ABSTRACT

In this thesis, some properties and basic definitions of fractional integral and derivatives of Riemann-Liouvill are presented , to construct the optimality conditions of mixed order unconstrained and constrained variational problems with continuous and discontinuous functional, on fixed and moving boundaries ,based on the classical product rule for Riemann-Liouvill , Several tested example are presented to demonstrate the implementation of the optimality necessary conditions.

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CHAPTER THREE

Optimality Necessary Conditions Of Fractional Variation Problem Along Movable Boundaries

3.1 Introduction

In this chapter we concern on the constructing the optimality and the necessary conditions for unconstrained and constrained fractional variation problems with continuous and discontinuous functional where the independent variable along movable boundaries having one or different multi fractional order derivatives on one and different multi-dependent variables , by using the formula(1.10).

3.2The Functional Of Continuous With Movable Boundaries

In this section we shall construct the optimality necessary conditions, when the functional integrand is continuous having non-integers orders , also functional having non-integer and integer orders.

3.2.1 Unconstrained Problem Having Only Non-integer Order

***First*, we shall consider the problem of the form :**

$$V(y)=\int_{x_0}^{x_1} F(x, y, y^{(\alpha)})\, dx \qquad \ldots(3.1)$$

where;

$0 < \alpha < 1$, and with given prescribed boundaries conditions.

Consider one of the end points is variable (say (x_1,y_1)) , i.e. (x_1,y_1) can move turning into $(x_1+\delta x_1 , y_1+\delta y_1)$, then for α>0 noninteger

$$\Delta v = \int_{x_0}^{x_1+\delta x_1} F\big(x, y+\delta y , y^{(\alpha)}+\delta y^{(\alpha)}\big)dx - \int_{x_0}^{x_1} F(x,y,y^{(\alpha)})\, dx$$

$$= \int_{x_0}^{x_1}\big(x, y+\delta y , y^{(\alpha)}+\delta y^{(\alpha)}\big)\, dx + \int_{x_1}^{x_1+\delta x_1} F\big(x, y+\delta y , y^{(\alpha)}+ \delta y^{(\alpha)}\big)dx - \int_{x_0}^{x_1} F(x,y,y^{(\alpha)})\, dx$$

$$= \int_{x_1}^{x_1+\delta x_1} F\big(x, y+\delta y , y^{(\alpha)}+\delta y^{(\alpha)}\big)dx + \int_{x_0}^{x_1}[F\big(x, y+\delta y , y^{(\alpha)}+ \delta y^{(\alpha)}\big) - F(x,y,y^{(\alpha)})]\, dx \qquad \ldots(3.2)$$

The first term of the right - hand side of the equation (3.2) can be transform with the aid of the mean value theorem given we get:

$$\int_{x_1}^{x_1+\delta x_1} F\big(x, y+\delta y , y^{(\alpha)}+\delta y^{(\alpha)}\big)dx = F_{x=x_1+\theta\delta x_1}\delta x_1 ,$$

where $(0<\theta<1)$

Furthermore , by virtue of continuity of the function F ,

$$F\,\big|_{x=x_1+\theta\delta x_1} = F(x,y,y^{(\alpha)})\,\big|_{x=x_1} + \mathcal{E}_1 ,$$

Where;

$$\mathcal{E}_1 \to 0 \quad as \quad \delta x_1 \to 0 \quad and \quad \delta y_1 \to 0$$

Consequently;

$$\int_{x_0}^{x_1+\delta x_1} F\big(x, y+\delta y , y^{(\alpha)}+\delta y^{(\alpha)}\big)dx = F(x,y,y^{(\alpha)})\,\big|_{x=x_1}\delta x_1 \qquad \ldots(3.3)$$

The second term of the right-hand side of eq. (3.2) , can be transformed by using Taylor formula to get :

$$\int_{x_0}^{x_1}[F\big(x, y+\delta y , y^{(\alpha)}+\delta y^{(\alpha)}\big) - F(x,y,y^{(\alpha)})]\, dx$$

$$=\int_{x_0}^{x_1} F_y(x,y,y^{(\alpha)})\delta y + \mathrm{F}_{\mathrm{y}^{(\alpha)}}(\mathrm{x},\mathrm{y},\mathrm{y}^{(\alpha)})\delta y^{(\alpha)})dx + R_1$$

Where R_1 is infinitesimal of higher order than δy or$\delta y^{(\alpha)}$ then

$$=\int_{x_0}^{x_1}(F_y\,\delta y + \mathrm{F}_{\mathrm{y}^{(\alpha)}}\delta y^{(\alpha)})dx = \int_{x_0}^{x_1}(F_y\,\delta y + \mathrm{F}_{\mathrm{y}^{(\alpha)}}D^{(\alpha)}\delta y)dx$$

Using (1.10) , for the second term , in which δy α-differentiable , we have

$$\delta v = \mathrm{F}(\mathrm{x},\mathrm{y},\mathrm{y}^{(\alpha)})\big|_{\mathrm{x}=\mathrm{x}_1}\delta\mathrm{x}_1 + \int_{x_0}^{x_1}(F_y - \tfrac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}})\,\delta y\,dx$$

Since the value of the functional are only along extremals (i.e. $\delta v = 0$) consequently , we have the following necessary condition :

$$\mathrm{F}(\mathrm{x},\mathrm{y},\mathrm{y}^{(\alpha)})\big|_{\mathrm{x}=\mathrm{x}_1} = 0 \qquad \text{… (3.4 a)}$$

$$F_y - \tfrac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}} = 0 \qquad \text{…(3.4 b)}$$

***second*, (3.1) can be extended, to different multi fractional-order αi>0 , and non integer (i=1,2,3,…m) , by the following problem :**

$$v(y) = \int_{x_0}^{x_1} F(x,y,y^{(\alpha_1)},\dots,y^{(\alpha_m)})\,dx \qquad \text{…(3.5)}$$

consider one of the end points is variable (say (x_1,y_1)) , i.e. (x_1,y_1) can move turning into $(x_1+\delta x_1\,,y_1+\delta y_1)$, then

$$\Delta v = \int_{x_0}^{x_1} F(x,y+\delta y\,,y^{(\alpha_1)}+\delta y^{(\alpha_1)},\dots,y^{(\alpha_m)}+\delta y^{(\alpha_m)})\,dx + \int_{x_1}^{x_1+\delta x_1} F\big(x,y+\delta y\,,\delta y\,,y^{(\alpha_1)}+\delta y^{(\alpha_1)},\dots,y^{(\alpha_m)}+\delta y^{(\alpha_m)}\,\big)dx - \int_{x_0}^{x_1} F(x,y,y^{(\alpha_1)},\dots,y^{(\alpha_m)})\,dx$$

$$\Delta v = \int_{x_1}^{x_1+\delta x_1} F\big(x,y+\delta y\,,\delta y\,,y^{(\alpha_1)}+\delta y^{(\alpha_1)},\dots,y^{(\alpha_m)}+\delta y^{(\alpha_m)}\,\big)dx + \int_{x_0}^{x_1}[F(x,y+\delta y\,,y^{(\alpha_1)}+\delta y^{(\alpha_1)},\dots,y^{(\alpha_m)}+\delta y^{(\alpha_m)}) - \mathrm{F}(\mathrm{x},\mathrm{y},\mathrm{y}^{(\alpha)},\dots,y^{(\alpha_m)})]\,dx \qquad \text{…(3.6)}$$

The first term of the right- hand side of the equation (3.6) can be transform with the aid of the mean value theorem to get :

$$\int_{x_1}^{x_1+\delta x_1} F\left(x, y+\delta y, \delta y, y^{(\alpha_1)}+\delta y^{(\alpha_1)}, \ldots, y^{(\alpha_m)}+\delta y^{(\alpha_m)}\right) dx =$$

$$F\big|_{x=x_1+\theta\delta x_1} \delta x_1 ,$$

where $0 < \theta < 1$

Furthermore , by virtue of continuity of the function F ,

$$F\big|_{x=x_1+\theta\delta x_1} = F(x, y, y^{(\alpha)}, \ldots, y^{(\alpha_m)})\big|_{x=x_1} + \mathcal{E}_1 ,$$

Where;

$$\mathcal{E}_1 \to 0 \quad as \quad \delta x_1 \to 0 \quad and \quad \delta y_1 \to 0$$

Consequently;

$$\int_{x_1}^{x_1+\delta x_1} F\left(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)}, \ldots, y^{(\alpha_m)}+\delta y^{(\alpha_m)}\right) dx =$$

$$F(x, y, y^{(\alpha)}, \ldots, y^{(\alpha_m)})\big|_{x=x_1} \delta x_1$$

The second term of the right-hand side of eq. (3.6) , can be transformed by using Taylor formula to get :

$$\int_{x_0}^{x_1} (F\left(x, y+\delta y, y^{(\alpha_1)}+\delta y^{(\alpha_1)}, \ldots, y^{(\alpha_m)}+\delta y^{(\alpha_m)}\right) -$$

$$F(x, y, y^{(\alpha_1)}, \ldots, y^{(\alpha_m)})) \, dx$$

$$= \int_{x_0}^{x_1} F_y(x, y, y^{(\alpha_1)}, \ldots, y^{(\alpha_m)}) \delta y + F_{y^{(\alpha_1)}}(x, y, y^{(\alpha_1)}, \ldots, y^{(\alpha_m)}) \delta y^{(\alpha_1)} + \cdots +$$

$$F_{y^{(\alpha_m)}}(x, y, y^{(\alpha_1)}, \ldots, y^{(\alpha_m)}) \delta y^{(\alpha_m)} dx + R_1$$

Where R_1 **is infinitesimal of higher order than** $\delta y \, or \delta y^{(\alpha_1)}, \ldots,$ $\delta y^{(\alpha_m)}$ **then**

$$=\int_{x_0}^{x_1}(F_y\,\delta y+\mathrm{F}_{y^{(\alpha_1)}}\delta y^{(\alpha_1)}+\cdots+F_{y^{(\alpha_m)}}\delta y^{(\alpha_m)})dx=\int_{x_0}^{x_1}(F_y\,\delta y+\mathrm{F}_{y^{(\alpha_1)}}D^{(\alpha_1)}\delta y+\cdots+\mathrm{F}_{y^{(\alpha_m)}}D^{(\alpha_m)}\delta y)dx$$

Using (1.10) , for the second term , in which δy α-differentiable , we obtain :

$$\delta v=\mathrm{F}(\mathrm{x},\mathrm{y},\mathrm{y}^{(\alpha_1)},\ldots,y^{(\alpha_m)})\,\big|_{\mathrm{x}=\mathrm{x}_1}\delta\mathrm{x}_1+\int_{x_0}^{x_1}(F_y-\frac{d^{\alpha_1}}{dx^{\alpha_1}}F_{y^{(\alpha_1)}}-\cdots-\frac{d^{\alpha_m}}{dx^{\alpha_m}}F_{y^{(\alpha_m)}})\,\delta y\,dx$$

Since the value of the functional are only along extremals (i.e. $\delta v=0$) consequently , we have the following necessary conditions :

$$\mathrm{F}(\mathrm{x},\mathrm{y},\mathrm{y}^{(\alpha_1)},\ldots,y^{(\alpha_m)})\,\big|_{\mathrm{x}=\mathrm{x}_1}\equiv 0 \qquad \ldots(3.7\text{ a})$$

$$F_y-\sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}F_{y_j^{(\alpha_i)}}=0 \qquad \ldots(3.7\text{ b})$$

<u>Third</u>, the problem (3.5) can extended, further more to multi dependent variable , for the following problem : α_i>0 non integer

$$v(y_1,\ldots,y_n)=\int_{x_0}^{x_1}F(x,y,\ldots,y_n,y_1^{(\alpha_1)},\ldots,y_1^{(\alpha_m)},y_n^{(\alpha_1)},\ldots,y_n^{(\alpha_m)})\,dx \quad \ldots(3.8)$$

consider one of the end points is variable (say (x_1,y_1)) , i.e. (x_1,y_1) can move turning into $(x_1+\delta x_1\,,y_1+\delta y_1)$, by variants on dependent variable and fixing the remaining dependent variables , then

$$\Delta v=\int_{x_0}^{x_1+\delta x_1}F\Big(x,y_1+\delta y_1\,,y_2+\delta y_2,\ldots,y_n+\delta y_n\,,y_1^{(\alpha_1)}+\delta y_1^{(\alpha_1)},\ldots,y_1^{(\alpha_m)}+\delta y_1^{(\alpha_m)},\ldots,y_n^{(\alpha_1)}+\delta y_n^{(\alpha_1)},\ldots,y_n^{(\alpha_m)}\delta y_n^{(\alpha_m)}\Big)dx-\int_{x_0}^{x_1}F(x,y,\ldots,y_n,y_1^{(\alpha_1)},\ldots,y_1^{(\alpha_m)},y_n^{(\alpha_1)},\ldots,y_n^{(\alpha_m)})\,dx$$

$$\Delta v = \int_{x_0}^{x_1} F\left(x, y_1 + \delta y_1, \dots, y_n + \delta y_n, y_1^{(\alpha_1)} + \delta y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)} + \delta y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)} \delta y_n^{(\alpha_m)}\right) dx + \int_{x_0}^{x_1+\delta x_1} F\left(x, y_1 + \delta y_1, \dots, y_n + \delta y_n, y_1^{(\alpha_1)} + \delta y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)} + \delta y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)} \delta y_n^{(\alpha_m)}\right) dx - \int_{x_0}^{x_1} F(x, y, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})\, dx =$$

$$\int_{x_1}^{x_1+\delta x_1} F\left(x, y_1 + \delta y_1, \dots, y_n + \delta y_n, y_1^{(\alpha_1)} + \delta y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)} + \delta y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)} \delta y_n^{(\alpha_m)}\right) dx + \int_{x_0}^{x_1} [F\left(x, y_1 + \delta y_1, \dots, y_n + \delta y_n, + \delta y_1^{(\alpha_1)}, \dots, y_1^{\alpha_m} + \delta y_1^{\alpha_m}, \dots, y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)} \delta y_n^{(\alpha_m)}\right) - \mathrm{F}(\mathrm{x}, y, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})\, dx \qquad \dots(3.9)$$

The first term of the right- hand side of the equation (3.9) can be transform with the aid of the mean value theorem to get :

$$\int_{x_1}^{x_1+\delta x_1} F\left(x, y_1 + \delta y_1, \dots, y_n + \delta y_n, y_1^{(\alpha_1)} + \delta y_1^{\alpha_1}, \dots, y_1^{(\alpha_m)} + \delta y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)} + \delta y_n^{(\alpha_m)}\right) dx = F\big|_{\mathrm{x}=\mathrm{x}_1+\theta\delta\mathrm{x}_1} \delta x_1 ,$$

where $0 < \theta < 1$

Furthermore , by virtue of continuity of the function F ,

$$F\big|_{x=x_1+\theta\delta x_1} = F(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})\big|_{\mathrm{x}=\mathrm{x}_1} + \mathcal{E}_1 ,$$

Where;

$$\mathcal{E}_1 \to 0 \quad as \quad \delta x_1 \to 0 \quad and \quad \delta y_1 \to 0$$

Consequently;

$$\int_{x_1}^{x_1+\delta x_1} F\Big(x, y_1+\delta y_1, \dots, y_n+\delta y_n, y_1^{(\alpha_1)}+\delta y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}+\delta y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)}+\delta y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}+\delta y_n^{(\alpha_m)}\Big)dx = F(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})\big|_{x=x_1}\delta x_1 \qquad \dots(3.10)$$

The second term of the right-hand side of eq. (3.9) , can be transformed by using Taylor formula to get :

$$\int_{x_0}^{x_1} F(x, y_1+\delta y_1, \dots, y_n+\delta y_n, y_1^{(\alpha_1)}+\delta y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}+\delta y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)}+\delta y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}+\delta y_n^{(\alpha_m)}) - F(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})) \, dx$$

$$= \int_{x_0}^{x_1} F_y(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}) = \int_{x_0}^{x_1}(F_{y_1}\delta y_1 + \dots + F_{y_n}\delta y_n + F_{y_1^{(\alpha_1)}}\delta y_1^{(\alpha_1)} + \dots + F_{y_1^{(\alpha_m)}}\delta y_1^{(\alpha_m)} + \dots + F_{y_n^{(\alpha_1)}}\delta y_n^{(\alpha_1)} + \dots + F_{y_n^{(\alpha_m)}}\delta y_n^{(\alpha_m)})dx + R_1$$

Where R_1 is infinitesimal of higher order than δy or $\delta y^{(\alpha_i)}$ for all i

$$= \int_{x_0}^{x_1}(F_{y_1}\delta y_1 + \dots + F_{y_n}\delta y_n + F_{y_1^{(\alpha_1)}}\delta y_1^{(\alpha_1)} + \dots + F_{y_1^{(\alpha_m)}}\delta y_1^{(\alpha_m)} + \dots + F_{y_n^{(\alpha_1)}}\delta y_n^{(\alpha_1)} + \dots + F_{y_n^{(\alpha_m)}}\delta y_n^{(\alpha_m)})dx$$

$$= \int_{x_0}^{x_1}(F_{y_1}\delta y_1 + \dots + F_{y_n}\delta y_n + F_{y_1^{(\alpha_1)}}D^{(\alpha_1)}\delta y_1 + \dots + F_{y_1^{(\alpha_m)}}D^{(\alpha_m)}\delta y_1 + \dots + F_{y_n^{(\alpha_1)}}D^{(\alpha_1)}\delta y_n + \dots + F_{y_n^{(\alpha_m)}}D^{(\alpha_m)}\delta y_n)dx$$

Using (1.10) , for the second term , in which δy α-differentiable , we obtain :

$\delta v = F\left(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right) |_{x=x_1} \delta x_1 +$

$\int_{x_0}^{x_1}(F_{y_1} + \dots + F_{y_1} - \frac{d^{\alpha_1}}{dx^{\alpha_1}} F_{y_1^{\alpha_1}} - \dots - \frac{d^{\alpha_m}}{dx^{\alpha_m}} F_{y_1^{(\alpha_m)}} - \dots - \frac{d^{\alpha_1}}{dx^{\alpha_1}} F_{y_n^{(\alpha_1)}} - \dots -$

$\frac{d^{\alpha_m}}{dx^{\alpha_m}} F_{y_n^{(\alpha_m)}}) \, \delta y \, dx$

$\delta v = \int_{x_0}^{x_1}(F_{y_1} - \frac{d^{\alpha_1}}{dx^{\alpha_1}} F_{y_1^{(\alpha_1)}} - \dots - \frac{d^{\alpha_m}}{dx^{\alpha_m}} F_{y_1^{(\alpha_m)}}) \delta y_1 \dots (F_{y_n} - \frac{d^{\alpha_1}}{dx^{\alpha_1}} F_{y_n^{(\alpha_1)}} - \dots -$

$\frac{d^{\alpha_m}}{dx^{\alpha_m}} F_{y_n^{(\alpha_m)}}) \, \delta y_n \, dx$

Since the value of the functional are only along extremals (i.e. $\delta v = 0$) consequently , we have the following necessary condition :

$$F\left(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right) |_{x=x_1} = 0 \qquad \dots(3.11\ a)$$

$$F_{y_j} - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} F_{y_j^{(\alpha_i)}} = 0 \quad , \qquad (j=1,2,\dots,n) \qquad \dots(3.11\ b)$$

3.2.2 Unconstrained Problem Having Integer and Non-integer Order

***First*, we shall consider the problem of the form :**

$$v(y) = \int_{x_0}^{x_1} F(x, y, y', y^{(\alpha)}) \, dx \qquad \dots(3.12)$$

Where one of the end point is variable (say (x_1, y_1) i.e. (x_1, y_1) can move turning in to $(x_1 + \delta x_1, y_1 + \delta y_1)$,$0 < \alpha < 1$ and with given prescribed boundaries conditions.

