ABSTRACT

Fractional (or non-integer) differentiation is an important concept both from theoretical and applicational points of view. The study of problems of the calculus of variations with fractional derivatives is a rather recent subject.

In this work, some properties and basic definitions of fractional integral and derivatives of Riemann-Liouvill are presented. The optimality necessary conditions for fractional variational problems are constructed for different types of fractional problems of calculus of variations having one and different multi fractional order derivatives (FOD) on one and different multi-dependent variables with one independent variable, along fixed and moving boundaries. Several examples are presented to demonstrate the implementation of the optimality necessary conditions.

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1.1 INTRODUCTION:

In this chapter, we give some concepts about fractional calculus, and calculus of variation. Also some of theories and definitions are presented that needed then later.

1.2 FRACTIONAL CALCULUS:

This section presents some of the most basic and important concepts in fractional calculus which are necessary for understanding the subject of fractional calculus.

1.2.1 The Gamma Function:

The gamma function is represented by an improper integral and it's defined by [15].

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \qquad z > 0. \tag{1.1}$$

As will be clear later, the gamma function is intrinsically tied to fractional calculus by definition. The simple interpretation of the gamma function is simply the generalization of the fractional for all positive real numbers [9].

Some of the properties of the gamma function are:

1. $\Gamma(1) = 1$.

2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

3. $\Gamma(0) = \pm\infty$.

4. $\Gamma(-n) = \pm\infty \quad \forall\, n \in \mathbb{N}^+$.

5. $\Gamma(z+1) = z\,\Gamma(z), \qquad z \in \mathbb{R}^+$.

6. $\Gamma(n) = (n-1)!\,, \qquad n \in \mathbb{N}$.

7. $\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!\,\sqrt{\pi}}{4^n\, n!}$.

8. $\Gamma\left(\frac{1}{2}-n\right) = \frac{(-4)^n\, n!\,\sqrt{\pi}}{(2n)}$.

1.2.2 ***Definitions:***

Riemann's modified form of Liouville's fractional integral operator is a direct generalization of Cauchy's formula for an n-fold integral .

$$\int_a^x dx_1 \int_a^{x_1} dx_2 \ldots \int_a^{x_{n-1}} f(x_n)dx_n = \frac{1}{(n-1)!}\int_a^x \frac{f(t)}{(x-t)^{1-n}}\,dt \qquad (1.2)$$

By n-fold here means that the integration is deployed n-times. Since $(n-1)! = \Gamma(n)$, Riemann realized that the RHS of (1.2) might have meaning even when *n* takes non-integer values [16].

Thus perhaps it was natural to define fractional integration as follows:

Definition (1.2.2.1) [17], [18]:

Let $f(t) \in L_1[a,b]$, $\alpha \in \mathbb{R}^+$. The fractional (arbitrary) order integral of the function $f(t)$ of order α is defined as:

$$\mathrm{I}_a^\alpha f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds\,, \tag{1.3}$$

When $a = 0$ we can write, $\mathrm{I}^\alpha f(t) = \mathrm{I}_0^\alpha f(t) = f(t)\,\Phi_\alpha(t)$, where;

$$\Phi_\alpha(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & ,\text{for } t > 0 \\ 0 & ,\text{for } t \le 0 \end{cases}$$

Definition (1.2.2.2) [17], [18]:

The fractional derivative D^α of order $\alpha \in (0,1]$ of the absolutely continuous function $f(t)$ is defined as:

$$D_a^\alpha f(t) = \mathrm{I}_a^{1-\alpha} \frac{d}{dt} f(t)\,, \quad t \in [a,b] \tag{1.4}$$

1.2.3 Riemann-Liouville Fractional Integrals And Fractional Derivatives [10]:

We give the definitions of the Riemann-Liouville fractional integrals and fractional derivatives on a finite interval of the real line and present some of their properties in spaces of continuous functions.

Let $\Omega = [a,b]$ be a finite interval .

The Riemann-Liouville fractional integrals $\mathrm{I}_a^\alpha f$ and $\mathrm{I}_b^\alpha f$ of order;
$\alpha \in \mathbb{C}(Re(\alpha) > 0)$ are defined by;

$$(\mathrm{I}_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}}, \quad (x > a, Re(\alpha) > 0), \tag{1.5}$$

and

$$(\mathrm{I}_b^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha}}, \quad (x < b, Re(\alpha) > 0), \tag{1.6}$$

These integrals are called the left-sided and the right-sided fractional integrals.

The Riemann-Liouville fractional derivatives $D_a^\alpha y$ and $D_b^\alpha y$ of order $\alpha \in \mathbb{C}\ (Re(\alpha) > 0)$ are defined by:

$$(D_a^\alpha y)(x) = \frac{d^n}{dx^n} (\mathrm{I}_a^{n-\alpha} y)(x)$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{y(t)\,dt}{(x-t)^{\alpha-n+1}}, \quad (n = [Re(\alpha)] + 1, \quad x > a),$$

----------------------- (1.7)

and;

$$(D_b^\alpha y)(x) = (-1)^n \frac{d^n}{dx^n} (\mathrm{I}_b^{n-\alpha} y)(x)$$

$$= \frac{1}{\Gamma(n-\alpha)} (-1)^n \frac{d^n}{dx^n} \int_x^b \frac{y(t)\,dt}{(t-x)^{\alpha-n+1}}, \quad (n = [Re(\alpha)] + 1, \quad x < b),$$

------------------- (1.8)

where; $[Re(\alpha)]$ means the integer part of $Re(\alpha)$.

In particular, when $\alpha = n \in \mathbb{N}$, then;

$(D_\alpha^0 y)(x) = (D_b^0 y)(x) = y(x)$, $\qquad (D_a^n y)(x) = y^{(n)}(x)$, and;

$$(D_b^n y)(x) = (-1)^n y^{(n)}(x)\,, \qquad (n \in \mathbb{N})\,, \tag{1.9}$$

where $y^{(n)}(x)$ is the usual derivative of $y(x)$ of order n.

If $0 < Re(\alpha) < 1$, then;

$$(D_a^\alpha y)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{y(t)dt}{(x-t)^{\alpha-[Re(\alpha)]}}, (0 < Re(\alpha) < 1, x > a) \tag{1.10}$$

$$(D_b^\alpha y)(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{y(t)dt}{(t-x)^{\alpha-[Re(\alpha)]}}, (0 < Re(\alpha) < 1, x < b) \tag{1.11}$$

When $\alpha \in \mathbb{R}^+$, then equations (1.7) and (1.8) take the following forms:

$$(D_a^\alpha y)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{y(t)\,dt}{(x-t)^{\alpha-n+1}}\,, \quad (n = [\alpha]+1, \quad x > a)$$

--------------------- (1.12)

and;

$$(D_b^\alpha y)(x) =$$

$$\frac{1}{\Gamma(n-\alpha)} (-1)^n \frac{d^n}{dx^n} \int_x^b \frac{y(t)\,dt}{(t-x)^{\alpha-n+1}}\,, \qquad (n = [\alpha]+1, x < b), \tag{1.13}$$

while, equations (1.10) and (1.11) are given by:

$$(D_a^\alpha y)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{y(t)dt}{(x-t)^\alpha}, \quad (0 < \alpha < 1, x > a) \tag{1.14}$$

and;

$$(D_b^\alpha y)(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{y(t)dt}{(t-x)^\alpha}, \quad (0 < \alpha < 1, x < b) \tag{1.15}$$

1.2.4 Lemmas[10]:

In this subsection, we are presented the following two lemmas for fractional integral and fractional derivatives.

Let $L_p = L_p[a,b]$ be the class of Lebesgue integrable functions on $[a,b]$, $a < b < \infty$, $(1 \le p < \infty)$.

Lemma (1.2.4.1):

If $Re(\alpha) > 0$ and $Re(\beta) > 0$, then the equations;

$$\left(\mathrm{I}_a^\alpha\, \mathrm{I}_a^\beta\, f\right)(x) = \left(\mathrm{I}_a^{\alpha+\beta}\, f\right)(x) \quad \text{and} \quad \left(\mathrm{I}_b^\alpha\, \mathrm{I}_b^\beta\, f\right)(x) = \left(\mathrm{I}_b^{\alpha+\beta}\, f\right)(x). \qquad (1.16)$$

are satisfied at almost every point $x \in [a,b]$ for;

$f \in L_p(a,b)$ $(1 \le p < \infty)$.

If $\alpha + \beta > 1$, then the relations (1.16) hold at any point of $[a,b]$.

Lemma (1.2.4.2):

If $Re(\alpha) > 0$ and $f \in L_p(a,b)$, $(1 \le p < \infty)$, then

$$(D_a^\alpha\, \mathrm{I}_a^\alpha\, f)_{(x)} = f(x) \quad \text{and} \quad (D_b^\alpha\, \mathrm{I}_b^\alpha\, f)_{(x)} = f(x)\ ; \qquad (1.17)$$

1.2.5 Properties [10]:

In this subsection, we are presented some properties for fractional integral and fractional derivatives.

Property (1.2.5.1):

If $Re(\alpha) > Re(\beta) > 0$, then for $f \in L_p(a,b)$, $(1 \le p < \infty)$, the relations;

$$\left(D_a^\beta\, \mathrm{I}_a^\alpha\, f\right)(x) = \mathrm{I}_a^{\alpha-\beta} f(x)\ , \qquad (1.18a)$$

and;

$$\left(D_b^{\beta}\ \mathrm{I}_b^{\alpha} f\right)(x) = \mathrm{I}_b^{\alpha-\beta} f(x)\,, \tag{1.18b}$$

In particular, when $\beta = k \in \mathbb{N}$ and $Re(\alpha) > k$, then;

$$(D_a^k\ \mathrm{I}_a^{\alpha} f)(x) = \mathrm{I}_a^{\alpha-k} f(x)\,; \tag{1.19a}$$

and

$$\left(D_b^k\ \mathrm{I}_b^{\alpha} f\right)(x) = (-1)^k\, I_b^{\alpha-k} f(x)\,. \tag{1.19b}$$

Property (1.2.5.2):

Let $Re(\alpha) \geq 0, \qquad m \in \mathbb{N}$.

a. If the fractional derivatives $(D_a^{\alpha} y)(x)$ and $(D_a^{\alpha+m} y)(x)$ exist, then;

$$(D^m\, D_a^{\alpha}\, y)(x) = (D_a^{\alpha+m} y)(x)\,. \tag{1.20}$$

b. If the fractional derivatives $(D_b^{\alpha} y)(x)$ and $(D_b^{\alpha+m} y)(x)$ exist, then;

$$(D^m D_b^{\alpha} y)(x) = (-1)^m (D_b^{\alpha+m} y)(x)\,. \tag{1.21}$$

Now, some additional important properties of the fractional differential operator D_t^{α} are presented for completeness purpose[19]:

