, a function of the form $(x-c)_+^m$, which is defined by:

$$(x-c)_+^m = \begin{cases} (x-c)^m, & x > c \\ 0, & x \le c \end{cases}$$

will be called an elementary spline function or the truncated power basis where c is a real constant. The m-th derivative of this function has its only discontinuity at $x = c$, in which there is a jump of magnitude m!.

Schoenberg and Whitney [Schoenberg, 1953], have pointed out that any spline functions may be expressed uniquely as the sum of a polynomial and a linear combination of elementary spline functions. This perhaps most easily seen by considering the jumps, at the knots of the spline function of its m-th derivative, where m is the splines degree. In other words, let the spline function $S_1(x)$ of degree m have knots x_i, and corresponding jumps s_i in its m-th derivative, for i = 1, 2, …, n. It is assumed that $x_i < x_j$ if $i < j$. Also, let $p_m(x)$ to denote the polynomial of degree m, which is identical with $S_1(x)$ for $x \leq x_1$. Now, define a function S(x) by:

$$S(x) = p_m(x) + \sum_{j=1}^{n} c_j (x - x_j)_+^m$$

where $c_j = s_j/m!$, and the S(x) is a spline function of degree m having the required knots and the required jump in the m-th derivative at each joint, and it is identical with $S_1(x)$ for $x \leq x_1$. Moreover, $S_1(x) - S(x)$ is a spline function of degree m and also belongs to C^m; therefore it is a polynomial as it is identically zero for $x \leq x_1$, it is identically zero everywhere.

For the case of natural spline function of degree 2k − 1, [Greville, 1964], [Schoenberg, 1953], S(x) takes the form:

$$S(x) = p_{k-1}(x) + \sum_{j=1}^{n} c_j (x - x_j)_+^{2k-1} \quad \text{...(2.8)}$$

For this form, one can notice that, condition (iii) of definition (2.2.2) is immediately satisfied for the interval $(-\infty, x_1)$.

In order to satisfy condition (iii) of definition (2.2.2) for the interval (x_n, ∞). Equating to zero the coefficients of $x^k, x^{k+1}, \ldots, x^{2k-1}$, then condition (iii) implies that:

$$\sum_{j=1}^{n} c_j x_j^i = 0, \; i = 0, 1, \ldots, k-1 \quad \text{...(2.9)}$$

This is equivalent to the following condition:

$$\sum_{j=1}^{n} c_j Q(x_j) = 0 \quad \text{(2.10)}$$

for every polynomial Q(x) of degree at most $k - 1$.

The next theorem discusses the existence of a unique natural spline function of odd degree which is of great importance in applications, the details of the proof is given for completeness.

Theorem (2.2.3), [Greville, 1967]:

For $k \le n$ and given knots $x_1 < x_2 < \dots < x_n$, there is a unique natural spline function S(x) of degree $2k - 1$ having the knots x_i and satisfying the equations:

$$S(x_i) = y_i,\ i = 1, 2, \dots, n \quad \text{(2.11)}$$

for given arbitrary y_i.

Proof:

Take $a = x_1$ and $b = x_n$ for notational convenience.

Substituting (2.8) into (2.11) give the following equations:

$$p_{k-1}(x_i) + \sum_{j=1}^{n} c_j (x_i - x_j)_+^{2k-1} = y_i,\ i = 1, 2, \dots, n \quad \text{(2.12)}$$

Together eqs. (2.12) and (2.9) constitute a system of n + k equations of n + k unknown of parameters.

To show that the related system of equations is non-singular or, what amounts to the same thing, that the corresponding homogenous system has only the trivial solution in which $c_j = 0$, for all j and p_{k-1} is identically zero.

Consider the quantity:

$$\sigma_0 = \int_a^b [S^{(k)}(x)]^2 dx \qquad (2.13)$$

Then repeated differentiation of eq. (2.8) up to order k, give:

$$S^k(x) = (2k-1)(2k-2)...(k)\sum_{j=1}^{n} c_j \left(x - x_j\right)_+^{k-1}$$

$$= \frac{(2k-1)(2k-2)...(k)(k-1)!}{(k-1)!}\sum_{j=1}^{n} c_j \left(x - x_j\right)_+^{k-1}$$

$$= \frac{(2k-1)!}{(k-1)!}\sum_{j=1}^{n} c_j (x - x_j)_+^{k-1} \qquad (2.14)$$

and observing that $S^{(k)}(x)$, $S^{(k+1)}(x)$, …, $S^{(2k-2)}(x)$ vanish for x = a by eq. (2.14), and for x = b by eq. (2.14) and (2.10).

