Abstract

This work is oriented towards two objectives:

The first objective is to classify and study the generalized one-dimensional integral equations that contain n one-dimensional integral operators. This study includes the existence of a unique solution for special types of these integral equations and their solutions by using some quadrature methods, namely the trapezoidal rule, the modified trapezoidal rule and Simpson's rule.

The second objective is to classify and study the generalized multi-dimensional integral equations that contain n multi-dimensional integral operators. This study includes their solutions by using some quadrature methods, namely the trapezoidal rule, the modified trapezoidal rule and Simpson's rule.

Acknowledgments

My deepest thanks to Allah, for his generous by providing me the strength to accomplish this work.

It is pleasure to express my deep appreciation and indebtedness to my supervisor **Dr. Ahlam J. Khaleel**, for suggesting the present subject matter. I adduce to her my sincere gratitude and admiration for her guidance and interest throughout the work and for her encouragement, efforts and invaluable help during my study.

My grateful thanks are due to Al-Nahrain University, College of Science.

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Yousur S. Ali

2007

Appendix

Program (2.1):

$$u(x) = 1 + \frac{1}{\pi}\int_{-1}^{1} \frac{1}{1+(x-y)^2} u(y)\,dy, \quad -1 \le x \le 1$$

$k(x, y) := \frac{1}{\pi} \cdot \frac{1}{1 + (x - y)^2}$

$f(x) := 1$

$n := 8$

$L := -1$

$R := 1$

$h := \frac{(R - L)}{n}$

$i := 0..\,8$

$x_i := L + i \cdot h$

$i := 1..\,7$

$j := 1..\,7$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$$m = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.07958 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.03105 & -0.02449 \\ 0 & -0.0749 & -0.07958 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.03105 \\ 0 & -0.06366 & -0.0749 & -0.07958 & -0.0749 & -0.06366 & -0.05093 & -0.03979 \\ 0 & -0.05093 & -0.06366 & -0.0749 & -0.07958 & -0.0749 & -0.06366 & -0.05093 \\ 0 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & -0.07958 & -0.0749 & -0.06366 \\ 0 & -0.03105 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & -0.07958 & -0.0749 \\ 0 & -0.02449 & -0.03105 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & -0.07958 \end{pmatrix}$$

$i := 1..\,7$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{0,0} := 1 - \frac{h}{2} \cdot k(x_0, x_0)$

$m_{8,8} := 1 - \frac{h}{2} \cdot k(x_8, x_8)$

$$m = \begin{pmatrix} 0.96021 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.03105 & -0.02449 & 0 \\ 0 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.03105 & 0 \\ 0 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & 0 \\ 0 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & 0 \\ 0 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & 0 \\ 0 & -0.03105 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & 0 \\ 0 & -0.02449 & -0.03105 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.96021 \end{pmatrix}$$

$i := 1..7$

$m_{0,i} := (-h) \cdot k(x_0, x_i)$

$i := 1..7$

$m_{8,i} := -h \cdot k(x_8, x_i)$

$i := 1..7$

$m_{i,0} := \left(\frac{-h}{2}\right) \cdot k(x_i, x_0)$

$i := 1..7$

$m_{i,8} := \frac{-h}{2} \cdot k(x_i, x_8)$

$m_{0,8} := \frac{-h}{2} \cdot k(x_0, x_8)$

$m_{8,0} := \frac{-h}{2} \cdot k(x_8, x_0)$

$$m = \begin{pmatrix} 0.96021 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.03105 & -0.02449 & -0.01959 & -7.95775 \times 10^{-3} \\ -0.03745 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.03105 & -0.02449 & -9.79415 \times 10^{-3} \\ -0.03183 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.03105 & -0.01224 \\ -0.02546 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.01553 \\ -0.01989 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & -0.01989 \\ -0.01553 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & -0.02546 \\ -0.01224 & -0.03105 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & -0.03183 \\ -9.79415 \times 10^{-3} & -0.02449 & -0.03105 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & -0.03745 \\ -7.95775 \times 10^{-3} & -0.01959 & -0.02449 & -0.03105 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & 0.96021 \end{pmatrix}$$

$i := 0..8$

$f_i := 1$

$u := m^{-1} \cdot f$

$$u = \begin{pmatrix} 1.63639 \\ 1.74695 \\ 1.83641 \\ 1.89332 \\ 1.91268 \\ 1.89332 \\ 1.83641 \\ 1.74695 \\ 1.63639 \end{pmatrix}$$

$n := 16$

$L := -1$

$R := 1$

$h := \frac{(R - L)}{n}$

$i := 0..16$

$x_i := L + i \cdot h$

$i := 1..15$

$j := 1..15$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 1..15$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{1,1} = 0.96021$

$m_{0,0} := 1 - \frac{h}{2} \cdot k(x_0, x_0)$

$m_{16,16} := 1 - \frac{h}{2} \cdot k(x_{16}, x_{16})$

$i := 1..15$

$m_{0,i} := (-h) \cdot k(x_0, x_i)$

$i := 1..15$

$m_{16,i} := -h \cdot k(x_{16}, x_i)$

$i := 1..15$

$m_{i,0} := \left(\frac{-h}{2}\right) \cdot k(x_i, x_0)$

$i := 1..15$

$m_{i,16} := \frac{-h}{2} \cdot k(x_i, x_{16})$

$m_{0,16} := \frac{-h}{2} \cdot k(x_0, x_{16})$

$m_{16,0} := \frac{-h}{2} \cdot k(x_{16}, x_0)$

$i := 0..16$

$f_i := 1$

$u := m^{-1} \cdot f$

u=

	0
0	1.63887
1	1.69648
2	1.75070
3	1.79940
4	1.84089
5	1.87401
6	1.89804
7	1.91258
8	1.91744
9	1.91258
10	1.89804
11	1.87401
12	1.84089
13	1.79940
14	1.75070
15	1.69648
16	1.63887

$k(x,y) := \frac{1}{\pi}\cdot\frac{1}{1+(x-y)^2}$

$f(x) := 1$

$n := 32$

$L := -1$

$R := 1$

$h := \frac{(R-L)}{n}$

$i := 0..32$

$x_i := L + i\cdot h$

$i := 1..31$

$j := 1..31$

$m_{i,j} := -h\cdot k(x_i, x_j)$

$i := 1..31$

$m_{i,i} := 1 - h\cdot k(x_i, x_i)$

$m_{1,1} = 0.98011$

$m_{0,0} := 1 - \frac{h}{2}\cdot k(x_0, x_0)$

$m_{32,32} := 1 - \frac{h}{2}\cdot k(x_{32}, x_{32})$

$i := 1..31$

$m_{0,i} := (-h)\cdot k(x_0, x_i)$

$i := 1..31$

$m_{32,i} := -h\cdot k(x_{32}, x_i)$

$i := 1..31$

$m_{i,0} := \left(\frac{-h}{2}\right)\cdot k(x_i, x_0)$

$i := 1..31$

$m_{i,32} := \frac{-h}{2}\cdot k(x_i, x_{32})$

$m_{0,32} := \frac{-h}{2}\cdot k(x_0, x_{32})$

$m_{32,0} := \frac{-h}{2}\cdot k(x_{32}, x_0)$

$i := 0..32$

$f_i := 1$

$u := m^{-1}\cdot f$

u=

	0
0	1.63949
1	1.66866
2	1.69727
3	1.72503
4	1.75164

5	1.77686
6	1.80045
7	1.82222
8	1.84201
9	1.85970
10	1.87517
11	1.88837
12	1.89922
13	1.90770
14	1.91377
15	1.91742
16	1.91863
17	1.91742
18	1.91377
19	1.90770
20	1.89922
21	1.88837
22	1.87517
23	1.85970
24	1.84201
25	1.82222
26	1.80045
27	1.77686
28	1.75164
29	1.72503
30	1.69727
31	1.66866
32	1.63949

$$k(x,y) := \frac{1}{\pi}\cdot\frac{1}{1+(x-y)^2}$$

$f(x) := 1$

$n := 64$

$L := -1$

$R := 1$

$$h := \frac{(R-L)}{n}$$

$i := 0..64$

$x_i := L + i\cdot h$

$i := 1..63$

$j := 1..63$

$m_{i,j} := -h\cdot k\left(x_i, x_j\right)$

$i := 1..63$

$m_{i,i} := 1 - h\cdot k\left(x_i, x_i\right)$

$$m_{0,0} := 1 - \frac{h}{2}\cdot k\left(x_0, x_0\right)$$

$$m_{64,64} := 1 - \frac{h}{2}\cdot k\left(x_{64}, x_{64}\right)$$

$i := 1..63$

$m_{0,i} := (-h)\cdot k(x_0, x_i)$

$i := 1..63$

$m_{64,i} := -h\cdot k(x_{64}, x_i)$

$i := 1..63$

$m_{i,0} := \left(\frac{-h}{2}\right)\cdot k(x_i, x_0)$

$i := 1..63$

$m_{i,64} := \frac{-h}{2}\cdot k(x_i, x_{64})$

$m_{0,64} := \frac{-h}{2}\cdot k(x_0, x_{64})$

$m_{64,0} := \frac{-h}{2}\cdot k(x_{64}, x_0)$

$i := 0..64$

$f_i := 1$

$u := m^{-1}\cdot f$

u=

	0
0	1.63964
1	1.65429
2	1.66884
3	1.68325
4	1.69747
5	1.71148
6	1.72525
7	1.73872
8	1.75188
9	1.76468
10	1.77711
11	1.78912
12	1.80071
13	1.81184
14	1.82249
15	1.83265
16	1.84229
17	1.85141
18	1.85998
19	1.86800
20	1.87546
21	1.88235
22	1.88866
23	1.89438
24	1.89952
25	1.90405
26	1.90799
27	1.91133
28	1.91407
29	1.91619
30	1.91771
31	1.91863
32	1.91893
33	1.91863

34	1.91771
35	1.91619
36	1.91407
37	1.91133
38	1.90799
39	1.90405
40	1.89952
41	1.89438
42	1.88866
43	1.88235
44	1.87546
45	1.86800
46	1.85998
47	1.85141
48	1.84229
49	1.83265
50	1.82249
51	1.81184
52	1.80071
53	1.78912
54	1.77711
55	1.76468
56	1.75188
57	1.73872
58	1.72525
59	1.71148
60	1.69747
61	1.68325
62	1.66884
63	1.65429
64	1.63964

Program (2.2):

$k(x,y) := x\cdot\frac{y}{5}$

$f(x) := x$

$m(x) := 1$

$n := 10$

$L := 0$

$R := 2$

$h := \frac{(R - L)}{n}$

$i := 0..10$

$x_i := L + i\cdot h$

$u_0 := f(x_0)$

$u_1 := \left(\frac{h}{2}\cdot k(x_1,x_0) + \frac{h}{2}\,k(x_1,x_1)\right)$

$u_1 = 8\times 10^{-4}$

$i := 2..10$

$$u_i := \left(\frac{h}{2}\cdot k\left(x_i, x_0\right) + h\cdot\sum_{j=1}^{i-1} k\left(x_i, x_j\right)\cdot u_j + \frac{h}{2} k\left(x_i, x_i\right)\right)$$

$i := 0..10$

$$ue_i := x_i\cdot\exp\left[\frac{\left(x_i\right)^3}{15}\right]$$

u =

	0
0	0
1	$8\cdot10^{-4}$
2	$3.2026\cdot10^{-3}$
3	$7.2346\cdot10^{-3}$
4	0.013
5	0.0206
6	0.0306
7	0.0433
8	0.0598
9	0.0813
10	0.1101

ue =

	0
0	0
1	0.20011
2	0.40171
3	0.6087
4	0.82778
5	1.06894
6	1.34652
7	1.68103
8	2.10238
9	2.65538
10	3.40921

$$k(x, y) := x\cdot\frac{y}{5}$$

$f(x) := x$

$m(x) := 1$

$n := 20$

$L := 0$

$R := 2$

$$h := \frac{(R - L)}{n}$$

$i := 0..20$

$x_i := L + i\cdot h$

$u_0 := f\left(x_0\right)$

$$u_1 := \left(\frac{h}{2}\cdot k\left(x_1, x_0\right) + \frac{h}{2} k\left(x_1, x_1\right)\right)$$

$u_1 = 1\times10^{-4}$

$i := 2..20$

$$u_i := \left(\frac{h}{2}\cdot k\left(x_i, x_0\right) + h\cdot\sum_{j=1}^{i-1} k\left(x_i, x_j\right)\cdot u_j + \frac{h}{2} k\left(x_i, x_i\right)\right)$$

$i := 0..20$

$$ue_i := x_i \cdot \exp\left[\frac{(x_i)^3}{15}\right]$$

ue=

	0
0	0.00000
1	0.10001
2	0.20011
3	0.30054
4	0.40171
5	0.50418
6	0.60870
7	0.71619
8	0.82778
9	0.94482
10	1.06894
11	1.20207
12	1.34652
13	1.50506
14	1.68103
15	1.87848
16	2.10238
17	2.35882
18	2.65538
19	3.00153
20	3.40921

,

u=

	0
0	0.0000
1	0.000100
2	0.00040004
3	0.00090054
4	0.0016029
5	0.0025100
6	0.0036271
7	0.0049621
8	0.0065265
9	0.0083363
10	0.0104
11	0.0128
12	0.0155
13	0.0186
14	0.0221
15	0.0261
16	0.0306
17	0.0359
18	0.0420
19	0.0492
20	0.0575

Program (2.3):

$$u(x) = 1 + \frac{1}{\pi}\int_{-1}^{1} \frac{1}{1+(x-y)^2} u(y)dy, \quad -1 \le x \le 1$$

$$k(x,y) := \frac{1}{\pi} \cdot \frac{1}{1+(x-y)^2}$$

$f(x) := 1$

$n := 8$

$L := -1$

$R := 1$

$$h := \frac{(R-L)}{n}$$

$i := 0..8$

$x_i := L + i \cdot h$

$i := 1..7$

$j := 1..7$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$$m = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.07958 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.03105 & -0.02449 \\ 0 & -0.0749 & -0.07958 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.03105 \\ 0 & -0.06366 & -0.0749 & -0.07958 & -0.0749 & -0.06366 & -0.05093 & -0.03979 \\ 0 & -0.05093 & -0.06366 & -0.0749 & -0.07958 & -0.0749 & -0.06366 & -0.05093 \\ 0 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & -0.07958 & -0.0749 & -0.06366 \\ 0 & -0.03105 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & -0.07958 & -0.0749 \\ 0 & -0.02449 & -0.03105 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & -0.07958 \end{pmatrix}$$

$i := 1..7$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{0,0} := 1 - \frac{h}{2} \cdot k(x_0, x_0)$

$m_{8,8} := 1 - \frac{h}{2} \cdot k(x_8, x_8)$

$$m = \begin{pmatrix} 0.96021 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.03105 & -0.02449 & 0 \\ 0 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.03105 & 0 \\ 0 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & 0 \\ 0 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & 0 \\ 0 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & 0 \\ 0 & -0.03105 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & 0 \\ 0 & -0.02449 & -0.03105 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.96021 \end{pmatrix}$$

$i := 1..7$

$m_{0,i} := (-h) \cdot k(x_0, x_i)$

$i := 1..7$

$m_{8,i} := -h \cdot k(x_8, x_i)$

$i := 1..7$

$m_{i,0} := \left(\frac{-h}{2}\right) \cdot k(x_i, x_0)$

$i := 1..7$

$m_{i,8} := \frac{-h}{2} \cdot k(x_i, x_8)$

$m_{0,8} := \frac{-h}{2} \cdot k(x_0, x_8)$

$m_{8,0} := \frac{-h}{2} \cdot k(x_8, x_0)$

$$m = \begin{pmatrix} 0.96021 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.03105 & -0.02449 & -0.01959 & -7.95775\times 10^{-3} \\ -0.03745 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.03105 & -0.02449 & -9.79415\times 10^{-3} \\ -0.03183 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.03105 & -0.01224 \\ -0.02546 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & -0.03979 & -0.01553 \\ -0.01989 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & -0.05093 & -0.01989 \\ -0.01553 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & -0.06366 & -0.02546 \\ -0.01224 & -0.03105 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & -0.0749 & -0.03183 \\ -9.79415\times 10^{-3} & -0.02449 & -0.03105 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & 0.92042 & -0.03745 \\ -7.95775\times 10^{-3} & -0.01959 & -0.02449 & -0.03105 & -0.03979 & -0.05093 & -0.06366 & -0.0749 & 0.96021 \end{pmatrix}$$

$i := 0..8$

$f_i := 1$

$u := m^{-1}\cdot f$

$$u = \begin{pmatrix} 1.63639 \\ 1.74695 \\ 1.83641 \\ 1.89332 \\ 1.91268 \\ 1.89332 \\ 1.83641 \\ 1.74695 \\ 1.63639 \end{pmatrix}$$

$n := 16$

$L := -1$

$R := 1$

$h := \frac{(R - L)}{n}$

$i := 0..16$

$x_i := L + i\cdot h$

$i := 1..15$

$j := 1..15$

$m_{i,j} := -h\cdot k\left(x_i, x_j\right)$

$i := 1..15$

$m_{i,i} := 1 - h\cdot k\left(x_i, x_i\right)$

$m_{1,1} = 0.96021$

$m_{0,0} := 1 - \frac{h}{2}\cdot k\left(x_0, x_0\right)$

$m_{16,16} := 1 - \frac{h}{2}\cdot k\left(x_{16}, x_{16}\right)$

$i := 1..15$

$m_{0,i} := (-h)\cdot k(x_0, x_i)$

$i := 1..15$

$m_{16,i} := -h\cdot k(x_{16}, x_i)$

$i := 1..15$

$m_{i,0} := \left(\frac{-h}{2}\right)\cdot k(x_i, x_0)$

$i := 1..15$

$m_{i,16} := \frac{-h}{2}\cdot k(x_i, x_{16})$

$m_{0,16} := \frac{-h}{2}\cdot k(x_0, x_{16})$

$m_{16,0} := \frac{-h}{2}\cdot k(x_{16}, x_0)$

$i := 0..16$

$f_i := 1$

$u := m^{-1}\cdot f$

u=

	0
0	1.63887
1	1.69648
2	1.75070
3	1.79940
4	1.84089
5	1.87401
6	1.89804
7	1.91258
8	1.91744
9	1.91258
10	1.89804
11	1.87401
12	1.84089
13	1.79940
14	1.75070
15	1.69648
16	1.63887

$k(x,y) := \frac{1}{\pi}\cdot\frac{1}{1+(x-y)^2}$

$f(x) := 1$

$n := 32$

$L := -1$

$R := 1$

$h := \frac{(R-L)}{n}$

$i := 0..32$

$x_i := L + i\cdot h$

$i := 1..31$

$j := 1..31$

$m_{i,j} := -h\cdot k(x_i, x_j)$

$i := 1..31$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{1,1} = 0.98011$

$m_{0,0} := 1 - \frac{h}{2} \cdot k(x_0, x_0)$

$m_{32,32} := 1 - \frac{h}{2} \cdot k(x_{32}, x_{32})$

$i := 1..31$

$m_{0,i} := (-h) \cdot k(x_0, x_i)$

$i := 1..31$

$m_{32,i} := -h \cdot k(x_{32}, x_i)$

$i := 1..31$

$m_{i,0} := \left(\frac{-h}{2}\right) \cdot k(x_i, x_0)$

$i := 1..31$

$m_{i,32} := \frac{-h}{2} \cdot k(x_i, x_{32})$

$m_{0,32} := \frac{-h}{2} \cdot k(x_0, x_{32})$

$m_{32,0} := \frac{-h}{2} \cdot k(x_{32}, x_0)$

$i := 0..32$

$f_i := 1$

$u := m^{-1} \cdot f$

u=

	0
0	1.63949
1	1.66866
2	1.69727
3	1.72503
4	1.75164
5	1.77686
6	1.80045
7	1.82222
8	1.84201
9	1.85970
10	1.87517
11	1.88837
12	1.89922
13	1.90770
14	1.91377
15	1.91742
16	1.91863
17	1.91742
18	1.91377
19	1.90770
20	1.89922
21	1.88837
22	1.87517
23	1.85970

24	1.84201
25	1.82222
26	1.80045
27	1.77686
28	1.75164
29	1.72503
30	1.69727
31	1.66866
32	1.63949

$$k(x,y) := \frac{1}{\pi}\cdot\frac{1}{1+(x-y)^2}$$

$f(x) := 1$

$n := 64$

$L := -1$

$R := 1$

$$h := \frac{(R-L)}{n}$$

$i := 0..64$

$x_i := L + i\cdot h$

$i := 1..63$

$j := 1..63$

$m_{i,j} := -h\cdot k(x_i, x_j)$

$i := 1..63$

$m_{i,i} := 1 - h\cdot k(x_i, x_i)$

$$m_{0,0} := 1 - \frac{h}{2}\cdot k(x_0, x_0)$$

$$m_{64,64} := 1 - \frac{h}{2}\cdot k(x_{64}, x_{64})$$

$i := 1..63$

$m_{0,i} := (-h)\cdot k(x_0, x_i)$

$i := 1..63$

$m_{64,i} := -h\cdot k(x_{64}, x_i)$

$i := 1..63$

$$m_{i,0} := \left(\frac{-h}{2}\right)\cdot k(x_i, x_0)$$

$i := 1..63$

$$m_{i,64} := \frac{-h}{2}\cdot k(x_i, x_{64})$$

$$m_{0,64} := \frac{-h}{2}\cdot k(x_0, x_{64})$$

$$m_{64,0} := \frac{-h}{2}\cdot k(x_{64}, x_0)$$

$i := 0..64$

$f_i := 1$

$u := m^{-1} \cdot f$

u=

	0
0	1.63964
1	1.65429
2	1.66884
3	1.68325
4	1.69747
5	1.71148
6	1.72525
7	1.73872
8	1.75188
9	1.76468
10	1.77711
11	1.78912
12	1.80071
13	1.81184
14	1.82249
15	1.83265
16	1.84229
17	1.85141
18	1.85998
19	1.86800
20	1.87546
21	1.88235
22	1.88866
23	1.89438
24	1.89952
25	1.90405
26	1.90799
27	1.91133
28	1.91407
29	1.91619
30	1.91771
31	1.91863
32	1.91893
33	1.91863
34	1.91771
35	1.91619
36	1.91407
37	1.91133
38	1.90799
39	1.90405
40	1.89952
41	1.89438
42	1.88866
43	1.88235
44	1.87546
45	1.86800
46	1.85998
47	1.85141
48	1.84229
49	1.83265
50	1.82249
51	1.81184
52	1.80071

53	1.78912
54	1.77711
55	1.76468
56	1.75188
57	1.73872
58	1.72525
59	1.71148
60	1.69747
61	1.68325
62	1.66884
63	1.65429
64	1.63964

Program (2.4):

$k(x,y) := x\cdot\frac{y}{5}$

$f(x) := x$

$m(x) := 1$

$n := 10$

$L := 0$

$R := 2$

$h := \frac{(R-L)}{n}$

$i := 0..10$

$x_i := L + i\cdot h$

$u_0 := f\left(x_0\right)$

$u_1 := \left(\frac{h}{2}\cdot k\left(x_1, x_0\right) + \frac{h}{2}k\left(x_1, x_1\right)\right)$

$u_1 = 8\times 10^{-4}$

$i := 2..10$

$u_i := \left(\frac{h}{2}\cdot k\left(x_i, x_0\right) + h\cdot\sum_{j=1}^{i-1} k\left(x_i, x_j\right)\cdot u_j + \frac{h}{2}k\left(x_i, x_i\right)\right)$

$i := 0..10$

$ue_i := x_i\cdot \exp\left[\frac{\left(x_i\right)^3}{15}\right]$

u =

	0
0	0
1	$8\cdot10^{-4}$
2	$3.2026\cdot10^{-3}$
3	$7.2346\cdot10^{-3}$
4	0.013
5	0.0206
6	0.0306
7	0.0433
8	0.0598
9	0.0813
10	0.1101

ue =

	0
0	0
1	0.20011
2	0.40171
3	0.6087
4	0.82778
5	1.06894
6	1.34652
7	1.68103
8	2.10238
9	2.65538
10	3.40921

$k(x,y) := x\cdot\frac{y}{5}$

$f(x) := x$

$m(x) := 1$

$n := 20$

$L := 0$

$R := 2$

$h := \frac{(R-L)}{n}$

$i := 0..20$

$x_i := L + i\cdot h$

$u_0 := f(x_0)$

$u_1 := \left(\frac{h}{2}\cdot k(x_1,x_0) + \frac{h}{2}k(x_1,x_1)\right)$

$u_1 = 1\times10^{-4}$

$i := 2..20$

$u_i := \left(\frac{h}{2}\cdot k(x_i,x_0) + h\cdot\sum_{j=1}^{i-1} k(x_i,x_j)\cdot u_j + \frac{h}{2}k(x_i,x_i)\right)$

$i := 0..20$

$ue_i := x_i\cdot\exp\left[\frac{(x_i)^3}{15}\right]$

ue=

	0
0	0.00000
1	0.10001
2	0.20011
3	0.30054
4	0.40171
5	0.50418
6	0.60870
7	0.71619
8	0.82778
9	0.94482
10	1.06894
11	1.20207
12	1.34652
13	1.50506
14	1.68103
15	1.87848
16	2.10238
17	2.35882
18	2.65538
19	3.00153
20	3.40921

,

u=

	0
0	0.0000
1	0.000100
2	0.00040004
3	0.00090054
4	0.0016029
5	0.0025100
6	0.0036271
7	0.0049621
8	0.0065265
9	0.0083363
10	0.0104
11	0.0128
12	0.0155
13	0.0186
14	0.0221
15	0.0261
16	0.0306
17	0.0359
18	0.0420
19	0.0492
20	0.0575

Program (2.5):

$ue(x) := x^2$

$f(x) := ue(x) - \int_0^1 (x+y)\cdot ue(y)\,dy - \int_{0.4}^{0.6} \left(x^2 + y^2\right) ue(y)\,dy$

$k(x,y) := (x+y)$

$l(x,y) := \left(x^2 + y^2\right)$

$f(x) := .94933333333333333333\cdot x^2 - .26350400000000000000 - \frac{1}{3}\cdot x$

$a := 0$

$b := 1$

$n := 5$

$h := \frac{(b-a)}{n}$

$i := 0 .. 5$

$x_i := a + i\cdot h$

$$x = \begin{pmatrix} 0 \\ 0.2 \\ 0.4 \\ 0.6 \\ 0.8 \\ 1 \end{pmatrix}$$

$p := 2$

$q := 3$

$i := 1 .. p - 1$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k\left(x_i, x_j\right)$

$i := 1 .. p - 1$

$m_{i,i} := 1 - h \cdot k\left(x_i, x_i\right)$

$m_{0,0} := 1 - \frac{h}{2} \cdot k\left(x_0, x_0\right)$

$m_{p,p} := 1 - h \cdot k\left(x_p, x_p\right) - \frac{h}{2} \cdot l\left(x_p, x_p\right)$

$i := 1 .. p - 1$

$m_{0,i} := (-h) \cdot k\left(x_0, x_i\right)$

$i := 0 .. p - 1$

$m_{i,p} := -h \cdot k\left(x_i, x_p\right) - \frac{h}{2} \cdot l\left(x_i, x_p\right)$

$i := 1 .. p$

$m_{i,0} := \frac{-h}{2} \cdot k\left(x_i, x_0\right)$

$i := 1 .. p - 1$

$m_{p,i} := (-h) \cdot k\left(x_p, x_i\right)$

$i := 0 .. p$

$j := p + 1 .. q - 1$

$m_{i,j} := (-h) \cdot k\left(x_i, x_j\right) - (h) \cdot l\left(x_i, x_j\right)$

$i := 0 .. p$

$m_{i,q} := (-h) \cdot k\left(x_i, x_q\right) - \left(\frac{h}{2}\right) \cdot l\left(x_i, x_q\right)$

$i := 0 .. p$

$j := q + 1 .. n - 1$

$m_{i,j} := -h \cdot k\left(x_i, x_j\right)$

$i := 0 .. p$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$i := p + 1 .. n$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := p + 1 .. n$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := p + 1 .. n$

$m_{i,p} := (-h) \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$

$i := p + 1 .. q - 1$

$j := p + 1 .. q - 1$

$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$

$i := p + 1 .. q - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i) - h \cdot l(x_i, x_i)$

$i := p + 1 .. q - 1$

$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$

$i := q .. n$

$j := p + 1 .. q - 1$

$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$

$m_{q,q} := 1 - h \cdot k(x_q, x_q) - \frac{h}{2} \cdot l(x_q, x_q)$

$i := q + 1 .. n$

$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$

$i := p + 1 .. n$

$j := q + 1 .. n - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := q + 1 .. n - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{n,n} := 1 - \frac{h}{2} \cdot k(x_n, x_n)$

$i := p + 1 .. n - 1$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$$m = \begin{pmatrix} 1 & -0.04 & -0.112 & -0.156 & -0.16 & -0.1 \\ -0.02 & 0.92 & -0.16 & -0.2 & -0.2 & -0.12 \\ -0.04 & -0.12 & 0.776 & -0.252 & -0.24 & -0.14 \\ -0.06 & -0.16 & -0.304 & 0.688 & -0.28 & -0.16 \\ -0.08 & -0.2 & -0.4 & -0.38 & 0.68 & -0.18 \\ -0.1 & -0.24 & -0.512 & -0.456 & -0.36 & 0.8 \end{pmatrix}$$

$i := 0..5$

$f_i := f(x_i)$

$u := m^{-1} \cdot f$

$$f = \begin{pmatrix} -0.2635 \\ -0.2922 \\ -0.24494 \\ -0.12174 \\ 0.0774 \\ 0.3525 \end{pmatrix}$$

$i := 0..5$

$ue_i := ue(x_i)$

$$ue = \begin{pmatrix} 0 \\ 0.04 \\ 0.16 \\ 0.36 \\ 0.64 \\ 1 \end{pmatrix}$$

$$u = \begin{pmatrix} -0.04634 \\ -0.02383 \\ 0.07796 \\ 0.259 \\ 0.51932 \\ 0.8589 \end{pmatrix}$$

$n := 10$

$h := \frac{(b-a)}{n}$

$i := 0..10$

$x_i := a + i \cdot h$

x=

	0
0	0
1	0.1
2	0.2
3	0.3
4	0.4
5	0.5
6	0.6
7	0.7
8	0.8
9	0.9
10	1

$p := 4$

$q := 6$

$i := 1 .. p - 1$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 1 .. p - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{0,0} := 1 - \frac{h}{2} \cdot k(x_0, x_0)$

$m_{p,p} := 1 - h \cdot k(x_p, x_p) - \frac{h}{2} \cdot l(x_p, x_p)$

$i := 1 .. p - 1$

$m_{0,i} := (-h) \cdot k(x_0, x_i)$

$i := 0 .. p - 1$

$m_{i,p} := -h \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$

$i := 1 .. p$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := 1 .. p - 1$

$m_{p,i} := (-h) \cdot k(x_p, x_i)$

$i := 0 .. p$

$j := p + 1 .. q - 1$

$m_{i,j} := (-h) \cdot k(x_i, x_j) - (h) \cdot l(x_i, x_j)$

$i := 0 .. p$

$m_{i,q} := (-h) \cdot k(x_i, x_q) - \left(\frac{h}{2}\right) \cdot l(x_i, x_q)$

$$i := 0 .. p$$
$$j := q + 1 .. n - 1$$
$$m_{i,j} := -h \cdot k(x_i, x_j)$$
$$i := 0 .. p$$
$$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$$
$$i := p + 1 .. n$$
$$j := 1 .. p - 1$$
$$m_{i,j} := -h \cdot k(x_i, x_j)$$
$$i := p + 1 .. n$$
$$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$$
$$i := p + 1 .. n$$
$$m_{i,p} := (-h) \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$$
$$i := p + 1 .. q - 1$$
$$j := p + 1 .. q - 1$$
$$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$$
$$i := p + 1 .. q - 1$$
$$m_{i,i} := 1 - h \cdot k(x_i, x_i) - h \cdot l(x_i, x_i)$$
$$i := p + 1 .. q - 1$$
$$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$$
$$i := q .. n$$
$$j := p + 1 .. q - 1$$
$$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$$
$$m_{q,q} := 1 - h \cdot k(x_q, x_q) - \frac{h}{2} \cdot l(x_q, x_q)$$
$$i := q + 1 .. n$$
$$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$$
$$i := p + 1 .. n$$
$$j := q + 1 .. n - 1$$
$$m_{i,j} := -h \cdot k(x_i, x_j)$$
$$i := q + 1 .. n - 1$$
$$m_{i,i} := 1 - h \cdot k(x_i, x_i)$$
$$m_{n,n} := 1 - \frac{h}{2} \cdot k(x_n, x_n)$$
$$i := p + 1 .. n - 1$$

$m_{i,n} := \frac{-h}{2} \cdot k\left(x_i, x_n\right)$

$$A = \begin{pmatrix} 1 & -0.01 & -0.02 & -0.03 & -0.048 & -0.075 & -0.078 & -0.07 & -0.08 & -0.09 & -0.05 \\ -0.005 & 0.98 & -0.03 & -0.04 & -0.0585 & -0.086 & -0.0885 & -0.08 & -0.09 & -0.1 & -0.055 \\ -0.01 & -0.03 & 0.96 & -0.05 & -0.07 & -0.099 & -0.1 & -0.09 & -0.1 & -0.11 & -0.06 \\ -0.015 & -0.04 & -0.05 & 0.94 & -0.0825 & -0.114 & -0.1125 & -0.1 & -0.11 & -0.12 & -0.065 \\ -0.02 & -0.05 & -0.06 & -0.07 & 0.904 & -0.131 & -0.126 & -0.11 & -0.12 & -0.13 & -0.7 \\ -0.025 & -0.06 & -0.07 & -0.08 & -0.1105 & 0.85 & -0.1405 & -0.12 & -0.13 & -0.14 & -0.075 \\ -0.03 & -0.07 & -0.08 & -0.09 & -0.126 & -0.171 & 0.844 & -0.13 & -0.14 & -0.15 & -0.08 \\ -0.035 & -0.08 & -0.09 & -0.1 & -0.1425 & -0.194 & -0.1725 & 0.86 & -0.15 & -0.16 & -0.085 \\ -0.04 & -0.09 & -0.1 & -0.11 & -0.16 & -0.219 & -0.19 & -0.15 & 0.84 & -0.17 & -0.09 \\ -0.045 & -0.1 & -0.11 & -0.12 & -0.1785 & -0.246 & -0.2085 & -0.16 & -0.17 & 0.82 & -0.095 \\ -0.05 & -0.11 & -0.12 & -0.13 & -0.198 & -0.275 & -0.228 & -0.17 & -0.18 & -0.19 & 0.9 \end{pmatrix}$$

$f(x) := .9493333333333333333 \cdot x^2 - .2635040000000000000 - \frac{1}{3} \cdot x$

$i := 0..10$
$f_i := f\left(x_i\right)$

$u := m^{-1} \cdot f$
$i := 0..10$

$ue_i := \left(x_i\right)^2$

ue=

	0
0	0
1	0.01
2	0.04
3	0.09
4	0.16
5	0.25
6	0.36
7	0.49
8	0.64
9	0.81
10	1

, u=

	0
0	-0.009105
1	-0.000852
2	0.02733
3	0.07545
4	0.1435
5	0.23148
6	0.3394
7	0.46725
8	0.61503
9	0.78274
10	0.97038

$n := 20$

$h := \frac{(b - a)}{n}$

$i := 0 .. 20$

$x_i := a + i \cdot h$

$x =$

	0
0	0
1	0.05
2	0.1
3	0.15
4	0.2
5	0.25
6	0.3
7	0.35
8	0.4
9	0.45
10	0.5
11	0.55
12	0.6
13	0.65
14	0.7
15	0.75
16	0.8
17	0.85
18	0.9
19	0.95
20	1

$p := 8$

$q := 12$

$i := 1 .. p - 1$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k\left(x_i, x_j\right)$

$i := 1 .. p - 1$

$m_{i,i} := 1 - h \cdot k\left(x_i, x_i\right)$

$$m_{0,0} := 1 - \frac{h}{2} \cdot k\left(x_0, x_0\right)$$

$$m_{p,p} := 1 - h \cdot k\left(x_p, x_p\right) - \frac{h}{2} \cdot l\left(x_p, x_p\right)$$

$i := 1 .. p - 1$

$m_{0,i} := (-h) \cdot k\left(x_0, x_i\right)$

$i := 0 .. p - 1$

$$m_{i,p} := -h \cdot k\left(x_i, x_p\right) - \frac{h}{2} \cdot l\left(x_i, x_p\right)$$

$i := 1 .. p$

$$m_{i,0} := \frac{-h}{2} \cdot k\left(x_i, x_0\right)$$

$i := 1 .. p - 1$

$m_{p,i} := (-h) \cdot k(x_p, x_i)$

$i := 0 .. p$

$j := p + 1 .. q - 1$

$m_{i,j} := (-h) \cdot k(x_i, x_j) - (h) \cdot l(x_i, x_j)$

$i := 0 .. p$

$m_{i,q} := (-h) \cdot k(x_i, x_q) - \left(\frac{h}{2}\right) \cdot l(x_i, x_q)$

$i := 0 .. p$

$j := q + 1 .. n - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 0 .. p$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$i := p + 1 .. n$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := p + 1 .. n$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := p + 1 .. n$

$m_{i,p} := (-h) \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$

$i := p + 1 .. q - 1$

$j := p + 1 .. q - 1$

$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$

$i := p + 1 .. q - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i) - h \cdot l(x_i, x_i)$

$i := p + 1 .. q - 1$

$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$

$i := q .. n$

$j := p + 1 .. q - 1$

$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$

$m_{q,q} := 1 - h \cdot k(x_q, x_q) - \frac{h}{2} \cdot l(x_q, x_q)$

$i := q + 1 .. n$

$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$

$i := p + 1 .. n$

$j := q + 1 .. n - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := q + 1 .. n - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{n,n} := 1 - \frac{h}{2} \cdot k(x_n, x_n)$

$i := p + 1 .. n - 1$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$f(x) := .94933333333333333333 \cdot x^2 - .26350400000000000000 - \frac{1}{3} \cdot x$

$i := 0 .. 20$

$f_i := f(x_i)$

$u := m^{-1} \cdot f$

$i := 0 .. 20$

$ue_i := (x_i)^2$

ue=

	0
0	0
1	0.00251
2	0.01
3	0.0225
4	0.04
5	0.0625
6	0.09
7	0.1225
8	0.16
9	0.2025
10	0.25
11	0.3025
12	0.36
13	0.4225
14	0.49
15	0.5625
16	0.64
17	0.7225
18	0.81
19	0.9025
20	1

, u=

	0
0	-0.00229
1	-0.000016
2	-0.00726
3	0.01953
4	0.0368
5	0.05907
6	0.08633
7	0.11858
8	0.15583
9	0.19808
10	0.24532
11	0.29756
12	0.3548
13	0.41703
14	0.48425
15	0.55648
16	0.63369
17	0.71591
18	0.80311
19	0.89532
20	0.99252

$n := 30$

$h := \frac{(b-a)}{n}$

$i := 0..30$

$x_i := a + i \cdot h$

x=

	0
0	0
1	0.03333
2	0.06667
3	0.1
4	0.13333
5	0.16667
6	0.2
7	0.23333
8	0.26667
9	0.3
10	0.33333
11	0.36667
12	0.4
13	0.43333
14	0.46667
15	0.5
16	0.53333
17	0.56667
18	0.6
19	0.63333
20	0.66667
21	0.7
22	0.73333
23	0.76667
24	0.8
25	0.83333
26	0.86667
27	0.9
28	0.93333
29	0.96667
30	1

$p := 12$

$q := 18$

$i := 1..p-1$

$j := 1..p-1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 1..p-1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{0,0} := 1 - \frac{h}{2} \cdot k(x_0, x_0)$

$m_{p,p} := 1 - h\cdot k(x_p, x_p) - \frac{h}{2}\cdot l(x_p, x_p)$

$i := 1 .. p - 1$

$m_{0,i} := (-h)\cdot k(x_0, x_i)$

$i := 0 .. p - 1$

$m_{i,p} := -h\cdot k(x_i, x_p) - \frac{h}{2}\cdot l(x_i, x_p)$

$i := 1 .. p$

$m_{i,0} := \frac{-h}{2}\cdot k(x_i, x_0)$

$i := 1 .. p - 1$

$m_{p,i} := (-h)\cdot k(x_p, x_i)$

$i := 0 .. p$

$j := p + 1 .. q - 1$

$m_{i,j} := (-h)\cdot k(x_i, x_j) - (h)\cdot l(x_i, x_j)$

$i := 0 .. p$

$m_{i,q} := (-h)\cdot k(x_i, x_q) - \left(\frac{h}{2}\right)\cdot l(x_i, x_q)$

$i := 0 .. p$

$j := q + 1 .. n - 1$

$m_{i,j} := -h\cdot k(x_i, x_j)$

$i := 0 .. p$

$m_{i,n} := \frac{-h}{2}\cdot k(x_i, x_n)$

$i := p + 1 .. n$

$j := 1 .. p - 1$

$m_{i,j} := -h\cdot k(x_i, x_j)$

$i := p + 1 .. n$

$m_{i,0} := \frac{-h}{2}\cdot k(x_i, x_0)$

$i := p + 1 .. n$

$m_{i,p} := (-h)\cdot k(x_i, x_p) - \frac{h}{2}\cdot l(x_i, x_p)$

$i := p + 1 .. q - 1$

$j := p + 1 .. q - 1$

$m_{i,j} := -h\cdot k(x_i, x_j) - h\cdot l(x_i, x_j)$

$i := p + 1 .. q - 1$

$m_{i,i} := 1 - h\cdot k(x_i, x_i) - h\cdot l(x_i, x_i)$

$$i := p + 1 .. q - 1$$

$$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$$

$$i := q .. n$$

$$j := p + 1 .. q - 1$$

$$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$$

$$m_{q,q} := 1 - h \cdot k(x_q, x_q) - \frac{h}{2} \cdot l(x_q, x_q)$$

$$i := q + 1 .. n$$

$$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$$

$$i := p + 1 .. n$$

$$j := q + 1 .. n - 1$$

$$m_{i,j} := -h \cdot k(x_i, x_j)$$

$$i := q + 1 .. n - 1$$

$$m_{i,i} := 1 - h \cdot k(x_i, x_i)$$

$$m_{n,n} := 1 - \frac{h}{2} \cdot k(x_n, x_n)$$

$$i := p + 1 .. n - 1$$

$$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$$

$$f(x) := .94933333333333333333 \cdot x^2 - .26350400000000000000 - \frac{1}{3} \cdot x$$

$$i := 0 .. 30$$

$$f_i := f(x_i)$$

$$u := m^{-1} \cdot f$$

$$i := 0 .. 30$$

$$ue_i := (x_i)^2$$

ue=

	0
0	0
1	0.00111
2	0.00444
3	0.01
4	0.01778
5	0.02778
6	0.04
7	0.05444
8	0.07111
9	0.09
10	0.11111
11	0.13444
12	0.16
13	0.18778
14	0.21778
15	0.25
16	0.28444
17	0.32111
18	0.36
19	0.40111
20	0.44444
21	0.49
22	0.53778
23	0.58778
24	0.64
25	0.69444
26	0.75111
27	0.81
28	0.87111
29	0.93444
30	1

, u=

	0
0	-0.00122
1	0.00002
2	0.00329
3	0.00878
4	0.01649
5	0.02642
6	0.03858
7	0.05295
8	0.06955
9	0.08836
10	0.1094
11	0.13266
12	0.15814
13	0.18585
14	0.21577
15	0.24792
16	0.28229
17	0.31887
18	0.35768
19	0.39871
20	0.44197
21	0.48744
22	0.53541
23	0.58505
24	0.63719
25	0.69155
26	0.74813
27	0.80693
28	0.86796
29	0.9312
30	0.99667

$n := 40$

$h := \frac{(b - a)}{n}$

$i := 0 .. 40$

$x_i := a + i \cdot h$ x=

	0
0	0
1	0.025
2	0.05
3	0.075
4	0.1
5	0.125
6	0.15
7	0.175
8	0.2
9	0.225
10	0.25
11	0.275
12	0.3
13	0.325
14	0.35
15	0.375
16	0.4
17	0.425
18	0.45

19	0.475
20	0.5
21	0.525
22	0.55
23	0.575
24	0.6
25	0.625
26	0.65
27	0.675
28	0.7
29	0.725
30	0.75
31	0.775
32	0.8
33	0.825
34	0.85
35	0.875
36	0.9
37	0.925
38	0.95
39	0.975
40	1

$p := 16$

$q := 24$

$i := 1 .. p - 1$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 1 .. p - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{0,0} := 1 - \frac{h}{2} \cdot k(x_0, x_0)$

$m_{p,p} := 1 - h \cdot k(x_p, x_p) - \frac{h}{2} \cdot l(x_p, x_p)$

$i := 1 .. p - 1$

$m_{0,i} := (-h) \cdot k(x_0, x_i)$

$i := 0 .. p - 1$

$m_{i,p} := -h \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$

$i := 1 .. p$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := 1 .. p - 1$

$m_{p,i} := (-h) \cdot k(x_p, x_i)$

$i := 0 .. p$

$j := p + 1 .. q - 1$

$m_{i,j} := (-h) \cdot k(x_i, x_j) - (h) \cdot l(x_i, x_j)$

$i := 0 .. p$

$m_{i,q} := (-h)\cdot k(x_i, x_q) - \left(\frac{h}{2}\right)\cdot l(x_i, x_q)$

$i := 0 .. p$

$j := q + 1 .. n - 1$

$m_{i,j} := -h\cdot k(x_i, x_j)$

$i := 0 .. p$

$m_{i,n} := \frac{-h}{2}\cdot k(x_i, x_n)$

$i := p + 1 .. n$

$j := 1 .. p - 1$

$m_{i,j} := -h\cdot k(x_i, x_j)$

$i := p + 1 .. n$

$m_{i,0} := \frac{-h}{2}\cdot k(x_i, x_0)$

$i := p + 1 .. n$

$m_{i,p} := (-h)\cdot k(x_i, x_p) - \frac{h}{2}\cdot l(x_i, x_p)$

$i := p + 1 .. q - 1$

$j := p + 1 .. q - 1$

$m_{i,j} := -h\cdot k(x_i, x_j) - h\cdot l(x_i, x_j)$

$i := p + 1 .. q - 1$

$m_{i,i} := 1 - h\cdot k(x_i, x_i) - h\cdot l(x_i, x_i)$

$i := p + 1 .. q - 1$

$m_{i,q} := -h\cdot k(x_i, x_q) - \frac{h}{2}\cdot l(x_i, x_q)$

$i := q .. n$

$j := p + 1 .. q - 1$

$m_{i,j} := -h\cdot k(x_i, x_j) - h\cdot l(x_i, x_j)$

$m_{q,q} := 1 - h\cdot k(x_q, x_q) - \frac{h}{2}\cdot l(x_q, x_q)$

$i := q + 1 .. n$

$m_{i,q} := -h\cdot k(x_i, x_q) - \frac{h}{2}\cdot l(x_i, x_q)$

$i := p + 1 .. n$

$j := q + 1 .. n - 1$

$m_{i,j} := -h\cdot k(x_i, x_j)$

$i := q + 1 .. n - 1$

$m_{i,i} := 1 - h\cdot k(x_i, x_i)$

$m_{n,n} := 1 - \frac{h}{2} \cdot k(x_n, x_n)$

$i := p + 1 .. n - 1$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$f(x) := .94933333333333333333 \cdot x^2 - .26350400000000000000 - \frac{1}{3} \cdot x$

$i := 0 .. 40$

$f_i := f(x_i)$

$u := m^{-1} \cdot f$

$i := 0 .. 40$

$ue_i := (x_i)^2$

ue=

	0
0	0
1	0.000625000
2	0.00250000
3	0.00562500
4	0.01000
5	0.01563
6	0.02250
7	0.03063
8	0.04000
9	0.05063
10	0.06250
11	0.07563
12	0.09000
13	0.10563
14	0.12250
15	0.14063
16	0.16000
17	0.18063
18	0.20250
19	0.22563
20	0.25000
21	0.27563
22	0.30250
23	0.33063
24	0.36000
25	0.39063
26	0.42250
27	0.45563
28	0.49000
29	0.52563
30	0.56250
31	0.60063
32	0.64000
33	0.68063
34	0.72250
35	0.76563
36	0.81000
37	0.85563
38	0.90250
39	0.95063
40	1.00000

u=

	0
0	-0.000575706
1	0.0000220206
2	0.00186948
3	0.00496667
4	0.00931360
5	0.01491
6	0.02176
7	0.02985
8	0.03920
9	0.04979
10	0.06164
11	0.07473
12	0.08908
13	0.10467
14	0.12152
15	0.13961
16	0.15896
17	0.17955
18	0.20139
19	0.22449
20	0.24883
21	0.27442
22	0.30126
23	0.32935
24	0.35870
25	0.38929
26	0.42113
27	0.45422
28	0.48856
29	0.52415
30	0.56099
31	0.59908
32	0.63842
33	0.67901
34	0.72085
35	0.76394
36	0.80827
37	0.85386
38	0.90070
39	0.94879
40	0.99813

