ABSTRACT

The main objective of this work is to study the numerical solution of fractional ordinary differential equations using G-spline interpolation functions. Two numerical approaches are used, the first approach utilize the explicit linear multistep methods which can be applied easily for linear and nonlinear problems while the second approach is a modified approach by using the implicit linear multistep methods for solving nonlinear fractional ordinary differential equations which has so many difficulties in their solution. This is done by suggesting a new criterion by using the chain rule derivatives of fractional order.

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Muhammed Saleh Mehdi

April, 2008 ✍

SUPERVISOR CERTIFICATION

I certify that this thesis was prepared under my supervision at Al-Nahrain University, College of Science, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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We certify that we have read this thesis entitled *"Solutions of Fractional Differential Equations Using G-Spline Interpolation Functions"* and as examining committee examined the student *(Muhammed Saleh Mehdi)* in its contents and in what it connected with, and that is in our opinion it meets the standards of a thesis for the degree of Master of Science in Mathematics.

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CHAPTER ONE

FUNDAMENTAL CONCEPTS

There are several reasons for studying the subject of approximation theory and their methods, ranging from a need to represent functions in computer calculations to an interest in mathematical view of the subject and among such applications of the subject is to solve ordinary differential equations using certain types of spline functions.

Also, differential equations in general, and fractional differential equations in particular plays an important role in mathematical physics, therefore their solution is of great importance which may be so difficult in some cases, [Brauer, 1973]. Hence approximate methods by G-spline functions may be used to solve such types of equations.

This chapter presents the most fundamental concepts and notions in G-spline functions and fractional calculus theory including fractional ordinary differential equations.

1.1 G-SPLINE INTERPOLATION

Just over 138 years ago, since Lagrange in 1870 has constructed the polynomial of minimal degree such that the polynomial assumed has prescribed values at a given knots and the derivatives of certain orders of the polynomial also assumed to have prescribed values at the knots, [Burdern, 1997], [DeBoor, 1978].

In 1968, Schoenberg extended the idea of Hermite for splines to specify that the orders of the derivatives specified may vary from knot to knot. Schoenberg used the term "G-spline" instead of generalized splines, because the natural spline term "generalized spline" already described an extension in a different direction, [Schoenberg, 1968].

G-splines are used to interpolate the Heremite-Birkhoff data (problem), which is abbreviated by HB-problem, the data in this problem are the values of the function and its derivatives but without Hermite's condition that only consecutive derivatives be used at each knot, [Ahlberg, 1967].

Again, Schoenberg has defined G-splines as smooth piecewise polynomials, where the smoothness is governed by the incidence matrix E, and then he proved that the G-spline functions, satisfies what is called the "minimum norm property", [Powell, 1981], which is used for the optimality or best approximation of the G-spline functions, which is given mathematically by the following inequality:

$$\int_I \left[f^{(m)}(x)\right]^2 dx > \int_I \left[S^{(m)}(x)\right]^2 dx$$

where the function S is called a G-spline function and it is a polynomial spline of degree $2m - 1$ over the interval I and f is any interpolation function to the problem under consideration. If the only polynomial that solve the homogeneous HB-interpolation problem is identically the zero polynomial, then the problem is said to be m-poised, we will see later the consideration of the HB problem that the m-poised problem will play an important role for the uniqueness of the solution of the HB-problem (that is if the HB-problem is m-poised, then there is a unique G-spline of degree $2m - 1$ that solves the HB-problem).

1.1.1 The HB-Problem:

As it is mentioned above in the introduction of section 1.1, that the G-spline functions are calculated using the HB-problem. Therefore it is convenient to discuss the HB-problem. First of all, we give the tractable formal definition of the natural G-spline interpolation, with knots:

$$x_1 < x_2 < \ldots < x_k$$

to be distinct and reals and let α be the maximum of the orders of the derivatives to be specified at the knots.

Definition (1.1), [Schoenberg, 1968]:

The incidence matrix E is defined by:

$$a_{ij} = \begin{cases} 1, & \text{if} \quad (i, j) \in e \\ 0, & \text{if} \quad (i, j) \notin e \end{cases}$$

for all $i = 1, 2, \ldots, k$; $j = 0, 1, \ldots, \alpha$, where $e = \{(i, j)\}$ is chosen in such a way that i takes the values 1, 2, …, k; one or more times, while $j \in \{0, 1, \ldots, \alpha\}$ and $j = \alpha$ is attained in at least one element (i, j) of e.

Assume also that each row of the incidence matrix E and the last column of E should contain some element equals to 1. Let $y_i^{(j)}$ be prescribed real numbers for each $(i, j) \in e$. The HB-problem is to find $f(x) \in C^{\alpha}$, which satisfies the interpolatory condition:

$$f^{(j)}(x_i) = y_i^{(j)}, \text{ for } (i, j) \in e \quad \text{...(1.1)}$$

The matrix E will likewise describes the set of eqs. (1.1) if the set e defined by:

$$e = \{(i, j) \mid a_{ij} = 1\}$$

then the integer:

$$n = \sum_{i,j} a_{ij}$$

really is the number of interpolatory conditions required to constitute the linear system following from eqs.(1.1).

Therefore, at each knot x_i of the resulting linear system from eqs.(1.1), the value of $f(x_i)$ is prescribed and may be also a certain number of consecutive derivatives $f^{(j)}(x_i)$ for $j = 1, 2, \ldots, \alpha_i - 1$; where α_i denotes the number of the required derivatives at x_i, for each i, [Shoenberg, 1968].

<u>Definition (1.2), [Schoenberg, 1968]:</u>

Let m be a natural number, then the HB-problem (1.1) is said to be m-poised provided that:

$$p(x) \in \Pi_{m-1}$$

$$p^{(j)}(x_i) = 0, \text{ if } (i, j) \in e$$

then:

$$p(x) = 0$$

where Π_{m-1} is the set of all polynomials of degree at most $m \in$ □ .

<u>Lemma (1.1), [Schoenberg, 1968]:</u>

If the HB-problem (1.1) is m-poised and $1 \le m' < m$, then the HB-problem (1.1) is also m'-poised.

The next remark is of great importance in determining the poised number of HB-problem.

<u>Remarks (1.1), [Schoenberg, 1968]:</u>

1. If the system (1.1) is non normal then the system (1.1) may be m-poised for some value of $m < n$.
2. The condition that (1.1) is m-poised can be expressed as follows:

   If:

   $$p(x) = \sum_{v=0}^{m-1} a_v \frac{x^v}{v!}$$

   then $p^{(j)}(x_i) = 0$, for $(i, j) \in e$, becomes:

   $$\sum_{v=0}^{m-1} a_v \frac{x^{v-j}}{(v-j)!} = 0, \text{ for } (i, j) \in e$$

   where:

   $$\frac{x^{v-j}}{(v-j)!} = \begin{cases} 1, & \text{if } \quad v - j = 0 \\ 0, & \text{if } \quad v - j < 0 \end{cases}$$

Therefore, (1.1) is m-poised if and only if the matrix with entries $\dfrac{x^{v-j}}{(v-j)!}$ has rank m, where v = 0, 1, …, m – 1; refers to the column of the matrix of entries $\dfrac{x_i^{v-j}}{(v-j)!}$, while each $(i, j) \in e$ indicates a row of the same matrix.

1.1.2 Interpolation by G-Spline:

Here, we shall assume that the HB-problem given by eqs.(1.1) is m-poised and $\alpha < m < n$, where α, as it is mentioned previously in section (1.1.1), is the highest derivative that appears in the interpolation problem.

The definition of G-spline function is facilitated by defining a matrix E* which is obtained from the incidence matrix E by adding $m - \alpha - 1$ columns of zeros to the matrix E, i.e., let $E^* = [a^*_{ij}]$, where (i = 1, 2, …, k; j = 0, 1, …, m – 1), and:

$$a^*_{ij} = \begin{cases} a_{ij}, & \text{if} \quad j \le \alpha \\ 0, & \text{if} \quad j = \alpha+1, \alpha+2, \ldots, m-1 \end{cases}$$

If $j = \alpha + 1$, then $E^* = E$.

<u>Definition (1.3), [Ahlberg, 1967]:</u>

A function S(x) is called natural G-spline for the knots $x_1, x_2, \ldots, x_k$ and the matrix E* of order m provided that it satisfies the following conditions:

(1) $S(x) \in \Pi_{2m-1}$ in (x_i, x_{i+1}), $i = 1, 2, \ldots, k - 1$.

(2) $S(x) \in \Pi_{m-1}$ in $(-\infty, x_1)$ and in (x_k, ∞).

(3) $S(x) \in C^{m-1}(-\infty, \infty)$.

(4) If $a^*_{ij}=0$, then $S^{(2m-j-1)}(x)$ is continuous at $x=x_i$; that is, $S^{(2m-j-1)}(x_i-0) = S^{(2m-j-1)}(x_i+0)$, where $x_i + 0$ and $x_i - 0$ refers to the right and left hand limits of the function $S^{(2m-j-1)}$ at the knot x_i.

Next, we shall show that the set of natural spline functions is a special case of the set of G-spline functions. The set of all natural G-splines interpolation polynomials of a given function with knots x_1, x_2, …, x_k is denoted by:

$$\mathcal{S}_m = \mathcal{S}(E^*; x_1, x_2, \ldots, x_k)$$

and it is easily seen that $\mathcal{S}_m$ is a non empty set and this is shown by the inclusion relation:

$$\Pi_{m-1} \subset \mathcal{S}_m$$

Indeed, if $S(x) \in \Pi_{m-1}$, then $S(x)$ satisfies all conditions from (1) to (4) of definition (1.2) above. A special case when (1.1) is given by:

$$f(x_i) = y_i,\ f'(x_i) = y'_i,\ \ldots,\ f^{(\alpha_i-1)}(x_i) = y_i^{(\alpha_i-1)},\ i = 1, 2, \ldots, k$$

then the HB-problem is reduced to the usual Hermite problem in approximation theory.

It is clearly that $\alpha = \max\{\alpha_i - 1\}$, $i=1,2,\ldots,k$, and $\max\limits_i \alpha_i \le m \le n$; $S^{(2m-j-1)}(x)$ is continuous at $x = x_i$, for $j = \alpha_i, \ldots, m - 1$. In other words,

$S^{(v)}(x)$ is continuous at $x = x_i$, for $v = m, m + 1, .,.., 2m - \alpha_i - 1$ together with condition (3), of definition (1.3) we conclude that:

$$S(x) \in C^{2m-\alpha i-1}, \text{ near } x = x_i,\ i = 1, 2, \ldots, k \quad \text{...............................(1.2)}$$

Conditions (1), (2) of definition (1.3) and eq. (1.2) shows that $\mathcal{S}_m$ is identical with the natural spline function of degree $2m - 1$ having x_i $(i = 1, 2, \ldots, k)$ a multiple knot of multiplicity α_i, where $\alpha_i \leq m$, [Ahlberg, 1967].

Another special case, the Lagrange problem which occurs by assuming that $n = k + 1$ and $e = \{(i, 0), i = 0, 1, \ldots, k\}$. In this case, $m = k$ and we can show that $\mathcal{S}_m$ is identical with the class of natural spline functions of degree $2m - 1$ having knots $x_0, x_1, \ldots, x_k$, [Burden, 1997].

The uniqueness of the G-spline function is ensured by using the following theorem:

Theorem (1.1), [Schoenberg, 1968]:

If the HB-problem (1.1) is m-poised, then there exists a unique G-spline function:

$$S(x) \in \mathcal{S}_m (E^*; x_1, x_2, \ldots, x_k)$$

such that:

$$S^{(j)}(x_i) = y_i^{(j)}, \text{ for } (i, j) \in e$$

We can to summarize the above results as follows:

Under the assumption of theorem (1.1), define a G-spline fundamental function $L_{ij}(x) \in \mathcal{S}_m$ satisfying:

$$L_{ij}^{(s)}(x_r) = \begin{cases} 0, & \text{if} \quad (r,s) \neq (i,j) \\ 1, & \text{if} \quad (r,s) = (i,j) \end{cases}$$

If for $f(x) \in C^{\alpha}$, then one can write:

$$f(x) = \sum_{(i,j)\in e} f^{(j)}(x_i)L_{ij}(x) + Rf \quad \text{......(1.3)}$$

for which the right hand sum presents the G-spline interpolating f(x) at the data of the HB-problem (1.1), and Rf is the remainder or the error function occurred in the approximation.

