ABSTRACT

The Fractional Order Bounded Delay Differential Equations (FOBDDE's) has been studied in this work. The Existence and Uniqueness theorems of such type of differential equation have been proved, by using the successive approximation techniques. Also, the analytic solution of (FOBDDE's) are presented, using Laplace Transformation, and the numerical solutions are discussed, using general one-step methods and linear multi-step methods. The comparison, among these methods and the exact solutions are presented.

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Fager Adnan AbdulKhalik

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Supervisor Certification

I certify that this thesis was prepared under my supervision at Al-Nahrain University, College of Science, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics and Computer Applications.

Signature:

Name: Assist. Prof. Dr. Alaudeen N. Ahmed

Data: / / 2008

In view of the available recommendations, I forward this thesis for debate by the examining committee.

Signature:

Name: Assist. Prof. Dr. Akram M.Al-Abood

Head of the Department

Data: / / 2008

Examining Committee Certification

We certify that we have read this thesis entitled *"Solutions of Fractional Order Delay Differential Equations"* and as examining committee examined the student *(Fager Adnan AbdulKhalik)* in its contents and in what it connected with, and that is in our opinion it meets the standards of a thesis for the degree of Master of Science in Mathematics.

(Chairman)

Signature:

Name: Akram M. Al-Abood

Assist. Prof.

Date: / / 2008

(Member)

Signature:

Name: Dr. Saheb K. Jasim

Assist. Prof.

Date: / / 2008

(Member)

Signature:

Name: Dr. Zainab A. Salman

Lecturer

Date: / / 2008

(Member and Supervisor)

Signature:

Name: Dr. Alaudeen N. Ahmed

Assist. Prof.

Date: / / 2008

Approved by the Collage of Science

Signature

Name: Assist Prof. Dr. Laith Abdul Aziz Al-Ani

Dean of the Collage of Science

Data: / / 2008

CHAPTER ONE

FUNDAMENTAL CONCEPTS TO DELAY AND FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

In this chapter we shall deals with the basic of two kinds of differential equations known as delay and fractional order differential equation which are useful and have commonly used in the subject of this work.

1.1 DELAY DIFFERENTIAL EQUATION

Definition (1), [Bellman, 1963]:

Involve the delay differential equation “DDE” is defined as an unknown function y(t) and some of its derivatives, evaluated at arguments that differ by any of fixed number of values $\tau_{1,}\tau_2,\ldots,\tau_k$. The general form of the n-th order DDE is given by

$$F(t, y(t), y(t-\tau_1),\ldots, y(t-\tau_k), y'(t), y'(t-\tau_1),\ldots, y'(t-\tau_k),\ldots, y^{(n)}(t),$$
$$y^{(n)}(t-\tau_1),\ldots, y^{(n)}(t-\tau_k)) = 0 \quad \ldots\ldots(1.1)$$

where F is a given functional and τ_i, $\forall$ i = 1, 2, …, k; are given fixed positive number called the “time delay” [Bellman, 1963].

In some literature equation (1.1) is called a difference-differential equation or functional differential equation, [Bellman, 1963], or an equation with time lag [Halanay, 1966], or a differential equation with deviating arguments, [Driver, 1977].

The emphasis will be, in general, on the linear equations with constant coefficients of the first order and with one delay (because as in "ODE" any differential equation with higher order than one may be transformed into a linear system of differential equations of the first order)

For example:

$$a_0 y'(t) + a_1 y'(t-\tau) + b_0 y(t) + b_1 y(t-\tau) = f(t) \quad \ldots\ldots(1.2)$$

where f(t) is a given continuous function and τ is a positive constant and a_0, a_1, b_0, b_1 are constants (also if f(t) = 0, then equation (1.2) is said to be homogenous; otherwise it is non homogenous).

The kind of initial conditions that should be used in DDE's differ from ODE's so that one should specify in DDE's an initial function on some interval of length τ, say $[t_0 - \tau, t_0]$ and then try to find the solution of equation (1.2) for all $t \geq t_0$. Thus, we set y (t) = $\varphi_0(t)$, for $t_0 - \tau \leq t \leq t_0$ where $\varphi_0(t)$ is some given continuous function. Therefore the solution of DDE consist of finding a continuous extension of $\varphi_0(t)$ into a function y(t) which satisfies (1.2) for all $t \geq t_0$, [Halanay, 1966].

Delay differential equation given by equation (1.2) can be classified into three types which are retarded, neutral and mixed. The first type means an equation where the rate of change of state variable y is determined by the present and past states of the equation, for example equation (1.2) where the coefficient of $y'(t-\tau)$ is zero, i.e., ($a_0 \neq 0$, $a_1 = 0$). If the rate of change of state depends on its own past values as well on its derivatives, the equation is then of neutral type, for example equation (1.2) where the coefficient of $y(t-\tau)$ is zero, i.e.,($a_0 \neq 0$, $a_1 \neq 0$ and $b_1 = 0$), while the third type is a combination of the previous two types, i.e., ($a_0 \neq 0$, $a_1 \neq 0$, $b_0 \neq 0$ and $b_1 \neq 0$), [Al-Saady, 2000].

It is important to remark that, the theory of neutral differential equation is more complicated than of retarded type, [El'sgolt'c, 1973], [Al-Saady, 2000].

1.1.1 Solution of the First Order Delay Differential Equations, [Driver, 1977]:

Because of the initial condition which is given for a time step interval with length equals to τ, we must find this solution for $t \geq t_0$ divided into steps with length τ also.

1.1.1.1 The Method of Successive Integrations, [Driver, 1977]:

The best well known method for solving DDE's is the method of steps or the method of successive integrations which is used to solve a DDE of the form:

$$y'(t) = f(t, y(t), y(t-\tau), y'(t-\tau)), \ t \geq t_0 \quad \text{...(1.3)}$$

with initial condition $y(t) = \varphi_0(t)$, for $t_0 - \tau \leq t \leq t_0$.

For such equations the solution is constructed step by step as follows:

Given that a function $\varphi_0(t)$ continuous on $[t_0 - \tau, t_0]$, therefore one can obtain the solution in the next step interval $[t_0, t_0 + \tau]$ by solving the following equation:

$$y'(t) = f(t, y(t), \varphi_0(t-\tau), \varphi'_0(t-\tau)), \text{ for } t_0 \leq t \leq t_0+\tau$$

with the initial condition $y(t_0) = \varphi_0(t_0)$. If we consider that $\varphi_1(t)$ is the desired first step solution, which exists by virtue of continuity hypotheses.

Similarly, if $\varphi_1(t)$ is defined on the whole segment $[t_0, t_0+\tau]$ then, one can find the solution $\varphi_2(t)$ to the equation:

$$y'(t) = f(t, y(t), \varphi_1(t-\tau), \varphi'_1(t-\tau)), \text{ for } t_0+\tau \le t \le t_0 + 2\tau$$

with the initial condition $y(t_0 + \tau) = \varphi_1(t_0 + \tau)$

In general, by assuming that $\varphi_{k-1}(t)$, $\forall(k = 1,2,...)$ is defined on the interval $[t_0+ (k-2)\tau, t_0+ (k-1)\tau]$, then, one can find the solution $\varphi_k(t)$ to the equation:

$$y'(t) = f(t, y(t), \varphi_{k-1}(t-\tau), \varphi'_{k-1}(t-\tau)), \text{ for } t_0+(k-1)\tau \le t \le t_0 + k\tau$$

with the initial condition $y(t_0 + (k-1)\tau) = \varphi_{k-1}(t_0 + (k-1)\tau)$.

Now, we shall consider some illustrative examples for all types of DDE:

<u>Example (1.1.1), [Al-Saady, 2000]:</u>

Consider the retarded first order DDE:

$$y'(t) = y(t-1), \ t \ge 0$$

with the initial condition

$$y(t) = \varphi_0(t) = t, \text{ for } -1 \le t \le 0$$

To find the solution in the first step interval [0, 1] we have to solve the following equation:

$$y'(t) = \varphi_0(t-1)$$

$$= t - 1, \text{ for } 0 \le t \le 1$$

Integrating both sides from 0 to t where $0 \le t \le 1$, we have:

$$\int_0^t y'(s)ds = \int_0^t (s-1)ds$$

And hence after carrying some calculations we get the first time step solution:

$$y(t) = \frac{t^2}{2} - t\,,\ \text{for } 0 \le t \le 1$$

In order to find the solution in the second step interval, suppose that:

$$\varphi_1(t) = y_1\,(t) = \frac{t^2}{2} - t\,,\ 0 \le t \le 1$$

Since $\varphi_1(t)$ is defined on the whole segment [0, 1].

Hence by forming the new equation:

$$y'(t) = \varphi_1(t-1),\ \text{for } 0 \le t \le 1 \quad \text{..........}(1.4)$$

with the initial condition $\varphi_1(t) = \frac{t^2}{2} - t$, for $0 \le t \le 1$

One can find the solution in the next step interval [1, 2], and we shall solve equation (1.4)

$$y'(t) = \varphi_1(t-1),\ \text{for } 1 \le t \le 2$$

$$= \frac{(t-1)^2}{2} - (t-1)$$

$$= \frac{t^2}{2} - t + \frac{1}{2} - t + 1$$

$$= \frac{t^2}{2} - 2t + \frac{3}{2},\ \text{for } 1 \le t \le 2$$

Integrating both sides from 1 to t, where $t \in [1, 2]$, we get:

$$y(t) = -\frac{7}{6} + \frac{t^3}{6} - t^2 + \frac{3}{2}t, \text{ for } 1 \le t \le 2$$

Similarly, let:

$$y_2(t) = \frac{-7}{6} + \frac{t^3}{6} - t^2 + \frac{3}{2}t$$

and suppose $\varphi_2(t)$ is the desired second step solution, i.e.,

$$\varphi_2(t) = y_2(t) = \frac{-7}{6} + \frac{t^3}{6} - t^2 + \frac{3}{2}t$$

Since $\varphi_2(t)$ is defined on the whole segment [1, 2] hence by forming the new equation:

$$y'(t) = \varphi_2(t-1), \text{ for } 2 \le t \le 3$$

with the initial condition

$$\varphi_2(t) = \frac{-7}{6} + \frac{t^3}{6} - t^2 + \frac{3}{2}t$$

Similarly, one can find $y_3(t)$, $y_4(t)$ and so on.

Example (1.1.2), [Al-Saady, 2000]:

Consider the neutral first order DDE:

$$y'(t) = y'(t-1) + t, \ t \ge 0$$

with initial condition

$$\varphi_0(t) = t + 1, \text{ for } -1 \le t \le 0$$

To find solution of the first interval [0, 1]. We solve the following:

$$y'(t) = \varphi_0'(t-1) + t\text{, for } -1 \le t \le 0$$

$$= 1 + t\text{, for } -1 \le t \le 0$$

Integrating both sides from 0 to t where $0 \le t \le 1$, we have:

$$\int_0^t y'(s)ds = \int_0^t (1+s)ds$$

and hence:

$$y_1(t) = t + \frac{t^2}{2} + 1\text{, for } 0 \le t \le 1$$

In order to find the solution in the second step interval suppose that:

$$\varphi_1(t) = y_1(t) = t + \frac{t^2}{2} + 1$$

is the initial condition. Since $\varphi_1(t)$ is defined on the whole segment [0, 1]. Hence by forming the new equation:

$$y'(t) = \varphi_1'(t-1) + t\text{, for } 1 \le t \le 2 \quad \text{...(1.5)}$$

where $\varphi_1(t) = t + \frac{t^2}{2} + 1$, for $0 \le t \le 1$

One can find the solution in the next step interval [1, 2], and solving equation (1.5) for y (t), we have:

$$y'(t) = \varphi_1'(t-1) + t$$

$$= 2t\text{, for } 1 \le t \le 2$$

Integrating both sides from 1 to t where $1 \le t \le 2$, we get:

$$y(t) = t^2 + \frac{3}{2}\text{, for } 1 \le t \le 2$$

Therefore, y(t) is the desired second step solution which is denoted by

$$y(t) = \varphi_2(t) = t^2 + \frac{3}{2}, \text{ for } 1 \le t \le 2$$

Similarly, we proceed to the next intervals.