1. The operator D_t^{α} of order $\alpha = 0$ is the identity operator.
2. The operator D_t^{α} is linear, i.e.,

   $D_t^{\alpha}\left(c_1\, f(t) + c_2\, g(t)\right) = c_1\, D_t^{\alpha}\, f(t) + c_2\, D_t^{\alpha}\, g(t)$, where c_1 and c_2 are constants.
3. The operator D_t^{α} is homogenous;

   $D_t^{\alpha}\, \{c\, f(t)\} = c\, D_t^{\alpha}\, f(t)\,.$
4. $D_t^{\alpha} \sum_{i=1}^{n} f_i(t) = \sum_{i=1}^{n} D_t^{\alpha} f_i(t)$

1.2.6 Examples:

For the sake of illustration of the definition of Riemann-Liouville fractional derivative, the following examples are given:

Example (1.1):

The definition of fractional calculus due to Reimann-Liouville is considered for the function $f(x) = c$, where c is a constant.
then;

$$D_x^{\alpha} c = \frac{d^n}{dx^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}\, c\, dt\right]$$

$$= \frac{c}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\left[\frac{-(x-t)^{n-\alpha}}{n-\alpha}\bigg|_a^x\right]$$

$$= \frac{c}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\left[\frac{(x-a)^{n-\alpha}}{n-\alpha}\right]$$

Now, if $\alpha = 0.5$, $a = 0$ then $n = [Re(\alpha)] + 1 = 1$, then the above results become;

$$D_x^{0.5} c = \frac{c}{\Gamma(0.5)}\frac{d}{dx}\left(\frac{\sqrt{x}}{0.5}\right)$$

$$= \frac{c}{\sqrt{\pi x}}.$$

If $\boldsymbol{c = 1}$, then;

$$D_x^{0.5} 1 = \frac{1}{\sqrt{\pi x}}.$$

If $\boldsymbol{c} = \dfrac{\mathbf{1}}{\mathbf{2}}$, then;

$$D_x^{0.5}\,\frac{1}{2} = \frac{1}{2\sqrt{\pi x}}\,.$$

For $\alpha = 1.5$, $a = 0$ then $n = [Re(\alpha)] + 1 = 2$

$$D_x^{1.5}c = \frac{1}{\Gamma(0.5)}\frac{d^2}{dx^2}\int_0^x (x-t)^{-1/2}c\ dt$$

$$= \frac{c}{\Gamma(0.5)}\frac{d^2}{dx^2}\left[\frac{-\sqrt{x-t}}{1/2}\Bigg|_0^x\right]$$

$$= \frac{-c}{2\sqrt{\pi\,x^3}}\,.$$

If $\boldsymbol{c} = \mathbf{1}$, then;

$$D_x^{1.5}1 = \frac{-1}{2\sqrt{\pi\,x^3}}\,,$$

If $\boldsymbol{c} = \dfrac{\mathbf{1}}{\mathbf{2}}$ then;

$$D_x^{1.5}\,\frac{1}{2} = \frac{-1}{4\sqrt{\pi\,x^3}}\,.$$

<u>Example (1.2)</u>:

Consider $f(x) = x$, $a = 0$.

$$D_x^{-0.5}x = \frac{1}{\Gamma(0.5)}\int_0^x (x-t)^{-1/2}\ t\ dt$$

$$= \frac{4\sqrt{x^3}}{3\sqrt{\pi}}\,.$$

$$D_x^{0.5} x = \frac{1}{\Gamma(0.5)} \frac{d}{dx} \int_0^x (x-t)^{-1/2}\, t\, dt$$

$$= \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

$$D_x^{1.5} x = \frac{1}{\Gamma(0.5)} \frac{d^2}{dx^2} \int_0^x (x-t)^{-1/2}\, t\, dt$$

$$= \frac{1}{\sqrt{\pi x}}.$$

1.3 CALCULUS OF VARIATION [20]:

Functional are variable values which depend on a variable running through a set of functions, or on a finite number of such variables, and which are completely determined by a definite choice of these variable functions, means that the functional are variable quantities whose values are determined by the choice of one or several functions.

For instance, the length L of a curve joining two given points on the plane is a functional and the area γ of a surface is a functional.

The variational calculus gives methods for finding the maximal and minimal values of functional, and the variational problems are problems that consist in finding maxima or minima of a functional.

The variational calculus has been developing since 1696, and it became an independent mathematical discipline with its own research method after the fundamental discoveries of a member of the Patersburg Academy of Sciences

Euler (1707-1783),whom we can claim with good reason to be the founder of the calculus of variation.

There are three problems had considerable influence the development of the calculus of variations which are the brachistochrone problem in 1696 Johann Bernoulli, the problem of geodesics in 1697 Johann Bernoulli, and Isoperimetric problem by Euler.

1.3.1 Basic Definitions and Theories [20]:

Basic Definitions and the theories will be given in this subsection.

Definition (1.3.1.1):

The variable v is called functional depending on a function $y(x)$, in writing $v = v\big(y(x)\big)$, if to each function, $y(x)$ from a certain class of functions, there corresponds a certain value of v.

Definition (1.3.1.2):

The increment or variation δy of the argument $y(x)$ of a functional $v\big(y(x)\big)$ is the difference of two functions $\delta y = y(x) - y_1(x)$
where $y_1(x)$ is admissible curve .

Definition (1.3.1.3):

A functional $v\big(y(x)\big)$ is continuous along $y = y_0(x)$ in the sense of closeness of order k, if for arbitrary positive number ε there exists a $\delta > 0$ such that, $\big|v\big(y(x)\big) - v\big(y_0(x)\big)\big| < \varepsilon$, whenever;
$|y(x) - y_0(x)| < \delta$, $|y(x) - \acute{y}_0(x)| < \delta$, …,
$|y(x) - {y_0}^{(k)}\,(x)\,| < \delta$.

It is understood, that the function $y(x)$ is taken from the class of functions for which $v\big(y(x)\big)$ is defined.

Definition (1.3.1.4):

The functional $v(y(x))$ is called a linear functional, if it satisfies the conditions:

a. $v(cy(x)) = cv(y(x))$, where c is constant.

b. $v(y_1(x) + y_2(x)) = v(y_1(x)) + v(y_2(x))$.

Definition (1.3.1.5):

A functional $v(y(x))$ takes on a maximum value along the curve $y = y_o(x)$, if all the values of this functional $v(y(x))$ taken on along arbitrary neighboring to $y = y_o(x)$, curves are not greater than $v(y_o(x))$, i.e. $\Delta v = v(y(x)) - v(y_0(x)) \leq 0$. If $\Delta v \leq 0$ and $\Delta v = 0$ only when $y = y_0(x)$, then we say that the functional $v(y(x))$ takes on an absolute maximum along the curve $y = y_0(x)$. Similarly we define a curve $y = y_0(x)$ along which the functional takes on a minimum value.

Theorem(1.3.1.1)[20]:

If the variation of a functional $v(y(x))$ exists, and if v takes on a maximum or minimum along $y = y_0(x)$, then $\delta v = 0$ along $y = y_0(x)$.

Proof :

See [20]

Fundamental Lemma(1.3.1.1)[21]:

Let $G(x)$ be a fixed continuous function, defined on the interval $[x_1, x_2]$ and let:

$$\int_{x_1}^{x_2} \eta(x)\, G(x)\, dx = 0 \tag{1.22}$$

where $\eta(x)$ is any continuously differentiable function satisfying;

$$\eta(x_1) = \eta(x_2) = 0 \tag{1.23}$$

then G is identically zero on the interval $[x_1, x_2]$.

Proof :

See [21]

3.1 INTRODUCTION:

In this chapter we concerned with the constructing of the optimality necessary condition for the extremum of the fractional variational problems having one and different multi fractional order derivatives (FOD) on one and different multi dependent variables of one independent variable along movable boundaries.

3.2 VARIATIONAL PROBLEMS WITH SINGLE (FOD):

Consider the functional of the form:

$$v(x, y(x)) = \int_{x_o}^{x_1} F(x, y, y^{(\alpha)})\, dx, \tag{3.1}$$

where; $0 < \alpha < 1$ and one of the end points is variable (say (x_1, y_1)), i.e. (x_1, y_1) can move turning into $(x_1 + \delta x_1, y_1 + \delta y_1)$.

The functional taken only along the curve of the pencil turns into function of x_1 and y_1. It's variation turns into the differential of this function, with prescribed conditions on fixed boundary only if necessary.

$y(x_0) = y_0$ and $y(x_1) = y_1$.

$$\Delta v = \int_{x_0}^{x_1+\delta x_1} F(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})dx - \int_{x_0}^{x_1} F(x, y, y^{(\alpha)})dx$$

$$= \int_{x_1}^{x_1+\delta x_1} F(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})dx$$

$$+ \int_{x_0}^{x_1} \left(F(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}) - F(x, y, y^{(\alpha)})\right) dx. \tag{3.2}$$

The first term of the right – hand side of equation (3.2) will be transformed with the aid of the mean value theorem to get:

$$\int_{x_1}^{x_1+\delta x_1} F\left(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)}\right) dx = F|_{x=x_1+\theta\delta x_1}\delta x_1,$$

where $(0 < \theta < 1)$.