$\Delta v =$

$\int_{x_1}^{x_1+\delta x_1} F(x, y + \delta y, y' + \delta y', y^{(\alpha)} + \delta y^{(\alpha)}) \, dx - \int_{x_0}^{x_1} F(x, y, y', y^{(\alpha)}) \, dx$

$=$

$$\int_{x_0}^{x_1+\delta x_1} F(x, y+\delta y, y'+\delta y', y^{(\alpha)}+\delta y^{(\alpha)})\,dx + \int_{x_0}^{x_1}[F(x,y,y',y^{(\alpha)})\,dx - F(x,y,y',y^{(\alpha)})]dx \qquad \ldots(3.13)$$

First term of the right-hand side of equation (3.13) will be transformed with aid of the mean value theorem , we get :

$$\int_{x_1}^{x_1+\delta x_1} F(x, y+\delta y, y'+\delta y', y^{(\alpha)}+\delta y^{(\alpha)})\,dx = F(x,y,y',y^{(\alpha)})|_{x=x_1}\delta x_1 \qquad \ldots(3.14)$$

The second term of the right-hand side of equation (3.13) can be transformed by using Taylor formula to get

$$\int_{x_0}^{x_1}[F(x,y,y',y^{(\alpha)})\,dx - F(x,y,y',y^{(\alpha)})]dx =$$
$$\int_{x_0}^{x_1}[F_y(x,y,y',y^{(\alpha)})\delta y\,dx + F_{y'}(x,y,y',y^{(\alpha)})\delta y' + F_{y^{(\alpha)}}(x,y,y',y^{(\alpha)})\delta y^{(\alpha)}]dx$$

Integrate the second term by part , and using (1.10) for the third term in which δy is α-differentiable , we obtain

$$F_{y'}\delta y]_{x_0}^{x_1} + \int_{x_0}^{x_1}\left(F_y - \frac{d}{dx}F_{y'} + \frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}}\right)\delta y\ dx$$

Since the value of the functional are only along extremes (i.e. $\delta v = 0$) consequently we have

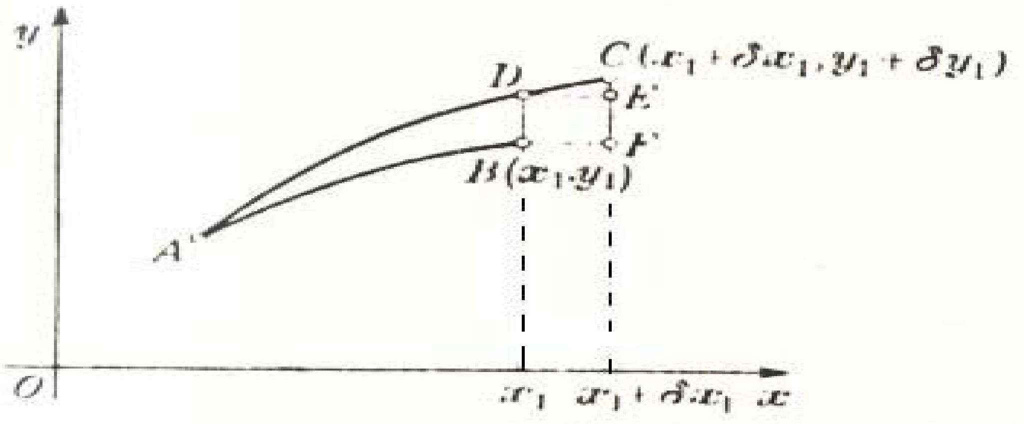
$$F_y - \frac{d}{dx}F_{y'} + \frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}} = 0 \qquad \ldots(3.15)$$

And therefore

$$\int_{x_0}^{x_1}(F_y\delta y + F_{y'}\delta y' + F_{y^{(\alpha)}}\delta y^{(\alpha)})dx \cong F_{y'}\delta y'|_{x=x_1} \qquad \ldots(3.16)$$

Observe that $\delta y|_{x=x_1}$ does not mean the same as δy_1 , the increment of y_1 , for δy_1 is the change of y-coordinate is afree end point , when it is moved from (x_1, y_1) to $(x_1+\delta x_1, y_1+\delta y_1)$ where ; $\delta y|_{x=x_1}$ is the change y-coordinate of an extremal produced at the point $(x = x_1)$ when this extremal changes from that passed through the points (x_0, y_0) and

(x_1, y_1) to another one passing through (x_0, y_0) and $(x_1 + \delta x_1, y_1 + \delta y_1)$.



Extremal Changes

Fig. (1)

BD=$Fc - Ec$

$$\delta y|_{x=x_1} \cong \delta y_1 - y'(x_1)\delta x_1$$

Consequently , since the fundamental necessary condition for extremum $\delta v = 0$ **. is**

$$\delta v = F|_{x=x_1}\delta x_1 + F_{y'}\delta y \equiv 0$$

$$= F|_{x=x_1}\delta x_1 + F_{y'}(\delta y_1 - y'\delta x_1) = 0$$

Therefore ,we have the following necessary condition :

$$\left(F - y'F_{y'}|_{x=x_1}\delta x_1 + F_{y'}|_{x=x_1}\delta y_1\right) = 0 \qquad \ldots (3.17)$$

If the variations δx_1 **and** δy_1 **are independent , then we have the following conditions for extremum .**

$$\left(F - y'F_{y'}\right)|_{x=x_1} = 0 \qquad \text{and} \qquad F_{y'}|_{x=x_1} = 0 \qquad \ldots (3.18)$$

If the variations δx_1 and δy_1 are dependent , for instance suppose that the end point (x_1, y_1) can move along a certain curve $y_1 = \phi(x_1)$ in equation (3.17) we get:

$$(F - y' F_{y'})|_{x=x_1} \delta x_1 + F_{y'}|_{x=x_1} (\phi(x_1)) \delta x_1 = 0$$

$$[F + (\phi' - y') F_{y'}|_{x=x_1}] \delta x_1 = 0$$

Since δx_1 is arbitrary then the necessary condition ,which is called "Transrersality Condition" becomes:

$$F + (\phi'(x) - y') F_{y'}|_{x=x_1} = 0 \qquad \ldots(3.19)$$

second **the problem (3.12) can be extended, to different multi integer and non integer order α>0 , of the following problem :**

$$v(y_1, \ldots, y_n) = \int_{x_0}^{x_1} F(x, y_1, \ldots, y_n, y_1', \ldots, y_n', y_1^{(\alpha)}, \ldots, y_n^{(\alpha)})\, dx \qquad \ldots(3.20)$$

Where one of the end points is variable (say(x_1, y_1)) , i.e. (x_1, y_1) can move turning in to ($x_1 + \delta\, x_1, y_1 + \delta\, y_1$)

$$\Delta v = \int_{x_0}^{x_1+\delta x_1} F(x, y_1 + \delta y_1, \ldots, y_n + \delta y_n + y_1' + \delta y_1', \ldots, y_n' + \delta y_n', y_1^{(\alpha)} + \delta y_1^{(\alpha)}, \ldots, y_n^{(\alpha)} + \delta y_n^{(\alpha)})\, dx - \int_{x_0}^{x_1} F(x, y_1, \ldots, y_n, y_1', \ldots, y_n', y_1^{(\alpha)}, \ldots, y_n^{(\alpha)})\, dx$$

$$= \int_{x_1}^{x_1+\delta x_1} F(x, y_1 + \delta y_1, \ldots, y_n + \delta y_n + y_1' + \delta y_1', \ldots, y_n' + \delta y_n', y_1^{(\alpha)} + \delta y_1^{(\alpha)}, \ldots, y_n^{(\alpha)} + \delta y_n^{(\alpha)})\, dx + \int_{x_0}^{x_1} [F(x, y_1 + \delta y_1, \ldots, y_n + \delta y_n + y_1' + \delta y_1', \ldots, y_n' + \delta y_n', y_1^{(\alpha)} + \delta y_1^{(\alpha)}, \ldots, y_n^{(\alpha)} + \delta y_n^{(\alpha)}) - F\left(x, y_1, \ldots, y_n, y_1', \ldots, y_n', y_1^{(\alpha)}, \ldots, y_n^{(\alpha)}\right)] dx \qquad \ldots(3.21)$$

The first term of the right-hand of equation (3.21) can be transformed with aid of mean value theorem to get :

$$= \int_{x_1}^{x_1+\delta x_1} F(x, y_1 + \delta y_1, \dots, y_n + \delta y_n + y_1' + \delta y_1', \dots, y_n' + \delta y_n', y_1^{(\alpha)} + \delta y_1^{(\alpha)}, \dots, y_n^{(\alpha)} + \delta y_n^{(\alpha)})\, dx = F\left(x, y_1, \dots, y_n, y_1', \dots, y_n', y_1^{(\alpha)}, \dots, y_n^{(\alpha)}\right)|_{x=x_1} \delta x_1$$

…(3.22)

The second term of the right-hand side of eq. (3.22) can be transformed by using Taylor formula , we get :

$$\int_{x_0}^{x_1} [F(x, y_1 + \delta y_1, \dots, y_n + \delta y_n + y_1' + \delta y_1', \dots, y_n' + \delta y_n', y_1^{(\alpha)} + \delta y_1^{(\alpha)}, \dots, y_n^{(\alpha)} + \delta y_n^{(\alpha)}) - F\left(x, y_1, \dots, y_n, y_1', \dots, y_n', y_1^{(\alpha)}, \dots, y_n^{(\alpha)}\right)] dx = \int_{x_0}^{x_1} F_{y_1} \delta y_1 + \dots + F_{y_n} \delta y_n + F_{y_1'} \delta y_1' + \dots + F_{y_n'} \delta y_n', F_{y_1^{\alpha}} \delta y_1^{(\alpha)} + \dots + F_{y_n^{(\alpha)}} \delta y_n^{(\alpha)})\, dx$$

Integrate the terms from $F_{y_1'} \delta y_1'$ to $F_{y_n'} \delta y_n'$ by part , and using (1.10) for terms $F_{y_1^{(\alpha)}} \delta y_1^{(\alpha)}$ to $F_{y_n^{(\alpha)}} \delta y_n^{(\alpha)}$ in which $\delta y_1, \dots, \delta y_n$ is α-differentiable we obtain:

$$\sum_{j=1}^{n} F_{y_j'} \delta y_j \Big]_{x_0}^{x_1} + \int_{x_0}^{x_1} \sum_{j=1}^{n} \left(F_{y_j} - \frac{d}{dx} F_{y_j'} - \frac{d^\alpha}{dx^\alpha} F_{y_j^{(\alpha)}}\right) dx$$

$$\delta v = F|_{x=x_1} \delta x_1 + \sum_{j=1}^{n} F_{y_j'} \delta y_j \Big]_{x_0}^{x_1} + \int_{x_0}^{x_1} \sum_{j=1}^{n} \left(F_{y_j} - \frac{d}{dx} F_{y_j'} - \frac{d^\alpha}{dx^\alpha} F_{y_j^{(\alpha)}}\right) dx$$

Since the value of the functional are only along extremes (i.e. $\delta v = 0$)

$$\sum_{j=1}^{n} F_{y_j} - \frac{d}{dx} F_{y_j'} - \frac{d^\alpha}{dx^\alpha} F_{y_j^{(\alpha)}} = 0$$

By the same argument as that given on first problem of eqution(1.3) . we obtain

$$\delta y_j|_{x=x_1} \cong \delta y_j - y_j'(x_1) \delta x_1 \quad , \qquad \text{(j=1,…,n)}$$

and consequently

$$\delta v = [F - \sum_{j=1}^{n} F_{y_j'} \delta y_j\Big]_{x=x_1} \delta x_1 + \sum_{j=1}^{n} F_{y_j'} \delta y_j\Big]_{x=x_1} = 0 \qquad \text{…(3.23)}$$

If the variation δx_1 , δy_1 , …, δy_n are independent then it follows from the condition $\delta v = 0$

$$F - \sum_{j=1}^{n} F_{y_j'} \delta y_j \Big]_{x=x_1} = 0 \quad , \quad F_{y_j'} \Big]_{x=x_1} = 0 \qquad ,(j=1,..,n) \quad …(3.24a)$$

If the boundary point can move along certain curve $y_j = \phi_j(x_1)$ for all (j=1,..,n) then $\delta y_j = \emptyset_j'(x_1)\delta x_1$ and the conditions $\delta v = 0$ or

$$F - \sum_{j=1}^{n} y_j' F_{y_j'} \Big]_{x=x_1} \delta x_1 + \sum_{j=1}^{n} F_{y_j'} \delta y_j = 0$$

Turns in to

$$F + \sum_{j=1}^{n} (\phi_j' - y_j') F_{y_j'} \Big]_{x=x_1} \delta x_1 = 0$$

and since δx_1 is arbitrary , we have

$$[F + \sum_{j=1}^{n} (\phi_j' - y_j') F_{y_j'} \Big]_{x=x_1} = 0 \qquad …(3.24b)$$

This condition is called the Tranrersality Condition .

<u>Third</u> , we consider the functional of the form

$$v(y) = \int_{x_0}^{x_1} F(x, y, y', y'', y^{(\alpha)})\, dx \qquad …(3.25)$$

Where one of the end points is variable (say(x_1, y_1)) i.e. (x_1, y_1) can move turning into ($x_1 + \delta\, x_1, y_1 + \delta\, y_1$)

$$\Delta v = \int_{x_0}^{x_1+\delta x_1} F(x, y+\delta y, y' + \delta\, y', y'' + \delta\, y'', y^{(\alpha)} + \delta y^{(\alpha)})dx - \int_{x_0}^{x_1} F(x, y, y', y'', y^{(\alpha)})\, dx$$

$$= \int_{x_1}^{x_1+\delta x_1} F(x, y+\delta y, y' + \delta\, y', y'' + \delta\, y'', y^{(\alpha)} + \delta y^{(\alpha)})dx + \int_{x_0}^{x_1} [F(x, y+\delta y, y' + \delta y', y'' + \delta\, y'', y^{(\alpha)} + \delta y^{(\alpha)}) - F(x, y, y', y'', y^{(\alpha)})]dx \qquad …(3.26)$$

The first term of the right-hand side of equation (3.26) can be transformed with aid the mean value theorem ,we get :

$$\int_{x_1}^{x_1+\delta x_1} F(x,y+\delta y,y'+\delta y',y''+\delta y'',y^{(\alpha)}+\delta y^{(\alpha)})dx = F(x,y,y',y'',y^{(\alpha)})|_{x=x_1}\delta x_1$$

The second term of the right-hand side of equation (3.26) can be transformed by using Taylor formula given by equation(1.5) we get :

$$\int_{x_0}^{x_1}[F(x,y+\delta y,y'+\delta y',y''+\delta y'',y^{(\alpha)}+\delta y^{(\alpha)}) - F(x,y,y',y'',y^{(\alpha)})]dx = \int_{x_0}^{x_1}(F_y\delta y + F_{y'}\delta y' + F_{y''}\delta y'' + F_{y^{(\alpha)}}\delta y^{(\alpha)})dx + R$$

Integrating by parts the second term of the integrated and doing the same twice with the third terms and using (1.10) for the forth terms

$$\int_{x_0}^{x_1} F_{y'}\delta y' dx = F_{y'}\delta y]_{x_0}^{x_1} - \int_{x_0}^{x_1}(\frac{d}{dx}F_{y'}\delta y)\,dx$$

$$\int_{x_0}^{x_1} F_{y''}\delta y'' dx = F_{y''}\delta y']_{x_0}^{x_1} - \int_{x_0}^{x_1}(\frac{d}{dx}F_{y''}\delta y')\,dx = -[F_{y''}\delta y]_{x_0}^{x_1} + \int_{x_0}^{x_1}(\frac{d^2}{dx^2}F_{y''}\delta y)\,dx$$

and then remembering that

$$\delta y|_{x=x_1} = 0\,,\ \delta y'|_{x=x_1} = 0\quad ,\text{and}\quad F_y - \frac{d}{dx}F_{y'} - \frac{d^2}{dx^2}F_{y''} - \frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}} = 0$$

We obtain:

$$\delta v = [F\,\delta x_1 + F_{y'}\delta y + F_{y'}\delta y' - \frac{d}{dx}(F_{y''})\,\delta y]_{x=x_1}$$

Making use of relation $\delta y_1 = y'(x_1)\delta x_1 + [\delta y]_{x=x_1}$ and applying this result also to δy_1

$$\delta y_1' = y''(x_1)\delta x_1 + [\delta y']_{x=x_1}$$

We obtain

$$\delta v = \left[F - y'F_{y'} - y''F_{y''} + y'\frac{d}{dx}(F_{y''})\right]_{x=x_1}\delta x_1 + [F_{y'} - \frac{d}{dx}F_{y''}]_{x=x_1}\delta y_1 + F_{y''}|_{x=x_1}\delta y_1'$$

Consequently , the fundamental condition of an extremum $\delta v = 0$ takes the form

$$I)\ [F - y' F_{y'} - y'' F_{y''} + y' \frac{d}{dx}(F_{y''})|_{x=x_1}\delta x_1 + [F_{y'} - \frac{d}{dx} F_{y''}]_{x=x_1}\delta y_1 + F_{y''}|_{x=x_1}\delta y_1' = 0$$

If δx_1 , δy_1 and $\delta y_1'$ are independent , then their coefficients should vanish at the point $x = x_1$, if these is some relation between them , $y_1 = \phi(x_1)$ and $y_1' = \psi(x_1)$, then $\delta y_1 = \phi'(x_1)\delta x_1$ and $\delta y_1' = \psi'(x_1)\delta x_1$ and substitution , these values into formula(I) we have :

$$[F - y' F_{y'} - y'' F_{y''} + y' \frac{d}{dx}(F_{y''}) + (F_{y'} - \frac{d}{dx} F_{y''}\phi') + F_{y''}\psi']_{x=x_1}\delta x_1 = 0$$

Hence $$[F - y' F_{y'} - y'' F_{y''} + y' \frac{d}{dx}(F_{y''}) + (F_{y'} - \frac{d}{dx} F_{y''}\phi') + F_{y''}\psi']_{x=x_1} = 0$$

II) If x_1 , y_1 and y_1' are related through one equation $\phi(x_1, y_1, y_1') = 0$, then two of the variations δx_1 , δy_1 and $\delta y_1'$ are arbitrary and the remaining one is give by the equation

$$\phi'_{x_1}\delta x_1 + \phi'_{y_1}\delta y_1 + \phi'_{y_1'}\delta y_1' = 0$$

3.2.3 Constrained Problem Having Only Non-integer Order

First , we shall considering the problem of the form :

$$V(y)=\int_{x_0}^{x_1} F(x, y, y^{(\alpha)})\, dx \ , \quad \text{such that} \quad \phi\left(x, y, y^{(\alpha)}\right) = 0 \qquad \ldots(3.27)$$

where;

$0 < \alpha < 1$, λ *is exist* and with given prescribed boundaries conditions. Our approach based on the theories presented in [15] ,we construct the following auxiliary functional :

$$Z\left(x, y(x), y^{(\alpha)}(x)\right) = F + \lambda\phi$$

Where λ is a Lagrange multiplier , then the problem (3.27) can be started as following:

$$V^*(x, y(x), y^{(\alpha)}) = \int_{x_0}^{x_1} Z(x, y, y^{(\alpha)})\, dx \qquad \text{…(3.28)}$$

and one of the end points is variable (say (x_1,y_1)) , i.e. (x_1,y_1) can move turning into $(x_1 + \delta x_1, y_1 + \delta y_1)$, then

$$\Delta v^* = \int_{x_0}^{x_1+\delta x_1} Z\big(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}\big)\, dx - \int_{x_0}^{x_1} Z(x, y, y^{(\alpha)})\, dx$$

$$= \int_{x_1}^{x_1+\delta x_1} Z\big(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}\big)\, dx + \int_{x_0}^{x_1} [Z\big(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}\big) - Z(x, y, y^{(\alpha)})]dx \qquad \text{…(3.29)}$$

The first term of the right - hand side of the equation (3.29) can be transform with the aid of the mean value theorem , we get :

$$\int_{x_1}^{x_1+\delta x_1} Z\big(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}\big)dx = Z_{x=x_1+\theta\delta x_1}\delta x_1 ,$$

where $(0 < \theta < 1)$

Furthermore , by virtue of continuity of the function F ,

$$Z\big|_{x=x_1+\theta\delta x_1} = Z(x, y, y^{(\alpha)})\big|_{x=x_1} + \mathcal{E}_1 ,$$

Where;

$$\mathcal{E}_1 \to 0 \quad as \quad \delta x_1 \to 0 \quad and \quad \delta y_1 \to 0$$

Consequently;

$$\int_{x_1}^{x_1+\delta x_1} Z\big(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}\big)dx = Z(x, y, y^{(\alpha)})\big|_{x=x_1}\delta x_1 \qquad \text{…(3.30)}$$

The second term of the right-hand side of eq. (3.29) , can be transformed by using Taylor formula we get :

$$\int_{x_0}^{x_1} Z\left(x, y+\delta y\,, y^{(\alpha)}+\delta y^{(\alpha)}\right) dx - \int_{x_0}^{x_1} Z(x,y,y^{(\alpha)})\,dx$$

$$=\int_{x_0}^{x_1} Z_y(x,y,y^{(\alpha)})\delta y + Z_{y^{(\alpha)}}(x,y,y^{(\alpha)})\delta y^{(\alpha)})dx + R_1$$

Where R_1 is infinitesimal of higher order than $\delta y\ or \delta y^{(\alpha)}$ then

$$=\int_{x_0}^{x_1}(Z_y\,\delta y + Z_{y^{(\alpha)}}\delta y^{(\alpha)})dx = \int_{x_0}^{x_1}(Z_y\,\delta y + Z_{y^{(\alpha)}}D^{(\alpha)}\delta y)dx$$

Using (1.10) , for the second term , in which δy α-differentiable, we obtain:

$$\delta v^* = Z(x,y,y^{(\alpha)})\Big|_{x=x_1}\delta x_1 + \int_{x_0}^{x_1}(Z_y - \frac{d^\alpha}{dx^\alpha} Z_{y^{(\alpha)}})\,\delta y\,dx$$

By using the Fundamental Lemma *(1.3.1.1)*and since ($\delta v^* = 0$) we get :

$$Z(x,y,y^{(\alpha)})\Big|_{x=x_1} = 0$$

$$Z_y - \frac{d^\alpha}{dx^\alpha} Z_{y^{(\alpha)}} = 0$$

$$F_y - \frac{d^\alpha}{dx^\alpha} F_{y^{(\alpha)}} = 0$$

Or $(F+\lambda\phi)\Big|_{x=x_1} = 0$ **…(3.31 a)**

$(F+\lambda\phi)_y - \frac{d^\alpha}{dx^\alpha}(F+\lambda\phi)_{y^\alpha} = 0$ **…(3.31 b)**

then

$$\phi(x,y,y^{(\alpha)}) = 0$$

<u>second</u>, the problem (3.27) can be extended to different multi-fractional order $\alpha i > 0$ (i=1,2,..,m) of the following problem :

$v(x,y,y^{(\alpha_1)},\dots,y^{(\alpha_m)}) = \int_{x_0}^{x_1} F(x,y,y^{(\alpha_1)},\dots,y^{(\alpha_m)})dx$ **…(3.32)**

Subject to

$\phi_k(x, y, y^{(\alpha_1)}, \dots, y^{(\alpha_m)}) = 0$, for (k=1,…,K)