Equation (1.3) is called the ***G-spline interpolation formula***, which is exact for all elements of $\mathcal{S}_m$ and in particular for the elements of Π_{m-1}.

The following theorem shows the optimal property of G-spline interpolation function is satisfied and it may be called the ***minimum norm property***. This theorem is given by [Schoenberg, 1968] and further illustrated by [Mohammed, 2006].

<u>Theorem (1.2):</u>

Let $I = [x_0, x_{k+1}]$ such that $x_0 < x_1 < \dots < x_{k+1}$ and let $f(x) \in C^m(I)$, with $f^{(m-1)}(x)$ is absolutely continuous and $f^{(m)}(x) \in L^2(I)$. If the HB-problem (1.1) is m-poised, and $\alpha < m \le n$, and let S(x) be the unique G-spline function satisfying the equations:

$$S^{(j)}(x_i) = f^{(j)}(x_i), \ (i, j) \in e$$

Then:

$$\int_I \left(f^{(m)}(x)\right)^2 dx > \int_I \left(S^{(m)}(x)\right)^2 dx$$

Proof:

Since $f(x) \in C^m(I)$, with $f^{(m-1)}(x)$ is absolutely continuous and $f^{(m)} \in L^2(I)$, then:

$$\int_I \left(f^{(m)}(x) - S^{(m)}(x)\right)^2 dx = \int_I \left(f^{(m)}(x)\right)^2 dx -$$

$$2\int_I f^{(m)}(x) S^{(m)}(x) dx + \int_I \left(S^{(m)}(x)\right)^2 dx$$

$$= \int_I \left(f^{(m)}(x)\right)^2 dx - 2\int_I \left(f^{(m)}(x) - S^{(m)}(x)\right) S^{(m)}(x) dx -$$

$$\int_I \left(S^{(m)}(x)\right)^2 dx \quad \text{...} (1.4)$$

To prove that:

$$J = \int_I \left(f^{(m)}(x) - S^{(m)}(x)\right) S^{(m)}(x) dx = 0 \quad \text{..................................} (1.5)$$

First, we write:

$$J = \int_{x_0}^{x_1} \left(f^{(m)}(x) - S^{(m)}(x)\right) S^{(m)}(x) dx +$$

$$\int_{x_1}^{x_2} \left(f^{(m)}(x) - S^{(m)}(x)\right) S^{(m)}(x) dx + \ldots +$$

$$\int_{x_k}^{x_{k+1}} \left(f^{(m)}(x) - S^{(m)}(x)\right) S^{(m)}(x) dx$$

and integrating by parts repeatedly each of those integrals according to the following scheme for each i = 0, 1, …, k:

$$\int_{x_i}^{x_{i+1}} S^{(m)}(x)\left(f^{(m)}(x)-S^{(m)}(x)\right)dx = S^{(m)}(x)\left(f^{(m-1)}(x)-S^{(m-1)}(x)\right)\Big|_{x_i}^{x_{i+1}}$$

$$-S^{(m+1)}(x)\left(f^{(m-2)}(x)-S^{(m-2)}(x)\right)\Big|_{x_i}^{x_{i+1}} + \ldots \pm$$

$$S^{(2m-1)}(x)\left(f(x)-S(x)\right)\Big|_{x_i}^{x_{i+1}} \pm \int_{x_i}^{x_{i+1}} S^{(2m)}(x)(f(x)-S(x))\,dx$$

The last integrals on the right of the last formula are vanishes since $S(x) \in \Pi_{2m-1}$ in each interval, by condition (1) and (2) in definition (1.2) from the "finite parts", we obtain at each x_i, i = 1, 2, …, k, a sum of terms:

$$\sum_{j=0}^{m-1} \mp \Delta_i^{(j)}$$

where:

$$\Delta_i^{(j)} = \text{jump at } x = x_i \text{ of } S^{(2m-j-1)}(x)(f^{(j)}(x) - S^{(j)}(x)),\ j = 0, 1, \ldots, m-1$$

Since $S(x) \in C^{m-1}(x)$, by condition (3) of definition (1.2), we obtain that:

$$\Delta_i^{(j)} = S^{(2m-j-1)}(x_i + 0)(f^{(j)}(x_i + 0) - S^{(j)}(x_i + 0)) -$$

$$S^{(2m-j-1)}(x_i - 0)(f^{(j)}(x_i - 0) - S^{(j)}(x_i - 0))$$

$$= (f^{(j)}(x_i) - S^{(j)}(x_i))\ \{S^{(2m-j-1)}(x_i + 0) - S^{(2m-j-1)}(x_i - 0)\}$$

Then $\Delta_i^{(j)} = 0$, because:

1. $f^{(j)}(x_i) = S^{(j)}(x_i)$, $(i, j) \in e$.
2. From definition (1.2), $S^{(2m-j-1)}(x)$ is continuous at $x = x_i$.

Hence, eq. (1.5) is established and therefore (1.4) becomes:

$$\int_I \left(f^{(m)}(x) - S^{(m)}(x)\right)^2 dx = \int_I \left(f^{(m)}(x)\right)^2 dx - \int_I \left(S^{(m)}(x)\right)^2 dx$$

Then:

$$\int_I \left(f^{(m)}(x)\right)^2 dx = \int_I \left(S^{(m)}(x)\right)^2 dx + \int_I \left(f^{(m)}(x) - S^{(m)}(x)\right)^2 dx$$

and since $\int_I \left(f^{(m)}(x) - S^{(m)}(x)\right)^2 dx > 0$. Therefore:

$$\int_I \left(f^{(m)}(x)\right)^2 dx > \int_I \left(S^{(m)}(x)\right)^2 dx. \quad \blacksquare$$

1.1.3 The Construction of G-Spline Functions, [Osama, 2007]:

The most difficulty in approximation by G-spline functions is the constructing of the G-spline itself, because most literatures give the results directly without details, therefore the method of construction will be illustrated in details.

Next, the construction of the G-spline interpolation formula in more efficient approach leading to a system of only m + n equations is given as follows:

From conditions (1), (2) and (3) of definition (1.3) it is clear that the most suitable form of the G-spline function S(x) must take the following form:

$$S(x) = P_{m-1}(x) + \sum_{i=1}^{k}\sum_{j=0}^{m-1} c_{ij}\frac{(x-x_i)_+^{2m-j-1}}{(2m-j-1)!} \quad \text{.................................(1.6)}$$

where $(x - .)_+$ is the truncated power basis defined by:

$$(x-x_i)_+^{2m-j-1} = \begin{cases} 0, & \text{if } x < x_i \\ (x-x_i)^{2m-j-1}, & \text{if } x \geq x_i \end{cases}$$

and $P_{m-1}(x) \in \Pi_{m-1}$, while c_{ij}'s are constants to be determined. Any function of the form given by eq.(1.6) satisfies the conditions (1), (2) and (3) of definition (1.3) except that:

$$S(x) \in \Pi_{m-1} \text{ if } x_k < x \text{..(1.7)}$$

and according to the definition of the truncated power basis, from eq.(1.6), we can see that $S^{(2m-j-1)}(x)$ is continuous at $x = x_i$ if and only if $c_{ij} = 0$, while condition (4) of definition (1.2) requires that $S^{(2m-j-1)}(x)$ is continuous if and only if $a_{ij}^{*} = 0$. Leaving out all such terms, one can obtain:

$$S(x) = p_{m-1}(x) + \sum_{(i,j)\in e} c_{ij}\frac{(x-x_i)_+^{2m-j-1}}{(2m-j-1)!} \quad \text{.................................(1.8)}$$

In order to satisfy eq.(1.8), expand all binomial terms and equating to zero those coefficients of x^m, x^{m+1}, …, x^{2m-1}, then the following equations are obtained:

$$\sum_{\substack{(i,j)\in e \\ j\leq v}} \frac{c_{ij}}{(2m-j-1)!}\binom{2m-j-1}{2m-v-1}(-x_i)^{v-j} = 0, \ v=0,1,\ldots, m-1 \text{...(1.9)}$$

and also have the following additional equations:

$$S^{(j)}(x_i) = y^{(j)}(x_i),\ (i, j) \in e \qquad (1.10)$$

Therefore, we get n + m equations from (1.9) and (1.10) in n + m unknowns and writing the solution of the unique G-spline so as to exhibit the $f^{(j)}(x_i)$, to get:

$$S(x) = \sum_{(i,j)\in e} y_i^{(j)} L_{ij}(x)$$

which is the final form of the G-spline approximation function. It is clear that the final form of the G-spline function depends on the fundamental G-spline functions $L_{ij}(x)$, $(i, j) \in e$.

Example (1.1), [Mohammed, 2006]:

As an illustration to the discussion given in section (1.1.3), consider the following HB-problem of the general form:

Given that:

$$f(-1) = y_1,\ f'(0) = y'_2,\ f(1) = y_3 \qquad (1.11)$$

and to find the G-spline function which interpolate (1.11). In this problem we have $\alpha = 1$, $n = 3$ and it is clear that it is a two-poised problem as given by remark (1.1)(1).

The incidence matrix is given by:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the HB-set e will take the form:

$$e = \{(1, 0), (2, 1), (3, 0)\}.$$

Therefore, the G-spline interpolation function has the form:

$$S(x) = a_0 + a_1x + \frac{1}{6}c_{10}(x+1)_+^3 + \frac{1}{2}c_{21}x_+^2 + \frac{1}{6}c_{30}(x-1)_+^3 \quad .....(1.12)$$

Now, to find the fundamental G-spline functions $L_{10}(x)$, $L_{21}(x)$ and $L_{30}(x)$, one must solve the following linear system of algebraic equations obtained from eq.(1.9) and eq.(1.10):

$$\frac{1}{6}c_{10} + \frac{1}{6}c_{30} = 0$$

$$\frac{1}{2}c_{10} + \frac{1}{2}c_{21} - \frac{1}{2}c_{30} = 0$$

$$a_0 - a_1 = y_1$$

$$a_1 + \frac{1}{2}c_{10} = y'_2$$

$$a_0 + a_1 + \frac{8}{6}c_{10} + \frac{1}{2}c_{21} = y_3$$

Hence, we get:

$$c_{10} = \frac{3}{2}y_1 + 3y'_2 - y_3$$

$$c_{21} = -3y_1 - 6y'_2 + 3y_3$$

$$c_{30} = -\frac{3}{2}y_1 - 3y'_2 + \frac{3}{2}y_3$$

$$a_0 = \frac{1}{4}y_1 - \frac{1}{2}y'_2 + \frac{3}{4}y_3$$

$$a_1 = -\frac{3}{4}y_1 - \frac{1}{2}y'_2 + \frac{3}{4}y_3$$

Therefore, upon substituting c_{10}, c_{21}, c_{30}, a_0 and a_1 back into eq.(1.12), and writing S(x) in terms of y_1, y'_2 and y_3, gives:

$$S(x) = y_1 L_{10}(x) + y'_2 L_{21}(x) + y_3 L_{30}(x)$$

where:

$$L_{10}(x) = \frac{1}{4}(1 - 3x) + \frac{1}{4}(x+1)_+^3 - \frac{3}{2}x_+^2 - \frac{1}{4}(x-1)_+^3$$

$$L_{21}(x) = -\frac{1}{2}(1 + x) + \frac{1}{4}(x+1)_+^3 - 3x_+^2 - \frac{1}{2}(x-1)_+^3$$

$$L_{30}(x) = \frac{3}{4}(1 + x) - \frac{1}{4}(x+1)_+^3 + \frac{3}{2}x_+^2 + \frac{1}{4}(x-1)_+^3$$

The approximate G-spline function for the function $f(x) = x^3$ with knots $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$; is illustrated in figure (1.1).