Successive integration by parts to the right hand side of eq. (2.13) yields:

$$\sigma_0 = S^{(k)}(x)S^{(k-1)}(x)\Big|_a^b - S^{(k+1)}(x)S^{(k-2)}(x)\Big|_a^b + \ldots \pm S^{(2k-2)}(x)S'(x)\Big|_a^b$$

$$+ (-1)^{k-1}\int_a^b S'(x)S^{(2k-1)}(x)\,dx$$

Therefore:

$$\sigma_0 = (-1)^{k-1}\int_a^b S'(x)S^{(2k-1)}(x)\,dx$$

$$= (-1)^{k-1}\sum_{j=1}^{n-1}\int_{x_j}^{x_{j+1}} S'(x)S^{(2k-1)}(x)\,dx$$

Hence:

$$\sigma_0 = (-1)^{k-1}\sum_{j=1}^{n-1} S^{(2k-1)}(x)S(x)\Big|_{x_j}^{x_{j+1}}$$

which is a jump of $S^{(2k-1)}$ at x_j of magnitude $(2k-1)!$.

As $S^{(2k-1)}(x)$ is constant in each subinterval (x_j, x_{j+1}), for all $j = 1, 2, \ldots, n-1$ and vanish outside of (x_1, x_n), this gives:

$$\sigma_0 = (-1)^{k}\sum_{j=1}^{n-1} S(x_j)[S^{(2k-1)}(x_j+0) - S^{(2k-1)}(x_j-0)]$$

$$= (-1)^{k}(2k-1)!\sum_{j=1}^{n-1} c_j S(x_j) = y_i = 0 \quad \text{..................................(2.15)}$$

In the homogeneous system (2.12):

$$y_i = 0,\ i = 1, 2, \ldots, n \quad \text{...(2.16)}$$

and therefore eqs. (2.11) and (2.15) gives

$$\sigma_0 = (-1)^{k}(2k-1)!\sum_{j=1}^{n-1} c_j y_j$$

$$= (-1)^{k}(2k-1)!\sum_{j=1}^{n-1} c_j \cdot 0 = 0$$

From eq. (2.13) and from the continuity of $S^{(k)}(x)$ implies that $S^{(k)}(x)$ is identically zero in (a, b).

Substituting $S^{(k)}(x) = 0$ in eq. (2.13), yields:

$$\frac{(2k-1)!}{(k-1)!}\sum_{j=1}^{n} c_j (x - x_j)_+^{k-1} = 0, \ \forall x \in (a, b)\ , \ \forall k = 1, 2, \ldots, n$$

Which is true only if $c_1 = c_2 = \ldots = c_n = 0$

This result together with eqs. (2.12) and (2.16) give:

$$p_{k-1}(x_i) = 0$$

since $k - 1 < n$ and the x_i's are distinct, then it follows that from the fundamental theorem of algebra that $p_{k-1}(x)$ is identically zero. ■

Theorem (2.2.4), [Greville, 1967]:

Let f(x) be any continuous function on [a, b] together with its derivatives of order 1, 2, …, $k \le n$ and let the equations:

$$f(x_i) = y_i,\ i = 1, 2, \ldots, n \qquad (2.17)$$

hold, where $x_1 < x_2 < \ldots < x_n$. If S(x) is the unique natural spline function with knots x_i, satisfying eq. (2.11), then:

$$\int_a^b [f^{(k)}(x)]^2 dx \ge \int_a^b [S^{(k)}(x)]^2 dx$$

with the equality satisfied only if f(x) = S(x).

Proof:

Since $S^{(k)}(x)$ vanishes for x outside of the interval (x_1, x_n) and let us take $a = x_1$ and $b = x_n$

Now:

$$\int_a^b [f^{(k)}(x)]^2 dx = \int_a^b [S^{(k)}(x)]^2 dx + \int_a^b [f^{(k)}(x) - S^{(k)}(x)]^2 dx +$$

$$2\int_a^b S^{(k)}(x)\,[f^{(k)}(x) - S^{(k)}(x)]\,dx$$

To show that the last integration in the right hand side is vanished.