Furthermore, by virtue of continuity of the function F,

$$F|_{x=x_1+\theta\delta x_1} = F\left(x, y, y^{(\alpha)}\right)\Big|_{x=x_1} + \varepsilon_1,$$

where;

$\varepsilon_1 \to 0$ as $\delta x_1 \to 0$ and $\delta y_1 \to 0$

Consequently;

$$\int_{x_1}^{x_1+\delta x_1} F\left(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)}\right) dx = F\left(x, y, y^{(\alpha)}\right)\Big|_{x=x_1}\delta x_1 + \varepsilon_1 dx_1. \quad (3.3)$$

To transform the second term of the right-hand side of equation (3.2), Taylor formula should be utilized to get:

$$\int_{x_0}^{x_1} F\left(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)}\right) - F\left(x, y, y^{(\alpha)}\right) dx$$

$$= \int_{x_0}^{x_1} \left(F_y\left(x, y, y^{(\alpha)}\right)\delta y + F_{y^{(\alpha)}}\left(x, y, y^{(\alpha)}\right)\delta y^{(\alpha)}\right) dx + R_1.$$

R_1 is an infinitesimal of higher order than δy or $\delta y^{(\alpha)}$

By using the definitions (1.2.2.1) and (1.2.2.2); it can be found that;

$$D^{\alpha}(\delta y) = \delta y^{(\alpha)} = \mathrm{I}^{1-\alpha}\frac{d}{dx}\delta y = \delta \acute{y}\left(\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right).$$

We have;

$$\int_{x_0}^{x_1} F\left(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)}\right) - F\left(x, y, y^{(\alpha)}\right) dx$$

$$= \int_{x_0}^{x_1} F_y\, \delta y\, dx + \frac{1}{\Gamma(1-\alpha)}\int_{x_0}^{x_1}\left(x^{-\alpha} F_{y^{(\alpha)}}\, \delta \acute{y}\right) dx.$$

Integrating by parts the second term to get:

$$= \int_{x_0}^{x_1} F_y\, \delta y\, dx + \frac{1}{\Gamma(1-\alpha)}\left(\left[x^{-\alpha} F_{y^{(\alpha)}}\, \delta y\right]_{x_0}^{x_1} - \int_{x_0}^{x_1} d\left(x^{-\alpha} F_{y^{(\alpha)}}\right)\delta y\ dx\right)$$

$$= \int_{x_0}^{x_1} F_y \delta y\, dx +$$

$$\frac{1}{\Gamma(1-\alpha)}\left(\left[x^{-\alpha} F_{y^{(\alpha)}}\, \delta y\right]_{x_0}^{x_1}\right.$$

$$\left. - \int_{x_0}^{x_1}\left(-\alpha\, x^{-\alpha-1} F_{y^{(\alpha)}} + x^{-\alpha}\frac{d}{dx}F_{y^{(\alpha)}}\right)\delta y\, dx\right). \tag{3.4}$$

$$= \frac{1}{\Gamma(1-\alpha)}\left[x^{-\alpha} F_{y^{\alpha}}\, \delta y\right]_{x_0}^{x_1}$$

$$+ \int_{x_0}^{x_1}\left(F_y\, \delta y - \frac{1}{\Gamma(1-\alpha)}\left(-\alpha\, x^{-\alpha-1} F_{y^{\alpha}} + x^{-\alpha}\frac{d}{dx}F_{y^{(\alpha)}}\right)\delta y\right) dx$$

$$= \frac{1}{\Gamma(1-\alpha)} \left[x^{-\alpha} F_{y^{(\alpha)}} \delta y \right]_{x_0}^{x_1} + \int_{x_0}^{x_1} \left(F_y - \frac{1}{\Gamma(1-\alpha)} \left(-\alpha\, x^{-\alpha-1} F_{y^{(\alpha)}} + x^{-\alpha} \frac{d}{dx} F_{y^{(\alpha)}} \right) \delta y \right) dx.$$

The values of the functional are taken only along fixed $x = x_0$ extremals.

Consequently;

$$F_y + \frac{\alpha}{\Gamma(1-\alpha)} x^{-(\alpha+1)} F_{y^{(\alpha)}} - \frac{1}{\Gamma(1-\alpha)} x^{-\alpha} \frac{d}{dx} F_{y^{(\alpha)}} = 0$$

Since the end point (x_0, y_0) is fixed ,it follows that $\delta y|_{\,x=x_0} = 0$ and therefore

$$\int_{x_0}^{x_1} \left(F_y\, \delta y + F_{y^{\alpha}}\, \delta y^{(\alpha)} \right) dx = \frac{1}{\Gamma(1-\alpha)} x^{-\alpha} F_{y^{(\alpha)}}\, \delta y \Big|_{x=x_1}. \qquad (3.5)$$

Observe that $\delta y|_{x=x_1}$ does not mean the same as δy_1, the increment of y_1, for δy_1 is the change of y-coordinate of the free end point, when it is moved from (x_1, y_1) to $(x_1 + \delta x_1, y_1 + \delta y_1)$, whereas; $\delta y|_{x=x_1}$ is the change of y-coordinate of an extremal produced at the point $x = x_1$ when this extremal changes from one that passes through the points (x_0, y_0) and (x_1, y_1) to another one passing through (x_0, y_0) and $(x_1 + \delta x_1, y_1 + \delta y_1)$ (Fig. 1).

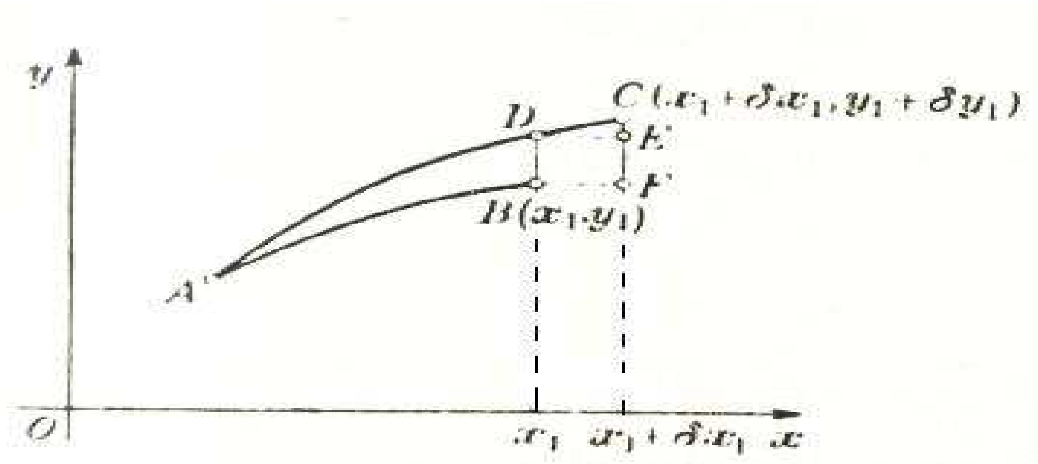

Fig. (1): Extremal Changes.

From Fig. (1):

$\text{BD} = \delta y|_{x=x_1}$ $\quad\quad$ $\text{FC} = \delta y_1$ $\quad\quad$ $\text{EC} \cong \acute{y}(x_1)\,\delta x_1$

$\text{BD} = \text{FC} - \text{EC}$

$\delta y|_{x=x_1} \cong \delta y_1 - \acute{y}(x_1)\delta x_1.$

Consequently;

$$\int_{x_1}^{x_1+\delta x_1} F\,dx \cong F|_{x=x_1}\,\delta x_1,$$

$$\int_{x_0}^{x_1} \left(F\left(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)}\right) - F\left(x, y, y^{(\alpha)}\right)\right) dx$$

$$\cong \frac{1}{\Gamma(1-\alpha)}\left[x^{-\alpha}\,F_{y^{(\alpha)}}\right]_{x=x_1}(\delta y_1 - \acute{y}(x_1)\delta x_1),$$

where all approximate equations hold apart from infinitesimal of fractional order with respect to δx_1 or δy_1, and it follows from equations (3.3) and (3.5), to get;

$$\delta v = F|_{x=x_1}\delta x_1 + \frac{1}{\Gamma(1-\alpha)}\left(x^{-\alpha}F_{y^{(\alpha)}}\right)\Big|_{x=x_1}(\delta y_1 - \acute{y}(x_1)\delta x_1),$$

then the fundamental necessary condition for an extremum $\delta v = 0$ is;

$$F|_{x=x_1}\delta x_1 + \frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{y^{(\alpha)}}\Big|_{x=x_1} \delta y_1 - \frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{y^{(\alpha)}}\Big|_{x=x_1} \acute{y}(x_1)\, \delta x_1 = 0.$$

$$F - \left(\frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{y^{(\alpha)}}\right) \acute{y}(x)\Big|_{x=x_1} \delta x_1 + \frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{y^{(\alpha)}}\Big|_{x=x_1} \delta y_1 = 0.$$

------------------ (3.6)

If the variations δx_1 and δy_1 are independent, then the necessary condition for the extremum is;

$$F - \left(\frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{y^{(\alpha)}}\right) \acute{y}(x)\Big|_{x=x_1} = 0, \qquad (3.7a)$$

and

$$\frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{y^{(\alpha)}}\Big|_{x=x_1} = 0. \qquad (3.7b)$$

If the variations δx_1 and δy_1 are dependent, for instance, suppose the end point (x_1, y_1) can move along a certain curve $y_1 = \varphi(x_1)$, then by substituting $\delta y_1 \cong \acute{\varphi}(x_1)\delta x_1$ in equation (3.6), the necessary condition for the extremal is;

$$F - \left(\frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{y^{(\alpha)}}\right) \acute{y}(x)\Big|_{x=x_1} \delta x_1 + \left(\frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{y^{(\alpha)}}\right) \acute{\varphi}(x)\Big|_{x=x_1} \delta x_1 = 0$$

$$F - \left(\frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{y^{(\alpha)}}\right) \acute{y}(x) + \left(\left(\frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{y^{(\alpha)}}\right) \acute{\varphi}(x)\right)\Big|_{x=x_1} \delta x_1 = 0$$

$$F - \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \left(\acute{y}(x) - \acute{\varphi}(x)\right) F_{y^{(\alpha)}}\Big|_{x=x_1} \delta x_1 = 0.$$

Since δx_1 is arbitrary, then the necessary condition becomes;

$$F - \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\left(\acute{y}(x) - \acute{\varphi}(x)\right)F_{y^{(\alpha)}}\Big|_{x=x_1} = 0. \tag{3.8}$$

3.3 VARIATIONAL PROBLEMS WITH MULTI DIFFERENT (FOD) ON ONE DEPENDENT VARIABLE:

Functional involving derivatives of multi fractional orders (α_i), $0 < \alpha_i < 1$.

$$v\big(x, y(x)\big) = \int_{x_0}^{x_1} F\left(x, y(x), y^{(\alpha_1)}(x), \dots, y^{(\alpha_m)}(x)\right) dx. \tag{3.9}$$

where one of the end points is variable, say (x_1, y_1), i.e. (x_1, y_1) can move turning into $\big((x_1 + \delta x_1, y_1 + \delta y_1)\big)$.

$$\Delta v = \int_{x_0}^{x_1+\delta x_1} F\big(x, y + \delta y, y^{(\alpha_1)} + \delta y^{(\alpha_1)}, y^{(\alpha_2)} + \delta y^{(\alpha_2)}, \dots, y^{(\alpha_m)} + \delta y^{(\alpha_m)})dx - \int_{x_0}^{x_1} F\big(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)}\big)dx.$$

$$= \int_{x_0}^{x_1+\delta x_1} F\big(x, y + \delta y, y^{(\alpha_1)} + \delta y^{(\alpha_1)}, \dots, y^{(\alpha_m)} + \delta y^{(\alpha_m)}\big)dx + \int_{x_0}^{x_1} \Big(F\big(x, y + \delta y, y^{(\alpha_1)} + \delta y^{(\alpha_1)}, y^{(\alpha_2)} + \delta y^{(\alpha_2)}, \dots, y^{(\alpha_m)} + \delta y^{(\alpha_m)}) - F\big(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)}\big)\Big) dx. \tag{3.10}$$

The first term of the right-hand side of equation (3.10) will be transformed with the aid of the mean value theorem to get;

$$\int_{x_0}^{x_1+\delta x_1} F\left(x, y+\delta y, y^{(\alpha_1)}+\delta y^{(\alpha_1)}, \dots, y^{(\alpha_m)}+\delta y^{(\alpha_m)}\right) dx$$

$$= F|_{x=x_1+\theta\delta x_1}\delta x_1.$$

where; $(0 < \theta < 1)$. Furthermore, by virtue of continuity of the function *F*.