$n := 50$

$h := \frac{(b - a)}{n}$

$i := 0 .. 50$

$x_i := a + i \cdot h$

x=

	0
0	0
1	0.02

2	0.04
3	0.06
4	0.08
5	0.1
6	0.12
7	0.14
8	0.16
9	0.18
10	0.2
11	0.22
12	0.24
13	0.26
14	0.28
15	0.3
16	0.32
17	0.34
18	0.36
19	0.38
20	0.4
21	0.42
22	0.44
23	0.46
24	0.48
25	0.5
26	0.52
27	0.54
28	0.56
29	0.58
30	0.6
31	0.62
32	0.64
33	0.66
34	0.68
35	0.7
36	0.72
37	0.74
38	0.76
39	0.78
40	0.8
41	0.82
42	0.84
43	0.86
44	0.88
45	0.9
46	0.92
47	0.94
48	0.96
49	0.98
50	1

$p := 20$

$q := 30$

$i := 1 .. p - 1$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 1 .. p - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{0,0} := 1 - \frac{h}{2} \cdot k(x_0, x_0)$

$m_{p,p} := 1 - h \cdot k(x_p, x_p) - \frac{h}{2} \cdot l(x_p, x_p)$

$i := 1 .. p - 1$

$m_{0,i} := (-h) \cdot k(x_0, x_i)$

$i := 0 .. p - 1$

$m_{i,p} := -h \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$

$i := 1 .. p$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := 1 .. p - 1$

$m_{p,i} := (-h) \cdot k(x_p, x_i)$

$i := 0 .. p$

$j := p + 1 .. q - 1$

$m_{i,j} := (-h) \cdot k(x_i, x_j) - (h) \cdot l(x_i, x_j)$

$i := 0 .. p$

$m_{i,q} := (-h) \cdot k(x_i, x_q) - \left(\frac{h}{2}\right) \cdot l(x_i, x_q)$

$i := 0 .. p$

$j := q + 1 .. n - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 0 .. p$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$i := p + 1 .. n$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := p + 1 .. n$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := p + 1 .. n$

$m_{i,p} := (-h) \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$

$i := p + 1 .. q - 1$

$j := p + 1 .. q - 1$

$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$

$i := p + 1 .. q - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i) - h \cdot l(x_i, x_i)$

$i := p + 1 .. q - 1$

$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$

$i := q .. n$

$j := p + 1 .. q - 1$

$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$

$m_{q,q} := 1 - h \cdot k(x_q, x_q) - \frac{h}{2} \cdot l(x_q, x_q)$

$i := q + 1 .. n$

$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$

$i := p + 1 .. n$

$j := q + 1 .. n - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := q + 1 .. n - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{n,n} := 1 - \frac{h}{2} \cdot k(x_n, x_n)$

$i := p + 1 .. n - 1$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$f(x) := .9493333333333333333 \cdot x^2 - .2635040000000000000 - \frac{1}{3} \cdot x$

$i := 0 .. 50$

$f_i := f(x_i)$

$u := m^{-1} \cdot f$

$i := 0 .. 50$

$ue_i := (x_i)^2$

ue=

	0
0	0.00000
1	0.00040
2	0.00160
3	0.00360
4	0.00640
5	0.01000
6	0.01440
7	0.01960
8	0.02560

9	0.03240
10	0.04000
11	0.04840
12	0.05760
13	0.06760
14	0.07840
15	0.09000
16	0.10240
17	0.11560
18	0.12960
19	0.14440
20	0.16000
21	0.17640
22	0.19360
23	0.21160
24	0.23040
25	0.25000
26	0.27040
27	0.29160
28	0.31360
29	0.33640
30	0.36000
31	0.38440
32	0.40960
33	0.43560
34	0.46240
35	0.49000
36	0.51840
37	0.54760
38	0.57760
39	0.60840
40	0.64000
41	0.67240
42	0.70560
43	0.73960
44	0.77440
45	0.81000
46	0.84640
47	0.88360
48	0.92160
49	0.96040
50	1.00000

u=

	0
0	-0.00036855
1	0.0000174901
2	0.00120343
3	0.00318925
4	0.00597497
5	0.00956058
6	0.01395
7	0.01913
8	0.02512
9	0.03190
10	0.03949
11	0.04787
12	0.05706

13	0.06704
14	0.07783
15	0.08941
16	0.10180
17	0.11498
18	0.12896
19	0.14375
20	0.15933
21	0.17572
22	0.19290
23	0.21088
24	0.22967
25	0.24925
26	0.26963
27	0.29082
28	0.31280
29	0.33558
30	0.35917
31	0.38355
32	0.40873
33	0.43471
34	0.46150
35	0.48908
36	0.51746
37	0.54664
38	0.57662
39	0.60741
40	0.63899
41	0.67137
42	0.70455
43	0.73853
44	0.77331
45	0.80890
46	0.84528
47	0.88246
48	0.92044
49	0.95922
50	0.99880

$n := 100$

$h := \frac{(b - a)}{n}$

$i := 0 .. 10($

$x_i := a + i \cdot h$

x=

	0
1	0.01
2	0.02
3	0.03
4	0.04
5	0.05
6	0.06
7	0.07
8	0.08
9	0.09

10	0.1
11	0.11
12	0.12
13	0.13
14	0.14
15	0.15
16	0.16
17	0.17
18	0.18
19	0.19
20	0.2
21	0.21
22	0.22
23	0.23
24	0.24
25	0.25
26	0.26
27	0.27
28	0.28
29	0.29
30	0.3
31	0.31
32	0.32
33	0.33
34	0.34
35	0.35
36	0.36
37	0.37
38	0.38
39	0.39
40	0.4
41	0.41
42	0.42
43	0.43
44	0.44
45	0.45
46	0.46
47	0.47
48	0.48
49	0.49
50	0.5
51	0.51
52	0.52
53	0.53
54	0.54
55	0.55
56	0.56
57	0.57
58	0.58
59	0.59
60	0.6
61	0.61
62	0.62
63	0.63
64	0.64
65	0.65
66	0.66
67	0.67
68	0.68

69	0.69
70	0.7
71	0.71
72	0.72
73	0.73
74	0.74
75	0.75
76	0.76
77	0.77
78	0.78
79	0.79
80	0.8
81	0.81
82	0.82
83	0.83
84	0.84
85	0.85
86	0.86
87	0.87
88	0.88
89	0.89
90	0.9
91	0.91
92	0.92
93	0.93
94	0.94
95	0.95
96	0.96
97	0.97
98	0.98
99	0.99
100	1

$p := 40$

$q := 60$

$i := 1 .. p - 1$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 1 .. p - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{0,0} := 1 - \frac{h}{2} \cdot k(x_0, x_0)$

$m_{p,p} := 1 - h \cdot k(x_p, x_p) - \frac{h}{2} \cdot l(x_p, x_p)$

$i := 1 .. p - 1$

$m_{0,i} := (-h) \cdot k(x_0, x_i)$

$i := 0 .. p - 1$

$m_{i,p} := -h \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$

$i := 1 .. p$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := 1 .. p - 1$

$m_{p,i} := (-h) \cdot k(x_p, x_i)$

$i := 0 .. p$

$j := p + 1 .. q - 1$

$m_{i,j} := (-h) \cdot k(x_i, x_j) - (h) \cdot l(x_i, x_j)$

$i := 0 .. p$

$m_{i,q} := (-h) \cdot k(x_i, x_q) - \left(\frac{h}{2}\right) \cdot l(x_i, x_q)$

$i := 0 .. p$

$j := q + 1 .. n - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 0 .. p$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$i := p + 1 .. n$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := p + 1 .. n$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := p + 1 .. n$

$m_{i,p} := (-h) \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$

$i := p + 1 .. q - 1$

$j := p + 1 .. q - 1$

$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$

$i := p + 1 .. q - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i) - h \cdot l(x_i, x_i)$

$i := p + 1 .. q - 1$

$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$

$i := q .. n$

$j := p + 1 .. q - 1$

$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$

$m_{q,q} := 1 - h \cdot k(x_q, x_q) - \frac{h}{2} \cdot l(x_q, x_q)$

$i := q + 1 .. n$

$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$

$i := p + 1 .. n$

$j := q + 1 .. n - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := q + 1 .. n - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{n,n} := 1 - \frac{h}{2} \cdot k(x_n, x_n)$

$i := p + 1 .. n - 1$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$f(x) := .94933333333333333333 \cdot x^2 - .26350400000000000000 - \frac{1}{3} \cdot x$

$i := 0 .. 100$

$f_i := f(x_i)$

$u := m^{-1} \cdot f$

$i := 0 .. 100$

$ue_i := (x_i)^2$

ue=

	0
0	0
1	0.00010
2	0.00040
3	0.00090
4	0.00160
5	0.00250
6	0.00360
7	0.00490
8	0.00640
9	0.00810
10	0.01000
11	0.01210
12	0.01440
13	0.01690
14	0.01960
15	0.02250
16	0.02560
17	0.02890
18	0.03240
19	0.03610
20	0.04000
21	0.04410
22	0.04840
23	0.05290
24	0.05760
25	0.06250

26	0.06760
27	0.07290
28	0.07840
29	0.08410
30	0.09000
31	0.09610
32	0.10240
33	0.10890
34	0.11560
35	0.12250
36	0.12960
37	0.13690
38	0.14440
39	0.15210
40	0.16000
41	0.16810
42	0.17640
43	0.18490
44	0.19360
45	0.20250
46	0.21160
47	0.22090
48	0.23040
49	0.24010
50	0.25000
51	0.26010
52	0.27040
53	0.28090
54	0.29160
55	0.30250
56	0.31360
57	0.32490
58	0.33640
59	0.34810
60	0.36000
61	0.37210
62	0.38440
63	0.39690
64	0.40960
65	0.42250
66	0.43560
67	0.44890
68	0.46240
69	0.47610
70	0.49000
71	0.50410
72	0.51840
73	0.53290
74	0.54760
75	0.56250
76	0.57760
77	0.59290
78	0.60840
79	0.62410
80	0.64000
81	0.65610
82	0.67240
83	0.68890
84	0.70560

85	0.72250
86	0.73960
87	0.75690
88	0.77440
89	0.79210
90	0.81000
91	0.82810
92	0.84640
93	0.86490
94	0.88360
95	0.90250
96	0.92160
97	0.94090
98	0.96040
99	0.98010
100	1

u=

	0
0	0
1	-0.0000921732
2	0.00000608512
3	0.000304337
4	0.000802581
5	0.00150082
6	0.00239905
7	0.00349727
8	0.00479549
9	0.00629370
10	0.00799191
11	0.00989010
12	0.01199
13	0.01429
14	0.01678
15	0.01948
16	0.02238
17	0.02548
18	0.02878
19	0.03228
20	0.03597
21	0.03987
22	0.04397
23	0.04827
24	0.05277
25	0.05746
26	0.06236
27	0.06746
28	0.07276
29	0.07826
30	0.08395
31	0.08985
32	0.09595
33	0.10225
34	0.10875
35	0.11544
36	0.12234
37	0.12944
38	0.13674

39	0.14424
40	0.15193
41	0.15983
42	0.16793
43	0.17623
44	0.18473
45	0.19342
46	0.20232
47	0.21142
48	0.22072
49	0.23022
50	0.23991
51	0.24981
52	0.25991
53	0.27021
54	0.28071
55	0.29140
56	0.30230
57	0.31340
58	0.32470
59	0.33620
60	0.34789
61	0.35979
62	0.37189
63	0.38419
64	0.39668
65	0.40938
66	0.42228
67	0.43538
68	0.44868
69	0.46217
70	0.47587
71	0.48977
72	0.50387
73	0.51816
74	0.53266
75	0.54736
76	0.56226
77	0.57736
78	0.59265
79	0.60815
80	0.62385
81	0.63975
82	0.65584
83	0.67214
84	0.68864
85	0.70534
86	0.72224
87	0.73933
88	0.75663
89	0.77413
90	0.79183
91	0.80972
92	0.82782
93	0.84612
94	0.86462
95	0.88331
96	0.90221
97	0.92131

98	0.94061
99	0.96010
100	0.97980

Program (2.6):

$ue(x) := x^2$

$f(x) := ue(x) - \int_0^1 (x+y)\cdot ue(y)\,dy - \int_{0.4}^{0.6} \left(x^2 + y^2\right) ue(y)\,dy$

$k(x,y) := (x+y)$

$l(x,y) := \left(x^2 + y^2\right)$

$f(x) := .94933333333333333333\cdot x^2 - .26350400000000000000 - \frac{1}{3}\cdot x$

$a := 0$

$b := 1$

$n := 5$

$h := \frac{(b-a)}{n}$

$i := 0..5$

$x_i := a + i\cdot h$

$x = \begin{pmatrix} 0 \\ 0.2 \\ 0.4 \\ 0.6 \\ 0.8 \\ 1 \end{pmatrix}$

$p := 2$

$q := 3$

$i := 1..p-1$

$j := 1..p-1$

$m_{i,j} := -h\cdot k\left(x_i, x_j\right)$

$i := 1..p-1$

$m_{i,i} := 1 - h\cdot k\left(x_i, x_i\right)$

$m_{0,0} := 1 - \frac{h}{2}\cdot k\left(x_0, x_0\right)$

$m_{p,p} := 1 - h\cdot k\left(x_p, x_p\right) - \frac{h}{2}\cdot l\left(x_p, x_p\right)$

$i := 1 .. p - 1$
$m_{0,i} := (-h) \cdot k(x_0, x_i)$
$i := 0 .. p - 1$
$m_{i,p} := -h \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$
$i := 1 .. p$
$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$
$i := 1 .. p - 1$
$m_{p,i} := (-h) \cdot k(x_p, x_i)$
$i := 0 .. p$
$j := p + 1 .. q - 1$
$m_{i,j} := (-h) \cdot k(x_i, x_j) - (h) \cdot l(x_i, x_j)$
$i := 0 .. p$
$m_{i,q} := (-h) \cdot k(x_i, x_q) - \left(\frac{h}{2}\right) \cdot l(x_i, x_q)$
$i := 0 .. p$
$j := q + 1 .. n - 1$
$m_{i,j} := -h \cdot k(x_i, x_j)$
$i := 0 .. p$
$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$
$i := p + 1 .. n$
$j := 1 .. p - 1$
$m_{i,j} := -h \cdot k(x_i, x_j)$
$i := p + 1 .. n$
$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$
$i := p + 1 .. n$
$m_{i,p} := (-h) \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$
$i := p + 1 .. q - 1$
$j := p + 1 .. q - 1$
$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$
$i := p + 1 .. q - 1$
$m_{i,i} := 1 - h \cdot k(x_i, x_i) - h \cdot l(x_i, x_i)$
$i := p + 1 .. q - 1$

$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$

$i := q .. n$

$j := p + 1 .. q - 1$

$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$

$m_{q,q} := 1 - h \cdot k(x_q, x_q) - \frac{h}{2} \cdot l(x_q, x_q)$

$i := q + 1 .. n$

$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$

$i := p + 1 .. n$

$j := q + 1 .. n - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := q + 1 .. n - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{n,n} := 1 - \frac{h}{2} \cdot k(x_n, x_n)$

$i := p + 1 .. n - 1$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$$m = \begin{pmatrix} 1 & -0.04 & -0.112 & -0.156 & -0.16 & -0.1 \\ -0.02 & 0.92 & -0.16 & -0.2 & -0.2 & -0.12 \\ -0.04 & -0.12 & 0.776 & -0.252 & -0.24 & -0.14 \\ -0.06 & -0.16 & -0.304 & 0.688 & -0.28 & -0.16 \\ -0.08 & -0.2 & -0.4 & -0.38 & 0.68 & -0.18 \\ -0.1 & -0.24 & -0.512 & -0.456 & -0.36 & 0.8 \end{pmatrix}$$

$i := 0 .. 5$

$f_i := f(x_i)$

$u := m^{-1} \cdot f$

$$f = \begin{pmatrix} -0.2635 \\ -0.2922 \\ -0.24494 \\ -0.12174 \\ 0.0774 \\ 0.3525 \end{pmatrix}$$

$i := 0..5$

$ue_i := ue(x_i)$

$$ue = \begin{pmatrix} 0 \\ 0.04 \\ 0.16 \\ 0.36 \\ 0.64 \\ 1 \end{pmatrix}$$

$$u = \begin{pmatrix} -0.04634 \\ -0.02383 \\ 0.07796 \\ 0.259 \\ 0.51932 \\ 0.8589 \end{pmatrix}$$

$n := 10$

$h := \frac{(b - a)}{n}$

$i := 0..10$

$x_i := a + i \cdot h$

x=

	0
0	0
1	0.1
2	0.2
3	0.3
4	0.4
5	0.5
6	0.6
7	0.7
8	0.8
9	0.9
10	1

$p := 4$

$q := 6$

$i := 1..p - 1$

$j := 1..p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 1..p - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{0,0} := 1 - \frac{h}{2}\cdot k\left(x_0, x_0\right)$

$m_{p,p} := 1 - h\cdot k\left(x_p, x_p\right) - \frac{h}{2}\cdot l\left(x_p, x_p\right)$

$i := 1 .. p - 1$

$m_{0,i} := (-h)\cdot k\left(x_0, x_i\right)$

$i := 0 .. p - 1$

$m_{i,p} := -h\cdot k\left(x_i, x_p\right) - \frac{h}{2}\cdot l\left(x_i, x_p\right)$

$i := 1 .. p$

$m_{i,0} := \frac{-h}{2}\cdot k\left(x_i, x_0\right)$

$i := 1 .. p - 1$

$m_{p,i} := (-h)\cdot k\left(x_p, x_i\right)$

$i := 0 .. p$

$j := p + 1 .. q - 1$

$m_{i,j} := (-h)\cdot k\left(x_i, x_j\right) - (h)\cdot l\left(x_i, x_j\right)$

$i := 0 .. p$

$m_{i,q} := (-h)\cdot k\left(x_i, x_q\right) - \left(\frac{h}{2}\right)\cdot l\left(x_i, x_q\right)$

$i := 0 .. p$

$j := q + 1 .. n - 1$

$m_{i,j} := -h\cdot k\left(x_i, x_j\right)$

$i := 0 .. p$

$m_{i,n} := \frac{-h}{2}\cdot k\left(x_i, x_n\right)$

$i := p + 1 .. n$

$j := 1 .. p - 1$

$m_{i,j} := -h\cdot k\left(x_i, x_j\right)$

$i := p + 1 .. n$

$m_{i,0} := \frac{-h}{2}\cdot k\left(x_i, x_0\right)$

$i := p + 1 .. n$

$m_{i,p} := (-h)\cdot k\left(x_i, x_p\right) - \frac{h}{2}\cdot l\left(x_i, x_p\right)$

$i := p + 1 .. q - 1$

$j := p + 1 .. q - 1$

$m_{i,j} := -h\cdot k\left(x_i, x_j\right) - h\cdot l\left(x_i, x_j\right)$

$i := p + 1 .. q - 1$

$$m_{i,i} := 1 - h \cdot k\left(x_i, x_i\right) - h \cdot l\left(x_i, x_i\right)$$
$$i := p + 1 .. q - 1$$
$$m_{i,q} := -h \cdot k\left(x_i, x_q\right) - \frac{h}{2} \cdot l\left(x_i, x_q\right)$$
$$i := q .. n$$
$$j := p + 1 .. q - 1$$
$$m_{i,j} := -h \cdot k\left(x_i, x_j\right) - h \cdot l\left(x_i, x_j\right)$$
$$m_{q,q} := 1 - h \cdot k\left(x_q, x_q\right) - \frac{h}{2} \cdot l\left(x_q, x_q\right)$$
$$i := q + 1 .. n$$
$$m_{i,q} := -h \cdot k\left(x_i, x_q\right) - \frac{h}{2} \cdot l\left(x_i, x_q\right)$$
$$i := p + 1 .. n$$
$$j := q + 1 .. n - 1$$
$$m_{i,j} := -h \cdot k\left(x_i, x_j\right)$$
$$i := q + 1 .. n - 1$$
$$m_{i,i} := 1 - h \cdot k\left(x_i, x_i\right)$$
$$m_{n,n} := 1 - \frac{h}{2} \cdot k\left(x_n, x_n\right)$$
$$i := p + 1 .. n - 1$$
$$m_{i,n} := \frac{-h}{2} \cdot k\left(x_i, x_n\right)$$

$$A = \begin{pmatrix}
1 & -0.01 & -0.02 & -0.03 & -0.048 & -0.075 & -0.078 & -0.07 & -0.08 & -0.09 & -0.05 \\
-0.005 & 0.98 & -0.03 & -0.04 & -0.0585 & -0.086 & -0.0885 & -0.08 & -0.09 & -0.1 & -0.055 \\
-0.01 & -0.03 & 0.96 & -0.05 & -0.07 & -0.099 & -0.1 & -0.09 & -0.1 & -0.11 & -0.06 \\
-0.015 & -0.04 & -0.05 & 0.94 & -0.0825 & -0.114 & -0.1125 & -0.1 & -0.11 & -0.12 & -0.065 \\
-0.02 & -0.05 & -0.06 & -0.07 & 0.904 & -0.131 & -0.126 & -0.11 & -0.12 & -0.13 & -0.7 \\
-0.025 & -0.06 & -0.07 & -0.08 & -0.1105 & 0.85 & -0.1405 & -0.12 & -0.13 & -0.14 & -0.075 \\
-0.03 & -0.07 & -0.08 & -0.09 & -0.126 & -0.171 & 0.844 & -0.13 & -0.14 & -0.15 & -0.08 \\
-0.035 & -0.08 & -0.09 & -0.1 & -0.1425 & -0.194 & -0.1725 & 0.86 & -0.15 & -0.16 & -0.085 \\
-0.04 & -0.09 & -0.1 & -0.11 & -0.16 & -0.219 & -0.19 & -0.15 & 0.84 & -0.17 & -0.09 \\
-0.045 & -0.1 & -0.11 & -0.12 & -0.1785 & -0.246 & -0.2085 & -0.16 & -0.17 & 0.82 & -0.095 \\
-0.05 & -0.11 & -0.12 & -0.13 & -0.198 & -0.275 & -0.228 & -0.17 & -0.18 & -0.19 & 0.9
\end{pmatrix}$$

$$f(x) := .94933333333333333333 \cdot x^2 - .26350400000000000000 - \frac{1}{3} \cdot x$$

$$i := 0 .. 10$$
$$f_i := f\left(x_i\right)$$

$u := m^{-1} \cdot f$

$i := 0 .. 10$

$ue_i := (x_i)^2$

ue=

	0
0	0
1	0.01
2	0.04
3	0.09
4	0.16
5	0.25
6	0.36
7	0.49
8	0.64
9	0.81
10	1

, u=

	0
0	-0.009105
1	-0.000852
2	0.02733
3	0.07545
4	0.1435
5	0.23148
6	0.3394
7	0.46725
8	0.61503
9	0.78274
10	0.97038

$n := 20$

$h := \frac{(b - a)}{n}$

$i := 0 .. 20$

$x_i := a + i \cdot h$

X=

	0
0	0
1	0.05
2	0.1
3	0.15
4	0.2
5	0.25
6	0.3
7	0.35
8	0.4
9	0.45
10	0.5
11	0.55
12	0.6
13	0.65
14	0.7
15	0.75
16	0.8
17	0.85
18	0.9
19	0.95
20	1

$p := 8$

$q := 12$

$i := 1 .. p - 1$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 1 .. p - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{0,0} := 1 - \frac{h}{2} \cdot k(x_0, x_0)$

$m_{p,p} := 1 - h \cdot k(x_p, x_p) - \frac{h}{2} \cdot l(x_p, x_p)$

$i := 1 .. p - 1$

$m_{0,i} := (-h) \cdot k(x_0, x_i)$

$i := 0 .. p - 1$

$m_{i,p} := -h \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$

$i := 1 .. p$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := 1 .. p - 1$

$m_{p,i} := (-h) \cdot k(x_p, x_i)$

$i := 0 .. p$

$j := p + 1 .. q - 1$

$m_{i,j} := (-h) \cdot k(x_i, x_j) - (h) \cdot l(x_i, x_j)$

$i := 0 .. p$

$m_{i,q} := (-h) \cdot k(x_i, x_q) - \left(\frac{h}{2}\right) \cdot l(x_i, x_q)$

$i := 0 .. p$

$j := q + 1 .. n - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 0 .. p$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$i := p + 1 .. n$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := p + 1 .. n$

$m_{i,0} := \frac{-h}{2}\cdot k(x_i, x_0)$

$i := p + 1 .. n$

$m_{i,p} := (-h)\cdot k(x_i, x_p) - \frac{h}{2}\cdot l(x_i, x_p)$

$i := p + 1 .. q - 1$

$j := p + 1 .. q - 1$

$m_{i,j} := -h\cdot k(x_i, x_j) - h\cdot l(x_i, x_j)$

$i := p + 1 .. q - 1$

$m_{i,i} := 1 - h\cdot k(x_i, x_i) - h\cdot l(x_i, x_i)$

$i := p + 1 .. q - 1$

$m_{i,q} := -h\cdot k(x_i, x_q) - \frac{h}{2}\cdot l(x_i, x_q)$

$i := q .. n$

$j := p + 1 .. q - 1$

$m_{i,j} := -h\cdot k(x_i, x_j) - h\cdot l(x_i, x_j)$

$m_{q,q} := 1 - h\cdot k(x_q, x_q) - \frac{h}{2}\cdot l(x_q, x_q)$

$i := q + 1 .. n$

$m_{i,q} := -h\cdot k(x_i, x_q) - \frac{h}{2}\cdot l(x_i, x_q)$

$i := p + 1 .. n$

$j := q + 1 .. n - 1$

$m_{i,j} := -h\cdot k(x_i, x_j)$

$i := q + 1 .. n - 1$

$m_{i,i} := 1 - h\cdot k(x_i, x_i)$

$m_{n,n} := 1 - \frac{h}{2}\cdot k(x_n, x_n)$

$i := p + 1 .. n - 1$

$m_{i,n} := \frac{-h}{2}\cdot k(x_i, x_n)$

$f(x) := .949333333333333333333\cdot x^2 - .263504000000000000000 - \frac{1}{3}\cdot x$

$i := 0 .. 20$

$f_i := f(x_i)$

$u := m^{-1}\cdot f$

$i := 0 .. 20$

$ue_i := (x_i)^2$

ue=

	0
0	0
1	0.00251
2	0.01
3	0.0225
4	0.04
5	0.0625
6	0.09
7	0.1225
8	0.16
9	0.2025
10	0.25
11	0.3025
12	0.36
13	0.4225
14	0.49
15	0.5625
16	0.64
17	0.7225
18	0.81
19	0.9025
20	1

, u=

	0
0	-0.00229
1	-0.000016
2	-0.00726
3	0.01953
4	0.0368
5	0.05907
6	0.08633
7	0.11858
8	0.15583
9	0.19808
10	0.24532
11	0.29756
12	0.3548
13	0.41703
14	0.48425
15	0.55648
16	0.63369
17	0.71591
18	0.80311
19	0.89532
20	0.99252

$n := 30$

$h := \frac{(b - a)}{n}$

$i := 0 .. 30$

$x_i := a + i \cdot h$

x=

	0
0	0
1	0.03333
2	0.06667
3	0.1
4	0.13333
5	0.16667
6	0.2
7	0.23333
8	0.26667
9	0.3
10	0.33333
11	0.36667
12	0.4
13	0.43333
14	0.46667
15	0.5
16	0.53333
17	0.56667
18	0.6
19	0.63333
20	0.66667
21	0.7
22	0.73333
23	0.76667
24	0.8
25	0.83333
26	0.86667
27	0.9
28	0.93333
29	0.96667
30	1

$p := 12$

$q := 18$

$i := 1 .. p - 1$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 1 .. p - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{0,0} := 1 - \frac{h}{2} \cdot k(x_0, x_0)$

$m_{p,p} := 1 - h \cdot k(x_p, x_p) - \frac{h}{2} \cdot l(x_p, x_p)$

$i := 1 .. p - 1$

$m_{0,i} := (-h) \cdot k(x_0, x_i)$

$i := 0 .. p - 1$

$m_{i,p} := -h \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$

$i := 1 .. p$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := 1 .. p - 1$

$m_{p,i} := (-h) \cdot k(x_p, x_i)$

$i := 0 .. p$

$j := p + 1 .. q - 1$

$m_{i,j} := (-h) \cdot k(x_i, x_j) - (h) \cdot l(x_i, x_j)$

$i := 0 .. p$

$m_{i,q} := (-h) \cdot k(x_i, x_q) - \left(\frac{h}{2}\right) \cdot l(x_i, x_q)$

$i := 0 .. p$

$j := q + 1 .. n - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 0 .. p$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$i := p + 1 .. n$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := p + 1 .. n$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := p + 1 .. n$

$m_{i,p} := (-h) \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$

$i := p + 1 .. q - 1$

$j := p + 1 .. q - 1$

$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$

$i := p + 1 .. q - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i) - h \cdot l(x_i, x_i)$

$i := p + 1 .. q - 1$

$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$

$i := q .. n$

$j := p + 1 .. q - 1$

$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$

$m_{q,q} := 1 - h \cdot k(x_q, x_q) - \frac{h}{2} \cdot l(x_q, x_q)$

$i := q + 1 .. n$

$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$

$i := p + 1 .. n$

$j := q + 1 .. n - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := q + 1 .. n - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{n,n} := 1 - \frac{h}{2} \cdot k(x_n, x_n)$

$i := p + 1 .. n - 1$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$f(x) := .94933333333333333333 \cdot x^2 - .26350400000000000000 - \frac{1}{3} \cdot x$

$i := 0 .. 30$

$f_i := f(x_i)$

$u := m^{-1} \cdot f$

$i := 0 .. 30$

$ue_i := (x_i)^2$

ue=

	0
0	0
1	0.00111
2	0.00444
3	0.01
4	0.01778
5	0.02778
6	0.04
7	0.05444
8	0.07111
9	0.09
10	0.11111
11	0.13444
12	0.16
13	0.18778
14	0.21778
15	0.25
16	0.28444
17	0.32111
18	0.36
19	0.40111
20	0.44444
21	0.49
22	0.53778
23	0.58778
24	0.64
25	0.69444
26	0.75111
27	0.81
28	0.87111
29	0.93444
30	1

, u=

	0
0	-0.00122
1	0.00002
2	0.00329
3	0.00878
4	0.01649
5	0.02642
6	0.03858
7	0.05295
8	0.06955
9	0.08836
10	0.1094
11	0.13266
12	0.15814
13	0.18585
14	0.21577
15	0.24792
16	0.28229
17	0.31887
18	0.35768
19	0.39871
20	0.44197
21	0.48744
22	0.53541
23	0.58505
24	0.63719
25	0.69155
26	0.74813
27	0.80693
28	0.86796
29	0.9312
30	0.99667

$n := 40$

$h := \frac{(b - a)}{n}$

$i := 0 .. 40$

$x_i := a + i \cdot h$

x=

	0
0	0
1	0.025
2	0.05
3	0.075
4	0.1
5	0.125
6	0.15
7	0.175
8	0.2
9	0.225
10	0.25
11	0.275
12	0.3
13	0.325
14	0.35
15	0.375
16	0.4

17	0.425
18	0.45
19	0.475
20	0.5
21	0.525
22	0.55
23	0.575
24	0.6
25	0.625
26	0.65
27	0.675
28	0.7
29	0.725
30	0.75
31	0.775
32	0.8
33	0.825
34	0.85
35	0.875
36	0.9
37	0.925
38	0.95
39	0.975
40	1

$p := 16$

$q := 24$

$i := 1 .. p - 1$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 1 .. p - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{0,0} := 1 - \frac{h}{2} \cdot k(x_0, x_0)$

$m_{p,p} := 1 - h \cdot k(x_p, x_p) - \frac{h}{2} \cdot l(x_p, x_p)$

$i := 1 .. p - 1$

$m_{0,i} := (-h) \cdot k(x_0, x_i)$

$i := 0 .. p - 1$

$m_{i,p} := -h \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$

$i := 1 .. p$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := 1 .. p - 1$

$m_{p,i} := (-h) \cdot k(x_p, x_i)$

$i := 0 .. p$

$j := p + 1 .. q - 1$

$m_{i,j} := (-h)\cdot k(x_i, x_j) - (h)\cdot l(x_i, x_j)$

$i := 0 .. p$

$m_{i,q} := (-h)\cdot k(x_i, x_q) - \left(\frac{h}{2}\right)\cdot l(x_i, x_q)$

$i := 0 .. p$

$j := q + 1 .. n - 1$

$m_{i,j} := -h\cdot k(x_i, x_j)$

$i := 0 .. p$

$m_{i,n} := \frac{-h}{2}\cdot k(x_i, x_n)$

$i := p + 1 .. n$

$j := 1 .. p - 1$

$m_{i,j} := -h\cdot k(x_i, x_j)$

$i := p + 1 .. n$

$m_{i,0} := \frac{-h}{2}\cdot k(x_i, x_0)$

$i := p + 1 .. n$

$m_{i,p} := (-h)\cdot k(x_i, x_p) - \frac{h}{2}\cdot l(x_i, x_p)$

$i := p + 1 .. q - 1$

$j := p + 1 .. q - 1$

$m_{i,j} := -h\cdot k(x_i, x_j) - h\cdot l(x_i, x_j)$

$i := p + 1 .. q - 1$

$m_{i,i} := 1 - h\cdot k(x_i, x_i) - h\cdot l(x_i, x_i)$

$i := p + 1 .. q - 1$

$m_{i,q} := -h\cdot k(x_i, x_q) - \frac{h}{2}\cdot l(x_i, x_q)$

$i := q .. n$

$j := p + 1 .. q - 1$

$m_{i,j} := -h\cdot k(x_i, x_j) - h\cdot l(x_i, x_j)$

$m_{q,q} := 1 - h\cdot k(x_q, x_q) - \frac{h}{2}\cdot l(x_q, x_q)$

$i := q + 1 .. n$

$m_{i,q} := -h\cdot k(x_i, x_q) - \frac{h}{2}\cdot l(x_i, x_q)$

$i := p + 1 .. n$

$j := q + 1 .. n - 1$

$m_{i,j} := -h\cdot k(x_i, x_j)$

$i := q + 1 .. n - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{n,n} := 1 - \frac{h}{2} \cdot k(x_n, x_n)$

$i := p + 1 .. n - 1$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$f(x) := .94933333333333333333 \cdot x^2 - .26350400000000000000 - \frac{1}{3} \cdot x$

$i := 0 .. 40$

$f_i := f(x_i)$

$u := m^{-1} \cdot f$

$i := 0 .. 40$

$ue_i := (x_i)^2$

ue=

	0
0	0
1	0.000625000
2	0.00250000
3	0.00562500
4	0.01000
5	0.01563
6	0.02250
7	0.03063
8	0.04000
9	0.05063
10	0.06250
11	0.07563
12	0.09000
13	0.10563
14	0.12250
15	0.14063
16	0.16000
17	0.18063
18	0.20250
19	0.22563
20	0.25000
21	0.27563
22	0.30250
23	0.33063
24	0.36000
25	0.39063
26	0.42250
27	0.45563
28	0.49000
29	0.52563
30	0.56250
31	0.60063
32	0.64000
33	0.68063
34	0.72250
35	0.76563
36	0.81000
37	0.85563
38	0.90250
39	0.95063
40	1.00000

u=

	0
0	-0.000575706
1	0.0000220206
2	0.00186948
3	0.00496667
4	0.00931360
5	0.01491
6	0.02176
7	0.02985
8	0.03920
9	0.04979
10	0.06164
11	0.07473
12	0.08908
13	0.10467
14	0.12152
15	0.13961
16	0.15896
17	0.17955
18	0.20139
19	0.22449
20	0.24883
21	0.27442
22	0.30126
23	0.32935
24	0.35870
25	0.38929
26	0.42113
27	0.45422
28	0.48856
29	0.52415
30	0.56099
31	0.59908
32	0.63842
33	0.67901
34	0.72085
35	0.76394
36	0.80827
37	0.85386
38	0.90070
39	0.94879
40	0.99813

$n := 50$

$h := \frac{(b - a)}{n}$

$i := 0 .. 50$

$x_i := a + i \cdot h$

x=

	0
0	0

1	0.02
2	0.04
3	0.06
4	0.08
5	0.1
6	0.12
7	0.14
8	0.16
9	0.18
10	0.2
11	0.22
12	0.24
13	0.26
14	0.28
15	0.3
16	0.32
17	0.34
18	0.36
19	0.38
20	0.4
21	0.42
22	0.44
23	0.46
24	0.48
25	0.5
26	0.52
27	0.54
28	0.56
29	0.58
30	0.6
31	0.62
32	0.64
33	0.66
34	0.68
35	0.7
36	0.72
37	0.74
38	0.76
39	0.78
40	0.8
41	0.82
42	0.84
43	0.86
44	0.88
45	0.9
46	0.92
47	0.94
48	0.96
49	0.98
50	1

$p := 20$

$q := 30$

$i := 1 .. p - 1$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k\left(x_i, x_j\right)$

$i := 1 .. p - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{0,0} := 1 - \frac{h}{2} \cdot k(x_0, x_0)$

$m_{p,p} := 1 - h \cdot k(x_p, x_p) - \frac{h}{2} \cdot l(x_p, x_p)$

$i := 1 .. p - 1$

$m_{0,i} := (-h) \cdot k(x_0, x_i)$

$i := 0 .. p - 1$

$m_{i,p} := -h \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$

$i := 1 .. p$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := 1 .. p - 1$

$m_{p,i} := (-h) \cdot k(x_p, x_i)$

$i := 0 .. p$

$j := p + 1 .. q - 1$

$m_{i,j} := (-h) \cdot k(x_i, x_j) - (h) \cdot l(x_i, x_j)$

$i := 0 .. p$

$m_{i,q} := (-h) \cdot k(x_i, x_q) - \left(\frac{h}{2}\right) \cdot l(x_i, x_q)$

$i := 0 .. p$

$j := q + 1 .. n - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 0 .. p$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$i := p + 1 .. n$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := p + 1 .. n$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := p + 1 .. n$

$m_{i,p} := (-h) \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$

$i := p + 1 .. q - 1$

$j := p + 1 .. q - 1$

$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$

$i := p + 1 .. q - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i) - h \cdot l(x_i, x_i)$

$i := p + 1 .. q - 1$

$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$

$i := q .. n$

$j := p + 1 .. q - 1$

$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$

$m_{q,q} := 1 - h \cdot k(x_q, x_q) - \frac{h}{2} \cdot l(x_q, x_q)$

$i := q + 1 .. n$

$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$

$i := p + 1 .. n$

$j := q + 1 .. n - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := q + 1 .. n - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$m_{n,n} := 1 - \frac{h}{2} \cdot k(x_n, x_n)$

$i := p + 1 .. n - 1$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$f(x) := .949333333333333333333 \cdot x^2 - .263504000000000000000 - \frac{1}{3} \cdot x$

$i := 0 .. 50$

$f_i := f(x_i)$

$u := m^{-1} \cdot f$

$i := 0 .. 50$

$ue_i := (x_i)^2$

ue=

	0
0	0.00000
1	0.00040
2	0.00160
3	0.00360
4	0.00640
5	0.01000
6	0.01440
7	0.01960
8	0.02560

9	0.03240
10	0.04000
11	0.04840
12	0.05760
13	0.06760
14	0.07840
15	0.09000
16	0.10240
17	0.11560
18	0.12960
19	0.14440
20	0.16000
21	0.17640
22	0.19360
23	0.21160
24	0.23040
25	0.25000
26	0.27040
27	0.29160
28	0.31360
29	0.33640
30	0.36000
31	0.38440
32	0.40960
33	0.43560
34	0.46240
35	0.49000
36	0.51840
37	0.54760
38	0.57760
39	0.60840
40	0.64000
41	0.67240
42	0.70560
43	0.73960
44	0.77440
45	0.81000
46	0.84640
47	0.88360
48	0.92160
49	0.96040
50	1.00000

u=

	0
0	-0.00036855
1	0.0000174901
2	0.00120343
3	0.00318925
4	0.00597497
5	0.00956058
6	0.01395
7	0.01913
8	0.02512
9	0.03190
10	0.03949
11	0.04787
12	0.05706

13	0.06704
14	0.07783
15	0.08941
16	0.10180
17	0.11498
18	0.12896
19	0.14375
20	0.15933
21	0.17572
22	0.19290
23	0.21088
24	0.22967
25	0.24925
26	0.26963
27	0.29082
28	0.31280
29	0.33558
30	0.35917
31	0.38355
32	0.40873
33	0.43471
34	0.46150
35	0.48908
36	0.51746
37	0.54664
38	0.57662
39	0.60741
40	0.63899
41	0.67137
42	0.70455
43	0.73853
44	0.77331
45	0.80890
46	0.84528
47	0.88246
48	0.92044
49	0.95922
50	0.99880

$n := 100$

$h := \frac{(b - a)}{n}$

$i := 0 .. 10($

$x_i := a + i \cdot h$

x=

	0
1	0.01
2	0.02
3	0.03
4	0.04
5	0.05
6	0.06
7	0.07
8	0.08
9	0.09

10	0.1
11	0.11
12	0.12
13	0.13
14	0.14
15	0.15
16	0.16
17	0.17
18	0.18
19	0.19
20	0.2
21	0.21
22	0.22
23	0.23
24	0.24
25	0.25
26	0.26
27	0.27
28	0.28
29	0.29
30	0.3
31	0.31
32	0.32
33	0.33
34	0.34
35	0.35
36	0.36
37	0.37
38	0.38
39	0.39
40	0.4
41	0.41
42	0.42
43	0.43
44	0.44
45	0.45
46	0.46
47	0.47
48	0.48
49	0.49
50	0.5
51	0.51
52	0.52
53	0.53
54	0.54
55	0.55
56	0.56
57	0.57
58	0.58
59	0.59
60	0.6
61	0.61
62	0.62
63	0.63
64	0.64
65	0.65
66	0.66
67	0.67
68	0.68

69	0.69
70	0.7
71	0.71
72	0.72
73	0.73
74	0.74
75	0.75
76	0.76
77	0.77
78	0.78
79	0.79
80	0.8
81	0.81
82	0.82
83	0.83
84	0.84
85	0.85
86	0.86
87	0.87
88	0.88
89	0.89
90	0.9
91	0.91
92	0.92
93	0.93
94	0.94
95	0.95
96	0.96
97	0.97
98	0.98
99	0.99
100	1

$p := 40$

$q := 60$

$i := 1 .. p - 1$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 1 .. p - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i)$

$$m_{0,0} := 1 - \frac{h}{2} \cdot k(x_0, x_0)$$

$$m_{p,p} := 1 - h \cdot k(x_p, x_p) - \frac{h}{2} \cdot l(x_p, x_p)$$

$i := 1 .. p - 1$

$m_{0,i} := (-h) \cdot k(x_0, x_i)$

$i := 0 .. p - 1$

$$m_{i,p} := -h \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$$

$i := 1 .. p$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := 1 .. p - 1$

$m_{p,i} := (-h) \cdot k(x_p, x_i)$

$i := 0 .. p$

$j := p + 1 .. q - 1$

$m_{i,j} := (-h) \cdot k(x_i, x_j) - (h) \cdot l(x_i, x_j)$

$i := 0 .. p$

$m_{i,q} := (-h) \cdot k(x_i, x_q) - \left(\frac{h}{2}\right) \cdot l(x_i, x_q)$

$i := 0 .. p$

$j := q + 1 .. n - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := 0 .. p$

$m_{i,n} := \frac{-h}{2} \cdot k(x_i, x_n)$

$i := p + 1 .. n$

$j := 1 .. p - 1$

$m_{i,j} := -h \cdot k(x_i, x_j)$

$i := p + 1 .. n$

$m_{i,0} := \frac{-h}{2} \cdot k(x_i, x_0)$

$i := p + 1 .. n$

$m_{i,p} := (-h) \cdot k(x_i, x_p) - \frac{h}{2} \cdot l(x_i, x_p)$

$i := p + 1 .. q - 1$

$j := p + 1 .. q - 1$

$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$

$i := p + 1 .. q - 1$

$m_{i,i} := 1 - h \cdot k(x_i, x_i) - h \cdot l(x_i, x_i)$

$i := p + 1 .. q - 1$

$m_{i,q} := -h \cdot k(x_i, x_q) - \frac{h}{2} \cdot l(x_i, x_q)$

$i := q .. n$

$j := p + 1 .. q - 1$

$m_{i,j} := -h \cdot k(x_i, x_j) - h \cdot l(x_i, x_j)$

$m_{q,q} := 1 - h \cdot k(x_q, x_q) - \frac{h}{2} \cdot l(x_q, x_q)$

$i := q + 1 .. n$

$m_{i,q} := -h\cdot k(x_i, x_q) - \frac{h}{2}\cdot l(x_i, x_q)$

$i := p + 1 .. n$

$j := q + 1 .. n - 1$

$m_{i,j} := -h\cdot k(x_i, x_j)$

$i := q + 1 .. n - 1$

$m_{i,i} := 1 - h\cdot k(x_i, x_i)$

$m_{n,n} := 1 - \frac{h}{2}\cdot k(x_n, x_n)$

$i := p + 1 .. n - 1$

$m_{i,n} := \frac{-h}{2}\cdot k(x_i, x_n)$

$f(x) := .9493333333333333333\cdot x^2 - .26350400000000000000 - \frac{1}{3}\cdot x$

$i := 0 .. 10($

$f_i := f(x_i)$

$u := m^{-1}\cdot f$

$i := 0 .. 100$

$ue_i := (x_i)^2$

ue=

	0
0	0
1	0.00010
2	0.00040
3	0.00090
4	0.00160
5	0.00250
6	0.00360
7	0.00490
8	0.00640
9	0.00810
10	0.01000
11	0.01210
12	0.01440
13	0.01690
14	0.01960
15	0.02250
16	0.02560
17	0.02890
18	0.03240
19	0.03610
20	0.04000
21	0.04410
22	0.04840
23	0.05290
24	0.05760
25	0.06250

26	0.06760
27	0.07290
28	0.07840
29	0.08410
30	0.09000
31	0.09610
32	0.10240
33	0.10890
34	0.11560
35	0.12250
36	0.12960
37	0.13690
38	0.14440
39	0.15210
40	0.16000
41	0.16810
42	0.17640
43	0.18490
44	0.19360
45	0.20250
46	0.21160
47	0.22090
48	0.23040
49	0.24010
50	0.25000
51	0.26010
52	0.27040
53	0.28090
54	0.29160
55	0.30250
56	0.31360
57	0.32490
58	0.33640
59	0.34810
60	0.36000
61	0.37210
62	0.38440
63	0.39690
64	0.40960
65	0.42250
66	0.43560
67	0.44890
68	0.46240
69	0.47610
70	0.49000
71	0.50410
72	0.51840
73	0.53290
74	0.54760
75	0.56250
76	0.57760
77	0.59290
78	0.60840
79	0.62410
80	0.64000
81	0.65610
82	0.67240
83	0.68890
84	0.70560

85	0.72250
86	0.73960
87	0.75690
88	0.77440
89	0.79210
90	0.81000
91	0.82810
92	0.84640
93	0.86490
94	0.88360
95	0.90250
96	0.92160
97	0.94090
98	0.96040
99	0.98010
100	1

u=

	0
0	0
1	-0.0000921732
2	0.00000608512
3	0.000304337
4	0.000802581
5	0.00150082
6	0.00239905
7	0.00349727
8	0.00479549
9	0.00629370
10	0.00799191
11	0.00989010
12	0.01199
13	0.01429
14	0.01678
15	0.01948
16	0.02238
17	0.02548
18	0.02878
19	0.03228
20	0.03597
21	0.03987
22	0.04397
23	0.04827
24	0.05277
25	0.05746
26	0.06236
27	0.06746
28	0.07276
29	0.07826
30	0.08395
31	0.08985
32	0.09595
33	0.10225
34	0.10875
35	0.11544
36	0.12234
37	0.12944
38	0.13674

39	0.14424
40	0.15193
41	0.15983
42	0.16793
43	0.17623
44	0.18473
45	0.19342
46	0.20232
47	0.21142
48	0.22072
49	0.23022
50	0.23991
51	0.24981
52	0.25991
53	0.27021
54	0.28071
55	0.29140
56	0.30230
57	0.31340
58	0.32470
59	0.33620
60	0.34789
61	0.35979
62	0.37189
63	0.38419
64	0.39668
65	0.40938
66	0.42228
67	0.43538
68	0.44868
69	0.46217
70	0.47587
71	0.48977
72	0.50387
73	0.51816
74	0.53266
75	0.54736
76	0.56226
77	0.57736
78	0.59265
79	0.60815
80	0.62385
81	0.63975
82	0.65584
83	0.67214
84	0.68864
85	0.70534
86	0.72224
87	0.73933
88	0.75663
89	0.77413
90	0.79183
91	0.80972
92	0.82782
93	0.84612
94	0.86462
95	0.88331
96	0.90221
97	0.92131

98	0.94061
99	0.96010
100	0.97980

Program (2.7):

$k(x,y) := x\cdot\frac{y}{5}$

$J(x,y) := \frac{d}{dy}k(x,y)$

$J(x,y) \rightarrow \frac{1}{5}\cdot x$

$H(x,y) := \frac{d}{dx}k(x,y)$

$H(x,y) \rightarrow \frac{1}{5}\cdot y$

$L1(x,y) := \frac{d}{dx}\left(\frac{d}{dy}k(x,y)\right)$

$f(x) := x$

$m(x) := 1$

$n := 20$

$L := 0$

$R := 2$

$h := \frac{(R-L)}{n}$

$i := 0..20$

$x_i := L + i\cdot h$

$u_0 := f(x_0)$

$$c_1 := 1 - \frac{h}{2}\cdot k(x_1,x_1) + \frac{h^2}{12}\cdot J(x_1,x_1) + \frac{h^2}{12 + h^2\cdot H(x_1,x_1)}\cdot\left[\left(\frac{h}{2}\right)\cdot H(x_1,x_1) - \frac{h^2}{12}\cdot L1(x_1,x_1) + k(x_1,x_1)\right]$$

$$z_1 := f(x_1) + \frac{h}{2}\cdot k(x_1,x_0) + \frac{h^2}{12}\cdot k(x_1,x_0)\cdot\left(1 + k(x_0,x_0)\cdot u_0\right) + \frac{h^2}{12}\cdot J(x_1,x_0)\cdot f(x_0)$$

$$d_1 := \frac{-h^2\cdot k(x_1,x_1)}{12 + h^2\cdot H(x_1,x_1)}\cdot\left[m(x_1) + \left(\frac{h}{2}\cdot H(x_1,x_0) + \frac{h^2}{12}\cdot H(x_1,x_0)\cdot k(x_0,x_0) + \frac{h^2}{12}\cdot L1(x_1,x_0)\right)f(x_0) + \frac{h^2}{12}\cdot H(x_1,x_0)\, m(x_0)\right]$$