Our approach based on the theories presented in [15] , we construct the following auxiliary functional :

$$Z(x, y, y^{(\alpha_1)}, \dots, y^{(\alpha_m)}) = F + \lambda\phi$$

Where λ is a Lagrange multiplier , then the problem (3.32) can be started as following :

$$v^*(y) = \int_{x_0}^{x_1} Z(x, y, y^{(\alpha_1)}, \dots, y^{(\alpha_m)})dx \qquad \dots(3.33)$$

and one of the end points is variable (say (x_1,y_1)) , i.e. (x_1,y_1) can move turning into $(x_1 + \delta x_1 , y_1 + \delta y_1)$, then

$$\Delta v^* = \int_{x_0}^{x_1+\delta x_1} Z(x, y + \delta y , y^{(\alpha_1)} + \delta y^{(\alpha_1)}, \dots, y^{(\alpha_1)} + \delta y^{(\alpha_1)})\, dx - \int_{x_0}^{x_1} Z(x, y, y^{\alpha_1}, \dots, y^{\alpha_m})dx$$

$$\Delta v^* = \int_{x_1}^{x_1+\delta x_1} Z(x, y + \delta y , \delta y , y^{(\alpha_1)} + \delta y^{(\alpha_1)}, \dots, y^{(\alpha_m)} + \delta y^{(\alpha_m)})dx + \int_{x_0}^{x_1}[Z(x, y + \delta y , y^{(\alpha_1)} + \delta y^{(\alpha_1)}, \dots, y^{(\alpha_m)} + \delta y^{(\alpha_m)}) - Z(x, y, y^{(\alpha_1)}, \dots, y^{(\alpha_m)})]\, dx \qquad \dots(3.34)$$

The first term of the right- hand side of the equation (3.34) can be transform with the aid of the mean value theorem , we get :

$$\int_{x_1}^{x_1+\delta x_1} Z(x, y + \delta y , \delta y , y^{(\alpha_1)} + \delta y^{(\alpha_1)}, \dots, y^{(\alpha_m)} + \delta y^{(\alpha_m)})dx = Z\big|_{x=x_1+\theta\delta x_1} \delta x_1 ,$$

where $0 < \theta < 1$

Furthermore , by virtue of continuity of the function F ,

$$Z\Big|_{x=x_1+\theta\delta x_1} = Z(\mathrm{x},\mathrm{y},\mathrm{y}^{(\alpha)},\dots,y^{(\alpha_m)})\Big|_{\mathrm{x}=\mathrm{x}_1} + \mathcal{E}_1 \ ,$$

Where;

$$\mathcal{E}_1 \to 0 \quad as \quad \delta x_1 \to 0 \quad and \quad \delta y_1 \to 0$$

Consequently;

$$\int_{x_1}^{x_1+\delta x_1} Z\big(x, y+\delta y\,, y^{(\alpha)}+\delta y^{(\alpha)},\dots, y^{(\alpha_m)}+\delta y^{(\alpha_m)}\big)dx =$$

$$Z(\mathrm{x},\mathrm{y},\mathrm{y}^{(\alpha)},\dots,y^{(\alpha_m)})\Big|_{\mathrm{x}=\mathrm{x}_1}\delta\mathrm{x}_1 \qquad \dots (3.35)$$

The second term of the right-hand side of eq. (3.34) , can be transformed by using Taylor formula , we get :

$$\int_{x_0}^{x_1}(Z\big(x, y+\delta y\,, y^{(\alpha_1)}+\delta y^{(\alpha_1)},\dots, y^{(\alpha_m)}+\delta y^{(\alpha_m)}\big) -$$

$$Z(x,y,\mathrm{y}^{(\alpha_1)},\dots,y^{(\alpha_m)}))\ dx$$

$$=\int_{x_0}^{x_1}(Z_y\,\delta y + Z_{\mathrm{y}^{(\alpha_1)}}\delta y^{(\alpha_1)} + \dots + Z_{\mathrm{y}^{(\alpha_m)}}\delta y^{(\alpha_m)})dx + R1$$

Where R_1 is infinitesimal of higher order than $\delta y\ or \delta y^{(\alpha_1)},\dots,$ $\delta y^{(\alpha_m)}$ (i=1,…,m) then

$$\int_{x_0}^{x_1}(Z_y\,\delta y + Z_{\mathrm{y}^{(\alpha_1)}}\delta y^{(\alpha_1)} + \dots + Z_{\mathrm{y}^{(\alpha_m)}}\delta y^{(\alpha_m)})dx = \int_{x_0}^{x_1}(Z_y\,\delta y +$$

$$Z_{y^{(\alpha_1)}}D^{(\alpha_1)}\delta y + \dots + Z_{\mathrm{y}^{(\alpha_m)}}D^{(\alpha_m)}\delta y)dx$$

Using (1.10) , for the second term , in which δy α-differentiable, we have:

$$\delta v^* = Z(\mathrm{x},\mathrm{y},\mathrm{y}^{(\alpha_1)},\dots,y^{(\alpha_m)})\Big|_{\mathrm{x}=\mathrm{x}_1}\delta\mathrm{x}_1 + \int_{x_0}^{x_1}(Z_y - \sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}Z_{y_j^{(\alpha_i)}})\ \delta y\ dx$$

, By using the Fundamental *(1.3.1.1)*and since ($\delta v \;\; = 0$) ,we have the following necessary condition :

$$F + \lambda\phi\ \Big|_{\mathrm{x}=\mathrm{x}_1} \equiv 0 \qquad \dots (3.36\ a)$$

$$(F+\lambda\phi)_y-\sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F+\lambda\phi)_{y_j^{(\alpha_i)}}=0 \qquad \ldots(3.36\text{ b})$$

As well as

$\phi_k(\mathrm{x},\mathrm{y},\mathrm{y}^{(\alpha_1)},\ldots,y^{(\alpha_m)})=0$ **, for (k=1,…,K)**

<u>Third</u> , we shall discussed the necessary conditions for the general form of the problem (3.32) including many dependent variables , multi-fractional order derivatives , and fractional order constrains, in which such problems can be started as follows :

$$v(y_1,\ldots,y_n)=\int_{x_0}^{x_1}F(x,y_1,\ldots,y_n,y_1^{(\alpha_1)},\ldots,y_1^{(\alpha_m)},y_n^{(\alpha_1)},\ldots,y_n^{(\alpha_m)})\,dx \qquad \ldots(3.37)$$

Subject to

$$\phi_k=\int_{x_0}^{x_1}F(x,y_1,\ldots,y_n,y_1^{(\alpha_1)},\ldots,y_1^{(\alpha_m)},y_n^{(\alpha_1)},\ldots,y_n^{(\alpha_m)})\,dx=0 ,$$

(k=1,…,K)

by variants one dependent variable and fixing the remaining , dependent variables , , and extend to our problems , therefore , we construct the following auxiliary functional :

$$Z(x,y_1,\ldots,y_n,y_1^{(\alpha_1)},\ldots,y_1^{(\alpha_m)},y_n^{(\alpha_1)},\ldots,y_n^{(\alpha_m)})=F+\sum_{k=1}^{m}\lambda_k\,\phi_k$$

Where λ_k is a Lagrange multiplier , then the problem (3.37) can be started as following:

$$v^*(y_1,\ldots,y_n)=\int_{x_0}^{x_1}Z\left(x,y_1,\ldots,y_n,y_1^{(\alpha_1)},\ldots,y_1^{(\alpha_m)},\ldots,y_n^{(\alpha_1)},\ldots,y_n^{(\alpha_m)}\right) \qquad \ldots(3.38)$$

And one of the end points is variable (say (x_1,y_1)) , i.e. (x_1,y_1) can move turning into $(x_1 + \delta x_1 , y_1 + \delta y_1)$, then

$$\Delta v^* = \int_{x_0}^{x_1+\delta x_1} Z(x, y_1 + \delta y_1 , \dots , y_n + \delta y_n + y_1^{(\alpha_1)} + \delta y_1^{(\alpha_m)}, \dots , y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots , y_n^{(\alpha_1)} + \delta y_n^{(\alpha_m)})\, dx - \int_{x_0}^{x_1} Z(x, y_1, \dots , y_n, y_1^{(\alpha_1)}, \dots , y_1^{(\alpha_m)}, \dots , y_n^{(\alpha_1)}, \dots , y_n^{(\alpha_m)})\, dx$$

$$= \int_{x_1}^{x_1+\delta x_1} Z(x, y_1 + \delta y_1 , \dots , y_n + \delta y_n + y_1^{(\alpha_1)} + \delta y_1^{\alpha_m}, \dots , y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots , y_n^{(\alpha_1)} + \delta y_n^{(\alpha_m)})\, dx + \int_{x_0}^{x_1} [Z(x, y_1 + \delta y_1 , \dots , y_n + \delta y_n + y_1^{(\alpha_1)} + \delta y_1^{(\alpha_m)}, \dots , y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots , y_n^{(\alpha_1)} + \delta y_n^{(\alpha_m)}) - Z\left(x, y_1, \dots , y_n, y_1^{(\alpha_1)}, \dots , y_1^{(\alpha_m)}, \dots , y_n^{(\alpha_1)}, \dots , y_n^{(\alpha_m)}\right)]dx \qquad \dots(3.39)$$

The first term of the right- hand side of the equation (3.39) will be transform with the aid of the mean value theorem , we get :

$$\int_{x_1}^{x_1+\delta x_1} Z(x, y_1 + \delta y_1 , \dots , y_n + \delta y_n + y_1^{(\alpha_1)} + \delta y_1^{(\alpha_m)}, \dots , y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots , y_n^{(\alpha_m)} + \delta y_n^{(\alpha_m)})\, dx = Z \big|_{x=x_1+\theta\delta x_1} \delta x_1 \text{ , where } 0 < \theta < 1$$

Furthermore , by virtue of continuity of the function F ,

$$Z \big|_{x=x_1+\theta\delta x_1} = Z\left(x, y_1, \dots , y_n, y_1^{(\alpha_1)}, \dots , y_1^{(\alpha_m)}, \dots , y_n^{(\alpha_1)}, \dots , y_n^{(\alpha_m)}\right) \big|_{x=x_1} + \mathcal{E}_1 ,$$

Where;

$$\mathcal{E}_1 \to 0 \quad as \quad \delta x_1 \to 0 \quad and \quad \delta y_1 \to 0$$

Consequently;

$$\int_{x_1}^{x_1+\delta x_1} Z(x, y_1+\delta y_1, \dots, y_n+\delta y_n + y_1^{(\alpha_1)} + \delta y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_1)} + \delta y_n^{(\alpha_m)})\, dx =$$
$$Z\left(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right)\Big|_{x=x_1} \delta x_1 \qquad \dots(3.40)$$

The second term of the right-hand side of eq. (3.39) , can be transformed by using Taylor formula , we get :

$$\int_{x_0}^{x_1} [Z\left(x, y_1+\delta y_1, \dots, y_n+\delta y_n + y_1^{(\alpha_1)} + \delta y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_1)} + \delta y_n^{(\alpha_m)}\right) dx -$$
$$Z\left(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right)]\, dx$$

$$=\int_{x_0}^{x_1}(Z_{y_1}\delta y_1 + \dots + Z_{y_n}\delta y_n + Z_{y_1^{(\alpha_1)}}\delta y_1^{(\alpha_1)} + \dots + Z_{y_1^{(\alpha_m)}}\delta y_1^{(\alpha_m)} + Z_{y_n^{(\alpha_1)}}\delta y_n^{(\alpha_1)} + \dots + Z_{y_n^{(\alpha_m)}}\delta y_n^{(\alpha_m)})dx + R1$$

Where R_1 is infinitesimal of higher order than $\delta y_j \, or \delta y_j^{\alpha_i}$ for (i=1,…,m,j,1,…,n) then

$$\int_{x_0}^{x_1}(Z_{y_1}\delta y_1 + \dots + Z_{y_n}\delta y_n + Z_{y_1^{(\alpha_1)}}\delta y_1^{(\alpha_1)} + \dots + Z_{y_1^{(\alpha_m)}}\delta y_1^{(\alpha_m)} + Z_{y_n^{(\alpha_1)}}\delta y_n^{(\alpha_1)} + \dots + Z_{y_n^{(\alpha_m)}}\delta y_n^{(\alpha_m)})dx$$

$$= \int_{x_0}^{x_1}(Z_{y_1}\delta y_1 + \dots + Z_{y_n}\delta y_n + Z_{y_1^{(\alpha_1)}} D^{\alpha_1}\delta y_1 + \dots + Z_{y_1^{(\alpha_m)}} D^{\alpha_m}\delta y_1 + Z_{y_n^{(\alpha_1)}} D^{\alpha_1}\delta y_n + \dots + Z_{y_n^{(\alpha_m)}} D^{\alpha_m}\delta y_n)dx$$

Using (1.10) in which δy α-differentiable , we obtain :

$$\delta v^* = Z\Big|_{x=x_1}\delta x_1 + \int_{x_0}^{x_1}[(Z_{y_1} - \sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_{y_j^{(\alpha_i)}}))\,\delta y_1 \dots (Z_{y_n} - \sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_{y_j^{(\alpha_i)}}))\,\delta y_n]dx$$

using the Fundamental Lemma*(1.3.1.1)* and since ($\delta v = 0$) ,we have the following necessary condition :

$$(F + \sum_{k=1}^{m} \lambda_k \phi_k) \mid_{x=x_1} = 0 \qquad \ldots(3.41\ a)$$

$$[F_{y_j}] - [\sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} F_{y_j^{(\alpha_i)}}] = 0 \quad , (j=1,\ldots,n) \qquad \ldots(3,41\ b)$$

As well as

$$\phi_k \left(x, y_1, \ldots, y_n, y_1^{(\alpha_1)}, \ldots, y_1^{(\alpha_m)}, \ldots, y_n^{(\alpha_1)}, \ldots, y_n^{(\alpha_m)}\right) \mid_{x=x_1} = 0 \ ,$$

for (k=1,…,K)

<u>3.2.4 Constrained Problem Having Integer Non-integer Orders</u>

<u>First</u>, we shall consider the problem of the form :

$$v(y) = \int_{x_0}^{x_1} F(x, y, y', y^{(\alpha)})\, dx$$

Such that $\phi(x, y, y', y^{(\alpha)}) = 0$ **…(3.42)**

Where one of end points is variable say (x_1, y_1) i.e. (x_1, y_1) can move turning in to ($x_1 + \delta\, x_1, y_1 + \delta\, y_1$).

we construct the following auxiliary functional

$$Z(x, y, y', y^{(\alpha)}) = F + \lambda\phi$$

Where λ is a Lagrange multiplier , then the problem (3.42) can be stated as following:

$$v^*(x, y, y', y^{(\alpha)}) = \int_{x_0}^{x_1} Z(x, y, y', y^{(\alpha)})\, dx \qquad \ldots(3.43)$$

and one of end points is variable say (x_1, y_1) i.e. (x_1, y_1) can move turning in to ($x_1 + \delta\, x_1, y_1 + \delta\, y_1$).then

$$\Delta v^* = \int_{x_0}^{x_1+\delta x_1} Z(x, y+\delta y, y' + \delta y', y^{(\alpha)} + \delta y^{(\alpha)})\, dx - \int_{x_0}^{x_1} Z(x, y, y', y^{(\alpha)})\, dx$$

$$= \int_{x_1}^{x_1+\delta x_1} Z(x, y+\delta y, y' + \delta y', y^{(\alpha)} + \delta y^{(\alpha)})\, dx + \int_{x_0}^{x_1} [Z(x, y, y', y^{(\alpha)})\, dx - Z\left(x, y, y', y^{(\alpha)}\right)]dx \quad \ldots(3.44)$$

The first term of the right-hand side of equation (3.44) can be transformed with aid of the mean value theorem , we get :

$$\int_{x_1}^{x_1+\delta x_1} Z(x, y+\delta y, y' + \delta y', y^{(\alpha)} + \delta y^{(\alpha)})\, dx = Z|_{x=x_1+\theta\delta x_1} \delta x_1 ,$$

where $(0 < \theta < 1)$

Furthermore , by value of continuity of the function

$$Z|_{x=x_1+\theta\delta x_1} = Z\left(x, y, y', y^{(\alpha)}\right)|_{x=x_1} + \varepsilon_1$$

Where $\varepsilon_1 \to 0$ **as** $\delta x_1 \to 0$ **and** $\delta y_1 \to 0$

Consequently

$$\int_{x_0}^{x_1+\delta x_1} Z(x, y+\delta y, y' + \delta y', y^{(\alpha)} + \delta y^{(\alpha)})\, dx = Z\left(x, y, y', y^{(\alpha)}\right)|_{x=x_1} \delta x_1 \quad \ldots(3.45)$$

The second term of the right-hand side of equation (3.44) can be transformed by using Taylor formula given , we get:

$$\int_{x_0}^{x_1} [Z(x, y+\delta y, y' + \delta y', y^{(\alpha)} + \delta y^{(\alpha)}) - Z\left(x, y, y', y^{(\alpha)}\right)]dx = \int_{x_0}^{x_1} (Z_y \delta y + Z_{y'} \delta y' + Z_{y^{(\alpha)}} \delta y^{(\alpha)})dx + R1$$

Where R_1 **is infinitesimal of higher order than** δy **or** $\delta y^{(\alpha)}$

Integrating by part for the second term and using (1.10) for the third term , on which δy **is** α**-differentiable. we obtain :**

$$\int_{x_0}^{x_1}(Z_y\delta y + Z_{y'}\delta y' + Z_{y^{(\alpha)}}\delta y^{(\alpha)})dx = [Z_{y'}\delta y]|_{x_0}^{x_1} + \int_{x_0}^{x_1}\left(Z_y - \frac{d}{dx}Z_{y'}\delta - \frac{d^\alpha}{dx^\alpha}Z_{y^{(\alpha)}}\right)\delta y \, \mathrm{dx}$$

The values of the functional are taken only along extremal

$$Z_y - \frac{d}{dx}Z_{y'}\delta - \frac{d^\alpha}{dx^\alpha}Z_{y^{(\alpha)}} = 0$$

Then

$$\int_{x_0}^{x_1}(Z_y\delta y + Z_{y'}\delta y' + Z_{y^{(\alpha)}}\delta y^{(\alpha)})dx = Z_{y'}\delta y|_{x=x_1}$$

The same procedure in the moving boundary we get:

$$\delta y|_{x=x_1} \cong \delta y_1 - y'(x_1)\delta x_1$$

$$\delta v^* = Z|_{x=x_1}\delta x_1 + Z_{y'}|_{x=x_1}(\delta y_1 - y'(x_1))\delta x_1 = (Z - y'Z_{y'})|_{x=x_1}\delta x_1 + Z_{y'}|_{x=x_1}\delta y_1$$

The fundamental necessary condition for an extremum $\delta v = 0$ takes the form

$$(Z - y'Z_{y'})|_{x=x_1}\delta x_1 + Z_{y'}|_{x=x_1}\delta y_1 = 0 \qquad \ldots(3.46)$$

If the variations δx_1 and δy_1 are independent , then it follows that

$(Z - y'Z_{y'})|_{x=x_1} = 0$ **and** $Z_{y'}|_{x=x_1} = 0$

$$Z(x, y, y', y^{(\alpha)})|_{x=x_1} = 0$$

$$Z_y - \frac{d}{dx}Z_{y'} - \frac{d^\alpha}{dx^\alpha}Z_{y^{(\alpha)}} = 0$$

As well as

$$\phi(x, y, y', y^{(\alpha)}) = 0$$

<u>second</u>, we shall consider the problem of the form :

$$v(y_1, \ldots, y_n) = \int_{x_0}^{x_1} F(x, y_1, \ldots, y_n, y_1', \ldots, y_n', y_1^{(\alpha)}, \ldots, y_n^{(\alpha)})\, dx$$

Subject to

$$\phi_k\left(x, y_1, \dots, y_n, y_1', \dots, y_n', y_1^{(\alpha)}, \dots, y_n^{(\alpha)}\right) = 0 \quad , \quad (k=1,\dots k) \qquad \dots(3.47)$$

Where one of end points is variable say (x_1, y_1) i.e. (x_1, y_1) can move turning in to $(x_1 + \delta x_1, y_1 + \delta y_1)$.

we construct the following auxiliary functional

$$Z\left(x, y_1, \dots, y_n, y_1', \dots, y_n', y_1^{(\alpha)}, \dots, y_n^{(\alpha)}\right) = F \sum_{k=1}^{K} \lambda_k \phi_k$$

Where λ_k is Lagrange multiplier , then the problem (3.47) can be stated as following:

$$v^*(y_1, \dots, y_n) = \int_{x_0}^{x_1} Z(x, y_1, \dots, y_n, y_1', \dots, y_n', y_1^{(\alpha)}, \dots, y_n^{(\alpha)})\, dx \qquad \dots(3.48)$$

And one of end points is variable say (x_1, y_1) i.e. (x_1, y_1) can move turning in to $(x_1 + \delta x_1, y_1 + \delta y_1)$ then .

$$\Delta v^* = \int_{x_0}^{x_1+\delta x_1} Z(x, y_1 + \delta y_1, \dots, y_n + \delta y_n + y_1' + \delta y_1', \dots, y_n' + \delta y_n', y_1^{(\alpha)} + \delta y_1^{(\alpha)}, \dots, y_n^{(\alpha)} + \delta y_n^{(\alpha)})\, dx - \int_{x_0}^{x_1} Z(x, y_1, \dots, y_n, y_1', \dots, y_n', y_1^{(\alpha)}, \dots, y_n^{(\alpha)})\, dx$$

$$= \int_{x_1}^{x_1+\delta x_1} Z(x, y_1 + \delta y_1, \dots, y_n + \delta y_n + y_1' + \delta y_1', \dots, y_n' + \delta y_n', y_1^{(\alpha)} + \delta y_1^{(\alpha)}, \dots, y_n^{(\alpha)} + \delta y_n^{(\alpha)})\, dx + \int_{x_0}^{x_1} [Z(x, y_1 + \delta y_1, \dots, y_n + \delta y_n + y_1' + \delta y_1', \dots, y_n' + \delta y_n', y_1^{(\alpha)} + \delta y_1^{(\alpha)}, \dots, y_n^{(\alpha)} + \delta y_n^{(\alpha)}) - Z\left(x, y_1, \dots, y_n, y_1', \dots, y_n', y_1^{(\alpha)}, \dots, y_n^{(\alpha)}\right)]dx \qquad \dots(3.49)$$