$F|_{x=x_1+\theta\delta x_1}\delta x_1 = F|_{x=x_1} + \varepsilon_1,$

where; $\varepsilon_1 \to 0$, as $\delta x_1 \to 0$ and $\delta y_1 \to 0$.

Consequently;

$$\int_{x_1}^{x_1+\delta x_1} F\left(x, y+\delta y, y^{(\alpha_1)}+\delta y^{(\alpha_1)}, \dots, y^{(\alpha_m)}+\delta y^{(\alpha_m)}\right) dx$$

$$= F|_{x=x_1}\delta x_1 + \varepsilon_1 dx_1. \tag{3.11}$$

To transform the second term of the right-hand side of equation (3.10), Taylor formula should be utilized to get;

$$\int_{x_0}^{x_1} \left(F\left(x, y+\delta y, y^{(\alpha_1)}+\delta y^{(\alpha_1)}, y^{(\alpha_2)}+\delta y^{(\alpha_2)}, \dots, y^{(\alpha_m)}+\delta y^{(\alpha_m)}\right)\right.$$

$$\left. - F\left(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)}\right)\right) dx$$

$$= \int_{x_0}^{x_1} \left(F_y\, \delta y + F_{y^{(\alpha_1)}}\, \delta y^{(\alpha_1)} + F_{y^{(\alpha_2)}}\, \delta y^{(\alpha_2)} + \dots + F_{y^{(\alpha_m)}}\, \delta y^{(\alpha_m)}\right) dx + R_1$$

$$= \int_{x_0}^{x_1} \left(F_y\, \delta y + \sum_{i=1}^{m} F_{y^{(\alpha_i)}}\, \delta y^{(\alpha_i)}\right) dx + R_1,$$

where; R_1 is an infinitesimal of higher order than δy or $\delta y^{(\alpha_i)}$, then by using the definitions (1.2.2.1) and (1.2.2.2);

$$D^{\alpha_i}(\delta y) = \delta y^{\alpha_i} = \mathrm{I}^{1-\alpha_i}\frac{d}{dx}\delta y = \delta \acute{y}\left(\frac{x^{-\alpha_i}}{\Gamma(1-\alpha_i)}\right).$$

We have;

$$\int_{x_0}^{x_1} \Big(F\big(x, y+\delta y, y^{(\alpha_1)}+\delta y^{(\alpha_1)}, y^{(\alpha_2)}+\delta y^{(\alpha_2)}, \dots, y^{(\alpha_m)}+\delta y^{(\alpha_m)}\big) - F\big(x, y, y^{(\alpha_1)}, y^{(\alpha_2)}, \dots, y^{(\alpha_m)}\big)\Big)\,dx$$

$$= \int_{x_0}^{x_1} F_y\, \delta y dx + \int_{x_0}^{x_1} \left(\sum_{i=1}^{m} \frac{x^{-\alpha_i}}{\Gamma(1-\alpha_i)}\, F_{y^{(\alpha_i)}}\, \delta \acute{y}\right) dx. \tag{3.12}$$

By the same approach of section (3.2), to integrate by part the second term of equation (3.12), we get;

$$\int_{x_0}^{x_1} F_y\, \delta y dx + \int_{x_0}^{x_1} \left(\sum_{i=1}^{m} \frac{x^{-\alpha_i}}{\Gamma(1-\alpha_i)}\, F_{y^{(\alpha_i)}}\, \delta \acute{y}\right) dx = \int_{x_0}^{x_1} F_y\, \delta y dx +$$

$$\sum_{i=1}^{m} \left(\frac{1}{\Gamma(1-\alpha_i)} \left\{\left[x^{-\alpha_i}\, F_{y^{(\alpha_i)}}\, \delta y\right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \left(-\alpha_i\, x^{-(\alpha_i+1)}\, F_{y^{(\alpha_i)}} + x^{-(\alpha_i)} \frac{d}{dx} F_{y^{(\alpha_i)}}\right) \delta y\, dx\right\}\right)$$

$$= \sum_{i=1}^{m} \frac{1}{\Gamma(1-\alpha_i)} \left[x^{-\alpha_i}\, F_{y^{(\alpha_i)}}\, \delta y\right]_{x=x_1} +$$

$$\int_{x_0}^{x_1} \left(F_y + \sum_{i=1}^{m} \frac{1}{\Gamma(1-\alpha_i)} \left(-\alpha_i\, x^{-(\alpha_i+1)}\, F_{y^{(\alpha_i)}} + x^{-(\alpha_i)} \frac{d}{dx} F_{y^{(\alpha_i)}}\right)\right) \delta y\, dx$$

The values of functional are taken only along fixed $x = x_0$ extremuls.

$$F_y + \sum_{i=1}^{m} \frac{1}{\Gamma(1-\alpha_i)} \left(-\alpha_i\, x^{-(\alpha_i+1)}\, F_{y^{(\alpha_i)}} + x^{-(\alpha_i)} \frac{d}{dx} F_{y^{(\alpha_i)}}\right) = 0. \tag{3.13}$$

We get;

$$\int_{x_0}^{x_1} F_y\,\delta y dx + \int_{x_0}^{x_1}\left(\sum_{i=1}^{m}\frac{x^{-\alpha_i}}{\Gamma(1-\alpha_i)}\;F_{y^{(\alpha_i)}}\,\delta\acute{y}\right)dx$$

$$= \sum_{i=1}^{m}\frac{1}{\Gamma(1-\alpha_i)}\left[x^{-\alpha_i}\,F_{y^{(\alpha_i)}}\,\delta y\right]_{x=x_1}. \tag{3.14}$$

Now, from the Fig. (1) in section (3.2), we get;

$$\delta y|_{x=x_1} \cong \delta y_1 - \acute{y}(x_1)\,\delta x_1$$

$$\delta v = F|_{x=x_1}\delta x_1 + \sum_{i=1}^{m}\frac{1}{\Gamma(1-\alpha_i)}\left[x^{-\alpha_i}\,F_{y^{(\alpha_i)}}\right]_{x=x_1}(\delta y_1 - \acute{y}(x_1)\,\delta x_1)$$

Then, the fundamental necessary condition for an extremal $\delta v = 0$ is;

$$\left[F_y - \sum_{i=1}^{m}\frac{x^{-\alpha_i}}{\Gamma(1-\alpha_i)}F_{y^{(\alpha_i)}}\acute{y}(x)\right]_{x=x_1}\delta x_1 + \left[\sum_{i=1}^{m}\frac{x^{-\alpha_i}}{\Gamma(1-\alpha_i)}F_{y^{(\alpha_i)}}\right]_{x=x_1}\delta y_1 = 0$$

-------------------- (3.15)

3.4 <u>VARIATIONAL PROBLEMS WITH MULTI DIFFERENT (FOD) ON MANY DEPENDENT VARIABLES</u>:

***<u>FIRST</u>*:**

We consider the functional dependence only on two functions $y(x)$, and $z(x)$

$$v = \int_{x_0}^{x_1} F\left(x, y, z, y^{(\alpha)}, z^{(\beta)}\right)dx, \tag{3.16}$$

where; $0 < \alpha < 1$ and $0 < \beta < 1$.

By similar calculations to that carried out in section (3.2), then;

$$\Delta v = \int_{x_0}^{x_1+\delta x_1} F\big(x, y+\delta y, z+\delta z, y^{(\alpha)}+\delta y^{(\alpha)}, z^{(\beta)}+\delta z^{(\beta)}\big)dx - \int_{x_0}^{x_1} F\big(x, y, z, y^{(\alpha)}, z^{(\beta)}\big)dx$$

$$= \int_{x_0}^{x_1+\delta x_1} F\big(x, y+\delta y, z+\delta z, y^{(\alpha)}+\delta y^{(\alpha)}, z^{(\beta)}+\delta z^{(\beta)}\big)dx + \int_{x_0}^{x_1} \big[F\big(x, y+\delta y, z+\delta z, y^{(\alpha)}+\delta y^{(\alpha)}, z^{(\beta)}+\delta z^{(\beta)}\big) - F\big(x, y, z, y^{(\alpha)}, z^{(\beta)}\big)\big]dx, \quad (3.17)$$

We apply the mean value theorem to the first term of the right hand side of equation (3.17), and refer to its continuity, and by the Taylor formula we separate the main liner part from the second part of equation (3.17).We then have;

$$\delta v = F|_{x=x_1}\delta x_1 + \int_{x_0}^{x_1} \Big(F_y\delta y + F_z\delta z + F_{y^{(\alpha)}}\delta y^{(\alpha)} + F_{z^{(\beta)}}\delta z^{(\beta)}\Big)\,dx. \quad (3.18)$$

By using the definitions (1.2.2.1) and (1.2.2.2); it can be found that;

$$D^{\alpha}(\delta y) = \delta y^{(\alpha)} = I^{1-\alpha}\frac{d}{dx}\delta y = \delta\acute{y}\left(\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right),$$

and

$$D^{\beta}(\delta z) = \delta z^{(\beta)} = I^{1-\beta}\frac{d}{dx}\delta z = \delta\acute{z}\left(\frac{x^{-\beta}}{\Gamma(1-\beta)}\right).$$

Then;

$$\delta v = F|_{x=x_1}\delta x_1 + \int_{x_0}^{x_1}\left(F_y\delta y + F_z\delta z + F_{y(\alpha)}\left(\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right)\delta\acute{y} + F_{z(\beta)}\left(\frac{x^{-\beta}}{\Gamma(1-\beta)}\right)\delta\acute{z}\right)dx.$$

-------------------- (3.19)

Integrate the last two terms of equation (3.19) by parts to get:

$$\int_{x_0}^{x_1} F_{y(\alpha)}\left(\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right)\delta\acute{y}\,dx = \frac{1}{\Gamma(1-\alpha)}\left(\left(x^{-\alpha}F_{y(\alpha)}\right)\Big|_{x=x_1}\delta y - \int_{x_0}^{x_1}\frac{d}{dx}\left(x^{-\alpha}F_{y(\alpha)}\right)\delta y\,dx\right)$$

$$\int_{x_0}^{x_1} F_{z(\beta)}\left(\frac{x^{-\beta}}{\Gamma(1-\beta)}\right)\delta\acute{z}\,dx = \frac{1}{\Gamma(1-\beta)}\left(\left(x^{-\beta}F_{z(\beta)}\right)\Big|_{x=x_1}\delta z - \int_{x_0}^{x_1}\frac{d}{dx}\left(x^{-\beta}F_{z(\beta)}\right)\delta z\,dx\right)$$