<u>Example (1.1.3), [Al-Saady, 2000]:</u>

Consider the mixed DDE:

$$y'(t) = y(t-1) + 2y'(t-1), \ t \ge 1$$

with initial condition

$$\varphi_0(t) = 1, \text{ for } 0 \le t \le 1$$

To find the solution in the first step interval [1, 2], we will to solve the following equation:

$$y'(t) = \varphi_0(t-1) + 2\varphi_0'(t-1), \text{ for } 1 \le t \le 2$$

$$y'(t) = 1$$

By integrating from 1 to t, where $1 \le t \le 2$, we have:

$$y(t) = t, \text{ for } 1 \le t \le 2$$

and suppose that $\varphi_1(t)$ is the desired first step solution

$$y_1(t) = \varphi_1(t) = t, \text{ for } 1 \le t \le 2$$

Since $\varphi_1(t)$ is defined on the whole segment [1, 2], hence by forming the new equation:

$$y'(t) = \varphi_1(t-1) + 2\varphi_1'(t-1), \text{ for } 2 \le t \le 3$$

with initial condition

$y_1(t) = \varphi_1(t) = t$, for $1 \le t \le 2$

and so on, we proceed to the next intervals.

The next example considers the solution of DDE with variable delay which can be solved by successive integration method.

Example (1.1.4):

Consider the retarded first order DDE

$$y'(t) = -y(t - e^t), \text{ for } 0 \le t \le 1$$

with initial condition:

$$y(t) = \varphi_0(t) = 1, \text{ for } -1 \le t \le 0$$

To find the solution in the first step interval [0, 1] we have to solve the following equation:

$$y'(t) = -\varphi_0(t - e^t)$$

$$= -1, \text{ for } 0 \le t \le 1$$

Integrating both sides from 0 to t where $0 \le t \le 1$, we have:

$$\int_0^t y'(s)ds = \int_0^t -ds$$

Hence

$$y(t) = 1 - t, \text{ for } 0 \le t \le 1$$

In order to find the solution in the second step interval suppose that:

$$\varphi_1(t) = y_1(t) = 1 - t$$

Therefore:

$$y_1(t) = 1 - t, \text{ for } 0 \le t \le 1$$

Since $\varphi_1(t)$ is defined on the whole segment [0, 1].

Hence by forming the new equation:

$$y'(t) = -\varphi_1(t - e^t)$$

$$= -1 + (t - e^t)$$

Integrating both sides from 1 to t, where $t \in [1, 2]$, yields:

$$y(t) = 3.2 - t + \frac{t^2}{2} - e^t, \text{ for } 1 \le t \le 2$$

Similarly, let:

$$y_2(t) = 3.2 - t + \frac{t^2}{2} - e^t, \text{ for } 1 \le t \le 2$$

and suppose $\varphi_2(t)$ is the desired second step solution, i.e.,

$$\varphi_2(t) = y_2(t)$$

$$= 3.2 - t + \frac{t^2}{2} - e^t, \text{ for } 1 \le t \le 2$$

Since $\varphi_2(t)$ is defined on the whole segment [1, 2], hence by forming the new equation:

$$y'(t) = -\varphi_2(t - e^t), \text{ for } 2 \le t \le 3$$

with initial condition

$$\varphi_2(t) = 3.2 - t + \frac{t^2}{2} - e^t, \text{ for } 1 \le t \le 2$$

similarly, one can find $y_3(t)$, $y_4(t)$ and so on.

1.1.1.2 Laplace Transformation Method, [Ross, 1984]:

Laplace transformation method is also, one of the most widely use methods for solving DDE's. It is important here to review the Laplace transformation of a given function.

Suppose that f is a real-valued function of the real variable x defined for x > 0. Let s be a parameter that we shall assume to be real, and consider the function F defined by

$$\mathcal{L}\{f\} = \int_0^\infty e^{-sx} f(x)dx \qquad \text{.......(1.7)}$$

For all values of s for which this integral exists. The function $\mathcal{L}\{f\}$ defined by the integral (1.7) is called the Laplace transformation of the function f and we shall denote the Laplace transform $\mathcal{L}\{f\}$ of f by F(s).

Also, as it is known, Laplace transformation method may be used to solve linear ODE's and we can use it also to solve DDE by two approaches. The first approach is by mixing between method of steps and Laplace transform method and the other approach is by applying directly the Laplace transform method to the original DDE.

1.1.1.2.1 The First Approach, [Brauer, 1973]:

This approach depends mainly on applying first the method of steps to transform the DDE into ODE and then applying Laplace transformation method to solve the resulting equation. This approach can be explained in the following examples:

Example (1.1.5), [Marie, 2001]:

Consider the following neutral DDE:

$$y'(t) = y'(t-1) + t \text{, for } 0 \le t \le 1$$

with initial condition:

$$y(t) = \varphi_0(t) = t+1 \text{, for } -1 \le t \le 0$$

To find the solution in the first step interval [0, 1], we apply the method of steps, to get:

$$y'(t) = \varphi_0'(t-1) + t$$

$$= 1 + t \text{, for } 0 \le t \le 1$$

which is an ODE of the first order

Now, taking the Laplace transformation approach:

$$\mathcal{L}\{ y'(t) \} = \mathcal{L}\{1\} + \mathcal{L}\{t\}$$

$$sY(s) - y(0) = \frac{1}{s} + \frac{1}{s^2}$$

and so the Laplace transform of the solution y(t) into Y(s) is given by:

$$Y(s) = \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s}$$

Taking inverse Laplace transform, we have:

$$y(t) = \mathcal{L}^{-1}\left[\frac{1!}{s^2}\right] + \frac{1}{2!}\mathcal{L}^{-1}\left[\frac{2!}{s^3}\right] + \mathcal{L}^{-1}\left[\frac{1}{s}\right]$$

$$y(t) = t + \frac{t^2}{2} + 1 \text{, for } 0 \le t \le 1$$

Hence, the solution in the first step interval is given by:

$$y(t) = \varphi_1(t) = t + \frac{t^2}{2} + 1, \text{ for } 0 \le t \le 1$$

In order to find the solution in the second step interval [1, 2], we proceed similarly as in the first step with initial condition

$$\varphi_1(t) = t + \frac{t^2}{2} + 1, \text{ for } 0 \le t \le 1$$

and hence:

$$y'(t) = \varphi_1'(t-1) + t, \text{ for } 0 \le t \le 1$$

with the equivalent ODE $y'(t) = 2t, \text{ for } 1 \le t \le$ ٢ with initial condition, $y(1) = \frac{5}{2}$

By making changing independent variable $w = t - 1$ then $w \in [0, 1]$, so that

$$y'(w+1) = 2(w+1),\ y(1-1) = \frac{5}{2}$$

and by considering:

$$z(w) = y(w + 1)$$

Implies that:

$$z'(w) - 2(w+1) = 0, \text{ with } z(0) = \frac{5}{2},\ w \in [0, 1]$$

Taking the Laplace transform of both sides, we have:

$$sZ(s) - z(0) = \frac{2}{s^2} + \frac{2}{s}$$

where Z(s) is the Laplace transform of z(w) hence:

$$Z(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{5}{2s}$$

Taking inverse Laplace, we have:

$$z(w) = w^2 + 2w + \frac{5}{2}$$

Hence the solution in the second step interval [1, 2] is given by:

$$z(w) = y(t) = (t-1)^2 + 2(t-1) + \frac{5}{2}$$

Similarly, we proceed to the next intervals.

Similarly, as in subsection (1.1.2.1) we can use Laplace transformation method to solve DDE with variable delay:

Example (1.1.6):

Consider the following DDE:

$$y'(t) = y'(t - e^t) + t, \text{ for } 0 \le t \le 1$$

with initial condition

$$y(t) = \varphi_0(t) = t+1, \text{ for } -1 \le t \le 0$$

To find the solution in the first step interval [0, 1], we apply the method of steps, to get:

$$y'(t) = \varphi_0'(t - e^t) + t$$

$$= 1 + t - e^t, \text{ for } 0 \le t \le 1$$

and this is an ODE of the first order.

Now, taking the Laplace transform produces:

$$\mathcal{L}\{ y'(t) \} = \mathcal{L}\{1\} + \mathcal{L}\{t\} - \mathcal{L}\{e^t\}$$

$$sY(s) - y(0) = \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s-1}$$

and so the Laplace transform of the solution y(t) into Y(s) is given by:

$$Y(s) = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s(s-1)}$$

Taking inverse Laplace transform, we have:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1!}{s^2}\right\} + \frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\}$$

$$y(t) = 2 + t + \frac{t^2}{2} - e^t, \text{ for } 0 \le t \le 1$$

1.1.1.2.2 Second Approach, [Brauer, 1973]:

This approach is to solve DDE's by using Laplace transform method directly without using the method of steps. Laplace transformation method is extremely useful in obtaining the solution of the linear DDE's with constant coefficients. Let us illustrative this method by considering the following example:

<u>Example (1.1.7):</u>

Consider the following DDE:

$$u'(t) = u(t-1)$$

with initial condition:

$$u(t) = \varphi_0(t) = t, \text{ for } -1 \le t \le 0$$

such that u (0) = 0, $u'(0) = 1$

Applying the Laplace transform method to both sides of the equation, we get:

$$sU(s) = \int_0^\infty u(t-1)e^{-st}dt$$

Using the transform z = t − 1, yields:

$$\int_0^\infty u(t-1)e^{-st}dt = \int_{-1}^\infty u(z)e^{-s(z+1)}dz$$

$$= e^{-s}\int_{-1}^0 u(z)e^{-sz}dz + e^{-s}\int_0^\infty u(z)e^{-sz}dz$$

$$= e^{-s}\int_{-1}^0 (z)e^{-sz}dz + e^{-s}\int_0^\infty u(z)^{-sz}dz$$

Since u(z) = z , for −1 ≤ z ≤ 0.

Finally:

$$U(s) = \left[\frac{-1}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]\left[\frac{1}{s-e^{-s}}\right] \quad \text{...(1.8)}$$

From equation (1.8), it follows that:

$$U(s) = \left[\frac{-1}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]\left[\frac{1}{s-e^{-s}}\right]$$

and upon taking the inverse Laplace one can find the solution u(t), where it is so difficult to obtain, which in force us to prefer using the numerical method.

Now, after we started the basic needed about delay differential equations we shall start the next section, with another important type of differential equation, which is so called fractional order differential equations abbreviated by FODE's.

١,٢ FRACTIONAL CALCULUS

1.2.1 Basic Concepts:

Understanding of definitions and use of fractional calculus will be made more clear by quickly discussing some necessary but relatively simple mathematical definitions that will arise in the study of these concepts.

1.2.1.1 The Gamma Function:

The gamma function is intrinsically tied to fractional calculus by definition. The simplest interpretation of the gamma function is simply the generalization of the factorial for all real numbers.

Also, gamma function $\Gamma(z)$ plays an important role in the theory of differintegration the term "differintegration" means derivative or integral to arbitrary order. The definition of the gamma function is given by

$$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du, \text{ for all } z > 0 \qquad (1.9)$$

and integrating by parts, yields:

$$\Gamma(z+1) = z\Gamma(z), \text{ when } z \in \mathbb{R}$$

If z = n, where n is a positive integer, then:

(i) $\Gamma(n + 1) = n!$.

(ii) $\Gamma\left(z+\frac{1}{2}\right) = \frac{(2z-1)!}{2^n}\Gamma\left(\frac{1}{2}\right)$, where $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

1.2.1.2 Definitions:

Riemann's modified the Cauchy's formula for an n-fold integral of a function f, to get its own definition of fractional integral operator:

$$\int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} f(x_n)dx_n dx_{n-1} \cdots dx_1 = \frac{1}{(n-1)!}\int_a^x \frac{f(t)}{(x-t)^{1-n}}dt \;\;............(1.10)$$

By n-fold here means that the integration is deployed n-times. Since $(n - 1)! = \Gamma(n)$, Riemann realized that the right hand side of (1.10) might have meaning even when *n* takes non-integer values, [Samko, 1993].