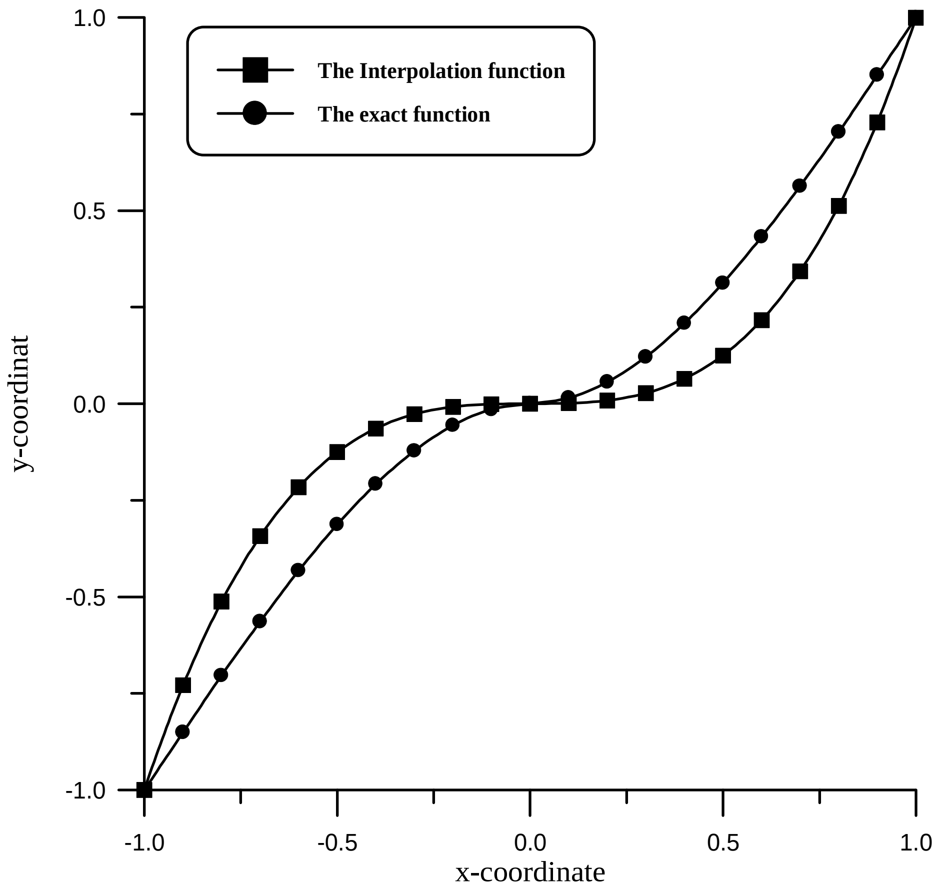

Figure (1.1) Approximation by G-spline function for $f(x) = x^3$.

It is important to notice that, more accurate results may be obtained if the functions derivatives are given and required at the end points or at other knot points.

1.2 FRACTIONAL DIFFERENTIAL EQUATIONS

The subject of fractional calculus has a long history whose infancy dates back to the beginning of classical calculus and it is an area having interesting applications in real life problems.

This type of calculus has its origin in the generalization of the differential and integral calculus, [Nishimoto, 1997].

1.2.1 Basic Concepts:

Here, some fundamental and necessary concepts related to the subject of fractional calculus are given for completeness purpose in order to avoid vague notions in this subject.

1- Gamma and Beta Functions, [Oldham, 1974]:

Undoubtedly, one of the basic functions encountered in fractional calculus is the Euler's gamma function $\Gamma(x)$, which generalizes the ordinary definition of factorial of a positive integer number n and allows n to take also any non- integer positive or negative and even complex values.

The gamma function $\Gamma(x)$ is defined using the following improper integral:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \;\; x > 0 \text{ ....(1.13)}$$

First of all, it is easy to show that the gamma function for a natural number can be proved also to satisfy:

1. $\Gamma(1) = 1$.

2. $\Gamma(x + 1) = x\Gamma(x)$.

3. $\Gamma(x + 1) = x!$.

4. $\Gamma\left(\frac{1}{2} - n\right) = \frac{(-4)^n n! \sqrt{\pi}}{(2n)!}$.

5. $\Gamma\left(\frac{1}{2} + n\right) = \frac{(2n)! \sqrt{\pi}}{4^n n!}$.

6. $\Gamma(-x) = \frac{-\pi \csc(\pi x)}{\Gamma(x+1)}$.

7. $\Gamma(nx) = \sqrt{\frac{2\pi}{n}} \left[\frac{n^x}{\sqrt{2\pi}} \right] \prod_{k=0}^{n-1} \left(n + \frac{k}{n} \right), \; \forall \; n \in \mathbb{N}$.

which enable us to calculate for any positive real x the gamma function in terms of the fractional part of x.

Also, an important function in fractional calculus is the beta function defined by:

$$B(p,q) = \int_0^1 y^{p-1} (1-y)^{q-1} dy, \;\; p, q>0 \text{.....................................(1.14)}$$

2- Riemann-Liouville Formula of Fractional Derivatives, [Oldham, 1974]:

Riemann and Liouville in (1832) introduced a differential operator of fractional order q > 0 of the from:

$$D^q{}_t y(t) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_{t_\circ}^{t} \frac{y(u)}{(t-u)^{q-m+1}} du \quad \text{............................(1.15)}$$

where m is the integer defined by $m - 1 < q \leq m$.

The case of $0 < q < 1$ seems to be particularly important, but there are also some applications for $q > 1$.

1.2.2 Fractional Calculus:

Fractional differentiation and integration may be defined using several approaches depending on the used definition of differentiations. Therefore, presented next some of such types of differentiation are:

1- Fractional Derivative:

The usual formulation of the fractional derivative, given in standard references such as [Samko, 1993], [Oldham, 1974] is the Riemann-Liouville differential equations which require initial values expressed as fractional derivatives.

The Grünwald definition of fractional derivatives is:

$$\frac{d^q f(t)}{dt^q} = \lim_{N\to\infty} \frac{\left(\frac{t}{N}\right)^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(t - j\left(\frac{t}{N}\right)\right) \quad \text{...................(1.16)}$$

where $q < 0$ indicates fractional integration and $q > 0$ indicates fractional differentiation.

The Reiman-Liouvilli definition of fractional derivative is given by:

$$D^{q}_{x_0} y(x) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_{x_0}^{x} \frac{y(u)}{(x-u)^{q-m+1}} du \qquad (1.17)$$

where $m - 1 < q \leq m$.

Other types of fractional derivatives may be found in [Oldham, 1974], [Al-Azawi, 2004].

2- Fractional Integration:

The common formulation for the fractional integral can derive directly from a traditional expression of the repeated integration of a function. This approach is commonly referred to as Riemann – Liouville approach.

The Riemann-Liouville definition of fractional integral is given by:

$$f_q^{+}(a,x) = \frac{1}{\Gamma(q)} \int_{a}^{x} (x-t)^{q-1} f(t) dt \quad \text{(right hand integration)} \qquad (1.18)$$

$$f_q^{-}(x,b) = \frac{1}{\Gamma(q)} \int_{x}^{b} (t-x)^{q-1} f(t) dt \quad \text{(left hand integration)} \qquad (1.19)$$

where $q < 0$. Equations (1.18) and (1.19) may be combined together to give the following formula:

$$I^{q} y(x) = \frac{1}{\Gamma(q)} \int_{x_0}^{x} \frac{y(u)}{(x-u)^{1-q}} du \qquad (1.20)$$

1.2.3 Fractional Derivative of Certain Functions, [Oldham, 1974]:

Here, some fractional derivatives using Grüuwald definition will be evaluated as an illustrative examples of fractional differentiations. Other functions derivative may be derived, such as the fractional derivative of sinh($\sqrt{x}$), sin($\sqrt{x}$), etc.

1- The Unit Function, [Oldham, 1974]:

Consider first the differintegral to order q of the function f = 1, for which it is convenient to reserve the special notation, this function will be referred to as the unit function.

From Riemann-Liouville formula with m = 0, $x_0 = 0$

$$\frac{d^q(1)}{dx^q} = \frac{1}{\Gamma(-q)}\int_0^x \frac{1}{(x-u)^{q+1}}du$$

$$= \frac{1}{\Gamma(-q)}\int_0^x (x-u)^{-q-1}du$$

$$= \frac{1}{\Gamma(-q)}\left[-\frac{(x-u)^{-q}}{-q}\right]_0^x$$

$$= \frac{1}{q\Gamma(-q)}\left[0-x^{-q}\right] = \frac{x^{-q}}{\Gamma(1-q)}$$

Hence:

$$\frac{d^q(1)}{dx^q} = \frac{x^{-q}}{\Gamma(1-q)} \quad \text{.......................................}(1.21)$$

2- The constant Function, [Oldham, 1974]::

For the constant function f = c, where c is any constant including zero, we have:

$$\frac{d^q}{dx^q}(c) = c\frac{d^q}{dx^q}(1) = c\frac{x^{-q}}{\Gamma(1-q)} \quad \text{...(1.22)}$$

3- The function x^p, p > –1:

The function of fractional degree we consider here f = x^p, where p is initially arbitrary, we shall see, however, that p must exceed –1 for differintegration to have the properties we demand of the operator.

For integer n of either sign, one can show that:

$$\frac{d^n x^p}{dx^n} = p(p-1)...(p-n+1)x^{p-n}, \ n = 1, 2, \ldots$$

and from the classical theory of calculus our first encounter with non-integer q, will be restricted to negative q so that one may exploit the Riemann-Liouville definition, thus:

$$\frac{d^q x^p}{dx^q} = \frac{1}{\Gamma(-q)}\int_0^x \frac{y^p}{(x-y)^{q+1}}\,dy, \ q < 0$$

letting y = xu, then the integral may be cast into the standard form of beta function:

$$\frac{d^q x^p}{dx^p} = \frac{x^{p-q}}{\Gamma(-q)}\int_0^1 u^p(1-u)^{-q-1}\,du, \ q < 0 \quad \text{................................(1.23)}$$

To define the integral in eq. (1.23) is recognized as the beta function B(p + 1, −q) provided that both arguments are positive and therefore:

$$\frac{d^q x^p}{dx^q} = \frac{x^{p-q}}{\Gamma(-q)} B(p+1, -q)$$

$$= \frac{\Gamma(p+1)x^{p-q}}{\Gamma(p-q+1)} \quad \text{...(1.24)}$$

4- The Exponential Function, exp(k − cx):

With k and c are arbitrary constants, the power series expansion:

$$\exp(k - cx) = \exp(k - cq)\sum_{j=0}^{\infty} \frac{[-c(x-a)]^j}{\Gamma(j+1)}$$

which is valid for all x − a.

Differintegration term by term with respect to c(x − a), yields:

$$\frac{d^q \exp(k - cx)}{d(x - ca)^q} = \{c(x-a)\}^{-q} \exp(k - ca)\sum_{j=0}^{\infty} \frac{[-c(x-a)]^j}{\Gamma(j-q+1)} \quad \text{.....(1.25)}$$

Since the incomplete gamma function is defined by:

$$\gamma^*(-n, y) = \frac{c^{-x}}{\Gamma(x)} \frac{d^m}{dx^m} \int_0^c y^{x-1} \exp(-y)\, dy$$

$$= \exp(-x)\sum_{j=0}^{\infty} \frac{x^j}{\Gamma(j+c+1)}$$

Hence, the sum in eq. (1.25) may be expressed as an incomplete gamma function of argument −c(x − a) and parameter −q. The final result appears as:

$$\frac{d^q \exp(k-cx)}{[d(x-a)]^q} = \frac{\exp(k-cx)}{[x-a]^q}\gamma^*(-q,-c(x-a))$$

where $\gamma^*(-n,y) = y^n$ for non negative integer n. The above result seems to be reduced to the well-known formula for multiple differentiation of an exponential function:

$$\frac{d^q \exp(\mp x)}{dx^q} = \frac{\exp(\mp x)}{x^q}\gamma^*(-q,\mp x) \text{ ....(1.26)}$$

occurs on substituting k = a = 0 and c = ±1 into the general result.

5- The Functions $\frac{x^q}{1-x}$ and $\frac{x^p}{1-x}$:

By using the Maclaurian expansion of $(1-x)^{-1}$ and the technique of term-by-term differentegration, one can arrive at:

$$\frac{d^q}{dx^q}\left(\frac{x^q}{1-x}\right) = \sum_{j=0}^{\infty}\frac{d^q}{dx^q}x^{j+q}$$

As a formula expressing the effect of $\frac{d^q}{dx^q}$ operator with the lower limit zero on the $\frac{x^q}{1-x}$ function, subject to the condition that x not exceeds unity in magnitude. Provided also that q exceeds −1, the rules of subsection (1.2.3)(3) permit differintegration of the powers of x and lead to:

$$\frac{d^q}{dx^q}\left(\frac{x^q}{1-x}\right)=\sum_{j=0}^{\infty}\frac{\Gamma(j+q+1)}{\Gamma(j+1)}x^j$$

$$=\Gamma(q+1)\sum_{j=0}^{\infty}\binom{-q-1}{j}(-x)^j$$

Identification of the sum as a binomial expansion produces:

$$\frac{d^q}{dx^q}\left(\frac{x^q}{1-x}\right)=\frac{\Gamma(q+1)}{(1-x)^{q+1}}$$

as the simple final result.