Substitute in equation (3.19) to get:

$$\delta v = F|_{x=x_1}\delta x_1 + \int_{x_0}^{x_1}\left(F_y\delta y + F_z\delta z\right)dx + \frac{1}{\Gamma(1-\alpha)}\left(\left(x^{-\alpha}F_{y(\alpha)}\right)\Big|_{x=x_1}\delta y - \int_{x_0}^{x_1}\frac{d}{dx}\left(x^{-\alpha}F_{y(\alpha)}\right)\delta y\,dx\right) + \frac{1}{\Gamma(1-\beta)}\left(\left(x^{-\beta}F_{z(\beta)}\right)\Big|_{x=x_1}\delta z - \int_{x_0}^{x_1}\frac{d}{dx}\left(x^{-\beta}F_{z(\beta)}\right)\delta z\,dx\right).$$

$$\delta v = F|_{x=x_1}\delta x_1 + \frac{1}{\Gamma(1-\alpha)}\left(x^{-\alpha}F_{y^{(\alpha)}}\right)\Big|_{x=x_1}\delta y + \frac{1}{\Gamma(1-\beta)}\left(x^{-\beta}F_{z^{(\beta)}}\right)\Big|_{x=x_1}\delta z + \int_{x_0}^{x_1}\left(\left(F_y - \left(\frac{1}{\Gamma(1-\alpha)}\right)\frac{d}{dx}\left(x^{-\alpha}F_{y^{(\alpha)}}\right)\right)\delta y + \left(F_z - \left(\frac{1}{\Gamma(1-\beta)}\right)\frac{d}{dx}\left(x^{-\beta}F_{z^{(\beta)}}\right)\right)\delta z\right)dx.$$

$$\delta v = F|_{x=x_1}\delta x_1 + \frac{1}{\Gamma(1-\alpha)}\left(x^{-\alpha}F_{y^{(\alpha)}}\right)\Big|_{x=x_1}\delta y + \frac{1}{\Gamma(1-\beta)}\left(x^{-\beta}F_{z^{(\beta)}}\right)\Big|_{x=x_1}\delta z + \int_{x_0}^{x_1}\left(\left(F_y + \left(\frac{\alpha x^{-(\alpha+1)}}{\Gamma(1-\alpha)}\right)F_{y^{(\alpha)}} - \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\frac{d}{dx}F_{y^{(\alpha)}}\right)\delta y + \left(F_z + \frac{\beta x^{-(\beta+1)}}{\Gamma(1-\beta)}F_{z^{(\beta)}} - \frac{x^{-\beta}}{\Gamma(1-\beta)}\frac{d}{dx}F_{z^{(\beta)}}\right)\delta z\right)dx. \quad (3.20)$$

The values of functional are taken only along fixed $x = x_o$ extremals.

$$F_y + \left(\frac{\alpha x^{-(\alpha+1)}}{\Gamma(1-\alpha)}\right)F_{y^{(\alpha)}} - \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\frac{d}{dx}F_{y^{(\alpha)}} = 0, \quad (3.21a)$$

and

$$F_z + \frac{\beta x^{-(\beta+1)}}{\Gamma(1-\beta)}F_{z^{(\beta)}} - \frac{x^{-\beta}}{\Gamma(1-\beta)}\frac{d}{dx}F_{z^{(\beta)}} = 0. \quad (3.21b)$$

Consequently;

$$\delta v = F|_{x=x_1}\delta x_1 + \frac{1}{\Gamma(1-\alpha)}\left(x^{-\alpha}F_{y^{(\alpha)}}\right)\Big|_{x=x_1}\delta y + \frac{1}{\Gamma(1-\beta)}\left(x^{-\beta}F_{z^{(\beta)}}\right)\Big|_{x=x_1}\delta z. \qquad (3.22)$$

By the same argument as that given in section (3.2) and from Fig. (1), it can be obtained that;

$\delta y|_{x=x_1} \cong \delta y_1 - \acute{y}(x_1)\delta x_1$ and $\delta z|_{x=x_1} \cong \delta z_1 - \acute{z}(x_1)\delta x_1,$

Consequently;

$$\left(F - \frac{x^{-\alpha}}{\Gamma(1-\alpha)}F_{y^{(\alpha)}}\,\acute{y}(x) - \frac{x^{-\beta}}{\Gamma(1-\beta)}F_{z^{(\beta)}}\,\acute{z}(x)\right)\Bigg|_{x=x_1}\delta x_1 + \frac{x^{-\alpha}}{\Gamma(1-\alpha)}F_{y^{(\alpha)}}\Big|_{x=x_1}\delta y_1 + \frac{x^{-\beta}}{\Gamma(1-\beta)}F_{z^{(\beta)}}\Big|_{x=x_1}\delta z_1 = 0.$$

--------------------- (3.23)

If the variations δx_1, δy_1 and δz_1 are independent, then the necessary condition for the extremum is;

$$\left(F - \frac{x^{-\alpha}}{\Gamma(1-\alpha)}F_{y^{(\alpha)}}\,\acute{y}(x) - \frac{x^{-\beta}}{\Gamma(1-\beta)}F_{z^{(\beta)}}\,\acute{z}(x)\right)\Bigg|_{x=x_1} = 0, \qquad (3.24a)$$

$$\frac{x^{-\alpha}}{\Gamma(1-\alpha)}F_{y^{(\alpha)}}\Big|_{x=x_1} = 0, \qquad (3.24b)$$

$$\frac{x^{-\beta}}{\Gamma(1-\beta)}F_{z^{(\beta)}}\Big|_{x=x_1} = 0. \qquad (3.24c)$$

If the boundary point $B(x_1, y_1, z_1)$ can move along a certain curve $y_1 = \varphi(x_1)$ and $z_1 = \eta(x_1)$, then $\delta y_1 = \acute{\varphi}(x_1)\delta x_1$ and $\delta z_1 = \acute{\eta}(x_1)$.

Hence equation (3.23) becomes

$$\left(F - \frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{y^{(\alpha)}} \acute{y}(x) - \frac{x^{-\beta}}{\Gamma(1-\beta)} F_{z^{(\beta)}} \acute{z}(x) \right)\Bigg|_{x=x_1} \delta x_1 +$$

$$\frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{y^{(\alpha)}} \acute{\varphi}(x)|_{x=x_1} \delta x_1 + \frac{x^{-\beta}}{\Gamma(1-\beta)} F_{z^{(\beta)}} \acute{\eta}(x)|_{x=x_1} \delta x_1 = 0,$$

Or

$$\left(F - \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \big(\acute{y}(x) - \acute{\varphi}(x_1)\big) F_{y^{(\alpha)}} - \frac{x^{-\beta}}{\Gamma(1-\beta)} \big(\acute{z}(x) - \acute{\eta}(x_1)\big) F_{z^{(\beta)}} \right)\Bigg|_{x=x_1} \delta x_1 = 0.$$

Since δx_1 is arbitrary, then the necessary condition becomes;

$$\left(F - \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \big(\acute{y}(x) - \acute{\varphi}(x_1)\big) F_{y^{(\alpha)}} - \frac{x^{-\beta}}{\Gamma(1-\beta)} \big(\acute{z}(x) - \acute{\eta}(x_1)\big) F_{z^{(\beta)}} \right)\Bigg|_{x=x_1} = 0. \qquad (3.25)$$

***<u>SECOND</u>*:**

We construct the necessary conditions for the functional v has the form;

$$v\big(y(x), z(x)\big) = \int_{x_o}^{x_1} F\big(x, y, y^{(\alpha_1)}, \dots, y^{(\alpha_m)}, z, z^{(\alpha_1)}, \dots, z^{(\alpha_m)}\big) dx. \qquad (3.26)$$

With one variable boundary end point, then by a similar argument, we find the necessary condition;

$$F - \sum_{i=1}^{m} \left[\frac{x^{-\alpha_i}}{\Gamma(1-\alpha_i)} F_{y^{(\alpha_i)}}\, \acute{y}(x) - \frac{x^{-\beta_i}}{\Gamma(1-\beta_i)}\, F_{z^{(\beta_i)}}\, \acute{z}(x) \right] \Bigg|_{x=x_1} \delta x_1 +$$

$$\sum_{i=1}^{m} \frac{x^{-\alpha_i}}{\Gamma(1-\alpha_i)}\, F_{y^{(\alpha_i)}} \Bigg|_{x=x_1} \delta y_1 + \sum_{i=1}^{m} \frac{x^{-\beta_i}}{\Gamma(1-\beta_i)}\, F_{z^{(\beta_i)}} \Bigg|_{x=x_1} \delta z_1 = 0. \quad (3.27)$$

Now, we consider the general form;

$$v = \int_{x_o}^{x_1} F\big(x, y_1, y_1{}^{(\alpha_{11})}, \dots, y_1{}^{(\alpha_{1m})},$$

$$y_2, y_2{}^{(\alpha_{21})}, \dots, y_2{}^{(\alpha_{2m})}, \dots, y_n, y_n{}^{(\alpha_{n1})}, \dots, y_n{}^{(\alpha_{nn})}\big) dx$$

With one variable boundary end point, then we can find the general necessary conditions;

$$\left[F - \sum_{j=1}^{n} \sum_{i=1}^{m} \frac{x^{-\alpha_i}}{\Gamma(1-\alpha_i)}\, F_{y_j{}^{\alpha_i}}\, \acute{y}_j(x) \right]_{x=x_1} \delta x_1$$

$$+ \left[\sum_{j=1}^{n} \sum_{i=1}^{m} \frac{x^{-\alpha_i}}{\Gamma(1-\alpha_i)}\, F_{y_j{}^{\alpha_i}} \right]_{x=x_1} \delta y_j = 0. \quad (3.28)$$

3.5 EXAMPLES:

To explain the approaches, the example will be considered to find the extremal of the functional.