Thus perhaps it was natural to define fractional integration as follows:

Definition (1), [Gorenflo, 1997]:

Let $f(t) \in L_1[a, b]$, $q \in \mathbb{R}^{+}$. The fractional (arbitrary) order integral of the function f(t) of order q is defined as:

$$I_a^q f(t) = \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(s)ds \;\;..(1.11)$$

when $a = 0$ we can write, $I^q f(t) = I_0^q f(t) = f(t) * \varphi_q(t)$, where:

$$\varphi_q(t) = \begin{cases} \dfrac{t^{q-1}}{\Gamma(q)}, & \text{for} \quad t > 0 \\ 0 \quad\quad, & \text{for} \quad t \le 0 \end{cases}$$

and "*" denoted the convolution operator.

Definition (3), [Gorenflo, 1997]:

The fractional derivative D^q of order $q \in (0, 1]$ of the absolutely continuous function f(t) is defined as:

$$D_a^q f(t) = \frac{d}{dt} I_a^{1-q} f(t), t \in [a, b] \qquad (1.12)$$

1.2.1.3 Riemann-Liouville Fractional Integrals and Fractional Derivatives, [Kilbas, 2006]:

We give the definitions of the Riemann-Liouville fractional integrals and fractional derivatives on a finite interval of the real line and present some of their properties in spaces of continuous functions.

Let $\Omega = [a, b]$ be a finite interval. The Riemann-Liouville fractional integrals $I_a^q f$ and I_b^q of order $q \in \mathbb{C}$ $(\mathrm{Re}(q) > 0)$, are defined by:

$$(I_a^q f)(x) = \frac{1}{\Gamma(q)} \int_a^x \frac{f(t)dt}{(x-t)^{1-q}}, (x > a, \mathrm{Re}(q) > 0) \qquad (1.13)$$

and

$$(I_b^q f)(x) = \frac{1}{\Gamma(q)} \int_x^b \frac{f(t)dt}{(t-x)^{1-q}}, (x < b, \mathrm{Re}(q) > 0) \qquad (1.14)$$

These integrals are called the left-sided and the right-sided fractional integrals.

The Riemann-Liouville fractional derivatives $D_a^q y$ and $D_b^q y$ of order $q \in \mathbb{C}$ $(\mathrm{Re}(q) > 0)$ are defined by:

$$({}_aD_x^q y)(x) = \frac{d^n}{dx^n}(I_a^{n-q}y)(x)$$

$$= \frac{1}{\Gamma(n-q)}\frac{d^n}{dx^n}\int_a^x \frac{y(t)dt}{(x-t)^{q-n+1}}, (n = [\mathrm{Re}(q)]+1, x > a, n-1 < q \le n)$$

.....................................(1.15)

and

$$({}_xD_b^q y)(x) = (-1)^n \frac{d^n}{dx^n}(I_a^{n-q}y)(x)$$

$$= \frac{1}{\Gamma(n-q)}(-1)^n \frac{d^n}{dx^n}\int_x^b \frac{y(t)dt}{(x-t)^{q-n+1}}, (n = [\mathrm{Re}(q)]+1, x < b)$$

.....................................(1.16)

where [Re(q)] means the integer part of Re (q).

In particular, when $q = n \in \mathbb{N}$, then:

$$(D_a^0 y)(x) = (D_b^0 y)(x) = y(x),\ \ (D_a^n y)(x) = y^{(n)}(x),$$

and;

$$(D_b^n y)(x) = (-1)^n y^{(n)}(x), \quad (n \in \mathbb{N}) \quad \text{...(1.17)}$$

where $y^{(n)}(x)$ is the usual derivative of y(x) of order n.

1.2.1.4 Properties, [Kilbas, 2006]:

In this subsection, we are presenting an important property of fractional derivatives:

Let Re $(q) \geq 0$, $m \in$. If the fractional derivatives $(D_a^q y)(x)$ and $(D_a^{q+m} y)(x)$ exist, then:

$$(D^m D_a^q y)(x) = (D_a^{q+m} y)x \quad \text{.......}(1.18)$$

If the fractional derivatives $(D_b^q y)(x)$ and $(D_b^{q+m} y)(x)$ exist, then:

$$(D^m D_b^q y)(x) = (-1)^m (D_b^{q+m} y)x \quad \text{.......}(1.19)$$

Now, some additional important properties of the fractional order differential operator D_t^q are presented for completeness purpose, [Oldham, 1974]:

1. The operator D_t^q of order q = 0 is the identity operator.
2. The operator D_t^q is linear, i.e.,

   $$D_t^q (c_1 f(t) + c_2 g(t)) = c_1 D_t^q f(t) + c_2 D_t^q g(t)$$

   where c_1 and c_2 are constants.
3. $D_t^q \sum_{i=1}^{n} f_i(t) = \sum_{i=1}^{n} D_t^q f_i(t)$.
4. (i) $D_t^q (1) = \dfrac{1}{\Gamma(1-q)} t^{-q}$.

   (ii) $D_t^q (c) = \dfrac{c}{\Gamma(1-q)} t^{-q}$.

   (iii) $D_t^q (t^p) = \dfrac{\Gamma(p+1)}{\Gamma(p-q+1)} t^{p-q}$.

1.2.2 Analytic Methods for Solving Fractional Order Differential Equations, [Oldham, 1974]:

In this present subsection, some analytical methods are proposed for solving fractional order differential equations, and among such methods:

1.2.2.1 Inverse operator method:

Let f be an unknown function and let q be an arbitrary real number, F is known function, then we can construct the simplest of all fractional order differential equations by:

$$\frac{d^q f}{dx^q} = F \qquad (1.20)$$

hence upon taking the inverse operator $\frac{d^{-q}}{dx^{-q}}$, gives:

$$f = \frac{d^{-q}F}{dx^{-q}}$$

where it is clear that it is not always the case that they are equal, but this is not the most general solution, [Oldham, 1974]:

$$f - \frac{d^{-q}}{dx^{-q}}\frac{d^q}{dx^q}f = 0 \qquad (1.21)$$

additional terms must be added to equation (1.21), which are $c_1x^{q-1}, c_2x^{q-2}, \ldots, c_mx^{q-m}$ and hence:

$$f - \frac{d^{-q}}{dx^{-q}}\frac{d^q}{dx^q}f = c_1x^{q-1} + c_2x^{q-2} + \ldots + c_mx^{q-m}$$

where $c_1, c_2, \ldots, c_m$ are an arbitrary constants to be determined from the initial conditions and $q \leq m < q + 1$.

Thus:

$$f - c_1 x^{q-1} - c_2 x^{q-2} - \ldots - c_m x^{q-m} = \frac{d^{-q}}{dx^{-q}} \frac{d^q}{dx^q} f = \frac{d^{-q}}{dx^{-q}} F$$

Hence, the most general solution of eq. (1.20) is given by:

$$f = \frac{d^{-q}}{dx^{-q}} F + c_1 x^{q-1} + c_2 x^{q-2} + \ldots + c_m x^{q-m}$$

where $0 < q \leq m < q + 1$.

As an illustration, we consider the following example:

<u>Example (1.2.1), [Oldham, 1974]:</u>

Consider the fractional order differential equation:

$$\frac{d^{\frac{3}{2}} f(x)}{dx^{\frac{3}{2}}} = x^5 \qquad \text{.......(1.22)}$$

with the initial condition:

$$\frac{d^{\frac{1}{2}} f(0)}{dx^{\frac{1}{2}}} = k_0$$

Applying $\frac{d^{-\frac{3}{2}}}{dx^{-\frac{3}{2}}}$ to both sides of equation (1.22), we get:

$$f(x)=\frac{d^{-\frac{3}{2}}x^5}{dx^{-\frac{3}{2}}}+c_1x^{\frac{1}{2}}$$

and from the initial condition, we have $c_1=\frac{k_0}{\Gamma\left(\frac{3}{2}\right)}$, therefore:

$$f(x)=\frac{\Gamma(6)x^{\frac{13}{2}}}{\Gamma(\frac{15}{2})}+\frac{k_0x^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}$$

1.2.2. ٢. Laplace transform method, [Oldham, 1974]:

In this subsection, we seek a Laplace transform of d^qf/dx^q for all q and differintegrable f, i.e., we wish to relate:

$$L\left\{\frac{d^qf}{dx^q}\right\}=\int_0^\infty \exp(-sx)\frac{d^qf}{dx^q}dx$$

to the Laplace transform $\ell\{f\}$ of the differintegrable function. Let us first recall the well-known transforms on integer-order derivatives:

$$L\left\{\frac{d^qf}{dx^q}\right\}=s^qL\{f\}-\sum_{k=0}^{q-1}s^{q-1-k}\frac{d^kf(0)}{dx^k},q=1,2,...$$

and multiple integrals:

$$L\left\{\frac{d^qf}{dx^q}\right\}=s^qL\{f\},q=0,-1,... \quad ....(1.23)$$

and note that both formulas are embraced by:

$$L\left\{\frac{d^q f}{dx^q}\right\}=s^q L\{f\}-\sum_{k=0}^{q-1}s^k\frac{d^{q-1-k}f(0)}{dx^{q-1-k}},q=0,\mp1,..., \quad ....(1.24)$$

Also, formula (1.24), can be generalized to include non integer q ∈ □ , as:

$$L\left\{\frac{d^q f}{dx^q}\right\}=s^q L\{f\}-\sum_{k=0}^{n-1}s^k\frac{d^{q-1-k}f(0)}{dx^{q-1-k}}, \text{ for all } q \ ....(1.25)$$

where n is integer such than $n-1<q\le n$. The sum vanishes when $q\le 0$. In proving (1.25), we first consider $q<0$, so that the Riemann-Liouville definition:

$$\frac{d^q f}{dx^q}=\frac{1}{\Gamma(-q)}\int_0^x\frac{f(y)}{[x-y]^{q+1}}dy, q<0$$

may be adopted and upon direct application of the convolution theorem [Churchill, 1948]:

$$L\left\{\int_0^x f_1(x-y)f_2(y)dy\right\}=L\{f_1\}L\{f_2\}$$

Then gives:

$$L\left\{\frac{d^q f}{dx^q}\right\}=\frac{1}{\Gamma(-q)}L\left\{x^{-1-q}\right\}L\{f\}=s^q L\{f\}, q<0 \ ....(1.26)$$

For positive non integer q, we use the result, [Oldham, 1974]:

$$\left[\frac{d^q f}{dx^q}\right]=\frac{d^n}{dx^n}\left[\frac{d^{q-n}f}{dx^{q-n}}\right] \ ....(1.27)$$

$$\frac{d^q f}{dx^q}=\frac{d^n}{dx^n}\frac{d^{q-n}f}{dx^{q-n}}$$

where n is an integer number such that n − 1 < q ≤ n. Now, on application of the formula (1.24), we find that:

$$L\left\{\frac{d^q f}{dx^q}\right\} = L\left\{\frac{d^n}{dx^n}\left[\frac{d^{q-n} f}{dx^{q-n}}\right]\right\}$$

$$= s^n L\left\{\frac{d^{q-n} f}{dx^{q-n}}\right\} - \sum_{k=0}^{n-1} s^k \frac{d^{n-1-k}}{dx^{n-1-k}}\left[\frac{d^{q-n} f}{dx^{q-n}}\right](0)$$

The difference q − n being negative, the first right-hard term may be evaluated by use of equation (1.26), since q − n < 0, the composition rule may be applied to the terms within the summation. The result:

$$L\left\{\frac{d^q f}{dx^q}\right\} = s^q L\{f\} - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k} f(0)}{dx^{q-1-k}}, 0 < q \neq 1, 2, \ldots$$

follows from these two operations and is seen to be incorporated in (1.25). The transformation (1.25) is a very simple generalization of the classical formula for the Laplace transform of the derivative or integral of f. No similar generalization exists, however, for the classical formulas, [Oldham, 1974]:

$$L\left\{\frac{-f}{x}\right\} = \frac{d^{-1} L\{f\}}{ds^{-1}}(s) - \frac{d^{-1} L\{f\}}{ds^{-1}}(\infty)$$

$$L\{-xf\} = \frac{dL\{f\}}{ds}$$

$$L\left\{[-x]^n f\right\} = \frac{d^n L\{f\}}{ds^n}, \ n = 1, 2, \ldots \qquad (1.28)$$

As a final result of this section we shall establish the useful formula:

$$L\left\{\exp(-kx)\frac{d^q}{dx^q}[fe^{kx}]\right\} = [s+k]^q L\{f\}, q \leq 0 \qquad (1.29)$$

in which equation (1.26), may be regarded as a special case, when k = 0 in equation (1.29).