In a similar manner, the technique for differintegrating $\frac{x^p}{1-x}$ that it will suffice to cite one intermediate and the final result:

$$\frac{d^q}{dx^q}\left(\frac{x^p}{1-x}\right)=x^{p-q}\sum_{j=0}^{\infty}\frac{\Gamma\left(j+p+1\right)x^j}{\Gamma(j+p-q+1)}$$

$$=\frac{\Gamma(p+1)B_x(p-q,q+1)}{\Gamma(p-q)[1-x]^{q+1}}$$

together with the restriction, name

ly, 0 < x < 1 and p > –1, which were assumed during the derivation.

CHAPTER TWO

SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS USING EXPLICIT LINEAR MULTISTEP METHODS

Numerical and approximate methods may be sometimes considered as the most suitable methods for solving differential equations.

Finite difference equations may be used effectively in solving differential equations which has the utility of its simplicity in programming and its ability in solving various types of differential equations, involving ordinary, partial, delay, fractional of linear and non linear types, etc., [Lambert, 1973].

2.1 FRACTIONAL DIFFERENTIAL EQUATIONS

A relationship involving one or more derivatives of the unknown function y with respect to its independent variable x is known as an ordinary differential equation. Similar relationships involving at least one differential of non integer order may be termed as extraordinary or fractional differential equations, [Al-Authab, 2005].

As with ordinary differential equations, the situation of extraordinary (fractional) differential equations often involves integrals and contains arbitrary constants.

The differential equations may involve Riemann-Liouville differential operators of fractional order $q > 0$, which takes the form:

$$D_{x_0}^{q} y(x) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_{x_0}^{x} \frac{y(u)}{(x-u)^{q-m+1}} du \text{ ....(2.1)}$$

where m is an integer number and $m - 1 < q \leq m$. Differential equations involving these fractional derivatives have proved to be valuable tools in the modeling of many physical problems. Also, D^q has an m-dimensional singular kernel, and therefore one need to specify m initial conditions in order to obtain a unique solution to the fractional differential equation:

$$D^q y(x) = f(x, y(x)),\ x \geq x_0 \text{ ....(2.2)}$$

with some given function f. In the standard mathematical theory, the initial conditions corresponding to eq. (2.2) must consists of the following m-equations:

$$\frac{d^{q-k}}{dt^{q-k}} y(x)\Big|_{x=x_0} = b_k,\ k = 1, 2, \ldots, m \text{ ....(2.3)}$$

where b_k, $\forall$ k are given values.

In practical applications, these values are frequently not available and so Caputo in 1967 suggested that one should incorporate derivatives of integer-order of the function y as they are commonly used in initial value problems with positive integer-order equations, into the fractional-order equation, where the equivalent problem is, [Kalil, 2006]:

$$D^q[y - T_{m-1}(y)]\,(x) = f\,(x, y(x)) \text{ ....(2.4)}$$

where $T_{m-1}(y)$ is the Taylor polynomial of order (m – 1) for y, centered at 0. Then, one can specify the initial conditions in the classical form, as:

$$y^{(k)}(0) = y_0^{(k)},\ k = 0, 1, \ldots, m-1 \qquad (2.5)$$

As in ordinary differential equations with positive integer order, the classification for fractional differential equations may be given with respect to several aspects, which is to be either linear or nonlinear, homogeneous or non-homogeneous, etc.

2.1.1 The Existence and Uniqueness Theorem, [Diethelm, 1997]:

Looking at the questions of existence and uniqueness of the solution of fractional differential equations, the following result may be presented that are very similar to the corresponding classical theorems known in the case of first-order ordinary differential equations. Only the scalar setting will be discussed explicitly; the generalization to vector-valued functions is straight forward.

Theorem (2.1) (The Existence Theorem):

Assume that $D = [0, \chi^*] \times [y_0^{(0)} - \alpha, y_0^{(0)} + \alpha]$ with some real number

$\chi^* > 0$ and some $\alpha > 0$, and let the function $f : D \longrightarrow \mathbb{R}$, be a continuous function. Furthermore, define:

$$\chi = \min\left\{\chi^*, (\alpha\Gamma(q+1) / \|f\|_\infty)^{1/q}\right\}$$

Then, there exists a function $y : [0, \chi] \longrightarrow \mathbb{R}$, solving the initial value problem (2.4)-(2.5).

<u>Theorem (2.2) (The Uniqueness Theorem):</u>

Assume that $D = [0, \chi^*] \times [y_0^{(0)} - \alpha, y_0^{(0)} + \alpha]$, with some real number $\chi^* > 0$ and some $\alpha > 0$. Furthermore, let the function $f : D \longrightarrow \mathbb{R}$, be bounded function on D and fulfill a Lipschitze condition with respect to the second variable y, i.e.,

$$|f(x, y) - f(x, z)| \leq L|y - z|$$

with some constant $L > 0$ independent of x, y and z which is called the Lipschitz constant. Then there exists at most one function $y : [0, \chi] \longrightarrow \mathbb{R}$, solving the initial value problem (2.4)-(2.5).

2.1.2 Properties of Fractional Differentiation and Integration:

Here, some properties related to fractional differentiation and integration are explained, those properties which will provide our primary means of understanding and utilizing fractional differential equations.

We start with those properties of most importance:

1- Linearity, [Oldham, 1974]:

By linearity of the differintegral operator, by which we mean:

$$D^q(c_1f_1 + c_2f_2) = c_1D^qf_1 + c_2D^qf_2 \qquad (2.6)$$

where f_1 and f_2 are an arbitrary functions while c_1 and c_2 are an arbitrary constants, since by using the Riemann-Liouville formula:

$$D^q(c_1f_1 + c_2f_2) = \frac{1}{\Gamma(m-q)}\frac{d^m}{dx^m}\int_{x_0}^{x}\frac{c_1f_1 + c_2f_2}{(x-u)^{q-m+1}}du$$

$$= \frac{1}{\Gamma(m-q)}\frac{d^m}{dx^m}\left(\int_{x_0}^{x}\frac{c_1f_1}{(x-u)^{q-m+1}}du + \int_{x_0}^{x}\frac{c_1f_1}{(x-u)^{q-m+1}}du\right)$$

$$= \frac{1}{\Gamma(m-q)}\frac{d^m}{dx^m}\int_{x_0}^{x}\frac{c_1f_1}{(x-u)^{q-m+1}}du +$$

$$\frac{1}{\Gamma(m-q)}\frac{d^m}{dx^m}\int_{x_0}^{x}\frac{c_1f_1}{(x-u)^{q-m+1}}du = c_1D^qf_1 + c_2D^qf_2$$

2- Scale Change, [Oldham, 1974]:

By a scale change of the function f with respect to the lower limit a, we mean its replacement by $f(\beta x - \beta a + a)$, where β is a constant termed the scaling factor, and hence the factorial derivative of order q with $Y = \beta y - \beta a + a$, and $X = x + (a - \beta a)/\beta$, is given by:

$$\frac{d^q f(\beta x)}{[d(x-a)]^q} = \frac{d^q f(\beta x - \beta a + a)}{[d(x-a)]^q}$$

$$= \frac{1}{\Gamma(-q)}\int_{a}^{x}\frac{f(\beta y - \beta a + a)}{[x-y]^{q+1}}dy$$

$$= \frac{1}{\Gamma(-q)}\int_{a}^{\beta x}\frac{f(Y)[dY/\beta]}{\{[\beta X - Y]/\beta\}^{q+1}}$$

$$= \frac{\beta^q}{\Gamma(-q)}\int_{a}^{\beta x}\frac{f(Y)}{[\beta X - Y]^{q+1}}dY$$

$$= \beta^q\frac{d^q f(\beta X)}{[d(\beta X - a)]^q} \quad \text{.. (2.7)}$$

3- Leibniz's Rule:

The rule for differentiation of a product of two functions f and g is a familiar result in elementary calculus. It states that for a positive integer n, then the differenitegration of the product function is defined in general using the following binomial rule:

$$\frac{d^n[fg]}{dx^n} = \sum_{j=0}^{n} \binom{n}{j} \frac{d^{n-j}f}{dx^{n-j}} \cdot \frac{d^j g}{dxj} \quad \text{...(2.8)}$$

The following product rule for multiple integrals is also satisfied

$$\frac{d^{-n}[fg]}{[d(x-a)]^{-n}} = \sum_{j=0}^{\infty} \binom{-n}{j} \frac{d^{-n-j}f}{[d(x-a)]^{-n-j}} \cdot \frac{d^j g}{[d(x-a)]^j} \quad \text{....................(2.9)}$$

Now, when we observe that the finite sum in eq.(2.8) can equally well extended to infinity (since $\binom{n}{j} = 0$ for all j > n), we might expect the product rule to be generalized to an arbitrary order q as:

$$\frac{d^q[fg]}{[d(x-a)]^q} = \sum_{j=0}^{\infty} \binom{q}{j} \frac{d^{q-j}f}{[d(x-a)]^{q-j}} \cdot \frac{d^j g}{[d(x-a)]^j} \quad \text{.................... (2.10)}$$

Thus such a generalization is valid indeed for real order q and is called the Leibniz rule.

Further generalization of Leibniz's rule due to Osler (1972) is the integral form (see [Oldham, 1974]):

$$\frac{d^q[fg]}{dx^q} = \int_{-\infty}^{\infty} \frac{\Gamma(q+1)}{\Gamma(q-\gamma-\lambda+1)\Gamma(\gamma+\lambda+1)} \cdot \frac{d^{q-\gamma-\lambda}f}{dx^{q-\gamma-\lambda}} \cdot \frac{d^{\gamma+\lambda}g}{dx^{\gamma+\lambda}} d\lambda$$

in which a discrete sum is replaced by an integral.

4- The Chain Rule [Oldham, 1974]:

Among the most important derivatives of fractional order is the chain rule, which will play an important role later in solving non-liner fractional differential equations.

The Chain rule for the first order differentiation is given by:

$$\frac{d}{dx} g\left(f(x)\right) = \frac{d}{df(x)} g\left(f(x)\right) \frac{d}{dx} f(x)$$

which tacks a simple counterpart in the integral calculus.

Indeed if there were such a counterpart, the process of integration would pose no greater difficulty than does differentiation, since any general formula for a special case of little hope that can be held out for a useful chain rule for arbitrary real number q.

Nevertheless, a formal chain rule in fractional orders may be derived quit simply, which takes the form:

$$\frac{d^q \Phi}{[d(x-a)]^q} = \frac{[x-a]^{-q}}{\Gamma(1-q)} \Phi + \sum_{j=1}^{\infty} \binom{q}{j} \frac{[x-a]^{j-q}}{\Gamma(j-q+1)} . \frac{d^j \Phi}{dx^j}$$

Now, we consider $\Phi = \Phi(f(x))$ and evaluate $d^j \Phi\left(f(x)\right) / dx^j$, in the second term of the last equation as follows:

$$\frac{d^j}{dx^j} \Phi\left(f(x)\right) = j! \sum_{m=1}^{j} \Phi^{(m)} \sum \prod_{k=1}^{j} \frac{1}{p_k !} \left[\frac{f^{(k)}}{k!} \right]^{p_k}$$

where $\sum$ extends over all combinations of nonnegative integer values of $p_1, p_2, ..., p_j$, such that:

$$\sum_{k=1}^{j} k p_k = j \quad \text{and} \quad \sum_{k=1}^{j} p_k = m$$

Thus:

$$\frac{d^q}{[d(x-a)]^q}\Phi\big(f(x)\big) = \frac{[x-a]^{-q}}{\Gamma(1-q)}\Phi\big(f(x)\big) +$$
$$\sum_{j=1}^{\infty}\binom{q}{j}\frac{[x-a]^{j-q}}{\Gamma(j-q+1)} j! \sum_{m=1}^{j}\Phi^{(m)}\sum\prod_{k=1}^{j}\frac{1}{p_k!}\left[\frac{f^{(k)}}{k!}\right]^{p_k}$$

2.2 NUMERICAL SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS USING G-SPLINE INTERPOLATION

In this section, the linear multistep methods based on G-spline functions will be used for solving linear fractional differential equations. Therefore, first some basic concepts related to this trained of numerical analysis are given first.