Example (3.1):

Consider the functional of the form:

$$\int_0^{x_1} \left(x\acute{y} + y\, y^{(1/2)}\right) dx,$$

where x_1 is movable along the given known curve $\varphi(x)$

From equation (3.6), the necessary condition is;

$$F - \frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{y^{(\alpha)}}\, \acute{y}(x)\Big|_{x=x_1} \delta x_1 + \frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{y^{(\alpha)}}\Big|_{x=x_1} \delta y_1 = 0,$$

$$F - \frac{x^{-1/2}}{\Gamma(1-1/2)} F_{y^{(1/2)}}\, \acute{y}(x)\Big|_{x=x_1} \delta x_1 + \frac{x^{-1/2}}{\Gamma(1-1/2)} F_{y^{(1/2)}}\Big|_{x=x_1} \delta y_1 = 0,$$

$$F = x\,\acute{y} + y\, y^{(1/2)},$$

$$F_{y^{(1/2)}} = y$$

From the definitions (1.2.2.1) and (1.2.2.2)

$$y^{(1/2)} = D^{1/2} y = \acute{y}\frac{x^{(1/2)-1}}{\Gamma(1/2)} = \frac{\acute{y}}{\sqrt{\pi x}}$$

then;

$$x\acute{y} + \frac{\acute{y}\, y}{\sqrt{\pi x}} - \left(\frac{x^{-1/2}\, y}{\Gamma(1/2)}\right)\acute{y}(x)\Big|_{x=x_1} \delta x_1 + \frac{x^{-1/2}\, y}{\Gamma(1/2)}\Big|_{x=x_1} \delta y_1 = 0,$$

$$x\acute{y}|_{x=x_1}\delta x_1 + \left.\frac{y}{\sqrt{\pi x}}\right|_{x=x_1}\delta y_1 = 0.$$

<u>Case (1)</u>:

If the variations δx_1 and δy_1 are independent, then the above equation will be:

$x\acute{y}|_{x=x_1} = 0,$

$x_1\,\acute{y} = 0$

Since $x_1 \neq 0$, then $\acute{y} = 0$

$\therefore\ y = \text{constant.}$

and

$$\left.\frac{y}{\sqrt{\pi x}}\right|_{x=x_1} = 0 \qquad \Rightarrow \qquad y = 0, \qquad \text{trivial solution}$$

<u>Case (2)</u>:

If the variation δx_1 and δy_1 are dependent, and the end point (x_1, y_1) can move along a certain curve $y_1 = \varphi(x_1)$.

The necessary condition will be:

$$F - \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\left.\left(\left(\acute{y}(x) - \acute{\varphi}(x)\right) F_{y^{(\alpha)}}\right)\right|_{x=x_1} = 0$$

$$\left(x\acute{y} + \frac{y\,\acute{y}}{\sqrt{\pi x}}\right) - \frac{x^{-(1/2)}}{\Gamma\left(1-(1/2)\right)}\left.\left(\acute{y}(x) - \acute{\varphi}(x)\right)y\right|_{x=x_1} = 0$$

$$x\acute{y} + \frac{y\,\acute{y}}{\sqrt{\pi x}} - \frac{y\,\acute{y}}{\sqrt{\pi x}} + \left.\frac{y}{\sqrt{\pi x}}\acute{\varphi}(x)\right|_{x=x_1} = 0$$

$$x\acute{y} + \frac{y}{\sqrt{\pi x}}\acute{\varphi}(x)\Big|_{x=x_1} = 0$$

$$\frac{y(x_1)}{\sqrt{\pi x}} \ \acute{\varphi}(x_1) = -\, x_1\acute{y}(x_1)$$

$$\frac{\acute{\varphi}(x)}{\acute{y}(x_1)} = -\frac{x_1\sqrt{\pi x}}{y(x_1)}$$

Which is a relation between the directional coefficients $\acute{\varphi}$ and $\acute{y}$ at the end point. It is called transversality condition.

2.1 INTRODUCTION:

In this chapter, we restrict our attention to the use the Riemann-Liouville fractional derivative, to construct the optimality necessary condition for the extremum of the fractional variational problems having one and different multi fractional order derivative (FOD) on one and different multi-dependent variables of one independent variable along fixed boundaries.

2.2 VARIATIONAL PROBLEMS WITH SINGLE (FOD):

Let us examine for extreme a functional of the simplest form:

$$v\big(y(x)\big) = \int_{x_o}^{x_1} F\left(x, y(x), y^{(\alpha)}(x)\right) dx, \tag{2.1}$$

where;

$0 < \alpha < 1$, and with given prescribed boundaries conditions.

It is also assumed that the higher integer and fractional order derivatives of the function $F\big(x, y,\ y^{(\alpha)}\big)$ exists, where α is real.

We already know that a necessary condition for an extremum of a functional is that its first variation vanishes. We take any admissible curve $y = y^*(x)$, neighboring to $y = y(x)$ and we set up one-parameter family of curves;

$y(x, \psi) = y(x) + \psi\big(y^*(x) - y(x)\big) = y(x) + \psi\delta y.$

When $\psi = 0$, we have $y = y(x)$, and when $\psi = 1$ we have $y = y^*(x)$.

The variation $\delta y = y^*(x) - y(x)$ is a function of the variable ψ, this function can be differentiated once or more and we have:

$D^{\alpha}(\delta y) = (\delta y)^{(\alpha)} = (y^{*})^{(\alpha)} - y^{(\alpha)} = \delta y^{(\alpha)}$,

$D^{n\alpha}(\delta y) = (\delta y)^{(n\alpha)} = (y^{*})^{(n\alpha)} - y^{(n\alpha)} = \delta y^{(n\alpha)}$.

Take on along the curve of the family $y = y(x, \psi)$ only, then we have a function of the variable ψ :

$$v\big(y(x,\psi)\big) = \varphi(\psi).$$

It is well known, the necessary condition that the function $\varphi(\psi)$ has an extremum for $\psi = 0$ its derivative should vanish.

$$\delta v\big(y(x,\psi)\big) = \frac{\partial}{\partial \psi}\, v(y(x) + \delta y)\Big|_{\psi=0}$$

$\delta v = \acute{\varphi}(\psi) = \acute{\varphi}(0) = 0.$

Since;

$$\varphi(\psi) = \int_{x_0}^{x_1} F\left(x, y(x,\psi), y^{(\alpha)}(x,\psi)\right) dx, \tag{2.2}$$

we have;

$$\acute{\varphi}(\psi) = \int_{x_0}^{x_1} \left(F_y \frac{\partial}{\partial \psi} y(x,\psi) + F_{y^{(\alpha)}} \frac{\partial}{\partial \psi} y^{(\alpha)}(x,\psi)\right) dx, \tag{2.3}$$

where;

$$F_y = \frac{\partial}{\partial y} F\left(x, y(x,\psi), y^{(\alpha)}(x,\psi)\right), \tag{2.4}$$

$$F_{y^{(\alpha)}} = \frac{\partial}{\partial y^{(\alpha)}} F\left(x, y(x,\psi), y^{(\alpha)}(x,\psi)\right). \tag{2.5}$$

Because of the relations:

$$\frac{\partial}{\partial \psi} y(x,\psi) = \frac{\partial}{\partial \psi}(y(x) + \psi \delta y) = \delta y, \tag{2.6}$$

and

$$\frac{\partial}{\partial\psi}y^{(\alpha)}(x,\psi)=\frac{\partial}{\partial\psi}\left(y^{(\alpha)}(x)+\psi\delta y^{(\alpha)}\right)=\delta y^{(\alpha)}, \tag{2.7}$$

it follows that:

$$\acute{\varphi}(\psi)=\int_{x_0}^{x_1}\left(F_y\left(x,y(x,\psi),y^{(\alpha)}(x,\psi)\right)\delta y\right.$$
$$\left.+F_{y^{(\alpha)}}\left(x,y(x,\psi),y^{(\alpha)}(x,\psi)\right)\delta y^{(\alpha)}\right)dx, \tag{2.8}$$

$$\acute{\varphi}(0)=\int_{x_0}^{x_1}\left(F_y\left(x,y(x),y^{(\alpha)}(x)\right)\delta y+F_{y^{(\alpha)}}\left(x,y(x),y^{(\alpha)}(x)\right)\delta y^{(\alpha)}\right)dx.$$

----------------- (2.9)

As we have already remarked, $\acute{\varphi}(0)$ is called a variation of the functional and it is designated by δv. The necessary condition for a functional v to have an extremum is that its variation should vanish $\delta v=0$.

$$\delta v=\int_{x_0}^{x_1}\left(F_y\,\delta y+F_{y^{(\alpha)}}\,\delta y^{(\alpha)}\right)dx=0, \tag{2.10}$$

$$\delta v=\int_{x_0}^{x_1}\left(F_y\,\delta y\right)dx+\int_{x_0}^{x_1}\left(F_{y^{(\alpha)}}\,\delta y^{(\alpha)}\right)dx=0. \tag{2.11}$$

The second integral of (2.11):

$$\int_{x_0}^{x_1}\left(F_{y^{(\alpha)}}\,\delta y^{(\alpha)}\right)dx. \tag{2.12}$$

By using the definitions (1.2.2.1) and (1.2.2.2); it can be found that;

$$D^{\alpha}(\delta y)=\delta y^{(\alpha)}=\mathrm{I}^{1-\alpha}\frac{d}{dx}\delta y=\delta\acute{y}\left(\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right). \tag{2.13}$$

Substitute equation (2.13) in (2.12) to get:

$$\int_{x_0}^{x_1} \left(F_{y^{(\alpha)}}\, \delta y^{(\alpha)}\right) dx = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x_1} \left(x^{-\alpha}\, F_{y^{(\alpha)}}\, \delta \acute{y}\right) dx.$$

By using the integrating by parts:

$$\int_{x_0}^{x_1} \left(F_{y^{\alpha}}\, \delta y^{(\alpha)}\right) dx = \frac{1}{\Gamma(1-\alpha)} \left(x^{-\alpha}\, F_{y^{(\alpha)}}\, \delta y \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \left(d\left(x^{-\alpha}\, F_{y^{(\alpha)}}\right) \delta y \right) dx. \right.$$

Since:

$\delta y|_{x=x_0} = 0$ and $\delta y|_{x=x_1} = 0$

$$\int_{x_0}^{x_1} \left(F_{y^{(\alpha)}}\, \delta y^{(\alpha)}\right) dx = \frac{-1}{\Gamma(1-\alpha)} \int_{x_0}^{x_1} \left(-\alpha\, x^{-\alpha-1}\, F_{y^{(\alpha)}} + x^{-\alpha} \frac{d}{dx} F_{y^{(\alpha)}}\right) \delta y\, dx.$$

----------------- (2.14)

Substitute equation (2.14) in equation (2.11) to get:

$$\delta v = \int_{x_0}^{x_1} \left(F_y + \frac{\alpha}{\Gamma(1-\alpha)} x^{-(\alpha+1)} F_{y^{(\alpha)}} - \frac{1}{\Gamma(1-\alpha)} x^{-\alpha} \frac{d}{dx} F_{y^{(\alpha)}}\right) \delta y\, dx = 0.$$

Since δy is an arbitrary function that subject to some general condition, therefore; by using the fundamental lemma, it can be found that:

$$F_y + \frac{\alpha}{\Gamma(1-\alpha)} x^{-(\alpha+1)} F_{y^{(\alpha)}} - \frac{1}{\Gamma(1-\alpha)} x^{-\alpha} \frac{d}{dx} F_{y^{(\alpha)}} = 0, \qquad (2.15)$$

which is the necessary condition for extremum of the functional (2.1).