As an illustration, we consider the following example:

Example (1.2.2):

Consider the semi differential equation:

$$\frac{d^{\frac{1}{2}}f(x)}{dx^{\frac{1}{2}}}+\frac{d^{\frac{-1}{2}}f(x)}{dx^{\frac{-1}{2}}}+2f(x)=\frac{2}{\sqrt{\pi x}}+6\sqrt{\frac{x}{\pi}}+\frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}}+2x+4 \quad \text{...............} (1.30)$$

and in order to solve this equation using Laplace transformation method, first we take the Laplace transformation to the both sides of equation (1.30):

$$L\left\{\frac{d^{\frac{1}{2}}f(x)}{dx^{\frac{1}{2}}}\right\}+L\left\{\frac{d^{\frac{-1}{2}}f(x)}{dx^{\frac{-1}{2}}}\right\}+2L\{f(x)\}=\frac{2}{\sqrt{\pi}}L\left\{\frac{1}{\sqrt{x}}\right\}+\frac{6}{\sqrt{\pi}}L\left\{\sqrt{x}\right\}$$

$$+\frac{4}{3\sqrt{\pi}}L\left\{x^{\frac{3}{2}}\right\}+2L\{x\}+L\{4\}$$

use equation (126), leads to:

$$L(f)=\frac{2s^2+3s+1+2\sqrt{s}+4s\sqrt{s}}{s^2(s+1+2\sqrt{s})}$$

$$=\frac{(2s+1)+(s+1+2\sqrt{s})}{s^2(s+1+2\sqrt{s})}$$

$$=\frac{2}{s}+\frac{1}{s^2}$$

Then upon using the inverse Laplace transform, we have:

f(x) = 2 + x

as the solution of the fractional order differential equation.

CHAPTER THREE

ANALYTIC AND NUMERICAL SOLUTIONS FOR SOLVING THE FRACTIONAL ORDER-BOUNDED DELAY DIFFERENTIAL EQUATION

In this chapter, some analytic and numerical methods are presented in order to solve the fractional order-bounded delay differential equations, such as the Laplace transform method and linear multistep methods, numerical and approximate methods are used here, because some times such types of equations has a few difficulties in their methods of solutions, which could not be handled easily.

The numerical solution of a differential equation can consist of a set of tabulated values of the dependent variable y to the required number of significant figures or, more particularly in some real time applications, the solution can be produced directly on a video screen in the from of a graph.

However, it is important to recluse that the numerical solution of a differential equation is essentially a set of numbers which are approximations to the true solution at the corresponding values x_0, x_1, x_2,… of the independent variable x; in general these points are equidistant, so that $x_n = x. + nh$. The distance between any two successive points is:

$$x_{n+1} - x_n = x_0 + (n+1)h - x_0 - nh$$

$$= h$$

and the value of h is referred to as the step width.

This chapter consists of four sections. In section 3.1 we introduce the analytic solution of fractional order -bounded delay differential equations using Laplace transform method, while in section 3.2 we study basic concepts of the numerical methods (general one step method, linear mutistep methods).

In section 3.3 the solution of fractional order-bounded delay differential equations have been introduced using numerical methods which are presented in section 3.2, finally in section 3.4 an illustrative example is given in order to compare between the exact and approximate solution.

3.1 ANALYTIC METHOD FOR SOLVING FRACTIONAL ORDER-BOUNDED DELAY DIFFERENTIAL EQUATION

Several analytical methods are proposed for solving fractional order-bounded delay differential equations, and among such methods which we are used here in this work the Laplace transform method.

3.1.1 Laplace Transformation Method:

To explain the implementation of Laplace transformation method for solving fractional order-bounded delay differential equation we consider the following examples:

Example (3.1.1):

Consider the FOBDDE:

$$y^{(1/2)}(t) + ty(t-1) + y(t) = \frac{8}{3\sqrt{\pi}} t^{3/2} + t(t-1)^2 + t^2$$

with initial condition:

$$y(t) = t^2$$

In order to solve the above FOBDDE using Laplace transformation method, first we take the Laplace transformation to both sides:

$$L\left\{y^{(1/2)}(t)\right\} + L\{y(t)\} = \frac{8}{3\sqrt{\pi}} L\{t^{3/2}\} + L\{t^2\}$$

$$Y(s)\left\{1+\sqrt{s}\right\} = \frac{2}{s^{5/2}} + \frac{2}{s^3}$$

$$Y(s) = \frac{2(1+\sqrt{s})}{s^3(1+\sqrt{s})}$$

taking inverse Laplace transform. The solution is:

$$y(t) = t^2$$

Example (3.1.2):

Consider the FOBDDE:

$$y^{(1/2)}(t) + y(t-1) + y(t) = \sin\left(t + \frac{\pi}{4}\right) + \sin(t-1) + \sin(t)$$

with initial condition:

$$y(t) = \sin(t)$$

$$y^{(1/2)}(t) + \sin(t-1) + y(t) = \sin t \cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos t + \sin(t-1) + \sin(t)$$

in order to solve the above FOBDDE using Laplace transformation method, first we take the Laplace transformation to both sides:

$$\mathcal{L}\left\{y^{(1/2)}(t)\right\}+\mathcal{L}\left\{y(t)\right\}=\mathcal{L}\left\{\sin t\cos\frac{\pi}{4}\right\}-\mathcal{L}\left\{\sin\frac{\pi}{4}\cos t\right\}+\mathcal{L}\left\{\sin(t)\right\}$$

$$Y(s)\left\{1+\sqrt{s}\right\}=\frac{1}{s^2+1}\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\frac{s}{s^2+1}+\frac{1}{s^2+1}$$

$$Y(s)=\frac{1-s+\sqrt{2}}{\sqrt{2}(s^2+1)(1+\sqrt{s})}$$

taking inverse Laplace transform, we have:

$$y(t)=\mathcal{L}^{-1}\left[\frac{1-s+\sqrt{2}}{\sqrt{2}(s^2+1)(1+\sqrt{s})}\right]$$

Example (3.1.3):

Consider the FOBDDE:

$$y^{(1/2)}(t)+y(t-1)+y(t)=\frac{1}{\sqrt{\pi t}}+e^t\frac{\sqrt{t}}{\pi}\left[-t\frac{\left(-\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}\right]+e^{(t-1)}+e^t$$

with initial condition:

$$y(t)=e^t$$

$$y^{(1/2)}(t)+e^{(t-1)}+y(t)=\frac{1}{\sqrt{\pi t}}+e^t\frac{\sqrt{t}}{\pi}\left[-t\frac{\left(-\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}+\right]+e^{(t-1)}+e^t$$

in order to solve the above FOBDDE using Laplace transformation method, first we take the Laplace transformation to both sides:

$$L\left\{y^{(1/2)}(t)\right\}+L\{y(t)\}=\frac{1}{\sqrt{\pi}}L\left\{t^{-1/2}\right\}+\frac{1}{\pi}L\left\{e^{t}t^{3/2}\right\}+L\{e^{t}\}$$

$$Y(s)\left\{1+\sqrt{s}\right\}=\frac{1}{\sqrt{s}}+\frac{4}{3\sqrt{\pi}}\frac{1}{(s-1)^{5/2}}+\frac{1}{s-1}$$

$$Y(s)=\frac{3\sqrt{\pi}+4\sqrt{s}+3\sqrt{\pi}\sqrt{s}(s-1)^{3/2}}{3\sqrt{\pi}\sqrt{s}(1+\sqrt{s})(s-1)^{5/2}}$$

taking inverse Laplace transform, we have:

$$y(t)=L^{-1}\left[\frac{3\sqrt{\pi}+4\sqrt{s}+3\sqrt{\pi}\sqrt{s}(s-1)^{3/2}}{3\sqrt{\pi}\sqrt{s}(1+\sqrt{s})(s-1)^{5/2}}\right]$$

Because of the difficulties, in obtaining the Laplace inverse in some problems as in the preceding example, we are using the numerical methods, which are discussed in details in the next section.

3.2 BASIC CONCEPTS

In order to present, the numerical solution for solving fractional order bounded delay differential equation, we need to give some basic concepts of general one-step methods, and general linear multi -step methods, and with their derivation.

3.2.1 General One-Step Methods:

In this subsection we will describe a general one –step methods and hopping that we can use these formulas in order to solve the fractional order bounded delay differential equations in the next subsections.

In one-step method, only the current value y_n of the solution is used in calculating the next value y_{n+1}. Examples of one-step methods are Euler's method and the Runge-Kutta methods, both of which are described in section 3.2.1.

3.2.1.1 Euler's Method, [Lambert, 1973]:

The simplest 1-step method available for the integration of the first order differential equation $\frac{dy}{dx} = f(x, y)$, uses the relation:

$$y_{n+1} = y_n + hf(x_n, y_n) \quad \text{.......(3.1)}$$

in this formulation $x_n = x_0 + nh$ ($n = 1, 2, \ldots$), where x_0 is an initial given value of x and h is the step width, while $\{y_n\}$ constitutes the set of numbers which approximates the set $\{y(x_n)\}$, which represents the value of the exact solution to the initial value problem defined by:

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

the relation (3.1) can be seen to be derived from the Taylor expansion for the exact solution for the corresponding values of x:

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2!} y''(x_n) + \frac{h^3}{3!} y'''(x_n) + \ldots$$

where:

$$y'(x_n) = \left.\frac{dy}{dx}\right|_{\substack{x=x_n \\ y=y_n}} = f(x_n, y_n)$$

from the differential equation.

If all terms involving h^2 and higher powers are ignored, then:

$$y(x_{n+1}) = y(x_n) + hf(x_n, y_n)$$

3.2.1.2 Runge-Kutta Methods, [Lambert, 1973]:

Euler's method has a simple geometrical interpretation, since the more from y_n to y_{n+1} occurs in the direction of the gradient calculated at (x_n, y_n). the general class of Runge-Kutta methods attempt to improve upon this by using a weighted mean of the gradients at a set of points (x_i, y_i) in the neighborhood of the points (x_n, y_n).

A particular feature of these methods in that the y_i's them selves involve the evaluation of the function f and thus considerably more calculation is involved per step than in Euler's method. But these methods are used extensively in numerical applications mainly because they need no special starting arrangements, the step width h can be changed easily and storage requirements are minimal.

In order to get a formula of the modified Euler. The derivation of the second- order scheme of the Runge-Kutta method will now be derived.

The idea is to express y_{n+1} as a linear combination of the form

$$y(x_{n+1}) = y(x_n) + aK_1 + bK_2 + e \quad \text{..(3.2)}$$

where:

$$K_1 = hf(x_n, y(x_n))$$

$$K_2 = hf(x_n + \theta h, y(x_n) + \varphi K_1)$$

and a, b, θ and φ are parameters whose values are to be calculated in order that the error e in (3.2) be of order h^3 .