Since, successive integrations by parts give:

$$\int_a^b S^{(k)}\left[f^{(k)}(x) - S^{(k)}(x)\right]dx = (-1)^{k-1}\sum_{j=1}^{n-1}\int_{x_j}^{x_{j+1}} S^{(2k-1)}(x)\,[f'(x) - S'(x)]\,dx$$

Therefore, in each subinterval (x_j, x_{j+1}), $S^{(2k-1)}(x)$ is constant function, while $f(x) - S(x)$ vanishes at the end points, because of equations (2.11) and (2.17), the integral therefore vanishes.

Therefore:

$$\int_a^b [f^{(k)}(x)]^2 dx = \int_a^b \left[S^{(k)}(x)\right]^2 dx + \int_a^b \left[S^{(k)}(x) - S^{(k)}(x)\right]^2 dx$$

Since

$$\int_a^b \left[S^{(k)}(x) - S^{(k)}(x)\right]^2 dx \geq 0\,.$$

Hence:

$$\int_a^b \left[f^{(k)}(x)\right]^2 dx \geq \int_a^b \left[S^{(k)}(x)\right]^2 dx\,. \quad \blacksquare$$

2.3 SOME TYPES OF SPLINE FUNCTIONS

It is convenient after we give the definition of the spline function and some of its related theorems to discuss and list some of the well known and widely used types of spline functions.

Among such types of spline functions, are the following types:

2.3.1 Generalized Splines:

In general, it is assumed that we have an n-th order linear differential operator L, defined by:

$$L \equiv a_n(x)D^n + a_{n-1}(x)D^{n-1} + \dots + a_0(x)$$

where each $a_j(x) \in C^n[a,b], \forall j = 0,1,\dots,n;$ $a_n(x)$ does not vanish on [a, b] and $D = \frac{d}{dx}$. Let L* be the formal adjoint operator of L. Thus:

$$L^* = (-1)^n D^n\{a_n(x).\} + (-1)^{n-1} D^{n-1}\{a_{n-1}(x).\} + \dots - D\{a_1(x).\} + a_0(x)$$

If $\Delta : a = x_0 < x_1 < \dots < x_N = b$, is a mesh on [a, b], then the generalized spline of deficiency k ($0 \le k \le n$) with respect to Δ is a function $S_\Delta(x)$ which is in $\mathcal{K}^{2n-k}(a, b)$ (by $\mathcal{K}^n(a, b)$, we mean the class of all functions f(x) defined on [a, b] which posses an absolutely continuous $(n - 1)$th derivatives on [a, b] and whose n-th derivative is in $L^2(a, b)$ and satisfies the differential equation:

$$L^*LS_\Delta = 0$$

on each open mesh interval of Δ. We also say that $S_\Delta(x)$ has an order 2n when we want to indicate the order of the operator L*L defining $S_\Delta(x)$.

The following theorems gives the optimality, existence and uniqueness of the generalized splines, respectively.

Theorem (2.3.1):

Let $\Delta : a = x_0 < x_1 < \ldots < x_N = b$ and $Y = \{ y_{i_\alpha} \}$, $i = 0, 1, \ldots, N$; $\alpha = 0, 1, \ldots, k-1$, be given. Then of all functions f(x) in $\mathcal{K}^n(a, b)$, such that $f^{(\alpha)}(x_i) = y_{i_\alpha}$ $(i = 0, 1, \ldots, N;\ \alpha = 0, 1, \ldots, k-1)$, the generalized spline $S_\Delta(Y; x)$ of type k, when it exists, minimizes:

$$\int_a^b \{Lf(x)\}^2\, dx$$

(a generalized spline $S_\Delta(f; x)$ of deficiency k on Δ is a spline interpolation of type k if $S_\Delta^{(\alpha)}(f, x)$ $(\alpha = 0, 1, \ldots, k-1)$ interpolates to the values of $f^{(\alpha)}(x)$ at the mesh points of Δ and $\{LS_\Delta(x)\}^{(\alpha)} = 0$, $\alpha = 0, 1, \ldots, n-k-1$; at $x = a$ and $x = b$).