2.3 VARIATIONAL PROBLEMS WITH MULTI DIFFERENT (FOD) ON ONE DEPENDENT VARIABLE:

Functions involving derivatives of m-different fractional orders (α_i);

$$v(x, y(x)) = \int_{x_o}^{x_1} F\left(x, y(x), y^{(\alpha_1)}(x), y^{(\alpha_2)}(x), \dots, y^{(\alpha_m)}(x)\right) dx. \qquad (2.16)$$

where; $0 < \alpha_i < 1$, $i = 1,2, \dots, m$, and with given prescribed boundaries conditions.

Consider a one-parameter family of functions:

$y(x, \psi) = y(x) + \psi(y^*(x) - y(x))$,

or

$y(x, \psi) = y(x) + \psi\, \delta y$.

For $\psi = 0$, $y(x, \psi) = y(x)$, and for $\psi = 1$, $y(x, \psi) = y^*(x)$

If we consider the values taken by the functional $v(y(x))$ along the curves of the family, $y = y(x, \psi)$ only, then this functional turns into an ordinary function of the parameter ψ, that has an extremum for $\psi = 0$. Consequently,

$$\frac{d}{d\psi} v(y(x, \psi))\bigg|_{\psi=0} = 0.$$

The necessary condition for a functional v to have an extremum is that its variation should vanish $\delta v = 0$.

$$\delta v = \int_{x_o}^{x_1} \left(F_y\, \delta y + F_{y^{(\alpha_1)}} \delta y^{(\alpha_1)} + F_{y^{(\alpha_2)}} \delta y^{(\alpha_2)} + \dots + F_{y^{(\alpha_m)}} \delta y^{(\alpha_m)}\right) dx.$$

----------------- (2.17)

By using the definitions (1.2.2.1) and (1.2.2.2) , it can be found that;

$$D^{\alpha_i}\,\delta y = \delta y^{(\alpha_i)} = \mathrm{I}^{1-\alpha_i}\,\frac{d}{dx}\delta y = \mathrm{I}^{1-\alpha_i}\,\delta \acute{y} = \delta \acute{y}\frac{x^{-\alpha_i}}{\Gamma(1-\alpha_i)}. \qquad (2.18)$$

Substitute equation (2.18) in equation (2.17) to get:

$$\delta v = \int_{x_o}^{x_1}\left(F_y\,\delta y + F_{y^{(\alpha_1)}}\left(\delta \acute{y}\frac{x^{-\alpha_1}}{\Gamma(1-\alpha_1)}\right) + F_{y^{(\alpha_2)}}\left(\delta \acute{y}\frac{x^{-\alpha_2}}{\Gamma(1-\alpha_2)}\right) + \cdots + F_{y^{(\alpha_m)}}\left(\delta \acute{y}\frac{x^{-\alpha_m}}{\Gamma(1-\alpha_m)}\right)\right)dx,$$

$$\therefore\ \delta v = \int_{x_o}^{x_1}\left(F_y\,\delta y + \left(\frac{F_{y^{(\alpha_1)}}\,x^{-\alpha_1}}{\Gamma(1-\alpha_1)} + \frac{F_{y^{(\alpha_2)}}\,x^{-\alpha_2}}{\Gamma(1-\alpha_2)} + \cdots + \frac{F_{y^{(\alpha_m)}}\,x^{-\alpha_m}}{\Gamma(1-\alpha_m)}\right)\delta \acute{y}\right)dx.$$

------------------ (2.19)

$$\delta v = \int_{x_o}^{x_1}\left(F_y\delta y + \left(\sum_{i=1}^{m}\frac{x^{-\alpha_i}}{\Gamma(1-\alpha_i)}F_{y^{(\alpha_i)}}\right)\delta \acute{y}\right)dx = 0$$

$$= \int_{x_o}^{x_1}F_y\delta y\,dx + \sum_{i=1}^{m}\int_{x_o}^{x_1}\left(\frac{x^{-\alpha_i}}{\Gamma(1-\alpha_i)}\,F_{y^{(\alpha_i)}}\right)\delta \acute{y}\,dx = 0. \qquad (2.20)$$

Integrating by part the second term of equation (2.20) to get;

$$\sum_{i=1}^{m} \frac{1}{\Gamma(1-\alpha_i)} \int_{x_o}^{x_1} x^{-\alpha_i} F_{y(\alpha_i)} \delta \acute{y} \, dx =$$

$$\sum_{i=1}^{m} \frac{1}{\Gamma(1-\alpha_i)} \left(\left| x^{-\alpha_i} F_{y(\alpha_i)} \delta y \right|_{x_o}^{x_1} - \int_{x_o}^{x_1} \left(d\left(x^{-\alpha_i} F_{y(\alpha_i)} \right) \delta y \right) dx \right). \quad (2.21)$$

Since $\delta y|_{x=x_o} = 0$ and $\delta y|_{x=x_1} = 0$, then equation (2.21) will be:

$$\sum_{i=1}^{m} \frac{1}{\Gamma(1-\alpha_i)} \int_{x_o}^{x_1} x^{-\alpha_i} F_{y(\alpha_i)} \delta \acute{y} \, dx =$$

$$\sum_{i=1}^{m} \frac{-1}{\Gamma(1-\alpha_i)} \int_{x_o}^{x_1} \left(-\alpha_i \, x^{-\alpha_i - 1} F_{y^{\alpha_i}} + x^{-\alpha_i} \frac{d}{dx} F_{y(\alpha_i)} \right) \delta y \, dx.$$

$$\therefore \delta v = \int_{x_o}^{x_1} F_y \, \delta y \, dx$$

$$+ \sum_{i=1}^{m} \frac{-1}{\Gamma(1-\alpha_i)} \int_{x_o}^{x_1} \left(-\alpha_i \, x^{-\alpha_i - 1} F_{y(\alpha_i)} + x^{-\alpha_i} \frac{d}{dx} F_{y(\alpha_i)} \right) \delta y \, dx$$

$$= \int_{x_o}^{x_1} \left(F_y + \sum_{i=1}^{m} \frac{-1}{\Gamma(1-\alpha_i)} \left(-\alpha_i \, x^{-\alpha_i - 1} F_{y(\alpha_i)} + x^{-\alpha_i} \frac{d}{dx} F_{y(\alpha_i)} \right) \right) \delta y \, dx.$$

------------------ (2.22)

Since, δy is an arbitrary function that subjected to conditions that satisfied the fundamental lemma, it can be found that the necessary condition is:

$$F_y + \sum_{i=1}^{m} \frac{(-1)}{\Gamma(1-\alpha_i)} \left(-\alpha_i \, x^{-\alpha_i - 1} F_{y(\alpha_i)} + x^{-\alpha_i} \frac{d}{dx} \left(F_{y(\alpha_i)} \right) \right) = 0. \quad (2.23)$$

2.4 VARIATIONAL PROBLEMS WITH MULTI DIFFERENT (FOD) ON MANY DEPENDENT VARIABLES:

***FIRST*:**

We consider the functional dependence only on two functions $y(x)$, and $z(x)$.

$$v\big(y(x), z(x)\big) = \int_{x_0}^{x_1} F\big(x, y, z, y^{(\alpha)}, z^{(\beta)}\big)dx, \tag{2.24}$$

where; $0 < \alpha < 1$ and $0 < \beta < 1$, and with given prescribed boundaries conditions.

Varying only $y(x)$ and having $z(x)$ being fixed. Then similarly the necessary conditions for a functional *v* to have an extremum is that its variation should vanish $\delta v = 0$.

$$\delta v = \int_{x_0}^{x_1} \left(F_y\ \delta y + F_{y^{(\alpha)}}\ \delta y^{(\alpha)} + F_z\ \delta z + F_{z^{(\beta)}}\ \delta z^{(\beta)}\right)dx. \tag{2.25}$$

By using the definitions (1.2.2.1) and (1.2.2.2), it can be found that;

$$D^\alpha\ \delta y = \delta y^{(\alpha)} = \mathrm{I}^{1-\alpha} \frac{d}{dx} \delta y = \mathrm{I}^{1-\alpha}\ \delta \acute{y} = \delta \acute{y} \frac{x^{-\alpha}}{\Gamma(1-\alpha)},$$

and

$$D^\beta\ \delta z = \delta z^{(\beta)} = \mathrm{I}^{1-\beta} \frac{d}{dx} \delta z = \mathrm{I}^{1-\beta}\ \delta \acute{z} = \delta \acute{z} \frac{x^{-\beta}}{\Gamma(1-\beta)}.$$

Then;

$$\delta v = \int_{x_0}^{x_1} \left(F_y\ \delta y + F_{y^{(\alpha)}}\ \delta \acute{y} \frac{x^{-\alpha}}{\Gamma(1-\alpha)} + F_z\ \delta z + F_{z^{(\beta)}}\ \delta \acute{z} \frac{x^{-\beta}}{\Gamma(1-\beta)}\right) dx.$$

By using the same argument in the above section, we get system of two necessary conditions.

$$F_y + \frac{\alpha x^{-(\alpha+1)}}{\Gamma(1-\alpha)} F_{y^{(\alpha)}} - \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{dx} F_{y^{(\alpha)}} = 0, \tag{2.26a}$$

and

$$F_z + \frac{\beta x^{-(\beta+1)}}{\Gamma(1-\beta)} F_{z^{(\beta)}} - \frac{x^{-\beta}}{\Gamma(1-\beta)} \frac{d}{dx} F_{z^{(\beta)}} = 0. \tag{2.26b}$$

***SECOND*:**

We construct the necessary conditions for the functional *v* has the form;

$$v\big(y(x), z(x)\big) = \int_{x_o}^{x_1} F\big(x, y, y^{(\alpha_1)}, \dots, y^{(\alpha_m)}, z, z^{(\alpha_1)}, \dots, z^{(\alpha_m)}\big) dx. \tag{2.27}$$

where; $0 < \alpha_i < 1$, $i = 1,2,\dots,m$, and with given prescribed boundaries conditions.