The calculation is done by expanding both sides in Taylor series and making use of the differential equation:

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2!} y''(x_n) + \ldots$$

$$= y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2!} \left\{ \frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right\}_{x_n} + \ldots$$

$$K_1 = hf(x_n, y(x_n)) = hf_n$$

$$K_2 = hf(x_n + \theta h, y(x_n) + \varphi K_1)$$

$$= hf(x_n, y(x_n)) + \theta h^2 \frac{\partial f}{\partial x} + h\varphi K_1 \frac{\partial f}{\partial y} + o(h^3)$$

if the coefficients of powers of h are now equated in (3.2)

$$h : 1 = a + b$$

$$h^2 : \frac{1}{2} = b\theta; \frac{1}{2} = b\varphi$$

Three of the parameters can be expressed in terms of the fourth, thus giving the resulting formula:

$$y_{n+1} = y_n + \left(1 - \frac{1}{2\theta}\right) hf_n + \frac{h}{2\theta} f(x_n + \theta h, y_n + \theta hf_n)$$

where $y_n = y(x_n)$ and $f_n = f(x_n, y(x_n))$, a particular value of θ lead to well known case:

$$\theta = \frac{1}{2}: \; y_{n+1} = y_n + hf(x_n + \frac{1}{2}h, y + \frac{1}{2}hf_n)$$

which is the modified Euler method.

3.2.2 General Linear Multi Step Methods, [Atkinson, 1985]:

In this subsection, we establish some basic notations that will be used in order to solve the fractional order bounded delay differential equation. Let y(t) be the exact solution of the initial value problems:

$$\left.\begin{aligned} & y'(t) = f\left(t, y(t)\right), \ \ a \le t \le b \\ & y(a) = y_0 \end{aligned}\right\} \quad \text{......(3.3)}$$

An approximate solution y at only a discrete set of nodes, say:

$$a \le t_1 < t_2 < \ldots < t_n \le b \quad \text{......(3.4)}$$

In this work, we will be taken these nodes to be evently spaced:

$t_n = a + nh$, $n = 0, 1, 2, \ldots, N$.

The following notations are all used for the approximate solution at the node points y_n to obtain a solution y at points in [a, b]. Other than those in (3.4), some form of interpolation must be used. In the present stage, such problems will be not considered.

The linear multistep method will be used to compute y_n, as an approximation to $y(t_n)$ as a linear combination of y_{n+j} and f_{n+j}, $\forall$ j = 0, 1, …, k.

This method is called Linear Multistep Methods (LMM) of step number k, or a linear k-step method.

The general form of LMM may thus be written as:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \quad \text{......(3.5)}$$

for some fixed numbers $\alpha_0, \alpha_1, \ldots, \alpha_k, \beta_0, \beta_1, \ldots, \beta_k$.

It may happen, of course, that α_0 or β_0 is zero, but we assume that this is not the case for both of them. Also since equation (3.5) can be multiplied on both sides by the same constant without altering the relationship, we can assume that $\alpha_k = 1$.

As a classification to the LMM, when $\beta_k = 0$, the method is called explicit or open when $\beta_k \neq 0$, the method is called implicit or closed, since y_{n+k} occurs on both sides of equation (3.5) and is determined only implicitly. Depending on f(t, y) to be either linear or non linear function, [Lambert, 1973].

Definition (3.1):

The Local Transaction Error (LTE) (denoted by T), defined as:

$$T = \sum_{j=0}^{k} \alpha_j y(t_n + jh) - h\sum_{j=0}^{k} \beta_j f(t_n + jh, y(t_n + jh)) \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.6)$$

In order to evaluate the constants $C_0, C_1, \ldots, C_p$ expand $y(t_n + jh)$, $y'(t_n + jh)$ using Taylor series expansion about t_n, one get

$$y(t_n + jh) = y(t_n) + jhy'(t_n) + \frac{j^2h^2}{2!}y''(t_n) + \frac{j^3h^3}{3!}y'''(t_n) + \ldots$$

$$y'(t_n + jh) = y'(t_n) + jhy''(t_n) + \frac{j^2h^2}{2!}y''(t_n) + \ldots$$

Hence substituting in the LTE:

$$T = \sum_{j=0}^{k} (\alpha_j y(t_n + jh) - h\beta_j f(t_n + jh, y(t_n + jh))$$

$$= \sum_{j=0}^{k}(\alpha_j\, y(t_n) + \alpha_j hj\, y'(t_n) + \frac{\alpha_j h^2 j^2}{2} y''(t_n) \ldots -$$

$$h\beta_j\, y'(t_n) - j\,\beta_j\, h^2\, y''(t_n) - \frac{j^2}{2}\beta_j h^3 y'''(t_n) - \ldots$$

$$= \sum_{j=0}^{k}\left(\alpha_j\, y(t_n) + (j\alpha_j - \beta_j)\, h\, y'(t_n) + \left(\frac{j^2\alpha_j}{2} - j\beta_j\right) h^2 y''(t_n) + \ldots\right)$$

$= C_0\, y(t_n) + C_1\, hy'(t_n) + \ldots + C_p h^p y^{(p)}(t_n)$ ....(3.7)

where:

$C_0 = \alpha_0 + \alpha_1 + \ldots + \alpha_k$

$C_1 = \alpha_1 + 2\,\alpha_2 + \ldots + k\,\alpha_{12} - (\beta_0 + \beta_1 + \beta_2 + \ldots + \beta_k)$

$\vdots$

$C_p = \frac{1}{p!}(\alpha_1 + 2^p\,\alpha_2 + \ldots + k^p\,\alpha_p) - \frac{1}{(p-1)!}(\beta_1 + 2^{p-1}\beta_2 + \ldots + k^{p-1}\,\beta_p)$

$p = 2, 3, \ldots.$

Definition (3.2), [Lambert, 1973]:

The order of LMM is p if, in (3.7)

$C_0 = C_1 = \ldots = C_p = 0$ but $C_{p+1} \neq 0$

Definition (3.3), [Lambert, 1973]:

The LMM is said to be consistent if it has an order $p \geq 1$, i.e., consistent methods implies $C_0 = C_1 = 0$ but $C_2 \neq 0$, or:

$$\sum_{j=0}^{k}\alpha_j = 0 \quad \text{and} \quad \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k}\beta_j$$

Definition (3.4), [Lambert, 1973]:

The first and second characteristic polynomials of LMM (3.5) are respectively, gives by:

$$\rho(r) = \sum_{j=0}^{k} \alpha_j r^j = \alpha_0 + \alpha_1 r^1 + \ldots + \alpha_k r^k$$

$$\sigma(r) = \sum_{j=0}^{k} \beta_j r^j = \beta_0 + \beta_1 r^1 + \ldots + \beta_k r^k .$$

Definition (3.5), [Jain, 1984]:

The LMM is said to be convergent, if, for all initial value problem (3.3) subject to the hypothesis of the existence and uniqueness theorem, we have that:

$$\lim_{\substack{h \to 0 \\ nh = t_n - a}} y_n = y(t_n)$$

holds for all $t \in [a, b]$, and for all solutions $\{y_n\}$ of the difference equation (3.5) satisfying starting conditions $y_v = y_{0v}(h)$ for which $\lim_{h \to 0} y_{0v}(h) = y_0$, $v = 0, 1, \ldots, k - 1$.

Definition (3.6), [Atkinson, 1989]:

The LMM's is said to be zero-stable (0-stable) if all the zeros (roots) r_j's, $j = 1, \ldots, k$; of $\rho(r) = 0$ satisfy $|r_j| \leq 1$ and if r_j is a multiple zero of $\rho(r)$ then $|r_j| < 1$.

Theorem (3.1), [Atkinson, 1989]:

Assume the consistency conditions, then the LMM is convergent if and only if its zero stability is satisfied.

Example (3.3.1):

Suppose we have a one step implicit LMM

$$y_{n+1} - y_n = \frac{h}{2}(f_n + f_{n+1})$$

i.e., $k = 1$, $\alpha_0 = -1$, $\alpha_1 = 1$, $\beta_0 = \frac{1}{2}$, $\beta_1 = \frac{1}{2}$

To test the convergence, the roots be evaluated first by using the first characteristic polynomial $\rho(r)$ of LMM, which is:

$$\rho(r) = \sum_{j=0}^{k} \alpha_j r^j = \alpha_0 + \alpha_1 r$$

therefore

$$\rho(r) = -1 + 1 \times r$$

which has the roots $r = 1$. Then by using the definition of the zero-stability. Hence the method is zero-stable.

To test the consistency for the method, one have:

$$C_0 = \sum_{j=0}^{1} \alpha_j = -1 + 1 = 0$$

$$C_1 = \sum_{j=0}^{1} j\alpha_j - \sum_{j=0}^{1} \beta_j$$

$$C_1 = \alpha_1 - \beta_0 - \beta_1$$

$$= 1 - \frac{1}{2} - \frac{1}{2} = 0$$

Therefore, the method is consistent, and as a result the method convergent.

3.2.3 Derivation of Linear Multistep Methods, [Lambert, 1973]:

In the days of desk computation, it was common in practice to write the right-hand side of a LMM in terms of a power series of difference operator. A typical example is

$$y_{n+1} - y_n = h\left(1 - \frac{1}{2}\nabla - \frac{1}{12}\nabla^2 - \frac{1}{24}\nabla^3 - \ldots\right)f_{n+1} \quad \ldots\ldots\ldots(3.8)$$

Truncating the series after the second term, gives:

$$y_{n+1} - y_n = \frac{1}{2}\, h\,(f_{n+1} + f_n),$$

which is the Trapezoidal rule.

While truncating of third term, gives

$$y_{n+1} - y_n = \frac{1}{12}\, h(5\, f_{n+1} + 8\, f_n - f_{n-1})$$

a method which is equivalent to the two-step Adams-Moulton method, is:

$$y_{n+2} - y_{n+1} = \frac{h}{12}(5\, f_{n+2} + 8\, f_{n+1} - f_n)$$

One reason for expressing a numerical algorithm in a form such as (3.8) lay in the technique common in desk computation, of including higher difference of f if they became significantly large at some stage of the calculation. This is equivalent to exchanging the LMM for one with a higher step number. Although this will not affect the zero-stability (which is a function only of the first characteristic polynomial p(r)).

Other stability phenomena which are functions of the second characteristic polynomial $\sigma(r)$ as well as of the first, and these will be affected in higher differences of f are arbitrarily introduced. Thus the practice can be dangerous unless supported by adequate analysis.

In any event, when a digital computer is used it is much more convenient to computer with a fixed LMM and alters the step length if a demand for greater accuracy required at a later stage of the calculation. The existence of formulae like (3.8) has resulted in family name being given to classes of LMM's, of different step number, which share a common form the first characteristic polynomial $\rho(r)$.

Thus methods for which $p(r) = r^k - r^{k-1}$ are called Adams methods.

They have the property that all the spurious roots of ρ are located at the origin, such methods are thus zero-stable.

Adams methods which are explicit are called Adams-Bashforth methods, while those which are implicit are called Adams-Moulton methods. Explicit methods for which $\rho(r) = r^k - r^{k-2}$ are called Nystrom methods, and implicit methods with the same form for ρ are called generalized Mile -Simpson methods both these families are clearly zero-stable, since they have one spurious root at -1 and the rest at the origin.

Clearly there exist many LMM's which do not belong to any of the families named above.

It is important to notice that, there are other methods for derivation of LMM, which are by using Taylor expansion method and numerical integration, see [Lambert, 1973].

3.2.4 Derivation of two-step Adam-Bash fourth Method, [Lambert, 1973]:

To begin the derivation of the Adam-Bash fourth two step methods, note that the solution to the initial-value problem

$$y' = f(x, y), a \le x \le b, y(a) = \alpha$$

if integrated over the interval [x_n, x_{n+1}] has the property that:

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(x, y)dx$$

Since, we can note integrate f(x, y(x)) without knowing y(x), the solution to the problem, we instead integrate an interpolating polynomial which is the Newton backward-difference formula. Hence:

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} \left(1 + t\nabla + \frac{t(t+1)}{2}\nabla^2 + ...\right) f_n dx$$

where $x = x_n + th$

Changing of variables, and integrate, we get the following result:

$$y(x_{n+1}) - y(x_n) = h\left[1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + ...\right] f_n$$

truncating after the second term, we get:

$$y_{n+1} - y_n = \frac{h}{2}[3f_n - f_{n-1}]$$

which implies

$$y_{n+2} - y_{n+1} = \frac{h}{2}[3f_{n+1} - f_n]$$

which is the two-step Adam-Bashfourth method.