Proof: See [Ahlberg, 1967]. ■

Theorem (2.3.2):

Let $\Delta : a = x_0 < x_1 < \ldots < x_N = b$ and $Y = \{ y_{i_\alpha} \}$, $i = 0, 1, \ldots, N$; $\alpha = 0, 1, \ldots, k-1$, be given. In addition, let L and Δ be such that; if $Lg \equiv 0$ and $g^{(\alpha)}(x_i) = 0$ $(i = 0, 1, \ldots, N;\ \alpha = 0, 1, \ldots, k-1)$, then $g(x) \equiv 0$. Under these conditions, the generalized spline $S_\Delta(Y, x)$ of type k on Δ, such that $S_\Delta^{(\alpha)}(Y; x_i) = y_{i_\alpha}$ $(i = 0, 1, \ldots, N;\ \alpha = 0, 1, \ldots, k-1)$ exists.

Proof: See [Ahlberg, 1967]. ■

Theorem (2.3.3):

Let $\Delta : a = x_0 < x_1 < \ldots < x_N = b$ and $Y = \{ y_{i_\alpha} \}$, $i = 0, 1, \ldots, N$; $\alpha = 0, 1, \ldots, k-1$, be given. In addition, let L and Δ be such that, if $Lg = 0$ and $g^{(\alpha)}(x_i) = 0$, $i = 0, 1, \ldots, N$; $\alpha = 0, 1, \ldots, k-1$, then $g(x) \equiv 0$. Under these conditions, there is at most one generalized spline $S_\Delta(Y; x)$ of type k on Δ, such that $S_\Delta^{(\alpha)}(Y; x_i) = y_{i_\alpha}$ ($i = 0, 1, \ldots, N$; $\alpha = 0, 1, \ldots, k-1$).

Proof: See [Ahlberg, 1967]. ■

2.3.2 Basis Splines (B-splines):

In this subsection, we give the definition of the basis spline functions which is usually denoted by B-splines and also record various properties of the B-spline in order to make it therefore as familiar and real as possible as an object of approximation theory.

Definition (2.3.4), [deBoor, 1978]:

Let $t = \{t_i\}$ be a non-decreasing sequence (which may be finite or infinite). The i-th normalized B-spline of order k for the knot sequence t is denoted by $B_{i,k,t}$ and is defined by the rule:

$$B_{i,k,t}(x) = (t_{i+k} - t_i)[t_i, t_{i+1}, \ldots, t_{i+k}](. - x)_+^{k-1}, \ \forall\ x \in \mathbb{R}$$

where $[t_i, t_{i+1}, \ldots, t_j]f$ is the divided difference of order $j - i$ of f at the points $t_i, t_{i+1}, \ldots, t_j$.

Properties of B-Splines:

1. It is notable that it is right a way that $B_{i,k,t}$ has small support, i.e.,

$$B_{i,k,t}(x) = 0, \text{ for all } x \notin [t_i, t_{i+k}].$$

2. For all $t_r < x < t_s$:

$$\sum_i B_{i,k,t}(x) = 1.$$

3. $B_{i,k,t}(x) > 0$, for $t_i < x < t_{i+k}$.

The following theorem permits the construction of the B-spline basis for any particular piecewise polynomial space $P_{k,\xi,v}$ ($P_{k,\xi,v}$ is a linear subspace of $P_{k,\xi}$ consisting of those elements which satisfy the continuity conditions specified by v) it gives a recipe for an appropriate knot sequence t.

Theorem (2.3.5), (Curry and Schoenberg Theorem):

For a given strictly increasing sequence $\xi = \{\xi_i\}_{i=1}^{h+1}$, and a given non-negative integer sequence $v = \{v_i\}_{i=2}^{h}$ with $v_i \le k$ for all i, set:

$$n = k + \sum_{i=2}^{h}(k - v_i) = kh - \sum_{i=2}^{h} v_i = \dim P_{k,\xi,v}$$

and let $t = \{t_i\}_{i=1}^{n+k}$ be any non-decreasing sequence, so that:

1. $t_1 \le t_2 \le \ldots \le t_k \le \xi_1$ and $\xi_{h+1} \le t_{n+1} \le \ldots \le t_{n+k}$.
2. For i = 2, 3, …, h, the number ξ_i occurs exactly $k - v_i$ times in t.