The first term of the right-hand of equation (3.49) will be transformed with aid of mean value theorem to get :

$$= \int_{x_1}^{x_1+\delta x_1} Z(x, y_1 + \delta y_1, \dots, y_n + \delta y_n + y_1' + \delta y_1', \dots, y_n' + \delta y_n', y_1^{(\alpha)} + \delta y_1^{(\alpha)}, \dots, y_n^{(\alpha)} + \delta y_n^{(\alpha)})\, dx = Z|_{x=x_1+\theta\delta x_1} \delta x_1 \quad \text{, where } (0<\theta<1)$$

Furthermore , by value of continuity of the function

$$Z|_{x=x_1+\theta\delta x_1} = Z|_{x=x_1} + \varepsilon_1$$

Where $\varepsilon_1 \to 0$ as $\delta x_1 \to 0$ and $\delta y_1 \to 0$

Consequently

$$\int_{x_1}^{x_1+\delta x_1} Z(x, y_1+\delta y_1, \dots, y_n+\delta y_n + y_1' + \delta y_1', \dots, y_n' + \delta y_n', y_1^{(\alpha)} + \delta y_1^{(\alpha)}, \dots, y_n^{(\alpha)} + \delta y_n^{(\alpha)})\, dx = Z|_{x=x_1}\delta x_1$$

The second term of the right-hand side of eq. (3.49) can be transformed by using Taylor formula to get :

$$\int_{x_0}^{x_1}[Z(x, y_1+\delta y_1, \dots, y_n+\delta y_n + y_1' + \delta y_1', \dots, y_n' + \delta y_n', y_1^{(\alpha)} + \delta y_1^{(\alpha)}, \dots, y_n^{(\alpha)} + \delta y_n^{(\alpha)}) - Z\left(x, y_1, \dots, y_n, y_1', \dots, y_n', y_1^{(\alpha)}, \dots, y_n^{(\alpha)}\right)]dx = \int_{x_0}^{x_1} Z_{y_1}\delta y_1 + \dots + Z_{y_n}\delta y_n + Z_{y_1'}\delta y_1' + \dots + Z_{y_n'}\delta y_n', Z_{y_1^{(\alpha)}}\delta y_1^{(\alpha)} + \dots + Z_{y_n^{(\alpha)}}\delta y_n^{(\alpha)})\, dx + R1$$

Where R_1 is infinitesimal of higher order than δy_j or $\delta y_j'$ or $\delta y_j^{(\alpha)}$ for (j=1,…,n)

Integrating the term's from$Z_{y_1'}\delta y_1'$ to $Z_{y_n'}\delta y_n'$ by part and using (1.10) in which δy is α_i-differentiable. we obtain :

$$\sum_{j=1}^{n} Z_{y_j'}\delta y_j\Big]_{x_0}^{x_1} + \int_{x_0}^{x_1}\sum_{j=1}^{n}\left(Z_{y_j} - \frac{d}{dx}Z_{y_j'} - \frac{d^\alpha}{dx^\alpha}Z_{y_j^{(\alpha)}}\right)dx$$

$$\delta v^* = Z|_{x=x_1}\delta x_1 + \sum_{j=1}^{n} Z_{y_j'}\delta y_j\Big]_{x_0}^{x_1} + \int_{x_0}^{x_1}\sum_{j=1}^{n}\left(Z_{y_j} - \frac{d}{dx}Z_{y_j'} - \frac{d^\alpha}{dx^\alpha}Z_{y_j^{(\alpha)}}\right)dx$$

Since $\delta v^* = 0$ along extremes , we have the following:

$$Z|_{x=x_1}\delta x_1 + \sum_{j=1}^{n} Z_{y_j'}\delta y_j\Big]_{x_0}^{x_1} + \int_{x_0}^{x_1}\sum_{j=1}^{n}\left(Z_{y_j} - \frac{d}{dx}Z_{y_j'} - \frac{d^\alpha}{dx^\alpha}Z_{y_j^{(\alpha)}}\right)dx = 0$$

$$\sum_{j=1}^{n}\left(Z_{y_j} - \frac{d}{dx}Z_{y_j'} - \frac{d^\alpha}{dx^\alpha}Z_{y_j^{(\alpha)}}\right)dx = 0$$

By the same argument as that given on first problem of this chapter ,we obtain

$$\delta_{y_j}|_{x=x_1} \cong \delta_{y_j} - y_j'(x_1)\delta x_1 \;, \qquad \text{(j=1,…,n)}$$

And consequently

$$\delta v^* = [Z - \sum_{j=1}^{n} y_j' F_{y_j'}]_{x=x_1}\delta x_1 + [\sum_{j=1}^{n} Z_{y_j'}\,\delta y_j]_{x=x_1} = 0$$

If the variations , $\delta x_1, \delta y_1$,..., δy_n are independent then it follows the condition $\delta v = 0$ that:

$$[Z - \sum_{j=1}^{n} y_j' Z_{y_j'}]_{x=x_1} = 0 \qquad , \qquad F_{y_j'}|_{x=x_1} = 0$$

If the boundary can move a long certain curve $y_j = \phi_j(x_1)$ for all (j=1,..,n) then $y_j' = \phi_j' \delta x_1$ and the condition $\delta v = 0$ then

$$Z - [\sum_{j=1}^{n} y_j' Z_{y_j'}]_{x=x_1} \delta x_1 + \sum_{j=1}^{n} Z_{y_j'} \delta y_j = 0$$

Turn in to

$$Z + \sum_{j=1}^{n} (y_j' - \phi_j') Z_{y_j'}]_{x=x_1} \delta x_1 = 0$$

And since δx_1 is arbitrary , we have

$$[Z + \sum_{j=1}^{n} (y_j' - \phi_j') Z_{y_j'}]_{x=x_1} = 0 \qquad \text{...(3.50)}$$

As well as

$$\phi_k\left(x, y_1, \dots, y_n, y_1', \dots, y_n', y_1^{(\alpha)}, \dots, y_n^{(\alpha)}\right) = 0$$

<u>Third</u> : we are consider the problem of the form

$$v(y) = \int_{x_0}^{x_1} F(x, y, y', y'', y^{(\alpha)})\, dx$$

Subject to

$$\phi(x, y, y', y'', y^{(\alpha)}) = 0 \qquad \text{...(3.51)}$$

Where one of the end points is variable (say(x_1, y_1)) i.e. (x_1, y_1) can move turning into ($x_1 + \delta x_1, y_1 + \delta y_1$).

therefore , our approach based on the theories presented in [15] ,we construct the following auxiliary functional

$$Z(x, y, y', y'', y^{(\alpha)}) = F + \lambda\phi$$

Where λ_k is Lagrange multiplier , then the problem (3.51) can be stated as following:

$$v^*(y) = \int_{x_0}^{x_1} Z(x, y, y', y'', y^{(\alpha)})\, dx \qquad \text{... (3.52)}$$

$$\Delta v^* = \int_{x_0}^{x_1+\delta x_1} Z(x, y+\delta y, y' + \delta y', y'' + \delta y'', y^\alpha + \delta y^\alpha)dx - \int_{x_0}^{x_1} Z(x, y, y', y'', y^{(\alpha)})\, dx$$

$$= \int_{x_1}^{x_1+\delta x_1} Z(x, y+\delta y, y' + \delta y', y'' + \delta y'', y^{(\alpha)} + \delta y^{(\alpha)})dx + \int_{x_0}^{x_1}[Z(x, y+\delta y, y' + \delta y', y'' + \delta y'', y^{(\alpha)} + \delta y^{(\alpha)}) - Z(x, y, y', y'', y^{(\alpha)})]dx \qquad \ldots(3.53)$$

Applying the mean value theorem and using the continuity of the functions Z and $y'(x), y''(x), y^\alpha(x)$, we have

$$\Delta v^* = Z\big(x, y, y', y'', y^{(\alpha)}\big)|_{x=x_1}\delta x_1 + \int_{x_0}^{x_1}(Z_y \delta y + Z_{y'}\delta y' + Z_{y''}\delta y'' + Z_{y^\alpha}\delta y^\alpha)dx + R$$

Where R is an infinitesimal of order higher than the maximum of the absolute values $|\delta x_1|$, $|\delta y_1|$. $|\delta y|$, $|\delta y'|$ and $|\delta y''|$. consequently ;

$$\delta v^* = Z|_{x=x_1}\delta x_1 + \int_{x_0}^{x_1}(Z_y \delta y + Z_{y'}\delta y' + Z_{y''}\delta y'' + Z_{y^{(\alpha)}}\delta y^{(\alpha)})dx$$

Integrating by parts the second term of the integrated and doing the same twice with the third terms and using (1.10) for the forth terms, and then remembering that

$$\delta y|_{x=x_1} = 0 \quad , \quad \delta y'|_{x=x_1} = 0 \quad \text{and} \quad Z_y - \frac{d}{dx} Z_{y'} + \frac{d^2}{dx^2} Z_{y''} - \frac{d^\alpha}{dx^\alpha} Z_{y''} = 0$$

We have :

$$\delta v = [Z\delta x_1 + Z_{y'}\delta y + Z_{y''}\delta y' - \frac{d}{dx}(Z_{y'})\delta y]_{x=x_1}$$

Making use of relation $\delta y_1 = y'(x_1)\delta x_1 + [\delta y]_{x=x_1}$ and applying this result also to $\delta y_1'$

$$\delta y_1' = y''(x_1)\delta x_1 + [\delta y']_{x=x_1}$$

We have:

$$\delta v = \Big[Z - y' Z_{y'} - y'' Z_{y''} + y' \frac{d}{dx}(Z_{y''})\Big]_{x=x_1} \delta x_1 + [Z_{y'} - \frac{d}{dx} Z_{y''}]_{x=x_1}\delta y_1 + Z_{y''}|_{x=x_1}\delta y_1' = 0 \qquad \ldots(3.54)$$

If δx_1 , δy_1 and $\delta y_1'$ are independent , then their coefficients should vanish at the point $x = x_1$

$$Z - y' Z_{y'} - y'' Z_{y''} + y' \frac{d}{dx} Z_{y''} = 0$$

$$Z_{y'} - \frac{d}{dx} Z_{y''} = 0$$

$$Z_{y''} |_{x=x_1} = 0$$

If there is some relation between them , for instance $y_1 = \phi(x_1)$ and $y_1' = \psi(x_1)$, then $\delta y_1 = \phi'(x_1)\delta x_1$ and $\delta y_1' = \psi'(x_1)\delta x_1$ and substitution , these values into (3.53) we have :

$$y' Z_{y'} - y'' Z_{y''} + y' \frac{d}{dx}(Z_{y''}) + (Z_{y'} - \frac{d}{dx} Z_{y''})\phi' + Z_{y''} \psi']_{x=x_1} = 0$$

3.3 The Functional Of Discontinuous With Movable boundaries

In this section , we shall construct optimality and necessary conditions , when functional is discontinuous on (x_k) , for (k=1,…,m) having non-integers orders , functional having non-integers orders and integers orders.

3.3.1 Unconstraint Problem

First, we consider the simplest problem

$$v(y) = \int_{x_s}^{x_f} F(x, y, y^{(\alpha)})dx$$

Where $F_k(x, y, y^{(\alpha)})$ is well defined functional on the interval (x_f, y_f) and (x_f, y_f) is moving i.e. (x_f, y_f) is moving i.e. (x_f, y_f) an move turning in to $(x_f + \delta x_f, y_f + \delta y_f)$,$0 < \alpha < 1$ and with given prescribed boundaries conditions.

Since the Fundamental Lemma of the calculus of variation*(1.3.1.1)* cannot be applied , because of the discontinuities , it is more convenient to calculate the extreme value of $v(x, y(x), y^{(\alpha)})$ along the curves approximately so it is convenient to replace the integral of eq(2.32)by the following

$$v(y) = \int_{x_s}^{x_f} F(x, y, y^{(\alpha)})dx$$

$$\cong \int_{x_s}^{x_1} F_1(x, y, y^{(\alpha)})dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} F_k(x, y, y^{(\alpha)})\, dx + \int_{x_n}^{x_f} F_n(x, y, y^{(\alpha)})dx \quad \ldots(3.55)$$

Then

$$\Delta v = \int_{x_s}^{x_1} F_1(x, y, y^{(\alpha)})dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} F_k(x, y, y^{(\alpha)})\, dx + \int_{x_n}^{x_f+\delta x_f} F_n(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})\, dx - \int_{x_n}^{x_f} F_n(x, y, y^{(\alpha)})dx$$

$$\Delta v = \int_{x_s}^{x_1} F_1(x, y, y^{(\alpha)})dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} F_k(x, y, y^{(\alpha)})\, dx + \int_{x_f}^{x_f+\delta x_f} F_n(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})\, dx + \int_{x_n}^{x_f} [F_n(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}) - F_n(x, y, y^{(\alpha)})]dx \quad \ldots(3.56)$$

The third integral term of the right-hand side of equation (3.56) will be transformed with aid of the mean value theorem , we get:

$$\int_{x_f}^{x_f+\delta x_f} F_n(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})\, dx = F_n|_{x=x_f+\theta\delta x_f} \delta x_f\ ,$$

where $(0 < \theta < 1)$

Furthermore , by virtue of continuity of the functional *F*

$$F_n|_{x=x_f+\theta\delta x_f} \delta x_f = F_n(x, y, y^{(\alpha)})|_{x=x_f} + \mathcal{E}_1\ ,$$

Where $\mathcal{E}_1 \to 0$ as $\delta x_f \to 0$ **and** $\delta y_f \to 0$ **consequently ;**

$$\int_{x_f}^{x_f+\delta x_f} F_n(x,y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)})\,dx = F_n(x,y,y^{(\alpha)})|_{x=x_f}\delta x_f \quad ...(3.57)$$

The 4th integral term of the right-hand side of equation (3.56) can be transformed by using Taylor formula theorem to we get:

$$\int_{x_n}^{x_f}[F_n(x,y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)}) - F_n(x,y,y^{(\alpha)})]dx$$

$$= \int_{x_n}^{x_f}[(F_n)_y(x,y,y^{(\alpha)})\delta y + (F_n)_{y^\alpha}(x,y,y^{(\alpha)})\delta y^{(\alpha)}]dx + R$$

$$= \int_{x_n}^{x_f}[(F_n)_y(x,y,y^{(\alpha)})\delta y + (F_n)_{y^\alpha}(x,y,y^{(\alpha)})D^{(\alpha)}\delta y]dx$$

And

$$\delta v = \int_{x_s}^{x_1}(F_{1_y}\,\delta y + F_{1_{y^\alpha}}\delta y^{(\alpha)})dx + \sum_{k=2}^{n-1}\int_{x_{k-1}}^{x_k}(F_{k_y}\,\delta y + F_{k_{y^\alpha}}\delta y^{(\alpha)})dx + F_n(x,y,y^{(\alpha)})|_{x=x_f}\delta x_f + \int_{x_n}^{x_f}[(F_n)_y(x,y,y^{(\alpha)})\delta y + (F_n)_{y^\alpha}(x,y,y^{(\alpha)})D^{(\alpha)}\delta y]dx$$

Using (1.10) in which δy α-differentiable , we have :

$$\therefore\ \delta v = \int_{x_s}^{x_1}(F_{1_y} - \frac{d^\alpha}{dx^\alpha}F_{1_{y^\alpha}})\,\delta y\,dx + \sum_{k=2}^{n-1}\int_{x_{k-1}}^{x_k}(F_{k_y} - \frac{d^\alpha}{dx^\alpha}F_{k_{y^\alpha}})\delta y\,dx + F_n(x,y,y^{(\alpha)})|_{x=x_f}\delta x_f + \int_{x_n}^{x_f}((F_n)_y - \frac{d^\alpha}{dx^\alpha}(F_n)_{y^\alpha})\delta y\,dx$$

Since the value of the functional are only along extremals (i.e. $\delta v = 0$) consequently , we have the following necessary conditions:

$$F_n(x,y,y^{(\alpha)})|_{x=x_f} \equiv 0 \quad ...(3.58\ a)$$

$$((F_k)_y - \frac{d^\alpha}{dx^\alpha}(F_k)_{y^\alpha}) \equiv 0\text{ , for all k} \quad ...(3.58\ b)$$

<u>Second</u>, (3.54) can be extended to different multi-fractional order $\alpha_i > 0$ (i =1,2,…,m).

$$v = \int_{x_s}^{x_f} F(x,y,y^{(\alpha_1)},y^{(\alpha_2)},\dots,y^{(\alpha_m)})\,dx \quad ...(3.59)$$

Where $F_k(x,y,y^{(\alpha_1)},y^{(\alpha_2)},\dots,y^{(\alpha_m)})$ is well defined functional on the interval (x_f,y_f) and (x_f,x_f) is moving i.e. (x_f,y_f) is moving i.e. (x_f,y_f) an move turning in to $(x_f+\delta x_f, y_f+\delta y_f)$,

$$v(y) = \int_{x_s}^{x_f} F(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)})\, dx \cong \int_{x_s}^{x_1} F_1(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)})\, dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} F_k(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)})\, dx + \int_{x_n}^{x_f} F_n(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)})\, dx \quad \dots(3.60)$$

then

$$\Delta v = \int_{x_s}^{x_1} F_1(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)}) dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} F_k(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)})\, dx + \int_{x_n}^{x_f+\delta x_f} F_n(x, y + \delta y, y^{(\alpha_1)} + \delta y^{(\alpha_1)}, y^{(\alpha_2)} + \delta y^{(\alpha_2)}, \dots, y^{(\alpha_m)} + \delta y^{(\alpha_m)})\, dx - \int_{x_n}^{x_f} F_n(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)}) dx$$

$$= \int_{x_s}^{x_1} F_1(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)}) dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} F_k(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)})\, dx + \int_{x_f}^{x_f+\delta x_f} F_n(x, y + \delta y, y^{(\alpha_1)} + \delta y^{(\alpha_1)}, y^{(\alpha_2)} + \delta y^{(\alpha_2)}, \dots, y^{(\alpha_m)} + \delta y^{(\alpha_m)})\, dx + \int_{x_n}^{x_f} [F_n\left(x, y + \delta y, y^{(\alpha_1)} + \delta y^{(\alpha_1)}, y^{(\alpha_2)} + \delta y^{(\alpha_2)}, \dots, y^{(\alpha_m)} + \delta y^{(\alpha_m)}\right) - F_n\left(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)}\right)] dx \quad \dots(3.61)$$

The third integral term of the right-hand side of equation (3.61) will be transformed with aid of the mean value theorem , we get:

$$\int_{x_f}^{x_f+\delta x_f} F_n(x, y + \delta y, y^{(\alpha_1)} + \delta y^{(\alpha_1)}, y^{(\alpha_2)} + \delta y^{(\alpha_2)}, \dots, y^{(\alpha_m)} + \delta y^{(\alpha_m)})\, dx = F_n|_{x=x_f+\theta\delta x_f} \delta x_f \;, \quad \text{where } (0 < \theta < 1)$$

Furthermore , by virtue of continuity of the functional *F*

$$F_n|_{x=x_f+\theta\delta x_f} \delta x_f = F_n\left(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)}\right)|_{x=x_f} + \mathcal{E}_1 \,,$$

Where $\mathcal{E}_1 \to 0$ *as* $\delta x_f \to 0$ **and** $\delta y_f \to 0$ **consequently ;**

$$\int_{x_f}^{x_f+\delta x_f} F_n(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)})\, dx = F_n\left(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)}\right)|_{x=x_f} \delta x_f \quad \dots(3.62)$$

The 4th integral term of the right-hand side of equation (3.61) can be transformed by using Taylor formula theorem , we get:

$$\int_{x_n}^{x_f}[F_n\left(x,y+\delta y,y^{(\alpha_1)}+\delta y^{(\alpha_1)},y^{(\alpha_2)}+\delta y^{(\alpha_2)},\dots,y^{(\alpha_m)}+\delta y^{(\alpha_m)}\right)-F_n\left(x,y,y^{(\alpha_1)},y^{(\alpha_2)},\dots,y^{(\alpha_m)}\right)]dx$$

$$=$$

$$\int_{x_n}^{x_f}[(F_n)_y(x,y,y^{(\alpha_1)},y^{(\alpha_2)},\dots,y^{(\alpha_m)})\delta y+ (F_n)_{y^{\alpha_1}}\left(x,y,y^{(\alpha_1)},y^{(\alpha_2)},\dots,y^{(\alpha_m)}\right)\delta y^{(\alpha_1)}+ (F_n)_{y^{\alpha_2}}\left(x,y,y^{(\alpha_1)},y^{(\alpha_2)},\dots,y^{(\alpha_m)}\right)\delta y^{(\alpha_2)}+\cdots+ (F_n)_{y^{\alpha_m}}\left(x,y,y^{(\alpha_1)},y^{(\alpha_2)},\dots,y^{(\alpha_m)}\right)\delta y^{(\alpha_m)}]dx$$

$$\delta v= \int_{x_s}^{x_1}(F_{1_y}\,\delta y+F_{1_{y^{(\alpha_1)}}}\delta y^{(\alpha_1)}+F_{1_{y^{(\alpha_2)}}}\delta y^{(\alpha_2)}+\cdots+F_{1_{y^{(\alpha_m)}}}\delta y^{(\alpha_m)})dx+ \sum_{k=2}^{n-1}\int_{x_{k-1}}^{x_k}(F_{k_y}\,\delta y+F_{k_{y^{(\alpha_1)}}}\delta y^{(\alpha_1)}+F_{k_{y^{(\alpha_2)}}}\delta y^{(\alpha_2)}+\cdots+ F_{k_{y^{(\alpha_m)}}}\delta y^{(\alpha_m)})dx+F_n\left(x,y,y^{(\alpha_1)},y^{(\alpha_2)},\dots,y^{(\alpha_m)}\right)|_{x=x_f}\delta x_f+ \int_{x_n}^{x_f}[(F_n)_y\,\delta y+(F_n)_{y^{(\alpha_1)}}\delta y^{(\alpha_1)}+(F_n)_{y^{(\alpha_2)}}\delta y^{(\alpha_2)}+\cdots+ (F_n)_{y^{(\alpha_m)}}\delta y^{(\alpha_m)}]dx \quad \dots(3.62)$$