3.3 NUMERICAL SOLUTION FOR SOLVING FRACTIONAL ORDER BOUNDED DELAY DIFFERENTIAL EQUATION

In this section the numerical formulas which we are discussed in section 3.2 are used here in order to find the numerical solution for solving the fractional order bounded delay differential equation as follows:-

3.3.1 Euler's Method for solving FOBDDE's:

To use the Euler's method to solve fractional order bounded delay differential equation, the following approach is followed:

Consider the FOBDDE's:

$$y^{(q)}(t) = f(t, y(t), y(t-\tau))$$

$$y(t) = \varphi(t), t_0 - \tau \le t \le t_0$$

and since Euler method reads as follows:

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + O(h^2) \\ &= y_n + hD^{1-q}D^q y_n + O(h^2) \\ &= y_n + hD^{1-q}f(t_n, y(t_n), y(t_n - \tau)) + O(h^2) \\ &= y_n + hD^{1-q}f(t_n, y(t_n), \varphi(t_n - \tau)) + O(h^2) \end{aligned}$$

where $D^{1-q}f(t, y(t), \varphi(t_n - \tau)) + O(h^2)$ could be evaluated easily by using fractional calculus.

3.3.2 Modified Euler's Method for Solving Fractional order Bounded Delay Differential Equation:

As, we do in the subsection 3.3.1, the same approach will be considered in order to use the modified Euler's method to solve the fractional order bounded delay differential equation as follows:

Consider the FOBDDE's:

$$y^{(q)}(t) = f(t, y(t), y(t-\tau))$$

$$y(t) = \varphi(t), t_0 - \tau \leq t \leq t_0$$

Recall that the modified Euler's method written as

$$y_{n+1} = y_n + hK_2$$

where $K_1 = f(t_n, y_n)$ and $K_2 = f(t_n + 1/2h, y_n + 1/2hK_1)$. Hence:

$$y_{n+1} = y_n + hK^*_2$$

where:

$$K^*_1 = D^{1-q} f(t_n, y_n, y(t_n - \tau)),$$

$$K^*_2 = D^{1-q} f(t_n + 1/2h, y_n + 1/2hK_1, y(t_n - \tau) + 1/2hK_1)$$

where K^*_1 and K^*_2 could be evaluated directly by using fractional calculus.

3.3.3 Two-step Adam-Bashfourth method for solving fractional order bounded delay differential equation:

The following scheme is look likes the previous procedures in subsections (3.3.1) and (3.3.2), therefore:

Consider the FOBDDE's:

$$y^{(q)}(t) = f(t, y(t), y(t-\tau))$$

$$y(t) = \varphi(t), t_0 - \tau \le t \le t_0$$

Recall that, two-step Adam-Bashforth method are given in the subsection (3.2.4) as

$$y_{n+2} - y_{n+1} = \frac{h}{2}[3y'_{n+1} - y'_n]$$

Hence:

$$y_{n+2} - y_{n+1} = \frac{h}{2}\left[3D^{1-q}D^q y_{n+1} - D^{1-q}D^q y_n\right]$$

$$y_{n+2} - y_{n+1} = \frac{h}{2}\Big[3D^{1-q}f(t_{n+1}, y(t_{n+1}), y(t_{n+1} - \tau)) - D^{1-q}f(t_n, y(y_n), y(y_n - \tau))\Big]$$

where $D^{1-q}f(t_{n+1}, y(t_{n+1}), y(t_{n+1} - \tau))$ and $D^{1-q}f(t_n, y(t_n), y(t_n - \tau))$ are evaluated easily by fractional calculus.

3.4 ILLUSTRATIVE EXAMPLES

In this section, an illustrative example will be given as a comparison between the numerical methods and focus on the powerful approaches used in solving this new field in differential equations, which is a fractional order bounded delay differential equations.

Example (3.4.1):

Consider the FOBDDE:

$$y^{(1/2)}(t) + y(t-1) + y(t) = \frac{8}{3\sqrt{\pi}}t^{3/2} + (t-1)^2 + t^2$$

with initial condition:

$$y(t) = t^2, -1 \le t \le 0$$

In order to solve this equation, we given the following alterative form:

$$y' = D^{1/2}\left\{-y(t) - y(t-1) + \frac{8}{3\sqrt{\pi}}t^{3/2} + (t-1)^2 + t^2\right\} \quad \text{......................(3.9)}$$

the numerical methods will be used here with step size h = 0.1.

Using Euler's method on the interval [0, 1], equation (3.9) will be written as:

$$y' = -D^{1/2}y(t) - D^{1/2}y(t-1) + \frac{8}{3\sqrt{\pi}}D^{1/2}t^{3/2} + D^{1/2}(t-1)^2 + D^{1/2}t^2$$

and upon using the initial condition

$$y(t) = t^2, -1 \le t \le 0$$

in order to evaluate $D^{1/2}y(t-1)$, we get:

$$y' = -D^{1/2}y(t) - D^{1/2}(t-1)^2 + \frac{8}{3\sqrt{\pi}}D^{1/2}t^{3/2} + D^{1/2}(t-1)^2 + D^{1/2}t^2$$

hence, by using fractional calculus we get:

$$y' = y(t) + 2t - t^2 \quad \text{..(3.10)}$$

recall that Euler method was given as

$$y_{n+1} = y_n + hD^{1/2}f(t_n, y_n, y(t_n - \tau)) \quad \text{..(3.11)}$$

then from equations (3.10) and (3.11):

$$y_{n+1} = y_n + h[y_n + 2t_n - t^2{}_n]$$

and for the interval [1, 2], we shall used a linear approximation for evaluating $D^{1/2}y(t-1)$ in equation (3.9) depending on the information, which we get it in the interval [0, 1], therefore equation (3.9) will be written as:

$$y' = y(t) + y(t-1) - 2t^2 + 4t - 1 - D^{1/2}y(t-1) - \frac{4}{\sqrt{\pi}}t^{1/2} + \frac{1}{\sqrt{\pi}}t^{-1/2} + \frac{8}{3\sqrt{\pi}}t^{3/2} \quad \text{...(3.12)}$$

then from equation (3.11) and (3.12):

$$y_{n+1} = y_n + h\Big[y(t_n) + y(t_n - 1) - 2t_n{}^2 + 4t_n - 1 - D^{1/2}y(t_n - 1) - \frac{4}{\sqrt{\pi}}t_n{}^{1/2} + \frac{1}{\sqrt{\pi}}t_n{}^{-1/2} + \frac{8}{3\sqrt{\pi}}t_n{}^{3/2}$$

Similarly, (Rung-Kutta method, modified Euler method) can be applied on the interval [0, 1], which has the form:

$$y_{n+1} = y_n + hK_2$$

where:

$$K_1 = y_n + 2t_n - t_n{}^2$$

$$K_2 = y_n + \frac{1}{2}hK_1 + 2\left(t_n + \frac{1}{2}h\right) - \left(t_n + \frac{1}{2}h\right)^2$$

while for the interval [1, 2] takes the form:

$$y_{n+1} = y_n + hK_2$$

where:

$$K_1 = y_n + y(t_n - 1) - 2t_n{}^2 + 4t_n - 1 - D^{1/2}y(t_n - 1) - \frac{4}{\sqrt{\pi}}t_n{}^{1/2} + \frac{1}{\sqrt{\pi}}t_n{}^{-1/2} + \frac{8}{3\sqrt{\pi}}t_n{}^{3/2}$$

and

$$K_2 = y_n + \frac{1}{2}hK_1 + y(t_n - 1) + \frac{1}{2}hK_1 - 2\left(t_n + \frac{1}{2}h\right)^2 + 4\left(t_n + \frac{1}{2}h\right) - 1 - D^{1/2}y(t_n - 1) - \frac{4}{\sqrt{\pi}}\left(t_n + \frac{1}{2}h\right)^{1/2} + \frac{1}{\sqrt{\pi}}\left(t_n + \frac{1}{2}h\right)^{-1/2} + \frac{8}{3\sqrt{\pi}}\left(t_n + \frac{1}{2}h\right)^{3/2}$$

Finally, two-step Adam-Bash fourth method also can be applied which has the following form on [0, 1]:

$$y_{n+2} = y_{n+1} + \frac{h}{2}\left[3(y_{n+1} + 2t_{n+1} - t_{n+1}{}^2) - (y_n + 2t_n - t_n{}^2)\right]$$

while for the interval [1, 2]

$$y_{n+2} = y_{n+1} + \frac{h}{2}[3(y_{n+1} + y(t_{n+1} - 1) - 2t_{n+1}{}^2 + 4t_{n+1} - 1 - D^{1/2}y(t_{n+1} - 1) - \frac{4}{\sqrt{\pi}}t^{1/2}{}_{n+1} + \frac{1}{\sqrt{\pi}}t_{n+1}{}^{-1/2} + \frac{8}{3\sqrt{\pi}}t_{n+1}{}^{3/2}) - (y_n + y(t_{n-1}) - 2t_n{}^2 + 4t_n - 1 - D^{1/2}y(t_n - 1) - \frac{4}{\sqrt{\pi}}t_n{}^{1/2} + \frac{1}{\sqrt{\pi}}t_n{}^{-1/2} + \frac{8}{3\sqrt{\pi}}t_n{}^{3/2})]$$

The numerical results are illustrated in table (3.1) as follows:

Table (3.1)

The comparison between the numerical results of example (3.4.1).

t_i	Euler method	Runge-Kutta (Modified Euler)	Adam Bash fourth method
0.1	0	0.01	0.01
0.2	0.019	0.041	0.04
0.3	0.057	0.093	0.09
0.4	0.114	0.166	0.16
0.5	0.189	0.261	0.25
0.6	0.283	0.377	0.36
0.7	0.395	0.515	0.49
0.8	0.526	0.675	0.64
0.9	0.674	0.856	0.81
1	0.841	1.06	1
1.1	1.17	1.196	1.21
1.2	1.342	1.396	1.435
1.3	1.516	1.6	1.667
1.4	1.689	1.807	1.893
1.5	1.862	2.015	2.122
1.6	2.031	2.222	2.353
1.7	2.194	2.427	2.584
1.8	2.351	2.628	2.814
1.9	2.498	2.822	3.041
2	2.631	3.006	3.263

Example (3.4.2):

Consider the neutral FOBDDE:

$$y^{(1/2)}(t) + y'(t-1) = \frac{8}{3\sqrt{\pi}} t^{3/2} + 2(t-1)$$

with initial condition:

$$y(t) = t^2, -1 \le t \le 0$$

and carrying similar calculations as in example (3.4.1), we get the results presented in table (3.2).

Table (3.2)

The comparison between the numerical results of example (3.4.2).

t_i	*Euler method*	*Runge-Kutta (Modified Euler)*	*Adam Bashfourth method*
0.1	0	0.01	0.01
0.2	0.02	0.04	0.04
0.3	0.06	0.09	0.09
0.4	0.12	0.16	0.16
0.5	0.2	0.25	0.25
0.6	0.3	0.36	0.36
0.7	0.42	0.49	0.49
0.8	0.56	0.64	0.64
0.9	0.72	0.81	0.81
1	0.9	1	1
1.1	1.249	1.098	1.326
1.2	1.623	1.413	1.678
1.3	2.022	1.753	2.054
1.4	2.445	2.12	2.456
1.5	2.892	2.511	2.882
1.6	3.364	2.927	3.333
1.7	3.86	3.369	3.808
1.8	4.38	3.835	4.307
1.9	4.924	4.326	4.831
2	5.492	4.841	5.379

CHAPTER TWO
EXISTENCE AND UNIQUENESS THEOREM OF FRACTIONAL ORDER-BOUNDED DELAY DIFFERENTIAL EQUATION

In this chapter, a new type of differential equation is formulated by mixing two well known types of differential equations which are the fractional order differential equations and bounded delay differential equations. This type of equations will be called fractional order-bounded delay differential equations “FOBDDE’s” and has the following form:

$$y^{(q)}(t) = f(t, y(g_1(t)), \ldots, y(g_m(t)))$$

where $g_j(t)$ is a retarded argument , i.e., $g_j(t) \leq t$ for $j = 1,\ldots,m$, $0 < q < 1$. With the initial condition:

$$y(t) = \theta(t), \text{ for } -\tau \leq t \leq 0$$

where θ is a given initial function mapping $[-\tau, 0] \longrightarrow D$.

and the following problems may be consider:

1- The solution existence and uniqueness theorem of the such type equations, is discussed in this chapter.