By varying only $y(x)$ and having kept $z(x)$ fixed, and perform the same approach as in section (2.3), we find that any pair of functions $y(x)$, $z(x)$ that gives an extremum of this functional.

$$F_y + \sum_{i=1}^{m} \frac{(-1)}{\Gamma(1-\alpha_i)} \left(-\alpha_i \, x^{-\alpha_i - 1} \, F_{y^{(\alpha_i)}} + x^{-\alpha_i} \frac{d}{dx} F_{y^{(\alpha_i)}} \right) = 0, \tag{2.28}$$

and by varying *z*(*x*), having kept *y*(*x*) fixed, we obtain;

$$F_z + \sum_{i=1}^{m} \frac{(-1)}{\Gamma(1-\beta_i)} \left(-\beta_i \, x^{-\beta_i - 1} \, F_{z^{(\beta_i)}} + x^{-\beta_i} \frac{d}{dx} F_{z^{(\beta_i)}} \right) = 0. \tag{2.29}$$

Now, we consider the general form;

$$v = \int_{x_o}^{x_1} F\big(x, y_1, y_1{}^{(\alpha_{11})}, \dots, y_1{}^{(\alpha_{1m})},$$
$$y_2, y_2{}^{(\alpha_{21})}, \dots, y_2{}^{(\alpha_{2m})}, \dots, y_n, y_n{}^{(\alpha_{n1})}, \dots, y_n{}^{(\alpha_{nn})}\big)dx$$

with given prescribed boundaries conditions.

The same line of argument applies in the discussion of extrema of similar functional depending on an arbitrary number *f* functions; then varying any function $y_j(x)$ and keeping the remaining ones fixed, we find that the necessary conditions for an extremum as;

$$F_{y_j} + \sum_{i=1}^{m} \frac{(-1)}{\Gamma(1-\alpha_i)} \left(-\alpha_i x^{-\alpha_i - 1} F_{y_j^{(\alpha_i)}} + x^{-\alpha_i} \frac{d}{dx} F_{y_j^{(\alpha_i)}} \right) = 0, \qquad (2.30)$$

for all j=1,…,n.

2.5 EXAMPLES:

To explain the approaches, considering the example, to find the extremal of the functional.

Example (2.1):

Consider the functional of the form;

$$v = \int_0^1 (y\acute{y} + x^{\alpha+1} D^{\alpha} y) dx$$

From equation (2.15), the necessary condition is;

$$F_y + \frac{\alpha}{\Gamma(1-\alpha)} x^{-(\alpha+1)} F_{y^{(\alpha)}} - \frac{1}{\Gamma(1-\alpha)} x^{-\alpha} \frac{d}{dx} F_{y^{(\alpha)}} = 0.$$

$$F_y = \acute{y}\,, \quad F_{y^{(\alpha)}} = x^{\alpha+1}$$

$$\acute{y} + \frac{\alpha}{\Gamma(1-\alpha)} x^{-(\alpha+1)} x^{\alpha+1} - \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{dx} x^{\alpha+1} = 0$$

$$\acute{y} + \frac{\alpha}{\Gamma(1-\alpha)} - \frac{x^{-\alpha}}{\Gamma(1-\alpha)} (\alpha+1) x^{\alpha} = 0$$

$$\acute{y} = \frac{-\alpha}{\Gamma(1-\alpha)} + \frac{(\alpha+1)}{\Gamma(1-\alpha)}$$

$$\acute{y} = \frac{-\alpha+\alpha+1}{\Gamma(1-\alpha)}$$

$$\acute{y} = \frac{1}{\Gamma(1-\alpha)}$$

Which is a first order nonhomogenous ordinary differential equation.

$$y = \frac{1}{\Gamma(1-\alpha)} x + k$$

Which is a straight line and k is constant, its value depend on the given prescribed boundary condition.

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Dedication

To My Family

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

إِنَّمَا يَخْشَى اللَّهَ مِنْ عِبَادِهِ الْعُلَمَاءُ

صدق الله العظيم

(فاطر:٢٨)

DISCUSSION AND FUTURE WORKS

***DISCUSSION*:**

In this work, the optimality necessary conditions are constructed for fractional variational problems with multi-dependent variables along fixed and movable boundaries, in which from of the necessary conditions are constructed depending on the structures from of the considered example. Where first ordinary differential equation has been obtained as in the example (2.1) in chapter two and second ordinary differential equation has been obtained as in the example (3.1) in chapter three.

***FUTURE WORK*:**

We may look to construct;

1. The optimality necessary conditions for fractional variational problems with multi-independent variables.

2. The optimality sufficient conditions, for fractional variational problems with one and multi-independent variables.

3. The optimality necessary and sufficient conditions for fractional variational problems with additional constraints (may have integer or fractional order derivatives).

Examining Committee's Certification

We certify that we read this thesis entitled ***"Optimality Necessary Condition For Fractional order Variational Problems"*** and as examining committee examined the student, ***Nadia Jasim Ismair*** in its contents and in what it connected with, and that is in our opinion it meet the standard of thesis for the degree of Master of Science in Mathematics.

(Chairman)	**(Member)**
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Name: Dr.Tariq Salih Abdul-Razaq	**Name: Dr.Ahlam J.Khaleel**
Data: / /2007	**Data: / /2007**
(Member)	**(Member and Supervisor)**
Signature:	**Signature:**
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Dean of College of Science of Al-Nahrain University

Date: / /2007

Fractional calculus is a branch of mathematics which deals with the investigation and applications of integrals and derivatives of arbitrary order. fractional calculus may be considered as old and yet a novel topic, actually, it is an old topic since starting from some spectrum of Leibniz (1695–1697) and Euler (1730) who said "When n is an integer, the ratio d^nP, P is a function of x, to dx^n can be made if n is fraction?", it has been developed up to nowadays. In fact, the idea of generalizing the notion of derivative to non – integer order, in particular to the order of 1/2 (which is called semi – integral or semi – derivative) is found in the correspondence of Leibniz and Bernoulli, L'Hopital and Wallis. Euler took the first step by observing that the result of the derivative evaluation of the power function has a meaning for non integer order thanks to his Gamma function [1].

There are wide areas of applications for the fractional calculus, such as viscoplasticity [2] and viscoelastic constitutive equations [3] which are good applications. That is the constitutive equations governing these phenomenon involve differential equations fractional order. It is also applied in potential field data [4] where the use of fractional gradients provides a much greater flexibility which is generating enhanced analytic signal data. Also any application which uses the computation of velocity and acceleration is an application of fractional differ integration [5]. In physics there are wide applications such as the pressure behavior of transport of different Medias [6] and the diffusion equations [7] and [8]. In engineering, the fractional calculus is applied in Tensili–Flexral strength of disorder materials and signal processing [9].

In addition, of course, to the theories of differential, integral, and integro–differential equations, of mathematical physics as well as their extensions and generalization in one and more variables, some of the areas of present day applications of fractional calculus include Fluid Flow, Porous

Structures, Diffusive Transport Akin to Diffusion, Electrical Networks, Probability and Statistics, Control Theory of Dynamical System, Viscoelasticity, Electrochemistry of Corrosion, Chemical Physics, Optics and Signal Processing, and so on [10].

The calculus of variations essentially is an extension of minimizing or maximizing a function of one variable to problems involving minimizing or maximizing a functional. Typically, a functional is an integral whose integrand involves an unknown function and its derivatives; the objective is to find the (not necessarily unique) function that makes the integral stationary within a given class of functions [11].

The study of problems of the calculus of variations with fractional derivatives is a rather recent subject, the main result being the fractional necessary optimality condition of Euler – Lagrange to be obtained [11].

Riewe [12], [13] obtained a version of the Euler – Lagrange equations for problem of the Calculus of Variations with fractional derivatives, that combines the conservative and non – conservative cases. More recently, Agrawal [14] gave a formulation for variational problems with right and left fractional derivatives in the Riemann – Liouville sense.

This work, concerns with fractional varational problems, in which the optimality necessary conditions are obtained, for problems having one and different multi-fractional order derivatives (FOD), on one and multi-dependent variables of one independent variable, along fixed and moving boundaries, with examples . This work consists of three chapters.

Chapter one presents the basic concepts of fractional calculus such as Gamma function, Beta function, the Riemann-Liouville definition and some properties and lemmas. It also presents the basic concepts of calculus of variation such as fundamental lemma. Some Examples are given in this chapter.

Chapter two presents the optimality necessary conditions for fractional variational problems of calculus of variation having one and different multi fractional order derivatives (FOD) on one and different multi-dependent variable of one independent variable along fixed boundaries. Solved examples had been presented for each case.

Chapter three presents the optimality necessary conditions for fractional variational problems of calculus of variation having one and different multi fractional order derivatives (FOD) on one and different multi-dependent variable of one independent variable along movable boundaries. Solved examples had been presented for each case.

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SUPERVISOR CERTIFICATION

I certify that this thesis was prepared under my supervision at the department of mathematics and computer applications, ***College of Science, Al-Nahrain University*** as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

Signature:

Name: Assist. Prof. Dr. Alaudin N. Ahmed

Date: /10/2007

In view of the available recommendations; I forward this thesis for debate by examination committee.

Signature:

Name: Assist. Prof. Dr. Akram M. Al-Abood

Head of the Department Mathematics and Computer Applications

Date: /10/2007

الخلاصة

تعد المشتقة ذات الرتب الكسرية فكرة جيدة من الناحيتين النظرية والتطبيقية. وتعد دراسة حسبان مسائل التغاير المتضمنة مشتقات ذات رتب كسرية (fractional variational problems) من المواضيع الحديثة.

في هذا العمل، تم عرض بعض الخواص والتعاريف الأساسية للمشتقات والتكاملات ذات الرتب الكسرية لريمان لوفيل (Riemann Liouville). كما تم استنباط الشروط الضرورية لأمثلية أنواع مختلفة من المسائل التغاير ذات الرتب الكسرية التي تتضمن مشتقات ذات رتب كسرية ومتغيرات معتمدة (dependent variables) لمتغير مستقل (independent variable) واحد فقط على طول حدود ثابتة ومتحركة. كما تم عرض بعض الأمثلة توضح تحقيق أمثلية الشروط الضرورية.

Ministry of Higher Educationand
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Optimality Necessary Conditions For Fractional order Variational Problems

A Thesis

Submitted to the Department of Mathematics, College of Science, Al-Nahrain University, as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

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وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم

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بإشراف
أ. م. د. علاء الدين نوري احمد

تشرين الثاني ٢٠٠٧ ذي القعدة ١٤٢٨

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CHAPTER TWO

OPTIMALITY NECESSARY CONDITIONS OF FRACTIONAL VARIATINAL PROBLEMS ALONG FIXED BONDARIES

CHAPTER THREE

OPTIMALITY NECESSARY CONDITIONS OF FRACTIONAL VARIATINAL PROBLEMS ALONG MOVABLE BONDARIES

DISCUSSION

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