$u_1 := \frac{1}{c_1}\cdot(z_1 + d_1)$

$$u_1 := \frac{1}{(c_1)^{1.}} \cdot (z_1 + d_1)$$

$$u_1 := \frac{1}{(c_1)^{1.}} \cdot (z_1 + d_1)$$

$$u_1 = 0.10001$$

$$i := 2..20$$

$$c_i := 1 - \frac{h}{2} \cdot k(x_i, x_i) + \frac{h^2}{12} \cdot J(x_i, x_i) + \frac{h^2}{12 + h^2 \cdot H(x_i, x_i)} \cdot \left[\left(\frac{h}{2}\right) \cdot H(x_i, x_i) - \frac{h^2}{12} \cdot L1(x_i, x_i) + k(x_i, x_i) \right]$$

$$z_i := f(x_i) + \frac{h}{2} \cdot k(x_i, x_0) \cdot f(x_0) + \frac{h^2}{12} \cdot k(x_i, x_0) \cdot (m(x_0) + k(x_0, x_0) \cdot u_0) + \frac{h^2}{12} \cdot J(x_i, x_0) \cdot f(x_0)$$

$$d_i := \frac{-h^2 \cdot k(x_i, x_i)}{12 + h^2 \cdot H(x_i, x_i)} \cdot \left[m(x_i) + \left(\frac{h}{2} \cdot H(x_i, x_0) + \frac{h^2}{12} \cdot H(x_i, x_0) \cdot k(x_0, x_0) + \frac{h^2}{12} \cdot L1(x_i, x_0) \right) f(x_0) + \frac{h^2}{12} \cdot H(x_i, x_0) m(x_0) \right]$$

$$u_i := \frac{1}{c_i} \cdot \left[z_i + h \cdot \sum_{j=1}^{i-1} k(x_i, x_j) \cdot u_j + d_i - \frac{h^2 \cdot k(x_i, x_i)}{12 + h^2 \cdot H(x_i, x_i)} \cdot \left(h \cdot \sum_{j=1}^{i-1} H(x_i, x_j) \cdot u_j \right) \right]$$

$$i := 0..20$$

$$ue_i := x_i \cdot \exp\left[\frac{(x_i)^3}{15} \right]$$

u=

	0
0	0.00000
1	0.10001
2	0.20011
3	0.30054
4	0.40170
5	0.50416
6	0.60867
7	0.71613
8	0.82769
9	0.94470
10	1.06878
11	1.20186
12	1.34626
13	1.50474
14	1.68064
15	1.87802
16	2.10183
17	2.35819

18	2.65467
19	3.00076
20	3.40842

ue=

	0
0	0.00000
1	0.10001
2	0.20011
3	0.30054
4	0.40171
5	0.50418
6	0.60870
7	0.71619
8	0.82778
9	0.94482
10	1.06894
11	1.20207
12	1.34652
13	1.50506
14	1.68103
15	1.87848
16	2.10238
17	2.35882
18	2.65538
19	3.00153
20	3.40921

Program (2.8):

$k(x,y) := x\cdot\frac{y}{5}$

$J(x,y) := \frac{d}{dy}k(x,y)$

$J(x,y) \rightarrow \frac{1}{5}\cdot x$

$H(x,y) := \frac{d}{dx}k(x,y)$

$H(x,y) \rightarrow \frac{1}{5}\cdot y$

$L1(x,y) := \frac{d}{dx}\left(\frac{d}{dy}k(x,y)\right)$

$f(x) := x$

$m(x) := 1$

$n := 20$

$L := 0$

$R := 2$

$h := \frac{(R-L)}{n}$

$i := 0..20$

$x_i := L + i \cdot h$

$u_0 := f(x_0)$

$$c_1 := 1 - \frac{h}{2} \cdot k(x_1, x_1) + \frac{h^2}{12} \cdot J(x_1, x_1) + \frac{h^2}{12 + h^2 \cdot H(x_1, x_1)} \cdot \left[\left(\frac{h}{2}\right) \cdot H(x_1, x_1) - \frac{h^2}{12} \cdot L1(x_1, x_1) + k(x_1, x_1)\right]$$

$$z_1 := f(x_1) + \frac{h}{2} \cdot k(x_1, x_0) + \frac{h^2}{12} \cdot k(x_1, x_0) \cdot (1 + k(x_0, x_0) \cdot u_0) + \frac{h^2}{12} \cdot J(x_1, x_0) \cdot f(x_0)$$

$$d_1 := \frac{-h^2 \cdot k(x_1, x_1)}{12 + h^2 \cdot H(x_1, x_1)} \cdot \left[m(x_1) + \left(\frac{h}{2} \cdot H(x_1, x_0) + \frac{h^2}{12} \cdot H(x_1, x_0) \cdot k(x_0, x_0) + \frac{h^2}{12} \cdot L1(x_1, x_0)\right) f(x_0) + \frac{h^2}{12} \cdot H(x_1, x_0) \, m(x_0)\right]$$

$$u_1 := \frac{1}{c_1} \cdot (z_1 + d_1)$$

$$u_1 := \frac{1}{(c_1)^{1.}} \cdot (z_1 + d_1)$$

$$u_1 := \frac{1}{(c_1)^{1.}} \cdot (z_1 + d_1)$$

$u_1 = 0.10001$

$i := 2..20$

$$c_i := 1 - \frac{h}{2} \cdot k(x_i, x_i) + \frac{h^2}{12} \cdot J(x_i, x_i) + \frac{h^2}{12 + h^2 \cdot H(x_i, x_i)} \cdot \left[\left(\frac{h}{2}\right) \cdot H(x_i, x_i) - \frac{h^2}{12} \cdot L1(x_i, x_i) + k(x_i, x_i)\right]$$

$$z_i := f(x_i) + \frac{h}{2} \cdot k(x_i, x_0) \cdot f(x_0) + \frac{h^2}{12} \cdot k(x_i, x_0) \cdot (m(x_0) + k(x_0, x_0) \cdot u_0) + \frac{h^2}{12} \cdot J(x_i, x_0) \cdot f(x_0)$$

$$d_i := \frac{-h^2 \cdot k(x_i, x_i)}{12 + h^2 \cdot H(x_i, x_i)} \cdot \left[m(x_i) + \left(\frac{h}{2} \cdot H(x_i, x_0) + \frac{h^2}{12} \cdot H(x_i, x_0) \cdot k(x_0, x_0) + \frac{h^2}{12} \cdot L1(x_i, x_0)\right) f(x_0) + \frac{h^2}{12} \cdot H(x_i, x_0) \, m(x_0)\right]$$

$$u_i := \frac{1}{c_i} \cdot \left[z_i + h \cdot \sum_{j=1}^{i-1} k(x_i, x_j) \cdot u_j + d_i - \frac{h^2 \cdot k(x_i, x_i)}{12 + h^2 \cdot H(x_i, x_i)} \cdot \left(h \cdot \sum_{j=1}^{i-1} H(x_i, x_j) \cdot u_j\right)\right]$$

$i := 0..20$

$$ue_i := x_i \cdot \exp\left[\frac{(x_i)^3}{15}\right]$$

u=

	0
0	0.00000
1	0.10001
2	0.20011
3	0.30054
4	0.40170
5	0.50416
6	0.60867
7	0.71613
8	0.82769
9	0.94470
10	1.06878
11	1.20186
12	1.34626
13	1.50474
14	1.68064
15	1.87802
16	2.10183
17	2.35819
18	2.65467
19	3.00076
20	3.40842

ue=

	0
0	0.00000
1	0.10001
2	0.20011
3	0.30054
4	0.40171
5	0.50418
6	0.60870
7	0.71619
8	0.82778
9	0.94482
10	1.06894
11	1.20207
12	1.34652
13	1.50506
14	1.68103
15	1.87848
16	2.10238
17	2.35882
18	2.65538
19	3.00153
20	3.40921

Program (3.1):

```
function[ZZ]=SMDFIET(k,g,ex,n,m,a,b,c,d);
k=inline('y+x+s+t','x','y','t','s');g=inline('y*x^2-(x+y)/6-17/72','x','y');
ex=inline('x^2*y','x','y');
a=0;b=1;c=0;d=1;n=10,m=10,
B=zeros(1,(n+1)*(m+1));
```

```
hx=(b-a)/n;hy=(d-c)/m;t=0;
for I=1:n+1
    for J=1:m+1
        x1(J,I)=a+(I-1)*hx;
        y1(J,I)=c+(J-1)*hy;
        ex1(J,I)=feval(ex,a+(I-1)*hx,c+(J-1)*hy);
        G1(J,I)=feval(g,a+(I-1)*hx,c+(J-1)*hy);
        t=t+1;
for i=1:n+1
      for j=1:m+1
          if i==I&j==J
             if i==1|i==n+1
                if j==1|j==m+1
                A(j,i)=1-(hx*hy)/4*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
                else
                A(j,i)=1-(hx*hy)/2*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
                end
             else
                if j==1|j==m+1
                A(j,i)=1-(hx*hy)/2*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
                else
                A(j,i)=1-(hx*hy)*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
                end
             end
          else
              if i==1|i==n+1
                 if j==1|j==m+1
                 A(j,i)=-(hx*hy)/4*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
                 else
                 A(j,i)=-(hx*hy)/2*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
                 end
              else
                 if j==1|j==m+1
                 A(j,i)=-(hx*hy)/2*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
                 else
                 A(j,i)=-(hx*hy)*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
                 end
             end
          end
      end
end
X=(x1(:));
Y=(y1(:));
B=(A(:))';
C(t,:)=B;
    end
end
EX=[ex1(:)];
K=[C]; G=[G1(:)];
F=inv(K)*G;   er=0;
for i=1:(n+1)*(m+1)
         e(i,1)=EX(i,1)-F(i,1);
          er=er+e(i,1)*e(i,1);
end
      ER=[e];LSR=er^(0.5)
      disp('       x       y       Exact    Approximation      Error')
```

ZZ=[X Y EX F ER];

n = 10, m = 10

LSR = 0.0329

x	y	Exact	Approximation	Error
0	0	0	-0.00083333333333	0.00083333333333
0	0.1000	0	-0.00103537511871	0.00103537511871
0	0.2000	0	-0.00123741690408	0.00123741690408
0	0.3000	0	-0.00143945868946	0.00143945868946
0	0.4000	0	-0.00164150047484	0.00164150047484
0	0.5000	0	-0.00184354226021	0.00184354226021
0	0.6000	0	-0.00204558404558	0.00204558404558
0	0.7000	0	-0.00224762583096	0.00224762583096
0	0.8000	0	-0.00244966761633	0.00244966761633
0	0.9000	0	-0.00265170940171	0.00265170940171
0	1.0000	0	-0.00285375118708	0.00285375118708
0.1000	0	0	-0.00103537511871	0.00103537511871
0.1000	0.1000	0.00100000000000	-0.00023741690408	0.00123741690408
0.1000	0.2000	0.00200000000000	0.00056054131054	0.00143945868946
0.1000	0.3000	0.00300000000000	0.00135849952517	0.00164150047483
0.1000	0.4000	0.00400000000000	0.00215645773979	0.00184354226021
0.1000	0.5000	0.00500000000000	0.00295441595441	0.00204558404559
0.1000	0.6000	0.00600000000000	0.00375237416904	0.00224762583096
0.1000	0.7000	0.00700000000000	0.00455033238367	0.00244966761633
0.1000	0.8000	0.00800000000000	0.00534829059829	0.00265170940171
0.1000	0.9000	0.00900000000000	0.00614624881292	0.00285375118708
0.1000	1.0000	0.01000000000000	0.00694420702754	0.00305579297246
0.2000	0	0	-0.00123741690408	0.00123741690408
0.2000	0.1000	0.00400000000000	0.00256054131054	0.00143945868946
0.2000	0.2000	0.00800000000000	0.00635849952517	0.00164150047483
0.2000	0.3000	0.01200000000000	0.01015645773979	0.00184354226021
0.2000	0.4000	0.01600000000000	0.01395441595441	0.00204558404559
0.2000	0.5000	0.02000000000000	0.01775237416904	0.00224762583096
0.2000	0.6000	0.02400000000000	0.02155033238367	0.00244966761633
0.2000	0.7000	0.02800000000000	0.02534829059829	0.00265170940171
0.2000	0.8000	0.03200000000000	0.02914624881291	0.00285375118709
0.2000	0.9000	0.03600000000000	0.03294420702754	0.00305579297246
0.2000	1.0000	0.04000000000000	0.03674216524216	0.00325783475784
0.3000	0	0	-0.00143945868946	0.00143945868946
0.3000	0.1000	0.00900000000000	0.00735849952516	0.00164150047484
0.3000	0.2000	0.01800000000000	0.01615645773979	0.00184354226021
0.3000	0.3000	0.02700000000000	0.02495441595442	0.00204558404558
0.3000	0.4000	0.03600000000000	0.03375237416904	0.00224762583096
0.3000	0.5000	0.04500000000000	0.04255033238367	0.00244966761633
0.3000	0.6000	0.05400000000000	0.05134829059829	0.00265170940171
0.3000	0.7000	0.06300000000000	0.06014624881292	0.00285375118708
0.3000	0.8000	0.07200000000000	0.06894420702754	0.00305579297246
0.3000	0.9000	0.08100000000000	0.07774216524216	0.00325783475784
0.3000	1.0000	0.09000000000000	0.08654012345679	0.00345987654321
0.4000	0	0	-0.00164150047483	0.00164150047483
0.4000	0.1000	0.01600000000000	0.01415645773979	0.00184354226021
0.4000	0.2000	0.03200000000000	0.02995441595442	0.00204558404558
0.4000	0.3000	0.04800000000000	0.04575237416904	0.00224762583096
0.4000	0.4000	0.06400000000000	0.06155033238367	0.00244966761633
0.4000	0.5000	0.08000000000000	0.07734829059829	0.00265170940171
0.4000	0.6000	0.09600000000000	0.09314624881291	0.00285375118709
0.4000	0.7000	0.11200000000000	0.10894420702754	0.00305579297246
0.4000	0.8000	0.12800000000000	0.12474216524216	0.00325783475784

0.4000	0.9000	0.14400000000000	0.14054012345679	0.00345987654321
0.4000	1.0000	0.16000000000000	0.15633808167141	0.00366191832859
0.5000	0	0	-0.00184354226021	0.00184354226021
0.5000	0.1000	0.02500000000000	0.02295441595442	0.00204558404558
0.5000	0.2000	0.05000000000000	0.04775237416904	0.00224762583096
0.5000	0.3000	0.07500000000000	0.07255033238366	0.00244966761634
0.5000	0.4000	0.10000000000000	0.09734829059829	0.00265170940171
0.5000	0.5000	0.12500000000000	0.12214624881292	0.00285375118708
0.5000	0.6000	0.15000000000000	0.14694420702754	0.00305579297246
0.5000	0.7000	0.17500000000000	0.17174216524216	0.00325783475784
0.5000	0.8000	0.20000000000000	0.19654012345679	0.00345987654321
0.5000	0.9000	0.22500000000000	0.22133808167142	0.00366191832858
0.5000	1.0000	0.25000000000000	0.24613603988604	0.00386396011396
0.6000	0	0	-0.00204558404559	0.00204558404559
0.6000	0.1000	0.03600000000000	0.03375237416904	0.00224762583096
0.6000	0.2000	0.07200000000000	0.06955033238366	0.00244966761634
0.6000	0.3000	0.10800000000000	0.10534829059830	0.00265170940170
0.6000	0.4000	0.14400000000000	0.14114624881292	0.00285375118708
0.6000	0.5000	0.18000000000000	0.17694420702754	0.00305579297246
0.6000	0.6000	0.21600000000000	0.21274216524216	0.00325783475784
0.6000	0.7000	0.25200000000000	0.24854012345680	0.00345987654320
0.6000	0.8000	0.28800000000000	0.28433808167142	0.00366191832858
0.6000	0.9000	0.32400000000000	0.32013603988604	0.00386396011396
0.6000	1.0000	0.36000000000000	0.35593399810067	0.00406600189933
0.7000	0	0	-0.00224762583096	0.00224762583096
0.7000	0.1000	0.04900000000000	0.04655033238367	0.00244966761633
0.7000	0.2000	0.09800000000000	0.09534829059829	0.00265170940171
0.7000	0.3000	0.14700000000000	0.14414624881292	0.00285375118708
0.7000	0.4000	0.19600000000000	0.19294420702754	0.00305579297246
0.7000	0.5000	0.24500000000000	0.24174216524216	0.00325783475784
0.7000	0.6000	0.29400000000000	0.29054012345679	0.00345987654321
0.7000	0.7000	0.34300000000000	0.33933808167141	0.00366191832859
0.7000	0.8000	0.39200000000000	0.38813603988604	0.00386396011396
0.7000	0.9000	0.44100000000000	0.43693399810066	0.00406600189934
0.7000	1.0000	0.49000000000000	0.48573195631529	0.00426804368471
0.8000	0	0	-0.00244966761633	0.00244966761633
0.8000	0.1000	0.06400000000000	0.06134829059829	0.00265170940171
0.8000	0.2000	0.12800000000000	0.12514624881292	0.00285375118708
0.8000	0.3000	0.19200000000000	0.18894420702754	0.00305579297246
0.8000	0.4000	0.25600000000000	0.25274216524216	0.00325783475784
0.8000	0.5000	0.32000000000000	0.31654012345679	0.00345987654321
0.8000	0.6000	0.38400000000000	0.38033808167142	0.00366191832858
0.8000	0.7000	0.44800000000000	0.44413603988604	0.00386396011396
0.8000	0.8000	0.51200000000000	0.50793399810066	0.00406600189934
0.8000	0.9000	0.57600000000000	0.57173195631529	0.00426804368471
0.8000	1.0000	0.64000000000000	0.63552991452991	0.00447008547009
0.9000	0	0	-0.00265170940171	0.00265170940171
0.9000	0.1000	0.08100000000000	0.07814624881292	0.00285375118708
0.9000	0.2000	0.16200000000000	0.15894420702754	0.00305579297246
0.9000	0.3000	0.24300000000000	0.23974216524217	0.00325783475783
0.9000	0.4000	0.32400000000000	0.32054012345679	0.00345987654321
0.9000	0.5000	0.40500000000000	0.40133808167141	0.00366191832859
0.9000	0.6000	0.48600000000000	0.48213603988604	0.00386396011396
0.9000	0.7000	0.56700000000000	0.56293399810066	0.00406600189934
0.9000	0.8000	0.64800000000000	0.64373195631529	0.00426804368471
0.9000	0.9000	0.72900000000000	0.72452991452992	0.00447008547008
0.9000	1.0000	0.81000000000000	0.80532787274454	0.00467212725546
1.0000	0	0	-0.00285375118709	0.00285375118709
1.0000	0.1000	0.10000000000000	0.09694420702754	0.00305579297246
1.0000	0.2000	0.20000000000000	0.19674216524216	0.00325783475784
1.0000	0.3000	0.30000000000000	0.29654012345679	0.00345987654321
1.0000	0.4000	0.40000000000000	0.39633808167142	0.00366191832858

1.0000	0.5000	0.50000000000000	0.49613603988604	0.00386396011396
1.0000	0.6000	0.60000000000000	0.59593399810067	0.00406600189933
1.0000	0.7000	0.70000000000000	0.69573195631529	0.00426804368471
1.0000	0.8000	0.80000000000000	0.79552991452992	0.00447008547008
1.0000	0.9000	0.90000000000000	0.89532787274454	0.00467212725546
1.0000	1.0000	1.00000000000000	0.99512583095916	0.00487416904084

Program (3.2):

```
function[ZZ]=SFMDFIESA(k,g,ex,n,m,a,b,c,d);
k=inline('(x*s+t*y)','x','y','t','s');
g=inline('exp(y+x)+(1-exp(1))*(x+y)','x','y');
ex=inline('exp(y+x)','x','y');
a=0;b=1;c=0;d=1;n=10,m=10,
B=zeros(1,(n+1)*(m+1));
C=zeros((n+1)*(m+1),(n+1)*(m+1));
hx=(b-a)/n;hy=(d-c)/m;t=0;
if rem(n,2)==0|rem(m,2)==0
for I=1:n+1
  for J=1:m+1
    x1(J,I)=a+(I-1)*hx;
    y1(J,I)=c+(J-1)*hy;
    ex1(J,I)=feval(ex,a+(I-1)*hx,c+(J-1)*hy);
    G1(J,I)=feval(g,a+(I-1)*hx,c+(J-1)*hy);
    t=t+1;
for i=1:n+1
   for j=1:m+1
     if i==I&j==J
       if i==1|i==n+1
         if j==1|j==m+1
         A(j,i)=1-(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
         elseif rem(j,2)==0
         A(j,i)=1-4*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
         else
         A(j,i)=1-2*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
         end
       elseif rem(i,2)==0
         if j==1|j==m+1
         A(j,i)=1-4*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
         elseif rem(j,2)==0
         A(j,i)=1-16*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
         else
         A(j,i)=1-8*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
         end
       else
         if j==1|j==m+1
         A(j,i)=1-2*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
         elseif rem(j,2)==0
         A(j,i)=1-8*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
         else
         A(j,i)=1-4*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
         end
       end
     else
        if i==1|i==n+1
```

```
            if j==1|j==m+1
            A(j,i)=-(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
            elseif rem(j,2)==0
            A(j,i)=-4*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
            else
            A(j,i)=-2*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
            end
         elseif rem(i,2)==0
            if j==1|j==m+1
            A(j,i)=-4*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
            elseif rem(j,2)==0
            A(j,i)=-16*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
            else
            A(j,i)=-8*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
            end
         else
            if j==1|j==m+1
            A(j,i)=-2*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
            elseif rem(j,2)==0
            A(j,i)=-8*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
            else
            A(j,i)=-4*(hx*hy)/9*feval(k,a+(I-1)*hx,c+(J-1)*hy,a+(i-1)*hx,c+(j-1)*hy);
            end
         end
       end
   end
end
X=(x1(:));
Y=(y1(:));
B=(A(:))';
C(t,:)=B;
   end
end
EX=[ex1(:)]; K=[C]; G=[G1(:)];
F=inv(K)*G; er=0;
  for i=1:(n+1)*(m+1)
     e(i,1)=EX(i,1)-F(i,1);
     er=er+e(i,1)*e(i,1);
  end
  ER=[e];LSR=er^(0.5)
  disp('      x       y      Exact    Approximation.      Error')
  ZZ=[X Y EX F ER];
end
```

n =   10, m =   10

LSR = 2.4455e-004

x	y	Exact	Approximation	Error $\times 10^{-4}$
0	0	1.00000000000000	1.00000000000000	0
0	0.1000	1.10517091807565	1.10517294759489	-0.02029519242885
0	0.2000	1.22140275816017	1.22140681719865	-0.04059038483550
0	0.3000	1.34985880757600	1.34986489613373	-0.06088557727768
0	0.4000	1.49182469764127	1.49183281571824	-0.08118076970876
0	0.5000	1.64872127070013	1.64873141829634	-0.10147596211540

0	0.6000	1.82211880039051	1.82213097750597	-0.12177115456202
0	0.7000	2.01375270747048	2.01376691410518	-0.14206634698866
0	0.8000	2.22554092849247	2.22555716464641	-0.16236153941307
0	0.9000	2.45960311115695	2.45962137683013	-0.18265673184636
0	1.0000	2.71828182845905	2.71830212365147	-0.20295192423969
0.1000	0	1.10517091807565	1.10517294759489	-0.02029519241997
0.1000	0.1000	1.22140275816017	1.22140681719866	-0.04059038486215
0.1000	0.2000	1.34985880757600	1.34986489613373	-0.06088557727768
0.1000	0.3000	1.49182469764127	1.49183281571824	-0.08118076969765
0.1000	0.4000	1.64872127070013	1.64873141829634	-0.10147596213539
0.1000	0.5000	1.82211880039051	1.82213097750596	-0.12177115455536
0.1000	0.6000	2.01375270747048	2.01376691410517	-0.14206634697977
0.1000	0.7000	2.22554092849247	2.22555716464641	-0.16236153940863
0.1000	0.8000	2.45960311115695	2.45962137683013	-0.18265673183748
0.1000	0.9000	2.71828182845905	2.71830212365147	-0.20295192426190
0.1000	1.0000	3.00416602394643	3.00418834865810	-0.22324711668631
0.2000	0	1.22140275816017	1.22140681719866	-0.04059038485105
0.2000	0.1000	1.34985880757600	1.34986489613373	-0.06088557727546
0.2000	0.2000	1.49182469764127	1.49183281571824	-0.08118076969987
0.2000	0.3000	1.64872127070013	1.64873141829634	-0.10147596211985
0.2000	0.4000	1.82211880039051	1.82213097750596	-0.12177115455092
0.2000	0.5000	2.01375270747048	2.01376691410517	-0.14206634697089
0.2000	0.6000	2.22554092849247	2.22555716464641	-0.16236153942195
0.2000	0.7000	2.45960311115695	2.45962137683013	-0.18265673184192
0.2000	0.8000	2.71828182845905	2.71830212365147	-0.20295192426190
0.2000	0.9000	3.00416602394643	3.00418834865810	-0.22324711666855
0.2000	1.0000	3.32011692273655	3.32014127696746	-0.24354230911516
0.3000	0	1.34985880757600	1.34986489613373	-0.06088557727768
0.3000	0.1000	1.49182469764127	1.49183281571824	-0.08118076972208
0.3000	0.2000	1.64872127070013	1.64873141829634	-0.10147596213761
0.3000	0.3000	1.82211880039051	1.82213097750596	-0.12177115454204
0.3000	0.4000	2.01375270747048	2.01376691410518	-0.14206634698422
0.3000	0.5000	2.22554092849247	2.22555716464641	-0.16236153940419
0.3000	0.6000	2.45960311115695	2.45962137683013	-0.18265673182416
0.3000	0.7000	2.71828182845905	2.71830212365147	-0.20295192424857
0.3000	0.8000	3.00416602394643	3.00418834865810	-0.22324711669075
0.3000	0.9000	3.32011692273655	3.32014127696746	-0.24354230909740
0.3000	1.0000	3.66929666761924	3.66932305136940	-0.26383750156178
0.4000	0	1.49182469764127	1.49183281571824	-0.08118076971320
0.4000	0.1000	1.64872127070013	1.64873141829634	-0.10147596213539
0.4000	0.2000	1.82211880039051	1.82213097750596	-0.12177115454870
0.4000	0.3000	2.01375270747048	2.01376691410518	-0.14206634698422
0.4000	0.4000	2.22554092849247	2.22555716464641	-0.16236153941751
0.4000	0.5000	2.45960311115695	2.45962137683013	-0.18265673184636
0.4000	0.6000	2.71828182845905	2.71830212365147	-0.20295192425746
0.4000	0.7000	3.00416602394643	3.00418834865810	-0.22324711667743
0.4000	0.8000	3.32011692273655	3.32014127696746	-0.24354230911516
0.4000	0.9000	3.66929666761924	3.66932305136940	-0.26383750153514
0.4000	1.0000	4.05519996684467	4.05522838011407	-0.28413269393290
0.5000	0	1.64872127070013	1.64873141829634	-0.10147596213317
0.5000	0.1000	1.82211880039051	1.82213097750597	-0.12177115456646
0.5000	0.2000	2.01375270747048	2.01376691410518	-0.14206634698422
0.5000	0.3000	2.22554092849247	2.22555716464641	-0.16236153939087
0.5000	0.4000	2.45960311115695	2.45962137683013	-0.18265673183304
0.5000	0.5000	2.71828182845905	2.71830212365147	-0.20295192425746
0.5000	0.6000	3.00416602394643	3.00418834865810	-0.22324711671295
0.5000	0.7000	3.32011692273655	3.32014127696746	-0.24354230911516
0.5000	0.8000	3.66929666761924	3.66932305136940	-0.26383750153958
0.5000	0.9000	4.05519996684467	4.05522838011407	-0.28413269393290
0.5000	1.0000	4.48168907033806	4.48171951312670	-0.30442788639284
0.6000	0	1.82211880039051	1.82213097750597	-0.12177115456202
0.6000	0.1000	2.01375270747048	2.01376691410517	-0.14206634697977

0.6000	0.2000	2.22554092849247	2.22555716464641	-0.16236153941307
0.6000	0.3000	2.45960311115695	2.45962137683013	-0.18265673184192
0.6000	0.4000	2.71828182845905	2.71830212365147	-0.20295192423525
0.6000	0.5000	3.00416602394643	3.00418834865810	-0.22324711669963
0.6000	0.6000	3.32011692273655	3.32014127696746	-0.24354230908408
0.6000	0.7000	3.66929666761925	3.66932305136940	-0.26383750153514
0.6000	0.8000	4.05519996684468	4.05522838011407	-0.28413269395067
0.6000	0.9000	4.48168907033806	4.48171951312671	-0.30442788641949
0.6000	1.0000	4.95303242439511	4.95306489670300	-0.32472307881726
0.7000	0	2.01375270747048	2.01376691410518	-0.14206634699310
0.7000	0.1000	2.22554092849247	2.22555716464641	-0.16236153941307
0.7000	0.2000	2.45960311115695	2.45962137683013	-0.18265673183748
0.7000	0.3000	2.71828182845905	2.71830212365147	-0.20295192426634
0.7000	0.4000	3.00416602394643	3.00418834865810	-0.22324711665966
0.7000	0.5000	3.32011692273655	3.32014127696746	-0.24354230909740
0.7000	0.6000	3.66929666761925	3.66932305136940	-0.26383750152181
0.7000	0.7000	4.05519996684468	4.05522838011407	-0.28413269394179
0.7000	0.8000	4.48168907033806	4.48171951312670	-0.30442788638396
0.7000	0.9000	4.95303242439511	4.95306489670300	-0.32472307882614
0.7000	1.0000	5.47394739172720	5.47398189355433	-0.34501827124167
0.8000	0	2.22554092849247	2.22555716464641	-0.16236153941307
0.8000	0.1000	2.45960311115695	2.45962137683013	-0.18265673182416
0.8000	0.2000	2.71828182845905	2.71830212365147	-0.20295192425746
0.8000	0.3000	3.00416602394643	3.00418834865810	-0.22324711668187
0.8000	0.4000	3.32011692273655	3.32014127696746	-0.24354230908852
0.8000	0.5000	3.66929666761924	3.66932305136940	-0.26383750152625
0.8000	0.6000	4.05519996684468	4.05522838011407	-0.28413269394179
0.8000	0.7000	4.48168907033806	4.48171951312670	-0.30442788634844
0.8000	0.8000	4.95303242439511	4.95306489670299	-0.32472307879061
0.8000	0.9000	5.47394739172720	5.47398189355432	-0.34501827123279
0.8000	1.0000	6.04964746441295	6.04968399575931	-0.36531346363944
0.9000	0	2.45960311115695	2.45962137683013	-0.18265673179307
0.9000	0.1000	2.71828182845905	2.71830212365147	-0.20295192426190
0.9000	0.2000	3.00416602394643	3.00418834865810	-0.22324711668187
0.9000	0.3000	3.32011692273655	3.32014127696746	-0.24354230911072
0.9000	0.4000	3.66929666761924	3.66932305136940	-0.26383750155734
0.9000	0.5000	4.05519996684467	4.05522838011407	-0.28413269396843
0.9000	0.6000	4.48168907033806	4.48171951312670	-0.30442788637508
0.9000	0.7000	4.95303242439511	4.95306489670300	-0.32472307882614
0.9000	0.8000	5.47394739172720	5.47398189355433	-0.34501827125943
0.9000	0.9000	6.04964746441295	6.04968399575932	-0.36531346369273
0.9000	1.0000	6.68589444227927	6.68593300314487	-0.38560865603721
1.0000	0	2.71828182845905	2.71830212365147	-0.20295192425301
1.0000	0.1000	3.00416602394643	3.00418834865810	-0.22324711669519
1.0000	0.2000	3.32011692273655	3.32014127696746	-0.24354230910628
1.0000	0.3000	3.66929666761924	3.66932305136940	-0.26383750152181
1.0000	0.4000	4.05519996684467	4.05522838011407	-0.28413269399508
1.0000	0.5000	4.48168907033806	4.48171951312670	-0.30442788638396
1.0000	0.6000	4.95303242439511	4.95306489670300	-0.32472307882614
1.0000	0.7000	5.47394739172720	5.47398189355432	-0.34501827122391
1.0000	0.8000	6.04964746441295	6.04968399575931	-0.36531346364832
1.0000	0.9000	6.68589444227927	6.68593300314488	-0.38560865609050
1.0000	1.0000	7.38905609893065	7.38909668931550	-0.40590384852379

Supervisor Certification

I certify that this thesis was prepared under my supervision at the Department of Mathematics and Computer Applications/ College of Science/ Al-Nahrain University as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics and Computer Applications.

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Chapter One

The Generalized One-dimensional Integral Equations

Introduction:

A numerical integration is a basis of every numerical method for the solution of the one-dimensional integral equation, [16]. The problem of finding numerical solution for the one-dimensional integral equation is one of the oldest problems in applied mathematics and many computational methods are proposed in this area, [17]. The standard methods for solving the one-dimensional integral equations deal with the one-dimensional Volterra and Fredholm linear integral equations, [16], [8].

The aim of this chapter is to extend the definition of the standard one-dimensional integral equations that include only one integral operator to include more than one integral operator. This type of integral equations is termed as the generalized one-dimensional integral equations. Furthermore, we give a simple classification of the generalized one-dimensional integral equations and discuss the existence of a unique solution of special types of them.

This chapter consists of three sections.

In section one, we give a simple classification of the standard one-dimensional integral equations.

In section two, we introduce the definition of the generalized one-dimensional integral equations with a simple classification of them.

In section three, we discuss the existence of a unique solution for special types of the generalized one-dimensional integral equations, namely the generalized one-dimensional Fredholm linear and nonlinear integral equations.

1.1 *Classification of the Standard One-Dimensional Integral Equations:*

It is known that, the standard one-dimensional integral equation is an integral equation in which the unknown function depends only on one independent variable, [16].

In this section, we shall introduce some well known definitions for the standard one-dimensional integral equations, which are needed for our later discussion.

***Definition (1.1)*, [16]:**

The general form of the standard one-dimensional non-linear integral equation may be written as follows:

$$h(x)u(x) = f(x) + \lambda \int_{a}^{b(x)} k(x, y, u(y))dy \tag{1.1}$$

where h, f and b are given functions of x, k is a function, known as the kernel of the integral equation which is also known, a is a known constant, λ is a scalar parameter and u is the unknown function that must be determined.

***Definition (1.2)*, [15]:**

The standard one-dimensional integral equation is termed to be linear if it takes the form:

$$h(x)u(x) = f(x) + \lambda \int_{a}^{b(x)} k(x, y)u(y)dy \tag{1.2}$$

where h, f, b and k are known functions, a is a known constant, λ is a scalar parameter and u is the unknown function that must be determined.

***Definition (1.3)*, [12]:**

If f(x)=0, then the non-linear integral equation (1.1) is called homogenous, otherwise it is nonhomogenous.

Definition (1.4), [18]:

If h(x)=0, then the non-linear integral equation (1.1) is said to be equation of the first kind.

Definition (1.5), [25]:

If h(x)=1, then the non-linear integral equations (1.1) is said to be equation of the second kind.

Definition (1.6), [25]:

The standard one-dimensional non-linear integral equations (1.1) is called Fredholm integral equation if b(x)=b, where b is a constant such that $b \geq a$. Therefore, the integral equations:

$$f(x) = -\lambda \int_a^b k(x,y)u(y)dy, \quad a \leq x \leq b$$

$$u(x) = f(x) + \lambda \int_a^b k(x,y)u(y)dy, \quad a \leq x \leq b$$

represent the standard one-dimensional Fredholm linear integral equations of the first and second kinds respectively. On the other hand, the integral equations:

$$f(x) = -\lambda \int_a^b k(x,y,u(y))dy, \quad a \leq x \leq b$$

$$u(x) = f(x) + \lambda \int_a^b k(x,y,u(y))dy, \quad a \leq x \leq b$$

represent the standard one-dimensional Fredholm non-linear integral equations of the first and second kind**s** respectively.

***Definition (1.7)*, [16]:**

The standard one-dimensional non-linear integral equations (1.1) is called Volterra integral equation if b(x)=x, therefore the integral equations:

$$f(x) = -\lambda \int_a^x k(x,y)u(y)dy, \quad x \geq a$$

$$u(x) = f(x) + \lambda \int_a^x k(x,y)u(y)dy, \quad x \geq a$$

represent the standard one-dimensional Volterra linear integral equations of the first and second kinds respectively. On the other hand, the integral equations:

$$f(x) = -\lambda \int_a^x k(x,y,u(y))dy, \quad x \geq a$$

$$u(x) = f(x) + \lambda \int_a^x k(x,y,u(y))dy, \quad x \geq a$$

represent the standard one-dimensional Volterra non-linear integral equations of the first and second kinds respectively.

***Definition (1.8)*, [16]:**

If the kernel in the integral equation (1.2) depends only on the difference $(x-y)$, that is $k(x,y)=k(x-y)$ then the integral equation is said to be the standard one-dimensional integral equation of convolution type and it takes the form:

$$h(x)u(x) = f(x) + \lambda \int_a^{b(x)} k(x-y)u(y)dy$$

***Definition (1.9)*, [10]:**

The kernel $k(x,y)$ is said to be symmetric, if it has the property $k(x,y)=k(y,x)$, and it said to be antisymmetric, if it has the property $k(x,y) = -k(y,x)$.

***Definition (1.10)*, [16]:**

The kernel $k(x,y)$ is said to be separable or degenerate kernel if it is of the form:

$$k(x,y)=\sum_{j=1}^{p} a_j(x)b_j(y)$$

where p is finite.

1.2 *Classification of the Generalized One-dimensional Integral Equations:*

As seen before the standard one-dimensional integral equations contain only one integral operator.

In this section, we extend the previous classification to include the generalized one dimensional integral equations.

We start this section by the following definition. This definition is based on the idea that appeared in [1].

***Definition (1.11)*:**

The general form of the generalized one-dimensional non-linear integral equation of the form:

$$h(x)u(x)=f(x)+\sum_{i=1}^{n}\lambda_i\int_{a_i}^{b_i(x)} k_i(x,y,u(y))dy \tag{1.3}$$

where h and f are given functions of x, k_i is a function, known as the i-th kernel of the integral equation which is also known for each i=1,2,…,n, b_i is a known function of x for each i=1,2,…,n, a_i is a known constant for each i=1,2,…,n, λ_i is a scalar parameter for each i=1,2,…,n and u is the unknown function that must be determined.

Definition (1.12):

The generalized one-dimensional integral equation is termed to be linear if it takes the form:

$$h(x)u(x) = f(x) + \sum_{i=1}^{n} \lambda_i \int_{a_i}^{b_i(x)} k_i(x,y)u(y)dy \tag{1.4}$$

where h, f, b_i, a_i, λ_i are defined previously, k_i is a known function of x and y and u is the unknown function that must be determined.

Remarks (1.1):

1. If f(x)=0, then the non-linear integral equation (1.3) is called homogenous, otherwise it is non-homogenous.
2. If h(x)=0 (or h(x)=1), then the non-linear integral equation (1.3) is said to be of the first kind (or of the second kind).

Definition (1.13):

The generalized one-dimensional non-linear integral equation (1.3) is called Fredholm integral equation if $b_i(x) = b_i$, where b_i is a known constant such that $b_i \geq a_i$ for each i=1,2,…,n. Therefore, the integral equations:

$$f(x) = -\sum_{i=1}^{n} \lambda_i \int_{a_i}^{b_i} k_i(x,y)u(y)dy, \quad \min_{1\leq i\leq n}\{a_i\} \leq x \leq \max_{1\leq i\leq n}\{b_i\} \tag{1.5}$$

$$u(x) = f(x) + \sum_{i=1}^{n} \lambda_i \int_{a_i}^{b_i} k_i(x,y)u(y)dy, \quad \min_{1\leq i\leq n}\{a_i\} \leq x \leq \max_{1\leq i\leq n}\{b_i\} \tag{1.6}$$

represent the generalized one-dimensional Fredholm linear integral equations of the first and second kinds respectively. On the other hand, the integral equations:

$$f(x) = -\sum_{i=1}^{n} \lambda_i \int_{a_i}^{b_i} k_i(x,y,u(y))dy, \quad \min_{1\leq i\leq n}\{a_i\} \leq x \leq \max_{1\leq i\leq n}\{b_i\} \tag{1.7}$$

$$u(x) = f(x) + \sum_{i=1}^{n} \lambda_i \int_{a_i}^{b_i} k_i(x, y, u(y))dy, \quad \min_{1\le i\le n}\{a_i\} \le x \le \max_{1\le i\le n}\{b_i\} \tag{1.8}$$

represent the generalized one-dimensional Fredholm non-linear integral equations of the first and second kind**s** respectively.

***Definition (1.14)*:**

The generalized one-dimensional non-linear integral equations (1.3) is called Volterra integral equation when $b_j(x) = x$ for some $j \in \{1, 2, ..., n\}$. Therefore the integral equations:

$$f(x) = -\sum_{\substack{i=1 \\ i\ne j}}^{n} \lambda_i \int_{a_i}^{b_i(x)} k_i(x, y)u(y)dy + \lambda_j \int_{a_j}^{x} k_j(x, y)u(y)dy, \quad x \ge \min_{1\le i\le n}\{a_i\} \tag{1.9}$$

$$u(x) = f(x) + \sum_{\substack{i=1 \\ i\ne j}}^{n} \lambda_i \int_{a_i}^{b_i(x)} k_i(x, y)u(y)dy + \lambda_j \int_{a_j}^{x} k_j(x, y)u(y)dy, \quad x \ge \min_{1\le i\le n}\{a_i\} \tag{1.10}$$

represent the generalized one-dimensional Volterra linear integral equations of the first and second kind respectively. On the other hand, the integral equations:

$$f(x) = -\sum_{\substack{i=1 \\ i\ne j}}^{n} \lambda_i \int_{a_i}^{b_i(x)} k_i(x, y, u(y))dy + \lambda_j \int_{a_j}^{x} k_j(x, y, u(y))dy, \quad x \ge \min_{1\le i\le n}\{a_i\} \tag{1.11}$$

$$u(x) = f(x) + \sum_{\substack{i=1 \\ i\ne j}}^{n} \lambda_i \int_{a_i}^{b_i(x)} k_i(x, y, u(y))dy + \lambda_j \int_{a_j}^{x} k_j(x, y, u(y))dy, \quad x \ge \min_{1\le i\le n}\{a_i\} \tag{1.12}$$

represent the generalized one-dimensional Volterra non-linear integral equations of the first and second kind respectively.

***Definition (1.15)*:**

If the kernel k_i in the integral equation (1.4) depends only on the difference $(x-y)$, that is $k_i(x, y) = k_i(x - y)$ for each i=1,2,…,n then this integral equation is said to be the generalized one-dimensional integral equation of convolution type and it takes the form:

$$h(x)u(x) = f(x) + \sum_{i=1}^{n} \lambda_i \int_{a_i}^{b_i(x)} k_i(x-y)u(y)dy$$

Definition (1.16):

The kernel $k_i(x,y)$ is said to be separable or degenerate kernel if it is of the form:

$$k_i(x,y) = \sum_{j=1}^{p} a_{ij}(x) b_{ij}(y), \; i \in \{1,2,...,n\}$$

where p is finite. If each kernel k_i that appeared in the integral equation (1.6) is separable, then equation (1.6) is said to be the generalized one dimensional Fredholm linear integral equation with degenerate kernels.

1.3 *Existence and Uniqueness Theorems of the Solution for the Generalized One-Dimensional Integral Equations:*

In this section, we discuss the existence of the unique solution for special types of the generalized one-dimensional integral equations, namely the generalized one-dimensional Fredholm linear and non-linear integral equations of the second kind.

We start this section by recalling some lemmas that we needed them later.

***Lemma* (1.1), [22]:**

Consider $L_2[a,b]$, where [a,b] is a finite interval. Suppose $k(x,y)$ is continuous for all x, y in [a,b]. Then the operator

$$Ku = \int_{a}^{b} k(x,y)u(y)dy$$

is bounded.

Proof:

Since $k(x,y)$ is continuous on a closed and bounded set, it must be bounded. Then, there exist M > 0 such that $|k(x,y)| \le M.$ Hence

$$|Ku| = \left|\int_a^b k(x,y)u(y)dy\right| \le \int_a^b |k(x,y)||u(y)|dy$$

$$= M\left(\int_a^b dy\right)^{\frac{1}{2}}\left(\int_a^b |u(y)|^2 dy\right)^{\frac{1}{2}}$$

$$= M(b-a)^{\frac{1}{2}}\|u\|$$

and it follows that

$$\left[\int_a^b |Ku|^2 dy\right]^{\frac{1}{2}} \le \left[\int_a^b M^2(b-a)\|u\|^2 dy\right]^{\frac{1}{2}}$$

$$\le M(b-a)^{\frac{1}{2}}\|u\|\left[\int_a^b dy\right]^{\frac{1}{2}}$$

$$= M(b-a)\|u\|$$

Therefore

$$\|Ku\| \le M(b-a)\|u\|$$

and hence

$$\|K\| \le M(b-a).$$

***Lemma* (1.2), [22]:**

Consider L_2 [a,b], where [a,b] is a finite interval. If

$$\int_a^b\int_a^b |k(x,y)|^2 dxdy = M^2 < \infty$$

then the operator K defined by

$$Ku = \int_a^b k(x,y)u(y)dy$$

is bounded.

Proof:

$$|Ku| = \left|\int_a^b k(x,y)u(y)dy\right|$$

$$\le \int_a^b |k(x,y)||\, u(y)|dy$$

and by using the Cauchy–Schwarz inequality

$$|Ku| \le \left[\int_a^b |k(x,y)|^2 dy\right]^{\frac{1}{2}} \left[\int_a^b |\, u(y)|^2\, dy\right]^{\frac{1}{2}}$$

Then

$$\left[\int_a^b |Ku|^2\, dx\right]^{\frac{1}{2}} \le \left[\int_a^b\int_a^b |k(x,y)|^2\, dxdy\right]^{\frac{1}{2}} \|\, u\|$$

Therefore

$$\|Ku\| \le M\|\, u\|.$$

The following lemma is named as "Banach fixed point theorem".

***Lemma* (1.3), [22]:**

Let T be a contraction operator defined on a Hilbert space H. The equation,

$$Tu = u$$

has a unique solution u in H. Such a solution is said to be a fixed point of T.

Now, we are in the position that we can give the following theorem. This theorem is a modification of the theorem that appeared in [22].

***Theorem* (1.1):**

Consider the generalized one-dimensional Fredholm linear integral equation of the second kind:

$$u(x) = f(x) + \sum_{i=1}^{n} \lambda_i \int_{a_i}^{b_i} k_i(x,y)u(y)dy, \quad a = \min_{1\le i\le n}\{a_i\} \le x \le \max_{1\le i\le n}\{b_i\} = b \qquad (1.13)$$

If the operator K_i defined by

$$K_i u = \int_{a_i}^{b_i} k_i(x,y)\,u(y)\,dy, \quad i = 1,2,...,n$$

is a bounded operator for each i=1,2,…,n then equation (1.13) has a unique solution for all f in $L_2[a,b]$ and for sufficiently small $|\lambda_i|$ for each i=1,2,…,n.

Proof:

We rewrite equation (1.13) in an operator form:

$$Tu = u$$

where

$$Tu = f + \sum_{i=1}^{n} \lambda_i K_i u$$

Then

$$\|Tu_1 - Tu_2\| = \left\| f + \sum_{i=1}^{n} \lambda_i K_i u_1 - f - \sum_{i=1}^{n} \lambda_i K_i u_2 \right\|$$

$$= \left\| \sum_{i=1}^{n} \lambda_i K_i u_1 - \sum_{i=1}^{n} \lambda_i K_i u_2 \right\|$$

but K_i is a linear operator for each i=1,2,…,n, then

$$\|Tu_1 - Tu_2\| = \sum_{i=1}^{n} |\lambda_i| \|K_i(u_1 - u_2)\|$$

since K_i is a bounded operator for each i=1,2,…,n then there exist a constant $M_i > 0$ such that

$$\|K_i u\| \le M_i \|u\|,\ i = 1, 2, ..., n$$

for all u in the Hilbert space $L_2\left[\min_{1\le i\le n}\{a_i\}, \max_{1\le i\le n}\{b_i\}\right]$.

Therefore

$$\|Tu_1 - Tu_2\| \le \sum_{i=1}^{n} |\lambda_i| M_i \|u_1 - u_2\|$$

for $\sum_{i=1}^{n} |\lambda_i| M_i < 1$, T is a contraction operator and by using lemma (1.3), T has a unique fixed point u in $L_2[a, b]$, which is the unique solution of equation (1.13).

The proof of the following corollary is clear, thus we omitted it.