Using (1.10) in which δy α-differentiable , we get :

$$\delta v=\int_{x_s}^{x_1}((F_1)_y-\frac{d^{\alpha}}{dx^{\alpha}}(F_1)_{y^{(\alpha)}}-\frac{d^{\alpha_1}}{dx^{\alpha_1}}(F_1)_{y^{(\alpha_1)}}-\frac{d^{\alpha_2}}{dx^{\alpha_2}}(F_1)_{y^{\alpha_2}}-\cdots- \frac{d^{\alpha_m}}{dx^{\alpha_m}}(F_1)_{y^{(\alpha_m)}})\,\delta y\,dx+ \sum_{k=2}^{n-1}\int_{x_{k-1}}^{x_k}((F_k)_y-\frac{d^{\alpha_1}}{dx^{\alpha_1}}(F_k)_{y^{(\alpha_1)}}-\frac{d^{\alpha_2}}{dx^{\alpha_2}}(F_k)_{y^{(\alpha_2)}}-\cdots- \frac{d^{\alpha_m}}{dx^{\alpha_m}}(F_k)_{y^{(\alpha_m)}})\delta y\,dx+F_n\left(x,y,y^{(\alpha_1)},y^{(\alpha_2)},\dots,y^{(\alpha_m)}\right)|_{x=x_f}\delta x_f+ \int_{x_{s_n}}^{x_f}((F_n)_y--\frac{d^{\alpha_1}}{dx^{\alpha_1}}(F_n)_{y^{(\alpha_1)}}-\frac{d^{\alpha_2}}{dx^{\alpha_2}}(F_n)_{y^{(\alpha_2)}}-\cdots-\frac{d^{\alpha_m}}{dx^{\alpha_m}}(F_n)_{y^{(\alpha_m)}})\delta y\,dx$$

Since the value of the functional are only along extremals (i.e. $\delta v\ =0$) consequently , we have the following necessary conditions:

$$F_n\left(x,y,y^{(\alpha_1)},y^{(\alpha_2)},\dots,y^{(\alpha_m)}\right)|_{x=x_f}\delta x_f\equiv 0 \quad \dots(3.63\text{ a})$$

$$((F_k)_y - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} (F_k)_{y^{(\alpha_i)}}) \equiv 0\ , \qquad \text{for all k} \qquad \dots(3.63\ b)$$

<u>Third</u> (3.59) can be extended further more to multi-dependent variable .

$$v(y_1, \dots, y_n) = \int_{x_s}^{x_f} F(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})dx$$

...(3.64)

by variant one dependent variable and fixing the retaining dependent

$$v(y_1, \dots, y_n) = \int_{x_s}^{x_f} F(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})dx$$

...(3.65)

$$\cong \int_{x_s}^{x_1} F_1(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} F_k(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})\, dx + \int_{x_n}^{x_f} F_n(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})dx \qquad \dots(3.66)$$

Where $F_k(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})$ is well defined functional on the interval (x_f, y_f) and (x_f, y_f) is moving i.e. (x_f, y_f) is moving i.e. (x_f, y_f) an move turning in to $(x_f + \delta x_f, y_f + \delta y_f)$, then

$$\Delta v = \int_{x_s}^{x_1} F_1(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} F_k(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})\, dx + \int_{x_n}^{x_f+\delta x_f} F_n(x, y_1 + \delta y_1, \dots, y_n + \delta y_n, y_1^{(\alpha_1)} + \delta y_1^{(\alpha_1)}, \dots, y_1^{\alpha_m} + \delta y_1^{(\alpha_m)}, y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)} + \delta y_n^{(\alpha_m)})\, dx - \int_{x_n}^{x_f} F_n\left(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right) dx$$

$$\Delta v = \int_{x_s}^{x_1} F_1(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} F_k(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})\, dx + \int_{x_f}^{x_f+\delta x_f} F_n(x, y_1 + \delta y_1, \dots, y_n + \delta y_n, y_1^{(\alpha_1)} + \delta y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)} + \delta y_1^{(\alpha_m)}, y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)} + \delta y_n^{(\alpha_m)})\, dx + \int_{x_n}^{x_f} [F_n(x, y_1 + \delta y_1, \dots, y_n + \delta y_n, y_1^{(\alpha_1)} + \delta y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)} + \delta y_1^{(\alpha_m)}, y_n^{(\alpha_1)} +$$

$$\delta y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)} + \delta y_n^{(\alpha_m)}) -$$
$$F_n\left(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right)]dx \qquad \dots(3.67)$$

The third integral term of the right-hand side of equation (3.67) will be transformed with aid of the mean value theorem , we get:

$$\int_{x_f}^{x_f+\delta x_f} F_n(x, y_1 + \delta y_1, \dots, y_n + \delta y_n, y_1^{(\alpha_1)} + \delta y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)} + \delta y_1^{(\alpha_m)}, y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)} + \delta y_n^{(\alpha_m)})\, dx = F_n|_{x=x_f+\theta\delta x_f}\delta x_f \quad ,$$

where $(0 < \theta < 1)$

Furthermore , by virtue of continuity of the functional F

$$F_n|_{x=x_f+\theta\delta x_f}\delta x_f = F_n\left(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right)|_{x=x_f} + \mathcal{E}_1 \,,$$

Where $\mathcal{E}_1 \to 0$ as $\delta x_f \to 0$ **and** $\delta y_f \to 0$ **consequently ;**

$$\int_{x_f}^{x_f+\delta x_f} F_n(x, y_1 + \delta y_1, \dots, y_n + \delta y_n, y_1^{(\alpha_1)} + \delta y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)} + \delta y_1^{(\alpha_m)}, y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)} + \delta y_n^{(\alpha_m)})\, dx =$$
$$F_n\left(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)1}, \dots, y_n^{(\alpha_m)}\right) dx =$$
$$F_n\left(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right)|_{x=x_f}\delta x_f \qquad \dots(3.68)$$

The 4th integral term of the right-hand side of equation (3.67) can be transformed by using Taylor formula theorem to get:

$$\int_{x_n}^{x_f}[F_n\left(, y_1 + \delta y_1, \dots, y_n + \delta y_n, y_1^{(\alpha_1)} + \delta y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)} + \delta y_1^{(\alpha_m)}, y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)} + \delta y_n^{(\alpha_m)}\right) -$$
$$F_n\left(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right)]dx$$

$$=$$
$$\int_{x_n}^{x_f}[(F_n)_{y_1}\delta y_1 + \dots + (F_n)_{y_n}\delta y_n + (F_n)_{y_1^{(\alpha_1)}}\delta y_1^{(\alpha_1)} + \dots +$$
$$(F_n)_{y_1^{(\alpha_m)}}\delta y_1^{(\alpha_m)} + \dots + (F_n)_{y_n^{(\alpha_1)}}\delta y_n^{(\alpha_1)} + \dots + (F_n)_{y_n^{(\alpha_m)}}\delta y_n^{(\alpha_m)}]dx$$

Then

$$\delta v = \int_{x_s}^{x_f}[F_{1_{y_1}}\delta y_1 + \cdots + F_{1_{y_n}}\delta y_n + F_{1_{y_1^{(\alpha_1)}}}\delta y_1^{(\alpha_1)} + \cdots + F_{1_{y_1^{(\alpha_m)}}}\delta y_1^{(\alpha_m)} + \cdots + F_{n_{y_n^{(\alpha_1)}}}\delta y_n^{(\alpha_1)} + \cdots + F_{n_{y_n^{(\alpha_m)}}}\delta y_n^{(\alpha_m)}]dx + \sum_{k=2}^{n-1}\int_{x_{k-1}}^{x_k}(F_{k_{y_1}}\delta y_1 + \cdots + F_{k_{y_n}}\delta y_n + F_{k_{y_1^{(\alpha_1)}}}\delta y_1^{(\alpha_1)} + \cdots + F_{k_{y_1^{(\alpha_m)}}}\delta y_1^{(\alpha_m)} + \cdots + F_{k_{y_n^{(\alpha_1)}}}\delta y_n^{(\alpha_1)} + \cdots + F_{k_{y_n^{(\alpha_m)}}}\delta y_n^{(\alpha_m)})dx + F_n\left(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right)|_{x=x_f}\delta x_f + \int_{x_n}^{x_f}(F_{1_{y_1}}\delta y_1 + \cdots + F_{1_{y_n}}\delta y_n + F_{1_{y_1^{(\alpha_1)}}}\delta y_1^{(\alpha_1)} + \cdots + F_{1_{y_1^{(\alpha_m)}}}\delta y_1^{(\alpha_m)} + \cdots + F_{n_{y_n^{(\alpha_1)}}}\delta y_n^{(\alpha_1)} + \cdots + F_{n_{y_n^{(\alpha_m)}}}\delta y_n^{(\alpha_m)})dx$$

Using (1.10) in which δy α-differentiable , we get :

$$\delta v = \int_{x_s}^{x_1}(F_{1_{y_1}} - \frac{d^{\alpha_1}}{dx^{\alpha_1}}F_{1_{y_1^{(\alpha_1)}}} - \cdots - \frac{d^{\alpha_m}}{dx^{\alpha_m}}F_{1_{y_1^{(\alpha_m)}}})\,\delta y_1\,dx + \cdots + (F_{1_{y_n}} - \frac{d^{\alpha_1}}{dx^{\alpha_1}}F_{1_{y_n^{(\alpha_1)}}} - \cdots - \frac{d^{\alpha_m}}{dx^{\alpha_m}}F_{1_{y_n^{(\alpha_m)}}})\delta y_n\,dx + \sum_{k=2}^{n-1}\int_{x_{k-1}}^{x_k}(F_{k_{y_1}} - \frac{d^{\alpha_1}}{dx^{\alpha_1}}F_{k_{y_1^{(\alpha_1)}}} - \cdots - \frac{d^{\alpha_m}}{dx^{\alpha_m}}F_{k_{y_1^{(\alpha_m)}}})\delta y_1 + \cdots + \int_{x_{k-1}}^{x_k}(F_{k_{y_{1n}}} - \frac{d^{\alpha_1}}{dx^{\alpha_1}}F_{k_{y_n^{(\alpha_1)}}} - \cdots - \frac{d^{\alpha_m}}{dx^{\alpha_m}}F_{k_{y_n^{(\alpha_m)}}})\delta y_n + F_n|_{x=x_f}\delta x_f + \int_{x_n}^{x_f}(F_{n_{y_1}} - \frac{d^{\alpha_1}}{dx^{\alpha_1}}F_{n_{y_1^{(\alpha_1)}}} - \cdots - \frac{d^{\alpha_m}}{dx^{\alpha_m}}F_{n_{y_1^{(\alpha_m)}}})\,\delta y_1\,dx + \cdots + (F_{n_{y_n}} - \frac{d^{\alpha_1}}{dx^{\alpha_1}}F_{n_{y_n^{(\alpha_1)}}} - \cdots - \frac{d^{\alpha_m}}{dx^{\alpha_m}}F_{n_{y_n^{(\alpha_m)}}})\delta y_n\,dx$$

Since the value of the functional are only along extremals (i.e. $\delta v = 0$) consequently , we have the following necessary conditions:

$$F_n\left(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right)|_{x=x_f}\delta x_f \equiv 0 \quad \dots(3.67\text{ a})$$

$$[(F_h)_{y_j}] - \sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_h)_{y_j^{(\alpha_i)}}) \equiv 0\,, \quad \text{for all h and j} \quad \dots(3.67\text{ b})$$

CHAPTER TWO

Optimality Necessary Conditions Of Fractional Variation Problem Along Fixed Boundaries

2.1 Introduction

In this chapter we concern on with the constructing of the optimality the necessary conditions for unconstrained and constrained fractional variation problems with continuous and discontinuous functional where the independent variable along fixed boundaries having one or different multi fractional order derivatives on one and different multi-dependent variables ,by using the formula(1.10).

2.2 The Functional Of Continuous With Fixed Boundaries

In this section we shall constructed the optimality necessary condition, when the functional integrand is continuous having non-integers orders ,also functional having non-integers and integer orders.

2.2.1 Unconstrained Problem Having only Non-integer Order

***First*, we shall consider the problem of the form :**

$$v(y)=\int_{x_0}^{x_1} F(x,y,\mathrm{y}^{(\alpha)})\,dx \qquad \dots(2.1)$$

where;

$0<\alpha<1$, and with given prescribed boundaries conditions.

It is also assumed that the higher integer and fractional order derivatives of the function $F(x, y, y^{(\alpha)})$ exists, where α is real.

We already know that a necessary condition for an extremum of a functional is that its first variation vanishes. We take any admissible curve $y = y^*(x)$, neighboring to $y = y(x)$ and we set up one-parameter family of curves;

$y(x,\psi) = y(x) + \psi\big(y^*(x) - y(x)\big) = y(x) + \psi\delta y.$

When $\psi = 0$, we have $y = y(x)$, and when $\psi = 1$ we have $y = y^*(x)$.

The variation $\delta y = y^*(x) - y(x)$ is a function of the variable ψ, this function can be differentiated once or more and we have:

$D^{\alpha}(\delta y) = (\delta y)^{(\alpha)} = (y^*)^{(\alpha)} - y^{(\alpha)} = \delta y^{(\alpha)},$

$D^{n\alpha}(\delta y) = (\delta y)^{(n\alpha)} = (y^*)^{(n\alpha)} - y^{(n\alpha)} = \delta y^{(n\alpha)}.$

Take on along the curve of the family $y = y(x,\psi)$ only, then we have a function of the variable ψ :

$$v\big(y(x,\psi)\big) = \varphi(\psi).$$

It is well known, the necessary condition that the function $\varphi(\psi)$ has an extremum for $\psi = 0$ its derivative should vanish.

$$\delta v\big(y(x,\psi)\big) = \frac{\partial}{\partial\psi}\, v(y(x) + \delta y)\Big|_{\psi=0}$$

$\delta v = \acute{\varphi}(\psi) = \acute{\varphi}(0) = 0.$

Since;

$$\varphi(\psi) = \int_{x_0}^{x_1} F\left(x, y(x,\psi), y^{(\alpha)}(x,\psi)\right) dx, \qquad \ldots(2.2)$$

we have;

$$\acute{\varphi}(\psi) = \int_{x_0}^{x_1} \left(F_y \frac{\partial}{\partial\psi} y(x,\psi) + F_{y^{(\alpha)}} \frac{\partial}{\partial\psi} y^{(\alpha)}(x,\psi)\right) dx, \qquad \ldots(2.3)$$

where;

$$F_y = \frac{\partial}{\partial y} F\left(x, y(x,\psi), y^{(\alpha)}(x,\psi)\right), \quad \dots (2.4)$$

$$F_{y^{(\alpha)}} = \frac{\partial}{\partial y^{(\alpha)}} F\left(x, y(x,\psi), y^{(\alpha)}(x,\psi)\right). \quad \dots (2.5)$$

Because of the relations:

$$\frac{\partial}{\partial \psi} y(x,\psi) = \frac{\partial}{\partial \psi}(y(x) + \psi \delta y) = \delta y, \quad \dots (2.6)$$

and

$$\frac{\partial}{\partial \psi} y^{(\alpha)}(x,\psi) = \frac{\partial}{\partial \psi}\left(y^{(\alpha)}(x) + \psi \delta y^{(\alpha)}\right) = \delta y^{(\alpha)}, \quad \dots (2.7)$$

it follows that:

$$\acute{\varphi}(\psi) = \int_{x_0}^{x_1} \left(F_y\left(x, y(x,\psi), y^{(\alpha)}(x,\psi)\right)\delta y + F_{y^{(\alpha)}}\left(x, y(x,\psi), y^{(\alpha)}(x,\psi)\right)\delta y^{(\alpha)}\right) dx, \quad \dots (2.8)$$

$$\acute{\varphi}(0) = \int_{x_0}^{x_1} \left(F_y\left(x, y(x), y^{(\alpha)}(x)\right)\delta y + F_{y^{(\alpha)}}\left(x, y(x), y^{(\alpha)}(x)\right)\delta y^{(\alpha)}\right) dx. \quad \dots (2.9)$$

As we have already remarked, $\acute{\varphi}(0)$ is called a variation of the functional and it is designated by δv. The necessary condition for a functional v to have an extremum is that its variation should vanish $\delta v = 0$.

Then

$$\delta v = \int_{x_0}^{x_1} F_y\, \delta y + F_{y^{(\alpha)}} \delta y^{(\alpha)}\ )dx = \int_{x_0}^{x_1} F_y\, \delta y + F_{y^{(\alpha)}} D^{\alpha} \delta y\,)dx \quad \dots (2.10)$$

Using (1.10) for the second term , in which δy is α differentiable we have

$$\delta v = \int_{x_0}^{x_1} (F_y - \frac{d^{\alpha}}{dx^{\alpha}} F_{y^{(\alpha)}})\delta y\ \ dx$$

Since the value of the functional are only along extremals (i.e. $\delta v = 0$) consequently $\delta v = \int_{x_0}^{x_1}(F_y - \frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}})\delta y\, dx = 0$

By using the Fundamental Lemma*(1.3.1.1)* we have the following necessary condition

$$F_y - \frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}} = 0 \qquad \ldots(2.11)$$

second, we consider the problem of the form :

$$v(y) = \int_{x_0}^{x_1} F(x, y, y^{(\alpha)}, y^{(\beta)})\, dx$$

for non integer, α *and* β, $\alpha > 0$, $\beta > 0$, with given prescribed boundaries conditions.

Then

$$\delta v = \int_{x_0}^{x_1} F_y\,\delta y + F_{y^{(\alpha)}}\delta y^{(\alpha)} + F_{y^{(\beta)}}\delta y^{(\beta)}\,)dx = \int_{x_0}^{x_1}(F_y\,\delta y + F_{y^{(\alpha)}}D^\alpha\delta y + F_{y^{(\beta)}}D^\beta\delta y\,)\, dx \qquad \ldots(2.12)$$

Using (1.10) , for the second and third terms of (2.12) which δy α-differentiable we obtain :

$$\delta v = \int_{x_0}^{x_1}(F_y - \frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}} - \frac{d^\beta}{dx^\beta}F_{y^{(\beta)}})\delta y\, dx$$

Since the value of the functional are only along extremals (i.e. $\delta v = 0$) consequently we have :

$$F_y - \frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}} - \frac{d^\beta}{dx^\beta}F_{y^{(\beta)}} = 0 \qquad \ldots(2.13)$$

Third, (2.1) can be extended, to different multi fractional-orderα_j >0 , and non integer (j=1,2,3,…m) , of the following problem :

$$v(y) = \int_{x_0}^{x_1} F(x, y, y^{(\alpha_1)}, \dots, y^{(\alpha_m)})\, dx \qquad \dots(2.14)$$

Then

$$\delta v = \int_{x_0}^{x_1} F_y\, \delta y + \sum_{j=1}^{m} F_{y^{(\alpha_j)}} \delta y^{(j)}\ )dx =$$

$$\int_{x_0}^{x_1} (F_y\, \delta y + \sum_{j=1}^{m} F_{y^{(\alpha_j)}} D^{\alpha_j} \delta y\ )\, dx \qquad \dots(2.15)$$

Using (1.10) , for the $F_{y^{(\alpha_j)}} D^{\alpha_j} \delta_y$, for all j, in which δy is α-differentiable, we obtain:

$$\delta v = \int_{x_0}^{x_1} (F_y - \frac{d^{\alpha_1}}{dx^{\alpha_1}} F_{y^{(\alpha_1)}} - \dots - \frac{d^{\alpha_m}}{dx^{\alpha_m}} F_{y^{(\alpha_m)}}) \delta y\, dx \qquad \dots(2.16)$$

Since the value of the functional are only along extremals (i. e. $\delta v = 0$) consequently , we have the following necessary condition :

$$F_y - \sum_{j=1}^{m} \frac{d^{\alpha_j}}{dx^{\alpha_j}} F_{y^{(\alpha_j)}} = 0 \qquad \dots(2.17)$$

***<u>Forth</u>*, the problem (2.14) can extended, further more to multi dependent variable , for the following problem : α_i>0 non integer**

$$v(y_1, \dots, y_n) = \int_{x_0}^{x_1} F\left(x, y_1, \dots, y_n,\ y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\ \right) dx$$

$$\dots(2.18)$$

By variants one dependent variable and fixing the remaining dependent variables ,we have

Then

$$\int_{x_0}^{x_1} (F_{y_1}\, \delta y_1, \dots, F_{y_n} \delta y_n\ , \dots, F y_1^{(\alpha_1)} \delta y_1^{(\alpha_1)}, \dots, F y_1^{(\alpha_m)} \delta y_1^{(\alpha_m)}, \dots, F y_n^{(\alpha_1)} \delta y_n^{(\alpha_1)}, \dots, F y_n^{(\alpha_m)} \delta y_n^{(\alpha_m)})\, dx$$

$$=$$

$$\int_{x_0}^{x_1} (F_{y_1}\, \delta y_1, \dots\ , F_{y_n} \delta y_n, \dots, F y_1^{(\alpha_1)} D^{\alpha_1} \delta y_1, \dots, F y_1^{(\alpha_m)} D^{\alpha_m} \delta y_1, \dots, F y_n^{(\alpha_1)} D^{\alpha_1} \delta y_n, \dots,$$

$$F y_n^{(\alpha_m)} D^{\alpha_m} \delta y_n) dx \qquad \dots(2.19)$$

Using (1.10) , for the $F_{y_i} D^{\alpha_j} \delta y_i$, for all i,j , in which δy is α_i-differentiable, we obtain:

$$\delta v =$$

$$\int_{x_0}^{x_1} ((F_{y_1} - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} F_{y_1^{(\alpha_i)}}) \delta y_1 + \cdots + \left(F_{y_n} - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} F_{y_n^{(\alpha_i)}}\right) \delta y_n \, dx$$