2- The analytical and numerical solutions for solving such type of equations, are discussed in the next chapter.

2.1 BASIC DEFINITIONS

Some needed definitions and notations that used to prove the existence and uniqueness theorem of FOBDDE are

Many practical problems give rise to bounded differential equations having constant or not constant delays.

$$y'(t) = f(t, y(g_1(t)), \dots, y(g_m(t)))$$

$$t - \tau \le g_j(t) \le t\,,\ t \ge 0$$

for some constant $\tau \ge 0$, then the initial condition

$$y(t) = \theta(t)\,,\ \text{for } -\tau \le t \le 0$$

The set $C([-\tau,0], \mathbb{R}^n)$ of all continuous functions mapping $[-\tau, 0] \longrightarrow \mathbb{R}^n$ will be denoted by $\wp$ and if A is any set in $\mathbb{R}^n$, we will let $\wp_A = C([-\tau,0], A)$.

<u>Definition (2.1), [Driver, 1977]:</u>

For a function $\varphi \in \wp_D$, it is convenient to define a measure of magnitude of φ by $\|\varphi\|_r = \sup\limits_{-r \le \delta \le 0} \|\varphi(\delta)\|$

In the special case when $D = \mathbb{R}^n$, $\wp_D = \wp$ is a linear space and $\|\,.\,\|_r$ is a norm on $\wp$.this means that $\|\,.\,\|_r$ satisfies the following conditions:

1- $\|\varphi\|_r \ge 0$, for all $\varphi \in \wp$.

2- $\|\varphi\|_r = 0 \Leftrightarrow \varphi = 0$ (the zero function).

3- $\|c\varphi\|_r = |c|\|\varphi\|_r$, for all $\varphi \in \wp$ and all $c \in R$.

4- $\|\varphi + \tilde{\varphi}\|_r \le \|\varphi\|_r + \|\tilde{\varphi}\|_r$, for all $\varphi, \tilde{\varphi} \in \wp$ (the triangle inequality).

(we shall refer to $\| \,.\, \|_r$ as the r-norm).

2.2 EXISTENCE AND UNIQUENESS THEOREM FOR FRACTIONAL ORDER-BOUNDED DELAY DIFFERENTIAL EQUATION

In this section, we shall state and prove the existence and uniqueness theorem of fractional order differential equations to the case of delay differential systems with bounded delays.

Consider a FOBDD system, such as:

$$y^{(q)}(t) = f(t, y(g_1(t)), \ldots y(g_m(t))) \quad \text{.......(2.1)}$$

We shall assume that

$t - \tau \le g_j(t) \le t$, for $t \ge 0$, $j = 1, 2, \ldots, m$, $0 < q < 1$

For some constant $\tau \ge 0$ .then the initial condition takes the form

$$y(t) = \theta(t), \text{ for } -\tau \le t \le 0 \quad \text{.......(2.2)}$$

We assume that f is defined on $[0, \beta) \times D^n \longrightarrow \mathbb{R}^n$ for some $\beta > 0$ and some open set $D \subset \mathbb{R}^n$.

Equation (2.1) can be rewritten as:

$$y^{(q)}(t) = F(t, y_t) \quad \text{.......(2.3)}$$

where $F(t, y_t) = f(t, y(g_1(t)), \ldots, y(g_m(t)))$.

Definition (2.2), [Driver, 1977]:

If y is a function defined at least on $[t-\tau,\ t] \longrightarrow \mathbb{R}^n$, then we define a new function $y_t : [-\tau, 0] \longrightarrow \mathbb{R}^n$, by:

$$y_t(\delta) = y(t+\delta), \quad \text{for } -\tau \le \delta \le 0$$

When we using the notation of equation (2.3), we shall also rewrite the initial condition (2.2) .

Equation (2.2) is equivalent to $y(\delta) = \theta(\delta)$ for $-\tau \le \delta \le 0$ or simply $y_0 = \theta_0$ introducing $\varphi = \theta_0$ this becomes:

$$y_0 = \varphi \quad \text{.........(2.4)}$$

it is important to recognize that (2.4) means $y(\delta) = \varphi(\delta)$ or, by letting $t = \delta$ then

$$y(t) = \varphi(t), \text{ for } -\tau \le t \le 0$$

and in particular

$$y(0) = \varphi(0)$$

Continuity Condition, [Driver, 1977]:

If $F:[0,\ \beta) \times \wp_D \longrightarrow \mathbb{R}^n$ satisfied that $F(t, y_t)$ is continuous with respect to t in $[0,\ \beta)$ for each given continuous function:

$$y : [-\tau, \beta) \longrightarrow D$$

then a continuous function

$$y : [-\tau,\ \beta_1) \longrightarrow D, \text{ for some } \beta_1 \in (0, \beta]$$

is a solution of:

$$\left.\begin{array}{l} y^{(q)}(t) = F(t, y_t) \\ y_0 = \varphi \end{array}\right\} \quad \text{...(2.5)}$$

if and only if:

$$y(t) = \begin{cases} \varphi(t), -\tau \le t \le 0 \\ \varphi(0) + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} F(s, y_s) ds, 0 \le t < \beta_1 \end{cases} \quad \text{...........................(2.6)}$$

Definition (2.3), [Driver, 1977]:

The functional $F : J \longrightarrow \mathbb{R}^n$ is locally Lipschitzian if for each given $(\bar{t}, \bar{\psi}) \in [0, \beta) \times \wp_D$, there exist numbers a > 0 and b > 0, such that:

$$\xi = ([\bar{t} - a, \bar{t} + a] \cap J = [0, \beta)) \times \{\psi \in \wp : \|\psi - \bar{\psi}\|_r \le b\}$$

is a subset of $J = [0, \beta) \times \wp_D$ and F is Lipschitzian on ξ.

In other words, for some number K (a Lipschitz constant depending on ξ),

$$\|F(t, \psi) - F(t, \tilde{\psi})\| \le K \|\psi - \tilde{\psi}\|_r$$

whenever (t, ψ) and (t, $\tilde{\psi}$) $\in \xi$.

Theorem (2.1) (The Existence and Uniqueness Theorem):

Let $F : [0, \beta) \times \wp_D \longrightarrow \mathbb{R}^n$ satisfy the continuity condition and let it be locally Lipschitzian. Then, for each $\varphi \in \wp_D$, there exists a solution to equation (2.1) on $[-\tau, \Delta)$ for some $\Delta > 0$, and this solution is unique:

Proof:

Choose any a > 0 and b > 0 sufficiently small, so that:

$\xi = [0, a] \times \{\psi \in \wp : \|\psi - \varphi\|_r < b\}$

is a subset of $[0, \beta) \times \wp_D$ and F is Lipschitzian on ξ (say with Lipschitz constant K).

Define a continuous function $\bar{y}$ on $[0 - \tau, a] = [-\tau, a] \longrightarrow \mathbb{R}^n$, by:

$$\bar{y}(t) = \begin{cases} \varphi(t), -\tau \le t \le 0 \\ \varphi(0), 0 \le t \le a \end{cases}$$

Then $F(t, \bar{y}_t)$ depends continuously on t, and hence $\|F(t, \bar{y}_t)\| \le \beta_1$ on [0,a] for some constant β_1

Now define $\beta = kb + \beta_1$ choose $a_1 \in (0, a]$, such that:

$\|\bar{y}_t - \varphi\|_r = \|\bar{y}_t - \bar{y}_0\|_r \le b$, for $0 \le t \le a_1$

choose $\Delta > 0$, such that:

$$\Delta \le \min\left\{ a_1, \left(\frac{\Gamma(q+1)b}{\beta} \right)^{\frac{1}{q}} \right\} \text{ and } \left(\frac{\Delta^q k}{\Gamma(q+1)} \right) < 1$$

this condition may be neglected.

Let S be the set of all continuous functions $y : [-\tau, \Delta] \longrightarrow \mathbb{R}^n$, such that:

$y(t) = \varphi(t)$, for $-\tau \le t \le 0$

$\|y(t) - \varphi(0)\| \le b$, for $0 \le t \le \Delta$

Note that if $y \in S$ and $t \in [0, \Delta]$, then:

$$\|y_t - \bar{y}_t\|_r \le b$$

so that:

$$\|F(t,y_t)\| \le \|F(t,y_t) - F(t,\bar{y}_t)\| + \|F(t,\bar{y}_t)\|$$

$$\le k\|y_t - \bar{y}_t\| + \beta_1 = \beta$$

To prove that S is invariant, for each $y \in S$ define a function Ty on $[-\tau, \Delta]$ by:

$$(Ty)(t) = \begin{cases} \varphi(t), & -\tau \le t \le 0 \\ \varphi(0) + \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1}F(s,y_s)ds, & 0 \le t \le \Delta \end{cases}$$

$$\|(Ty)(t) - \varphi(0)\| \le \frac{\|F(t,y_t)\|\, t^q}{\Gamma(q+1)}$$

$$\le \frac{\|F(t,y_t)\|\Delta^q}{\Gamma(q+1)}$$

$$\le \frac{\beta\Delta^q}{\Gamma(q+1)}$$

$$\le \frac{\beta\Gamma(q+1)b}{\Gamma(q+1)\beta} = b, \text{ for } 0 \le t \le \Delta$$

Also Ty is continuous, then $Ty \in S$ and we can say that T maps S on to S.