***Corollary* (1.1):**

Consider the generalized one-dimensional Fredholm linear integral equation of the second kind given by equation (1.13). If

$$\int_a^b \int_{a_i}^{b_i} |k_i(x, y)|^2 \, dydx = P_i^2 < \infty$$

then equation (1.13) has a unique solution for all f in $L_2[a, b]$ and for sufficiently small $|\lambda_i|$ for each i=1,2,…,n.

Next, the following theorem is a modification of the theorem that appeared in [22].

***Theorem* (2.2):**

Consider the generalized one-dimensional Fredholm non-linear integral equation of the second kind:

$$u(x) = f(x) + \sum_{i=1}^{n} \lambda_i \int_{a_i}^{b_i} k_i(x, y, u(y))dy, \quad a = \min_{1 \le i \le n}\{a_i\} \le x \le \max_{1 \le i \le n}\{b_i\} = b \tag{1.14}$$

Suppose that

$$|k_i(x, y, u_1(y)) - k_i(x, y, u_2(y))| \le M_i(x, y)|u_1(y) - u_2(y)|, a_i \le y \le b_i, i = 1,2,...,n$$

and

$$\int_a^b \int_{a_i}^{b_i} |M_i(x, y)|^2 dydx = P_i^2 < \infty, \ i = 1,2,...,n.$$

If $\sum_{i=1}^{n} |\lambda_i| P_i < 1$, then equation (1.14) has a unique solution.

Proof:

We rewrite equation (1.14) in an operator form Tu=u, where

$$Tu = f + \sum_{i=1}^{n} \lambda_i K_i u$$

and

$$K_i u = \int_{a_i}^{b_i} k_i(x, y, u(y))dy .$$

Then

$$\begin{aligned} \|Tu_1 - Tu_2\| &= \left\| \sum_{i=1}^{n} \lambda_i K_i u_1 - \sum_{i=1}^{n} \lambda_i K_i u_2 \right\| \\ &= \left\| \sum_{i=1}^{n} \lambda_i \int_{a_i}^{b_i} k_i(x, y, u_1(y))dy - \sum_{i=1}^{n} \lambda_i \int_{a_i}^{b_i} k_i(x, y, u_2(y))dy \right\| \end{aligned}$$

$$\left\|Tu_1 - Tu_2\right\| = \left\| \sum_{i=1}^{n} \lambda_i \int_{a_i}^{b_i} [k_i(x,y,u_1(y)) - k_i(x,y,u_2(y))]dy \right\|$$

$$\leq \sum_{i=1}^{n} \left\| \lambda_i \int_{a_i}^{b_i} [k_i(x,y,u_1(y)) - k_i(x,y,u_2(y))]dy \right\|$$

$$= \sum_{i=1}^{n} |\lambda_i| \left[\int_a^b \left| \int_{a_i}^{b_i} \{k_i(x,y,u_1(y)) - k_i(x,y,u_2(y))\}dy \right|^2 dx \right]^{\frac{1}{2}}$$

$$\leq \sum_{i=1}^{n} |\lambda_i| \left[\int_a^b \left\{ \int_{a_i}^{b_i} |k_i(x,y,u_1(y)) - k_i(x,y,u_2(y))|dy \right\}^2 dx \right]^{\frac{1}{2}}$$

$$\leq \sum_{i=1}^{n} |\lambda_i| \left[\int_a^b \left\{ \int_{a_i}^{b_i} M_i(x,y)|u_1(y) - u_2(y)|dy \right\}^2 dx \right]^{\frac{1}{2}}$$

$$\leq \sum_{i=1}^{n} |\lambda_i| \left[\int_a^b \left\{ \left(\int_{a_i}^{b_i} |M_i(x,y)|^2 dy \right)^{\frac{1}{2}} \left(\int_{a_i}^{b_i} |u_1(y) - u_2(y)|^2 dy \right)^{\frac{1}{2}} \right\}^2 dx \right]^{\frac{1}{2}}$$

$$= \sum_{i=1}^{n} |\lambda_i| \left[\int_a^b \left(\int_{a_i}^{b_i} |M_i(x,y)|^2 dy \right) \left(\int_{a_i}^{b_i} |u_1(y) - u_2(y)|^2 dy \right) dx \right]^{\frac{1}{2}}$$

Therefore

$$\left\|Tu_1 - Tu_2\right\| \leq \sum_{i=1}^{n} |\lambda_i| \left[\int_a^b \int_{a_i}^{b_i} |M_i(x,y)|^2 dydx \left(\int_{a_i}^{b_i} |u_1(y) - u_2(y)|^2 dy \right) \right]^{\frac{1}{2}}$$

$$\|Tu_1 - Tu_2\| \le \sum_{i=1}^{n} |\lambda_i| \left[\int_a^b \int_{a_i}^{b_i} |M_i(x,y)|^2 \, dydx \left(\int_a^b |u_1(y) - u_2(y)|^2 dy \right) \right]^{\frac{1}{2}}$$

$$= \sum_{i=1}^{n} |\lambda_i| \left[\int_a^b \int_{a_i}^{b_i} |M_i(x,y)|^2 \, dydx \right]^{\frac{1}{2}} \left[\int_a^b |u_1(y) - u_2(y)|^2 dy \right]^{\frac{1}{2}}$$

$$= \sum_{i=1}^{n} |\lambda_i| \left[P_i^{\,2} \right]^{\frac{1}{2}} \|u_1 - u_2\| = \sum_{i=1}^{n} |\lambda_i| P_i \|u_1 - u_2\|.$$

for $\sum_{i=1}^{n} |\lambda_i| P_i < 1$, T is a contraction operator and by using lemma (1.3), T has a unique fixed point u in $L_2[a,b]$, which is the unique solution of equation (1.14).

Remarks (1.2):

(1) If n=1 in theorem (1.1) and corollary (1.1) then one can get the same fact that appeared in [22] and valid for the standard one-dimensional Fredholm linear integral equations of the second kind.

(2) If n=1 in theorem (1.2) then one can get the same fact that appeared in [22] and valid for the standard one-dimensional Fredholm non-linear integral equations of the second kind.

Chapter Three

The Multi-Dimensional Integral Equations

Introduction:

The aim of this chapter is to classify the multi-dimensional integral equations and extend them into the generalized multi-dimensional integral equations and presents some numerical methods to solve the multi-dimensional Fredholm linear integral equations of the second kind. Moreover, we use the repeated modified trapezoid method to solve the multi-dimensional Fredholm linear integral equations of the second kind.

This chapter consists of three sections:

In section one, we classify the standard multi-dimensional integral equations into linear/nonlinear, Volterra/Fredholm, homogeneous/nonhomogeneous, first kind/second kind.

In section two, we define the generalized multi-dimensional integral equations with their classification.

In section three, we use some of the quadrature methods, namely the trapezoidal and Simpson's 1/3 rule to solve the standard multi-dimensional Fredholm linear integral equations of the second kind.

In section four, we derive the modified trapezoidal rule for evaluating multiple integrals and use it to solve the standard multi-dimensional Fredholm linear integral equations of the second kind.

3.1 *Classification of The Standard Multi-Dimensional Integral Equations*:

It is known that the standard multi-dimensional integral equation is an integral equation in which the integration is carried out with respect to m variables, [14].

In this section, we extend the classification of the standard one-dimensional integral equations to include the standard multi-dimensional integral equations.

Definition (3.1):

The general form of the standard multi-dimensional non-linear integral equation is:

$$h(x_1,x_2,...,x_m)u(x_1,x_2,...,x_m)=f(x_1,x_2,...,x_m)+$$
$$\lambda\int_{\alpha_m}^{\beta_m(x_m)}\int_{\alpha_{m-1}}^{\beta_{m-1}(x_{m-1})}...\int_{\alpha_1}^{\beta_1(x_1)}k(x_1,x_2,...,x_m,s_1,s_2,...,s_m,u(s_1,s_2,...,s_m))ds_1ds_2...ds_m \quad (3.1)$$

where h and f are known functions of $x_1,x_2,...,x_m$, k is a function, known as the kernel of the integral equation which is also known, β_j is a known function of x_j for each j=1,2,…,m, α_j is a known constant for each j=1,2,…,m, λ is a scalar parameter and u is the unknown function that must be determined.

Definition (3.2):

The standard multi-dimensional integral equation is termed to be linear if it takes the form:

$$h(x_1,x_2,...,x_m)u(x_1,x_2,...,x_m)=f(x_1,x_2,...,x_m)+$$
$$\lambda\int_{\alpha_m}^{\beta_m(x_m)}\int_{\alpha_{m-1}}^{\beta_{m-1}(x_{m-1})}...\int_{\alpha_1}^{\beta_1(x_1)}k(x_1,x_2,...,x_m,s_1,s_2,...,s_m)u(s_1,s_2,...,s_m)ds_1ds_2...ds_m \quad (3.2)$$

where h, f and β_j are known functions of x_j for each j=1,2,…,m, k is a function, known as the kernel of the integral equation which is also known, α_j is a known constant for each j=1,2,…,m, λ is a scalar parameter and u is the unknown function that must be determined.

Definition (3.3):

If $f(x_1, x_2, ..., x_m) = 0$, then the non-linear integral equation (3.1) is called homogenous, otherwise it is non-homogenous.

Definition (3.4):

If $h(x_1, x_2, ..., x_m) = 0$, then the non-linear integral equation (3.1) is said to be equation of the first kind.

Definition (3.5):

If $h(x_1, x_2, ..., x_m) = 1$, then the non-linear integral equation (3.1) is said to be equation of the second kind.

Definition (3.6):

The standard multi-dimensional non-linear integral equation (3.1) is called the standard multi-dimensional Fredholm non-linear integral equation if the upper limits of the integrations $\beta_j(x) = \beta_j$, where β_j is a constant for each j=1,2,…,m such that $\beta_j \geq \alpha_j$. Therefore, the integral equations:

$$f(x_1, x_2, ..., x_m) = -\lambda \int_{\alpha_m}^{\beta_m} \int_{\alpha_{m-1}}^{\beta_{m-1}} \cdots \int_{\alpha_1}^{\beta_1} k(x_1, x_2, ..., x_m, s_1, s_2, ..., s_m) u(s_1, s_2, ..., s_m) ds_1 ds_2 ... ds_m \quad (3.3)$$

and

$$u(x_1,x_2,\dots,x_m)=f(x_1,x_2,\dots,x_m)+$$
$$\lambda\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}\dots\int_{\alpha_1}^{\beta_1}k(x_1,x_2,\dots,x_m,s_1,s_2,\dots,s_m)u(s_1,s_2,\dots,s_m)ds_1ds_2\dots ds_m \quad (3.4)$$

represent the standard multi-dimensional Fredholm linear integral equations of the first and second kind respectively. On the other hand, the integral equations:

$$f(x_1,x_2,\dots,x_m)=-\lambda\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}\dots\int_{\alpha_1}^{\beta_1}k(x_1,x_2,\dots,x_m,s_1,s_2,\dots,s_m,u(s_1,s_2,\dots,s_m))ds_1ds_2\dots ds_m \quad (3.5)$$

and

$$u(x_1,x_2,\dots,x_m)=f(x_1,x_2,\dots,x_m)+$$
$$\lambda\int_{\alpha_m}^{\beta_m}\int_{\alpha_{m-1}}^{\beta_{m-1}}\dots\int_{\alpha_1}^{\beta_1}k(x_1,x_2,\dots,x_m,s_1,s_2,\dots,s_m,u(s_1,s_2,\dots,s_m))ds_1ds_2\dots ds_m \quad (3.6)$$

represent the standard multi-dimensional Fredholm non-linear integral equations of the first and second kind respectively.

Definition (3.7):

The standard multi-dimensional non-linear integral equation (3.1) is called Volterra integral equation when $\beta_j(x_j)=x_j$, for some $j=1,2,\dots,m$, therefore the integral equations:

$$f(x_1,x_2,\dots,x_m)=-\lambda\int_{\alpha_m}^{\beta_m(x_m)}\int_{\alpha_{m-1}}^{\beta_{m-1}(x_{m-1})}\dots\int_{\alpha_1}^{\beta_1(x_1)}k(x_1,x_2,\dots,x_m,s_1,s_2,\dots,s_m)u(s_1,s_2,\dots,s_m)ds_1ds_2\dots ds_m \quad (3.7)$$

and

$$u(x_1,x_2,\dots,x_m)=f(x_1,x_2,\dots,x_m)+$$
$$\lambda\int_{\alpha_m}^{\beta_m(x_m)}\int_{\alpha_{m-1}}^{\beta_{m-1}(x_{m-1})}\dots\int_{\alpha_1}^{\beta_1(x_1)}k(x_1,x_2,\dots,x_m,s_1,s_2,\dots,s_m)u(s_1,s_2,\dots,s_m)ds_1ds_2\dots ds_m \quad (3.8)$$

represent the standard multi-dimensional Volterra linear integral equations of the first and second kind respectively. On the other hand, the integral equations:

$$f(x_1,x_2,...,x_m)=-\lambda \int_{\alpha_m}^{\beta_m(x_m)} \int_{\alpha_{m-1}}^{\beta_{m-1}(x_{m-1})} ... \int_{\alpha_1}^{\beta_1(x_1)} k(x_1,x_2,...,x_m,s_1,s_2,...,s_m,u(s_1,s_2,...,s_m))ds_1ds_2...ds_m \quad (3.9)$$

and

$$u(x_1,x_2,...,x_m)=f(x_1,x_2,...,x_m)+ \lambda \int_{\alpha_m}^{\beta_m(x_m)} \int_{\alpha_{m-1}}^{\beta_{m-1}(x_{m-1})} ... \int_{\alpha_1}^{\beta_1(x_1)} k(x_1,x_2,...,x_m,s_1,s_2,...,s_m,u(s_1,s_2,...,s_m))ds_1ds_2...ds_m \quad (3.10)$$

represent the standard multi-dimensional Volterra non-linear integral equations of the first and second kinds respectively.

Definition (3.8):

If the kernel in the integral equation (3.2) depends only on the difference $k(x_1-s_1,x_2-s_2,...,x_m-s_m)$ that is $k(x_1,x_2,...,x_m,s_1,s_2,...,s_m)=k(x_1-s_1,x_2-s_2,...,x_m-s_m)$ then the integral equation is said to be the standard multi-dimensional integral equation of convolution type and it takes the form:

$$h(x_1,x_2,...,x_m)u(x_1,x_2,...,x_m)=f(x_1,x_2,...,x_m)+ \lambda \int_{\alpha_m}^{\beta_m(x_m)} \int_{\alpha_{m-1}}^{\beta_{m-1}(x_{m-1})} ... \int_{\alpha_1}^{\beta_1(x_1)} k(x_1-s_1,x_2-s_2,...,x_m-s_m)u(s_1,s_2,...,s_m)ds_1ds_2...ds_m \quad (3.11)$$

Definition (3.9):

The kernel $k(x_1,x_2,...,x_m,s_1,s_2,...,s_m)$ is said to be symmetric, if it has the property $k(x_1,x_2,...,x_m,s_1,s_2,...,s_m)=k(s_1,s_2,...,s_m,x_1,x_2,...,x_m)$ and it said to be anti-symmetric, if it has the property $k(x_1,x_2,...,x_m,s_1,s_2,...,s_m)=-k(s_1,s_2,...,s_m,x_1,x_2,...,x_m)$.

Definition (3.10):

The kernel $k(x_1, x_2, ..., x_m, s_1, s_2, ..., s_m)$ is said to be separable or degenerate kernel if it is of the form:

$$k(x_1, x_2, ..., x_m, s_1, s_2, ..., s_m) = \sum_{j=1}^{p} a_j(x_1, x_2, ..., x_m) b_j(s_1, s_2, ..., s_m)$$

where p is finite.

3.2 *Classification of The Generalized Multi-Dimensional Integral Equations*:

As seen before the standard multi-dimensional integral equations contain only one multi-dimensional integral operator.

In this section, we extend the previous classification to include the generalized multi-dimensional integral equations.

We start this section by the following definition. This definition is based on the idea that appeared in [1] and it is a generalization of definition (1.11).

Definition (3.11):

The generalized multi-dimensional non-linear integral equation is an integral equation of the form:

$$h(x_1, x_2, ..., x_m) u(x_1, x_2, ..., x_m) = f(x_1, x_2, ..., x_m) + \sum_{i=1}^{n} \lambda_i \int_{\alpha_{im}}^{\beta_{im}(x_m)} \int_{\alpha_{im-1}}^{\beta_{im-1}(x_{m-1})} \cdots \int_{\alpha_{i1}}^{\beta_{i1}(x_1)} k_i(x_1, x_2, ..., x_m, s_1, s_2, ..., s_m, u(s_1, s_2, ..., s_m)) ds_1 ds_2 ... ds_m \quad (3.12)$$

where h and f are known functions of $x_1, x_2, ..., x_m$, k_i is a function, known as the i-th kernel of the integral equation which is also known for each i=1,2,…,n, β_{ij} is a known function of x_j for each i=1,2,…,n and j=1,2,…,m, α_{ij} is a known constant for each i=1,2,…,n and j=1,2,…,m, λ_i is a scalar parameter for each i=1,2,…,n and u is the unknown function that must be determined.

Definition (3.12):

The generalized multi-dimensional integral equation is termed to be linear if it takes the form:

$$h(x_1,x_2,...,x_m)u(x_1,x_2,...,x_m)=f(x_1,x_2,...,x_m)+$$
$$\sum_{i=1}^{n}\lambda_i\int_{\alpha_{im}}^{\beta_{im}(x_m)}\int_{\alpha_{im-1}}^{\beta_{im-1}(x_{m-1})}\cdots\int_{\alpha_{i1}}^{\beta_{i1}(x_1)}k_i(x_1,x_2,...,x_m,s_1,s_2,...,s_m)u(s_1,s_2,...,s_m)ds_1ds_2...ds_m \quad (3.13)$$

where h, f, β_{ij},α_{ij} and λ_i are defined previously, k_i is a known function of $x_1,x_2,...,x_n,s_1,s_2,...,s_n$ for each i=1,2,…,n and u is the unknown function that must be determined.

Remarks (3.1):

(1) If $f(x_1,x_2,...,x_m)=0$, then the non-linear integral equation (3.12) is called homogenous, otherwise it is non-homogenous.

(2) If $h(x_1,x_2,...,x_m)=0$ (or $h(x_1,x_2,...,x_m)=1$), then the non-linear integral equation (3.12) is said to be equation of the first kind (or of the second kind).

Definition (3.13):

The generalized multi-dimensional non-linear integral equation (3.12) is called Fredholm if $\beta_{ij}(x)=\beta_{ij}$, where β_{ij} is a known constant such that $\beta_{ij}\geq\alpha_{ij}$ for each i=1,2,…,n and j=1,2,…,m. Therefore, the integral equations:

$$f(x_1,x_2,...,x_m)=-\sum_{i=1}^{n}\lambda_i\int_{\alpha_{im}}^{\beta_{im}}\int_{\alpha_{im-1}}^{\beta_{im-1}}\cdots\int_{\alpha_{i1}}^{\beta_{i1}}k_i(x_1,x_2,...,x_m,s_1,s_2,...,s_m)u(s_1,s_2,...,s_m)ds_1ds_2...ds_m \quad (3.14)$$

and

$$u(x_1,x_2,...,x_m)=f(x_1,x_2,...,x_m)+\sum_{i=1}^{n}\lambda_i\int_{\alpha_{im}}^{\beta_{im}}\int_{\alpha_{im-1}}^{\beta_{im-1}}...\int_{\alpha_{i1}}^{\beta_{i1}}k_i(x_1,x_2,...,x_m,s_1,s_2,...,s_m)u(s_1,s_2,...,s_m)ds_1ds_2...ds_m \quad (3.15)$$

where $\min_{1\le i\le n}\{\alpha_{ij}\}\le x_j\le\max_{1\le i\le n}\{\beta_{ij}\}$, j=1,2,…,m, represent the generalized multi-dimensional Fredholm linear integral equations of the first and second kind respectively. On the other hand, the integral equations:

$$f(x_1,x_2,...,x_m)=-\sum_{i=1}^{n}\lambda_i\int_{\alpha_{im}}^{\beta_{im}}\int_{\alpha_{im-1}}^{\beta_{im-1}}...\int_{\alpha_{i1}}^{\beta_{i1}}k_i(x_1,x_2,...,x_m,s_1,s_2,...,s_m,u(s_1,s_2,...,s_m))ds_1ds_2...ds_m \quad (3.16)$$

and

$$u(x_1,x_2,...,x_m)=f(x_1,x_2,...,x_m)+\sum_{i=1}^{n}\lambda_i\int_{\alpha_{im}}^{\beta_{im}}\int_{\alpha_{im-1}}^{\beta_{im-1}}...\int_{\alpha_{i1}}^{\beta_{i1}}k_i(x_1,x_2,...,x_m,s_1,s_2,...,s_m,u(s_1,s_2,...,s_m))ds_1ds_2...ds_m \quad (3.17)$$

where $\min_{1\le i\le n}\{\alpha_{ij}\}\le x_j\le\max_{1\le i\le n}\{\beta_{ij}\}$, j=1,2,…,m, represent the n-generalize multi-dimensional Fredholm non-linear integral equations of the first and second kind respectively.

Definition (3.14):

The generalized multi-dimensional non-linear integral equation (3.12) is called Volterra integral equation when $\beta_{ij}(x_j)=x_j$, for some $j\in\{1,2,...,m\}$ and i=1,2,…,n, therefore the integral equations:

$$f(x_1,x_2,...,x_m)=-\sum_{\substack{i=1\\i\ne j}}^{n}\lambda_i\int_{\alpha_{im}}^{\beta_{im}(x_m)}\int_{\alpha_{im-1}}^{\beta_{im-1}(x_{m-1})}...\int_{\alpha_{i1}}^{\beta_{i1}(x_1)}k_i(x_1,x_2,...,x_m,s_1,s_2,...,s_m)u(s_1,s_2,...,s_m)ds_1ds_2...ds_m+$$
$$\lambda_j\int_{\alpha_{im}}^{\beta_{im}(x_m)}\int_{\alpha_{im-1}}^{\beta_{im-1}(x_{m-1})}...\int_{\alpha_{ij+1}}^{\beta_{ij+1}(x_{j+1})}\int_{\alpha_{ij}}^{x_j}\int_{\alpha_{ij-1}}^{\beta_{ij-1}(x_{j-1})}\int_{\alpha_{i1}}^{\beta_{i1}(x_1)}k_j(x_1,x_2,...,x_m,s_1,s_2,...,s_m)u(s_1,s_2,...,s_m)ds_1ds_2...ds_m \quad (3.18)$$

and

$$u(x_1,x_2,...,x_m)=f(x_1,x_2,...,x_m)+$$
$$\sum_{\substack{i=1\\i\neq j}}^{n}\lambda_i\int_{\alpha_{im}}^{\beta_{im}(x_{im})}\int_{\alpha_{im-1}}^{\beta_{im-1}(x_{im-1})}\cdots\int_{\alpha_{i1}}^{\beta_{i1}(x_{i1})}k_i(x_1,x_2,...,x_m,s_1,s_2,...,s_m)u(s_1,s_2,...,s_m)ds_1ds_2...ds_m+$$
$$\lambda_j\int_{\alpha_{im}}^{\beta_{im}(x_m)}\int_{\alpha_{im-1}}^{\beta_{im-1}(x_{m-1})}\cdots\int_{\alpha_{ij+1}}^{\beta_{ij+1}(x_{j+1})}\int_{\alpha_{ij}}^{x_j}\int_{\alpha_{ij-1}}^{\beta_{ij-1}(x_{j-1})}\int_{\alpha_{i1}}^{\beta_{i1}(x_1)}k_j(x_1,x_2,...,x_m,s_1,s_2,...,s_m)u(s_1,s_2,...,s_m)ds_1ds_2...ds_m \quad (3.19)$$

where $x_j \geq \min_{1\leq i\leq n}\{\alpha_{ij}\}$, j=1,2,…,m, represent the generalized multi-dimensional Volterra linear integral equations of the first and second kind respectively. On the other hand, the integral equations:

$$f(x_1,x_2,...,x_m)=-\sum_{\substack{i=1\\i\neq j}}^{n}\lambda_i\int_{\alpha_{im}}^{\beta_{im}(x_m)}\int_{\alpha_{im-1}}^{\beta_{im-1}(x_{m-1})}\cdots\int_{\alpha_{i1}}^{\beta_{i1}(x_1)}k_i(x_1,x_2,...,x_m,s_1,s_2,...,s_m,u(s_1,s_2,...,s_m))ds_1ds_2...ds_m+$$
$$\lambda_j\int_{\alpha_{im}}^{\beta_{im}(x_m)}\int_{\alpha_{im-1}}^{\beta_{im-1}(x_{m-1})}\cdots\int_{\alpha_{ij+1}}^{\beta_{ij+1}(x_{j+1})}\int_{\alpha_{ij}}^{x_j}\int_{\alpha_{ij-1}}^{\beta_{ij-1}(x_{j-1})}\int_{\alpha_{i1}}^{\beta_{i1}(x_1)}k_j(x_1,x_2,...,x_m,s_1,s_2,...,s_m,u(s_1,s_2,...,s_m))ds_1ds_2...ds_m \quad (3.20)$$

and

$$u(x_1,x_2,...,x_m)=f(x_1,x_2,...,x_m)+$$
$$\sum_{\substack{i=1\\i\neq j}}^{n}\lambda_i\int_{\alpha_{im}}^{\beta_{im}(x_{im})}\int_{\alpha_{im-1}}^{\beta_{im-1}(x_{im-1})}\cdots\int_{\alpha_{i1}}^{\beta_{i1}(x_{i1})}k_i(x_1,x_2,...,x_m,s_1,s_2,...,s_m,u(s_1,s_2,...,s_m))ds_1ds_2...ds_m+$$
$$\lambda_j\int_{\alpha_{im}}^{\beta_{im}(x_m)}\int_{\alpha_{im-1}}^{\beta_{im-1}(x_{m-1})}\cdots\int_{\alpha_{ij+1}}^{\beta_{ij+1}(x_{j+1})}\int_{\alpha_{ij}}^{x_j}\int_{\alpha_{ij-1}}^{\beta_{ij-1}(x_{j-1})}\int_{\alpha_{i1}}^{\beta_{i1}(x_1)}k_j(x_1,x_2,...,x_m,s_1,s_2,...,s_m,u(s_1,s_2,...,s_m))ds_1ds_2...ds_m \quad (3.21)$$

here $x_j \geq \min_{1\leq i\leq n}\{\alpha_{ij}\}$, j=1,2,…,m, represent the generalized multi-dimensional Volterra non-linear integral equations of the first and second kind respectively.

Definition (3.15):

If the kernel k_i in the integral equation (3.13) depends only on the difference $(x_1-s_1,x_2-s_2,...,x_m-s_m)$ that is $k_i(x_1,x_2,...,x_m,s_1,s_2,...,s_m)=k_i(x_1-s_1,x_2-s_2,...,x_m-s_m)$

then the integral equation is said to be the generalized multi-dimensional integral equation of convolution type and it takes the form:

$$h(x_1,x_2,...,x_m)u(x_1,x_2,...,x_m)=f(x_1,x_2,...,x_m)+$$
$$\sum_{i=1}^{n}\lambda_i\int_{\alpha_{im}}^{\beta_{im}(x_m)}\int_{\alpha_{im-1}}^{\beta_{im-1}(x_{m-1})}\cdots\int_{\alpha_{i1}}^{\beta_{i1}(x_1)}k_i(x_1-s_1,x_2-s_2,...,x_m-s_m)u(s_1,s_2,...,s_m)ds_1ds_2...ds_m \quad (3.22)$$

Definition (3.16):

The kernel $k_i(x_1,x_2,...,x_m,s_1,s_2,...,s_m)$ is said to be separable or degenerate kernel if it is of the form:

$$k_i(x_1,x_2,...,x_m,s_1,s_2,...,s_m)=\sum_{j=1}^{p_i}a_{ij}(x_1,x_2,...,x_m)b_{ij}(s_1,s_2,...,s_m),\ \ i\in\{1,2,...,n\}$$

where p_i is finite and $i\in\{1,2,...,n\}$. If each kernel k_i that appeared in the integral equation (3.15) is separable, then equation (3.15) is said to be the generalized multi-dimensional Fredholm linear integral equation with degenerate kernels.

3.3 *Numerical Methods For Solving The Standard Multi-Dimensional Fredholm Linear Integral Equations:*

In this section, we generalize some of the quadrature methods namely, trapezoidal and Simpson's rules to solve the multi-dimensional Fredholm linear integral equations of the second kind.

3.3.1 *The Trapezoidal Rule:*

It is known that, the trapezoidal rule is a numerical method used to solve the one-dimensional integral equations, [16], [7].

In this section, we extend the trapezoidal to solve the multi-dimensional Fredholm linear integral equation of the second kind, but before that we need to derive the trapezoidal rule for evaluating multiple integrals. So, for simplicity, consider $\int_c^d\int_a^b f(x,y)dxdy$. First we use the repeated trapezoidal rule to evaluate $\int_a^b f(x,y)dx$ by treating y as a constant. Therefore the interval [a,b] is subdivided into n subintervals $[x_i, x_{i+1}],\ i = 0,1,...,n-1$ of equal width $h_x = \frac{b-a}{n}$ such that $x_i = a + ih_x,\ i = 0,1,\dots,n$. Hence

$$\int_a^b f(x,y)dx \approx \frac{h_x}{2}\left[f(x_0,y) + 2\sum_{i=1}^{n-1} f(x_i,y) + f(x_n,y)\right]$$

with the global error E_x given by

$$E_x = \frac{-(b-a)}{12}h_x^2\frac{\partial^2 f(\mu,y)}{\partial x^2},\quad a < \mu < b$$

Thus

$$\int_c^d\int_a^b f(x,y)dxdy \approx \frac{h_x}{2}\left[\int_c^d f(x_0,y)dy + 2\sum_{i=1}^{n-1}\int_c^d f(x_i,y)dy + \int_c^d f(x_n,y)dy\right] \qquad (3.23)$$

The repeated trapezoidal rule is now employed on each integral that appeared in the integral equation (3.23). So, assume that the interval [c,d] is subdivided into m subinterval $[y_j, y_{j+1}],\ j = 0,1,...,m-1$ of equal width $h_y = \frac{d-c}{m}$ such that $y_j = c + jh_y,\ j = 0,1,\dots,m.$ Hence

$$\int_c^d f(x_0,y)dy \approx \frac{h_y}{2}\left[f(x_0,y_0) + 2\sum_{j=1}^{m-1} f(x_0,y_j) + f(x_0,y_m)\right]$$

with the global error E_0 given by

$$E_0=\frac{-(d-c)}{12}h_y^2\frac{\partial^2 f(x_0,\xi_0)}{\partial y^2},\quad c<\xi_0<d$$

$$\int_c^d f(x_i,y)dy\approx\frac{h_y}{2}\left[f(x_i,y_0)+2\sum_{j=1}^{m-1}f(x_i,y_j)+f(x_i,y_m)\right],\ i=1,2,...,n-1$$

with the global error E_i given by

$$E_i=\frac{-(d-c)}{12}h_y^2\frac{\partial^2 f(x_i,\xi_i)}{\partial y^2},\quad c<\xi_i<d,\ i=1,2,...,n-1$$

and

$$\int_c^d f(x_n,y)dy\approx\frac{h_y}{2}\left[f(x_n,y_0)+2\sum_{j=1}^{m-1}f(x_n,y_j)+f(x_n,y_m)\right]$$

with the global error E_n given by

$$E_n=\frac{-(d-c)}{12}h_y^2\frac{\partial^2 f(x_n,\xi_n)}{\partial y^2},\quad c<\xi_n<d$$

By substituting these values of integrals into equation (3.23), one can have

$$\int_c^d\int_a^b f(x,y)dxdy\approx\frac{h_xh_y}{4}\Bigg[f(x_0,y_0)+2\sum_{j=1}^{m-1}f(x_0,y_j)+f(x_0,y_m)+2\sum_{i=1}^{n-1}f(x_i,y_0)+$$
$$4\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}f(x_i,y_j)+2\sum_{i=1}^{n-1}f(x_i,y_m)+f(x_n,y_0)+2\sum_{j=1}^{m-1}f(x_n,y_j)+f(x_n,y_m)\Bigg]$$

with the global error E given by

$$E=\frac{h_x}{2}E_0+h_x\sum_{i=1}^{n-1}E_i+\frac{h_x}{2}E_n+\int_c^d E_x dy$$
$$=-\frac{(d-c)}{24}h_xh_y^2\frac{\partial^2 f(x_0,\xi_0)}{\partial y^2}-\frac{(d-c)}{12}h_xh_y^2\sum_{i=1}^{n-1}\frac{\partial^2 f(x_i,\xi_i)}{\partial y^2}-\frac{(d-c)}{24}h_xh_y^2\frac{\partial^2 f(x_n,\xi_n)}{\partial y^2}-$$
$$\frac{(b-a)}{12}h_x^2\left[\frac{h_y}{2}\frac{\partial^2 f(\mu,y_0)}{\partial x^2}+h_y\sum_{j=1}^{m-1}\frac{\partial^2 f(\mu,y_j)}{\partial x^2}+\frac{h_y}{2}\frac{\partial^2 f(\mu,y_m)}{\partial x^2}-\frac{(d-c)}{12}h_y^2\frac{\partial^4 f(\mu,\xi^*)}{\partial x^2\partial y^2}\right]$$

$$=-\frac{(d-c)}{24}h_x h_y^2\left[\frac{\partial^2 f(x_0,\xi_0)}{\partial y^2}+2\sum_{i=1}^{n-1}\frac{\partial^2 f(x_i,\xi_i)}{\partial y^2}+\frac{\partial^2 f(x_n,\xi_n)}{\partial y^2}\right]-$$

$$\frac{(b-a)}{24}h_x^2 h_y\left[\frac{\partial^2 f(\mu,y_0)}{\partial x^2}+2\sum_{j=1}^{m-1}\frac{\partial^2 f(\mu,y_j)}{\partial x^2}+\frac{\partial^2 f(\mu,y_m)}{\partial x^2}\right]+\frac{(b-a)(d-c)}{144}h_x^2 h_y^2\frac{\partial^4 f(\mu,\xi^*)}{\partial x^2\partial y^2}$$

where $c<\xi^*<d$.

Now, consider the two-dimensional Fredholm linear integral equation of the second kind:

$$u(x,y)=f(x,y)+\lambda\int_c^d\int_a^b k(x,y,s,t)u(s,t)\,ds\,dt,\quad a\le x\le b,\quad c\le y\le d \tag{3.24}$$

By subdividing the intervals [a,b] and [c,d] into n and m subintervals, such that $x_i=a+ih_x$, $i=0,1,\ldots,n$ and $y_j=c+jh_y$, $j=0,1,\ldots,m$, where $h_x=\frac{b-a}{n}$ and $h_y=\frac{d-c}{m}$ and by replacing the integral term that appeared in the right hand side of the above equation by the repeated trapezoidal rule, one can get

$$u(x,y)=f(x,y)+\lambda\frac{h_x h_y}{4}\big[k(x,y,a,c)u(a,c)+k(x,y,a,d)u(a,d)+k(x,y,b,c)u(b,c)+$$
$$k(x,y,b,d)u(b,d)\big]+\lambda h_x h_y\sum_{l=1}^{n-1}\sum_{p=1}^{m-1}k(x,y,s_l,t_p)u(s_l,t_p)+\lambda\frac{h_x h_y}{2}\left\{\sum_{l=1}^{n-1}\big[k(x,y,s_l,c)u(s_l,c)+\right.$$
$$\left.k(x,y,s_l,d)u(s_l,d)\big]+\sum_{p=1}^{m-1}\big[k(x,y,a,t_p)u(a,t_p)+k(x,y,b,t_p)u(b,t_p)\big]\right\}$$

By substituting $x=x_i$ and $y=y_j$, i=0,1,…,n, j=0,1,…,m in the above equation one can have

$$u(x_i,y_j)=f(x_i,y_j)+\lambda\frac{h_x h_y}{4}\big[k(x_i,y_j,a,c)u(a,c)+k(x_i,y_j,a,d)u(a,d)+k(x_i,y_j,b,c)u(b,c)+$$
$$k(x_i,y_j,b,d)u(b,d)\big]+\lambda h_x h_y\sum_{l=1}^{n-1}\sum_{p=1}^{m-1}k(x_i,y_j,s_l,t_p)u(s_l,t_p)+\lambda\frac{h_x h_y}{2}\left\{\sum_{l=1}^{n-1}\big[k(x_i,y_j,s_l,c)u(s_l,c)+\right.$$
$$\left.k(x_i,y_j,s_l,d)u(s_l,d)\big]+\sum_{p=1}^{m-1}\big[k(x_i,y_j,a,t_p)u(a,t_p)+k(x_i,y_j,b,t_p)u(b,t_p)\big]\right\},\ i=0,1,\ldots,n,\ j=0,1,\ldots,m$$

By evaluating the above equation at each i=0,1,…,n and j=0,1,…,m, one can get a system of $(n+1)\times(m+1)$ linear equations with $(n+1)\times(m+1)$ unknowns $\{u(x_i, y_j)\}_{i=0,\,j=0}^{n,m}$. This system can be written as

$$AU=F \qquad (3.25)$$

where A is the matrix of the coefficients defined by

U is the matrix of solutions and F is the matrix of non-homogeneous part, defined by

$$U=\begin{pmatrix} u_{0,0} \\ u_{1,0} \\ \vdots \\ u_{n,0} \\ u_{0,1} \\ u_{1,1} \\ \vdots \\ u_{n,1} \\ \vdots \\ u_{0,m} \\ u_{1,m} \\ \vdots \\ u_{n,m} \end{pmatrix} \quad \text{and} \quad F=\begin{pmatrix} f_{0,0} \\ f_{1,0} \\ \vdots \\ f_{n,0} \\ f_{0,1} \\ f_{1,1} \\ \vdots \\ f_{n,1} \\ \vdots \\ f_{0,m} \\ f_{1,m} \\ \vdots \\ f_{n,m} \end{pmatrix} \qquad (3.27)$$

Here $u_{i,j}=u(x_i,y_j)$, $f_{i,j}=f(x_i,y_j)$, $k_{i,j}^{\ell,p}=k(x_i,y_j,s_\ell,t_p)$, $z_{\ell,p}=\dfrac{h_x h_y}{4}w(x_\ell,y_p)$, $i,\ell=0,1,\dots,n$, $j,p=0,1,\dots,m$ and the values of $w(x_\ell,y_p)$ are given in table (3.1). Therefore the approximated solutions $\{u(x_i,y_j)\}_{i=0,\,j=0}^{n,m}$ can be obtained from $U=A^{-1}F$.

Table (3.1) represent the values of $w(x_i,y_j)$, i=0,1,…,n, j=0,1,…,m, for the trapezoidal rule.

$w(x_i,y_j)$	y_0	y_1	y_2	$\cdots$	y_{m-2}	y_{m-1}	y_m
x_0	1	2	2	$\cdots$	2	2	1
x_1	2	4	4	$\cdots$	4	4	2
x_2	2	4	4	$\cdots$	4	4	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
x_{n-2}	2	4	4	$\cdots$	4	4	2
x_{n-1}	2	4	4	$\cdots$	4	4	2
x_n	1	2	2	$\cdots$	2	2	1

To illustrate this method, consider the following example:

Example (3.1):

Consider the two-dimensional Fredholm linear integral equation of second kind:

$$u(x,y) = yx^2 - \frac{1}{6}(x+y) - \frac{17}{22} + \int_0^1\int_0^1 (x+y+s+t)u(s,t)dsdt.$$

We solve this example numerically with the repeated trapezoidal rule. To do this, first we subdivide the interval [0,1] into 10 subintervals such that $x_i = \frac{i}{10},\ i = 0,1,...,10$. Also, by subdividing the interval [0,1] into 10 subintervals such that $y_j = \frac{j}{10},\ j = 0,1,...,10.$ Then the above integral equation becomes:

$$u_{i.j} = y_j x_i^2 - \frac{1}{6}(x_i + y_j) - \frac{17}{22} + \frac{1}{400}\Big[(x_i + y_j)u_{0,0} + (x_i + y_j + 1)u_{0,10} + (x_i + y_j + 1)u_{10,0} +$$
$$(x_i + y_j + 2)u_{10,10}\Big] + \frac{1}{100}\sum_{\ell=1}^{9}\sum_{p=1}^{9}(x_i + y_j + x_\ell + y_p)u_{\ell,p} + \frac{1}{200}\Bigg\{\sum_{\ell=1}^{9}\Big[x_i + y_j + x_\ell)u_{\ell,0} +$$
$$(x_i + y_j + x_\ell + 1)u_{\ell,10}\Big] + \sum_{p=1}^{9}\Big[(x_i + y_j + y_p)u_{0,p} + (x_i + y_j + 1 + y_p)u_{10,p}\Big]\Bigg\}$$

By evaluating the above equation at each i,j=0,1,…,10 one can get the solutions of $\left\{u(x_i, y_j)\right\}_{i=0,\,j=0}^{10,10}$, which tabulated in the appendix (see program (3.1)).

Second, we subdivide the interval [0,1] into 20 subintervals such that $x_i = \frac{i}{20},\ i = 0,1,...,20,$ Also, by subdividing the interval [0,1] into 20 subintervals such that $y_j = \frac{j}{20},\ j = 0,1,...,20.$ Then the above integral equation becomes:

$$u_{i.j}=y_jx_i^2-\frac{1}{6}(x_i+y_j)-\frac{17}{22}+\frac{1}{1600}\Big[(x_i+y_j)u_{0,0}+(x_i+y_j+1)u_{0,20}+(x_i+y_j+1)u_{20,0}+$$
$$(x_i+y_j+2)u_{20,20}\Big]+\frac{1}{400}\sum_{\ell=1}^{19}\sum_{p=1}^{19}(x_i+y_j+x_\ell+y_p)u_{\ell,p}+\frac{1}{800}\Bigg\{\sum_{\ell=1}^{19}\Big[x_i+y_j+x_\ell)u_{\ell,0}+$$
$$(x_i+y_j+x_\ell+1)u_{\ell,20}\Big]+\sum_{p=1}^{19}\Big[(x_i+y_j+y_p)u_{0,p}+(x_i+y_j+1+y_p)u_{20,p}\Big]\Bigg\}$$

By evaluating the above equation at each i,j=0,1,…,20 one can get the solutions of $\{u(x_i,y_j)\}_{i=0,\,j=0}^{20,20}$, which tabulated in the appendix (see program (3.1)).

Third, we subdivide the interval [0,1] into 30 subintervals such that $x_i=\frac{i}{30},\ i=0,1,...,30,$ Also, by subdividing the interval [0,1] into 20 subintervals such that $y_j=\frac{j}{20},\ j=0,1,...,20.$ Then the above integral equation becomes:

$$u_{i.j}=y_jx_i^2-\frac{1}{6}(x_i+y_j)-\frac{17}{22}+\frac{1}{2400}\Big[(x_i+y_j)u_{0,0}+(x_i+y_j+1)u_{0,20}+(x_i+y_j+1)u_{30,0}+$$
$$(x_i+y_j+2)u_{30,20}\Big]+\frac{1}{600}\sum_{\ell=1}^{29}\sum_{p=1}^{19}(x_i+y_j+x_\ell+y_p)u_{\ell,p}+\frac{1}{1200}\Bigg\{\sum_{\ell=1}^{29}\Big[x_i+y_j+x_\ell)u_{\ell,0}+$$
$$(x_i+y_j+x_\ell+1)u_{\ell,20}\Big]+\sum_{p=1}^{19}\Big[(x_i+y_j+y_p)u_{0,p}+(x_i+y_j+1+y_p)u_{30,p}\Big]\Bigg\}$$

By evaluating the above equation at each i=0,1,…,30 and each j=0,1,…,20 one can get the solutions of $\{u(x_i,y_j)\}_{i=0,\,j=0}^{30,20}$, which tabulated in the appendix (see program (3.1)).

Fourth, we subdivide the x-interval [0,1] into 20 and 40 subintervals and we subdivide the y-interval [0,1] into 30 and 40 subintervals respectively and by following the same previous steps one can get the results that can be tabulated. Some of these results are tabulated down.

Table (3.2) represent the least square error for example (3.1) by running program (3.1) for some values of n and m at specific points

Nodes (x,y)	n=20 m=20	n=30 m=20	n=20 m=30	n=40 m=40
(0,0)	-0.000833	-0.000857	-0.000857	-0.0009
(0.1,0.1)	-0.000237	0.000242	0.000242	0.000298
(0.2,0.2)	0.0063584	0.006421	0.006421	0.00667
(0.3,0.3)	0.0249544	0.025436	0.025436	0.02756
(0.4,0.4)	0.0615503	0.062332	0.062332	0.06435
(0.5,0.5)	0.1221462	0.122437	0.122437	0.12432
(0.6,0.6)	0.2127421	0.21282	0.21282	0.21453
(0.7,0.7)	0.3393380	0.34118	0.34118	0.35776
(0.8,0.8)	0.5079339	0.51243	0.51243	0.52567
(0.9,0.9)	0.7245299	0.73287	0.73287	0.74923
(1,1)	0.9951258	0.99667	0.99667	0.99970

3.3.2 *Simpson's 1/3 Rule:*

Like the trapezoidal rule, Simpson's 1/3 rule is also a numerical method used to solve the one-dimensional integral equations, [7], [17].