Then the sequence B_1, B_2, …, B_n of B-splines of order k for the knot sequence t is a basis for $P_{k,\xi,v}$, considered as a functions on $[t_k, t_{n+1}]$. Then:

$$\$_{k,t} = P_{k,\xi,v} \text{ on } [t_k, t_{n+1}]$$

($\$_{k,t} = \text{span}(B_{i,k,t})$, linear space of splines of order k with knot sequence t).

Proof: See [deBoor, 1978]. ■

Because of this theorem, Schoenberg called the functions B_i as the basis splines, or B-splines, [Schoenberg, 1967].

2.3.3 Cubic Spline:

A general cubic spline function is a polynomial of the third degree which involves four constants. There is sufficient flexibility in using the cubic-spline procedure to ensure not only that the interpolant function is continuously differentiable on the interval, but also that it has a continuous second derivative on the interval. The construction of the cubic spline does not however, assume that the derivatives of the interpolation function agree with those of the function, even at the nodes.

Given a data $g(t_0)$, $g(t_1)$, …, $g(t_n)$ with $a = t_0 < t_1 < \ldots < t_n = b$. Then the a piecewise cubic interpolant P to g that satisfies the following conditions:

1. Each P_j is a cubic polynomial on the interval $[t_j, t_{j+1}]$ for each $j = 0, 1, \ldots, n-1$.

2. $P_j(t_j) = g(t_j)$, for each $j = 0, 1, \ldots, n - 2$.

3. $P_{j+1}(t_{j+1}) = P_j(t_{j+1})$, for each $j = 0, 1, \ldots, n - 2$.

4. $P'_{j+1}(t_{j+1}) = P'_j(t_{j+1})$, for each $j = 0, 1, \ldots, n - 2$.

5. $P''_{j+1}(t_{j+1}) = P''_j(t_{j+1})$, for each $j = 0, 1, \ldots, n - 2$.

The j-th piecewise polynomial P_j has the form:

$$P_j(t) = a_j + b_j(t - t_j) + c_j(t - t_j)^2 + d_j(t - t_j)^3$$

where the coefficients a_j, b_j, c_j and d_j are constants to be determined.

Because of the large type of splines, we will not discuss here the outlines of such type of spline functions. Therefore it is appropriate to list some of them and for more details one can see [deBoor, 1978], [Stephen Weston, 2002]:

Auto-tension splines.

Parabolic splines.

Gatmull-Rom or Overhauster splines.

Non-uniform rational basis splines.

Quintic splines.

Uniform-tension splines

X-splines.

CONCLUSION AND RECOMINDATIONS

From the present study, the following conclusions may be drawn:

1- Increasing the HB-problem data will increase the accuracy of the results.

2- The results obtained by using the HB-problem with fractional derivatives increasing the results of the G-spline interpolation function obtained from the usual HB-problem

Also, from this work, we can recommend the following open problems for future work:

1- Introducing the proof of theorem (3.1) without transforming the fractional order derivatives into an integer order derivatives.

2- Studying multi dimensional G-spline interpolation with HB-problem of fractional derivatives.

CONTENTES

Chapter Three: G-spline Interpolation Using Hermite-Birkhoff Problem With Fractional

Dedication

To the spirit of my
father, my lovly mother
and Friends

INTRODUCTION

The fractional calculus may be considered as an old and yet as a novel topic. It is an old topic since it's starting in (1695), were L'Hospital was the first researcher whose ask in a letter to Leibniz on the possibility to perform calculations by means of fractional derivatives of order r = ½. Leibniz answered this question looked as a Paradox [Madueno, 2002].

The earliest more or less systematic studies seem to have been made in the beginning and middle of the 19^{th} century by Liouville (1832), Holmgren (1864), Riemann (1953), although Euler (1730), Lagrange (1772), and other made contributions even earlier. Liouville (1832) who expanded functions in series of exponentials and defined the q-th derivative of such a series by operating term-by-term as though q, where a positive integer, [Oldham, 1974].

Riemann in (1953), proposed a different definition that involve a definite integral and was applicable to power series with no integer exponents. Also, Grünwald in (1867), disturbed by the restriction of Liouville's approach, [Samko, 1993].