Let us now construct "successive approximation" technique choosing $y_{(0)} \in S$ and then defining:

$$y_{(1)} = Ty_{(0)},\ y_{(2)} = Ty_{(1)},\ \ldots$$

bear in mind that each $y_{(\ell)}(t) = \varphi(t)$ on $[-\tau, 0]$. It is clear that to show that:

$$\left\|y_{(1)}(t)-y_{(0)}(t)\right\| \le 2b$$

$$\left\|y_{(1)}(t)-y_{(0)}(t)\right\| \le b+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left\|F(s,y_{(0)}(s))\right\|ds$$

$$\le b+\frac{\beta}{\Gamma(q+1)}\Delta^q = 2b$$

$$\left\|y_{(2)}(t)-y_{(1)}(t)\right\| = \left\|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}F(s,y_{(1)}(s))ds-\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}F(s,y_{(0)}(s))ds\right\|$$

$$=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left\|F(s,y_{(1)}(s))-F(s,y_{(0)}(s))\right\|ds$$

$$\le\frac{k}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left\|y_{(1)}(s)-y_{(0)}(s)\right\|ds$$

$$\le\frac{2bkt^q}{\Gamma(q+1)}$$

$$\le 2b\left(\frac{\Delta^q k}{\Gamma(q+1)}\right)$$

$$\left\|y_{(3)}(t)-y_{(2)}(t)\right\| = \left\|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(F(s,y_{(2)}(s))-F(s,y_{(1)}(s)))ds\right\|$$

$$=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left\|F(s,y_{(2)}(s))-F(s,y_{(1)}(s))\right\| ds$$

$$\le\frac{k}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left\|y_{(2)}(s)-y_{(1)}(s)\right\| ds$$

$$\leq 2b\frac{k\Delta^q}{\Gamma(q+1)}\left(\frac{\Delta^q k}{\Gamma(q+1)}\right)$$

$$\leq 2b\left(\frac{k\Delta^q}{\Gamma(q+1)}\right)^2$$

one can prove by induction that:

$$\left\|y_{(\ell+1)}(t)-y_{(\ell)}(t)\right\|\leq 2b\left(\frac{k\Delta^q}{\Gamma(q+1)}\right)^{\ell}, \text{ for } \ell=0,1,2,\ldots$$

Now the series:

$$\sum_{p=0}^{\infty}\left\|y_{(p+1)}(t)-y_{(p)}(t)\right\|\leq\sum_{p=0}^{\infty}2b\left(\frac{k\Delta^q}{\Gamma(q+1)}\right)^p$$

$$=2b\sum_{p=0}^{\infty}\left(\frac{k\Delta^q}{\Gamma(q+1)}\right)^p$$

converges, the convergence comes from the condition $\left(\frac{k\Delta^q}{\Gamma(q+1)}\right) < 1$.

put y(t) = $\lim_{\ell\to\infty} y_{(\ell)}(t)$, which is the desired solution and this complete the proof. ■

Now, to prove the uniqueness, we need the following lemma:

Lemma (2.1), [Driver, 1977]:

Let $y:[-\tau,\ \beta)\longrightarrow \mathbb{R}^n$ be continuous, then, given any $\bar{t}\in[0,\beta)$ and any $\varepsilon>0$, there exist $\delta>0$, such that:

$$\|y_t - y_{\bar{t}}\|_r < \varepsilon, \text{ whenever } t\in[0,\beta) \text{ and } |t-\bar{t}|<\delta$$

Now, to continue the proof of the uniqueness, let $F:[0,\beta)\times\wp_D\longrightarrow\mathbb{R}^n$ satisfy the continuity condition and $\frac{k(t-s)^{q-1}}{\Gamma(q)}$ must be non-negative and continuous and let F be locally Lipschitzian, and suppose (for contradiction) that for some $\beta_1\in(0,\beta]$ there are two solutions y and $\tilde{y}$ mapping $[-\tau,\ \beta_1)\longrightarrow D$, with $y\neq\tilde{y}$

Let:

$$t_1=\inf\{t\in(0,\beta_1): y(t)\neq\tilde{y}(t)\} \quad\ldots\ldots(2.7)$$

then $0\le t<\beta_1$ and

$$y(t)=\tilde{y}(t), \text{for } -\tau\le t\le t_1$$

since $(t_1, y_{t_1})\in[0,\beta_1)\times\wp_D$, there exist numbers a>0 and b>0 such that the set:

$$\xi=[t_1, t_1+a]\times\{\psi\in\wp : \|\psi - y_{t_1}\|_r\le b\}$$

is contained in $[0,\beta)\times\wp_D$ and F is Lipschitzian on ξ (with Lipschitz constant k). By above lemma, $\exists\,\delta\in(0,a]$ such that $(t, y_t)\in\xi$ and $(t,\tilde{y}_t)\in\xi$ for $t_1\le t\le t_1+\delta$, moreover, both y and $\tilde{y}$ satisfy (2.4) for $-\tau\le t\le t_1+\delta$.thus, for $t_1\le t\le t_1+\delta$

$$\|y(t)-\tilde{y}(t)\| = \left\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} (F(s, y_s) - F(s, \tilde{y}_s) ds \right\|$$

$$\leq \frac{k}{\Gamma(q)} \int_{t_1}^{t} (t-s)^{q-1} \|y_s - \tilde{y}_s\|_r \, ds$$

Now, since the right hand side is an increasing function of t and since $\|y(t)-\tilde{y}(t)\| = 0$ for $t_1 - \tau \leq t \leq t_1$

By Reid's lemma [Driver, 1977] it follows that $y(t) = \tilde{y}(t)$ on $[t_1, t_1 + \delta)$ which contradiction for the definition of equation (2.7). ■

CONCLUSIONS AND RECOMMENDATION

From the present study, we can conclude the following:

1. From the results of table (3.1), one can notice that the Adam-Bashforth method give more accurate results than Euler's method and Modified Eulers method.
2. From the results of table (3.2), one can notice that the Modified Euler method give more accurate results than Adam-Bashforth method and Euler's method, this is due to the instability of the Adam-Bashforth method.
3. The accumulation error resulting in tables (3.1) and (3.2) for the second interval [1, 2] is due to the approximation of $D^{1/2}y(t-1)$ and $D^{1/2}y'(t-1)$, using the linear approximation.

Also, we can recommend the following problems for future work:

1. Studying the solution of partial fractional order delay differential equations, with boundaries conditions.
2. Studying the fractional order bounded delay differential equation by converting it into equivalence integral equation.
3. Studying the stability of fractional order bounded delay differential equation.

CONTENTS

Dedication

To My Family

To My Husband

With all Love and Respects

INTRODUCTION

The concept of fractional calculus is popularly be lived to have stemmed from a question raised in the year 1695 by L'hopital (1661-1704) to Leibniz (1646-1716), which sought the meaning of Leibniz's (currently popular) notation $\frac{d^n y}{dx^n}$ for the derivative of order $n \in N_0 := \{0, 1, \ldots\}$ when $n = \frac{1}{2}$ (what if $n = \frac{1}{2}$?) In his reply, dated 30 September 1695, Leibniz wrote to L'hopital as follows: "… This is an apparent paradox from which, one day, useful consequences will be drawn. …" .

We shall introduce some literature survey concern the delay differential equations together with fractional calculus.

Delay differential equations were initially introduced in the 18th century by Laplace and Condorect, [Ulsoy, 2003]. However, the rapid development of the theory and applications of those equations did not come until after the Second World War, and continues till today. The basic theory concerning the stability of systems described by equations of this type was developed by Pontryagin in 1942. Important works have been written by Smith in 1957, Pinney in 1958, Bellman and Cooke in 1963, Halanay in 1966, Myshkis in 1972, Hale 197٧, Yanusherski in 1978 and Marshal in 1979, [Ulsoy, 2003].

On the other hand, many complicated physical problems described in terms of partial differential equations can be approximated by much simpler problems described in terms of delay differential equations, [Pinney, 1958].

The impetus has mainly been due to the developments in many fields, such as the control theory, mathematical biology, and mathematical economics, etc. Minorsky, [Hale, 1977] was one of the first investigators of modern times to study the delay differential equation:

$$y'(t) = f(t, y(t), y(t-\tau))$$

and its effect on simple feed-back control systems in which the communication time can not be neglected.

The abundance of applications is stimulating a rapid development of the theory of differential equations with deviating argument and, at present, this theory is one of the most rapidly developing branches of mathematical analysis.

Equations with a deviating argument describe many processes with an effect; such equations appear, for example, any time when in physics or technology we consider a problem of a force, acting on a material point, that depends on the velocity and position of the point not only at the given moment but at some moment preceding the given moment, [El'sgolt'c, 1973].

Fractional calculus is a field of mathematical study that grows out of the traditional definitions of the calculus integral and derivative operators in the same way fractional exponents is an outgrowth of exponents with integer value, [Loverro, 2004].

Many found, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative. The most famous of these definitions that have been popularized in the world of fractional calculus are the Riemann-Liouville and Grünwald-Letnikov definition. Also, Caputo, [Podlubny, 1999] reformulated the more "classic" definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions to solve his fractional order differential equations. Recently, [Kolowankar, 1996] reformulated again, the Riemann-Liouville fractional order derivative, in order to, differentiate no-where differentiable fractal functions.

In recent years, considerable interest in fractional calculus have been simulated by the applications that this subject finds in numerical analysis, differential equations and different areas of applied sciences, especially in physics and engineering, possibly including fractal phenomena, [Al-Husseiny, 2006].

Fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decodes or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering.

This subject, devoted exclusively to the subject of fractional calculus in the book by Oldham and Spanier [Oldham, 1974] published in 1974. One of the most recent works on the subject of fractional calculus in the book of Podlubny [Podlubny, 1999], published in 1999 which deals principally with fractional order differential equations, and today there exist at least two international journals which are devoted almost entirely to the subject of fractional calculus; (i) Journal of fractional calculus and (ii) Fractional calculus and Applied Analysis.

Delay differential equations, (DDEs) which is arise in many areas of mathematical modeling: for example population dynamics (taking into account the gestation times), infections diseases (accounting for the incubation periods), physiological and pharmaceutical kinetics (modeling for example, the body's reaction to CO_2, etc. in circulating blood) and chemical kinetics (such as mixing reactants), the navigational control of ships and aircraft (with respectively large and short lags), and more general control problems.

There exists now collection of books that indicate applications areas form DDEs and we cite in particular, the books [Driver, 1977], [Halanay, 1966], and [Kuang, 1993]. Whilst ordinary differential equations (ODEs) model problems in which the variables react to current conditions, DDEs (and related functional differential equations) model problems where there is an after- effect affecting at least one of the variables.

Also, many authors and researchers concerned with the fractional differential equations say Al-Shather A. in 2003, presented some approximated solutions for the fractional delay integro-differential equation, Abdul-Razzak B.T. in 2004, gave new algorithm for solving fractional order Fredholm integro-differential equation, Al-Azawi S., in 2004, presented some results in fractional calculus, Al-Rahhal D. in 2005, used the numerical solutions for the fractional integro-differential equation, Gorial I. in 2005, used the finite difference method to solve the eigenvalue problems for the partial fractional differential equation, Abdul-Jabber in 2005, discussed the inverse problem of the fractional integro-differential equation, and Abdul-Jabbar R.S. in 2005 studied the inverse problems of some fractional order Integro-differential equations, and Khalil E. in 2006, used linear multi-step methods to solve some fractional order ordinary differential equations, and Aziz S. in 2006 used some approximated methods for solving partial fractional differential equations, and Al-

Husseiny R. in 2006, discussed the existence of uniqueness solution of some fuzzy fractional order ordinary differential equations.

While in delay differential equations, several authors studied this subject, and upon them, Al-Saady A. S. 2000, studied the approximate solution of delay differential equations using cubic spline interpolation techniques. Nadia K. M. in 2001 studied the variational formulation of delay differential equations, Thikra A. in 2001 studied the approximate solution of delay integral equations using variational approach. Maha A. in 2003 studied the inverse problem of delay integral equations using variational approach. Haifaa M. B. in 2004 studied the variational formulation of partial delay differential equations. Finally, Gadeer J. M in 2007 studied the numerical solution of linear partial delay differential equations using the finite difference methods.

The purpose of this work is to combine between fractional and delay differential equations to obtain the so called fractional delay differential equations.

This work consists of three chapters, as well as, this introduction. In chapter one, the fundamental concepts for delay and fractional order differential equation is given, while in chapter two, the existences and uniqueness solution theorem of fractional order bounded delay differential equation is stated and proved. Finally in chapter three the analytic and numerical solution for such type of differential equations is presented, as well as, the comparison between these methods and the exact solutions are presented.

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بسم الله الرحمن الرحيم

﴿ يرفع الله الذين آمنوا منكم والذين

أوتوا العلم درجات والله بما تعملون خبير ﴾

صدق الله العظيم

﴿المجادلة / 11﴾

المستخلص

في هذا العمل تمت دراسة المعادلات التفاضلية التباطؤية ذات الرتب الكسرية (FOBDDE’s) حيث تم اثبات نظرية الوجود والوحدانية لحل هذا النوع من المعادلات التفاضلية باستخدام طريقة التقريبات المتتابعة.

وكذلك تم عرض اسلوب حل المعادلات التفاضلية التباطؤية ذات الرتب الكسرية (FOBDDE’s) تحليليا (Analytically) باستخدام تحويلات لابلاس (Laplace Transformation) وعدديا (Numerically) باستخدام طرائق متعددة الخطوات (linear multi-steps methods) وطرائق ذات الخطوة الواحدة (general-one step methods) وتمت مقارنة نتائج هذه الطرق مع قيم الحل المضبوطة (Exact solutions values).