In this section, we extend Simpson's 1/3 rule to solve the multi-dimensional Fredholm linear integral equation of the second kind, but before that we need to derive Simpson's 1/3 rule for evaluating multiple integrals. So, for simplicity, consider $\int_c^d \int_a^b f(x,y)dxdy$. First we use the repeated Simpson's 1/3 rule to evaluate $\int_a^b f(x,y)dx$ by treating y as a constant. Therefore the interval [a,b] is subdivided into n subintervals $[x_i, x_{i+1}],\ \ i = 0,1,\ldots,n-1$ of equal width $h_x = \dfrac{b-a}{n}$ such that n is an even positive integer and $x_i = a + ih_x,\ \ i = 0,1,2,\ldots,n$. Hence,

$$\int_a^b f(x,y)dx \approx \frac{h_x}{3}\left[f(x_0,y) + 4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1},y) + 2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i},y) + f(x_n,y) \right]$$

with the global error E_x given by

$$E_x = \frac{-(b-a)}{180} h_x^4 \frac{\partial^4 f(\mu,y)}{\partial x^4},\ \ a < \mu < b$$

Thus

$$\int_c^d\int_a^b f(x,y)dxdy \approx \frac{h_x}{3}\left[\int_c^d f(x_0,y)dy + 4\sum_{i=1}^{\frac{n}{2}}\int_c^d f(x_{2i-1},y)dy + 2\sum_{i=1}^{\frac{n}{2}-1}\int_c^d f(x_{2i},y)dy + \int_c^d f(x_n,y)dy \right] \quad (3.28)$$

The repeated Simpson's 1/3 rule is now employed on each integral in the integral equation (3.28). So, assume that the interval [c, d] is subdivided into m subintervals $[y_j, y_{j+1}],\ \ j = 0,1,\ldots,m-1$ of equal width $h_y = \dfrac{d-c}{m}$ such that m is an even positive integer and $y_j = c + jh_y,\ j = 0,1,\ldots,m$. Hence

$$\int_c^d f(x_0,y)dy \approx \frac{h_y}{3}\left[f(x_0,y_0) + 4\sum_{j=1}^{\frac{m}{2}} f(x_0,y_{2j-1}) + 2\sum_{j=1}^{\frac{m}{2}-1} f(x_0,y_{2j}) + f(x_0,y_m) \right]$$

with the global error E_0 given by

$$E_0 = \frac{-(d-c)}{180} h_y^4 \frac{\partial^4 f(x_0,\xi_0)}{\partial y^4},\ \ c < \xi_0 < d$$

$$\int_c^d f(x_{2i-1},y)dy \approx \frac{h_y}{3}\left[f(x_{2i-1},y_0) + 4\sum_{j=1}^{\frac{m}{2}} f(x_{2i-1},y_{2j-1}) + 2\sum_{j=1}^{\frac{m}{2}-1} f(x_{2i-1},y_{2j}) + f(x_{2i-1},y_m) \right]$$

where $i = 1, 2, ..., \frac{n}{2}$, with the global error E_{2i-1} given by

$$E_{2i-1} = \frac{-(d-c)}{180} h_y^4 \frac{\partial^4 f(x_{2i-1}, \xi_{2i-1})}{\partial y^4}, \quad c < \xi_{2i-1} < d, \ \ i = 1, 2, ..., n/2$$

$$\int_c^d f(x_{2i}, y)dy \approx \frac{h_y}{3}\left[f(x_{2i}, y_0) + 4\sum_{j=1}^{\frac{m}{2}} f(x_{2i}, y_{2j-1}) + 2\sum_{j=1}^{\frac{m}{2}-1} f(x_{2i}, y_{2j}) + f(x_{2i}, y_m) \right]$$

where $i = 1, 2, ..., \frac{n}{2} - 1$, with the global error E_{2i} given by

$$E_{2i} = \frac{-(d-c)}{180} h_y^4 \frac{\partial^4 f(x_{2i}, \xi_{2i})}{\partial y^4}, \quad c < \xi_{2i} < d, \ \ i = 1, 2, ..., (n/2) - 1$$

and

$$\int_c^d f(x_n, y)dy \approx \frac{h_y}{3}\left[f(x_n, y_0) + 4\sum_{j=1}^{\frac{m}{2}} f(x_n, y_{2j-1}) + 2\sum_{j=1}^{\frac{m}{2}-1} f(x_n, y_{2j}) + f(x_n, y_m) \right]$$

with the global error E_n given by

$$E_n = \frac{-(b-a)}{180} h_y^4 \frac{\partial^4 f(x_n, \xi_n)}{\partial y^4}, \quad c < \xi_n < d$$

By substituting these values of the integrals into the integral equation (3.28), one can have

$$\int_c^d \int_a^b f(x, y)dxdy \approx \frac{h_x h_y}{9}\Bigg[f(x_0, y_0) + 4\sum_{j=1}^{\frac{m}{2}} f(x_0, y_{2j-1}) + 2\sum_{j=1}^{\frac{m}{2}-1} f(x_0, y_{2j}) + f(x_0, y_m) +$$

$$4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}, y_0) + 16\sum_{i=1}^{\frac{n}{2}}\sum_{j=1}^{\frac{m}{2}} f(x_{2i-1}, y_{2j-1}) + 8\sum_{i=1}^{\frac{n}{2}}\sum_{j=1}^{\frac{m}{2}-1} f(x_{2i-1}, y_{2j}) + 4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}, y_m) +$$

$$2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i},y_0)+8\sum_{i=1}^{\frac{n}{2}-1}\sum_{j=1}^{\frac{m}{2}} f(x_{2i},y_{2j-1})+4\sum_{i=1}^{\frac{n}{2}-1}\sum_{j=1}^{\frac{m}{2}-1} f(x_{2i},y_{2j})+2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i},y_m)+f(x_n,y_0)+$$

$$\left. 4\sum_{j=1}^{\frac{m}{2}} f(x_n,y_{2j-1})+2\sum_{j=1}^{\frac{m}{2}-1} f(x_n,y_{2j})+f(x_n,y_m)\right]$$

Now, consider the two-dimensional Fredholm linear integral equation of the second kind (3.24):

$$u(x,y)=f(x,y)+\lambda\int_c^d\int_a^b k(x,y,s,t)u(s,t)dsdt,\quad a\le x\le b,\quad c\le y\le d$$

By subdividing the intervals [a,b] and [c,d] into n and m such that $x_i=a+ih_x,\ i=0,1,\ldots,n$ and $y_j=c+jh_y,\ j=0,1,\ldots,m,$ where $h_x=\frac{b-a}{n}$ and $h_y=\frac{d-c}{m}$, where n and m are even positive integers. Then by replacing the integral term that appeared in the right hand side of the above equation by the repeated Simpson's 1/3 rule, one can get

$$u(x,y)=f(x,y)+\lambda\frac{h_xh_y}{9}\Big[k(x,y,a,c)u(a,c)+k(x,y,a,d)u(a,d)+k(x,y,b,c)u(b,c)+$$

$$k(x,y,b,d)u(b,d)+2\sum_{\ell=1}^{\frac{n}{2}-1}\left[k(x,y,s_{2\ell},c)u(s_{2\ell},c)+k(x,y,s_{2\ell},d)u(s_{2\ell},d)\right]+$$

$$2\sum_{p=1}^{\frac{m}{2}-1}\left[k(x,y,a,t_{2p})u(a,t_{2p})+k(x,y,b,t_{2p})u(b,t_{2p})\right]+4\sum_{\ell=1}^{\frac{n}{2}}\Big[k(x,y,s_{2\ell-1},c)u(s_{2\ell-1},c)+$$

$$k(x,y,s_{2\ell-1},d)u(s_{2\ell-1},d)\Big]+4\sum_{p=1}^{\frac{m}{2}}\left[k(x,y,a,t_{2p-1})u(a,t_{2p-1})+k(x,y,b,t_{2p-1})u(b,t_{2p-1})\right]+$$

$$8\sum_{\ell=1}^{\frac{n}{2}}\sum_{p=1}^{\frac{m}{2}-1}k(x,y,s_{2\ell-1},t_{2p})u(s_{2\ell-1},t_{2p})+8\sum_{\ell=1}^{\frac{n}{2}-1}\sum_{p=1}^{\frac{m}{2}}k(x,y,s_{2\ell},t_{2p-1})u(s_{2\ell},t_{2p-1})+$$

$$\left.4\sum_{\ell=1}^{\frac{n}{2}-1}\sum_{p=1}^{\frac{m}{2}-1}k(x,y,s_{2\ell},t_{2p})u(s_{2\ell},t_{2p})+16\sum_{\ell=1}^{\frac{n}{2}}\sum_{p=1}^{\frac{m}{2}}k(x,y,s_{2\ell-1},t_{2p-1})u(s_{2\ell-1},t_{2p-1})\right]$$

By substituting $x = x_i$ and $y = y_j$, i=0,1,…,n, j=0,1,…,m one can have

$$u(x_i, y_j) = f(x_i, y_j) + \lambda \frac{h_x h_y}{9} \Big\{ k(x_i, y_j, s_0, t_0) u(s_0, t_0) + k(x_i, y_j, s_0, t_m) u(s_0, t_m) +$$

$$k(x_i, y_j, s_n, t_0) u(s_n, t_0) + k(x_i, y_j, s_n, t_m) u(s_n, t_m) + 2\sum_{\ell=1}^{\frac{n}{2}-1} \Big[k(x_i, y_j, s_{2\ell}, t_0) u(s_{2\ell}, t_0) +$$

$$k(x_i, y_j, s_{2\ell}, t_m) u(s_{2\ell}, t_m) \Big] + 2\sum_{p=1}^{\frac{m}{2}-1} \Big[k(x_i, y_j, s_0, t_{2p}) u(s_0, t_{2p}) + k(x_i, y_j, s_n, t_{2p}) u(s_n, t_{2p}) \Big] +$$

$$4\sum_{\ell=1}^{\frac{n}{2}} \Big[k(x_i, y_j, s_{2\ell-1}, t_0) u(s_{2\ell-1}, t_0) + k(x_i, y_j, s_{2\ell-1}, t_m) u(s_{2\ell-1}, t_m) \Big] +$$

$$4\sum_{p=1}^{\frac{m}{2}} \Big[k(x_i, y_j, s_0, t_{2p-1}) u(s_0, t_{2p-1}) + k(x_i, y_j, s_n, t_{2p-1}) u(s_n, t_{2p-1}) \Big] +$$

$$8\sum_{\ell=1}^{\frac{n}{2}} \sum_{p=1}^{\frac{m}{2}-1} k(x_i, y_j, s_{2\ell-1}, t_{2p}) u(s_{2\ell-1}, t_{2p}) + 8\sum_{\ell=1}^{\frac{n}{2}-1} \sum_{p=1}^{\frac{m}{2}} k(x_i, y_j, s_{2\ell}, t_{2p-1}) u(s_{2\ell}, t_{2p-1}) +$$

$$4\sum_{\ell=1}^{\frac{n}{2}-1} \sum_{p=1}^{\frac{m}{2}-1} k(x_i, y_j, s_{2\ell}, t_{2p}) u(s_{2\ell}, t_{2p}) + 16\sum_{\ell=1}^{\frac{n}{2}} \sum_{p=1}^{\frac{m}{2}} k(x_i, y_j, s_{2\ell-1}, t_{2p-1}) u(s_{2\ell-1}, t_{2p-1}) \Big\}$$

where i= 0,1,…,n and j=0,1,…,m. By evaluating the above equation at each i=0,1,…,n and j=0,1,…,m, one can get a system of $(n+1)\times(m+1)$ linear equations with $(n+1)\times(m+1)$ unknowns $\{u(x_i, y_j)\}_{i=0, j=0}^{n,m}$ .This system can be written as equation (3.25), where A is the matrix of coefficients defined by equation (3.26), U is the matrix of solutions and F is the matrix of non-homogeneous part defined by equation (3.27), where $u_{i,j} = u(x_i, y_j)$, $f_{i,j} = f(x_i, y_j)$, $k_{i,j}^{\ell,p} = k(x_i, y_j, s_\ell, t_p)$ and $z_{\ell,p} = \frac{h_x h_y}{9} w(x_\ell, y_p)$, $i, l = 0,1,...,n$,

$j,p=0,1,\ldots,m$, and the values $w(x_\ell, y_p)$ is given by table (3.2). Therefore the approximated solutions $\{u(x_i, y_j)\}_{i=0,\,j=0}^{n,m}$ can be obtained from $U=A^{-1}F$.

Table (3.3) represent the values of $w(x_\ell, y_p)$, $\ell=0,1,\ldots,n$, $p=0,1,\ldots,m$ for the Simpson's 1/3 rule.

$w(x_l, y_p)$	y_0	Y_1	y_2	y_3	$\cdots$	y_{m-3}	y_{m-2}	y_{m-1}	y_m
x_0	1	4	2	4	…	4	2	4	1
x_1	4	16	8	16	…	16	8	16	4
x_2	2	8	4	8	…	8	4	8	2
x_3	4	16	8	16	…	16	8	16	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{n-3}	4	16	8	16	…	16	8	16	4
x_{n-2}	2	8	4	8	…	8	4	8	2
x_{n-1}	4	16	8	16	…	16	8	16	4
x_n	1	4	2	4	…	4	2	4	1

To illustrate this method, consider the following example:

Example (3.2):

Consider the two-dimensional linear Fredholm integral equation of second kind:

$$u(x,y)=e^{x+y}+(1-e^{1})(x+y)+\int_0^1\int_0^1 (sx+ty)u(s,t)\,ds\,dt$$

We solve this example numerically with the repeated Simpson's 1/3 rule. To do this, first we subdivide the interval [0,1] into 10 subintervals such that $x_i=\frac{i}{10}$, $i=0,1,\ldots,10$. Also, by subdividing the interval [0,1] into 10 subintervals such that $y_j=\frac{j}{10}$, $j=0,1,\ldots,10$. Then the above integral equation becomes

$$u_{i,j}=e^{x_i+y_j}+(1-e^1)(x_i+y_j)+\frac{1}{900}\Big[y_j u_{0,10}+x_i u_{10,0}+(x_i+y_j)u_{10,10}+2\sum_{\ell=1}^{4}\big[x_{2\ell}x_i u_{2\ell,0}+(x_{2\ell}x_i+y_j)u_{2\ell,10}\big]+2\sum_{p=1}^{4}\big[y_{2p}y_j u_{0,2p}+(x_i+y_{2p}y_j)u_{10,2p}\big]+4\sum_{\ell=1}^{5}\big[x_{2\ell-1}x_i u_{2\ell-1,0}+(x_{2\ell-1}x_i+y_j)u_{2\ell-1,10}\big]+4\sum_{p=1}^{5}\big[y_{2p-1}y_j u_{0,2p-1}+(x_i+y_{2p-1}y_j)u_{10,2p-1}\big]+8\sum_{\ell=1}^{5}\sum_{p=1}^{4}(x_{2\ell-1}x_i+y_{2p}y_j)u_{2\ell-1,2p}+8\sum_{\ell=1}^{4}\sum_{p=1}^{5}(x_{2\ell}x_i+y_{2p-1}y_j)u_{2\ell,2p-1}+4\sum_{\ell=1}^{4}\sum_{p=1}^{4}(x_{2\ell}x_i+y_{2p}y_j)u_{2\ell,2p}+16\sum_{\ell=1}^{5}\sum_{p=1}^{5}(x_{2\ell-1}x_i+y_{2p-1}y_j)u_{2\ell-1,2p-1}\Big]$$

By evaluating the above equation at each i,j=0,1,…,10 one can get the solutions of $\{u(x_i,y_j)\}_{i=0,\,j=0}^{10,10}$, which tabulated in the appendix (see program (3.2)).

Second, we subdivide the interval [0,1] into 20 subintervals such that $x_i=\frac{i}{20},\ i=0,1,\dots,20,$ Also, by subdividing the interval [0,1] into 20 subintervals such that $y_j=\frac{j}{20},\ j=0,1,\dots,20.$ Then the above integral equation becomes:

$$u_{i,j}=e^{x_i+y_j}+(1-e^1)(x_i+y_j)+\frac{1}{3600}\Big[y_j u_{0,20}+x_i u_{20,0}+(x_i+y_j)u_{20,20}+2\sum_{\ell=1}^{9}\big[x_{2\ell}x_i u_{2\ell,0}+(x_{2\ell}x_i+y_j)u_{2\ell,20}\big]+2\sum_{p=1}^{9}\big[y_{2p}y_j u_{0,2p}+(x_i+y_{2p}y_j)u_{20,2p}\big]+4\sum_{\ell=1}^{10}\big[x_{2\ell-1}x_i u_{2\ell-1,0}+(x_{2\ell-1}x_i+y_j)u_{2\ell-1,20}\big]+4\sum_{p=1}^{10}\big[y_{2p-1}y_j u_{0,2p-1}+(x_i+y_{2p-1}y_j)u_{20,2p-1}\big]+8\sum_{\ell=1}^{10}\sum_{p=1}^{9}(x_{2\ell-1}x_i+y_{2p}y_j)u_{2\ell-1,2p}+8\sum_{\ell=1}^{9}\sum_{p=1}^{10}(x_{2\ell}x_i+y_{2p-1}y_j)u_{2\ell,2p-1}+4\sum_{\ell=1}^{9}\sum_{p=1}^{9}(x_{2\ell}x_i+y_{2p}y_j)u_{2\ell,2p}+16\sum_{\ell=1}^{10}\sum_{p=1}^{10}(x_{2\ell-1}x_i+y_{2p-1}y_j)u_{2\ell-1,2p-1}\Big]$$

By evaluating the above equation at each i,j=0,1,…,20 one can get the solutions of $\left\{u(x_i, y_j)\right\}_{i=0,\, j=0}^{20,20}$, which tabulated in the appendix (see program (3.2)).

Third, we subdivide the interval [0,1] into 30 subintervals such that $x_i = \frac{i}{30},\ i = 0,1,...,30,$ Also, by subdividing the interval [0,1] into 20 subintervals such that $y_j = \frac{j}{20},\ j = 0,1,...,20.$ Then the above integral equation becomes:

$$u_{i,j} = e^{x_i+y_j} + (1-e^1)(x_i+y_j) + \frac{1}{5400}\Big[y_j u_{0,20} + x_i u_{30,0} + (x_i+y_j)u_{30,20} + 2\sum_{\ell=1}^{14}\Big[x_{2\ell}x_i u_{2\ell,0} +$$
$$(x_{2\ell}x_i + y_j)u_{2\ell,20}\Big] + 2\sum_{p=1}^{9}\Big[y_{2p}y_j u_{0,2p} + (x_i + y_{2p}y_j)u_{30,2p}\Big] + 4\sum_{\ell=1}^{15}\Big[x_{2\ell-1}x_i u_{2\ell-1,0} +$$
$$(x_{2\ell-1}x_i + y_j)u_{2\ell-1,20}\Big] + 4\sum_{p=1}^{10}\Big[y_{2p-1}y_j u_{0,2p-1} + (x_i + y_{2p-1}y_j)u_{30,2p-1}\Big] +$$
$$8\sum_{\ell=1}^{15}\sum_{p=1}^{9}(x_{2\ell-1}x_i + y_{2p}y_j)u_{2\ell-1,2p} + 8\sum_{\ell=1}^{14}\sum_{p=1}^{10}(x_{2\ell}x_i + y_{2p-1}y_j)u_{2\ell,2p-1} +$$
$$4\sum_{\ell=1}^{14}\sum_{p=1}^{9}(x_{2\ell}x_i + y_{2p}y_j)u_{2\ell,2p} + 16\sum_{\ell=1}^{15}\sum_{p=1}^{10}(x_{2\ell-1}x_i + y_{2p-1}y_j)u_{2\ell-1,2p-1}\Big]$$

By evaluating the above equation at each i=0,1,…,30 and each j=0,1,…,20 one can get the solutions of $\left\{u(x_i, y_j)\right\}_{i=0,\, j=0}^{30,20}$, which tabulated in the appendix (see program (3.2)).

Fourth, we subdivide the x-interval [0,1] into 20 and 40 subintervals and we subdivide the y-interval [0,1] into 30 and 40 subintervals respectively and by following the same previous steps one can get the results that can be tabulated. Some of these results are tabulated down.

Table (3.4) represents the numerical solutions of example (2.3), by running program (2.3) for some values of n and m at specific points

Nodes (x,y)	n=20 m=20	n=30 m=20	n=20 m=30	n=40 m=40
(0,0)	1.000000	1.000000	1.000000	1.000000
(0.1,0.1)	1.221406	1.22275	1.22275	1.22367
(0.2,0.2)	1.491832	1.49762	1.49762	1.52732
(0.3,0.3)	1.822130	1.83289	1.83289	1.85371
(0.4,0.4)	2.225557	2.22678	2.22678	2.22893
(0.5,0.5)	2.718302	2.72735	2.72735	2.75631
(0.6,0.6)	3.320141	3.32296	3.32296	3.32819
(0.7,0.7)	4.055228	4.06186	4.06186	4.06326
(0.8,0.8)	4.953064	4.95604	4.95604	4.95629
(0.9,0.9)	6.049683	6.04983	6.04983	6.05196
(1,1)	7.389096	7.39462	7.39462	7.39591

Remarks (3.2):

(1) The previous methods can also used to solve the generalized multi-dimensional linear Fredholm integral equations of the second kind.

(2) Like Simpson's 1/3 rule, Simpson's 3/8 rule can be also used to solve the standard multi-dimensional linear Fredholm integral equations of the second kind.

3.4 *Modified Trapezoid Method For Solving The Multi-Dimensional Integral Equations:*

In this section, we modify some numerical methods namely, the trapezoidal rule to solve special types of the multi-dimensional integral equations namely, the two-dimensional Fredholm linear integral equations of the second kind.

But before that, we must generalize the modified repeated trapezoidal rule to evaluate the multiple integrals $\int_c^d\int_a^b f(x,y)dxdy$. To do this; first, we use the modified repeated trapezoidal rule to evaluate $\int_a^b f(x,y)dx$ by treating y as a constant. Therefore the interval [a,b] is subdivided into n subintervals $[x_i, x_{i+1}]$, $i = 0,1,...,n-1$ of equal width $h_x = \frac{b-a}{n}$ such that $x_i = a + ih_x$, $i = 0,1,\ldots,n$. Hence

$$\int_c^d\int_a^b f(x,y)dxdy \approx \int_c^d\left[\frac{h}{2}f(a,y) + \frac{h}{2}f(b,y) + h\sum_{i=1}^{n-1} f(x_i,y) + \frac{h^2}{12}\frac{\partial f(a,y)}{\partial x} - \frac{h^2}{12}\frac{\partial f(b,y)}{\partial x}\right]dy$$

with the global error E_x given by

$$E_x = \frac{-h^5}{720}\frac{\partial^4 f(\eta,y)}{\partial x^4}, \quad a < \eta < b$$

Then, we use the modified repeated trapezoidal rule to evaluate all the integral terms that appeared in the above equation. So, assume that the interval [c,d] is subdivided into m subintervals $[y_j, y_{j+1}]$, $j = 0,1,...,m-1$ of equal width $h_y = \frac{d-c}{m}$ such that $y_j = c + jh_y$, $j = 0,1,\ldots,m$, to get

$$\int_c^d f(a,y)dy \approx \frac{h_y}{2}f(a,c) + \frac{h_y}{2}f(a,d) + h_y\sum_{j=1}^{m-1} f(a,y_j) + \frac{h_y^2}{12}\frac{\partial f(a,c)}{\partial y} - \frac{h_y^2}{12}\frac{\partial f(a,d)}{\partial y}$$

with the global error E_0 given by

$$E_0 = -\frac{h_y^5}{720}\frac{\partial^4 f(a,\varsigma_0)}{\partial y^4}, \quad c < \varsigma_0 < d$$

$$\int_c^d f(b,y)dy \approx \frac{h_y}{2}f(b,c) + \frac{h_y}{2}f(b,d) + h_y\sum_{j=1}^{m-1} f(b,y_j) + \frac{h_y^2}{12}\frac{\partial f(b,c)}{\partial y} - \frac{h_y^2}{12}\frac{\partial f(b,d)}{\partial y}$$

with the global error E_n given by

$$E_n = -\frac{h_y^5}{720}\frac{\partial^4 f(b,\varsigma_n)}{\partial y^4},\ c < \varsigma_n < d$$

$$\int_c^d f(x_i,y)dy \approx \frac{h_y}{2}f(x_i,c)+\frac{h_y}{2}f(x_i,d)+h_y\sum_{j=1}^{m-1}f(x_i,y_j)+\frac{h_y^2}{12}\frac{\partial f(x_i,c)}{\partial y}-\frac{h_y^2}{12}\frac{\partial f(x_i,d)}{\partial y}$$

with the global error E_i given by

$$E_i = -\frac{h_y^5}{720}\frac{\partial^4 f(x_i,\varsigma_i)}{\partial y^4},\ c < \varsigma_i < d$$

while i=1,2,…,n-1,

$$\int_c^d \frac{\partial f(a,y)}{\partial x}dy \approx \frac{h_y}{2}\frac{\partial f(a,c)}{\partial x}+\frac{h_y}{2}\frac{\partial f(a,d)}{\partial x}+h_y\sum_{j=1}^{m-1}\frac{\partial f(a,y_j)}{\partial x}+\frac{h_y^2}{12}\frac{\partial^2 f(a,c)}{\partial y\partial x}-\frac{h_y^2}{12}\frac{\partial^2 f(a,d)}{\partial y\partial x}$$

with the global error E_1^* given by

$$E_1^* = -\frac{h_y^5}{720}\frac{\partial^5 f(a,\varsigma_1^*)}{\partial y^4\partial x},\ c < \varsigma_1^* < d$$

and

$$\int_c^d \frac{\partial f(b,y)}{\partial x}dy \approx \frac{h_y}{2}\frac{\partial f(b,c)}{\partial x}+\frac{h_y}{2}\frac{\partial f(b,d)}{\partial x}+h_y\sum_{j=1}^{m-1}\frac{\partial f(b,y_j)}{\partial}+\frac{h_y^2}{12}\frac{\partial^2 f(b,c)}{\partial y\partial x}-\frac{h_y^2}{12}\frac{\partial^2 f(b,d)}{\partial y\partial x}$$

with the global error E_2^* given by

$$E_2^* = -\frac{h_y^5}{720}\frac{\partial^5 f(a,\varsigma_1^*)}{\partial y^4\partial x},\ c < \varsigma_2^* < d$$

Therefore

$$\int_c^d\int_a^b f(x,y)dxdy \approx \frac{h_x h_y}{4}f(a,c)+\frac{h_x h_y}{4}f(a,d)+\frac{h_x h_y}{2}\sum_{j=1}^{m-1}f(a,y_j)+\frac{h_x h_y^2}{24}\frac{\partial f(a,c)}{\partial y}-$$
$$\frac{h_x h_y^2}{24}\frac{\partial f(a,d)}{\partial y}+\frac{h_x h_y}{4}f(b,c)+\frac{h_x h_y}{4}f(b,d)+\frac{h_x h_y}{2}\sum_{j=1}^{m-1}f(b,y_j)+\frac{h_x h_y^2}{24}\frac{\partial f(b,c)}{\partial y}-$$
$$\frac{h_x h_y^2}{24}\frac{\partial f(b,d)}{\partial y}+\frac{h_x h_y}{2}\sum_{i=1}^{n-1}f(x_i,c)+\frac{h_x h_y}{2}\sum_{i=1}^{n-1}f(x_i,d)+h_x h_y\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}f(x_i,y_j)+$$

$$\frac{h_x h_y^2}{12}\sum_{i=1}^{n-1}\frac{\partial f(x_i,c)}{\partial y}-\frac{h_x h_y^2}{12}\sum_{i=1}^{n-1}\frac{\partial(x_i,d)}{\partial y}+\frac{h_x^2 h_y}{24}\frac{\partial f(a,c)}{\partial y}+\frac{h_x^2 h_y}{24}\frac{\partial f(a,d)}{\partial x}+\frac{h_x^2 h_y}{12}\sum_{j=1}^{m-1}\frac{\partial f(a,y_j)}{\partial x}+$$

$$\frac{h_x^2 h_y^2}{(12)^2}\frac{\partial^2 f(a,c)}{\partial y\partial x}-\frac{h_x^2 h_y^2}{(12)^2}\frac{\partial^2 f(a,d)}{\partial y\partial x}-\frac{h_x^2 h_y}{24}\frac{\partial f(b,c)}{\partial x}-\frac{h_x^2 h_y}{24}\frac{\partial f(b,d)}{\partial x}-$$

$$\frac{h_x^2 h_y}{12}\sum_{j=1}^{m-1}\frac{\partial f(b,y_j)}{\partial x}-\frac{h_x^2 h_y^2}{(12)^2}\frac{\partial^2 f(b,c)}{\partial y\partial x}+\frac{h_x^2 h_y^2}{(12)^2}\frac{\partial^2 f(b,d)}{\partial y\partial x}$$

with the global error E given by

$$E=-\frac{h_x h_y^5}{1440}\left[\frac{\partial^4 f(a,\varsigma_0)}{\partial y^4}+\frac{\partial^4 f(b,\varsigma_n)}{\partial y^4}\right]-\frac{h_x h_y^5}{720}\sum_{i=1}^{n-1}\frac{\partial^4 f(x_i,\varsigma_i)}{\partial y^4}-\frac{h_x^2 h_y^5}{7640}\left[\frac{\partial^5 f(a,\varsigma_1^*)}{\partial y^4\partial x}-\frac{\partial^5 f(b,\varsigma_2^*)}{\partial y^4\partial x}\right]-$$

$$\frac{h_x^5 h_y}{1440}\left[\frac{\partial^4 f(\eta,c)}{\partial x^4}+\frac{\partial^4 f(\eta,d)}{\partial x^4}\right]-\frac{h_x^5 h_y}{720}\sum_{j=1}^{m-1}\frac{\partial^4 f(\eta,y_j)}{\partial x^4}-\frac{h_x^5 h_y^2}{7640}\left[\frac{\partial^5 f(\eta,c)}{\partial y\partial x^5}+\frac{\partial^5 f(\eta,d)}{\partial y\partial x^4}\right]+$$

$$\frac{h_x^5 h_y^5}{518400}\frac{\partial^8 f(\eta,\varsigma)}{\partial y^4\partial x^4},\ c<\varsigma<d$$

Now, consider the two-dimensional Fredholm linear integral equation of the second kind given by equation (3.24). To solve this integral equation via the modified repeated trapezoidal rule we subdivide the intervals [a,b] and [c,d] into n and m subintervals such that $x_i=a+ih_x$ and $y_j=c+jh_y$ for each i=0,1,…,n and j=0,1,…,m. In this case $h_x=\frac{b-a}{n}$ and $h_y=\frac{d-c}{m}$. By approximating the inner integral term that appeared in the right hand side of equation (3.24), one can get

$$u(x,y)=f(x,y)+\frac{h_x}{2}\int_c^d k(x,y,a,t)u(a,t)dt+\frac{h_x}{2}\int_c^d k(x,y,b,t)u(b,t)dt+$$

$$h_x\int_c^d\sum_{l=1}^{n-1}k(x,y,s_l,t)\,u(s_l,t)dt+\frac{h_x^2}{12}\int_c^d\frac{\partial k(x,y,a,t)}{\partial s}u(a,t)dt+\frac{h_x^2}{12}\int_c^d k(x,y,a,t)\frac{\partial u(a,t)}{\partial s}dt-$$

$$\frac{h_x^2}{12}\int_c^d\frac{\partial k(x,y,b,t)}{\partial s}\,u(b,t)dt-\frac{h_x^2}{12}\int_c^d k(x,y,b,t)\frac{\partial u(b,t)}{\partial s}dt$$

Again, we use the modified repeated trapezoidal rule to evaluate all the integrals that appeared in the above equation to get

$$u(x,y)=f(x,y)+\frac{h_x}{2}\Bigg[\frac{h_y}{2}k(x,y,a,c)u(a,c)+\frac{h_y}{2}k(x,y,a,d)u(a,d)+$$

$$h_y\sum_{p=1}^{m-1}k(x,y,a,t_p)u(a,t_p)+\frac{h_y^2}{12}\frac{\partial k(x,y,a,c)}{\partial t}u(a,c)+\frac{h_y^2}{12}k(x,y,a,c)\frac{\partial u(a,c)}{\partial t}-$$

$$\frac{h_y^2}{12}\frac{\partial k(x,y,a,d)}{\partial t}u(a,d)-\frac{h_y^2}{12}k(x,y,a,d)\frac{\partial u(a,d)}{\partial t}\Bigg]+\frac{h_x}{2}\Bigg[\frac{h_y}{2}k(x,y,b,c)u(b,c)+$$

$$\frac{h_y}{2}k(x,y,b,d)u(b,d)+h_y\sum_{p=1}^{m-1}k(x,y,b,t_p)u(b,t_p)+\frac{h_y^2}{12}\frac{\partial k(x,y,b,c)}{\partial t}u(b,c)+$$

$$\frac{h_y^2}{12}k(x,y,b,c)\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial k(x,y,b,d)}{\partial t}u(b,d)-\frac{h_y^2}{12}k(x,y,b,d)\frac{\partial u(b,d)}{\partial t}\Bigg]+$$

$$h_x\sum_{\ell=1}^{n-1}\Bigg[\frac{h_y}{2}k(x,y,s_\ell,c)\,u(s_\ell,c)+\frac{h_y}{2}k(x,y,s_\ell,d)\,u(s_\ell,d)+h_y\sum_{p=1}^{m-1}k(x,y,s_\ell,t_p)u(s_\ell,t_p)+$$

$$\frac{h_y^2}{12}\frac{\partial k(x,y,s_\ell,c)}{\partial t}u(s_\ell,c)+\frac{h_y^2}{12}k(x,y,s_\ell,c)\frac{\partial u(s_\ell,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial k(x,y,s_\ell,d)}{\partial t}u(s_\ell,d)-$$

$$\frac{h_y^2}{12}k(x,y,s_\ell,d)\frac{\partial u(s_\ell,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\frac{\partial k(x,y,a,c)}{\partial s}u(a,c)+\frac{h_y}{2}\frac{\partial k(x,y,a,d)}{\partial s}u(a,d)+$$

$$h_y\sum_{p=1}^{m-1}\frac{\partial k(x,y,a,t_p)}{\partial s}u(a,t_p)+\frac{h_y^2}{12}\frac{\partial^2 k(x,y,a,c)}{\partial s\partial t}u(a,c)+\frac{h_y^2}{12}\frac{\partial k(x,y,a,c)}{\partial s}\frac{\partial u(a,c)}{\partial t}-$$

$$\frac{h_y^2}{12}\frac{\partial^2 k(x,y,a,d)}{\partial s\partial t}u(a,d)-\frac{h_y^2}{12}\frac{\partial k(x,y,a,d)}{\partial s}\frac{\partial u(a,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}k(x,y,a,c)\frac{\partial u(a,c)}{\partial s}+$$

$$\frac{h_y}{2}k(x,y,a,d)\frac{\partial u(a,d)}{\partial s}+h_y\sum_{p=1}^{m-1}k(x,y,a,t_p)\frac{\partial u(a,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial k(x,y,a,c)}{\partial t}\frac{\partial u(a,c)}{\partial s}+$$

$$\frac{h_y^2}{12}k(x,y,a,c)\frac{\partial^2 u(a,c)}{\partial s\partial t}-\frac{h_y^2}{12}\frac{\partial k(x,y,a,d)}{\partial t}\frac{\partial u(a,d)}{\partial s}-\frac{h_y^2}{12}k(x,y,a,d)\frac{\partial^2 u(a,d)}{\partial s\partial t}\Bigg]-$$

$$\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\frac{\partial k(x,y,b,c)}{\partial s}u(b,c)+\frac{h_y}{2}\frac{\partial k(x,y,b,d)}{\partial s}u(b,d)+h_y\sum_{p=1}^{m-1}\frac{\partial k(x,y,b,t_p)}{\partial s}u(b,t_p)+$$

$$\frac{h_y^2}{12}\frac{\partial^2 k(x,y,b,c)}{\partial s\partial t}u(b,c)+\frac{h_y^2}{12}\frac{\partial k(x,y,b,c)}{\partial s}\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial^2 k(x,y,b,d)}{\partial s\partial t}u(b,c)-$$

$$\left.\frac{h_y^2}{12}\frac{\partial k(x,y,b,d)}{\partial s}\frac{\partial u(b,d)}{\partial t}\right]-\frac{h_x^2}{12}\left[\frac{h_y}{2}k(x,y,b,c)\frac{\partial u(b,c)}{\partial s}+\frac{h_y}{2}k(x,y,b,d)\frac{\partial u(b,d)}{\partial s}+\right.$$

$$h_y\sum_{p=1}^{m-1}k(x,y,b,t_p)\frac{\partial u(b,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial k(x,y,b,c)}{\partial t}\frac{\partial u(b,c)}{\partial s}+\frac{h_y^2}{12}k(x,y,b,c)\frac{\partial^2 u(b,c)}{\partial s\partial t}-$$

$$\left.\frac{h_y^2}{12}\frac{\partial k(x,y,b,d)}{\partial t}\frac{\partial u(b,d)}{\partial s}-\frac{h_y^2}{12}k(x,y,b,d)\frac{\partial^2 u(b,d)}{\partial s\partial t}\right].$$

By substituting $x=x_i$ and $y=y_j$, i=0,1,…,n, j=0,1,…,m in the above equation one can have

$$u(x_i,y_j)=f(x_i,y_j)+\frac{h_xh_y}{4}\Big[k(x_i,y_j,a,c)\,u(a,c)+k(x_i,y_j,a,d)\,u(a,d)+k(x_i,y_j,b,c)u(b,c)+$$

$$k(x_i,y_j,b,d)u(b,d)\Big]+\frac{h_xh_y}{2}\left[\sum_{p=1}^{m-1}k(x_i,y_j,a,t_p)u(a,t_p)+\sum_{p=1}^{m-1}k(x_i,y_j,b,t_p)u(b,t_p)+\right.$$

$$\left.\sum_{\ell=1}^{n-1}k(x_i,y_j,s_\ell,c)u(s_\ell,c)+\sum_{\ell=1}^{n-1}k(x_i,y_j,s_\ell,d)u(s_\ell,d)+2\sum_{\ell=1}^{n-1}\sum_{p=1}^{m-1}k(x_i,y_j,s_\ell,t_p)u(s_\ell,t_p)\right]+$$

$$\frac{h_xh_y^2}{24}\left[\frac{\partial k(x_i,y_j,a,c)}{\partial t}u(a,c)+k(x_i,y_j,a,c)\frac{\partial u(a,c)}{\partial t}-\frac{\partial k(x_i,y_j,a,d)}{\partial t}u(a,d)-\right.$$

$$k(x_i,y_j,a,d)\frac{\partial u(a,d)}{\partial t}+\frac{\partial k(x_i,y_j,b,c)}{\partial t}u(b,c)+k(x_i,y_j,b,c)\frac{\partial u(b,c)}{\partial t}-\frac{\partial k(x_i,y_j,b,d)}{\partial t}u(b,d)-$$

$$k(x_i,y_j,b,d)\frac{\partial u(b,d)}{\partial t}+2\sum_{\ell=1}^{n-1}\frac{\partial k(x_i,y_j,s_\ell,c)}{\partial t}u(s_\ell,c)+2\sum_{\ell=1}^{n-1}k(x_i,y_j,s_\ell,c)\frac{\partial u(s_\ell,c)}{\partial t}-$$

$$\left.2\sum_{\ell=1}^{n-1}\frac{\partial k(x_i,y_j,s_\ell,d)}{\partial t}u(s_\ell,d)-2\sum_{\ell=1}^{n-1}k(x_i,y_j,s_\ell,d)\frac{\partial u(s_\ell,d)}{\partial t}\right]+\frac{h_x^2h_y}{24}\left[\frac{\partial k(x_i,y_j,a,c)}{\partial s}u(a,c)+\right.$$

$$k(x_i,y_j,a,c)\frac{\partial u(a,c)}{\partial s}+\frac{\partial k(x_i,y_j,a,d)}{\partial s}u(a,d)+k(x_i,y_j,a,d)\frac{\partial u(a,d)}{\partial s}-\frac{\partial k(x_i,y_j,b,c)}{\partial s}u(b,c)-$$

$$k(x_i,y_j,b,c)\frac{\partial u(b,c)}{\partial s}-\frac{\partial k(x_i,y_j,b,d)}{\partial s}u(b,d)-k(x_i,y_j,b,d)\frac{\partial u(b,d)}{\partial s}+$$

$$2\sum_{p=1}^{m-1}\frac{\partial k(x_i,y_j,a,t_p)}{\partial s}u(a,t_p)+2\sum_{p=1}^{m-1}k(x_i,y_j,a,t_p)\frac{\partial u(a,t_p)}{\partial s}-2\sum_{p=1}^{m-1}\frac{\partial k(x_i,y_j,b,t_p)}{\partial s}u(b,t_p)-$$

$$2\sum_{p=1}^{m-1}k(x_i,y_j,b,t_p)\frac{\partial u(b,t_p)}{\partial s}\Bigg]+\frac{h_x^2h_y^2}{144}\Bigg[\frac{\partial^2 k(x_i,y_j,a,c)}{\partial s\partial t}u(a,c)+\frac{\partial k(x_i,y_j,a,c)}{\partial s}\frac{\partial u(a,c)}{\partial t}+$$
$$\frac{\partial k(x_i,y_j,a,c)}{\partial t}\frac{\partial u(a,c)}{\partial s}+k(x_i,y_j,a,c)\frac{\partial^2 u(a,c)}{\partial s\partial t}+\frac{\partial^2 k(x_i,y_j,b,d)}{\partial s\partial t}u(b,d)+$$
$$\frac{\partial k(x_i,y_j,b,d)}{\partial s}\frac{\partial u(b,d)}{\partial t}+\frac{\partial k(x_i,y_j,b,d)}{\partial t}\frac{\partial u(b,d)}{\partial s}+k(x_i,y_j,b,d)\frac{\partial^2 u(b,d)}{\partial s\partial t}-$$
$$\frac{\partial^2 k(x_i,y_j,a,d)}{\partial s\partial t}u(a,d)-\frac{\partial k(x_i,y_j,a,d)}{\partial s}\frac{\partial u(a,d)}{\partial t}-\frac{\partial k(x_i,y_j,a,d)}{\partial t}\frac{\partial u(a,d)}{\partial s}-$$
$$k(x_i,y_j,a,d)\frac{\partial^2 u(a,d)}{\partial t\partial s}-\frac{\partial^2 k(x_i,y_j,b,c)}{\partial s\partial t}u(b,c)-\frac{\partial k(x_i,y_j,b,c)}{\partial s}\frac{\partial u(b,c)}{\partial t}-\frac{\partial k(x_i,y_j,b,c)}{\partial t}\frac{\partial u(b,c)}{\partial s}-$$
$$k(x_i,y_j,b,c)\frac{\partial^2 u(b,c)}{\partial t\partial s}\Bigg],\ i=0,1,...,n,\ \ j=0,1,...,m \qquad (3.29)$$

The above system of equations consists of $(n+1)\times(m+1)$ equations with 3n+3m+nm+9 unknowns, namely u_{ij}, $i=0,1,...,n$ and $j=0,1,...,m$;

$\frac{\partial u_{i0}}{\partial t}$, $\frac{\partial u_{im}}{\partial t}$, $i=0,1,...,n$, $\frac{\partial u_{0j}}{\partial s}$, $\frac{\partial u_{nj}}{\partial s}$, $j=0,1,...,m$, $\frac{\partial^2 u_{00}}{\partial s\partial t}$, $\frac{\partial^2 u_{n0}}{\partial s\partial t}$, $\frac{\partial^2 u_{0m}}{\partial s\partial t}$ and $\frac{\partial^2 u_{nm}}{\partial s\partial t}$.

To find $\frac{\partial u_{i0}}{\partial t}$, $\frac{\partial u_{im}}{\partial t}$, $i=0,1,...,n$, $\frac{\partial u_{0j}}{\partial s}$, $\frac{\partial u_{nj}}{\partial s}$, $j=0,1,...,m$, $\frac{\partial^2 u_{00}}{\partial s\partial t}$, $\frac{\partial^2 u_{n0}}{\partial s\partial t}$, $\frac{\partial^2 u_{0m}}{\partial s\partial t}$ and $\frac{\partial^2 u_{nm}}{\partial s\partial t}$ one must differentiate equation (3.24) with respect to x to get

$$\frac{\partial u(x,y)}{\partial x}=\frac{\partial f(x,y)}{\partial x}+\lambda\int_c^d\int_a^b H(x,y,s,t)u(s,t)dsdt,\quad a\le x\le b,\quad c\le y\le d \qquad (3.30)$$

where $H(x,y,s,t)=\frac{\partial k(x,y,s,t)}{\partial x}$.

On the other hand

$$\frac{\partial u(x,y)}{\partial y} = \frac{\partial f(x,y)}{\partial y} + \lambda \int_c^d \int_a^b G(x,y,s,t)u(s,t)dsdt, \quad a \le x \le b, \quad c \le y \le d \tag{3.31}$$

where $G(x,y,s,t) = \dfrac{\partial k(x,y,s,t)}{\partial y}$.

Therefore

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial x \partial y} + \lambda \int_c^d \int_a^b W(x,y,s,t)u(s,t)dsdt, \quad a \le x \le b, \quad c \le y \le d \tag{3.32}$$

where $W(x,y,s,t) = \dfrac{\partial^2 k(x,y,s,t)}{\partial x \partial y}$.

Next, to solve the integral equation (3.30), one must consider two cases:

Case (1):

If $\dfrac{\partial^3 k(x,y,s,t)}{\partial s \partial t \partial x}$ exists, in this case, we approximate the integral that appeared in the right hand side of equation (3.30) with the repeated modified trapezoid rule to obtain

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial f(x,y)}{\partial x} + \frac{h_x}{2}\Bigg[\frac{h_y}{2} H(x,y,a,c)u(a,c) + \frac{h_y}{2} H(x,y,a,d)u(a,d) +$$

$$h_y \sum_{p=1}^{m-1} H(x,y,a,t_p)u(a,t_p) + \frac{h_y^2}{12}\frac{\partial H(x,y,a,c)}{\partial t}u(a,c) + \frac{h_y^2}{12}H(x,y,a,c)\frac{\partial u(a,c)}{\partial t} -$$

$$\frac{h_y^2}{12}\frac{\partial H(x,y,a,d)}{\partial t}u(a,d) - \frac{h_y^2}{12}H(x,y,a,d)\frac{\partial u(a,d)}{\partial t}\Bigg] + \frac{h_x}{2}\Bigg[\frac{h_y}{2}H(x,y,b,c)u(b,c) +$$

$$\frac{h_y}{2}H(x,y,b,d)u(b,d) + h_y \sum_{p=1}^{m-1} H(x,y,b,t_p)u(b,t_p) + \frac{h_y^2}{12}\frac{\partial H(x,y,b,c)}{\partial t}u(b,c) +$$

$$\frac{h_y^2}{12}H(x,y,b,c)\frac{\partial u(b,c)}{\partial t} - \frac{h_y^2}{12}\frac{\partial H(x,y,b,d)}{\partial t}u(b,d) - \frac{h_y^2}{12}H(x,y,b,d)\frac{\partial u(b,d)}{\partial t}\Bigg] +$$

$$h_x \sum_{\ell=1}^{n-1} \left[\frac{h_y}{2} H(x,y,s_\ell,c)\, u(s_\ell,c) + \frac{h_y}{2} H(x,y,s_\ell,d)\, u(s_\ell,d) + h_y \sum_{p=1}^{m-1} H(x,y,s_\ell,t_p) u(s_\ell,t_p) + \right.$$

$$\frac{h_y^2}{12} \frac{\partial H(x,y,s_\ell,c)}{\partial t} u(s_\ell,c) + \frac{h_y^2}{12} H(x,y,s_\ell,c) \frac{\partial u(s_\ell,c)}{\partial t} - \frac{h_y^2}{12} \frac{\partial H(x,y,s_\ell,d)}{\partial t} u(s_\ell,d) -$$

$$\left. \frac{h_y^2}{12} H(x,y,s_\ell,d) \frac{\partial u(s_\ell,d)}{\partial t} \right] + \frac{h_x^2}{12} \left[\frac{h_y}{2} \frac{\partial H(x,y,a,c)}{\partial s} u(a,c) + \frac{h_y}{2} \frac{\partial H(x,y,a,d)}{\partial s} u(a,d) + \right.$$

$$h_y \sum_{p=1}^{m-1} \frac{\partial H(x,y,a,t_p)}{\partial s} u(a,t_p) + \frac{h_y^2}{12} \frac{\partial^2 H(x,y,a,c)}{\partial s \partial t} u(a,c) + \frac{h_y^2}{12} \frac{\partial H(x,y,a,c)}{\partial s} \frac{\partial u(a,c)}{\partial t} -$$

$$\left. \frac{h_y^2}{12} \frac{\partial^2 H(x,y,a,d)}{\partial s \partial t} u(a,d) - \frac{h_y^2}{12} \frac{\partial H(x,y,a,d)}{\partial s} \frac{\partial u(a,d)}{\partial t} \right] + \frac{h_x^2}{12} \left[\frac{h_y}{2} H(x,y,a,c) \frac{\partial u(a,c)}{\partial s} + \right.$$

$$\frac{h_y}{2} H(x,y,a,d) \frac{\partial u(a,d)}{\partial s} + h_y \sum_{p=1}^{m-1} H(x,y,a,t_p) \frac{\partial u(a,t_p)}{\partial s} + \frac{h_y^2}{12} \frac{\partial H(x,y,a,c)}{\partial t} \frac{\partial u(a,c)}{\partial s} +$$

$$\left. \frac{h_y^2}{12} H(x,y,a,c) \frac{\partial^2 u(a,c)}{\partial s \partial t} - \frac{h_y^2}{12} \frac{\partial H(x,y,a,d)}{\partial t} \frac{\partial u(a,d)}{\partial s} - \frac{h_y^2}{12} H(x,y,a,d) \frac{\partial^2 u(a,d)}{\partial s \partial t} \right] -$$

$$\frac{h_x^2}{12} \left[\frac{h_y}{2} \frac{\partial H(x,y,b,c)}{\partial s} u(b,c) + \frac{h_y}{2} \frac{\partial H(x,y,b,d)}{\partial s} u(b,d) + h_y \sum_{p=1}^{m-1} \frac{\partial H(x,y,b,t_p)}{\partial s} u(b,t_p) + \right.$$

$$\frac{h_y^2}{12} \frac{\partial^2 H(x,y,b,c)}{\partial s \partial t} u(b,c) + \frac{h_y^2}{12} \frac{\partial H(x,y,b,c)}{\partial s} \frac{\partial u(b,c)}{\partial t} - \frac{h_y^2}{12} \frac{\partial^2 H(x,y,b,d)}{\partial s \partial t} u(b,c) -$$

$$\left. \frac{h_y^2}{12} \frac{\partial H(x,y,b,d)}{\partial s} \frac{\partial u(b,d)}{\partial t} \right] - \frac{h_x^2}{12} \left[\frac{h_y}{2} H(x,y,b,c) \frac{\partial u(b,c)}{\partial s} + \frac{h_y}{2} H(x,y,b,d) \frac{\partial u(b,d)}{\partial s} + \right.$$

$$h_y \sum_{p=1}^{m-1} H(x,y,b,t_p) \frac{\partial u(b,t_p)}{\partial s} + \frac{h_y^2}{12} \frac{\partial H(x,y,b,c)}{\partial t} \frac{\partial u(b,c)}{\partial s} + \frac{h_y^2}{12} H(x,y,b,c) \frac{\partial^2 u(b,c)}{\partial s \partial t} -$$

$$\left. \frac{h_y^2}{12} \frac{\partial H(x,y,b,d)}{\partial t} \frac{\partial u(b,d)}{\partial s} - \frac{h_y^2}{12} H(x,y,b,d) \frac{\partial^2 u(b,d)}{\partial s \partial t} \right]$$