...(2.20)

Since the value of the functional are only along extremals (i. e. $\delta v = 0$) consequently , we have the following necessary condition :

$$\left[F_{y_j}\right] - \left[\sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} F_{y_j^{(\alpha_i)}}\right] = 0 \quad , \text{(j=1,2,…,n)} \qquad \text{...(2.21)}$$

2.2.2 Unconstrained Problem Having Integer And Non-integer Order

First, **we shall consider the problem of the form :**

$$v(y) = \int_{x_0}^{x_1} F(x, y, y', y^{(\alpha)}) \, dx \qquad \text{...(2.22)}$$

where;

$0 < \alpha < 1$, and with given prescribed boundaries conditions.

$$\delta v = \int_{x_0}^{x_1} F(x, y + \delta y, y' + \delta y', y^{(\alpha)} + \delta y^{(\alpha)}) \, dx$$

$$= \int_{x_0}^{x_1} F_y \delta y + F_{y'} \delta y' + F_{y^{(\alpha)}} \delta y^{(\alpha)}) \, dx$$

Integrate the second term by part , and using the (1.10) for the third term in which δy α-differentiable ,we have

$$\int_{x_0}^{x_1} F_y \delta y - \frac{d}{dx}(F_{y'}) \delta y - \frac{d^\alpha}{dx^\alpha} F_{y^{(\alpha)}} \delta y) \, dx$$

$$\int_{x_0}^{x_1} (F_y - \frac{d}{dx} F_{y'} - \frac{d^\alpha}{dx^\alpha} F_{y^{(\alpha)}}) \delta y \, dx = 0$$

Since $\delta v = 0$, along extremals , By using the Fundamental Lemma*(1.3.1.1)*. we have the following necessary condition

$$F_y - \frac{d}{dx}F_{y'} - \frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}} = 0 \qquad \ldots(2.23)$$

second **the problem (2.24) can be extended, to different multi integer and non integer order α_i>0 , (i=1,2,3,…n) , of the following problem :**

$$v(y) = \int_{x_0}^{x_1} F(x, y, y', \ldots, y^n, y^{(\alpha_1)}, \ldots, y^{(\alpha_m)}) \qquad \ldots(2.24)$$

$$\delta v = \int_{x_0}^{x_1} F(x, y + \delta y, y' + \delta y', y^n + \delta y^n, y^{(\alpha_1)} + \delta y^{(\alpha_1)}, \ldots, y^{(\alpha_m)} + \delta y^{(\alpha_m)})\, dx$$

$$\delta v = \int_{x_0}^{x_1} F_y\, \delta y \;+ \sum_{k=1}^{n} \; F_{y^k}\delta y^k + \sum_{i=1}^{m} F_{y^{(\alpha_i)}}\delta y^{(\alpha_i)} \,)\, dx$$

Integrate the term's from $F_{y'}\delta y'$ to $F_{y^n}\delta y^n$ by parts and using the (1.10) for the term in which δy is α_1-differentiable $F_{y^{\alpha_1}}\delta y^{\alpha_1}$ to $F_{y^{(\alpha_m)}}\delta y^{(\alpha_m)}$, we have:

$$\int_{x_0}^{x_1}(F_y + \sum_{k=1}^{n}(-1)^k \frac{d^k}{dx^k}F_{y^n} - \sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}F_{y^{(\alpha_i)}})\delta y\, dx = 0$$

Since $\delta v = 0$, along extremals , By using the Fundamental Lemma*(1.3.1.1)*. we have the following necessary condition

$$F_y + \sum_{k=1}^{n}(-1)^k \frac{d^k}{dx^k}F_{y^n} - \sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}F_{y^{(\alpha_i)}} = 0 \qquad \ldots(2.25)$$

Third **the problem (2.22) can be extended to multi-depended variables**

$$v(y_1, \ldots, y_n) = \int_{x_0}^{x_1} F(x, y_1, \ldots, y_n, y_1', \ldots, y_1^k, \ldots, y_n', \ldots, y_n^{(k)}, y_1^{(\alpha_1)}, \ldots, y_1^{(\alpha_m)}, \ldots, y_n^{(\alpha_1)}, \ldots, y_m^{(\alpha_m)})\, dx$$

$\delta v = \int_{x_0}^{x_1} F(x, y_1 + \delta y_1, \dots, y_n + \delta y_n, y_1' + \delta y_1', \dots, y_1^{(k)} + \delta y_1^{(k)}, \dots, y_n' +$
$\delta y_n', \dots, y_n^{(k)} + \delta y_n^{(k)}, y_1^{(\alpha_1)} + \delta y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)} + \delta y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)} +$
$\delta y_n^{(\alpha_1)}, \dots, y_m^{(\alpha_m)} + \delta y_m^{(\alpha_m)})\, dx$

$\delta v =$

$\int_{x_0}^{x_1} \sum_{i=1}^{n} F_{y_i} \delta y_i + \sum_{i=1}^{n} \sum_{j=1}^{k} F_{y_i^j} \delta y_i^j + \sum_{i=1}^{n} \sum_{j=1}^{m} F_{y_i^{(\alpha_j)}} \delta y_i^{(\alpha_j)} F_{y_i^{(\alpha_j)}})\, dx$

Integrate the term's from $F_{y_i'} \delta y_i'$ *to* $F_{y_i^k} \delta y_i^k$ by part , and using the (1.10) for the term's from $F_{y_i^{(\alpha_1)}} \delta y_i^{(\alpha_1)}$ to $F_{y_i^{(\alpha_m)}} \delta y_i^{(\alpha_m)}$ in which δy_i α_i-differentiable for all (i=1,…,n). We have

$$\delta v = \int_{x_0}^{x_1} [\left(F_{y_1} + \sum_{k=1}^{K} (-1)^k \frac{d^k}{dx^k} F_{y_1^k} + \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} F_{y_1^{(\alpha_i)}}\right) \delta y_1 + \cdots (F_{y_n} +$$
$$+ \sum_{k=1}^{K} (-1)^k \frac{d^k}{dx^k} F_{y_n^k} - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} F_{y_n^{(\alpha_i)}}) \delta y_n] dx$$

Since $\delta v = 0$ along extremals By using the Fundamental Lemma*(1.3.1.1)*. we have the following necessary condition

$$F_{y_j} + \sum_{k=1}^{K} (-1)^k \frac{d^k}{dx^k} F_{y_j^k} - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} F_{y_j^{\alpha_i}} \equiv 0 \qquad \dots(2.26)$$

2.2.3 Constrained Problem Having only Non-integer Order

First , we shall considering the problem of the form :

$v(y)=\int_{x_0}^{x_1} F(x, y, y^{(\alpha)})\, dx$, such that $\phi(x, y, y^{(\alpha)}) = 0$, where λ *is* exist

…(2.27)

where;

$0 < \alpha < 1$, and with given prescribed boundaries conditions.

Our approach based on the theories presented in [15] , and extend to our problems , therefore , we construct the following auxiliary functional :

$$Z\left(x, y(x), y^{(\alpha)}(x)\right) = F + \lambda\phi$$

Where λ is a Lagrange multiplier exists , then the problem (2.27) can be started as following :

$$v^* = \int_{x_0}^{x_1} Z\, dx$$

$$v^* = \int_{x_0}^{x_1} Z(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}) dx$$

$$= \int_{x_0}^{x_1} Z_y \delta y + Z_{y^{(\alpha)}} \delta y^{(\alpha)}) dx$$

$$= \int_{x_0}^{x_1} [(F_y \delta y + F_{y^{(\alpha)}} \delta y^{(\alpha)}) + \lambda(\phi_y \delta y + \phi_{y^{(\alpha)}} \delta y^{(\alpha)})] dx$$

$$= \int_{x_0}^{x_1} [(F_y + \lambda\phi_y)\delta y + (F_{y^{(\alpha)}} + \lambda\phi_{y^{(\alpha)}})\delta y^{(\alpha)}] dx$$

$$= \int_{x_0}^{x_1} [(F_y + \lambda\phi_y)\delta y + (F_{y^{(\alpha)}} + \lambda\phi_{y^{(\alpha)}}) D^\alpha \delta y] dx$$

Using (1.10) in which δy is α-differentiable , we obtain :

$$\delta v^* = \int_{x_0}^{x_1} [(F_y + \lambda\phi_y)\delta y - \frac{d^\alpha}{dx^\alpha}(F_{y^{(\alpha)}} + \lambda\phi_{y^{(\alpha)}})\delta y] dx$$

$$\delta v^* = \int_{x_0}^{x_1} [(Z_y - \frac{d^\alpha}{dx^\alpha} Z_{y^{(\alpha)}})\delta y] dx = \int_{x_0}^{x_1} [(F_y + \lambda\phi_y) - \frac{d^\alpha}{dx^\alpha}(F_{y^{(\alpha)}} + \lambda\phi_{y^{(\alpha)}})] \delta y\, dx \quad \text{...(2.28)}$$

Since $(\delta v^* = 0)$, along extremals , then

$$(F_y + \lambda\phi_y) - \frac{d^\alpha}{dx^\alpha}(F_{y^{(\alpha)}} + \lambda\phi_{y^{(\alpha)}}) = 0, \text{ and}$$

$$\phi(x, y, y^{(\alpha)}) = 0 \quad \text{...(2.29)}$$

<u>second</u> , we shall discussed the necessary conditions for the general form of the problem (2.27) including many dependent variables , multi-fractional order derivatives , and fractional order constrains, in which such problems can be started as follows :

$$v(y_1,\dots,y_n)=\int_{x_0}^{x_1} F\left(x,y_1,\dots,y_n,y_1^{(\alpha_1)},\dots,y_1^{(\alpha_m)},\dots,y_n^{(\alpha_1)},\dots,y_n^{(\alpha_m)}\right)dx \quad \dots(2.30)$$

With

$$\phi_k\left(x,y_1,\dots,y_n,y_1^{(\alpha_1)},\dots,y_1^{(\alpha_m)},y_n^{(\alpha_1)},\dots,y_n^{(\alpha_m)}\right)=0\,,\quad k=1,\dots,K$$

By variants one dependent variable and fixing the remaining dependent variables, then we shall construct the following auxiliary functional :

$$Z\left(x,y_1,\dots,y_n,y_1^{(\alpha_1)},\dots,y_1^{(\alpha_m)},\dots,y_n^{(\alpha_1)},\dots,y_n^{(\alpha_m)}\right)=F+\sum_{k=1}^{k}\lambda_k\phi_k$$

Where λ is a Lagrange multiplier , then the problem (2.30) can be started as follows :

$$v^* = \int_{x_0}^{x_1} Z\,dx$$

$$\delta v^* = \int_{x_0}^{x_1} Z(x,y_1+\delta y_1,\dots,y_n+\delta y_n,y_1^{(\alpha_1)}+\delta y_1^{(\alpha_1)},\dots,y_1^{(\alpha_m)}+\delta y_1^{\alpha_m},\dots,y_n^{(\alpha_1)}+\delta y_n^{(\alpha_1)},\dots,y_n^{(\alpha_m)}+\delta y_n^{(\alpha_m)})dx$$

$$=\int_{x_0}^{x_1}[(Z_{y_1}\delta y_1+\cdots+Z_{y_n}\delta y_n+Z_{y_1^{(\alpha_1)}}\delta y_1^{(\alpha_1)}+\cdots+Z_{y_1^{(\alpha_m)}}\delta y_1^{(\alpha_m)}+\cdots+Z_{y_n^{(\alpha_1)}}\delta y_n^{(\alpha_1)}+\cdots+Z_{y_n^{(\alpha_m)}}\delta y_n^{(\alpha_m)})]dx$$

$$=$$

$$\int_{x_0}^{x_1}[(F_{y_1}\delta y_1+\sum_{i=1}^{m}F_{y_1^{(\alpha_i)}}\frac{d^{\alpha_i}}{dx^{\alpha_i}}\delta y_1)+\cdots+(F_{y_n}\delta y_n+$$

$\sum_{i=1}^{m} F_{y_n^{(\alpha_i)}} \frac{d^{\alpha_i}}{dx^{\alpha_i}} \delta y_n) + \cdots + \lambda_1(\phi_{y_1}\delta y_1 + \sum_{i=1}^{m} \phi_{y_1^{(\alpha_i)}} \frac{d^{\alpha_i}}{dx^{\alpha_i}} \delta y_1) +$

$\cdots + \lambda_n(\phi_{y_n}\delta y_n + \sum_{i=1}^{m} \phi_{y_n^{(\alpha_i)}} \frac{d^{\alpha_i}}{dx^{\alpha_i}} \delta y_n)]dx$

Using (1.10) in which δy is α-differentiable and since $(\delta v^* = 0)$, along extremals , then, we obtain :

$$\delta v^* = \int_{x_0}^{x_1} [(F_{y_1} - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} F_{y_1^{(\alpha_i)}})\delta y_1 + \cdots + (F_{y_n} - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} F_{y_n^{(\alpha_i)}})\delta y_n +$$

$$\cdots + \lambda_1(\phi_{y_1} - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} \phi_{y_1^{(\alpha_i)}})\delta y_1 + \cdots + \lambda_n(\phi_{y_n} - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} \phi_{y_n^{(\alpha_i)}})\delta y_n]dx = 0$$

$$\delta v^* = \int_{x_0}^{x_1} [(F_{y_1} + \lambda_1\phi_{y_1}) - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} \left(F_{y_1^{(\alpha_i)}} + \lambda_1 \phi_{y_1^{(\alpha_i)}}\right) \delta y_1 +$$

$$\cdots \left(F_{y_n} + \lambda_n \phi_{y_n^{(\alpha_i)}}\right) - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} (F_{y_n^{(\alpha_i)}} + \lambda_n \phi_{y_n^{(\alpha_i)}})\delta y_n \, dx = 0$$

By using the Fundamental Lemma*(1.3.1.1)*. we have the following necessary condition

$$F_j - \sum_{i=1}^{m} \lambda_k \phi_k \mid_{x_1} = 0 \qquad \ldots(2.31\text{ a})$$

$$F_{y_j} + \lambda_j \phi_j - \sum_{i=1}^{m} \left(\frac{d^{\alpha_i}}{dx^{\alpha_i}} F_{y_j^{(\alpha_i)}} + \lambda_j \phi_{y_j^{(\alpha_i)}}\right) = 0 \quad , (j=1,\ldots,n) \qquad \ldots(2.31\text{b})$$

As well as

$$\phi_k\left(x, y_1, \ldots, y_n, y_1^{(\alpha_1)}, \ldots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \ldots, y_n^{(\alpha_m)}\right) = 0, (k=1,\ldots,K) \qquad \ldots(2.31\text{ c})$$

2.2.4 Constrained Problem Having integer And Non-integer Orders

First, we shall consider the problem of the form :

$$v(y) = \int_{x_0}^{x_1} F(x, y, y', y^{(\alpha)})\, dx \qquad \ldots(2.32)$$

where;

$0 < \alpha < 1 , \lambda \; is \; exist$ **and with given prescribed boundaries conditions. Such that**

$$\phi(x, y, y', y^{(\alpha)}) = 0$$

Our approach based on the theories presented in [15] , and extend to our problems , therefore , we construct the following auxiliary functional :

$$Z\,(x, y(x), y'(x), y^{(\alpha)}(x)) = F + \lambda\phi$$

Where λ is a Lagrange multiplier , then the problem (2.33) can be started as follows :

$$v^* = \int_{x_0}^{x_1} Z\, dx$$

$$\delta v^* = \int_{x_0}^{x_1} Z(x, y + \delta y, y' + \delta y', y^{(\alpha)} + \delta y^{(\alpha)})\, dx$$

$$=\int_{x_0}^{x_1} [(F_y \delta y + F_{y'} \delta y' + F_{y^{(\alpha)}} \delta y^{(\alpha)}) + \lambda(\phi_y \delta y + \phi_{y'} \delta y' + \phi_{y^{(\alpha)}} \delta y^{(\alpha)}]\, dx$$

Integrate the term's from $F_{y'}\delta y'$ to $\phi_{y'}\delta y'$ by part and using the (1.10) for the terms $F_{y^{(\alpha)}}\delta y^{(\alpha)}$ and $\phi_{y^{(\alpha)}}\delta y^{(\alpha)} in \; which \; \delta y$ is α-differentiable ,we have

$$\int_{x_0}^{x_1} (F_y - \frac{d}{dx} F_{y'} - \frac{d^\alpha}{dx^\alpha} F_{y^{(\alpha)}}) + \lambda(\phi_y - \frac{d}{dx}\phi_{y'} - \frac{d^\alpha}{dx^\alpha}\phi_{y^{(\alpha)}})\delta y\, dx \quad \text{...(2.34)}$$

Since $\delta v^* = 0$ along extremals and By using the Fundamental Lemma*(1.3.1.1)*. we have the following necessary condition

$$(F_y + \lambda\phi_y) - \frac{d}{dx}(F_{y'} + \lambda\phi_{y'}) - \frac{d^\alpha}{dx^\alpha}(F_{y^{(\alpha)}} + \lambda\phi_{y^{(\alpha)}}) = 0$$

with

$$\phi(x, y, y', y^{(\alpha)}) = 0 \quad \text{...(2.35)}$$

<u>second</u> , we discuss the necessary conditions for the general form of the problem (2.32) including different multi-fractional and integers orders . with constrain

$$v(y) = \int_{x_0}^{x_1} F(x, y, y', \dots, y^n, y^{(\alpha_1)}, \dots, y^{(\alpha_m)})\, dx$$

Such that

$$\phi\left(x, y, y', \dots, y^n, y^{(\alpha_1)}, \dots, y^{(\alpha_m)}\right) = 0 \qquad \dots(2.36)$$

Our approach based on the theories presented in [15] , and extend to our problems , therefore , we construct the following auxiliary functional :

$$Z\left(x, y, y', \dots, y^n, y^{(\alpha_1)}, \dots, y^{(\alpha_m)}\right) = F + \lambda\phi$$

Where λ is a Lagrange multiplier , then the problem (2.36) can be started as following :

$$v^* = \int_{x_0}^{x_1} Z\, dx$$

$$\delta v^* = \int_{x_0}^{x_1} Z(x, y + \delta y, y' + \delta y', y^n + \delta y^n, y^{(\alpha_1)} + \delta y^{(\alpha_1)}, \dots y^{(\alpha_m)} + \delta y^{(\alpha_m)})\, dx$$

$$= \int_{x_0}^{x_1} (Z_y \delta y + Z_{y'} \delta y' + \dots + Z_{y^n} \delta y^n + Z_{y^{(\alpha_1)}} \delta y^{(\alpha_1)} + \dots + Z_{y^{(\alpha_m)}} \delta y^{(\alpha_m)})\, dx$$

$$= \int_{x_0}^{x_1} [(F_y \delta y + F_{y'} \delta y' + \dots + F_{y^n} \delta y^n + F_{y^{(\alpha_1)}} \delta y^{(\alpha_1)} + \dots + F_{y^{(\alpha_m)}} \delta y^{(\alpha_m)}) + \lambda(\phi_y \delta y + \phi_{y'} \delta y' + \dots + \phi_{y^n} \delta y^n + \phi_{y^{(\alpha_1)}} \delta y^{(\alpha_1)} + \dots + \phi_{y^{(\alpha_m)}} \delta y^{(\alpha_m)})]\, dx$$

Integrate the term's from $F_{y^{(i)}}\delta y^{(i)}$ to $\phi_{y^{(i)}}\delta y^{(i)}$ by part and using the (1.10) for the terms $F_{y^{(\alpha_j)}}\delta y^{(\alpha_j)}$ and $\phi_{y^{(\alpha_j)}}\delta y^{(\alpha_j)}$ is α_j-differentiable ,we have

$$\int_{x_0}^{x_1}(Z_y - \sum_{k=1}^{n}(-1)^k \frac{d^k}{dx^k} Z_{y^k} - \sum_{j=1}^{m} \frac{d^{\alpha_j}}{dx^{\alpha_j}} Z_{y^{(\alpha_j)}})\delta y\, dx$$

$$\int_{x_0}^{x_1}[(F_y + \lambda\phi_y) - \sum_{k=1}^{n}(-1)^k \frac{d^k}{dx^k}(F_{y^k} + \lambda^k\phi_{y^k}) - \sum_{j=1}^{m} \frac{d^{\alpha_j}}{dx^{\alpha_j}}(F_{y^{(\alpha_j)}} + \lambda_j\phi_{y^{(\alpha_j)}})]\delta y\, dx$$

Since $\delta v^* = 0$ *and* By using the Fundamental Lemma*(1.3.1.1)*. we have the following necessary condition

$$(F_y + \lambda\phi_y - \sum_{k=1}^{n}(-1)^k \frac{d^k}{dx^k}(F_{y^k} + \lambda^k\phi_{y^k}) - \sum_{j=1}^{m} \frac{d^{\alpha_j}}{dx^{\alpha_j}}(F_{y^{(\alpha_j)}}) + \lambda_j\phi_{y^{(\alpha_j)}}) = 0 \qquad \text{...(2.37)}$$

$$\phi\left(x, y, y', \dots, y^n, y^{(\alpha_1)}, \dots, y^{(\alpha_m)}\right) = 0$$

<u>Third</u> , we shall discuss the necessary conditions for the general form of the problem (2.37) including many dependent variables , multi-fractional , integer order derivatives and multi-fractional and order constraints in which such problems can stated as follows

$$v(y_1, \dots, y_n) = \int_{x_0}^{x_1} F(x, y_1, \dots, y_n, y_1', \dots, y_1^n, \dots, y_n', \dots, y_n^{(n)}, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)})\, dx$$