2.2.1 Linear Multistep Methods, [Byren, 1972], [Lambert, 1973]:

Consider the initial value problem for a single first-order differential equation:

$$y'(x) = f(x, y(x)),\ y(x_0) = y_0,\ x \geq x_0 \quad \text{...(2.11)}$$

where f is a given continuous function and x_0, y_0 are fixed. We seek for a solution in the range $x \geq x_0$, where x_0 is a finite scalar and assume that:

$$f(x, y) = O(\phi(x)),\ \text{as}\ x \longrightarrow x_0$$

which means that there exist a positive constant k, such that $|f(x)| \leq k|\phi(x)|$, for x sufficiently close to x_0. Our most frequent use of this notation will be in the context as $f(h) = o(h^p)$ and letting $h \longrightarrow 0$, where h is the step length associated with some numerical method. So frequently, we shall write this, that will be impracticable always to included the phrase "as $h \longrightarrow 0$", which is consequently to be taken as read. However, it is important to quard against the temptation mentally to debase the notation $f(h) = o(h^p)$ to mean "f(h) is roughly the same size as h^p". Then the problem has a unique continuously differentiable solution, which will be denoted by y(x).

Consider the sequence of points $\{x_n\}$ defined by $x_n = x_0 + nh$, $n = 0, 1, \ldots, N$, where N is a prespecified natural number, the parameter h, which will always be regarded as constant, except where otherwise indicated, is called the step length. An essential property of the majority of computational method for the solution of eq.(2.11) is that of discritization, that is, seeking for an approximate solution, not on the continuous interval $x_0 \leq x \leq b$, but on the discrete point set $\{x_n \mid n = 0, 1, \ldots, (b - x_0)/h\}$. Let y_n be an approximation to the theoretical solution at x_n, that is, to $y(x_n)$, and let $f_n \equiv f(x_n, y_n)$. If certain computational method used for determining the sequence $\{y_n\}$ takes the form of a linear relationship between y_{n+j} and f_{n+j}, for all $j = 0, 1, \ldots, k$, where k is called the step size of the method. This method is called the linear multistep method with step number k, or a linear k-step method.

Then the general form of a linear multistep method with k steps may thus be written as:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} \quad \text{...(2.12)}$$

where α_j and β_j, are constants to be determined. It is assumed that $\alpha_k \neq 0$ and that not both of α_0 and β_0, equals zero. Since eq. (2.12) can be multiplied on both sides by the same constant without altering the relationship, the coefficients α_j and β_j, are arbitrary to the extent of a constant multiplier. This arbitrariness was removed by assuming throughout that $\alpha_k = 1$.

The linear multistep method (2.12) is said to be explicit if $\beta_k = 0$ and implicit if $\beta_k \neq 0$, and the difference between the two methods is in their accuracy in solving certain problem.

The local truncation error of (2.12), has the form:

$$T = \sum_{j=0}^{k}\alpha_j y(x_{n+j}) - h\sum_{j=0}^{k}\beta_j y'(x_{n+j})$$

where $y(x_{n+j})$ is the exact solution at $x_{n+j} = x_n + jh$. Expanding $y(x_{n+j})$ and $y'(x_{n+j})$ in a Taylor series expansion about x_n, yields:

$$T = C_0 y(x_n) + C_1 h y'(x_n) + \ldots + C_p h^p y^{(p)}(x_n) + \ldots$$

Then the order of of the linear multistep method (2.12) is p if $C_0 = C_1 = \ldots = C_p = 0$, but $C_{p+1} \neq 0$.

Now, the first and second characteristic polynomials of the linear multistep method (2.12) are introduced for completeness purposes which are defined as $\rho(r)$ and $\sigma(r)$, respectively by:

$$\left.\begin{aligned} \rho(r) &= \sum_{j=0}^{k} \alpha_j r^j \\ \sigma(r) &= \sum_{j=0}^{k} \beta_j r^j \end{aligned}\right\} \qquad (2.13)$$

Thus, for consistent methods, the first characteristic polynomial $\rho(r)$ always has a root at +1, where the consistency of eq.(2.12) is equivalent to:

$$\sum_{j=0}^{k} \alpha_j = 0 \quad \text{and} \quad \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j \qquad (2.14)$$

The root r = +1 is called the principle root and always labeled as r_1. The remaining roots r_s, s = 2, 3, …, k; are called the spurious roots and arise only when the step size of the method is greater than one, that is, when one choose to replace a first order differential equation by a difference equation of order greater than one.

Now, the classification of linear multistep methods depends mainly on the first characteristic polynomial $\rho(r)$, and the classification is as follows:

(1) If the zeros of ρ are at r = l, r = 0, then the method is called of Adam's type and if the LMM is explicit, then it is called of Adam Bashforth type, while if it is implicit then it is called of Adam-Moulton type, i.e., in Adam's methods, we have the following:

$$\begin{aligned} \rho(r) &= r^k - r^{k-1} \\ &= r^{k-1}(r-1) = 0 \end{aligned}$$

(2) If the zeros of ρ are at $r = -1$, $r = 0$ and $r = 1$, then the method is called of Nystrom type if it is explicit and if the method is implicit, then it is called of Milne-Simpson type, i.e., we have:

$$\begin{aligned}\rho(r) &= r^k - r^{k-2} \\ &= r^{k-2}(r^2 - 1) \\ &= r^{k-2}(r-1)(r+1)\end{aligned}$$

The technique that will be used involves of writing $y(x_k)$, the exact solution of $y' = f(x, y)$, $x \in [a, b]$, $y(a) = y_0$ evaluated at $x_k = a + kh$, $k = 0, 1, \ldots, N$; as:

$$y(x_k) - y(x_q) = \int_{x_q}^{x_k} f(x, y(x))\, dx \quad \text{...(2.15)}$$

and then replacing f by its G-spline interpolant.

Now, as a construction of linear multistep formula for solving fractional ordinary differential equations in connection with G-spline interpolation functions, suppose we want to construct an k^{th} order linear multistep formula of the general type, [Byren, 1972], [Osama, 2007]:

$$y_{n+k} - y_{n+p} = \sum_{j=0}^{p}\sum_{i=0}^{k} \beta_{ij} h^{j+1} f^{(j)}(x_{n+i}, y_{n+i}) \quad \text{..............................(2.16)}$$

where β_{ij} are obtained by approximation of linear functionals with the sense of G-spline function, as:

Let I = [a, b] be a bounded interval containing the node points x_1, x_2, …, x_k and let us consider a linear functional $\mathcal{L}f : C^{\alpha}[a, b] \longrightarrow \mathbb{R}$, of the form:

$$\mathcal{L}f = \sum_{j=0}^{\alpha}\int_a^b a_j(x) f^{(j)}(x)\,dx + \sum_{j=0}^{\alpha}\sum_{i=1}^{n_j} b_{ji} f^{(j)}(x_{ji}) \quad \text{......................(2.17)}$$

where the $a_j(x)$ are piecewise continuous functions in I, $x_{ji} \in$ I and b_{ji} are real constants. Then the functional (2.13) may be approximated using the formula:

$$\mathcal{L}f = \sum_{(i,j)\in e} \beta_{ij} f^{(j)}(x_i) + Rf \quad \text{..(2.18)}$$

Therefore, in order to find the approximation $\mathcal{L}f$ given by eq.(2.18), one can propose to determine the reals β_{ij}, where e is an HB-set.

The procedure associated with the names of Newton and Cotes will be followed for the evaluation of the coefficients β_{ij} as follows:

There are $n = \sum_{i,j} a_{ij}$ of unknown parameters β_{ij}, then formula (2.18) is exact if $f \in \Pi_{n-1}$ (i.e., Rf = 0) and this is what we have to require. Under this condition, formula (2.18) is the best approximation to $\mathcal{L}f$.

For the derivation of this approximation, substitute into formula (2.18), with Rf = 0, yields:

$$f(x) = \frac{x^v}{v!}, \; v = 0, 1, \ldots, n-1$$

to get:

$$\mathcal{L}\frac{x^v}{v!} = \sum_{(i,j)\in e} \beta_{ij} \frac{x_i^{v-j}}{(v-j)!}, \ v = 0, 1, \ldots, n-1 \ \ldots\ldots\ldots\ldots(2.19)$$

and recall that the HB-interpolation formula is given by:

$$f(x) = \sum_{(i,j)\in e} f^{(j)}(x_i)L_{ij}(x), \text{ if } f \in \Pi_{n-1}$$

and upon taking $\mathcal{L}$ to the both sides of the last formula, gives:

$$\mathcal{L}f = \sum_{(i,j)\in e} f^{(j)}(x_i)\mathcal{L}L_{ij}(x), \text{ if } f \in \Pi_{n-1}$$

and hence in comparison with eq.(2.18), one may have:

$$\beta_{ij} = \mathcal{L}L_{ij}(x), (i, j) \in e$$

which produce the coefficient in the Newton-Cots are the best approximation to $\mathcal{L}f$.

Now, for eq. (2.16), y_j is an approximation to $y(x_j)$ and $[x_n, x_{n+k}] \subset [a, b]$ to this objective, one can pick k and α along with the m-poised HB-problem corresponding to the n values in the set:

$$\{ \varphi_i^{(j)} = \varphi^{(j)}(i), (i, j) \in e\}$$

where $\varphi(s) = f(x_n + sh, y(x_n + sh))$, for $0 \le s \le k$.

Here $e = \{(i, j)\}$ is chosen in such a way that it takes on each of the values 0, 1, …, k one or more times and j = p is attained in at least one element (i, j) of e, the HB-problem is to find a function $\psi \in C^p$ which satisfies the N interpolatory conditions:

$$\psi_{(i)}^{(j)} = \phi_i^{(j)}, \ \forall \ (i\ j) \in e \ \ldots\ldots\ldots\ldots(2.20)$$

Instating that the HB-problem is m-poised, means that if $p \in \Pi_{m-1}$, where Π_{m-1} is the class of all polynomial is of degree $m - 1$ or less, and if $p_{(i)}^{(j)} = 0$ for $(i, j) \in e$, then $p(s) = 0$ for reasons will be com more apparent, we now require that $p < m < N$.

At this point, the G-spline interpolant to φ may be given in terms of the fundamental G-splines L_{ij}, by:

$$S_m(s) = \sum_{(i,j)\in e} L_{ij}(s)\phi_i^{(j)} \quad \text{...(2.21)}$$

where $L_{ij}^{(r)}(t) = \delta_{it}\delta_{ir}$, for all $(t, r) \in e$, and at this point, the set of knots Δ is still $\Delta = \{0, 1, \ldots, k\}$. Now, in order to determine the coefficients β_{ij} in eq.(2.16), replace f in eq.(2.15) by its G-spline interpolation function and make a change of variables, integrate and compare the results with eq.(2.15). As a consequence of the uniqueness of the G-spline interpolant, it then follows that:

$$\beta_{ij} = \int_{\ell}^{k} L_{ij}(s)\, ds \quad \text{...(2.22)}$$

Next, some illustrative examples are given to illustrate the method of solution and the accuracy of the results:

<u>Example (2.1):</u>

In this example, the 2-step explicit method based on G-spline interpolation will be used to solve the linear fractional differential equation:

$$y^{(1/2)}(x) = -y(x) + x^2 + \frac{2x^{3/2}}{\Gamma(5/2)}, \ y(0) = 0$$

where the exact solution is given for comparison purpose by $y(x) = x^2$.