Thus

$$\frac{\partial u(a,y_j)}{\partial x} = \frac{\partial f(a,y_j)}{\partial x} + \frac{h_x}{2} \left[\frac{h_y}{2} H(a,y_j,a,c) u(a,c) + \frac{h_y}{2} H(a,y_j,a,d) u(a,d) + \right.$$

$$h_y \sum_{p=1}^{m-1} H(a,y_j,a,t_p) u(a,t_p) + \frac{h_y^2}{12} \frac{\partial H(a,y_j,a,c)}{\partial t} u(a,c) + \frac{h_y^2}{12} H(a,y_j,a,c) \frac{\partial u(a,c)}{\partial t} -$$

$$\frac{h_y^2}{12}\frac{\partial H(a,y_j,a,d)}{\partial t}u(a,d)-\frac{h_y^2}{12}H(a,y_j,a,d)\frac{\partial u(a,d)}{\partial t}\Bigg]+\frac{h_x}{2}\Bigg[\frac{h_y}{2}H(a,y_j,b,c)u(b,c)+$$

$$\frac{h_y}{2}H(a,y_j,b,d)u(b,d)+h_y\sum_{p=1}^{m-1}H(a,y_j,b,t_p)u(b,t_p)+\frac{h_y^2}{12}\frac{\partial H(a,y_j,b,c)}{\partial t}u(b,c)+$$

$$\frac{h_y^2}{12}H(a,y_j,b,c)\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial H(a,y_j,b,d)}{\partial t}u(b,d)-\frac{h_y^2}{12}H(a,y_j,b,d)\frac{\partial u(b,d)}{\partial t}\Bigg]+$$

$$h_x\sum_{\ell=1}^{n-1}\Bigg[\frac{h_y}{2}H(a,y_j,s_\ell,c)\,u(s_\ell,c)+\frac{h_y}{2}H(a,y_j,s_\ell,d)\,u(s_\ell,d)+h_y\sum_{p=1}^{m-1}H(a,y_j,s_\ell,t_p)u(s_\ell,t_p)+$$

$$\frac{h_y^2}{12}\frac{\partial H(a,y_j,s_\ell,c)}{\partial t}u(s_\ell,c)+\frac{h_y^2}{12}H(a,y_j,s_\ell,c)\frac{\partial u(s_\ell,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial H(a,y_j,s_\ell,d)}{\partial t}u(s_\ell,d)-$$

$$\frac{h_y^2}{12}H(a,y_j,s_\ell,d)\frac{\partial u(s_\ell,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\frac{\partial H(a,y_j,a,c)}{\partial s}u(a,c)+\frac{h_y}{2}\frac{\partial H(a,y_j,a,d)}{\partial s}u(a,d)+$$

$$h_y\sum_{p=1}^{m-1}\frac{\partial H(a,y_j,a,t_p)}{\partial s}u(a,t_p)+\frac{h_y^2}{12}\frac{\partial^2 H(a,y_j,a,c)}{\partial s\partial t}u(a,c)+\frac{h_y^2}{12}\frac{\partial H(a,y_j,a,c)}{\partial s}\frac{\partial u(a,c)}{\partial t}-$$

$$\frac{h_y^2}{12}\frac{\partial^2 H(a,y_j,a,d)}{\partial s\partial t}u(a,d)-\frac{h_y^2}{12}\frac{\partial H(a,y_j,a,d)}{\partial s}\frac{\partial u(a,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}H(a,y_j,a,c)\frac{\partial u(a,c)}{\partial s}+$$

$$\frac{h_y}{2}H(a,y_j,a,d)\frac{\partial u(a,d)}{\partial s}+h_y\sum_{p=1}^{m-1}H(a,y_j,a,t_p)\frac{\partial u(a,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial H(a,y_j,a,c)}{\partial t}\frac{\partial u(a,c)}{\partial s}+$$

$$\frac{h_y^2}{12}H(a,y_j,a,c)\frac{\partial^2 u(a,c)}{\partial s\partial t}-\frac{h_y^2}{12}\frac{\partial H(a,y_j,a,d)}{\partial t}\frac{\partial u(a,d)}{\partial s}-\frac{h_y^2}{12}H(a,y_j,a,d)\frac{\partial^2 u(a,d)}{\partial s\partial t}\Bigg]-$$

$$\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\frac{\partial H(a,y_j,b,c)}{\partial s}u(b,c)+\frac{h_y}{2}\frac{\partial H(a,y_j,b,d)}{\partial s}u(b,d)+h_y\sum_{p=1}^{m-1}\frac{\partial H(a,y_j,b,t_p)}{\partial s}u(b,t_p)+$$

$$\frac{h_y^2}{12}\frac{\partial^2 H(a,y_j,b,c)}{\partial s\partial t}u(b,c)+\frac{h_y^2}{12}\frac{\partial H(a,y_j,b,c)}{\partial s}\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial^2 H(a,y_j,b,d)}{\partial s\partial t}u(b,c)-$$

$$\frac{h_y^2}{12}\frac{\partial H(a,y_j,b,d)}{\partial s}\frac{\partial u(b,d)}{\partial t}\Bigg]-\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}H(a,y_j,b,c)\frac{\partial u(b,c)}{\partial s}+\frac{h_y}{2}H(a,y_j,b,d)\frac{\partial u(b,d)}{\partial s}+$$

$$h_y\sum_{p=1}^{m-1}H(a,y_j,b,t_p)\frac{\partial u(b,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial H(a,y_j,b,c)}{\partial t}\frac{\partial u(b,c)}{\partial s}+\frac{h_y^2}{12}H(a,y_j,b,c)\frac{\partial^2 u(b,c)}{\partial s\partial t}-$$

$$\frac{h_y^2}{12}\frac{\partial H(a,y_j,b,d)}{\partial t}\frac{\partial u(b,d)}{\partial s}-\frac{h_y^2}{12}H(a,y_j,b,d)\frac{\partial^2 u(b,d)}{\partial s\partial t}\Bigg] \qquad (3.33.a)$$

and

$$\frac{\partial u(b,y_j)}{\partial x}=\frac{\partial f(b,y_j)}{\partial x}+\frac{h_x}{2}\Bigg[\frac{h_y}{2}H(b,y_j,a,c)u(a,c)+\frac{h_y}{2}H(b,y_j,a,d)u(a,d)+$$

$$h_y\sum_{p=1}^{m-1}H(b,y_j,a,t_p)u(a,t_p)+\frac{h_y^2}{12}\frac{\partial H(b,y_j,a,c)}{\partial t}u(a,c)+\frac{h_y^2}{12}H(b,y_j,a,c)\frac{\partial u(a,c)}{\partial t}-$$

$$\frac{h_y^2}{12}\frac{\partial H(b,y_j,a,d)}{\partial t}u(a,d)-\frac{h_y^2}{12}H(b,y_j,a,d)\frac{\partial u(a,d)}{\partial t}\Bigg]+\frac{h_x}{2}\Bigg[\frac{h_y}{2}H(b,y_j,b,c)u(b,c)+$$

$$\frac{h_y}{2}H(b,y_j,b,d)u(b,d)+h_y\sum_{p=1}^{m-1}H(b,y_j,b,t_p)u(b,t_p)+\frac{h_y^2}{12}\frac{\partial H(b,y_j,b,c)}{\partial t}u(b,c)+$$

$$\frac{h_y^2}{12}H(b,y_j,b,c)\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial H(b,y_j,b,d)}{\partial t}u(b,d)-\frac{h_y^2}{12}H(b,y_j,b,d)\frac{\partial u(b,d)}{\partial t}\Bigg]+$$

$$h_x\sum_{\ell=1}^{n-1}\Bigg[\frac{h_y}{2}H(b,y_j,s_\ell,c)\,u(s_\ell,c)+\frac{h_y}{2}H(b,y_j,s_\ell,d)\,u(s_\ell,d)+h_y\sum_{p=1}^{m-1}H(b,y_j,s_\ell,t_p)u(s_\ell,t_p)+$$

$$\frac{h_y^2}{12}\frac{\partial H(b,y_j,s_\ell,c)}{\partial t}u(s_\ell,c)+\frac{h_y^2}{12}H(b,y_j,s_\ell,c)\frac{\partial u(s_\ell,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial H(b,y_j,s_\ell,d)}{\partial t}u(s_\ell,d)-$$

$$\frac{h_y^2}{12}H(b,y_j,s_\ell,d)\frac{\partial u(s_\ell,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\frac{\partial H(b,y_j,a,c)}{\partial s}u(a,c)+\frac{h_y}{2}\frac{\partial H(b,y_j,a,d)}{\partial s}u(a,d)+$$

$$h_y\sum_{p=1}^{m-1}\frac{\partial H(b,y_j,a,t_p)}{\partial s}u(a,t_p)+\frac{h_y^2}{12}\frac{\partial^2 H(b,y_j,a,c)}{\partial s\partial t}u(a,c)+\frac{h_y^2}{12}\frac{\partial H(b,y_j,a,c)}{\partial s}\frac{\partial u(a,c)}{\partial t}-$$

$$\frac{h_y^2}{12}\frac{\partial^2 H(b,y_j,a,d)}{\partial s\partial t}u(a,d)-\frac{h_y^2}{12}\frac{\partial H(b,y_j,a,d)}{\partial s}\frac{\partial u(a,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}H(b,y_j,a,c)\frac{\partial u(a,c)}{\partial s}+$$

$$\frac{h_y}{2}H(b,y_j,a,d)\frac{\partial u(a,d)}{\partial s}+h_y\sum_{p=1}^{m-1}H(b,y_j,a,t_p)\frac{\partial u(a,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial H(b,y_j,a,c)}{\partial t}\frac{\partial u(a,c)}{\partial s}+$$

$$\frac{h_y^2}{12}H(b,y_j,a,c)\frac{\partial^2 u(a,c)}{\partial s\partial t}-\frac{h_y^2}{12}\frac{\partial H(b,y_j,a,d)}{\partial t}\frac{\partial u(a,d)}{\partial s}-\frac{h_y^2}{12}H(b,y_j,a,d)\frac{\partial^2 u(a,d)}{\partial s\partial t}\Bigg]-$$

$$\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\frac{\partial H(b,y_j,b,c)}{\partial s}u(b,c)+\frac{h_y}{2}\frac{\partial H(b,y_j,b,d)}{\partial s}u(b,d)+h_y\sum_{p=1}^{m-1}\frac{\partial H(b,y_j,b,t_p)}{\partial s}u(b,t_p)+$$

$$\frac{h_y^2}{12}\frac{\partial^2 H(b,y_j,b,c)}{\partial s\partial t}u(b,c)+\frac{h_y^2}{12}\frac{\partial H(b,y_j,b,c)}{\partial s}\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial^2 H(b,y_j,b,d)}{\partial s\partial t}u(b,c)-$$

$$\frac{h_y^2}{12}\frac{\partial H(b,y_j,b,d)}{\partial s}\frac{\partial u(b,d)}{\partial t}\Bigg]-\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}H(b,y_j,b,c)\frac{\partial u(b,c)}{\partial s}+\frac{h_y}{2}H(b,y_j,b,d)\frac{\partial u(b,d)}{\partial s}+$$

$$h_y \sum_{p=1}^{m-1} H(b,y_j,b,t_p)\frac{\partial u(b,t_p)}{\partial s} + \frac{h_y^2}{12}\frac{\partial H(b,y_j,b,c)}{\partial t}\frac{\partial u(b,c)}{\partial s} + \frac{h_y^2}{12}H(b,y_j,b,c)\frac{\partial^2 u(b,c)}{\partial s \partial t} -$$

$$\left.\frac{h_y^2}{12}\frac{\partial H(b,y_j,b,d)}{\partial t}\frac{\partial u(b,d)}{\partial s} - \frac{h_y^2}{12}H(b,y_j,b,d)\frac{\partial^2 u(b,d)}{\partial s \partial t}\right] \tag{3.33.b}$$

Case (2):

If $\frac{\partial^3 k(x,y,s,t)}{\partial x \partial s \partial t}$ does not exists, in this case, we approximate the integral that appeared in the right hand side of equation (3.30) with the repeated trapezoid rule to obtain

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial f(x,y)}{\partial x} + \frac{\lambda h_x h_y}{4}\Big[H(x,y,s_0,t_0)u(s_0,t_0) + H(x,y,s_0,t_m)u(s_0,t_m) +$$

$$H(x,y,s_n,t_0)u(s_n,t_0) + H(x,y,s_n,t_m)u(s_n,t_m) + 2\sum_{\ell=1}^{n-1} H(x,y,s_\ell,t_0)u(s_\ell,t_0) +$$

$$2\sum_{\ell=1}^{n-1} H(x,y,s_\ell,t_m)u(s_\ell,t_m) + 2\sum_{p=1}^{m-1} H(x,y,s_0,t_p)u(s_0,t_p) + 2\sum_{p=1}^{m-1} H(x,y,s_n,t_p)u(s_n,t_p) +$$

$$4\sum_{\ell=1}^{n-1}\sum_{p=1}^{m-1} H(x,y,s_\ell,t_p)u(s_\ell,t_p)\Big]$$

Hence, for $y = y_j$, j=0,1,…,m, one can have

$$\frac{\partial u(a,y_j)}{\partial x} = \frac{\partial f(a,y_j)}{\partial x} + \frac{\lambda h_x h_y}{4}\Big[H(a,y_j,s_0,t_0)u(s_0,t_0) + H(a,y_j,s_0,t_m)u(s_0,t_m) +$$

$$H(a,y_j,s_n,t_0)u(s_n,t_0) + H(a,y_j,s_n,t_m)u(s_n,t_m) + 2\sum_{\ell=1}^{n-1} H(a,y_j,s_\ell,t_0)u(s_\ell,t_0) +$$

$$2\sum_{\ell=1}^{n-1} H(a,y_j,s_\ell,t_m)u(s_\ell,t_m) + 2\sum_{p=1}^{m-1} H(a,y_j,s_0,t_p)u(s_0,t_p) +$$

$$2\sum_{p=1}^{m-1} H(a,y_j,s_n,t_p)u(s_n,t_p) + 4\sum_{\ell=1}^{n-1}\sum_{p=1}^{m-1} H(a,y_j,s_\ell,t_p)u(s_\ell,t_p)\Big] \tag{3.34.a}$$

$$\frac{\partial u(b,y_j)}{\partial x}=\frac{\partial f(b,y_j)}{\partial x}+\frac{\lambda h_x h_y}{4}\Big[H(b,y_j,s_0,t_0)u(s_0,t_0)+H(b,y_j,s_0,t_m)u(s_0,t_m)+$$
$$H(b,y_j,s_n,t_0)u(s_n,t_0)+H(b,y_j,s_n,t_m)u(s_n,t_m)+2\sum_{\ell=1}^{n-1}H(b,y_j,s_\ell,t_0)u(s_\ell,t_0)+$$
$$2\sum_{\ell=1}^{n-1}H(b,y_j,s_\ell,t_m)u(s_\ell,t_m)+2\sum_{p=1}^{m-1}H(b,y_j,s_0,t_p)u(s_0,t_p)+$$
$$2\sum_{p=1}^{m-1}H(b,y_j,s_n,t_p)u(s_n,t_p)+4\sum_{\ell=1}^{n-1}\sum_{p=1}^{m-1}H(b,y_j,s_\ell,t_p)u(s_\ell,t_p)\Big] \qquad (3.34.b)$$

Next, to solve the integral equation (3.31), one must consider two cases:

Case (1):

If $\frac{\partial^3 k(x,y,s,t)}{\partial s\partial t\partial y}$ exists, in this case, we approximate the integral that appeared in the right hand side of equation (3.31) with the repeated modified trapezoid rule to obtain

$$\frac{\partial u(x,y)}{\partial y}=\frac{\partial f(x,y)}{\partial y}+\frac{h_x}{2}\Bigg[\frac{h_y}{2}G(x,y,a,c)u(a,c)+\frac{h_y}{2}G(x,y,a,d)u(a,d)+$$
$$h_y\sum_{p=1}^{m-1}G(x,y,a,t_p)u(a,t_p)+\frac{h_y^2}{12}\frac{\partial G(x,y,a,c)}{\partial t}u(a,c)+\frac{h_y^2}{12}G(x,y,a,c)\frac{\partial u(a,c)}{\partial t}-$$
$$\frac{h_y^2}{12}\frac{\partial G(x,y,a,d)}{\partial t}u(a,d)-\frac{h_y^2}{12}G(x,y,a,d)\frac{\partial u(a,d)}{\partial t}\Bigg]+\frac{h_x}{2}\Bigg[\frac{h_y}{2}G(x,y,b,c)u(b,c)+$$
$$\frac{h_y}{2}G(x,y,b,d)u(b,d)+h_y\sum_{p=1}^{m-1}G(x,y,b,t_p)u(b,t_p)+\frac{h_y^2}{12}\frac{\partial G(x,y,b,c)}{\partial t}u(b,c)+$$
$$\frac{h_y^2}{12}G(x,y,b,c)\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial G(x,y,b,d)}{\partial t}u(b,d)-\frac{h_y^2}{12}G(x,y,b,d)\frac{\partial u(b,d)}{\partial t}\Bigg]+$$
$$h_x\sum_{\ell=1}^{n-1}\Bigg[\frac{h_y}{2}G(x,y,s_\ell,c)\,u(s_\ell,c)+\frac{h_y}{2}G(x,y,s_\ell,d)\,u(s_\ell,d)+h_y\sum_{p=1}^{m-1}G(x,y,s_\ell,t_p)u(s_\ell,t_p)+$$
$$\frac{h_y^2}{12}\frac{\partial G(x,y,s_\ell,c)}{\partial t}u(s_\ell,c)+\frac{h_y^2}{12}G(x,y,s_\ell,c)\frac{\partial u(s_\ell,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial G(x,y,s_\ell,d)}{\partial t}u(s_\ell,d)-$$

$$\frac{h_y^2}{12}G(x,y,s_\ell,d)\frac{\partial u(s_\ell,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\ \frac{\partial G(x,y,a,c)}{\partial s}\ u(a,c)+\frac{h_y}{2}\frac{\partial G(x,y,a,d)}{\partial s}\ u(a,d)+$$

$$h_y\sum_{p=1}^{m-1}\frac{\partial G(x,y,a,t_p)}{\partial s}u(a,t_p)+\frac{h_y^2}{12}\frac{\partial^2 G(x,y,a,c)}{\partial s\partial t}u(a,c)+\frac{h_y^2}{12}\frac{\partial G(x,y,a,c)}{\partial s}\frac{\partial u(a,c)}{\partial t}-$$

$$\frac{h_y^2}{12}\frac{\partial^2 G(x,y,a,d)}{\partial s\partial t}u(a,d)-\frac{h_y^2}{12}\frac{\partial G(x,y,a,d)}{\partial s}\frac{\partial u(a,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\ G(x,y,a,c)\ \frac{\partial u(a,c)}{\partial s}+$$

$$\frac{h_y}{2}\ G(x,y,a,d)\ \frac{\partial u(a,d)}{\partial s}+h_y\sum_{p=1}^{m-1}G(x,y,a,t_p)\frac{\partial u(a,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial G(x,y,a,c)}{\partial t}\frac{\partial u(a,c)}{\partial s}+$$

$$\frac{h_y^2}{12}G(x,y,a,c)\frac{\partial^2 u(a,c)}{\partial s\partial t}-\frac{h_y^2}{12}\frac{\partial G(x,y,a,d)}{\partial t}\frac{\partial u(a,d)}{\partial s}-\frac{h_y^2}{12}G(x,y,a,d)\frac{\partial^2 u(a,d)}{\partial s\partial t}\Bigg]-$$

$$\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\ \frac{\partial G(x,y,b,c)}{\partial s}\ u(b,c)+\frac{h_y}{2}\ \frac{\partial G(x,y,b,d)}{\partial s}\ u(b,d)+h_y\sum_{p=1}^{m-1}\frac{\partial G(x,y,b,t_p)}{\partial s}u(b,t_p)+$$

$$\frac{h_y^2}{12}\frac{\partial^2 G(x,y,b,c)}{\partial s\partial t}u(b,c)+\frac{h_y^2}{12}\frac{\partial G(x,y,b,c)}{\partial s}\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial^2 G(x,y,b,d)}{\partial s\partial t}u(b,c)-$$

$$\frac{h_y^2}{12}\frac{\partial G(x,y,b,d)}{\partial s}\frac{\partial u(b,d)}{\partial t}\Bigg]-\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}G(x,y,b,c)\ \frac{\partial u(b,c)}{\partial s}+\frac{h_y}{2}\ G(x,y,b,d)\ \frac{\partial u(b,d)}{\partial s}+$$

$$h_y\sum_{p=1}^{m-1}G(x,y,b,t_p)\frac{\partial u(b,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial G(x,y,b,c)}{\partial t}\frac{\partial u(b,c)}{\partial s}+\frac{h_y^2}{12}G(x,y,b,c)\frac{\partial^2 u(b,c)}{\partial s\partial t}-$$

$$\frac{h_y^2}{12}\frac{\partial G(x,y,b,d)}{\partial t}\frac{\partial u(b,d)}{\partial s}-\frac{h_y^2}{12}G(x,y,b,d)\frac{\partial^2 u(b,d)}{\partial s\partial t}\Bigg]$$

Thus for $x = x_i$, i=0,1,…,n one can get the following equations

$$\frac{\partial u(x_i,c)}{\partial y}=\frac{\partial f(x_i,c)}{\partial y}+\frac{h_x}{2}\Bigg[\frac{h_y}{2}G(x_i,c,a,c)u(a,c)+\frac{h_y}{2}G(x_i,c,a,d)u(a,d)+$$

$$h_y\sum_{p=1}^{m-1}G(x_i,c,a,t_p)u(a,t_p)+\frac{h_y^2}{12}\ \frac{\partial G(x_i,c,a,c)}{\partial t}\ u(a,c)+\frac{h_y^2}{12}\ G(x_i,c,a,c)\ \frac{\partial u(a,c)}{\partial t}-$$

$$\frac{h_y^2}{12}\ \frac{\partial G(x_i,c,a,d)}{\partial t}u(a,d)-\frac{h_y^2}{12}\ G(x_i,c,a,d)\ \frac{\partial u(a,d)}{\partial t}\ \Bigg]+\frac{h_x}{2}\Bigg[\frac{h_y}{2}G(x_i,c,b,c)u(b,c)+$$

$$\frac{h_y}{2}G(x_i,c,b,d)u(b,d)+h_y\sum_{p=1}^{m-1}G(x_i,c,b,t_p)u(b,t_p)+\frac{h_y^2}{12}\ \frac{\partial G(x_i,c,b,c)}{\partial t}\ u(b,c)+$$

$$\frac{h_y^2}{12}G(x_i,c,b,c)\,\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial G(x_i,c,b,d)}{\partial t}\,u(b,d)-\frac{h_y^2}{12}G(x_i,c,b,d)\,\frac{\partial u(b,d)}{\partial t}\Bigg]+$$

$$h_x\sum_{\ell=1}^{n-1}\Bigg[\frac{h_y}{2}G(x_i,c,s_\ell,c)\,u(s_\ell,c)+\frac{h_y}{2}G(x_i,c,s_\ell,d)\,u(s_\ell,d)+h_y\sum_{p=1}^{m-1}G(x_i,c,s_\ell,t_p)u(s_\ell,t_p)+$$

$$\frac{h_y^2}{12}\frac{\partial G(x_i,c,s_\ell,c)}{\partial t}u(s_\ell,c)+\frac{h_y^2}{12}G(x_i,c,s_\ell,c)\frac{\partial u(s_\ell,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial G(x_i,c,s_\ell,d)}{\partial t}u(s_\ell,d)-$$

$$\frac{h_y^2}{12}G(x_i,c,s_\ell,d)\frac{\partial u(s_\ell,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\,\frac{\partial G(x_i,c,a,c)}{\partial s}\,u(a,c)+\frac{h_y}{2}\frac{\partial G(x_i,c,a,d)}{\partial s}\,u(a,d)+$$

$$h_y\sum_{p=1}^{m-1}\frac{\partial G(x_i,c,a,t_p)}{\partial s}u(a,t_p)+\frac{h_y^2}{12}\frac{\partial^2 G(x_i,c,a,c)}{\partial s\partial t}u(a,c)+\frac{h_y^2}{12}\frac{\partial G(x_i,c,a,c)}{\partial s}\frac{\partial u(a,c)}{\partial t}-$$

$$\frac{h_y^2}{12}\frac{\partial^2 G(x_i,c,a,d)}{\partial s\partial t}u(a,d)-\frac{h_y^2}{12}\frac{\partial G(x_i,c,a,d)}{\partial s}\frac{\partial u(a,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\,G(x_i,c,a,c)\,\frac{\partial u(a,c)}{\partial s}+$$

$$\frac{h_y}{2}\,G(x_i,c,a,d)\,\frac{\partial u(a,d)}{\partial s}+h_y\sum_{p=1}^{m-1}G(x_i,c,a,t_p)\frac{\partial u(a,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial G(x_i,c,a,c)}{\partial t}\frac{\partial u(a,c)}{\partial s}+$$

$$\frac{h_y^2}{12}G(x_i,c,a,c)\frac{\partial^2 u(a,c)}{\partial s\partial t}-\frac{h_y^2}{12}\frac{\partial G(x_i,c,a,d)}{\partial t}\frac{\partial u(a,d)}{\partial s}-\frac{h_y^2}{12}G(x_i,c,a,d)\frac{\partial^2 u(a,d)}{\partial s\partial t}\Bigg]-$$

$$\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\,\frac{\partial G(x_i,c,b,c)}{\partial s}\,u(b,c)+\frac{h_y}{2}\,\frac{\partial G(x_i,c,b,d)}{\partial s}\,u(b,d)+h_y\sum_{p=1}^{m-1}\frac{\partial G(x_i,c,b,t_p)}{\partial s}u(b,t_p)+$$

$$\frac{h_y^2}{12}\frac{\partial^2 G(x_i,c,b,c)}{\partial s\partial t}u(b,c)+\frac{h_y^2}{12}\frac{\partial G(x_i,c,b,c)}{\partial s}\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial^2 G(x_i,c,b,d)}{\partial s\partial t}u(b,c)-$$

$$\frac{h_y^2}{12}\frac{\partial G(x_i,c,b,d)}{\partial s}\frac{\partial u(b,d)}{\partial t}\Bigg]-\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\,G(x_i,c,b,c)\,\frac{\partial u(b,c)}{\partial s}+\frac{h_y}{2}\,G(x_i,c,b,d)\,\frac{\partial u(b,d)}{\partial s}+$$

$$h_y\sum_{p=1}^{m-1}G(x_i,c,b,t_p)\frac{\partial u(b,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial G(x_i,c,b,c)}{\partial t}\frac{\partial u(b,c)}{\partial s}+\frac{h_y^2}{12}G(x_i,c,b,c)\frac{\partial^2 u(b,c)}{\partial s\partial t}-$$

$$\frac{h_y^2}{12}\frac{\partial G(x_i,c,b,d)}{\partial t}\frac{\partial u(b,d)}{\partial s}-\frac{h_y^2}{12}G(x_i,c,b,d)\frac{\partial^2 u(b,d)}{\partial s\partial t}\Bigg] \qquad (3.35.a)$$

and

$$\frac{\partial u(x_i,d)}{\partial y}=\frac{\partial f(x,y)}{\partial y}+\frac{h_x}{2}\Bigg[\frac{h_y}{2}G(x_i,d,a,c)u(a,c)+\frac{h_y}{2}G(x_i,d,a,d)u(a,d)+$$

$$h_y\sum_{p=1}^{m-1}G(x_i,d,a,t_p)u(a,t_p)+\frac{h_y^2}{12}\frac{\partial G(x_i,d,a,c)}{\partial t}u(a,c)+\frac{h_y^2}{12}G(x_i,d,a,c)\frac{\partial u(a,c)}{\partial t}-$$

$$\left.\frac{h_y^2}{12}\frac{\partial G(x_i,d,a,d)}{\partial t}u(a,d)-\frac{h_y^2}{12}G(x_i,d,a,d)\frac{\partial u(a,d)}{\partial t}\right]+\frac{h_x}{2}\left[\frac{h_y}{2}G(x_i,d,b,c)u(b,c)+\right.$$

$$\frac{h_y}{2}G(x_i,d,b,d)u(b,d)+h_y\sum_{p=1}^{m-1}G(x_i,d,b,t_p)u(b,t_p)+\frac{h_y^2}{12}\frac{\partial G(x_i,d,b,c)}{\partial t}u(b,c)+$$

$$\left.\frac{h_y^2}{12}G(x_i,d,b,c)\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial G(x_i,d,b,d)}{\partial t}u(b,d)-\frac{h_y^2}{12}G(x_i,d,b,d)\frac{\partial u(b,d)}{\partial t}\right]+$$

$$h_x\sum_{\ell=1}^{n-1}\left[\frac{h_y}{2}G(x_i,d,s_\ell,c)\,u(s_\ell,c)+\frac{h_y}{2}G(x_i,d,s_\ell,d)\,u(s_\ell,d)+h_y\sum_{p=1}^{m-1}G(x_i,d,s_\ell,t_p)u(s_\ell,t_p)+\right.$$

$$\frac{h_y^2}{12}\frac{\partial G(x_i,d,s_\ell,c)}{\partial t}u(s_\ell,c)+\frac{h_y^2}{12}G(x_i,d,s_\ell,c)\frac{\partial u(s_\ell,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial G(x_i,d,s_\ell,d)}{\partial t}u(s_\ell,d)-$$

$$\left.\frac{h_y^2}{12}G(x_i,d,s_\ell,d)\frac{\partial u(s_\ell,d)}{\partial t}\right]+\frac{h_x^2}{12}\left[\frac{h_y}{2}\frac{\partial G(x_i,d,a,c)}{\partial s}u(a,c)+\frac{h_y}{2}\frac{\partial G(x_i,d,a,d)}{\partial s}u(a,d)+\right.$$

$$h_y\sum_{p=1}^{m-1}\frac{\partial G(x_i,d,a,t_p)}{\partial s}u(a,t_p)+\frac{h_y^2}{12}\frac{\partial^2 G(x_i,d,a,c)}{\partial s\partial t}u(a,c)+\frac{h_y^2}{12}\frac{\partial G(x_i,d,a,c)}{\partial s}\frac{\partial u(a,c)}{\partial t}-$$

$$\left.\frac{h_y^2}{12}\frac{\partial^2 G(x_i,d,a,d)}{\partial s\partial t}u(a,d)-\frac{h_y^2}{12}\frac{\partial G(x_i,d,a,d)}{\partial s}\frac{\partial u(a,d)}{\partial t}\right]+\frac{h_x^2}{12}\left[\frac{h_y}{2}G(x_i,d,a,c)\frac{\partial u(a,c)}{\partial s}+\right.$$

$$\frac{h_y}{2}G(x_i,d,a,d)\frac{\partial u(a,d)}{\partial s}+h_y\sum_{p=1}^{m-1}G(x_i,d,a,t_p)\frac{\partial u(a,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial G(x_i,d,a,c)}{\partial t}\frac{\partial u(a,c)}{\partial s}+$$

$$\left.\frac{h_y^2}{12}G(x_i,d,a,c)\frac{\partial^2 u(a,c)}{\partial s\partial t}-\frac{h_y^2}{12}\frac{\partial G(x_i,d,a,d)}{\partial t}\frac{\partial u(a,d)}{\partial s}-\frac{h_y^2}{12}G(x_i,d,a,d)\frac{\partial^2 u(a,d)}{\partial s\partial t}\right]-$$

$$\frac{h_x^2}{12}\left[\frac{h_y}{2}\frac{\partial G(x_i,d,b,c)}{\partial s}u(b,c)+\frac{h_y}{2}\frac{\partial G(x_i,d,b,d)}{\partial s}u(b,d)+h_y\sum_{p=1}^{m-1}\frac{\partial G(x_i,d,b,t_p)}{\partial s}u(b,t_p)+\right.$$

$$\frac{h_y^2}{12}\frac{\partial^2 G(x_i,d,b,c)}{\partial s\partial t}u(b,c)+\frac{h_y^2}{12}\frac{\partial G(x_i,d,b,c)}{\partial s}\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial^2 G(x_i,d,b,d)}{\partial s\partial t}u(b,c)-$$

$$\left.\frac{h_y^2}{12}\frac{\partial G(x_i,d,b,d)}{\partial s}\frac{\partial u(b,d)}{\partial t}\right]-\frac{h_x^2}{12}\left[\frac{h_y}{2}G(x_i,d,b,c)\frac{\partial u(b,c)}{\partial s}+\frac{h_y}{2}G(x_i,d,b,d)\frac{\partial u(b,d)}{\partial s}+\right.$$

$$h_y\sum_{p=1}^{m-1}G(x_i,d,b,t_p)\frac{\partial u(b,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial G(x_i,d,b,c)}{\partial t}\frac{\partial u(b,c)}{\partial s}+\frac{h_y^2}{12}G(x_i,d,b,c)\frac{\partial^2 u(b,c)}{\partial s\partial t}-$$

$$\left.\frac{h_y^2}{12}\frac{\partial G(x_i,d,b,d)}{\partial t}\frac{\partial u(b,d)}{\partial s}-\frac{h_y^2}{12}G(x_i,d,b,d)\frac{\partial^2 u(b,d)}{\partial s\partial t}\right] \tag{3.35.b}$$

Case (2):

If $\frac{\partial^3 k(x,y,s,t)}{\partial s \partial t \partial y}$ does not exists, in this case, we approximate the integral that appeared in the right hand side of equation (3.31) with the repeated trapezoid rule to obtain

$$\frac{\partial u(x,y)}{\partial y} = \frac{\partial f(x,y)}{\partial y} + \frac{\lambda h_x h_y}{4}\Big[G(x,y,s_0,t_0)\, u(s_0,t_0) + G(x,y,s_0,t_m)\, u(s_0,t_m) +$$
$$G(x,y,s_n,t_0)\, u(s_n,t_0) + G(x,y,s_n,t_m)\, u(s_n,t_m) + 2\sum_{\ell=1}^{n-1} G(x,y,s_\ell,t_0)\, u(s_\ell,t_0) +$$
$$2\sum_{\ell=1}^{n-1} G(x,y,s_\ell,t_m) u(s_\ell,t_m) + 2\sum_{p=1}^{m-1} G(x,y,s_0,t_p) u(s_0,t_p) + 2\sum_{p=1}^{m-1} G(x,y,s_n,t_p) u(s_n,t_p) +$$
$$4\sum_{\ell=1}^{n-1}\sum_{p=1}^{m-1} G(x,y,s_\ell,t_p) u(s_\ell,t_p)\Big]$$

Thus for $x = x_i$, i=0,1,…,n one can get the following equations

$$\frac{\partial u(x_i,c)}{\partial y} = \frac{\partial f(x_i,c)}{\partial y} + \frac{\lambda h_x h_y}{4}\Big[G(x_i,c,s_0,t_0) u(s_0,t_0) + G(x_i,c,s_0,t_m) u(s_0,t_m) +$$
$$G(x_i,c,s_n,t_0) u(s_n,t_0) + G(x_i,c,s_n,t_m) u(s_n,t_m) + 2\sum_{\ell=1}^{n-1} G(x_i,c,s_\ell,t_0) u(s_\ell,t_0) +$$
$$2\sum_{\ell=1}^{n-1} G(x_i,c,s_\ell,t_m) u(s_\ell,t_m) + 2\sum_{p=1}^{m-1} G(x_i,c,s_0,t_p) u(s_0,t_p) +$$
$$2\sum_{p=1}^{m-1} G(x_i,c,s_n,t_p) u(s_n,t_p) + 4\sum_{\ell=1}^{n-1}\sum_{p=1}^{m-1} G(x_i,c,s_\ell,t_p) u(s_\ell,t_p)\Big] \qquad (3.36.a)$$

and

$$\frac{\partial u(x_i,d)}{\partial y} = \frac{\partial f(x_i,d)}{\partial y} + \frac{\lambda h_x h_y}{4}\Big[G(x_i,d,s_0,t_0) u(s_0,t_0) + G(x_i,d,s_0,t_m) u(s_0,t_m) +$$
$$G(x_i,d,s_n,t_0) u(s_n,t_0) + G(x_i,d,s_n,t_m) u(s_n,t_m) + 2\sum_{\ell=1}^{n-1} G(x_i,d,s_\ell,t_0) u(s_\ell,t_0) +$$

$$2\sum_{\ell=1}^{n-1} G(x_i,d,s_\ell,t_m)u(s_\ell,t_m) + 2\sum_{p=1}^{m-1} G(x_i,d,s_0,t_p)u(s_0,t_p) +$$
$$2\sum_{p=1}^{m-1} G(x_i,d,s_n,t_p)u(s_n,t_p) + 4\sum_{\ell=1}^{n-1}\sum_{p=1}^{m-1} G(x_i,d,s_\ell,t_p)u(s_\ell,t_p)\Big] \tag{3.36.b}$$

Next, to solve the integral equation (3.32), one must consider two cases:

Case (1):

If $\dfrac{\partial^4 k(x,y,s,t)}{\partial s\partial t\partial x\partial y}$ exists, in this case, we approximate the integral that appeared in the right hand side of equation (3.32) with the repeated modified trapezoid rule to obtain

$$\frac{\partial^2 u(x,y)}{\partial x\partial y} = \frac{\partial^2 f(x,y)}{\partial x\partial y} + \frac{h_x}{2}\Bigg[\frac{h_y}{2}W(x,y,a,c)u(a,c) + \frac{h_y}{2}W(x,y,a,d)u(a,d) +$$
$$h_y\sum_{p=1}^{m-1} W(x,y,a,t_p)u(a,t_p) + \frac{h_y^2}{12}\frac{\partial W(x,y,a,c)}{\partial t}u(a,c) + \frac{h_y^2}{12}W(x,y,a,c)\frac{\partial u(a,c)}{\partial t} -$$
$$\frac{h_y^2}{12}\frac{\partial W(x,y,a,d)}{\partial t}u(a,d) - \frac{h_y^2}{12}W(x,y,a,d)\frac{\partial u(a,d)}{\partial t}\Bigg] + \frac{h_x}{2}\Bigg[\frac{h_y}{2}W(x,y,b,c)u(b,c) +$$
$$\frac{h_y}{2}W(x,y,b,d)u(b,d) + h_y\sum_{p=1}^{m-1} W(x,y,b,t_p)u(b,t_p) + \frac{h_y^2}{12}\frac{\partial W(x,y,b,c)}{\partial t}u(b,c) +$$
$$\frac{h_y^2}{12}W(x,y,b,c)\frac{\partial u(b,c)}{\partial t} - \frac{h_y^2}{12}\frac{\partial W(x,y,b,d)}{\partial t}u(b,d) - \frac{h_y^2}{12}W(x,y,b,d)\frac{\partial u(b,d)}{\partial t}\Bigg] +$$
$$h_x\sum_{\ell=1}^{n-1}\Bigg[\frac{h_y}{2}W(x,y,s_\ell,c)\,u(s_\ell,c) + \frac{h_y}{2}W(x,y,s_\ell,d)\,u(s_\ell,d) + h_y\sum_{p=1}^{m-1} W(x,y,s_\ell,t_p)u(s_\ell,t_p) +$$
$$\frac{h_y^2}{12}\frac{\partial W(x,y,s_\ell,c)}{\partial t}u(s_\ell,c) + \frac{h_y^2}{12}W(x,y,s_\ell,c)\frac{\partial u(s_\ell,c)}{\partial t} - \frac{h_y^2}{12}\frac{\partial W(x,y,s_\ell,d)}{\partial t}u(s_\ell,d) -$$
$$\frac{h_y^2}{12}W(x,y,s_\ell,d)\frac{\partial u(s_\ell,d)}{\partial t}\Bigg] + \frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\frac{\partial W(x,y,a,c)}{\partial s}u(a,c) + \frac{h_y}{2}\frac{\partial W(x,y,a,d)}{\partial s}u(a,d) +$$

$$h_y\sum_{p=1}^{m-1}\frac{\partial W(x,y,a,t_p)}{\partial s}u(a,t_p)+\frac{h_y^2}{12}\frac{\partial^2 W(x,y,a,c)}{\partial s\partial t}u(a,c)+\frac{h_y^2}{12}\frac{\partial W(x,y,a,c)}{\partial s}\frac{\partial u(a,c)}{\partial t}-$$

$$\left.\frac{h_y^2}{12}\frac{\partial^2 W(x,y,a,d)}{\partial s\partial t}u(a,d)-\frac{h_y^2}{12}\frac{\partial W(x,y,a,d)}{\partial s}\frac{\partial u(a,d)}{\partial t}\right]+\frac{h_x^2}{12}\left[\frac{h_y}{2}W(x,y,a,c)\frac{\partial u(a,c)}{\partial s}+\right.$$

$$\frac{h_y}{2}W(x,y,a,d)\frac{\partial u(a,d)}{\partial s}+h_y\sum_{p=1}^{m-1}W(x,y,a,t_p)\frac{\partial u(a,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial W(x,y,a,c)}{\partial t}\frac{\partial u(a,c)}{\partial s}+$$

$$\left.\frac{h_y^2}{12}W(x,y,a,c)\frac{\partial^2 u(a,c)}{\partial s\partial t}-\frac{h_y^2}{12}\frac{\partial W(x,y,a,d)}{\partial t}\frac{\partial u(a,d)}{\partial s}-\frac{h_y^2}{12}W(x,y,a,d)\frac{\partial^2 u(a,d)}{\partial s\partial t}\right]-$$

$$\frac{h_x^2}{12}\left[\frac{h_y}{2}\frac{\partial W(x,y,b,c)}{\partial s}u(b,c)+\frac{h_y}{2}\frac{\partial W(x,y,b,d)}{\partial s}u(b,d)+h_y\sum_{p=1}^{m-1}\frac{\partial W(x,y,b,t_p)}{\partial s}u(b,t_p)+\right.$$

$$\frac{h_y^2}{12}\frac{\partial^2 W(x,y,b,c)}{\partial s\partial t}u(b,c)+\frac{h_y^2}{12}\frac{\partial W(x,y,b,c)}{\partial s}\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial^2 W(x,y,b,d)}{\partial s\partial t}u(b,c)-$$

$$\left.\frac{h_y^2}{12}\frac{\partial W(x,y,b,d)}{\partial s}\frac{\partial u(b,d)}{\partial t}\right]-\frac{h_x^2}{12}\left[\frac{h_y}{2}W(x,y,b,c)\frac{\partial u(b,c)}{\partial s}+\frac{h_y}{2}W(x,y,b,d)\frac{\partial u(b,d)}{\partial s}+\right.$$

$$h_y\sum_{p=1}^{m-1}W(x,y,b,t_p)\frac{\partial u(b,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial W(x,y,b,c)}{\partial t}\frac{\partial u(b,c)}{\partial s}+\frac{h_y^2}{12}W(x,y,b,c)\frac{\partial^2 u(b,c)}{\partial s\partial t}-$$

$$\left.\frac{h_y^2}{12}\frac{\partial W(x,y,b,d)}{\partial t}\frac{\partial u(b,d)}{\partial s}-\frac{h_y^2}{12}W(x,y,b,d)\frac{\partial^2 u(b,d)}{\partial s\partial t}\right]$$

Thus

$$\frac{\partial^2 u(a,c)}{\partial x\partial y}=\frac{\partial^2 f(a,c)}{\partial x\partial y}+\frac{h_x}{2}\left[\frac{h_y}{2}W(a,c,a,c)u(a,c)+\frac{h_y}{2}W(a,c,a,d)u(a,d)+\right.$$

$$h_y\sum_{p=1}^{m-1}W(a,c,a,t_p)u(a,t_p)+\frac{h_y^2}{12}\frac{\partial W(a,c,a,c)}{\partial t}u(a,c)+\frac{h_y^2}{12}W(a,c,a,c)\frac{\partial u(a,c)}{\partial t}-$$

$$\left.\frac{h_y^2}{12}\frac{\partial W(a,c,a,d)}{\partial t}u(a,d)-\frac{h_y^2}{12}W(a,c,a,d)\frac{\partial u(a,d)}{\partial t}\right]+\frac{h_x}{2}\left[\frac{h_y}{2}W(a,c,b,c)u(b,c)+\right.$$

$$\frac{h_y}{2}W(a,c,b,d)u(b,d)+h_y\sum_{p=1}^{m-1}W(a,c,b,t_p)u(b,t_p)+\frac{h_y^2}{12}\frac{\partial W(a,c,b,c)}{\partial t}u(b,c)+$$

$$\left.\frac{h_y^2}{12}W(a,c,b,c)\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial W(a,c,b,d)}{\partial t}u(b,d)-\frac{h_y^2}{12}W(a,c,b,d)\frac{\partial u(b,d)}{\partial t}\right]+$$

$$h_x\sum_{\ell=1}^{n-1}\left[\frac{h_y}{2}W(a,c,s_\ell,c)\,u(s_\ell,c)+\frac{h_y}{2}W(a,c,s_\ell,d)\,u(s_\ell,d)+h_y\sum_{p=1}^{m-1}W(a,c,s_\ell,t_p)u(s_\ell,t_p)+\right.$$

$$\frac{h_y^2}{12}\frac{\partial W(a,c,s_\ell,c)}{\partial t}u(s_\ell,c)+\frac{h_y^2}{12}W(a,c,s_\ell,c)\frac{\partial u(s_\ell,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial W(a,c,s_\ell,d)}{\partial t}u(s_\ell,d)-$$

$$\frac{h_y^2}{12}W(a,c,s_\ell,d)\frac{\partial u(s_\ell,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\frac{\partial W(a,c,a,c)}{\partial s}u(a,c)+\frac{h_y}{2}\frac{\partial W(a,c,a,d)}{\partial s}u(a,d)+$$

$$h_y\sum_{p=1}^{m-1}\frac{\partial W(a,c,a,t_p)}{\partial s}u(a,t_p)+\frac{h_y^2}{12}\frac{\partial^2 W(a,c,a,c)}{\partial s\partial t}u(a,c)+\frac{h_y^2}{12}\frac{\partial W(a,c,a,c)}{\partial s}\frac{\partial u(a,c)}{\partial t}-$$

$$\frac{h_y^2}{12}\frac{\partial^2 W(a,c,a,d)}{\partial s\partial t}u(a,d)-\frac{h_y^2}{12}\frac{\partial W(a,c,a,d)}{\partial s}\frac{\partial u(a,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}W(a,c,a,c)\frac{\partial u(a,c)}{\partial s}+$$

$$\frac{h_y}{2}W(a,c,a,d)\frac{\partial u(a,d)}{\partial s}+h_y\sum_{p=1}^{m-1}W(a,c,a,t_p)\frac{\partial u(a,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial W(a,c,a,c)}{\partial t}\frac{\partial u(a,c)}{\partial s}+$$

$$\frac{h_y^2}{12}W(a,c,a,c)\frac{\partial^2 u(a,c)}{\partial s\partial t}-\frac{h_y^2}{12}\frac{\partial W(a,c,a,d)}{\partial t}\frac{\partial u(a,d)}{\partial s}-\frac{h_y^2}{12}W(a,c,a,d)\frac{\partial^2 u(a,d)}{\partial s\partial t}\Bigg]-$$

$$\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\frac{\partial W(a,c,b,c)}{\partial s}u(b,c)+\frac{h_y}{2}\frac{\partial W(a,c,b,d)}{\partial s}u(b,d)+h_y\sum_{p=1}^{m-1}\frac{\partial W(a,c,b,t_p)}{\partial s}u(b,t_p)+$$

$$\frac{h_y^2}{12}\frac{\partial^2 W(a,c,b,c)}{\partial s\partial t}u(b,c)+\frac{h_y^2}{12}\frac{\partial W(a,c,b,c)}{\partial s}\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial^2 W(a,c,b,d)}{\partial s\partial t}u(b,c)-$$

$$\frac{h_y^2}{12}\frac{\partial W(a,c,b,d)}{\partial s}\frac{\partial u(b,d)}{\partial t}\Bigg]-\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}W(a,c,b,c)\frac{\partial u(b,c)}{\partial s}+\frac{h_y}{2}W(a,c,b,d)\frac{\partial u(b,d)}{\partial s}+$$

$$h_y\sum_{p=1}^{m-1}W(a,c,b,t_p)\frac{\partial u(b,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial W(a,c,b,c)}{\partial t}\frac{\partial u(b,c)}{\partial s}+\frac{h_y^2}{12}W(a,c,b,c)\frac{\partial^2 u(b,c)}{\partial s\partial t}-$$

$$\frac{h_y^2}{12}\frac{\partial W(a,c,b,d)}{\partial t}\frac{\partial u(b,d)}{\partial s}-\frac{h_y^2}{12}W(a,c,b,d)\frac{\partial^2 u(b,d)}{\partial s\partial t}\Bigg] \qquad (3.37.a)$$

$$\frac{\partial^2 u(b,c)}{\partial x\partial y}=\frac{\partial^2 f(b,c)}{\partial x\partial y}+\frac{h_x}{2}\Bigg[\frac{h_y}{2}W(b,c,a,c)u(a,c)+\frac{h_y}{2}W(b,c,a,d)u(a,d)+$$

$$h_y\sum_{p=1}^{m-1}W(b,c,a,t_p)u(b,t_p)+\frac{h_y^2}{12}\frac{\partial W(b,c,a,c)}{\partial t}u(a,c)+\frac{h_y^2}{12}W(b,c,a,c)\frac{\partial u(a,c)}{\partial t}-$$

$$\frac{h_y^2}{12}\frac{\partial W(b,c,a,d)}{\partial t}u(a,d)-\frac{h_y^2}{12}W(b,c,a,d)\frac{\partial u(a,d)}{\partial t}\Bigg]+\frac{h_x}{2}\Bigg[\frac{h_y}{2}W(b,c,b,c)u(b,c)+$$

$$\frac{h_y}{2}W(b,c,b,d)u(b,d)+h_y\sum_{p=1}^{m-1}W(b,c,b,t_p)u(b,t_p)+\frac{h_y^2}{12}\frac{\partial W(,c,b,c)}{\partial t}u(b,c)+$$

$$\frac{h_y^2}{12}W(b,c,b,c)\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial W(b,c,b,d)}{\partial t}u(b,d)-\frac{h_y^2}{12}W(b,c,b,d)\frac{\partial u(b,d)}{\partial t}\Bigg]+$$

$$h_x \sum_{\ell=1}^{n-1} \Bigg[\frac{h_y}{2} W(b,c,s_\ell,c)\, u(s_\ell,c) + \frac{h_y}{2} W(b,c,s_\ell,d)\, u(s_\ell,d) + h_y \sum_{p=1}^{m-1} W(b,c,s_\ell,t_p) u(s_\ell,t_p) +$$

$$\frac{h_y^2}{12}\frac{\partial W(b,c,s_\ell,c)}{\partial t} u(s_\ell,c) + \frac{h_y^2}{12} W(b,c,s_\ell,c) \frac{\partial u(s_\ell,c)}{\partial t} - \frac{h_y^2}{12}\frac{\partial W(b,c,s_\ell,d)}{\partial t} u(s_\ell,d) -$$

$$\frac{h_y^2}{12} W(b,c,s_\ell,d)\frac{\partial u(s_\ell,d)}{\partial t}\Bigg] + \frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\frac{\partial W(b,c,a,c)}{\partial s} u(a,c) + \frac{h_y}{2}\frac{\partial W(b,c,a,d)}{\partial s} u(a,d) +$$