Subject to

$$\phi_r\left(x, y_1, \dots, y_n, y_1', \dots, y_1^n, \dots, y_n', \dots, y_n^{(n)}, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right) = 0 \quad \text{, where (r=1,...,I)} \qquad \text{...(2.38)}$$

Our approach based on the theories presented in [15], and extend to our problems, therefore , we construct the following auxiliary functional:

$$Z\left(x, y_1, \dots, y_n, y_1', \dots, y_1^n, \dots, y_n', \dots, y_n^{(n)}, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right) = F + \sum_{r=1}^{I} \lambda_r \emptyset_r$$

Where λ_r is a Lagrange multiplier , then the problem (2.38) can be started as following :

$$v^* = \int_{x_0}^{x_1} Z\, dx$$

Variants one dependent variable and fixing the remaining dependent variables .

$$\delta v^* = \int_{x_0}^{x_1} Z(x, y_1 + \delta y_1, \dots, y_n + \delta y_n, y_1' + \delta y_1', \dots, y_1^n + \delta y_1^n, \dots, y_n' + \delta y_n', \dots, y_n^{(n)} + \delta y_n^{(n)}, y_1^{(\alpha_1)} + \delta y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)} + \delta y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)} + \delta y_n^{(\alpha_1)}, \dots, y_m^{(\alpha_m)} + \delta y_m^{(\alpha_m)})\, dx$$

$$\delta v^* = \int_{x_0}^{x_1} Z_{y_1}\delta y_1 + \dots + Z_{y_n}\delta y_n + Z_{y_1'}\delta y_1' + \dots + Z_{y_1^n}\delta y_1^{(n)} + \dots + Z_{y_n'}\delta y_n' + \dots + Z_{y_n^{(n)}}\delta y_n^{(n)} + Z_{y_1^{(\alpha_1)}}\delta y_1^{(\alpha_1)} + \dots + Z_{y_1^{(\alpha_m)}}\delta y_1^{(\alpha_m)}, \dots, Z_{y_n^{(\alpha_1)}}\delta y_n^{(\alpha_1)} + \dots + Z_{y_n^{(\alpha_m)}}\delta y_n^{(\alpha_m)})\, dx$$

Now by performing the integrations by parts and using (1.10) in which δy is α_i-differentiable , we obtain

$$\delta v^* = \int_{x_0}^{x_1} [(Z_{y_1} + \sum_{k=1}^{n} (-1)^k \frac{d^k}{dx^k}\left(Z_{y_1^k}\right) - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}}\left(Z_{y_1^{(\alpha_i)}}\right))\delta y_1 \dots + (Z_{y_n} + \sum_{k=1}^{n} (-1)^k \frac{d^k}{dx^k}\left(Z_{y_n^k}\right) - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}}\left(Z_{y_n^{(\alpha_i)}}\right))\delta y_n]\, dx$$

$$\delta v^* = \int_{x_0}^{x_1}[((F_{y_1} + \sum_{K=1}^{I}\lambda_r(\phi_r)_{y_1}) + \sum_{k=1}^{n}(-1)^k \frac{d^k}{dx^k}\left(F_{y_1^k} + \sum_{r=1}^{I}\lambda_r(\phi_r)_{y_1^k}\right) - \sum_{j=1}^{m}\frac{d^{\alpha_j}}{dx^{\alpha_j}}(F_{y_1^{(\alpha_j)}} + \sum_{r=1}^{I}\lambda_r(\phi_r)_{y_1^{(\alpha_j)}}))\delta y_1 \dots + (\int_{x_0}^{x_1}[(F_{y_n} + \sum_{r=1}^{I}\lambda_r(\phi_r)_{y_1}) + \sum_{k=1}^{n}(-1)^k \frac{d^k}{dx^k}\left(F_{y_n^k} + \sum_{r=1}^{I}\lambda_r(\phi_r)_{y_n^k}\right) - \sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_{y_n^{(\alpha_j)}} + \sum_{r=1}^{I}\lambda_r(\phi_r)_{y_i^{(\alpha_j)}})]\delta y_n\, dx$$

Since $\delta v^* = 0$ and By using the Fundamental Lemma*(1.3.1.1)*. we have the following necessary condition

$$[(F_{y_j} + \sum_{r=1}^{I}\lambda_r(\phi_r)_{y_j}) + \sum_{k=1}^{n}(-1)^k \frac{d^k}{dx^k}\left(F_{y_j^k} + \sum_{r=1}^{I}\lambda_r(\phi_r)_{y_j^k}\right) - \sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_{y_j^{(\alpha_j)}} + \sum_{r=1}^{I}\lambda_r(\phi_r)_{y_i^{(\alpha_j)}})] = 0 \qquad \dots(2.39)$$

with

$$\phi_k\left(x, y_1, \dots, y_n, y_1', \dots, y_1^n, \dots, y_n', \dots, y_n^{(n)}, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, \dots, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right) = 0$$

2.3 The Functional Of Discontinuous With Fixed Boundaries

In this section , we are constructed optimality necessary conditions , when the functional is discontinuous on (x_k) , for (k=1,…,m) having non-integers orders , functional having non-integers orders and integers orders.

2.3.1 Unconstraint Problem

First, We consider the following simplest variation problem

$$v\left(x, y(x), y^{(\alpha)}\right) = \int_{x_s}^{x_f} F(x, y, y^{(\alpha)})\, dx \qquad \ldots(2.40)$$

where;

$0 < \alpha < 1$, and with given prescribed boundaries conditions.

Since the Fundamental Lemma of the calculus of variation cannot be applied , because of the discontinuities , it is more convenient to calculate the extreme value of $v\left(x, y(x), y^{(\alpha)}\right)$ along the curves approximately so it is convenient to replace the integral of eq(2.40)by the following

$$\int_{x_s}^{x_f} F(x, y, y^{(\alpha)})dx \cong \int_{x_s}^{x_1} F_1(x, y, y^{(\alpha)})dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} F_k(x, y, y^{(\alpha)})\, dx + \int_{x_n}^{x_f} F_n(x, y, y^{(\alpha)})dx \qquad \ldots(2.41)$$

Then

$$\delta v = \int_{x_s}^{x_1} [(F_1)_y(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})\delta y + (F_1)_{y^{(\alpha)}}(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})\delta y^{(\alpha)}]dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} [(F_k)_y(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})\delta y + (F_k)_{y^{(\alpha)}}\left(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}\right)\delta y^{(\alpha)}]\, dx + \int_{x_s}^{x_f} [(F_n)_y(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})\delta y + (F_n)_{y^{(\alpha)}}\left(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}\right)\delta y^{(\alpha)}]dx$$

On each integrand , integrate their second term by parts using (1.10) in which δy is α-differentiable , we obtain.

$$\delta v = \int_{x_s}^{x_1} ((F_1)_y - \frac{d^\alpha}{dx^\alpha}(F_1)_{y^{(\alpha)}})\, \delta y\, dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} ((F_k)_y - \frac{d^\alpha}{dx^\alpha}(F_k)_{y^{(\alpha)}})\delta y\, dx + \int_{x_{s_n}}^{x_f} ((F_n)_y - \frac{d^\alpha}{dx^\alpha}(F_n)_{y^{(\alpha)}})\delta y\, dx \qquad \ldots(2.42)$$

Since the value of the functional are only along extremals (i.e.$\delta v = 0$) consequently , we have the following necessary conditions:

$((F_1)_y - \frac{d^\alpha}{dx^\alpha}(F_1)_{y^{(\alpha)}}) = 0$, along the interval $[x_s , x_1)$ ... (2.43a)

$((F_k)_y - \frac{d^\alpha}{dx^\alpha}(F_k)_{y^{(\alpha)}}) \equiv 0$, along the interval (x_{k-1} , x_k) for (k=2,…,n-1)

...(2.43 b)

$((F_n)_y - \frac{d^\alpha}{dx^\alpha}(F_n)_{y^{(\alpha)}}) \equiv 0$, along the interval $(x_n , x_f]$ ...(2.43 c)

second , we consider functional of the form

$v(x, y(x), y^{(\alpha)}, y^{(\beta)}) = \int_{x_s}^{x_f} F(x, y, y^{(\alpha)}, y^{(\beta)})\, dx$,where are $\alpha, \beta > 0$ and non-integer ...(2.44)

We proceed as before with only single fractional order derivative , to get the following:

$$v(y) = \int_{x_s}^{x_f} F(x, y, y^{(\alpha)}, y^{(\beta)})\, dx \cong$$
$$\int_{x_s}^{x_f} F_1(x, y, y^{(\alpha)}, y^{(\beta)})\, dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} F_k(x, y, y^{(\alpha)}, y^{(\beta)})\, dx +$$
$$\int_{x_n}^{x_f} F_n(x, y, y^{(\alpha)}, y^{(\beta)})\, dx \quad ...(2.45)$$

Then

$$\delta v = \int_{x_s}^{x_1} [(F_1)_y(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}, y^{(\beta)} + \delta y^{(\beta)})\delta y + (F_1)_{y^{(\alpha)}}(x, y +$$
$$\delta y, y^{(\alpha)} + \delta y^{(\alpha)}, y^{(\beta)} + \delta y^{(\beta)})\delta y^{(\alpha)} + (F_1)_{y^{(\beta)}}(x, y + \delta y, y^{(\alpha)} +$$
$$\delta y^{(\alpha)}, y^{(\beta)} + \delta y^{(\beta)})\delta y^{\beta}]dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} [(F_k)_y(x, y + \delta y, y^{(\alpha)} +$$
$$\delta y^{(\alpha)}, y^{(\beta)} + \delta y^{(\beta)})\delta y + (F_k)_{y^{(\alpha)}}(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}, y^{(\beta)} +$$
$$\delta y^{(\beta)})\delta y^{(\alpha)} + (F_k)_{y^{(\beta)}}(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}, y^{(\beta)} + \delta y^{(\beta)})\delta y^{(\beta)}]\, dx +$$
$$\int_{x_{s_n}}^{x_f} [(F_n)_y(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}, y^{(\beta)} + \delta y^{(\beta)})\delta y + (F_n)_{y^{(\alpha)}}(x, y +$$

$\delta y, y^{(\alpha)} + \delta y^{(\alpha)})\delta y^{(\alpha)} + (F_n)_{y^\beta}(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}, y^{(\beta)} + \delta y^{(\beta)})\delta y^{(\beta)}]dx$

On each integrand , integrate their second term by part using (1.10) in which δy is α-differentiable , we obtain.

$$\delta v = \int_{x_s}^{x_1}((F_1)_y - \frac{d^\alpha}{dx^\alpha}(F_1)_{y^{(\alpha)}} - \frac{d^\beta}{dx^\beta}(F_1)_{y^{(\beta)}})\,\delta y\, dx + \sum_{k=2}^{n-1}\int_{x_{k-1}}^{x_k}((F_k)_y - \frac{d^\alpha}{dx^\alpha}(F_k)_{y^{(\alpha)}} - \frac{d^\beta}{dx^\beta}(F_k)_{y^{(\beta)}})\delta y\, dx + \int_{x_{s_n}}^{x_f}((F_n)_y - \frac{d^\alpha}{dx^\alpha}(F_n)_{y^{(\alpha)}} - \frac{d^\beta}{dx^\beta}(F_n)_{y^{(\beta)}})\delta y\, dx \quad \ldots(2.46)$$

Since the value of the functional are only along extremals (i.e. $\delta v = 0$) consequently , we have the following necessary condition:

$((F_1)_y - \frac{d^\alpha}{dx^\alpha}(F_1)_{y^{(\alpha)}} - \frac{d^\beta}{dx^\beta}(F_1)_{y^{(\beta)}}) \equiv 0$,**along the interval**$[x_s, x_1)$

…(2.47 a)

$((F_k)_y - \frac{d^\alpha}{dx^\alpha}(F_k)_{y^{(\alpha)}} - \frac{d^\beta}{dx^\beta}(F_k)_{y^{(\beta)}}) \equiv 0$ **, along the interval** (x_{k-1}, x_k) **for (k=2,…,n-1) …(2.47b)**

$((F_n)_y - \frac{d^\alpha}{dx^\alpha}(F_n)_{y^{(\alpha)}} - \frac{d^\beta}{dx^\beta}(F_n)_{y^{(\beta)}}) \equiv 0$ **, along the interval** $(x_n, x_f]$

…(2.47c)

<u>Third</u> , we shall extended the problem (2.40) to different multi-fractional order $\alpha_i > 0$ (i=1,2,…,m) of the following problem

$$v(y) = \int_{x_s}^{x_f} F(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \ldots, y^{(\alpha_m)})\, dx \quad \ldots(2.48)$$

$$v = \int_{x_s}^{x_f} F(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \ldots, y^{(\alpha_m)})\, dx \cong$$

$$\int_{x_s}^{x_1} F_1(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \ldots, y^{(\alpha_m)})\, dx +$$

$\sum_{k=2}^{n-1}\int_{x_{k-1}}^{x_k} F_k(x,y,y^{(\alpha_1)},y^{(\alpha_2)},\dots,y^{(\alpha_m)})\,dx +$

$\int_{x_n}^{x_f} F_n(x,y,y^{(\alpha_1)},y^{(\alpha_2)},\dots,y^{(\alpha_m)})\,dx$ …(2.49)

Then

$$\delta v = \int_{x_s}^{x_1}[(F_1)_y(x,y+\delta y,y^{(\alpha_1)}+\delta y^{(\alpha_1)},y^{(\alpha_2)}+\delta y^{(\alpha_2)},\dots,y^{(\alpha_m)}+\delta y^{(\alpha_m)})\delta y + \sum_{i=1}^{m}(F_1)_{y^{\alpha_i}}(x,y+\delta y,y^{(\alpha_1)}+\delta y^{(\alpha_1)},y^{(\alpha_2)}+\delta y^{(\alpha_2)},\dots,y^{(\alpha_m)}+\delta y^{(\alpha_m)})\delta y^{\alpha_i}]dx + \sum_{k=2}^{n-1}\int_{x_{k-1}}^{x_k}[(F_k)_y(x,y+\delta y,y^{(\alpha_1)}+\delta y^{(\alpha_1)},y^{(\alpha_2)}+\delta y^{(\alpha_2)},\dots,y^{(\alpha_m)}+\delta y^{(\alpha_m)})\,\delta y + \sum_{i=1}^{m}(F_k)_{y^{\alpha_i}}(x,y+\delta y,y^{(\alpha_1)}+\delta y^{(\alpha_1)},y^{(\alpha_2)}+\delta y^{(\alpha_2)},\dots,y^{(\alpha_m)}+\delta y^{(\alpha_m)})\delta y^{\alpha_i}]\,dx + \int_{x_n}^{x_f}[(F_n)_y(x,y+\delta y,y^{(\alpha_1)}+\delta y^{(\alpha_1)},y^{(\alpha_2)}+\delta y^{(\alpha_2)},\dots,y^{(\alpha_m)}+\delta y^{(\alpha_m)})\,\delta y + \sum_{i=1}^{m}(F_n)_{y^{(\alpha_i)}}(x,y+\delta y,y^{(\alpha_1)}+\delta y^{(\alpha_1)},y^{(\alpha_2)}+\delta y^{(\alpha_2)},\dots,y^{(\alpha_m)}+\delta y^{(\alpha_m)})\delta y^{(\alpha_i)}]\,dx$$

On each integrand , integrate their second term by part using (1.10) in which δy is α-differentiable , we obtain.

$$\delta v = \int_{x_s}^{x_1}((F_1)_y - \sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_1)_{y^{(\alpha_i)}})\,\delta y\,dx + \sum_{k=2}^{n-1}\int_{x_{k-1}}^{x_k}((F_k)_y - \sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_k)_{y^{(\alpha_i)}})\delta y\,dx + \int_{x_{s_n}}^{x_f}((F_n)_y - \sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_n)_{y^{(\alpha_i)}})\delta y\,dx \quad \dots(2.50)$$

Since the value of the functional are only along extremals (i.e. $\delta v = 0$) consequently , we have the following necessary condition:

$((F_1)_y - \sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_1)_{y^{(\alpha_i)}}) \equiv 0$ **, along the interval $[x_s, x_1)$** …(2.51 a)

$((F_k)_y - \sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_k)_{y^{(\alpha_i)}}) \equiv 0$ **, along the interval (x_{k-1}, x_k) for (k=2,…,n-1)** …(2.51 b)

$((F_n)_y - \sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_n)_{y^{(\alpha_i)}}) \equiv 0$ **, along the interval $(x_n, x_f]$** …(2.51 c)

2.3.2 Constrained Problem

First , we are considering the functional of the form :

$$v(y) = \int_{x_s}^{x_f} F(x, y, y^{(\alpha)})\, dx \qquad \ldots(2.52)$$

$$\cong \int_{x_s}^{x_1} F_1(x, y, y^{(\alpha)})dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} F_k(x, y, y^{(\alpha)})\, dx + \int_{x_n}^{x_f} F_n(x, y, y^{(\alpha)})dx$$

Subject to

$$\phi\left(x, y, y^{(\alpha)}\right) = 0, \text{ where } \lambda \text{ is exist} \qquad \ldots(2.53)$$

Our approach based on the theories presented in [10] , and extend to our problems , therefore , we construct the following auxiliary functional :

$$Z_k\left(x, y(x), y^{(\alpha)}(x)\right) = F_k + \lambda\phi$$

Where λ is a Lagrange multiplier , then the problem (2.52) can be started as following :

$$v^*(y) = \int_{x_s}^{x_f} Z_k\, dx \qquad \ldots(2.54)$$

Where F is discontinuous on (x_k) , for (k=1,2,…,n) with $\alpha > 0$, then Z_k also discontinuous on (x_k) , and the fundamental Lemma of calculus of variation cannot be applied , then it is more convenient to calculate the value of $v\left(x, y(x), y^{(\alpha)}\right)$ along polygonal curves approximately , so it is convenient to replace the integral of(2.54)by the following:

$$v^* = \int_{x_s}^{x_f} Z_k\left(x, y, y^{(\alpha)}\right) dx$$

$$\cong \int_{x_s}^{x_1} Z_1(x, y, y^{(\alpha)})dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} Z_k(x, y, y^{(\alpha)})\, dx + \int_{x_n}^{x_f} Z_n(x, y, y^{(\alpha)})dx \qquad \ldots(2.55)$$

$$\delta v = \int_{x_s}^{x_1} [(Z_1)_y(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})\delta y + (Z_1)_{y^\alpha}(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})]dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} [(Z_k)_y(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}) + (Z_k)_{y^\alpha}(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})]dx + \int_{x_n}^{x_f} [(Z_n)_y(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})\delta y + (Z_n)_{y^\alpha}(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})]dx \quad \ldots(2.56)$$

Now , by performing integration by parts on each integrand , using (1.10) for each sub –interval for the extreme (2.56) , to obtain the following necessary conditions :

$(F_1)_y + \lambda\phi_y - \frac{d^\alpha}{dx^\alpha}(F_{y^{(\alpha)}} + \lambda\phi_{y^{(\alpha)}}) \equiv 0$, along the interval $[x_s, x_1)$

…(2.57 a)

$(F_k)_y + \lambda\phi_y - \frac{d^\alpha}{dx^\alpha}(F_{k^{(\alpha)}} + \lambda\phi_{y^{(\alpha)}}) \equiv 0$, along the interval $[x_{k-1}, x_k)$ for (k=2,…,n-1) …(2.57 b)

$(F_n)_y + \lambda\phi_y - \frac{d^\alpha}{dx^\alpha}(F_{k^{(\alpha)}} + \lambda\phi_{y^{(\alpha)}}) \equiv 0$, along the interval $(x_n, x_{f_n}]$

…(2.57 c)

As well as

$$\phi(x, y, y^{(\alpha)}) = 0$$

<u>second</u>, we shall discuss the necessary conditions to furthermore general form of the problem (2.51) multi-dependent variables , and multi-fractional order derivatives :

$$v(y_1, \ldots, y_n) = \int_{x_s}^{x_f} F\left(x, y_1, \ldots, y_n, y_1^{(\alpha_1)}, \ldots, y_1^{(\alpha_m)}, \ldots, y_n^{(\alpha_1)}, \ldots, y_n^{(\alpha_m)}\right) dx$$

With fractional order constraints …(2.58)

$$\phi_q = F\left(x, y_1, \ldots, y_n, y_1^{(\alpha_1)}, \ldots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \ldots, y_n^{(\alpha_m)}\right) = 0, \quad q=1,\ldots,I$$

Variants one dependent variable and fixing the remaining dependent variables , for all dependent variables

$$Z_k\left(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right) = F_k + \sum_{r=1}^{I} \lambda_r \emptyset_r$$

Where λ_r is a Lagrange multiplier for r=1,…,I

, then the problem (2.58) can be started as following :

$$v^* = \int_{x_s}^{x_f} Z_k\, dx \qquad \text{…(2.59)}$$

Where F is discontinuous on (x_k) , for (k=1,…,n) with $\alpha > 0$

Then (Z_k) also discontinuous on (x_k) , and the Fundamental Lemma of the calculus of variation cannot be applied , then it is more convenient to calculate the value of

$v\left(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right)$ along the polygonal curves approximately , so it convenient to replace the integral of (2.59)

By the following :

$$v^*(y_1, \dots, y_n) = \int_{x_s}^{x_f} Z_k\left(x, y_1, \dots, y_n, y_1^{(\alpha_1)}, \dots, y_1^{(\alpha_m)}, y_n^{(\alpha_1)}, \dots, y_n^{(\alpha_m)}\right) dx \cong$$
$$\int_{x_s}^{x_1} Z_1\, dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} Z_k\, dx + \int_{x_n}^{x_f} Z_n\, dx \qquad \text{…(2.60)}$$

$$\delta v^* =$$
$$\int_{x_s}^{x_1} [\sum_{i=1}^{n} (Z_1)_{y_i} \delta y_i + \sum_{i=1}^{n} \sum_{j=1}^{m} (Z_1)_{y_i^{(\alpha_j)}} \delta y_i^{(\alpha_j)}] dx +$$
$$\sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} [\sum_{i=1}^{n} (Z_k)_{y_i} \delta y_i\, (Z_k)_{y_1} + \sum_{i=1}^{n} \sum_{j=1}^{m} (Z_k)_{y_i^{(\alpha_j)}} \delta y_i^{(\alpha_j)}] dx +$$
$$\int_{x_n}^{x_f} \sum_{i=1}^{n} (Z_n)_{y_i} \delta y_i + \sum_{i=1}^{n} \sum_{j=1}^{m} (Z_n)_{y_i^{(\alpha_j)}} \delta y_i^{(\alpha_j)}] dx \qquad \text{…(2.61)}$$

By using (1.10) for each sub-interval , and since $\delta v = 0$

$$\delta v^* =$$
$$\int_{x_s}^{x_1} [(Z_1)_{y_1} - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} (Z_1)_{y_1^{(\alpha_j)}}] \delta y_1 \dots \left[(Z_1)_{y_n} - \sum_{i=1}^{m} \frac{d^{\alpha_i}}{dx^{\alpha_i}} (Z_1)_{y_n^{(\alpha_j)}}\right] \delta y_n +$$

$\sum_{k=2}^{n-1}\int_{x_{k-1}}^{x_k}[(Z_k)_{y_1}-\sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(Z_k)_{y_1^{(\alpha_j)}}]\delta y_1 +$

$$\left[(Z_k)_{y_2}-\sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(Z_k)_{y_2^{(\alpha_j)}}\right]\delta y_2 \dots \left[(Z_n)_{y_n}-\sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(Z_n)_{y_n^{(\alpha_j)}}\right]\delta y_n = 0$$

Then we obtain the following necessary conditions :

$$(F_1)_{y_j}+\sum_{r=1}^{I}\lambda_r(\phi_r)_{y_j}-[\sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_1)_{y_j^{(\alpha_i)}}+\sum_{r=1}^{I}\lambda_r(\phi_r)_{y_j^{(\alpha_i)}}]=0$$

along interval $[x_s, x_1)$ for all j **...(2.62 a)**

$$(F_k)_{y_j}+\sum_{r=1}^{I}\lambda_r(\phi_r)_{y_j}-[\sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_k)_{y_j^{(\alpha_i)}}+\sum_{r=1}^{I}\lambda_r(\phi_r)_{y_j^{(\alpha_i)}}]=0 \quad ,$$

along interval (x_{k-1}, x_k) for (k=2,…,n-1) and for all j **...(2.62 b)**

$$(F_n)_{y_j}+\sum_{r=1}^{I}\lambda_r(\phi_r)_{y_j}-[\sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_n)_{y_j^{(\alpha_i)}}+\sum_{r=1}^{I}\lambda_r(\phi_r)_{y_j^{(\alpha_i)}}]=0 \quad ,$$

along interval $[x_n, x_f]$ **...(2.62c)**

As well as the additional constraints

$\phi_r(x, y_1, \dots, y_n, y_1^{\alpha_1}, \dots, y_1^{\alpha_m}, y_n^{\alpha_1}, \dots, y_n^{\alpha_m}) = 0$, **for all r**

2.4 Examples

We obtain the Euler-Lagrange equations for unconstrained and constrained fractional variation problems.