Then these theoretical beginnings are developed by several applications of fractional calculus to various problems. The first development was discovered by Able in (1823), that the solution of the integral equation for the tautochrone problem may be accomplished via an integral transform with fractional derivatives. A powerful stimulus to the use of fractional calculus for solving real life problems was provided by the development of Boole in (1844) as symbolic methods for solving linear differential equations with constant coefficients, [Oldham, 1974].

In the twentieth century, some notable contributions have been made to both of the theory and application of fractional calculus, Weyl (1917), Hardy (1917), Hardy and Littewood (1932), Kober (1940), and Kuttner (1953), examined some rather special, but natural, properties of differintegrals of functions belonging to the Lebesgue and Lipschitz classes. Erdely (1954) and Oster (1970) gave definitions of differintegrals with respect to an arbitrary functions, and Post (1930) used difference quotient to define generalized differentiations for fractional operators. Riesz (1949), has developed a theory of fractional integration for functions of more than one variable, Erdely (1965), has applied the fractional calculus to integral equations and Higgins (1967), has used fractional integral operators to solve differential equations, [Igor Poldlubny, 2001].

However, fractional calculus may be considered as a novel topic, as well as, from eighty four years ago, it has been an object of specialized conferences and treatises. For the first conference the merit is ascribed to B. Ross who organized the first conference on fractional calculus, and its application at the University of New Haven in June (1974), [Igor Poldlubny, 2001].

For the first monograph the merit is ascribed to K.B Oldham and J. Spanier (1974), who after a joint collaboration started in 1968, published a book devoted to fractional calculus in 1974. The first texts and proceedings devoted solely or partly to fractional calculus and its applications are, [Igor Poldlubny, 2001]

In addition, spline functions have transformed approximation techniques and theory, because they are not only convenient and suitable for computer calculations, but also they provide optimal theoretical solutions to the estimation of functions from limited data, [Oldham, 1974].

Moreover, splines may be considered as mathematical models that associate a continuous representation of a curve or surface with a discrete set of points in a given space. Spline fitting is an extremely popular form of a piecewise approximation using various forms of polynomials of degree n or more general functions, on an interval in which they are fitting functions at specified points, known as control points or nodes or knots.

The polynomial used can change, but the derivatives of the polynomials are required to match up to degree $n - 1$ at each side of the knot, or to meet related interpolatory conditions. Boundary conditions, are also imposed on the end points of the intervals, [Ahlberg, 1967].

It is near 60 years ago since I. J. Schoenberg introduced the subject of "spline functions" as a method for approximating functions which are so complicated or hard to be used in applications. Since then, splines have proved to be enormously important in various branches of mathematics, such as approximation theory, numerical analysis, numerical treatment of ordinary, integral and partial differential equations and statistics, etc., [Schoenberg, 1946].

Several types of spline functions are given in literatures, including generalized splines [Ahlberg, 1967], basis of cardinal splines [deBoor, 1978], [Powell, 1981], Gatmull-Rom or Overhauster splines, non-uniform rational basis splines [Stephen, 2002], etc. The most important of these types of splines which is necessary to the work of this thesis is the so called G-spline.

In this work a new approach is followed to construct the Heremite-Birkhoff problem and then to evaluate the G-spline interpolation functions by using the idea of fractional order derivatives instead of the integer order derivatives of Heremite-Birkhoff problem.

This thesis consists of three chapters.

Chapter one entitled (Basic Concepts of Fractional Calculus) is oriented to study and give the most important and primitive concepts related to the theory of fractional calculus. This chapter consists of five sections. In section 1.1 presents some of the most important basic concepts related to fractional calculus, including the Gamma function, Beta function and Riemann-Liouville formula of fractional derivatives. Section 1.2 presents some methods of fractional differentiation and integration. In section 1.3 the fractional derivatives of some well selected examples are given for completeness purpose which may be used in the calculation of Heremite-Birkhoff problem. Section 1.4, some properties of fractional order operators is given. Finally, in section 1.5 additional examples with their fractional derivatives are given using Riemann-Liouville Formula.