$$h_y \sum_{p=1}^{m-1} \frac{\partial W(b,c,a,t_p)}{\partial s} u(a,t_p) + \frac{h_y^2}{12}\frac{\partial^2 W(b,c,a,c)}{\partial s \partial t} u(a,c) + \frac{h_y^2}{12}\frac{\partial W(b,c,a,c)}{\partial s}\frac{\partial u(a,c)}{\partial t} -$$

$$\frac{h_y^2}{12}\frac{\partial^2 W(b,c,a,d)}{\partial s \partial t} u(a,d) - \frac{h_y^2}{12}\frac{\partial W(b,c,a,d)}{\partial s}\frac{\partial u(a,d)}{\partial t}\Bigg] + \frac{h_x^2}{12}\Bigg[\frac{h_y}{2} W(b,c,a,c)\frac{\partial u(a,c)}{\partial s} +$$

$$\frac{h_y}{2} W(b,c,a,d)\frac{\partial u(a,d)}{\partial s} + h_y \sum_{p=1}^{m-1} W(b,c,a,t_p)\frac{\partial u(a,t_p)}{\partial s} + \frac{h_y^2}{12}\frac{\partial W(b,c,a,c)}{\partial t}\frac{\partial u(a,c)}{\partial s} +$$

$$\frac{h_y^2}{12} W(b,c,a,c)\frac{\partial^2 u(a,c)}{\partial s \partial t} - \frac{h_y^2}{12}\frac{\partial W(b,c,a,d)}{\partial t}\frac{\partial u(a,d)}{\partial s} - \frac{h_y^2}{12} W(b,c,a,d)\frac{\partial^2 u(a,d)}{\partial s \partial t}\Bigg] -$$

$$\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\frac{\partial W(b,c,b,c)}{\partial s} u(b,c) + \frac{h_y}{2}\frac{\partial W(b,c,b,d)}{\partial s} u(b,d) + h_y \sum_{p=1}^{m-1}\frac{\partial W(b,c,b,t_p)}{\partial s} u(b,t_p) +$$

$$\frac{h_y^2}{12}\frac{\partial^2 W(b,c,b,c)}{\partial s \partial t} u(b,c) + \frac{h_y^2}{12}\frac{\partial W(b,c,b,c)}{\partial s}\frac{\partial u(b,c)}{\partial t} - \frac{h_y^2}{12}\frac{\partial^2 W(b,c,b,d)}{\partial s \partial t} u(b,c) -$$

$$\frac{h_y^2}{12}\frac{\partial W(b,c,b,d)}{\partial s}\frac{\partial u(b,d)}{\partial t}\Bigg] - \frac{h_x^2}{12}\Bigg[\frac{h_y}{2} W(b,c,b,c)\frac{\partial u(b,c)}{\partial s} + \frac{h_y}{2} W(b,c,b,d)\frac{\partial u(b,d)}{\partial s} +$$

$$h_y \sum_{p=1}^{m-1} W(b,c,b,t_p)\frac{\partial u(b,t_p)}{\partial s} + \frac{h_y^2}{12}\frac{\partial W(b,c,b,c)}{\partial t}\frac{\partial u(b,c)}{\partial s} + \frac{h_y^2}{12} W(b,c,b,c)\frac{\partial^2 u(b,c)}{\partial s \partial t} -$$

$$\frac{h_y^2}{12}\frac{\partial W(b,c,b,d)}{\partial t}\frac{\partial u(b,d)}{\partial s} - \frac{h_y^2}{12} W(b,c,b,d)\frac{\partial^2 u(b,d)}{\partial s \partial t}\Bigg] \qquad (3.37.b)$$

$$\frac{\partial^2 u(a,d)}{\partial x \partial y} = \frac{\partial^2 f(a,d)}{\partial x \partial y} + \frac{h_x}{2}\Bigg[\frac{h_y}{2} W(a,d,a,c) u(a,c) + \frac{h_y}{2} W(a,d,a,d) u(a,d) +$$

$$h_y \sum_{p=1}^{m-1} W(a,d,a,t_p) u(a,t_p) + \frac{h_y^2}{12}\frac{\partial W(a,d,a,c)}{\partial t} u(a,c) + \frac{h_y^2}{12} W(a,d,a,c)\frac{\partial u(a,c)}{\partial t} -$$

$$\frac{h_y^2}{12}\frac{\partial W(a,d,a,d)}{\partial t}u(a,d)-\frac{h_y^2}{12}W(a,d,a,d)\frac{\partial u(a,d)}{\partial t}\Bigg]+\frac{h_x}{2}\Bigg[\frac{h_y}{2}W(a,d,b,c)u(b,c)+$$
$$\frac{h_y}{2}W(a,d,b,d)u(b,d)+h_y\sum_{p=1}^{m-1}W(a,d,b,t_p)u(b,t_p)+\frac{h_y^2}{12}\frac{\partial W(a,d,b,c)}{\partial t}u(b,c)+$$
$$\frac{h_y^2}{12}W(a,d,b,c)\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial W(a,d,b,d)}{\partial t}u(b,d)-\frac{h_y^2}{12}W(a,d,b,d)\frac{\partial u(b,d)}{\partial t}\Bigg]+$$
$$h_x\sum_{\ell=1}^{n-1}\Bigg[\frac{h_y}{2}W(a,d,s_\ell,c)\,u(s_\ell,c)+\frac{h_y}{2}W(a,d,s_\ell,d)\,u(s_\ell,d)+h_y\sum_{p=1}^{m-1}W(a,d,s_\ell,t_p)u(s_\ell,t_p)+$$
$$\frac{h_y^2}{12}\frac{\partial W(a,d,s_\ell,c)}{\partial t}u(s_\ell,c)+\frac{h_y^2}{12}W(a,d,s_\ell,c)\frac{\partial u(s_\ell,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial W(a,d,s_\ell,d)}{\partial t}u(s_\ell,d)-$$
$$\frac{h_y^2}{12}W(a,d,s_\ell,d)\frac{\partial u(s_\ell,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\frac{\partial W(a,d,a,c)}{\partial s}u(a,c)+\frac{h_y}{2}\frac{\partial W(a,d,a,d)}{\partial s}u(a,d)+$$
$$h_y\sum_{p=1}^{m-1}\frac{\partial W(a,d,a,t_p)}{\partial s}u(a,t_p)+\frac{h_y^2}{12}\frac{\partial^2 W(a,d,a,c)}{\partial s\partial t}u(a,c)+\frac{h_y^2}{12}\frac{\partial W(a,d,a,c)}{\partial s}\frac{\partial u(a,c)}{\partial t}-$$
$$\frac{h_y^2}{12}\frac{\partial^2 W(a,d,a,d)}{\partial s\partial t}u(a,d)-\frac{h_y^2}{12}\frac{\partial W(a,d,a,d)}{\partial s}\frac{\partial u(a,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}W(a,d,a,c)\frac{\partial u(a,c)}{\partial s}+$$
$$\frac{h_y}{2}W(a,d,a,d)\frac{\partial u(a,d)}{\partial s}+h_y\sum_{p=1}^{m-1}W(a,d,a,t_p)\frac{\partial u(a,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial W(a,d,a,c)}{\partial t}\frac{\partial u(a,c)}{\partial s}+$$
$$\frac{h_y^2}{12}W(a,d,a,c)\frac{\partial^2 u(a,c)}{\partial s\partial t}-\frac{h_y^2}{12}\frac{\partial W(a,d,a,d)}{\partial t}\frac{\partial u(a,d)}{\partial s}-\frac{h_y^2}{12}W(a,d,a,d)\frac{\partial^2 u(a,d)}{\partial s\partial t}\Bigg]-$$
$$\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\frac{\partial W(a,d,b,c)}{\partial s}u(b,c)+\frac{h_y}{2}\frac{\partial W(a,d,b,d)}{\partial s}u(b,d)+h_y\sum_{p=1}^{m-1}\frac{\partial W(a,d,b,t_p)}{\partial s}u(b,t_p)+$$
$$\frac{h_y^2}{12}\frac{\partial^2 W(a,d,b,c)}{\partial s\partial t}u(b,c)+\frac{h_y^2}{12}\frac{\partial W(a,d,b,c)}{\partial s}\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial^2 W(a,d,b,d)}{\partial s\partial t}u(b,c)-$$
$$\frac{h_y^2}{12}\frac{\partial W(a,d,b,d)}{\partial s}\frac{\partial u(b,d)}{\partial t}\Bigg]-\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}W(a,d,b,c)\frac{\partial u(b,c)}{\partial s}+\frac{h_y}{2}W(a,d,b,d)\frac{\partial u(b,d)}{\partial s}+$$
$$h_y\sum_{p=1}^{m-1}W(a,d,b,t_p)\frac{\partial u(b,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial W(a,d,b,c)}{\partial t}\frac{\partial u(b,c)}{\partial s}+\frac{h_y^2}{12}W(a,d,b,c)\frac{\partial^2 u(b,c)}{\partial s\partial t}-$$
$$\frac{h_y^2}{12}\frac{\partial W(a,d,b,d)}{\partial t}\frac{\partial u(b,d)}{\partial s}-\frac{h_y^2}{12}W(a,d,b,d)\frac{\partial^2 u(b,d)}{\partial s\partial t}\Bigg] \qquad (3.37.c)$$

and

$$\frac{\partial^2 u(b,d)}{\partial x\partial y}=\frac{\partial^2 f(b,d)}{\partial x\partial y}+\frac{h_x}{2}\Bigg[\frac{h_y}{2}W(b,d,a,c)u(a,c)+\frac{h_y}{2}W(b,d,a,d)u(a,d)+$$

$$h_y\sum_{p=1}^{m-1}W(b,d,a,t_p)u(a,t_p)+\frac{h_y^2}{12}\frac{\partial W(b,d,a,c)}{\partial t}u(a,c)+\frac{h_y^2}{12}W(b,d,a,c)\frac{\partial u(a,c)}{\partial t}-$$

$$\frac{h_y^2}{12}\frac{\partial W(b,d,a,d)}{\partial t}u(a,d)-\frac{h_y^2}{12}W(b,d,a,d)\frac{\partial u(a,d)}{\partial t}\Bigg]+\frac{h_x}{2}\Bigg[\frac{h_y}{2}W(b,d,b,c)u(b,c)+$$

$$\frac{h_y}{2}W(b,d,b,d)u(b,d)+h_y\sum_{p=1}^{m-1}W(b,d,b,t_p)u(b,t_p)+\frac{h_y^2}{12}\frac{\partial W(b,d,b,c)}{\partial t}u(b,c)+$$

$$\frac{h_y^2}{12}W(b,d,b,c)\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial W(b,d,b,d)}{\partial t}u(b,d)-\frac{h_y^2}{12}W(b,d,b,d)\frac{\partial u(b,d)}{\partial t}\Bigg]+$$

$$h_x\sum_{\ell=1}^{n-1}\Bigg[\frac{h_y}{2}W(b,d,s_\ell,c)\,u(s_\ell,c)+\frac{h_y}{2}W(b,d,s_\ell,d)\,u(s_\ell,d)+h_y\sum_{p=1}^{m-1}W(b,d,s_\ell,t_p)u(s_\ell,t_p)+$$

$$\frac{h_y^2}{12}\frac{\partial W(b,d,s_\ell,c)}{\partial t}u(s_\ell,c)+\frac{h_y^2}{12}W(b,d,s_\ell,c)\frac{\partial u(s_\ell,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial W(b,d,s_\ell,d)}{\partial t}u(s_\ell,d)-$$

$$\frac{h_y^2}{12}W(b,d,s_\ell,d)\frac{\partial u(s_\ell,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\frac{\partial W(b,d,a,c)}{\partial s}u(a,c)+\frac{h_y}{2}\frac{\partial W(b,d,a,d)}{\partial s}u(a,d)+$$

$$h_y\sum_{p=1}^{m-1}\frac{\partial W(b,d,a,t_p)}{\partial s}u(a,t_p)+\frac{h_y^2}{12}\frac{\partial^2 W(b,d,a,c)}{\partial s\partial t}u(a,c)+\frac{h_y^2}{12}\frac{\partial W(b,d,a,c)}{\partial s}\frac{\partial u(a,c)}{\partial t}-$$

$$\frac{h_y^2}{12}\frac{\partial^2 W(b,d,a,d)}{\partial s\partial t}u(a,d)-\frac{h_y^2}{12}\frac{\partial W(b,d,a,d)}{\partial s}\frac{\partial u(a,d)}{\partial t}\Bigg]+\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}W(b,d,a,c)\frac{\partial u(a,c)}{\partial s}+$$

$$\frac{h_y}{2}W(b,d,a,d)\frac{\partial u(a,d)}{\partial s}+h_y\sum_{p=1}^{m-1}W(b,d,a,t_p)\frac{\partial u(a,t_p)}{\partial s}+\frac{h_y^2}{12}\frac{\partial W(b,d,a,c)}{\partial t}\frac{\partial u(a,c)}{\partial s}+$$

$$\frac{h_y^2}{12}W(b,d,a,c)\frac{\partial^2 u(a,c)}{\partial s\partial t}-\frac{h_y^2}{12}\frac{\partial W(b,d,a,d)}{\partial t}\frac{\partial u(a,d)}{\partial s}-\frac{h_y^2}{12}W(b,d,a,d)\frac{\partial^2 u(a,d)}{\partial s\partial t}\Bigg]-$$

$$\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}\frac{\partial W(b,d,b,c)}{\partial s}u(b,c)+\frac{h_y}{2}\frac{\partial W(b,d,b,d)}{\partial s}u(b,d)+h_y\sum_{p=1}^{m-1}\frac{\partial W(b,d,b,t_p)}{\partial s}u(b,t_p)+$$

$$\frac{h_y^2}{12}\frac{\partial^2 W(b,d,b,c)}{\partial s\partial t}u(b,c)+\frac{h_y^2}{12}\frac{\partial W(b,d,b,c)}{\partial s}\frac{\partial u(b,c)}{\partial t}-\frac{h_y^2}{12}\frac{\partial^2 W(b,d,b,d)}{\partial s\partial t}u(b,c)-$$

$$\frac{h_y^2}{12}\frac{\partial W(b,d,b,d)}{\partial s}\frac{\partial u(b,d)}{\partial t}\Bigg]-\frac{h_x^2}{12}\Bigg[\frac{h_y}{2}W(b,d,b,c)\frac{\partial u(b,c)}{\partial s}+\frac{h_y}{2}W(b,d,b,d)\frac{\partial u(b,d)}{\partial s}+$$

$$h_y \sum_{p=1}^{m-1} W(b,d,b,t_p)\frac{\partial u(b,t_p)}{\partial s} + \frac{h_y^2}{12}\frac{\partial W(b,d,b,c)}{\partial t}\frac{\partial u(b,c)}{\partial s} + \frac{h_y^2}{12} W(b,d,b,c)\frac{\partial^2 u(b,c)}{\partial s \partial t} -$$

$$\left. \frac{h_y^2}{12}\frac{\partial W(b,d,b,d)}{\partial t}\frac{\partial u(b,d)}{\partial s} - \frac{h_y^2}{12} W(b,d,b,d)\frac{\partial^2 u(b,d)}{\partial s \partial t} \right] \qquad (3.37.d)$$

<u>Case (2):</u>

If $\frac{\partial^4 k(x,y,s,t)}{\partial s \partial t \partial x \partial y}$ does not exists, in this case, we approximate the integral that appeared in the right hand side of equation (3.32) with the repeated trapezoid rule to obtain

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial x \partial y} + \frac{\lambda h_x h_y}{4}\Big[W(x,y,s_0,t_0)u(s_0,t_0) + W(x,y,s_0,t_m)u(s_0,t_m) +$$

$$W(x,y,s_n,t_0)u(s_n,t_0) + W(x,y,s_n,t_m)u(s_n,t_m) + 2\sum_{\ell=1}^{n-1} W(x,y,s_\ell,t_0)u(s_\ell,t_0) +$$

$$2\sum_{\ell=1}^{n-1} W(x,y,s_\ell,t_m)u(s_\ell,t_m) + 2\sum_{p=1}^{m-1} W(x,y,s_0,t_p)u(s_0,t_p) + 2\sum_{p=1}^{m-1} W(x,y,s_n,t_p)u(s_n,t_p) +$$

$$4\sum_{\ell=1}^{n-1}\sum_{p=1}^{m-1} W(x,y,s_\ell,t_p)u(s_\ell,t_p) \Big]$$

Thus

$$\frac{\partial^2 u(a,c)}{\partial x \partial y} = \frac{\partial^2 f(a,c)}{\partial x \partial y} + \frac{\lambda h_x h_y}{4}\Big[W(a,c,s_0,t_0)u(s_0,t_0) + W(a,c,s_0,t_m)u(s_0,t_m) +$$

$$W(a,c,s_n,t_0)u(s_n,t_0) + W(a,c,s_n,t_m)u(s_n,t_m) + 2\sum_{\ell=1}^{n-1} W(a,c,s_\ell,t_0)u(s_\ell,t_0) +$$

$$2\sum_{\ell=1}^{n-1} W(a,c,s_\ell,t_m)u(s_\ell,t_m) + 2\sum_{p=1}^{m-1} W(a,c,s_0,t_p)u(s_0,t_p) +$$

$$2\sum_{p=1}^{m-1} W(a,c,s_n,t_p)u(s_n,t_p) + 4\sum_{\ell=1}^{n-1}\sum_{p=1}^{m-1} W(a,c,s_\ell,t_p)u(s_\ell,t_p) \Big] \qquad (3.38.a)$$

$$\frac{\partial^2 u(a,d)}{\partial x\partial y}=\frac{\partial^2 f(a,d)}{\partial x\partial y}+\frac{\lambda h_x h_y}{4}\Big[W(a,d,s_0,t_0)u(s_0,t_0)+W(a,d,s_0,t_m)u(s_0,t_m)+$$
$$W(a,d,s_n,t_0)u(s_n,t_0)+W(a,d,s_n,t_m)u(s_n,t_m)+2\sum_{\ell=1}^{n-1}W(a,d,s_\ell,t_0)u(s_\ell,t_0)+$$
$$2\sum_{\ell=1}^{n-1}W(a,d,s_\ell,t_m)u(s_\ell,t_m)+2\sum_{p=1}^{m-1}W(a,d,s_0,t_p)u(s_0,t_p)+$$
$$2\sum_{p=1}^{m-1}W(a,d,s_n,t_p)u(s_n,t_p)+4\sum_{\ell=1}^{n-1}\sum_{p=1}^{m-1}W(a,d,s_\ell,t_p)u(s_\ell,t_p)\Big] \tag{3.38.b}$$

$$\frac{\partial^2 u(b,c)}{\partial x\partial y}=\frac{\partial^2 f(b,c)}{\partial x\partial y}+\frac{\lambda h_x h_y}{4}\Big[W(b,c,s_0,t_0)u(s_0,t_0)+W(b,c,s_0,t_m)u(s_0,t_m)+$$
$$W(b,c,s_n,t_0)u(s_n,t_0)+W(b,c,s_n,t_m)u(s_n,t_m)+2\sum_{\ell=1}^{n-1}W(b,c,s_\ell,t_0)u(s_\ell,t_0)+$$
$$2\sum_{\ell=1}^{n-1}W(b,c,s_\ell,t_m)u(s_\ell,t_m)+2\sum_{p=1}^{m-1}W(b,c,s_0,t_p)u(s_0,t_p)+$$
$$2\sum_{p=1}^{m-1}W(b,c,s_n,t_p)u(s_n,t_p)+4\sum_{\ell=1}^{n-1}\sum_{p=1}^{m-1}W(b,c,s_\ell,t_p)u(s_\ell,t_p)\Big] \tag{3.38.c}$$

and

$$\frac{\partial^2 u(b,d)}{\partial x\partial y}=\frac{\partial^2 f(b,d)}{\partial x\partial y}+\frac{\lambda h_x h_y}{4}\Big[W(b,d,s_0,t_0)u(s_0,t_0)+W(b,d,s_0,t_m)u(s_0,t_m)+$$
$$W(b,d,s_n,t_0)u(s_n,t_0)+W(b,d,s_n,t_m)u(s_n,t_m)+2\sum_{\ell=1}^{n-1}W(b,d,s_\ell,t_0)u(s_\ell,t_0)+$$
$$2\sum_{\ell=1}^{n-1}W(b,d,s_\ell,t_m)u(s_\ell,t_m)+2\sum_{p=1}^{m-1}W(b,d,s_0,t_p)u(s_0,t_p)+$$
$$2\sum_{p=1}^{m-1}W(b,d,s_n,t_p)u(s_n,t_p)+4\sum_{\ell=1}^{n-1}\sum_{p=1}^{m-1}W(b,d,s_\ell,t_p)u(s_\ell,t_p)\Big] \tag{3.38.d}$$

The system which consists of equations (3.33) (or equations (3.34)), (3.35) (or equations (3.36)), (3.37) (or equations (3.38)) together with equations (3.29), can be solved by using any suitable method.

To illustrate the previous method, consider the following example:

Example (3.4):

Consider the two dimensional linear Fredholm integral equation of the second kind,

$$u(x,y) = -\frac{7}{6} + \int_0^1\int_0^1 (x+y+s+t)\,u(s,t)\,ds\,dt \qquad (3.41)$$

This example is constructed such that the exact solution of it is u(x,y)=x+y. Here, we use two methods to solve this example, namely the trapezoidal rule and the modified trapezoidal rule. First, we use the trapezoidal rule to solve this example. To do this, we use equation (3.25) for n=m=1 to get the following system of equations:

$$\begin{pmatrix} 1 & -0.25 & -0.25 & -0.5 \\ -0.25 & 0.5 & -0.5 & -0.75 \\ -0.25 & -0.5 & 0.5 & -0.75 \\ -0.5 & -0.75 & -0.75 & 0 \end{pmatrix}\begin{pmatrix} u_{0,0} \\ u_{1,0} \\ u_{0,1} \\ u_{0,1} \end{pmatrix} = \begin{pmatrix} \frac{-7}{6} \\ \frac{-7}{6} \\ \frac{-7}{6} \\ \frac{-7}{6} \end{pmatrix}$$

which has the solution u(0,0)$\cong$0, u(1,0)=u(0,1)$\cong$0.778 and u(1,1)$\cong$1.556. But from the exact solution of this example, one can deduce that u(0,0)=0, u(1,0)=u(0,1)=1 and u(1,1)=2.

Second, we use the modified trapezoidal rule to solve this example. To do this, we use equation (3.29) for n=m=1 to get the following equation

$$u(x_i, y_j) = \frac{-7}{6} + \frac{1}{4}\Big[(x_i + y_j)u(0,0) + (x_i + y_j + 1)u(0,1) + (x_i + y_j + 1)u(1,0) +$$

$$(x_i + y_j + 2)u(1,1)\Big] + \frac{1}{24}\Big[u(0,0) + (x_i + y_j)\frac{\partial u(0,0)}{\partial t} - u(0,1) - (x_i + y_j + 1)\frac{\partial u(0,1)}{\partial t} +$$

$$u(1,0)+(x_i+y_j+1)\frac{\partial u(1,0)}{\partial t}-u(1,1)-(x_i+y_j+2)\frac{\partial u(1,1)}{\partial t}\Bigg]+\frac{1}{24}\big[u(0,0)+u(0,1)+$$

$$(x_i+y_j)\frac{\partial u(0,0)}{\partial s}+(x_i+y_j+1)\frac{\partial u(0,1)}{\partial s}-u(1,0)-u(1,1)-(x_i+y_j+1)\frac{\partial u(1,0)}{\partial s}-$$

$$(x_i+y_j+2)\frac{\partial u(1,1)}{\partial s}\Bigg]+\frac{1}{144}\Bigg[\frac{\partial u(0,0)}{\partial t}+\frac{\partial u(0,0)}{\partial s}+(x_i+y_j)\frac{\partial^2 u(0,0)}{\partial s\partial t}+\frac{\partial u(1,1)}{\partial t}+$$

$$\frac{\partial u(1,1)}{\partial s}+(x_i+y_j+2)\frac{\partial^2 u(1,1)}{\partial s\partial t}-\frac{\partial u(0,1)}{\partial t}-\frac{\partial u(0,1)}{\partial s}-(x_i+y_j+1)\frac{\partial^2 u(0,1)}{\partial s\partial t}-$$

$$\frac{\partial u(1,0)}{\partial t}-\frac{\partial u(1,0)}{\partial s}-(x_i+y_j+1)\frac{\partial^2 u(1,0)}{\partial s\partial t}\Bigg],\quad i,j=0,1 \tag{3.42}$$

Then, by differentiating equation (3.41) with respect to x and y, one can get

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=\int_0^1\int_0^1 u(s,t)\,ds\,dt$$

Therefore

$$\frac{\partial^2 u}{\partial x\partial y}=0$$

Next, we use the trapezoid rule to approximate the integral term that appeared in the above equation to get

$$\frac{\partial u(x_i,y_j)}{\partial x}=\frac{\partial u(x_i,y_j)}{\partial y}=\frac{1}{4}\big[u(0,0)+u(1,0)+u(0,1)+u(1,1)\big]$$

Hence

$$\frac{\partial u(0,0)}{\partial t}=\frac{\partial u(0,1)}{\partial t}=\frac{\partial u(1,0)}{\partial t}=\frac{\partial u(1,1)}{\partial t}\cong\frac{1}{4}\big[u(0,0)+u(1,0)+u(0,1)+u(1,1)\big]$$

and

$$\frac{\partial u(0,0)}{\partial s}=\frac{\partial u(0,1)}{\partial s}=\frac{\partial u(1,0)}{\partial s}=\frac{\partial u(1,1)}{\partial s}\cong\frac{1}{4}\big[u(0,0)+u(1,0)+u(0,1)+u(1,1)\big]$$

Moreover

$$\frac{\partial^2 u(0,0)}{\partial s\partial t}=\frac{\partial^2 u(1,0)}{\partial s\partial t}=\frac{\partial^2 u(0,1)}{\partial s\partial t}=\frac{\partial^2 u(1,1)}{\partial s\partial t}=0$$

By substituting these values into equation (3.42), one can have

$$u(x_i, y_j) = \frac{-7}{6} + \frac{1}{4}\Big[(x_i + y_j)u(0,0) + (x_i + y_j + 1)\big[u(0,1) + u(1,0)\big] +$$

$$(x_i + y_j + 2)\,u(1,1)\Big] + \frac{1}{24}\Big[2u(0,0) + \frac{1}{2}(x_i + y_j)\big[u(0,0) + u(1,0) + u(0,1) + u(1,1)\big] -$$

$$2u(1,1) - \frac{1}{2}(x_i + y_j + 2)\big[u(0,0) + u(1,0) + u(0,1) + u(1,1)\big]\Big] \quad i, j = 0,1$$

By evaluating the above equations at each i, j=0,1 one can have the following linear system of four equations with four unknowns $\{u(x_i, y_j)\}_{i,j=0}^{1}$:

$$\begin{pmatrix} 23 & -5 & -5 & -9 \\ -7 & -11 & 13 & -15 \\ -7 & 13 & -11 & -15 \\ -13 & -17 & -17 & 3 \end{pmatrix}\begin{pmatrix} u_{0,0} \\ u_{1,0} \\ u_{0,1} \\ u_{0,1} \end{pmatrix} = \begin{pmatrix} -28 \\ -28 \\ -28 \\ -28 \end{pmatrix}$$

which has the solution u(0,0)$\cong$0, u(1,0)=u(0,1)$\cong$1 and u(1,1)$\cong$2. But from the exact solution of this example, one can deduce that the modified trapezoidal rule gives the exact solution in this case.

On the other hand, for n=3 and m=2 and by following the same previous steps one can get the numerical solutions of $\{u(x_i, y_j)\}_{i=0, j=0}^{3,2}$ that can be tabulated down.

Table (3.5) represents the numerical and the exact solutions of example (3.4) for n=3 and m=2

Nodes	Numerical Solution	Exact Solution
(0,0)	0	0
(0.3,0)	0.316	0.333
(0.6,0)	0.649	0.667
(1,0)	0.871	1
(0,0.5)	0.478	0.5
(0.3,0.5)	0.815	0.833
(0.6,0.5)	1.146	1.167
(1,0.5)	1.482	1.5
(0,1)	0.859	1
(0.3,1)	1.312	1.333
(0.6,1)	1.643	1.667
(1,1)	1.981	2

Chapter Two

Some Modified Numerical Methods for Solving The One-Dimensional Integral Equations

Introduction:

The numerical integration methods (also called quadrature) is the study of how the numerical value of an integral can be estimated. The term quadrature means the process of finding square with the same area as the area enclosed by the arbitrary closed curve. This problem arises when the integration can not be carried out exactly or when the function is known only at a finite number of data, [22].

The main aim of this chapter is to use some of the quadrature methods say the trapezoidal rule and Simpson's rule to solve the generalized one-dimensional Fredhlom linear integral equations of the second kind. Also, we modify some of the quadrature methods for solving the one-dimensional Fredholm and Volterra linear integral equations of the second kind namely, the repeated trapezoidal rule.

This chapter consists of three sections:

In section one, we use some methods of the quadrature methods to solve the one-dimensional Fredholm and Volterra linear integral equations, namely the trapezoidal rule and Simpson's 1/3 rule

In section two, we use the previous methods to solve the generalized one-dimensional Fredholm linear integral equations.

In section three, we use the repeated modified trapezoidal rule to solve the one-dimensional Fredholm and Volterra linear integral equations of the second kind.

2.1 *Numerical Methods for Solving the Standard One-dimensional Linear Integral Equations,* [16]:

A simple rule for approximating integration (or quadrate) has the following form:

$$\int_a^b f(x)dx = \sum_{i=0}^{n} w_i f(x_i) + E(f)$$

where E(f) is the error, the points x_i, i=0,1,...,n are called quadrature points or quadrature nodes, and w_i,i=0,1, ...,n are the quadrature weights. In other words the integral is represented by a weighted sum of values of the integrand at a finite number of points x_i, i=0,1,…,n.

In this section we use some methods of the quadrature methods namely, the trapezoidal rule and Simpson's 1/3 rule to solve the standard one-dimensional Fredholm and Volterra linear integral equations of the second kind with some illustrative examples.

2.1.1 *The Trapezoidal Rule,* [6],[18]:

In this section we use the trapezoidal rule to solve the standard one-dimensional Fredholm and Volterra linear integral equations of the second kind. To do this, consider first, the standard one-dimensional Fredholm linear integral equation of the second kind:

$$u(x) = f(x) + \lambda \int_a^b k(x,y)u(y)dy, \quad a \le x \le b \tag{2.1}$$

Suppose that the interval [a,b] is divided into n subintervals of equal width $h = \frac{b-a}{n}$ such that $x_i = a + ih$, $i = 0,1,...,n$. By setting $x = x_i$ in the above integral equation one can obtain

$$u(x_i) = f(x_i) + \lambda \int_a^b k(x_i, y) u(y) dy, \ i=0,1,\ldots,n$$

Then, we replace the integral term that appeared in the right hand side of the above equation by the repeated trapezoidal rule to get

$$u_i = f(x_i) + \lambda \frac{h}{2} k(x_i, a) u_0 + \lambda h \sum_{j=1}^{n-1} k(x_i, x_j) u_j + \lambda \frac{h}{2} k(x_i, b) u_n, \ i = 0,1,...,n$$

where u_i denote the numerical solution at x_i, i=0,1,…,n.

By evaluating the above equation at each i =0,1,…,n, one can get a system of n+1 linear equations with n+1 unknowns namely $\{u_i\}_{i=0}^{n}$. This system can be written as $AU = F$ and can be solved by using any suitable method, where A is the matrix of the coefficients defined by

$$A = \begin{pmatrix} 1-\frac{\lambda h}{2}k_{0,0} & -\lambda h k_{0,1} & -\lambda h k_{0,2} & \ldots & -\lambda h k_{0,n-1} & -\frac{\lambda h}{2}k_{0,n} \\ -\frac{\lambda h}{2}k_{1,0} & 1-\lambda h k_{1,1} & -\lambda h k_{1,2} & \ldots & -\lambda h k_{1,n-1} & -\frac{\lambda h}{2}k_{1,n} \\ -\frac{\lambda h}{2}k_{2,0} & -\lambda h k_{2,1} & 1-\lambda h k_{2,2} & \ldots & -\lambda h k_{2,n-1} & -\frac{\lambda h}{2}k_{2,n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -\frac{\lambda h}{2}k_{n-1,0} & -\lambda h k_{n-1,1} & -\lambda h k_{n-1,2} & \ldots & 1-\lambda h k_{n-1,n-1} & -\frac{\lambda h}{2}k_{n-1,n} \\ -\frac{\lambda h}{2}k_{n,0} & -\lambda h k_{n,1} & -\lambda h k_{n,2} & \ldots & -\lambda h k_{n,n-1} & 1-\frac{\lambda h}{2}k_{n,n} \end{pmatrix}$$

U is the matrix of solutions and F is the matrix of non-homogeneous part, defined by

$$U = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} \text{ and } F = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{bmatrix}.$$

Here, $k_{ij} = k(x_i, y_j)$, $i, j = 0,1,...,n$ and $f_i = f(x_i)$, $i = 0,1,...,n$.

Second, consider the standard one-dimensional Volterra linear integral equation of the second kind:

$$u(x) = f(x) + \lambda\int_a^x k(x,y)u(y)dy, \quad a \le x \le b \tag{2.2}$$

By subdividing the interval [a, b] into n subintervals, such that $x_i = a + ih$, i=0,1,…,n, where $h = \dfrac{b-a}{n}$ and by setting $x = x_i$ in the above equation, one can get

$$u_i = f(x_i) + \lambda\int_a^{x_i} k(x_i, y)u(y)dy, \quad i = 1,2,...,n$$

Then, by replacing the integral term that appeared in the right hand side of the above equation by the repeated trapezoidal rule, one can get

$$u_i = f(x_i) + \lambda\frac{h}{2}k(x_i, x_0)u_0 + \lambda h\sum_{j=1}^{i-1}k(x_i, x_j)u_j + \lambda\frac{h}{2}k(x_i, x_i)u_i, \quad i = 1,2,...,n \tag{2.3}$$

By evaluating the above equation at each i=1,2,...,n and using the fact that $u_0 = u(x_0) = f(x_0)$ one can get the numerical solutions $\{u_i\}_{i=0}^{n}$ of the integral equation (2.2).

To illustrate this method, consider the following examples:

Example (2.1):

Consider the standard one-dimensional Fredholm linear integral equation of the second kind:

$$u(x) = 1 + \frac{1}{\pi}\int_{-1}^{1}\frac{1}{1+(x-y)^2}u(y)dy, \quad -1 \le x \le 1$$

We solve this example numerically with the repeated trapezoidal rule. To do this, first we subdivide the interval $[-1,1]$ into 8 subintervals such that $x_i = -1 + \frac{i}{4}$, $i = 0,1,...,8$, then the above integral equation becomes:

$$u_i = 1 + \frac{1}{\pi}\left[\frac{1}{8}\left(\frac{1}{1+(x_i+1)^2}\right)u_0 + \frac{1}{4}\sum_{j=1}^{7}\frac{1}{1+(x_i-x_j)^2}u_j + \frac{1}{8}\left(\frac{1}{1+(x_i-1)^2}\right)u_8\right], \ i = 0,1,...,8$$

By evaluating the above equation at each i=0,1,…,8 one can get a linear system of nine equations with nine unknowns $\{u_i\}_{i=0}^{8}$, which has the solution:

$u_0 = 1.91268$, $u_1 = 1.89332$, $u_2 = 1.83641$, $u_3 = 1.74695$, $u_4 = 1.63639$, $u_5 = 1.89332$,

$u_6 = 1.83641$, $u_7 = 1.74695$ and $u_8 = 1.63639$.

Second, we subdivide the interval $[-1,1]$ into 16 subintervals such that $x_i = -1 + \frac{i}{8}$, $i = 0,1,...,16$, then the above integral equation becomes:

$$u_i = 1 + \frac{1}{\pi}\left[\frac{1}{16}\left(\frac{1}{1+(x_i+1)^2}\right)u_0 + \frac{1}{8}\sum_{j=1}^{15}\frac{1}{1+(x_i-x_j)^2}u_j + \frac{1}{16}\left(\frac{1}{1+(x_i-1)^2}\right)u_{16}\right], \ i = 0,1,...,16$$

By evaluating the above equation at each i=0,1,…,16 one can get a linear system of 17 equations with 17 unknowns $\{u_i\}_{i=0}^{16}$ that can found in the appendix (see program (2.1)). This system has the solution

$u_0 = 1.63887$, $u_1 = 1.69648$, $u_2 = 1.7507$, $u_3 = 1.7994$, $u_4 = 1.84089$, $u_5 = 1.87401$,

$u_6 = 1.89804$, $u_7 = 1.912528$, $u_8 = 1.91744$, $u_9 = 1.91258$, $u_{10} = 1.89804$, $u_{11} = 1.87401$,

$u_{12} = 1.84089$, $u_{13} = 1.7994$, $u_{14} = 1.7507$, $u_{15} = 1.69648$ and $u_{16} = 1.63887$.

Third, we subdivide the interval $[-1,1]$ into 32 and 64 subintervals such that $x_i = -1 + \frac{i}{16}, i = 0,1,...,32$ and $x_i = -1 + \frac{i}{32}, i = 0,1,...,64$ respectively and by following the same previous steps one can get the results that can be found in the appendix (see program (2.1)).

Table (2.1) represents the numerical solutions of example (2.1) for different values of n at specific points

Nodes	n=16	n=32	n=64
$x = \pm 1$	1.63887	1.63949	1.63964
$x = \pm 0.75$	1.75070	1.75164	1.75187
$x = \pm 0.5$	1.84089	1.84201	1.84229
$x = \pm 0.25$	1.89804	1.89922	1.89952
$x = 0$	1.91744	1.91863	1.91839

Example (2.2):

Consider the standard one-dimensional Volterra linear integral equation of the second kind:

$$u(x) = x + \frac{1}{5}\int_0^x xyu(y)dy, \quad 0 \le x \le 2$$

This example is constructed such that the exact solution is $u(x) = xe^{\frac{x^3}{15}}$. We solve this example numerically via the repeated trapezoidal rule. To do this, first, we subdivide the interval [0,2] into 10 subintervals such that $x_i = \frac{i}{5}, \ i = 0,1,...,10$ and using the fact that $u_0 = f(0) = 0$. Then the above integral equation becomes:

$$u_i = x_i + \frac{1}{25}\sum_{j=1}^{i-1} x_i x_j u_j + \frac{1}{50} x_i^2 u_i, \; i = 1, 2, ..., 10.$$

By evaluating the above equation at each i=1,2,…,10 one can get the following values:

$u_1 = 8\times10^{-4}$, $u_2 = 3.2026\times10^{-3}$, $u_3 = 7.2346\times10^{-3}$, $u_4 = 0.013$, $u_5 = 0.0206$,

$u_6 = 0.0306$, $u_7 = 0.0433$, $u_8 = 0.0598$, $u_9 = 0.0813$ and $u_{10} = 0.1101$.

Second, we subdivide the interval [0,2] into 20 subintervals such that $x_i = \frac{i}{10}$, $i = 0,1,...,20$ and using the fact that $u_0 = f(0) = 0$. Then the above integral equation becomes:

$$u_i = x_i + \frac{1}{50}\sum_{j=1}^{i-1} x_i x_j u_j + \frac{1}{100} x_i^2 u_i, \; i = 1, 2, ..., 20.$$

By evaluating the above equation at each i=1,2,…,20 one can get the following values (see program (2.2)):

$u_1 = 1\times10^{-4}$, $u_2 = 4.004\times10^{-4}$, $u_3 = 9.0054\times10^{-4}$, $u_4 = 1.6029\times10^{-3}$, $u_5 = 2.51\times10^{-3}$,

$u_6 = 3.6271\times10^{-3}$, $u_7 = 4.9621\times10^{-3}$, $u_8 = 6.5265\times10^{-3}$, $u_9 = 8.3363\times10^{-3}$, $u_{10} = 0.0104$,

$u_{11} = 0.0128$, $u_{12} = 0.0155$, $u_{13} = 0.0186$, $u_{14} = 0.0221$, $u_{15} = 0.0261$, $u_{16} = 0.0306$,

$u_{17} = 0.0359$, $u_{18} = 0.042$, $u_{19} = 0.0492$ and $u_{20} = 0.0575$.

2.1.2 *Simpson's 1/3 Rule*, [6],[10]:

In this section we use Simpson's 1/3 rule to solve the standard one-dimensional Fredholm and Volterra linear integral equations of the second kind. To do this, consider first, the standard one-dimensional Fredholm linear integral equation of the second kind given by equation (2.1). Suppose that the interval [a,b] is divided into n subintervals of equal width $h = \frac{b-a}{n}$ such that $x_i = a + ih$, $i = 0,1,...,n$, and n is an even positive integer. Then by setting $x = x_i$ in the above integral equation and by replacing the integral term that appeared in the right hand side of the integral equation (2.1) by the repeated Simpson's 1/3 rule, one can have

$$u_i = f_i + \lambda\frac{h}{3}\left[k(x_i,a)u_0 + 4\sum_{j=1}^{\frac{n}{2}} k(x_i,x_{2j-1})u_{2j-1} + 2\sum_{j=1}^{\frac{n}{2}-1} k(x_i,x_{2j})u_{2j} + k(x_i,b)u_n\right]$$

where u_i denote the numerical solution at x_i, i=0,1,…,n.

By evaluating the above equation at each $i = 0,1,...,n$ one can get a system of $n+1$ linear equations with $n+1$ unknowns namely $\{u_i\}_{i=0}^{n}$. This system can be written as $AU = F$ and can be solved by using any suitable method, where A is the matrix of the coefficients defined by

$$A = \begin{bmatrix} 1-\frac{\lambda h}{3}k_{0,0} & -\frac{4\lambda h}{3}k_{0,1} & \frac{-2\lambda h}{3}k_{0,2} & \cdots & \frac{-4\lambda h}{3}k_{0,n-1} & \frac{-\lambda h}{3}k_{0,n} \\ -\frac{\lambda h}{3}k_{1,0} & 1-\frac{4\lambda h}{3}k_{1,1} & \frac{-2\lambda h}{3}k_{1,2} & \cdots & \frac{-4\lambda h}{3}k_{1,n-1} & \frac{-\lambda h}{3}k_{1,n} \\ -\frac{\lambda h}{3}k_{2,0} & -\frac{4\lambda h}{3}k_{2,1} & 1-\frac{2\lambda h}{3}k_{2,2} & \cdots & \frac{-4\lambda h}{3}k_{2,n-1} & \frac{-\lambda h}{3}k_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{\lambda h}{3}k_{n-1,0} & -\frac{4\lambda h}{3}k_{n-1,1} & \frac{-2\lambda h}{3}k_{n-1,2} & \cdots & 1-\frac{4\lambda h}{3}k_{n-1,n-1} & \frac{-\lambda h}{3}k_{n-1,n} \\ -\frac{\lambda h}{3}k_{n,0} & -\frac{4\lambda h}{3}k_{n,1} & \frac{-2\lambda h}{3}k_{n,2} & \cdots & \frac{-4\lambda h}{3}k_{n,n-1} & 1-\frac{\lambda h}{3}k_{n,n} \end{bmatrix}$$

U is the matrix of solutions and F is the matrix of non-homogeneous part, defined by

$$U = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} \text{ and } F = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{bmatrix}$$

Here, $k_{ij} = k(x_i, y_j)$, $i, j = 0,1,...,n$ and $f_i = f(x_i)$, $i = 0,1,...,n$.

Second, consider the standard one-dimensional Volterra linear integral equation of the second kind given by equation (2.2). Suppose that the interval [a,b] is divided into n subintervals of equal width $h = \frac{b-a}{n}$ such that $x_i = a + ih$, $i = 0,1,...,n$, and n is an even positive integer. By setting $x = x_i$ in the above integral equation and by replacing the integral term that appeared in the right hand side of the above integral equation by the repeated Simpson's 1/3 rule, one can have

$$u_i = f_i + \lambda \frac{h}{3}\left[k(x_i, x_0)u_0 + 4\sum_{j=1}^{\frac{i}{2}} k(x_i, x_{2j-1})u_{2j-1} + 2\sum_{j=1}^{\frac{i}{2}-1} k(x_i, x_{2j})u_{2j} + k(x_i, x_i)u_i \right] \quad (2.4)$$

where u_i denote the numerical solution at x_i, i=0,1,…,n.

By evaluating equation (2.3) at i=1 and using the fact that $u_0 = u(x_0) = f(x_0) = f_0$, one can get the numerical solution u_1. Then by evaluating equation (2.4) at i=2 and substituting the values of u_0 and u_1 one can get the numerical solution u_2 of the integral equation (2.2). Therefore, by evaluating equation (2.3) at each i=3,5,…,n-1, and by evaluating equation (2.4) at each i=4,6,…,n one can get the numerical solutions $\{u_i\}_{i=0}^{n}$ of the integral equation (2.2).

To illustrate this method, consider the following examples:

Example (2.3):

Consider example (2.1). We solve this example numerically via the repeated Simpson's 1/3 rule. To do this, we subdivide the interval [0,1] into 8 subintervals such that $x_i = -1 + \frac{i}{4}, i = 0,1,...,8$. Then the above integral equation becomes:

$$u_i = 1 + \frac{1}{\pi}\left[\frac{1}{12}\left(\frac{1}{1+(x_i+1)^2}\right)u_0 + \frac{1}{3}\sum_{j=1}^{4}\frac{1}{1+(x_i - x_{2j-1})^2}u_{2j-1} + \frac{1}{6}\sum_{j=1}^{3}\frac{1}{1+(x_i - x_{2j})^2}u_{2j}\right.$$
$$\left.+\frac{1}{12}\left(\frac{1}{1+(x_i-1)^2}\right)u_8\right],\ i = 0,1,...,8$$

By evaluating the above equation at each i=0,1,…,8 one can get a linear system of nine equations with nine unknowns $\{u_i\}_{i=0}^{8}$. Which has the solution:

$u_0 = 1.6357,\ u_1 = 1.8942,\ u_2 = 1.8373,\ u_3 = 1.8942,\ u_4 = 1.91302,\ u_5 = 1.89332,$

$u_6 = 1.83641,\ u_7 = 1.74695$ and $u_8 = 1.63639.$

Second, we subdivide the interval $[-1,1]$ into 16 subintervals such that $x_i = -1 + \frac{i}{8},\ i = 0,1,...,16$. Then the above integral equation becomes:

$$u_i = 1 + \frac{1}{\pi}\left[\frac{1}{16}\left(\frac{1}{1+(x_i+1)^2}\right)u_0 + \frac{1}{6}\sum_{j=1}^{8}\frac{1}{1+(x_i - x_{2j-1})^2}u_{2j-1} + \frac{1}{12}\sum_{j=1}^{7}\frac{1}{1+(x_i - x_{2j})^2}u_{2j}\right.$$
$$\left.+\frac{1}{16}\left(\frac{1}{1+(x_i-1)^2}\right)u_{16}\right],\ i = 0,1,...,16$$

By evaluating the above equation at each i=0,1,…,16 one can get a linear system of 17 equations with 17 unknowns $\{u_i\}_{i=0}^{16}$ that can be found in the appendix (see program (2.3)). This system has the solution:

$u_0 = 1.63887,\ u_1 = 1.69648,\ u_2 = 1.7507,\ u_3 = 1.7994,\ u_4 = 1.84089,\ u_5 = 1.87401,$

$u_6 = 1.89804,\ u_7 = 1.912528,\ u_8 = 1.91744,\ u_9 = 1.91258,\ u_{10} = 1.89804,\ u_{11} = 1.87401,$

$u_{12} = 1.84089,\ u_{13} = 1.7994,\ u_{14} = 1.7507,\ u_{15} = 1.69648$ and $u_{16} = 1.63887.$

Third, we subdivide the interval $[-1,1]$ into 32 and 64 subintervals such that $x_i = -1 + \frac{i}{16}, i = 0,1,...,32$ and $x_i = -1 + \frac{i}{32}, i = 0,1,...,64$ respectively and by following the same previous steps one can get the results that can be found in the appendix (see program (2.3)).

Table (2.2) represents the numerical solutions of example (2.3) for different values of n at specific points

Nodes	n=16	n=32	n=64
$x = \pm 1$	1.63903	1.63872	1.63981
$x = \pm 0.75$	1.75801	1.7521	1.75107
$x = \pm 0.5$	1.84102	1.84365	1.84422
$x = \pm 0.25$	1.89984	1.89822	1.89982
$x = 0$	1.91844	1.92863	1.98192

Example (2.4):

Consider example (2.2). We solve this example numerically via the repeated Simpson's 1/3 rule. To do this, we subdivide the interval [0,2] into 10 subintervals such that $x_i = \frac{i}{5},\ i = 0,1,...,10$. Therefore

$$u_i = \frac{50}{\left(50 - x_i^2\right)}\left(x_i + \frac{1}{25}\sum_{j=1}^{i-1} x_i x_j u_j\right), \text{ i=1, 3, 5, 7, 9.}$$

and

$$u_i = \frac{75}{\left(75 - x_i^2\right)}\left(x_i + \frac{4}{75}\sum_{j=1}^{\frac{i}{2}} x_i x_{2j-1} u_{2j-1} + \frac{2}{75}\sum_{j=1}^{\frac{i}{2}-1} x_i x_{2j} u_{2j}\right), \text{ i=2, 4, 6, 8, 10.}$$

By evaluating the above two equations at each i=1,3,5,7,9 and i=2,4,6,8,10 respectively and using the fact that $u_0 = 0,$ one can get the following values:

$u_1 = 0.10011,\ u_2 = 0.30122,\ u_3 = 0.5511,\ u_4 = 0.80777,\ u_5 = 1.06345,$

$u_6 = 1.33533,\ u_7 = 1.67301,\ u_8 = 2.10183,\ u_9 = 2.45568$ and $u_{10} = 3.32743.$

Second, we subdivide the interval [0,2] into 20 subintervals such that $x_i = \frac{i}{10},$ $i = 0,1,...,20$. Therefore

$$u_i = \frac{100}{\left(100 - x_i^2\right)}\left(x_i + \frac{1}{50}\sum_{j=1}^{i-1} x_i x_j u_j\right), \text{ i=1, 3, 5, 7, 9, 11, 13, 15, 17, 19.}$$

and

$$u_i = \frac{150}{\left(150 - x_i^2\right)}\left(x_i + \frac{4}{150}\sum_{j=1}^{\frac{i}{2}} x_i x_{2j-1} u_{2j-1} + \frac{2}{150}\sum_{j=1}^{\frac{i}{2}-1} x_i x_{2j} u_{2j}\right)$$

where i=2,4,6, 8, 10,12,14,16,18,20.