An HB-problem must be first chosen. The following choice:

$$\Delta = \{0, 1, 2\}$$

is taken to be the knot points and let:

$$e = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0)\}$$

we shall seek for $S_4(s) \in \mathcal{S}_4(E^*, \Delta)$, with:

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and for which:

$$S_4^{(j)}(i) = \phi^{(j)}(i), \ (i, j) \in e$$

Integrating $S_4(s)$ over [1, 2] yields the following closed formula:

$$y_{n+2} = y_{n+1} + [h\{\beta_{00}g(x_n, y_n) + \beta_{10}g(x_{n+1}, y_{n+1})\} + h^2\{\beta_{01}g'(x_n, y_n) + \beta_{11}g'(x_{n+1}, y_{n+1})\}]$$

where:

$$\beta_{00} = \int_1^2 L_{00}(s) \, ds = \frac{503}{3072}$$

$$\beta_{10} = \int_1^2 L_{10}(s) \, ds = \frac{1748}{3072}$$

$$\beta_{01} = \int_1^2 L_{01}(s)\, ds = \frac{150}{3072}$$

$$\beta_{11} = \int_1^2 L_{11}(s)\, ds = \frac{1068}{3072}$$

and

$$g(x, y) = D^{1/2}\left[-y(x) + x^2 + \frac{2x^{3/2}}{\Gamma(5/2)}\right]$$

$$= y(x) - x^2 + 2x$$

$$g'(x, y) = y - x^2 + 2$$

and the fundamental G-spline functions are defined by:

$$L_{00}(s) = [128 - 494s^2 + 411s^3 + 25s_+^7 - 70s_+^6 - 20(s-1)_+^7 - 140(s-1)_+^6 - 5(s-2)_+^7]/128.$$

$$L_{01}(s) = [64s - 150s^2 + 95s^3 + 5s_+^7 - 14s_+^6 - 4(s-1)_+^7 - 28(s-1)_+^6 - (s-2)_+^7]/64.$$

$$L_{10}(s) = [118s^2 - 95s^3 - 5s_+^7 + 14s_+^6 + 4(s-1)_+^7 + 28(s-1)_+^6 + (s-2)_+^7]/32.$$

$$L_{11}(s) = [-54s^2 + 63s + 5s_+^7 - 14s_+^6 - 4(s-1)_+^7 - 28(s-1)_+^6 - (s-2)_+^7]/32.$$

where:

$$(s-a)_+^n = \begin{cases} (s-a)^n, & \text{if} \quad s>a \\ 0, & \text{if} \quad s \le a \end{cases}$$

Table (2.1) presents the approximate results and its comparison with the exact solution with step size $h = 0.1$:

Table (2.1)

Approximate and exact results of example (2.1).

x_i	y_i	$y(x_i)$	***Absolute error***
0	0	0	0
0.1	0.01	0.01	0
0.2	0.06	0.04	0.02
0.3	0.15	0.09	0.06
0.4	0.27	0.16	0.11
0.5	0.41	0.25	0.16
0.6	0.55	0.36	0.19
0.7	0.73	0.49	0.24
0.8	0.96	0.64	0.32
0.9	1.04	0.81	0.23
1	1.12	1	0.12

Example (2.2):

In this example, the 3-step explicit method based on G-spline interpolation method will be used to solve the linear fractional differential equation:

$$y^{(1/2)}(x) = -y(x) + x^2 + \frac{2x^{3/2}}{\Gamma(5/2)},\ y(0) = 0$$

where the exact solution is given by $y(x) = x^2$.

An HB-problem must be first chosen. The choice:

$\Delta = \{0, 1, 2\}$

is taken to be the knot points and let:

$e = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0)\}$

we shall seek for $S_4(s) \in \mathcal{S}_4(E^*, \Delta)$, with:

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and for which:

$$S_4^{(j)}(i) = \phi^{(j)}(i),\ (i, j) \in e$$

Integrating $S_4(s)$ over [1, 3] yields the closed formula:

$$y_{n+3} = y_{n+1} + [h\{\beta_{00}g(x_n, y_n) + \beta_{10}g(x_{n+1}, y_{n+1})\} + h^2\{\beta_{01}g'(x_n, y_n) + \beta_{11}g'(x_{n+1}, y_{n+1})\}]$$

where:

$$\beta_{00} = \int_1^2 L_{00}(s)\, ds = 2.859$$

$$\beta_{10} = \int_1^2 L_{10}(s)\, ds = 1.087$$

$$\beta_{01} = \int_1^2 L_{01}(s)\, ds = 0.877$$

$$\beta_{11} = \int_1^2 L_{11}(s)\, ds = 3.724$$

and

$$g(x, y) = D^{1/2}\left[-y(x) + x^2 + \frac{2x^{3/2}}{\Gamma(5/2)}\right]$$

$$= y(x) - x^2 + 2x$$

$$g'(x, y) = y - x^2 + 2$$

and the fundamental G-spline functions are defined by:

$$L_{00}(s) = [128 - 494s^2 + 411s^3 + 25s_+^7 - 70s_+^6 - 20(s-1)_+^7 - 140(s-1)_+^6 - 5(s-2)_+^7]/128.$$

$$L_{01}(s) = [64s - 150s^2 + 95s^3 + 5s_+^7 - 14s_+^6 - 4(s-1)_+^7 - 28(s-1)_+^6 - (s-2)_+^7]/64.$$

$$L_{10}(s) = [118s^2 - 95s^3 - 5s_+^7 + 14s_+^6 + 4(s-1)_+^7 + 28(s-1)_+^6 + (s-2)_+^7]/32.$$

$$L_{11}(s) = [-54s^2 + 63s + 5s_+^7 - 14s_+^6 - 4(s-1)_+^7 - 28(s-1)_+^6 - (s-2)_+^7]/32.$$

where:

$$(s-a)_+^n = \begin{cases} (s-a)^n, & \text{if} \quad s>a \\ 0, & \text{if} \quad s \le a \end{cases}$$

Table (2.2) presents the approximate results and its comparison with the exact solution with step length $h = 0.1$:

Table (2.2)

Approximate and exact results of example (2.2).

x_i	y_i	$y(x_i)$	***Absolute error***
0	0	0	0
0.1	0.01	0.01	0
0.2	0.04	0.04	0
0.3	0.09	0.09	0
0.4	0.16	0.16	0
0.5	0.27	0.25	0.02
0.6	0.37	0.36	0.01
0.7	0.51	0.49	0.02
0.8	0.66	0.64	0.02
0.9	0.84	0.81	0.03
1	1.02	1	0.02

Remark (2.1):

It is noticeable that, as expected, that the 3-step method give more accurate results than the 2-step method, and hence the accuracy of the results may be improved by either increasing the steps of he method or by altering the HB-set.

CHAPTER THREE

NUMERICAL SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS USING IMPLICIT LINEAR MULTISTEP METHOD

As it is known from the theory of numerical analysis, especially from the theory of difference equations for solving ordinary or partial differential equations that the implicit numerical methods are more accurate than explicit or semi-explicit methods, but still there is a major disadvantage in using the implicit methods, since it is more difficult in approximations and may take more computational time than in explicit or semi-explicit methods.

Therefore, this chapter is devoted to present the implicit methods for solving fractional ordinary differential equations, since such problems require numerical methods that produce more accurate results.

3.1 SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS USING IMPLICIT METHODS

As it is said above, in all numerical methods, implicit methods give better results than explicit methods, but they are either more complicated or requires more computational time. The solution obtained from implicit methods are found in the same way of explicit method when the ordinary fractional differential equation is linear but when the ordinary fractional differential equation is nonlinear, then additional computations are needed which will increases the complexity of the problem under consideration.

The next example illustrates the simple application of implicit methods in solving linear fractional differential equations which will be reduced to an explicit form.

Example (3.1):

Consider the linear fractional differential equation:

$$y^{(1/2)}(x) = -y(x) + x^2 + \frac{2x^{3/2}}{\Gamma(5/2)},\ y(0) = 0$$

where the exact solution is given by $y(x) = x^2$.

To construct such a method via G-spline interpolation, an HB-problem must be chosen. The choice:

$\Delta = \{0, 1, 2\}$

is taken to be knot points, and letting:

$e = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0)\}$

Then, to seek for $S_4(s) \in \mathcal{S}_4(E^*, \Delta)$, with:

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and for which:

$$S_4^{(j)}(i) = \phi^{(j)}(i),\ (i, j) \in e$$

Integrating $S_4(s)$ over [1, 2] yields the closed formula:

$$y_{n+2} = y_{n+1} + [h\{\beta_{00}g(x_n, y_n) + \beta_{10}g(x_{n+1}, y_{n+1}) + \beta_{20}g(x_{n+2}, y_{n+2})\} + h^2\{\beta_{01}g'(x_n, y_n) + \beta_{11}g'(x_{n+1}, y_{n+1})\}]$$

where the fundamental G-spline functions are defined by:

$$L_{00}(s) = [128 - 494s^2 + 411s^3 + 25s_+^7 - 70s_+^6 - 20(s-1)_+^7 - 140(s-1)_+^6 - 5(s-2)_+^7]/128.$$

$$L_{01}(s) = [64s - 150s^2 + 95s^3 + 5s_+^7 - 14s_+^6 - 4(s-1)_+^7 - 28(s-1)_+^6 - (s-2)_+^7]/64.$$

$$L_{10}(s) = [118s^2 - 95s^3 - 5s_+^7 + 14s_+^6 + 4(s-1)_+^7 + 28(s-1)_+^6 + (s-2)_+^7]/32.$$

$$L_{11}(s) = [-54s^2 + 63s + 5s_+^7 - 14s_+^6 - 4(s-1)_+^7 - 28(s-1)_+^6 - (s-2)_+^7]/32.$$

$$L_{20}(s) = [22s^2 - 31s^3 - 5s^7 + 14s_+^7 + 4(s-1)_+^7 + 28(s-1)_+^6 + (s-2)_+^7]/128.$$

where:

$$(s-a)_+^n = \begin{cases} (s-a)^n, & \text{if} \quad s > a \\ 0, & \text{if} \quad s \le a \end{cases}$$

and:

$$\beta_{00} = \int_1^2 L_{00}(s)\, ds = \frac{503}{3072}$$

$$\beta_{10} = \int_1^2 L_{10}(s)\, ds = \frac{1748}{3072}$$

$$\beta_{20} = \int_1^2 L_{20}(s)\, ds = \frac{821}{3072}$$

$$\beta_{01} = \int_1^2 L_{01}(s)\, ds = \frac{150}{3072}$$

$$\beta_{11} = \int_1^2 L_{11}(s)\, ds = \frac{1068}{3072}$$

and

$$g(x, y) = D^{1/2}\left[-y + x^2 + \frac{2x^{3/2}}{\Gamma(5/2)}\right]$$

$$= y(x) - x^2 + 2x$$

$$g'(x, y) = y - x^2 + 2$$

Table (3.1) presents the approximate results and its comparison with the exact solution, where the step size is taken to be h = 0.1:

Table (3.1)

The approximate and exact results of example (3.1).

x_i	y_i	$y(x_i)$	*Absolute error*
0	0	0	0
0.1	0.01	0.01	0
0.2	0.05	0.04	0.01
0.3	0.12	0.09	0.03
0.4	0.22	0.16	0.06
0.5	0.35	0.25	0.10
0.6	0.51	0.36	0.15
0.7	0.71	0.49	0.22
0.8	0.94	0.64	0.30
0.9	1.21	0.81	0.40
1	1.51	1	0.51

3.2 IMPLICIT METHODS FOR SOLVING NONLINEAR FRACTIONAL ORDINARY DIFFERENTIAL EQUATIONS

Difficulties may be encountered in solving fractional differential equations using implicit methods in connection with G-spline interpolation methods. These difficulties are due to the existence of fractional derivatives of nonlinear terms that appeared in the differential equation. This problem may be handled using the chain rule for fractional derivatives.

Two approaches are modified and used next for solving such type of problems:

3.2.1 Predictor Corrector Method, [Lambert, 1973]:

Let us suppose that one may intend to use an implicit linear k-steps method to solve the considered fractional ordinary initial value problem. AT each step, one must solve for y_{n+k}, the nonlinear equation:

$$y_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h\beta_k f(x_{n+k}, y_{n+k}) + h \sum_{j=0}^{k-1} \beta_j f_{n+j} \qquad \text{.....................(3.1)}$$

where y_{n+j} are known, in general. A unique solution y_{n+k} exists and can be approached arbitrarily closely by the iteration:

$$y_{n+k}^{s+1} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h\beta_k f(x_{n+k}, y_{n+k}^{s}) + h \sum_{j=0}^{k-1} \beta_j f_{n+j}, \ s=0,1,\ldots \qquad \text{.....(3.2)}$$

where y_{n+k}^{0} is arbitrary.