Example(2.4.1)

As the first example, consider the following unconstrained fractional variational problem:

$$\textit{minimize} \qquad J[y]=\frac{1}{2}\int_0^1\left({}_0D_x^{\alpha}y\right)^2dx$$

Such that

y (0)=0 and y(1)=1

this example with $\alpha = 1$, for which the solution is y(x)=x. It can be shown for this problem, the Euler-Lagrange equation is

$$ {}_xD_1^\alpha y({}_oD_x^\alpha y) = 0 $$

It can be shown that for $\alpha > 1/2$, the solution is given as

$$ y(x) = (\alpha - 1)\int_0^1 \frac{dt}{[(1-t)(x-t)]^{1-\alpha}} $$

Example(2.4.2)

As the second example, consider the following constrained fractional variational problem:

$$ minimize \qquad J[y] = \frac{1}{2}\int_0^1 [y_1^2 + y_2^2]dx $$

Such that

$$ {}_oD_x^\alpha y_1 = -y_1 + y_2 $$

$$ y_1 = 1 $$

It can be shown for this problem, the Euler-Lagrange equation is

$$ y_1 + l + {}_xD_1^\alpha l = 0 $$

$$ y_2 - l = 0 $$

CHAPTER ONE

Primarily

1.1Introduction

This chapter involves two sections. In section (1.2),we give some of the most basic and important concepts in fractional calculus and Calculus of variation. also some definitions, theorems, Lemmas and examples are presented that needed then later.

1.2 Fractional Calculus

Definition (1.1) , [26]

The gamma function is defined by the following improper integral and

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\, dt, \qquad z > 0 \qquad \ldots(1.1)$$

As will be clear later, the gamma function is intrinsically tied to fractional calculus by definition. The simple interpretation of the gamma function is simply the generalization of the fractional for all positive real numbers.

Remark (1.1) , [9]

Riemann's modified form of Liouville's fractional integral operator is a direct generalization of the following Cauchy's formula for an n-fold integral .

$$\int_a^x \int_a^{x_1} \dots \int_a^{x_{n-1}} f(x_n) dx_1\, dx_2 \dots dx_n = \frac{1}{(n-1)!} \int_a^x \frac{f(t)}{(x-t)^{1-n}} dt$$

…(1.2)

By n-fold here means that the integration is deployed n-times. Since $(n-1)! = \Gamma(n)$, Riemann realized that the right hand side of (1.2) might have meaning even when n takes non-integer values. *Definition(1.2), Riemann-Liouville Fractional Derivatives ,[26]*

Let f be a continuous function on [a , b], for all $x \in [a\,, b]$. The left (resp. right) Riemann-Liouville derivative at x is given by

$${}_aD_t^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)}{(x-t)^{\alpha+1-n}} dt \qquad \dots(1.3)$$

$${}_tD_b^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_x^b \frac{f(t)}{(x-t)^{\alpha+1-n}} dt \qquad \dots(1.4)$$

With $(n-1 < \alpha < n)$, and n is positive integer.

Definition (1.3) [23]

Let $f \in L_1[a,b]$, $\alpha \in \mathbb{R}^+$. The fractional (arbitrary) order integral of the function f of order α is defined as:

$$\mathrm{I}_a^{\alpha} f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds\,, \qquad \dots(1.5)$$

When $a = 0$ we can write, $\mathrm{I}^{\alpha} f(t) = \mathrm{I}_0^{\alpha} f(t) = f(t) * \Phi_{\alpha}(t)$, where;

$$\Phi_{\alpha}(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & , \text{for } t > 0 \\ 0 & , \text{for } t \le 0 \end{cases}$$

and (*) is the convolution operator.

Definition (1.4) [28]

The fractional derivative $_{a}D_{t}^{\alpha}$of order $\alpha \in (0,1]$ of the absolutely continuous function $f(t)$ is defined as:

$$_{a}D_{t}^{\alpha}f(t) = \mathrm{I}_{a}^{1-\alpha}\frac{d}{dt}f(t)\,, \quad t \in [a,b] \qquad \dots(1.6)$$

1.2.1 Lemmas, and Corollary

In this subsection, we are presented the following two lemmas for fractional integral and fractional derivatives.

Let $L_p = L_p(a,b)$ be the class of Lebesgue integrable functions on $[a,b]$, $a < b < \infty$, $(1 \le p < \infty)$.

Lemma (1.2.1.1) [18]

If $Re(\alpha) > 0$ and $Re(\beta) > 0$, then the equations;

$$\left(\mathrm{I}_{a}^{\alpha}\,\mathrm{I}_{a}^{\beta}\,f\right)(x) = \left(\mathrm{I}_{a}^{\alpha+\beta}\,f\right)(x) \ \text{ and } \ \left(\mathrm{I}_{b}^{\alpha}\,\mathrm{I}_{b}^{\beta}\,f\right)(x) = \left(\mathrm{I}_{b}^{\alpha+\beta}\,f\right)(x) \qquad \dots(1.7)$$

are satisfied at almost every point $x \in [a,b]$ for;

$f \in L_p(a,b)$ $(1 \le p < \infty)$.

If $\alpha + \beta > 1$.

Lemma (1.2.1.2) [18]

If $Re(\alpha) > 0$ and $f \in L_p(a,b)$, $(1 \le p < \infty)$, then

$$({}_{a}D_{t}^{\alpha}\,\mathrm{I}_{a}^{\alpha}\,f)_{(x)} = f(x) \text{and} ({}_{t}D_{b}^{\alpha}\,\mathrm{I}_{b}^{\alpha}\,f)_{(x)} = f(x). \qquad \dots(1.8)$$

Lemma(1.2.1.3)[7]

For all $f\ ,\ g \in {}_{a}^{\alpha}D_{b}^{\beta}$ *, we have*

$$\int_a^b D_\mu^{\alpha,\beta}\ f(t)g(t)dt = -\int_a^b f(t)D_\mu^{\alpha,\beta}\ g(t)dt \qquad \ldots(1.9)$$

Provided that $f(a) = f(b) = 0 \ or \ g(a) = g(b) = 0$

Corollary(1.2.1.1),[6]

$$\int_a^b D_\mu^{\alpha}\ f(t)g(t)dt = -\int_a^b f(t)\,{}_tD_b^{\alpha}\, g(t)dt \qquad \ldots(1.10)$$

As long as $f(a) = f(b) = 0 \ or \ g(a) = g(b) = 0$ **.**

This furmula gives a strong connection between ${}_tD_b^{\beta}\ and\ {}_tD_b^{\alpha}$ **via generalized integration by part. This relation is responsible for emergence of** ${}_tD_b^{\alpha}$ **in problem of fractional calculus of variation only dealing with** ${}_tD_b^{\beta}$

1.2.2 Properties of Fractional [18]

In this subsection, we are presented some properties for fractional integral and fractional derivatives.

Now, some additional important properties of the fractional differential operator ${}_aD_t^{\alpha}$ **are presented for completeness purpose[20]:**

1. **The operator** ${}_aD_t^{\alpha}$ **of order** $\alpha = 0$ **is the identity operator.**
2. **The operator** ${}_aD_t^{\alpha}$ **is linear, i.e.,**

   ${}_aD_t^{\alpha}\big(c_1\, f(t) + c_2\, g(t)\big) = c_1\, {}_aD_t^{\alpha}\, f(t) + c_2\, {}_aD_t^{\alpha}\, g(t)$ **, where** c_1 **and** c_2 **are constants.**

3. **The operator** ${}_aD_t^{\alpha}$ **is homogenous;**

   ${}_aD_t^{\alpha}\,\{c\, f(t)\} = c\,{}_aD_t^{\alpha}\, f(t)$.

4. $${}_aD_t^{\alpha}\sum_{i=1}^{n} f_i(t) = \sum_{i=1}^{n} {}_aD_t^{\alpha} f_i(t)$$

1.3 Calculus of Variation

1.3.1Basic Definitions and Theories

Basic Definitions and the theories will be given in this subsection

Definition (1.3.1.1),[9]

The variable v is called functional depending on a function $y(x)$, in writing $v = v(y(x))$, if to each function, $y(x)$ from a certain class of functions, there corresponds a certain value of v.

Definition (1.3.1.2),[9]

The increment or variation δy of the argument $y(x)$ of a functional $v(y(x))$ is the difference of two functions $\delta y = y(x) - y_1(x)$ where $y_1(x)$ is admissible curve .

Definition (1.3.1.3),[9]

A functional $v(y(x))$ is continuous along $y = y_0(x)$ in the sense of closeness of order k, if for arbitrary positive number ε there exists a $\delta > 0$ such that, $|v(y(x)) - v(y_0(x))| < \varepsilon$, whenever;

$|y(x) - y_0(x)| < \delta$, $|y(x) - \acute{y}_0(x)| < \delta$, …,

$|y(x) - {y_0}^{(k)}(x)| < \delta$.

It is understood, that the function $y(x)$ is taken from the class of functions for which $v(y(x))$ is defined.

Definition (1.3.1.4),[9]

The functional $v(y(x))$ is called a linear functional, if it satisfies the conditions:

a. $v(cy(x)) = cv(y(x))$ **, where c is constant.**

b. $v(y_1(x) + y_2(x)) = v(y_1(x)) + v(y_2(x))$ **.**

Definition (1.3.1.5),[9]

A functional $v(y(x))$ takes on a maximum value along the curve $y = y_o(x)$, if all the values of this functional $v(y(x))$ taken on along arbitrary neighboring to $y = y_o(x)$, curves are not greater than $v(y_o(x))$, i.e. $\Delta v = v(y(x)) - v(y_0(x)) \leq 0$. If $\Delta v \leq 0$ and $\Delta v = 0$ only when

$y = y_0(x)$, then we say that the functional $v(y(x))$ takes on an absolute maximum along the curve $y = y_0(x)$. Similarly we define a curve $y = y_0(x)$ along which the functional takes on a minimum value.

Theorem(1.3.1.1),[9]

If the variation of a functional $v(y(x))$ exists, and if v takes on a maximum or minimum along $y = y_0(x)$, then $\delta v = 0$ along $y = y_0(x)$.

Fundamental Lemma(1.3.1.1)[34]:

Let $G(x)$ be a fixed continuous function, defined on the interval $[x_1, x_2]$ and let:

$$\int_{x_1}^{x_2} \eta(x)\, G(x)\, dx = 0$$

where $\eta(x)$ is any continuously differentiable function satisfying;

$$\eta(x_1) = \eta(x_2) = 0$$

then G is identically zero on the interval $[x_1, x_2]$.

Conclusion

The necessary conditions have developed for unconstrained and constrained fractional variational problems. One can see that, the approaches and necessaries conditions for all the above cases problems are same as in an integer order derivatives with fixed, while the approaches and necessaries conditions are different from the variational problems containing integer order derivatives on moving boundaries, duo to using (1.10) instead of using standard integration by parts in usual integer order derivatives. The result of a fractional calculus of variations reduce to those obtained from calculus of variations, in which, many of concepts of variations. Given the fact that many systems can be modeled more accurately using fractional derivative models, it is hoped that future research will continue in this area.

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Future Work

We may look to construct;

1.The necessary optimality conditions fractional variational problems with multi-independent variables.

2.The optimality sufficient conditions, for fractional variational problems with one and multi-independent variables.

3.The optimality sufficient conditions for fractional variational problems with additional constraints (may have integer or fractional order derivatives).

4. The optimality conditions using another approach than we used in this thesis.

Introduction

The field of variations is of significant importance in various disciplines such as science , engineering , pure and applied mathematics. Calculus of variations has been the starting point for various approximate numerical schemes , [7] .

Recent investigations, one can imagine obtaining the formulations by minimizing certain functional which naturally contain functional order derivative, and mathematical tools analogous to calculus of variations will be needed to minimize these functional. However , very little work has been done in the area of fractional calculus of variations , [29] & [30].

The calculus of variations essentially is extension of minimizing or maximizing a function of one variable to problems involves an unknown function and its derivatives, the objective is to find a (not necessarily unique) function that makes the integral stationary within a given class of functions [16]. Functional are variable values which depend on a variable running through a set of functions, or on a finite number of such variables, which are completely determined by a definite choice of these variable functions, means that the functional are variable quantities whose values are determined by the choice of one or several functions.

For instance, the length L of a curve joining two given points on the plane is a functional and the area of a surface is a functional.

The variational calculus gives methods for finding the maximal and minimal values of functional, and the variational problems are problems that consist in finding maxima or minima of a functional.

The variational calculus has been developing since 1696, and it became an independent mathematical discipline with its own research method after the fundamental discoveries of a member of the Patersburg Academy of Sciences.

Fractional calculus is a branch of mathematics which deals with the investigation and application of integrals and derivatives order. Fractional calculus may be considered as an old and yet a novel topic, actually, it is an old topic since starting from some spectrum of Leibniz (1695-`1697) and Euler (1730) who said that the $\frac{d^n f}{dx^n}$ can be made when *n* is an integer as well as when *n* is fractional.

In fact, the idea of generalizing the notion of derivative to non-integer order is found in the correspondence of Leibniz and Bernoulli , L'Hopital and Wallis. Euler take the first step by observing that .the result of the derivative evaluation of the power function has meaning for non-integer order [21].

There are wide areas of applications for the fractional calculus, such as viscoplasticity [29] and viscoelastic constitutive equations [34] which are good applications. That is the constitutive equations governing these phenomenon involve differential equations fractional order. It is also applied in potential field data [7] where the use of fractional gradients provides a much greater flexibility which is generating enhanced analytic signal data. Also any application which uses the computation of velocity and acceleration is an application of fractional differ integration [27]. In physics there are wide applications such as the pressure behavior of transport of different Medias [15] and the diffusion equations [23] and [9]. In engineering,

the fractional calculus is applied in Tensili–Flexral strength of disorder materials and signal processing.

The study of problems of the calculus of variations with fractional derivatives is rather recent subject, the main result being the fractional necessary optimality condition of Euler-Lagrange to be obtained [31]. More detals could be found in [29], [30] obtained a version of Euler-Lagrange equations for problem of the Calculus of Variations with fractional derivatives. More recently, Agrawal [1] gave a formulation for varational problems with right and left fractional derivatives in the Riemann-Liouville sense, and constructed the optimality necessary conditions of fractional variational problem of fixed boundaries, with non-fractional constrain.

The fractional calculus of variations has born in 1996-1997 with the work of F. Riewe: he obtained a version of the Euler-Lagrange equations for problems of the Calculus of Variations with fractional derivatives, combining the conservative and non-conservative cases[29], [30].

Many authors and researchers studied fractional calculus of variations such as:O. Agrawal in 2002, proved a formulation for variational problems with right and left fractional derivatives in the Riemann-Liouville sense[1], In 2004 the Euler-Lagrange equations of Agrawal were used by D. Baleanu and T. Avkar to investigate problems with Lagrangians which are linear on the velocities[4], Klimek in 2005[19] , studied problems depending on symmetric fractional derivatives for which Euler-Lagrange equations include only the derivatives that appear in the formulation of the problem[7],

El-Nabulsi in 2005 [10], introduced Riemann-Liouville fractional integral functionals, depending on a parameter α, but not on fractional-order derivatives of order α, and respective fractional Euler-Lagrange type equations were obtained [10], Agrawal in 2006 investigate transversality conditions for fractional variational problems[26], Baleanu and Agrawal in 2006, study variational problems within Caputo's fractional derivatives[9], Agrawal in 2007, studies fractional variational problems in terms of Riesz fractional derivatives[2], El-Nabulsi and Torres in 2007 establish necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order (α, β)[11], Frederico and Torres in 2007, give a formulation of Noether's theorem for fractional problems of the calculus of variations[18], Nadia J. Ismair in 2007, constructed the optimality necessary conditions for fractional order calculus variational problems based on the following definitions (1.4) and (1.5) [25] , El-Nabulsi and Torres in 2008, study fractional actionlike variational problems[11], Bastos, Ferreira, and Torres in 2009, proved necessary optimality conditions for fractional difference problems of the calculus of variation[24], Mozyrska and Torres in 2009, introduced a new notion of controllability in the memory domain for fractional continuous-time linear control systems and solved the modifed energy fractional optimal control problem[25], Almeida, Malinowska and Torres in 2010, developed a fractional calculus of variations for multiple integrals with application to vibrating string[12], Almeida and Torres in 2010, investigated a direct method for fractional optimization problems[33], Frederico and Torres in 2010, proved a fractional Noether's theorem in the Riesz-Caputo sense[13] ,finally,to

the best of one knowledge Malinowska and Torres in 2010, proved generalized natural boundary conditions for fractional variation problem with Caputo derivatives[22].

This thesis consists of three chapters. chapter one presents the basic concepts of fractional calculus based on Riemann-Liouville definition and calculus of variation .

Chapter two presents the optimality necessary conditions for unconstrained and constrained fractional variational problems with continuous and discontinuous functional having one and different multi fractional order derivatives on one and different multi-dependent variables of one independent variable along fixed boundaries.

Chapter three presents the optimality necessary conditions for unconstrained and constrained fractional variational problems with continuous and discontinuous functional having one and different multi-fractional order derivatives on one and different multi-dependent variable of one independent variable along movable boundaries.

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Optimality Necessary Conditions For Continuous and discontinuous Fractional order Variational Problems

A Thesis

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وزارة التعليم العالي والبحث العلم ي
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كلية العلوم
قسم الرياضيات وتطبيقات الحاسوب

الشروط الضرورية للأمثلية
لمسائل التغاير ذوات الرتب الكسرية المستمرة وغير المستمرة

رسالة
مقدمه إلى كلية العلوم/ جامعة النهرين
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شروط الأمثلية لمسائل التغاير ذات الرتب الكسرية

الخلاصة

في هذا البحث، تم عرض بعض الخواص والتعاريف الأساسية للمشتقات والتكاملات ذات الرتب الكسرية لريمان لوفيل (Riemann Liouville). كما تم تقديم اسلوب رياضي لأستنباط الشروط الضرورية لأمثلية مسائل التغاير المقيدة و الغير المقيدة ذات الدالي (Functional) المستمرة و الغير مستمر على حدود ثابتة و متحركه وبأستخدام,قاعدة الضرب الكلاسكية ل (Riemann Liouville)كما تم عرض بعض الأمثلة توضح تحقيق أمثلية الشروط الضرورية.

Dedicated to my family

With all love and respects

بسم الله الرحمن الرحيم

قَالُواْ سُبْحَانَكَ لاَ عِلْمَ لَنَا إِلاَّ مَا عَلَّمْتَنَا إِنَّكَ أَنتَ

الْعَلِيمُ الْحَكِيمُ

صدق الله العظيم

سورة البقرة

الآية (32)