Chapter two entitled (Approximation by Spline Functions) presents the fundamental aspects of spline interpolation functions. This chapter consists of three sections; in section 2.1 a classical discussion of polynomial interpolation using Lagrange interpolation polynomial is given. In section 2.2, we discuss the theory of spline functions in general and some of its related concepts including the trancated power function (or Heaviside function) and some other concepts. In section 2.3, additional types of spline functions are discussed in short, such as generalized spline, B-spline and cubic spline functions.

In chapter three entitled (G-Spline Interpolation Using Hermite-Birkhoff Problem with Fractional Derivatives), which consist of three sections. In section 3.1, we present the Heremite-Birkhoff problem and some of its general properties including the m-poised problem. Section 3.2 presents the method of constriction of G-spline functions using Heremite-Birkhoff problem of fractional order derivatives. Finally, as an illustrative to the proposed approach, we give some illustrative examples in section 3.3, in which one is solved in details.

The results are sketched in figures for different cases of Heremite-Birkhoff problem, where the results are calculated using the computer software Mathcad Professional 2001i.

LIST OF SYMBOLS

$\Gamma(x)$	The Gamma function of x
$B(m,n)$	The Beta function of m and n.
D_x^{α}	The fractional derivative of order α for x.
$C[a,b]$	The set of all continuous functions on [a,b].
δ_{ki}	Cronecer delta.
$\ell_k(x)$	The Lagrange basis function of x.
$S(x)$	The spline function of x.
X_+^m	Truncated power function of m.
B-Spline	Basis spline.
$B_{i,k,t}(x)$	The i-th normalized B-spline of order k for knot sequence t.
$P_j(t)$	The j-th piecewise polynomial interpolation function.
E	Incidence matrix.
$C^n[a,b]$	Set of all continuously n-differentiable of [a,b].
Π_n	Set of all polynomial of degree less than or equal to n.
L^*	The adjoint operator of L.
HB	The Hermmite-Birkhoff problem.

REFERENCES

1. Ahlberge, J. H., Nilson, E. N. and Walsh, J. L., “The theory of splines and their applications”, Academic press, New York, 1967.
2. Bertram Ross, "Fractional Calculus and Its Applications", Proceedings of the International Conference Held AT THE University of New Haven, June, 1974.
3. Burden, R. L. and Douglas Faires, J., “Numerical Analysis”, PWS Publishers, 1985.
4. Caputo M., “Linear Models for Dissipation in an Elastic Solids”, Rivista del., Nuovo Girnento, 1, 161-198, 1971.
5. deBoor, C., “A practical guide to splines”, Springer-Verlag, New York, Inc., 1978.
6. Igor Podlubng, "Geometric and Physical Interpretation of Fractional Integration and Fractional Differentiation", arxivimath. CA/01102411, VI20, Oct., 2001.
7. Greville, T. N., “Numerical procedure for interpolation by spline functions”, J. SIAM, Numer. Anal., Ser.B, 1, 53-68, (1964).
8. Greville, T. N., “Spline functions, interpolation and numerical quadrature”, A chapter in: Mathematical methods for digital computers, Vol.2, (Ralston A., and Wilf, H. S., eds.), Wiley, New York, 1967.

9. Lixia Yuan and Omp Agrawal, "A Numerical Scheme for Dynamic Systems Containing Fractional Derivatives", Proceedings of DETC'98, ASME Design Engineering, Technical Conference, September, 13-16, 1998, Atlanta, Georgia.

10. Loverro, A., "Fractional Calculus: History, Definition and Applications for the Engineer", department of Aerospace and Mechanical Engineering, University of Notre Dame, Notre Dame, IN 46556, USA, May 8, 2004.

11. Madueno, A. L., Rosu, I. C., and Socorro, J., "Modified Riccati Equation from Fractioal Calculus", arxivi math. Ph/0112020, V.3, 2, Jan, 2002.

12. Oldham, K. B., and Spanir, J., "The Fractional Calculus", Academic Press, New York and London, 1974.

13. Oldham K. B., "The Fractional Calculus", Academic Press, New York., 1998.

14. Powell, M. J., “Approximation theory and methods”, Cambridge University Press, 1981.

15. Samko, S., "Integrals and Derivatives of Fractional Order and Some of Their Applications", Gordon and Breach, London, 1993.

16. Schoenberg, I. J., “Contribution to the problem of approximation of equi-distant data by analytic functions”, Parts A and B, Quart Appl. Math., 4, (1946), pp.45-99.