At each step of this iteration (given by eq.(3.2)) clearly involves an evaluation of $f(x_{n+k}, y_{n+k}^{s})$. Thus, we are concerned to keep to a minimum number of times of iterations in eq.(3.2) is applied particularly so when the evaluation of f at a given values of its arguments is time consuming. Therefore, we would like to make the initial guess y_{n+k}^{0} as accurate as possible. This is done by using separate explicit method which are discussed in chapter two to estimate y_{n+k}, and taking this predicted value to be the initial guess y_{n+k}^{0} in the implicit rule.

The explicit method is called the predictor, and the implicit method is called the corrector.

We can now proceed in one of two different ways, the first consists of continuing the iterations given by eq. (3.2) until the iterates have converged (in practice, until some criterion, such as $|y_{n+k}^{s+1} - y_{n+k}^{s}| < \varepsilon$, where ε is a pre-assigned tolerance, of the order of the local round-off error, say, is satisfied). Then regarding the value y_{n+k}^{s+1} so obtained as an acceptable approximation to the exact solution y_{n+k} of eq.(3.1).

Since each iteration corresponds to one application of the corrector, then this mode of operation will be called of the predictor-corrector method correcting to convergence. In this mode, one cannot tell in advance how much the iterations will be necessary, that is, how many function evaluations will be required at each step to reach a certain accuracy.

On the other hand, the accepted value of y_{n+k}^{s+1} being independent of the initial guess y_{n+k}^{0}, the local truncation error and the weak stability characteristics of the overall method are precisely those of the corrector alone; the properties of the predictor are of no importance.

Let P indicate an application of the predictor, C a single application of the corrector, and E an evaluation of f in terms of known values of its arguments.

Suppose that we compute y^{0}_{n+k} from the predictor method, evaluate $f^{0}_{n+k} \equiv f(x_{n+k}, y^{0}_{n+k})$ and apply the corrector one time to get y^{1}_{n+k}; the calculation so far is denoted by PEC. A further evaluation of $f^{1}_{n+k} \equiv f(x_{n+k}, y^{1}_{n+k})$ followed by a second application of the corrector yields y^{2}_{n+k}, and the calculation is now denoted by PECEC, or $P(EC)^2$. Applying the corrector m-times is similarly denoted by $P(EC)^m$. Since m is fixed, we accept y^{2}_{n+k} as the numerical solution at x_{n+k}. At this stage, the last computed value we have for f_{n+k} is $f^{m-1}_{n+k} \equiv f(x_{n+k}, y^{m-1}_{n+k})$, and we have a further decision to make, namely, whether or not to evaluate $f^{m}_{n+k} \equiv f(x_{n+k}, y^{m}_{n+k})$. If this final evaluation is made, the mode is now denoted by $P(EC)^mE$, and if not, by $P(EC)^m$. This choice clearly affects the next step of the calculation, since both predicted and corrected values for y_{n+k+1} will depend on whether f^{m}_{n+k} is taken to be f^{m}_{n+k} or f^{m-1}_{n+k}.

Note that, for a given m, both $P(EC)^mE$ and $P(EC)^m$ modes apply the corrector the same number of times, but the former calls for one more function evaluation per step than the latter.

We now define the above modes precisely; It will turn out be and vantage us if the predictor and the corrector are separately of the same order, and this requirement may wall make it necessary for the step number of the predictor to be greater than that of the corrector. The notationally simplest may to deal with this contingency is to let both predictor and corrector have the same step number k, but in the case of the corrector, to release the condition, that not both α_0 and β_0 shall vanish.

Let the linear multistep method used as predictor be defined by the characteristic polynomials:

$$\left.\begin{aligned} \rho^*(\xi) &= \sum_{j=0}^{k} \alpha_j^* \xi^j, \ \alpha_k^* = 1 \\ \sigma^*(\xi) &= \sum_{j=0}^{k-1} \beta_j^* \xi^j \end{aligned}\right\} \quad \text{..(3.3)}$$

and that used as corrector by:

$$\left.\begin{aligned} \rho(\xi) &= \sum_{j=0}^{k} \alpha_j \xi^j, \ \alpha_k = 1 \\ \sigma(\xi) &= \sum_{j=0}^{k-1} \beta_j \xi^j \end{aligned}\right\} \quad \text{...(3.4)}$$

Then the modes $P(EC)^mE$ and $P(EC)^m$ just described above are formally defined as follows for m = 1, 2, …:

The $P(EC)^mE$ by:

$$\left.\begin{aligned} & y_{n+k}^{0} + \sum_{j=0}^{k-1} \alpha_j^* y_{n+j}^{m} = h \sum_{j=0}^{k-1} \beta_j^* f_{n+j}^{m}, \ f_{n+k}^{s} \equiv f(x_{n+k}, y_{n+k}^{s}) \\ & y_{n+k}^{s+1} + \sum_{j=0}^{k-1} \alpha_j y_{n+j}^{m} = h\beta_k f_{n+k}^{s} + h \sum_{j=0}^{k-1} \beta_j f_{n+j}^{m}, f_{n+k}^{m} \equiv f(x_{n+k}, y_{n+k}^{m}) \end{aligned}\right\} \text{...(3.5)}$$

for all s = 0, 1, …, m – 1.

The $P(EC)^m$ by:

$$\left.\begin{aligned} & y^{0}_{n+k} + \sum_{j=0}^{k-1} \alpha^{*}_{j} y^{m}_{n+j} = h \sum_{j=0}^{k-1} \beta^{*}_{j} f^{m}_{n+j}\,,\ f^{s}_{n+k} \equiv f(x_{n+k}, y^{s}_{n+k}) \\ & y^{s+1}_{n+k} + \sum_{j=0}^{k-1} \alpha_{j} y^{m}_{n+j} = h\beta_{k} f^{s}_{n+k} + h \sum_{j=0}^{k-1} \beta_{j} f^{m-1}_{n+j} \end{aligned}\right\} \quad \ldots\ldots\ldots(3.6)$$

for all $s = 0, 1, \ldots, m - 1$.

Note that as $m \longrightarrow \infty$, the results of computing with either of the above modes will tend to those given by the mode of correcting to convergence. In practice, it is unusual to use a mode for which m is greater than 2, [Al-Authab, 2005].

Example (3.2):

In this example, the method of predictor corrector method will be used to solve nonlinear fractional differential equations.

Consider the nonlinear ordinary fractional differential equation:

$$y^{(1/2)}(x) = -y^{2}(x) + x^{4} + \frac{2x^{3/2}}{\Gamma(5/2)},\ y(0) = 0$$

where the exact solution is given by $y(x) = x^{2}$.

Consider the following set of knot points:

$\Delta = \{0, 1, 2\}$

and let:

$e = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0)\}$

and to seek for $S_4(s) \in \mathcal{S}_4(E^*, \Delta)$, with:

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and for which:

$$S_4^{(j)}(i) = \phi^{(j)}(i),\ (i, j) \in e$$

Integrating $S_4(s)$ over [1, 2] yields the following closed formula:

$$y_{n+2} = y_{n+1} + [h\{\beta_{00}g(x_n, y_n) + \beta_{10}g(x_{n+1}, y_{n+1}) + \beta_{20}g(x_{n+2}, y_{n+2})\} + h^2\{\beta_{01}g'(x_n, y_n) + \beta_{11}g'(x_{n+1}, y_{n+1})\}]$$

where β_{00}, β_{10}, β_{20}, β_{01} and β_{20} are as in example (3.1), and

$$g(x, y) = D^{1/2}\left[-y^2 + x^4 + \frac{2x^{3/2}}{\Gamma(5/2)}\right]$$

$$= D^{1/2}[-y^2(x)] + D^{1/2}(x^4) + D^{1/2}\left[\frac{2x^{3/2}}{\Gamma(5/2)}\right]$$

The most difficulty in this stage is the evaluation of $D^{1/2}[-y^2(x)]$, so the chain rule formula will be used to find the result, so:

Let $f(y) = y$, $\varphi(u) = u^2$, where $\varphi(f) = y^2$

Now

$$\frac{d^q}{dx^q}\varphi(f) = \frac{x^{-q}}{\Gamma(1-q)}\varphi(f) + \sum_{j=1}^{k}\binom{q}{j}\frac{x^{j-q}}{\Gamma(j-q+1)}\left(j!\sum_{m=1}^{j}\varphi(m)\sum\prod_{k=1}^{j}\frac{1}{p^k}\frac{(f_k)^k}{k!}\right)$$

Hence:

$$\frac{d^{1/2}}{dx^{1/2}}\varphi(f) = \frac{x^{-1/2}}{\Gamma(1-1/2)}y^2 + \binom{1/2}{1}\frac{x^{1-\frac{1}{2}}}{\Gamma(1-\frac{1}{2}+1)}(\varphi' f') +$$
$$\binom{1/2}{2}\frac{x^{2-\frac{1}{2}}}{\Gamma(2-\frac{1}{2}+1)}(\varphi' f^{(2)} + \varphi^{(2)}(f')^2) +$$
$$\binom{1/2}{3}\frac{x^{3-\frac{1}{2}}}{\Gamma(3-\frac{1}{2}+1)}(\varphi' f^{(3)} + 3\varphi^{(2)}f^{(1)}f^{(2)} + \varphi^{(3)}(f')^3)$$

$$= \frac{x^{-1/2}}{\Gamma(1/2)}y^2 + \frac{1}{2}\frac{x^{\frac{1}{2}}}{\Gamma(3/2)}(2y) - \frac{1}{8}\frac{x^{3/2}}{\Gamma(5/2)}(2y\times 0 + 2\times 1) +$$
$$\frac{1}{16}\frac{x^{5/2}}{\Gamma(7/2)}(2y\times 0 + 3\times 2 + 0\times 1)$$

$$= \frac{x^{-1/2}}{\Gamma(1/2)}y^2 + \frac{x^{\frac{1}{2}}}{\Gamma(3/2)}y - \frac{1}{4}\frac{x^{3/2}}{\Gamma(5/2)}$$

Therefore:

$$D^{1/2}(y^2(x)) = \frac{x^{-1/2}}{\Gamma(1/2)}y^2 + \frac{x^{\frac{1}{2}}}{\Gamma(3/2)}y - \frac{1}{4}\frac{x^{3/2}}{\Gamma(5/2)}$$

and as a result:

$$g(x, y) = \frac{-x^{-1/2}}{\Gamma(1/2)}y^2 - \frac{x^{\frac{1}{2}}}{\Gamma(3/2)}y + \frac{1}{4}\frac{x^{3/2}}{\Gamma(5/2)} + \frac{24x^{7/2}}{\Gamma(9/2)} + 2x$$

Also:

$$g'(x, y) = \frac{-1}{\Gamma(1/2)}\left(2x^{-1/2}yg(x,y) - \frac{1}{2}x^{-3/2}y^2\right) -$$
$$\frac{1}{\Gamma(3/2)}\left(x^{1/2}g(x,y) + \frac{1}{2}x^{1/2}y\right) + \frac{3/2}{4\Gamma(5/2)}x^{1/2} +$$
$$\frac{24\times\frac{7}{2}}{\Gamma(9/2)}x^{5/2} + 2$$

The fundamental G-spline functions L_{00}, L_{01}, L_{10}, L_{11} and L_{20} are defined as in example (3.1).

The following results which are presented in table (3.2) are obtained using Mathcad computer software and its comparison with the exact solution.

Table (3.2)

The approximate and exact results of example (3.2).

x_i	y_i	$y(x_i)$	***Absolute error***
0	0	0	0
0.1	0.01	0.01	0
0.2	0.04	0.04	0
0.3	0.11	0.09	0.02
0.4	0.20	0.16	0.04
0.5	0.32	0.25	0.07
0.6	0.47	0.36	0.11
0.7	0.64	0.49	0.15
0.8	0.84	0.64	0.20
0.9	1.06	0.81	0.25
1	1.31	1	0.31

3.2.2 Newton-Raphson Method:

The problem that is considered here is:

$$y^{(\alpha)}(x) = f(x, y),\ y(x_0) = y_0,\ x \geq x_0$$

where f is nonlinear, x_0 and y_0 are given. The problem is so difficult to solve and hence upon applying any implicit rule, the resulting difference equation obtained upon using k-steps linear multistep method is nonlinear in y_{n+k}, i.e., the difference equation may take the form:

$$g(x_n, y_n, y_{n+1}, \ldots, y_{n+k}) = 0,\ y(x_0) = y_0 \quad \text{.......................................(3.7)}$$

which must be solved using the usual methods for solving nonlinear equations of one variable y_{n+k} at each stage, y_n, y_{n+1}, …, y_{n+k-1} are given from the previous stages of the numerical solution. It is remarkable here to recall that the considerable method used for solving the nonlinear difference equation is the Newton-Raphson method and then the numerical solution of eq. (3.7) will take the form:

$$y_{n+k}^{s+1} = y_{n+k}^{s} - \frac{g(x_n^s, y_n^s, \ldots, y_{n+k}^s)}{g'(x_n^s, y_n^s, \ldots, y_{n+k}^s)},\ s = 0, 1, \ldots$$

The next example illustrate the above method of solution:

Example (3.3):

In this example, we will discuss the results of using the implicit method for solving nonlinear fractional differential equations by using Newton-Raphson method.