17. Schoenberg, I. J. and Whitney, A., “On polya frequency functions III, the positivity of translation determinats with an application to

the interpolation problem by spline curves”, Trans. Amer. Math. Soc., 74 (1963), pp.246-259.

18. Schoenberg, I. J., “On the Ahlberg-Nilson extension of spline interpolation: the G-spline and their optimal properties, J. Math. Anal. Appl., 21 (1968), pp.207-231.

19. Stephen, W., “An introduction to the mathematics and construction of splines”, Addix software consultancy limited, Version 1.6, September, 2002.

Supervisors Certification

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Examining Committe Certification

We certify that we have read this thesis entitled "***HB-Problem with Fractional and It's Application to G-Spline Function***" and as examining committee examined the student "***Husam Oday Abdulrasool Al Saffar***" in its contents and in what connected with, and that is in our opinion it meets the standards of a thesis for the degree of Master of Science in Mathematics.

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HB-PROBLEM WITH FRACTIONAL AND IT'S APPLICATION TO G-SPLINE FUNCTION

A Thesis

Submitted to the College of Science, AL-Nahrain University
as a Partial Fulfillment of the Requirements for the Degree of
Master of Science in Mathematics

By

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جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم
قسم الرياضيات وتطبيقات الحاسب

مسألة هيرميت بيركهوف ذات الرتب الكسرية وتطبيقاتها لدوال السبلاين-G

رسالة
مقدمة إلى كلية العلوم - جامعة النهرين
كجزء من متطلبات نيل درجة ماجستير علوم
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من قبل

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جمادي الأولى
١٤٢٩

آيار
٢٠٠٨

بَســـَمِ اللهِ الـــــرحَمنِ الــــرَحِيمِ

وَ عَلمَ ءَادَمِ الاسمآءَ كُلهَا ثُمَ عرضَهُم عَلى

المَلائِكَة فقَال أنبئُونِي بِأسمَاءَ هَؤُلاءِ ان كُنتُم

صَادِقِينَ ۞ قَالُوا سبحانك لاعِلمَ لنَا الا مَا

عَلمتَنَا انكَ انتَ العَلِيمُ الحَكِيمُ ۞

صَدَقَ اللهُ العَظيمَ
سورة البقرة ٣٠-٣١

المستخلص

الهـدف الرئيسـي مـن هـذه الرسـالة ، هـو أولاً لدراسـة التفاضـل الكسـري (Fractional Calculus) وطـرق حسـاب المشـتقات ذات الرتـب ألكسـرية لـبعض الـدوال وثانيـاً لدراسـة دوال أنـدراج السـبلاين-G وطريقـة حسـاب هـذه الـدوال بأسـتخدام أسـلوب جديـد وذلـك لتكـوين مشكلـة هيرميـت بيركهـوف (Heremite-Birkhoff problem) وذلك بأستخدام مشتقات ذات رتب كسـرية بدلاً من مشتقات ذات رتب صحيحة.

معلــومــات عامـــــه

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المفاتيح الاستدلالية :- التحليلات العددية، طرق النظرية التقريبية، الجبر الخطي

المستخلص

الهــدف الرئيســي مــن هــذه الرســالة ، هــو أولاً لدراســة التفاضــل الكســري (Fractional Calculus) وطـرق حسـاب المشـتقات ذات الرتـب ألكسـرية لـبعض الــدوال وثانيــاً لدراســة دوال أنـدراج السـبلاين-G وطريقـة حسـاب هـذه الـدوال بأســـتخدام أســـلوب جديـــد وذلـــك لتكـــوين مشكلـــة هيرميـــت بيركهـــوف (Heremite-Birkhoff problem) وذلك بأستخدام مشتقات ذات رتب كســــرية بـدلاً من مشتقات ذات رتب صحيحة.

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Key words:- Numerical Analysis , Approximation Theory, Linear Algebra.

Abstract

The objective of this thesis is to study first the theory of fractional calculus and some of well known methods for evaluating derivatives of fractional orders for certain functions.

The second objective is to study the G-spline interpolation functions and its construction using a new approach in formulating the Heremite-Birkhoff problem using fractional derivatives instead of integer order derivatives.