Consider the nonlinear ordinary fractional differential equation:

$$y^{(1/2)}(x) = -y^2(x) + x^4 + \frac{2x^{3/2}}{\Gamma(5/2)},\ y(0) = 0$$

where the exact solution is given by $y(x) = x^2$.

Let $\Delta = \{0, 1, 2\}$ are taken to be the knot points and let:

$e = \{(0, 0), (0, 1), (1, 1), (2, 0)\}$

in order to seek for $S_4(s) \in \mathcal{S}_4(E^*, \Delta)$, with:

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and for which:

$$S_4^{(j)}(i) = \phi^{(j)}(i),\ (i, j) \in e$$

Integrating $S_4(s)$ over [1, 2] yields the closed formula:

$$y_{n+2} = y_{n+1} + [h\{\beta_{00}g(x_n, y_n) + \beta_{10}g(x_{n+1}, y_{n+1}) + \beta_{20}g(x_{n+2}, y_{n+2})\} + h^2\{\beta_{01}g'(x_n, y_n) + \beta_{11}g'(x_{n+1}, y_{n+1})\}]$$

where β_{00}, β_{10}, β_{20}, β_{01} and β_{20} are as in example (3.1), and

$$g(x, y) = D^{1/2}\left[-y^2 + x^4 + \frac{2x^{3/2}}{\Gamma(5/2)}\right]$$

$$= \frac{-x^{-1/2}}{\Gamma(1/2)}y^2 - \frac{x^{\frac{1}{2}}}{\Gamma(3/2)}y + \frac{1}{4}\frac{x^{3/2}}{\Gamma(5/2)} + \frac{24x^{7/2}}{\Gamma(9/2)} + 2x$$

Therefore:

$$g'(x, y) = \frac{-1}{\Gamma(1/2)}\left(2x^{-1/2}yg(x,y) - \frac{1}{2}x^{-3/2}y^2\right) - \frac{1}{\Gamma(3/2)}\left(x^{1/2}g(x,y) + \frac{1}{2}x^{1/2}y\right) + \frac{3/2}{4\Gamma(5/2)}x^{1/2} + \frac{24\times\frac{7}{2}}{\Gamma(9/2)}x^{5/2} + 2$$

with the fundamental G-spline functions as in example (3.1).

The following results and its comparison with the exact solution are obtained using the Mathcad computer software.

Table (3.3)

The approximate and exact results of example (3.3).

x_i	y_i	$y(x_i)$	***Absolute error***
0	0	0	0
0.1	0.01	0.01	0
0.2	0.04	0.04	0
0.3	0.12	0.09	0.03
0.4	0.22	0.16	0.06
0.5	0.35	0.25	0.10
0.6	0.49	0.36	0.13
0.7	0.65	0.49	0.16
0.8	0.85	0.64	0.21
0.9	1.07	0.81	0.26
1	1.33	1	0.33

CONCLUSIONS AND RECOMMENDATIONS

From the present study, the following conclusions may be drown:

1. Implicit LMM's has more accurate results than explicit LMM's in solving fractional differential equations.
2. Theory of approximation theory may be used effectively for solving several problems given in operator form.

Also, we can recommend the following problems for future work:

1. Using other interpolating functions of approximation theory in solving fractional differential equations with LMM's methods, such as cubic spline functions, B-spline functions, tensor spline, etc.
2. Using Runge-Kutta methods instead of LMM's for solving fractional differential equations.
3. Solving fractional partial differential equations using similar approach followed in this work.
4. Solving integral equations with fractional integrals using similar approach followed in this work.

CONTENTS

Dedicated to My Family

With all Love and Respects

INTRODUCTION

Fractional calculus is that field of mathematics of study which grows out of the traditional definitions of the calculus integral and derivative operators in which the same by fractional exponents is an outgrowth of exponents with integral value. Consider the physical meaning of the exponent. According to our primary school teacher exponents provides a short notation for what is essentially a repeated multiplication of numerical value. This concept in itself is easy to grasp and straight forward. However, this physical definitions can clearly become confused when considering exponents of non-integer value, [Loverro, 2004].

When the analytical solution of fractional differential equations goes for more complicated problems, the procedure of solution becomes more and more complicated till a very complex situation is encountered in some problems. Thus, am analytical solution may be impossible to evaluated and one should consult a numerical technique to solve the fractional differential equation, [Kalil, 2006].

Fractional differential equations are used to describe many phenomenas of physical interest. Fractional differential equations contain derivatives of non integer order; therefore there is no general agreement on how the numerical and approximated methods should be interpreted and used in solving differential equations in general, and fractional differential equations, in particular. Some see this analysis as the key word and wish to embed the subject entirely in rigorous modern analysis, others suggest that the numerical methods are the vital word and the algorithm is the only respectable yield. Numerical methods usually produces errors and we say that any numerical technique is a good one if the error approach quickly or rapidly goes to zero and the method requires a minimum computer capacity and less time consuming as possible, [Oldham, 1974].

In resent years, the theory of this class of equations had become an independent trend and the literatures on this subject comprise over 1000 titles.

Sometimes, numerical methods for solving ordinary differential equation are more reliable than analytic methods, especially in fractional differential equations, since such type of equations has some difficulties in their methods of solution, which could not be handled easily, [Burden, 1997], [Atkinson, 1989].

Although, approximation algorithms are used throughout the subject of science and in many industrial and commertial fields. Some of this theory has become highly specialized and abstract. Work in numerical analysis and in mathematical software packages is one of the main links between these two extremes, for purpose is to provide computer users with efficient programs for general approximation calculations, in order that useful advances in the subject can be applied, [Kalil, 2006].

In addition, spline functions have transformed approximation techniques and theory, because they are not only convenient and suitable for computer calculations, but also they provide optimal theoretical solutions to the estimation of functions from limited data, [Kalil, 2006].

Moreover, splines can be considered as mathematical models that associate a continuous representation of a curve or surface with a discrete set of points in a given space. Spline fitting is an extremely popular form of a piecewise approximation using various forms of polynomials of degree n or more general functions, on an interval in which they are fitting functions at specified points, known as control points or nodes or knots.

The polynomial used can change, but the derivatives of the polynomials are required to match up to degree $n - 1$ at each side of the knot, or to meet related interpolatory conditions. Boundary conditions, are also imposed on the end points of the intervals, [Mohammed, 2006].

It is near 60 years ago since I. J. Schoenberg introduced the subject of "spline functions" as a method for approximating functions which are so complicated or hard to be used in applications. Since then, splines have proved to be enormously important in various branches of mathematics, such as approximation theory, numerical analysis, numerical treatment of ordinary, integral and partial differential equations and statistics, etc., [Schoenberg, 1968].

The purpose of this work is to use an approximate solution of ordinary fractional differential equations using linear multi-step methods with the cooperation of G-spline interpolation functions.

This thesis consists of three chapters.

In chapter one entitled “Fundamental Concepts” presents the most important used concepts and notations related to this work, where this chapter consists of two sections. In section (1.1), the theory of G-spline interpolation functions is given, including the Hermit-Berkhoff problem and the construction of the G-spline functions, as well as, with some of the most important related theorems and illustrative examples. In section 1.2, the theory of fractional calculus has been summarized with some examples illustrating the fractional derivatives.

In chapter two, which is entitled “Solution of Fractional Differential Equations Using Explicit Linear Multistep Methods”, we give the G-spline interpolation for approximating the solution of fractional differential equations using linear multi-step methods. This chapter consists of two sections, section 2.1 is devoted to study theoretically the fractional differential equations, while section 2.2 presents the basic concepts of linear multistep methods in general and in particular the explicit linear multistep method for solving the fractional ordinary differential equations with an illustration examples.

Chapter three, entitled, “Numerical Solution of Fractional Differential Equations Using Implicit Linear Multistep Method” is devoted to improve the accuracy of the results by applying the implicit linear multistep methods for solving fractional ordinary differential equations. This chapter consists of two sections, in section 3.1 the implicit methods are explained for solving fractional ordinary differential equations while section 3.2 presents two approaches for solving implicitly nonlinear fractional ordinary differential equations either by predictor corrector methods or by using Newton-Raphson method for solving nonlinear algebraic equations. Such type of nonlinear problems has so many difficulties in the evaluation of the fractional derivatives.

The results are given either by a tabulated form or illustrated by figures with their comparison with the exact results in both cases, where the results are obtained using the computer software Mathcad professional 2001i.

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Solutions of Fractional Differential Equations Using G-Spline Interpolation Functions

A Thesis
Submitted to the College of Science of Al-Nahrain University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By
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Supervised by
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April 2008

جمهـورية العـراق
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حلول المعادلات التفاضلية الكسرية باستخدام دوال السبلاين-G

رسالة
مقدمة إلى كلية العلوم – جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

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نيسان ٢٠٠٨

بسم الله الرحمن الرحيم

قالُوا سُبحانَكَ لا عِلمَ لَنا إِلا ما عَلَمْتَنا إِنَّكَ أنتَ العَلِيمُ الحَكِيمُ

صدق الله العظيم

سورة البقرة

الآية (٣٢)

المستخلص

الهدف الرئيس لهذه الاطروحة هو دراسة الحلول العددية للمعادلات التفاضلية الاعتيادية الكسرية (ordinary fractional differential equations) باستخدام دوال السبلاين−G (G-spline functions) للاستكمال. طبق أسلوبين في الطرائق العددية، الاسلوب الاول هو باستخدام طرائق متعددة الخطوات الصريحة (explicit linear multistep methods) والتي يمكن استخدامها وبسهولة لحل المعادلات التفاضلية الخطية وغير الخطية بينما الاسلوب الثاني هو اسلوب مُحَّسن وهو استخدام طرائق متعددة الخطوات الضمنية (implicit linear multistep methods) لحل معادلات تفاضلية كسرية اعتيادية وغير خطية والتي من الصعب حلها بالطرق الاعتيادية. وقد تم ذلك باقتراح اسلوب جديد ألا وهو باستخدام قاعدة السلسلة (the chain rule) للمشتقات الكسرية.

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مفاتيح الكلمات: المعادلات التفاضلية الكسرية، نظرية التقريب، دوال التقريب–G

المستخلص

الهدف الرئيس لهذه الاطروحة هو دراسة الحلول العددية للمعادلات التفاضلية الاعتديادية الكسرية (ordinary fractional differential equations) باستخدام دوال السبلاين–G (G- spline functions) للاستكمال. طبق أسلوبين في الطرائق العددية، الاسلوب الاول هو باستخدام طرائق متعددة الخطوات الصريحة (explicit linear multistep methods) والتي يمكن استخدامها وبسهولة لحل المعادلات التفاضلية الخطية وغير الخطية بينما الاسلوب الثاني هو اسلوب مُحَّسن وهو استخدام طرائق متعددة الخطوات الضمنية (implicit linear multistep methods) لحل معادلات تفاضلية كسرية اعتيادية وغير خطية والتي من الصعب حلها بالطرق الاعتيادية. وقد تم ذلك باقتراح اسلوب جديد ألا وهو باستخدام قاعدة السلسلة (the chain rule) للمشتقات الكسرية.

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ABSTRACT

The main objective of this work is to study the numerical solution of fractional ordinary differential equations using G-spline interpolation functions. Two numerical approaches are used, the first approach utilize the explicit linear multistep methods which can be applied easily for linear and nonlinear problems while the second approach is a modified approach by using the implicit linear multistep methods for solving nonlinear fractional ordinary differential equations which has so many difficulties in their solution. This is done by suggesting a new criterion by using the chain rule derivatives of fractional order.