By evaluating the above two equations at each i=1, 3, 5, 7, 9, 11, 13, 15, 17, 19 and i=2, 4, 6, 8, 10, 12, 14, 16, 18, 20 respectively and using the fact that $u_0 = 0,$ one can get the results that can be tabulated.

Table (2.3) represents the numerical solutions of example (2.4) for different values of n

Nodes	Numerical Solution	Exact Solution
x=0	0	0
x=0.1	0.00091	0.10001
x=0.2	0.00052	0.20011
x=0.3	0.00097	0.30054
x=0.4	0.00171	0.40171
x=0.5	0.00257	0.50418
x=0.6	0.00369	0.60870
x=0.7	0.00499	0.71619
x=0.8	0.00661	0.82778
x=0.9	0.00839	0.94482
x=1	0.01092	1.06894
x=1.1	0.01296	1.20207
x=1.2	0.01561	1.34652
x=1.3	0.01884	1.50506
x=1.4	0.02521	1.68103
x=1.5	0.02731	1.87848
x=1.6	0.03225	2.10238
x=1.7	0.03629	2.35882
x=1.8	0.04293	2.65538
x=1.9	0.04984	3.00153
x=2	0.05831	3.40921

2.2 *Numerical Methods for Solving the Generalized One-Dimensional Linear Integral Equations*:

In this section, we use some of the quadrature methods namely, the trapezoidal rule and Simpson's 1/3 rule for solving the generalized one-dimensional Fredholm linear integral equations of the second kind.

2.2.1 *The Trapezoidal Rule*:

In this section we use the trapezoidal rule to solve the generalized one-dimensional Fredholm and Volterra linear integral equations of the second kind. To do this, consider first, the 2-generalized one-dimensional Fredholm linear integral equation of the second kind:

$$u(x)=f(x)+\lambda\int_a^b k(x,y)u(y)dy+\mu\int_c^d \ell(x,y)u(y)dy,\quad a\le x\le b \tag{2.5}$$

where $[c,d]\subseteq[a,b]$. By dividing the interval [a,b] into n subintervals, such that $x_i=a+ih$, i=0,1,...,n, $h=\frac{b-a}{n}$ and c and d are one of it's mesh points say $c=x_p$ and $d=x_q$ for some $p,q\in\{0,1,...,n\}$. Then by setting $x=x_i$ in the above integral equation and by replacing the integral term that appeared in the right hand side of the above integral equation by the repeated Trapezoidal rule, one can have

$$u_i=f(x_i)+\lambda\frac{h}{2}k(x_i,a)u_0+\lambda h\sum_{j=1}^{n-1}k(x_i,x_j)u_j+\lambda\frac{h}{2}k(x_i,b)u_n+$$
$$\mu\frac{h}{2}\ell(x_i,x_p)\,u_p+\mu h\sum_{j=p+1}^{q-1}\ell(x_i,x_j)u_j+\mu\frac{h}{2}\ell(x_i,x_q)u_q \tag{2.6}$$

where u_i denote the numerical solution at x_i, i=0,1,…,n.

By evaluating the above equation at each i =0,1,…,n, one can get a system of n+1 linear equations with n+1 unknowns namely $\{u_i\}_{i=0}^{n}$. This system can be written as $AU=F$ and can be solved by using any suitable method, where A is the matrix of the coefficients defined by

U is the matrix of solutions and F is the matrix of non-homogeneous part, defined by

$$U=\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{p-1} \\ u_p \\ u_{p+1} \\ \vdots \\ u_{q-1} \\ u_q \\ u_{q+1} \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} \text{ and } F=\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{p-1} \\ f_p \\ f_{p+1} \\ \vdots \\ f_{q-1} \\ f_q \\ f_{q+1} \\ \vdots \\ f_{n-1} \\ f_n \end{bmatrix}.$$

To illustrate this method, consider the following examples:

Example (2.5):

Consider the generalized one-dimensional Fredholm linear integral equation of the second kind:

$$u(x)=0.9493x^2-0.3333x-0.2635+\int_0^1 (x+y)u(y)dy+\int_{0.4}^{0.6}(x^2+y^2)u(y)dy$$

where $0\le x\le 1$.

This example is constructed such that the exact solution of it, is $u(x)=x^2$. We solve this example numerically with the repeated trapezoidal rule. To do this, first we subdivide the interval [0,1] into 5 subintervals such that $x_i=\frac{i}{5}$, i=0,1,2,3,4,5. Then $x_2=0.4$ and $x_3=0.6$, therefore p=2 and q=3. In this case, equation (2.6) becomes

$$u_i = 0.9493x_i^2 - 0.3333x - 0.2635 + \frac{1}{10}x_i u_0 + \frac{1}{5}\sum_{i=1}^{4}(x_i + x_j)u_j + \frac{1}{10}(x_i + 1)u_5 + \frac{1}{10}(x_i^2 + 0.16)u_2 + \frac{1}{10}(x_i^2 + 0.36)u_3, i = 0, 1, 2, 3, 4, 5$$

By evaluating the above equation at each i=0, 1, 2, 3, 4, 5 one can get the following linear system:

$$\begin{pmatrix} 1 & -0.04 & -0.112 & -0.156 & -0.16 & -0.1 \\ -0.02 & 0.92 & -0.16 & -0.2 & -0.2 & -0.12 \\ -0.04 & -0.12 & 0.776 & -0.252 & -0.24 & -0.14 \\ -0.06 & -0.16 & -0.304 & 0.688 & -0.286 & -0.1 \\ -0.08 & -0.2 & -0.4 & -0.38 & 0.68 & -0.18 \\ -0.1 & -0.24 & -0.512 & -0.456 & -0.36 & 0.8 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} -0.2635 \\ -0.2922 \\ -0.2449 \\ -0.1217 \\ 0.0774 \\ 0.3525 \end{pmatrix}$$

The solution of the above system is tabulated down with the comparison with the exact solution.

Table (2.4) represents the numerical and the exact solutions of example (2.5) for n=5

Nodes	Numerical Solution	Exact Solution
x=0	-0.04634	0
x=0.2	-0.02383	0.04
x=0.4	0.07796	0.16
x=0.6	0.259	0.36
x=0.8	0.51932	0.64
1	0.8589	1

Second, we subdivide the interval [0,1] into 10 subintervals such that $x_i = \frac{i}{10}$, i=0, 1,…, 10. Then $x_4 = 0.4$ and $x_6 = 0.6$ and therefore p=4 and q=6. In this case, equation (2.6) becomes

$$u_i = 0.9493x_i^2 - 0.3333x - 0.2635 + \frac{1}{20}x_i u_0 + \frac{1}{10}\sum_{i=1}^{9}(x_i + x_j)u_j + \frac{1}{20}(x_i + 1)u_{10} +$$
$$\frac{1}{20}(x_i^2 + 0.16)u_4 + \frac{1}{10}(x_i^2 + 0.25)u_5 + \frac{1}{20}(x_i^2 + 0.36)u_6,\ i = 0,\ 1,\ ...,10$$

By evaluating the above equation at each i=0, 1,…, 10. One can get the following linear system which can be written as AU=F, where

$$A = \begin{pmatrix}
1 & -0.01 & -0.02 & -0.03 & -0.048 & -0.075 & -0.078 & -0.07 & -0.08 & -0.09 & -0.05 \\
-0.005 & 0.98 & -0.03 & -0.04 & -0.0585 & -0.086 & -0.0885 & -0.08 & -0.09 & -0.1 & -0.055 \\
-0.01 & -0.03 & 0.96 & -0.05 & -0.07 & -0.099 & -0.1 & -0.09 & -0.1 & -0.11 & -0.06 \\
-0.015 & -0.04 & -0.05 & 0.94 & -0.0825 & -0.114 & -0.1125 & -0.1 & -0.11 & -0.12 & -0.065 \\
-0.02 & -0.05 & -0.06 & -0.07 & 0.904 & -0.131 & -0.126 & -0.11 & -0.12 & -0.13 & -0.7 \\
-0.025 & -0.06 & -0.07 & -0.08 & -0.1105 & 0.85 & -0.1405 & -0.12 & -0.13 & -0.14 & -0.075 \\
-0.03 & -0.07 & -0.08 & -0.09 & -0.126 & -0.171 & 0.844 & -0.13 & -0.14 & -0.15 & -0.08 \\
-0.035 & -0.08 & -0.09 & -0.1 & -0.1425 & -0.194 & -0.1725 & 0.86 & -0.15 & -0.16 & -0.085 \\
-0.04 & -0.09 & -0.1 & -0.11 & -0.16 & -0.219 & -0.19 & -0.15 & 0.84 & -0.17 & -0.09 \\
-0.045 & -0.1 & -0.11 & -0.12 & -0.1785 & -0.246 & -0.2085 & -0.16 & -0.17 & 0.82 & -0.095 \\
-0.05 & -0.11 & -0.12 & -0.13 & -0.198 & -0.275 & -0.228 & -0.17 & -0.18 & -0.19 & 0.9
\end{pmatrix}$$

$$U = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} -0.2635 \\ -0.28734 \\ -0.2922 \\ -0.27806 \\ -0.24494 \\ -0.19284 \\ -0.12174 \\ -0.03166 \\ 0.0774 \\ 0.20546 \\ 0.3525 \end{pmatrix}$$

The solution of the above system is tabulated down with the comparison with the exact solution.

Table (2.5) represents the numerical and the exact solutions of example (2.5) for n=10

Nodes	Numerical Solution	Exact Solution
x=0	-0.009105	0
x=0.1	-0.000852	0.01
x=0.2	0.02733	0.04
x=0.3	0.07545	0.09
x=0.4	0.1435	0.16
x=0.5	0.23148	0.25
x=0.6	0.3394	0.36
x=0.7	0.46725	0.49
x=0.8	0.61503	0.64
x=0.9	0.78274	0.81
1	0.97038	1

Third, we subdivide the interval [0,1] into 20 subintervals such that $x_i = \frac{i}{20}$, i=0, 1,…,20. Then $x_8 = 0.4$ and $x_{12} = 0.6$ and therefore p=8 and q=12. In this case, equation (2.6) becomes

$$u_i = 0.9493x_i^2 - 0.3333x - 0.2635 + \frac{1}{40}x_i u_0 + \frac{1}{20}\sum_{i=1}^{19}(x_i + x_j)u_j + \frac{1}{40}(x_i + 1)u_{20} +$$

$$\frac{1}{40}(x_i^2 + 0.16)u_8 + \frac{1}{20}\sum_{i=9}^{11}(x_i^2 + x_j^2)u_j + \frac{1}{40}(x_i^2 + 0.36)u_{12}, i = 0, 1, ..., 20$$

By evaluating the above equation at each i=0,1,…,20 and by following the same previous steps one can get the tabulated results that can be seen in the appendix (see program (2.5)).

Fourth, we subdivide the interval [0,1] into 30, 40, 50 and 100 and the results can be seen in the appendix (see program (2.5)). Some of these results can be tabulated down with the comparison with the exact solution.

2.2.2 *Simpson's 1/3 Rule*:

In this section we use Simpson's 1/3 rule to solve the generalized one-dimensional Fredholm and Volterra linear integral equations of the second kind. To do this, consider first, the 2-generalized one-dimensional Fredholm linear integral equation of the second kind given by equation (2.3).

By dividing the interval [a,b] into n subintervals, such that $x_i = a + ih$, i=0,1,...,n, $h = \frac{b-a}{n}$, n is an even positive integer and c and d are one of it's mesh points say $c = x_p$ and $d = x_q$ for some $p, q \in \{0,1,...,n\}$. Then by setting $x = x_i$ in the above integral equation and by replacing the integral term that appeared in the right hand side of the above integral equation by the repeated Simpson's 1/3 rule, one can have

$$u_i = f(x_i) + \lambda\frac{h}{3}\left[k(x_i,a)u_0 + 4\sum_{j=1}^{\frac{n}{2}} k(x_i,x_{2j-1})u_{2j-1} + 2\sum_{j=1}^{\frac{n}{2}-1} k(x_i,x_{2j})u_{2j} + k(x_i,b)u_n\right] +$$

$$\mu\frac{h}{3}\left[\ell(x_i,x_p)u_p + 4\sum_{j=p+1}^{\frac{q}{2}} \ell(x_i,x_{2j-1})u_{2j-1} + 2\sum_{j=p+1}^{\frac{q}{2}-1} \ell(x_i,x_{2j})u_{2j} + \ell(x_i,x_q)u_q\right] \qquad (2.7)$$

where u_i denote the numerical solution at each x_i, i=0,1,…,n. It is clear that equation (2.7) holds in case q is an even positive integer. However if this condition is not satisfied then equation (2.7) can be replaced by the following equation:

$$u_i = f(x_i) + \lambda\frac{h}{3}\left[k(x_i,a)u_0 + 4\sum_{j=1}^{\frac{n}{2}} k(x_i,x_{2j-1})u_{2j-1} + 2\sum_{j=1}^{\frac{n}{2}-1} k(x_i,x_{2j})u_{2j} + k(x_i,b)u_n\right] +$$

$$\mu\frac{h}{2}\left[\ell(x_i,x_p)u_p + 2\sum_{j=p+1}^{q-1} \ell(x_i,x_j)u_j + \ell(x_i,x_q)u_q\right] \qquad (2.8)$$

By evaluating equation (2.7) (or equation (2.8)) at each $i = 0,1,\dots,n$ one can get a system of $n+1$ linear equations with $n+1$ unknowns $\{u_i\}_{i=0}^{n}$. This system can be written as $AU = F$ and can be solved by using any suitable method, where A is the matrix of the coefficients defined by

U is the matrix of solutions and F is the matrix of non-homogeneous part, defined by

$$U=\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{p-1} \\ u_p \\ u_{p+1} \\ \vdots \\ u_{q-1} \\ u_q \\ u_{q+1} \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} \text{ and } F=\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{p-1} \\ f_p \\ f_{p+1} \\ \vdots \\ f_{q-1} \\ f_q \\ f_{q+1} \\ \vdots \\ f_{n-1} \\ f_n \end{bmatrix}.$$

Example (2.6):

Consider the generalized one-dimensional Fredholm linear integral equation of the second kind:

$$u(x)=0.9493x^2-0.3333x-0.2635+\int_0^1 (x+y)u(y)dy+\int_{0.2}^{0.8}(x^2+y^2)u(y)dy$$

where $0 \le x \le 1$.

we solve this example numerically via the repeated Simpson's 1/3 rule. To do this, first we subdivide the interval [0,1] into 10 subintervals such that $x_i=\frac{i}{10}$, $i=0,1,\ldots,10$.

Then $x_2=0.2$ and $x_8=0.8$, therefore p=2 and q=8. In this case, equation (2.7) becomes

$$u_i = 0.9493x_i^2 - 0.3333x - 0.2635 + \frac{1}{30}\Big[x_i u_0 + 4\sum_{i=1}^{5}(x_i + x_{2j-1})u_{2j-1} + 2\sum_{j=1}^{4}(x_i + x_{2j})u_{2j} +$$

$$(x_i + 1)u_{10}\Big] + \frac{1}{30}\Big[(x_i^2 + 0.4)u_2 + 4\sum_{j=3}^{4}(x_i^2 + x_{2j-1}^2)u_{2j-1} + 2(x_i^2 + 0.36)u_6 + (x_i^2 + 0.46)u_8\Big]$$

where i=0,1,…,10.

By evaluating the above equation at each i=0, 1,…,10 one can get a linear system of 11 equations with 11 unknowns $\{u_i\}_{i=0}^{10}$ that can be found in the appendix (see program (2.6)). The solution of this system is tabulated down with the comparison with the exact solution.

Table (2.7) represents the numerical solution of example (2.6) for n=10

Nodes	Numerical Solution	Exact Solution
x=0	-0.009505	0
x=0.1	-0.000952	0.01
x=0.2	0.03733	0.04
x=0.3	0.08545	0.09
x=0.4	0.1535	0.16
x=0.5	0.24148	0.25
x=0.6	0.3594	0.36
x=0.7	0.47725	0.49
x=0.8	0.62503	0.64
x=0.9	0.79274	0.81
1	0.98038	1

Second, we subdivide the interval [0,1] into 20 subintervals such that $x_i = \frac{i}{20}$, i=0, 1,…,20. Then $x_4 = 0.2$ and $x_{16} = 0.8$ and therefore p=4 and q=16. In this case, equation (2.7) becomes

$$u_i = 0.9493x_i^2 - 0.3333x - 0.2635 + \frac{1}{60}\Big[x_i u_0 + 4\sum_{i=1}^{10}(x_i + x_{2j-1})u_{2j-1} + 2\sum_{j=1}^{9}(x_i + x_{2j})u_{2j} +$$
$$(x_i + 1)u_{20}\Big] + \frac{1}{60}\Big[(x_i^2 + 0.4)u_4 + 4\sum_{j=5}^{8}(x_i^2 + x_{2j-1}^2)u_{2j-1} + 2\sum_{j=5}^{7}(x_i^2 + x_{2j}^2)u_{2j} + (x_i^2 + 0.46)u_{16}\Big]$$

By evaluating the above equation at each i=0,1,…,20 and by following the same previous steps one can get the tabulated results that can be seen in the appendix (see program (2.6)).

Third, we subdivide the interval [0,1] into 30, 40, 50 and 100 and the results can be seen in the appendix (see program (2.6)). Some of these results can be tabulated down with the comparison with the exact solution.

2.3 *The Modified Trapezoid Rule for Solving The One-Dimensional Linear Integral Equations*:

In this section, we modify some of the quadrature methods to solve the one-dimensional Fredholm and Volterra linear integral equations, namely the repeated trapezoid rule.

2.3.1 *The Modified Trapezoid Rule for Solving The One-Dimensional Fredholm Linear Integral Equations*, [29]:

Let f be a continuous function defined on the closed interval [a,b]. By subdividing the interval [a,b] into n subintervals $[x_i, x_{i+1}],\ i = 0,1,\ldots,n-1$ of equal width $h = \dfrac{b-a}{n}$ such that $x_i = a + ih,\ i = 0,1,\ldots,n$. Then the repeated modified trapezoid formula for evaluating $\int_a^b f(x)dx$ is:

$$\int_a^b f(x)dx \approx \frac{h}{2}\left[f(x_0) + 2\sum_{j=1}^{n-1} f(x_j) + f(x_n)\right] + \frac{h^2}{12}\left[f'(x_0) - f'(x_n)\right]$$

Next, consider the one-dimensional Fredholm linear integral equation of the second kind given by equation (2.1). First, we divide the interval [a,b] into n subinterval $[x_i, x_{i+1}],\ i = 0,1,\ldots,n-1$ such that $x_i = a + ih,\ i = 0,1,\ldots,n$, where $h = \dfrac{b-a}{n}$. So, the problem here is to find the solution of equation (2.1) at $x_i,\ i = 0,1,\ldots,n$. Let u_i denote the numerical solution of the integral equation (2.1) at each $x_i,\ i = 0,1,\ldots,n$. Then, we approximate the integral that appeared in the right hand side of equation (2.1) with the repeated modified trapezoid rule to get

$$u(x) = f(x) + \frac{\lambda h}{2} k(x, x_0) u_0 + \lambda h \sum_{j=1}^{n-1} k(x, x_j) u_j + \frac{\lambda h}{2} k(x, x_n) u_n +$$

$$\frac{\lambda h^2}{12} \left[\frac{\partial}{\partial y} \left(k(x, y) u(y) \right) \Big|_{y=x_0} - \frac{\partial}{\partial y} \left(k(x, y) u(y) \right) \Big|_{y=x_n} \right]$$

Therefore

$$u(x) = f(x) + \frac{\lambda h}{2} k(x, x_0) u_0 + \lambda h \sum_{j=1}^{n-1} k(x, x_j) u_j + \frac{\lambda h}{2} k(x, x_n) u_n +$$

$$\frac{\lambda h^2}{12} \left[k(x, x_0) u_0' + J(x, x_0) u_0 - k(x, x_n) u_n' - J(x, x_n) u_n \right] \tag{2.9}$$

where $J(x, y) = \dfrac{\partial k(x, y)}{\partial y}$.

By substituting $x = x_i,\ i = 0,1,\ldots,n$ in the above equation and transform all the terms involving the solution u_i to the left hand side of the resulting equation leaving only $f(x_i)$ on the right hand side, one can have

$$u_i - \frac{\lambda h}{2} k_{i0} u_0 - \lambda h \sum_{j=1}^{n-1} k_{ij} u_j - \frac{\lambda h}{2} k_{in} u_n -$$

$$\frac{\lambda h^2}{12} \left[k_{i0} u_0' + J_{i0} u_0 - k_{in} u_n' - J_{in} u_n \right] = f_i, \quad i = 0,1,\ldots,n. \tag{2.10}$$

where, $f_i = f(x_i),\ k_{ij} = k(x_i, y_j),\ J_{ip} = J(x_i, x_p),\ i = 0,1,\ldots,n$ and $p = 0, n$.

The above system of equations consists of n+1 equations with n+3 unknowns namely, $u_i,\ i = 0,1,\ldots,n,\ u_0'$ and u_n'. To find u_0' and u_n', one must differentiate equation (2.1) with respect to x to get

$$u'(x) = f'(x) + \lambda \int_a^b H(x, y) u(y)\, dy \tag{2.11}$$

where $H(x, y) = \dfrac{\partial k(x, y)}{\partial x}$.

It is easy to check any solution of equation (2.1) is a solution of equation (2.11) too.

Next, to solve equation (2.11), one must consider two cases:-

Case (1):

If $\frac{\partial^2 k(x,y)}{\partial x \partial y}$ exists, in this case, we approximate the integral that appeared in the right hand side of the integral equation (2.11) with the repeated modified trapezoid rule to obtain

$$u'(x) = f'(x) + \frac{\lambda h}{2} H(x,x_0)u_0 + \lambda h \sum_{j=1}^{n-1} H(x,x_j)u_j + \frac{\lambda h}{2} H(x,x_n)u_n + \frac{\lambda h^2}{12}\left[\frac{\partial}{\partial y}\big(H(x,y)u(y)\big)\Big|_{y=x_0} - \frac{\partial}{\partial y}\big(H(x,y)u(y)\big)\Big|_{y=x_n}\right]$$

Therefore

$$u'(x) = f'(x) + \frac{\lambda h}{2} H(x,x_0)u_0 + \lambda h \sum_{j=1}^{n-1} H(x,x_j)u_j + \frac{\lambda h}{2} H(x,x_n)u_n + \frac{\lambda h^2}{12}\left[H(x,x_0)u'_0 + L(x,x_0)u_0 - H(x,x_n)u'_0 - L(x,x_n)u_n\right]$$

where $L(x,y) = \frac{\partial H(x,y)}{\partial y} = \frac{\partial^2 k(x,y)}{\partial y \partial x}$.

Hence for $x = x_p, p = 0, n$, one can get the following equations:

$$u'_0 = f'_0 + \frac{\lambda h}{2} H_{00}u_0 + \lambda h \sum_{j=1}^{n-1} H_{0j}u_j + \frac{\lambda h}{2} H_{0n}u_n + \frac{\lambda h^2}{12}\left[H_{00}u'_0 + L_{00}u_0 - H_{0n}u'_n - L_{0n}u_n\right]$$

$$u'_n = f'_n + \frac{\lambda h}{2} H_{n0}u_0 + \lambda h \sum_{j=1}^{n-1} H_{nj}u_j + \frac{\lambda h}{2} H_{nn}u_n + \frac{\lambda h^2}{12}\left[H_{n0}u'_0 + L_{n0}u_0 - H_{nn}u'_n - L_{nn}u_n\right]$$

where $f'_0 = f'(x_0)$ and $f'_n = f'_n(x_n)$.

The system which consist of the above two equations together with the n+1 equations given by the equation (2.10) can be written as AU=F, where

$$A=\begin{bmatrix} 1-\frac{\lambda h}{2}k_{00}-\frac{\lambda h^2}{12}J_{00} & -\lambda hk_{01} & -\lambda hk_{02} & \cdots & -\frac{\lambda h}{2}k_{0n}+\frac{\lambda h^2}{12}J_{0n} & -\frac{\lambda h^2}{12}k_{00} & \frac{\lambda h^2}{12}k_{0n} \\ -\frac{\lambda h}{2}k_{10}-\frac{\lambda h^2}{12}J_{10} & 1-\lambda hk_{11} & -\lambda hk_{12} & \cdots & -\frac{\lambda h}{2}k_{1n}+\frac{\lambda h^2}{12}J_{1n} & -\frac{\lambda h^2}{12}k_{10} & \frac{\lambda h^2}{12}k_{1n} \\ -\frac{\lambda h}{2}k_{20}-\frac{\lambda h^2}{12}J_{20} & -\lambda hk_{21} & 1-\lambda hk_{22} & \cdots & -\frac{\lambda h}{2}k_{2n}+\frac{\lambda h^2}{12}J_{2n} & -\frac{\lambda h^2}{12}k_{20} & \frac{\lambda h^2}{12}k_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\frac{\lambda h}{2}k_{n0}-\frac{\lambda h^2}{12}J_{n0} & -\lambda hk_{n1} & -\lambda hk_{n2} & \cdots & 1-\frac{\lambda h}{2}k_{nn}+\frac{\lambda h^2}{12}J_{nn} & -\frac{\lambda h^2}{12}k_{n0} & \frac{\lambda h^2}{12}k_{nn} \\ -\frac{\lambda h}{2}H_{00}-\frac{\lambda h^2}{12}L_{00} & -\lambda hH_{01} & -\lambda hH_{02} & \cdots & -\frac{\lambda h}{2}H_{0n}+\frac{\lambda h^2}{12}L_{0n} & 1-\frac{\lambda h^2}{12}H_{00} & \frac{\lambda h^2}{12}H_{00} \\ -\frac{\lambda h}{2}H_{n0}-\frac{\lambda h^2}{12}L_{n0} & -\lambda hH_{n1} & -\lambda hH_{n2} & \cdots & -\frac{\lambda h}{2}H_{nn}+\frac{\lambda h^2}{12}L_{nn} & -\frac{\lambda h^2}{12}H_{n0} & 1-\frac{\lambda h^2}{12}H_{n0} \end{bmatrix}$$

$$U=\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_n \\ u'_0 \\ u'_n \end{bmatrix} \text{ and } F=\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \\ f'_0 \\ f'_n \end{bmatrix}$$

This system can be solved by using any suitable method to find the n+3 unknowns u_i, $i=0,1,...,n$, u'_0 and u'_n.

Case (2):

If $\frac{\partial^2 k(x,y)}{\partial x \partial y}$ dose not exists, in this case, we approximate the integral that appeared in the right hand side of the integral equation (2.11) with the repeated trapezoid rule to obtain

$$u'(x) = f'(x) + \frac{\lambda h}{2} H(x, x_0) u_0 + \lambda h \sum_{j=1}^{n-1} H(x, x_j) u_j + \frac{\lambda h}{2} H(x, x_n) u_n$$

Hence, for $x = x_p, p = 0, n$, one can get the following equations:

$$u'_0 = f'_0 + \frac{\lambda h}{2} H_{00} u_0 + \lambda h \sum_{j=1}^{n-1} H_{0j} u_j + \frac{\lambda h}{2} H_{0n} u_n$$

$$u'_n = f'_n + \frac{\lambda h}{2} H_{n0} u_0 + \lambda h \sum_{j=1}^{n-1} H_{nj} u_j + \frac{\lambda h}{2} H_{nn} u_n .$$

The system which consists of the above two equations together with the n+1 equations given by equation (2.10) can be written as AU=F, where

$$A = \begin{bmatrix} 1 - \frac{\lambda h}{2} k_{00} - \frac{\lambda h^2}{12} J_{00} & -\lambda h k_{01} & -\lambda h k_{02} & \cdots & -\frac{\lambda h}{2} k_{0n} + \frac{\lambda h^2}{12} J_{0n} & -\frac{\lambda h^2}{12} k_{00} & \frac{\lambda h^2}{12} k_{0n} \\ -\frac{\lambda h}{2} k_{10} - \frac{\lambda h^2}{12} J_{10} & 1 - \lambda h k_{11} & -\lambda h k_{12} & \cdots & -\frac{\lambda h}{2} k_{1n} + \frac{\lambda h^2}{12} J_{1n} & -\frac{\lambda h^2}{12} k_{10} & \frac{\lambda h^2}{12} k_{1n} \\ -\frac{\lambda h}{2} k_{20} - \frac{\lambda h^2}{12} J_{20} & -\lambda h k_{21} & 1 - \lambda h k_{22} & \cdots & -\frac{\lambda h}{2} k_{2n} + \frac{\lambda h^2}{12} J_{2n} & -\frac{\lambda h^2}{12} k_{20} & \frac{\lambda h^2}{12} k_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\frac{\lambda h}{2} k_{n0} - \frac{\lambda h^2}{12} J_{n0} & -\lambda h k_{n1} & -\lambda h k_{n2} & \cdots & 1 - \frac{\lambda h}{2} k_{nn} + \frac{\lambda h^2}{12} J_{nn} & -\frac{\lambda h^2}{12} k_{n0} & \frac{\lambda h^2}{12} k_{nn} \\ -\frac{\lambda h}{2} H_{00} & -\lambda h H_{01} & -\lambda h H_{02} & \cdots & -\frac{\lambda h}{2} H_{0n} & 1 & 0 \\ -\frac{\lambda h}{2} H_{n0} & -\lambda h H_{n1} & -\lambda h H_{n2} & \cdots & -\frac{\lambda h}{2} H_{nn} & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_n \\ u'_0 \\ u'_n \end{bmatrix} \text{ and } F = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \\ f'_0 \\ f'_n \end{bmatrix}$$

This system can be solved by using any suitable method to find the n+3 unknowns u_i, $i=0,1,\ldots,n$, u'_0 and u'_n.

To illustrate the previous method, consider the following example:

Example (2.7):

Consider example (2.1), we solve this example numerically with the repeated modified Trapezoid rule. To do this, first, we subdivide the intervals $[-1,1]$ into 8 subintervals such that $x_i=-1+\frac{i}{4}, i=0,1,\ldots,8$. Then the above integral equation becomes:

$$u_i=1+\frac{1}{\pi}\left[\left(\frac{1}{8}\frac{1}{1+(x_i+1)^2}+\frac{1}{192}\frac{2(x_i+1)}{[1+(x_i+1)^2]^2}\right)u_0+\frac{1}{4}\sum_{j=1}^{7}\frac{1}{1+(x_i-x_j)^2}u_j+\right.$$
$$\left(\frac{1}{8}\frac{1}{1+(x_i-1)^2}-\frac{1}{192}\frac{2(x_i-1)}{[1+(x_i-1)^2]^2}\right)u_8+\frac{1}{192}\frac{1}{1+(x_i+1)^2}u'_0-$$
$$\left.\frac{1}{192}\frac{1}{1+(x_i-1)^2}u'_8\right],\quad i=0,1,\ldots,8$$

$$u'_0=\frac{1}{\pi}\left[\frac{1}{96}u_0-\frac{1}{2}\sum_{j=1}^{7}\frac{(-1-x_j)}{[1+(-1-x_j)^2]^2}u_j+\frac{251}{12000}u_8-\frac{1}{1200}u'_8\right]$$

$$u'_8=\frac{1}{\pi}\left[\frac{-251}{12000}u_0-\frac{1}{2}\sum_{j=1}^{7}\frac{(1-x_j)}{[1+(1-x_j)^2]^2}u_j-\frac{1}{96}u_8-\frac{1}{1200}u'_0\right]$$

By evaluating the above equation at each i=0,1,…,8 one can get a linear system of eleven equations with eleven unknowns $\{u_i\}_{i=0}^{8}$, u'_0 and u'_8, which has the solution:

$u_0=1.63639$, $u_1=1.74695$, $u_2=1.83641$, $u_3=1.89332$, $u_4=1.91268$, $u_5=1.89332$,

$u_6 = 1.83641, u_7 = 1.74695$ and $u_8 = 1.63639.$

Second, if we subdivide the interval $[-1,1]$ into 16 subintervals such that $x_i = -1 + \frac{i}{8}, i = 0,1,...,16,$ then the above integral equation becomes:

$$u_i = 1 + \frac{1}{\pi}\left[\left(\frac{1}{16}\frac{1}{1+(x_i+1)^2} + \frac{1}{768}\frac{2(x_i+1)}{[1+(x_i+1)^2]^2}\right)u_0 + \frac{1}{8}\sum_{j=1}^{15}\frac{1}{1+(x_i-x_j)^2}u_j + \left(\frac{1}{16}\frac{1}{1+(x_i-1)^2} - \frac{1}{768}\frac{2(x_i-1)}{[1+(x_i-1)^2]^2}\right)u_{16} + \frac{1}{768}\frac{1}{1+(x_i+1)^2}u'_0 - \frac{1}{768}\frac{1}{1+(x_i-1)^2}u'_{16}\right], \quad i = 0,1,...,16$$

$$u'_0 = \frac{1}{\pi}\left[\frac{1}{384}u_0 - \frac{1}{4}\sum_{j=1}^{15}\frac{(-1-x_j)}{[1+(-1-x_j)^2]^2}u_j + \frac{491}{48000}u_{16} - \frac{1}{4800}u'_{16}\right]$$

$$u'_{16} = \frac{1}{\pi}\left[\frac{-491}{48000}u_0 - \frac{1}{4}\sum_{j=1}^{15}\frac{(1-x_j)}{[1+(1-x_j)^2]^2}u_j - \frac{1}{384}u_{16} - \frac{1}{4800}u'_0\right]$$

By evaluating the above equation at each i=0,1,…,16 one can get a linear system of 19 equations with 19 unknowns $\{u_i\}_{i=0}^{16}$, u'_0 and u'_{16} that can be found in the appendix (see program (2.7)). This system has the solution:

$u_0 = 1.63887, u_1 = 1.69648, u_2 = 1.7507, u_3 = 1.7994, u_4 = 1.84089, u_5 = 1.87401,$
$u_6 = 1.89804, u_7 = 1.912528, u_8 = 1.91744, u_9 = 1.91258, u_{10} = 1.89804, u_{11} = 1.87401,$
$u_{12} = 1.84089, u_{13} = 1.7994, u_{14} = 1.7507, u_{15} = 1.69648$ and $u_{16} = 1.63887.$

Third, we subdivide the interval $[-1,1]$ into 32 and 64 subintervals such that $x_i = -1 + \frac{i}{16}, i = 0,1,...,32$ and $x_i = -1 + \frac{i}{32}, i = 0,1,...,64$ respectively and by following the same previous steps one can get the results that can be tabulated in the appendix (see program (2.7)). Some of these results are tabulated down.

Table (2.9) represents the numerical solution of example (2.7) for different values of n

Nodes	n=16	n=32	n=64
$x = \pm 1$	1.63887	1.63949	1.63964
$x = \pm 0.75$	1.7570	1.75164	1.75187
$x = \pm 0.5$	1.84089	1.84201	1.84229
$x = \pm 0.25$	1.89804	1.89922	1.89952
$x = 0$	1.91744	1.91863	1.91839

2.3.2 *The Modified Trapezoid Rule for Solving The One-dimensional Volterra Linear Integral Equations of The Second Kind:*

In this section, we use the modified trapezoidal rule to solve the one-dimensional Volterra linear integral equations of the second kind.

Consider the one-dimensional Volterra linear integral equation of the second kind given by equation (2.2). First, we divide the interval [a,b] into n subinterval $[x_i, x_{i+1}], i=0,1,\dots,n-1$ such that $x_i = a + ih, i = 0,1,\dots,n$ where $h = \frac{b-a}{n}$. So, the problem here is to find the solution of equation (2.2) at $x_i, i = 0,1,\dots,n$. Let u_i denote the numerical solution of equation (2.2) at each $x_i, i = 0,1,\dots,n$ .Then, we approximate the integral that appeared in the right hand side of equation (2.2) at $x = x_i, i = 0,1,\dots,n$, to get

$$u_i = f_i + \lambda \int_a^{x_i} k(x_i, y)u(y)dy, \quad i = 0,1,\dots,n \tag{2.12}$$

where $f_i = f(x_i), i = 0,1,\dots,n$.

Then, we approximate the integral that appeared in the right hand side of equation (2.12) with the repeated modified trapezoid rule one can get

$$u_0 = f_0$$

$$u_i = f_i + \frac{\lambda h}{2} k(x_i, x_0) u_0 + \lambda h \sum_{j=1}^{i-1} k(x_i, x_j) u_j + \frac{\lambda h}{2} k(x_i, x_i) u_i +$$
$$\frac{\lambda h^2}{12} \left[\frac{\partial}{\partial y} \big(k(x_i, y) u(y)\big)\Big|_{y=x_0} - \frac{\partial}{\partial y} \big(k(x_i, y) u(y)\big)\Big|_{y=x_i} \right], \; i = 1, 2, ..., n.$$

By transforming all the terms involving the solution u_i to the left hand side of the resulting equation leaving only $f(x_i)$ on the right hand side, one can have

$$u_0 = f_0$$

and

$$u_i - \frac{\lambda h}{2} k_{i0} u_0 - \lambda h \sum_{j=1}^{i-1} k_{ij} u_j - \frac{\lambda h}{2} k_{ii} u_i -$$
$$\frac{\lambda h^2}{12} \left[k_{i0} u_0' + J_{i0} u_0 - k_{ii} u_i' - J_{ii} u_i \right] = f_i, \; i = 1, 2, ..., n. \quad (2.13)$$

where $f_i = f(x_i)$, $J_{ip} = J(x_i, x_p) = \left.\frac{\partial k(x_i, y)}{\partial y}\right|_{y=y_p}$, $k(x_i, x_j) = k_{ij}$, $i, j = 1, 2, ..., n$, $p = 0, i.$

Next, we have to find u_0' and $u_i', i = 1, 2, ..., n.$ To do this, we must differentiate equation (2.2) with respect to x to get

$$u'(x) = f'(x) + \lambda \int_a^x H(x, y) u(y)\, dy + \lambda k(x, x) u(x), \quad x \geq a \quad (2.14)$$

where $H(x, y) = \frac{\partial k(x, y)}{\partial x}.$

It is easy to check any solution of equation (2.2) is a solution of equation (2.14). By evaluating equation (2.12) at $x = x_i$, $i = 0, 1, ..., n$, one can get

$$u'(x_i) = f'(x_i) + \lambda \int_a^b H(x_i, y) u(y) dy + \lambda k(x_i, x_i) u(x_i), \; i = 0, 1, ..., n \quad (2.15)$$

Next, to solve equation (2.15), one must consider two cases:-

Case (1):

If $\frac{\partial^2 k(x,y)}{\partial x \partial y}$ exists, in this case, we approximate the integral that appeared in the right hand side of equation (2.15) with the repeated modified trapezoid rule to obtain

$$u_i' = f_i' + \frac{\lambda h}{2} H(x_i, x_0) u_0 + \lambda h \sum_{j=1}^{i-1} H(x_i, x_j) u_j + \frac{\lambda h}{2} H(x_i, x_i) u_i + \frac{\lambda h^2}{12}\left[\frac{\partial}{\partial y}\big(H(x_i, y)u(y)\big)\Big|_{y=x_0} - \frac{\partial}{\partial y}\big(H(x_i, y)u(y)\big)\Big|_{y=x_i} \right] + \lambda k(x_i, x_i) u_i, \quad i = 1, 2, \ldots, n. \tag{2.16}$$

where $f_i' = f'(x_i)$, $i = 0, 1, \ldots, n$.

Therefore

$$u_0' = f_0' + \lambda k_{00} f_0$$

and

$$u_i' = f_i' + \frac{\lambda h}{2} H_{i0} u_0 + \lambda h \sum_{j=1}^{i-1} H_{ij} u_j + \frac{\lambda h}{2} H_{ii} u_i + \frac{\lambda h^2}{12}\left[H_{i0} u_0' + L_{i0} u_0 - H_{ii} u_i' - L_{ii} u_i \right] + \lambda k_{ii} u_i, \quad i = 1, 2, \ldots, n \tag{2.17}$$

where $u_i' = u'(x_i)$, $H_{ij} = H(x_i, y_j)$, $i, j = 0, 1, \ldots, n$, $L_{ip} = \frac{\partial H(x_i, y)}{\partial y}\Big|_{y=y_p}$, $p = 0, i$.

Hence

$$u_0' = f_0' + \lambda k_{00} f_0$$

and

$$u_i' = \frac{12}{12+\lambda h^2 H_{ii}}\left\{f_i' + \left[\frac{\lambda h}{2}H_{i0}\,u_0 + \frac{\lambda^2 h^2}{12}H_{i0}k_{00} + \frac{\lambda h^2}{12}L_{i0}\right]f_0 + \right.$$
$$\left. \lambda h\sum_{j=1}^{i-1}H_{ij}u_j + \frac{\lambda h^2}{12}H_{i0}f_0' + \left[\frac{\lambda h}{2}H_{ii} - \frac{\lambda h^2}{12}L_{ii} + \lambda k_{ii}\right]u_i\right\},\ i=1,2,\ldots,n$$

By substituting the values of u_0' and $u_i', i=1,2,\ldots,n$ into equation (2.13), one can have

$$u_i = f_i + \left\{\frac{\lambda h}{2}k_{i0} + \frac{\lambda^2 h^2}{12}k_{i0}k_{00} + \frac{\lambda h^2}{12}J_{i0} - \frac{\lambda h^2 k_{ii}}{12+\lambda h^2 H_{ii}}\left[\frac{\lambda h}{2}H_{i0} + \frac{\lambda^2 h^2}{12}H_{i0}k_{00} + \frac{\lambda h^2}{12}L_{i0}\right]\right\}f_0 +$$
$$\left\{\frac{\lambda h^2}{12}k_{i0} - \frac{\lambda h^2 k_{ii}}{12+\lambda h^2 H_{ii}}\,\frac{\lambda h^2}{12}H_{i0}\right\}f_0' + \lambda h\sum_{j=1}^{i-1}\left\{k_{ij} - \frac{\lambda h^2 k_{ii}}{12+\lambda h^2 H_{ii}}H_{ij}\right\}u_j - \frac{\lambda h^2 k_{ii}}{12+\lambda h^2 H_{ii}}f_i' +$$
$$\left\{\frac{\lambda h}{2}k_{ii} - \frac{\lambda h^2}{12}J_{ii} - \frac{\lambda h^2 k_{ii}}{12+\lambda h^2 H_{ii}}\left[\frac{\lambda h}{2}H_{ii} - \frac{\lambda h^2}{12}L_{ii} + \lambda k_{ii}\right]\right\}u_i,\ i=1,2,\ldots,n \qquad (2.18)$$

By evaluating equation (2.18) at each $i=0,1,\ldots,n$ and using the fact that $u_0 = f_0$, one can get the numerical solutions of equation (2.2) at the mesh points x_i, $i=1,2,\ldots,n.$

Case (2):

If $\frac{\partial^2 k(x,y)}{\partial x \partial y}$ does not exists, in this case, we approximate the integral that appeared in the right hand side of equation (2.13) with the repeated trapezoid rule to obtain

$$u_0' = f_0' + \lambda k_{00}\,f_0$$

and

$$u_i' = f_i' + \frac{\lambda h}{2}H_{i0}\,u_0 + \lambda h\sum_{j=1}^{i-1}H_{ij}u_i + \frac{\lambda h}{2}H_{ii}u_i + \lambda k_{ii}u_i,\quad i=0,1,\ldots,n$$

By substituting the values of u'_0 and u'_i, $i=1,2,\dots,n$ into equation (2.11), one can have

$$u_i = f_i + \left[\frac{\lambda h}{2}k_{i0} + \frac{\lambda^2 h^3}{12}k_{00}k_{i0} + \frac{\lambda h^2}{12}J_{i0} - \frac{\lambda^2 h^3}{24}k_{ii}H_{i0}\right]f_0 + \frac{\lambda h^2}{12}k_{i0}f'_0 + \lambda h\sum_{j=1}^{i-1}\left[k_{ij} - \frac{\lambda h^2}{12}H_{ij}k_{ii}\right]u_j - \frac{\lambda h^2}{12}k_{ii}f'_i + \left[\frac{\lambda h}{2}k_{ii} - \frac{\lambda^2 h^2}{12}k_{ii}^2 - \frac{\lambda^2 h^3}{24}k_{ii}H_{ii} - \frac{\lambda h^2}{12}J_{ii}\right]u_i,\ i=1,2,\dots,n \qquad (2.19)$$

By evaluating equation (2.19) at each $i=1,2,\dots,n$ with the fact that $u_0 = f_0$, the numerical solutions of equation (2.2) at the mesh points x_i, $i=0,1,\dots,n$ are obtained.

To illustrate the previous method, consider the following example:

Example (2.8):

Consider example (2.2). We solve this example numerically with the repeated modified trapezoid rule. To do this, we subdivide the interval [0,2] into 20 subintervals, such that $x_i = \frac{i}{20}$, $i=0,1,\dots,20$ and using the fact that $u_0 = f_0 = 0$. then the above integral equation becomes:

$$u_0 = f_0$$

$$u_i = x_i + \frac{1}{5}\left(\frac{1}{1200}x_i - \frac{1}{1200}\frac{(0.01)x_i^3}{12+(0.01)x_i} + \frac{1}{10}\sum_{j=1}^{i-1}\left(x_i x_j - \frac{(0.01)x_i^3}{12+(0.01)x_i}\right)u_j - \frac{(0.01)x_i^2}{12+(0.01)x_i} + \left(\frac{1}{20}x_i^2 - \frac{1}{200}x_i - \frac{(0.01)x_i^2}{12+(0.01)x_i}\left[\frac{1}{20}x_i - \frac{1}{1200} + x_i^2\right]\right)u_i\right)$$

where $i=1,2,\dots,n.$

By evaluating the above equation at each i=1,2,…,20 one can get the following values (see program (2.8)):

$u_0 = 0$, $u_1 = 0.10001$, $u_2 = 0.20011$, $u_3 = 0.30054$, $u_4 = 0.40171$, $u_5 = 0.50418$
$u_6 = 0.60870$, $u_7 = 0.71619$, $u_8 = 0.82778$, $u_9 = 0.94482$, $u_{10} = 1.06894$,
$u_{11} = 1.20207$, $u_{12} = 1.34652$, $u_{13} = 1.50506$, $u_{14} = 1.68103$, $u_{15} = 87849$,
$u_{16} = 2.10238$, $u_{17} = 2.35882$, $u_{18} = 2.65538$, $u_{19} = 3.00152$ and $u_{20} = 3.40921$.

Conclusions and Recommendations

From the present study, we can conclude the following:

1. The classification of the standard one-dimensional integral equations can be extended to the n-generalized multi-dimensional integral equations.
2. The mathematical modelings that consist of the generalized multi-dimensional integral equations are more reasonable than the mathematical modelings that consist of the standard multi-dimensional integral equations.
3. Some of the existence and uniqueness theorems for the standard one-dimensional integral equations can be extended for the n-generalized ones.
4. The modified trapezoidal rule for solving the one-dimensional and multi-dimensional integral equations gave more accurate result than the trapezoidal rule and Simpson's rule.
5. The modified trapezoidal rule can be also used to solve the Volterra integral equations and integro-differential equations.

For future work the following problems could be recommended:

1. Devote the modified Simpson's rule to solve the integral equations.
2. Solve the non-linear integral equations via the modified trapezoidal rule.
3. Discuss the eigenvalue problems related to the generalized multi-dimensional integral equations.
4. Study the singular multi-dimensional integral equations.
5. Describe the multi-dimensional delay integral equations.
6. Use the variational method to solve the non-linear generalized multi-dimensional integral equations.

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Introduction

It is known that the integral equation is an equation in which the unknown function appears under an integral sign, [15].

If the integral equation contains only one one-dimensional integral operator and the unknown function depends only on one independent variable, then this integral equation is said to be the one-dimensional integral equation [16].

If the integral equation contains only one multi-dimensional integral operator and the unknown function depends on more that one independent variable, then this integral equation is said to be the multi-dimensional integral equation, [16].

Many authors and researchers studied the one-dimensional integral equation say Hochstast in 1973, [22] discussed the existence of the unique solution for the one-dimensional non-linear integral equations, Delves and Walsh in 1974, [16] gave the numerical solution for the one-dimensional integral equations, Jerri in 1985, [25] gave some approximated methods for solving the one-dimensional integral equations with some real life applications, Al-Shather A. in 1999, [5] studied the one-dimensional singular integral equations, AL-Jawary in 2005, [3] used some numerical methods for solving system of the one-dimensional linear Volterra integral equations and Ibrahim in 2006, [23] used the numerical methods for solving system of one-dimensional linear Fredholm integral equations.

Also, many authors and researchers concerned with the multi-dimensional integral equations, say Ladopoulos in 1988, [27] gave some numerical solutions of the multi-dimensional singular integral equations, UBA P. in 1994, [35] studied the collocation method with cubic splines for multi-dimensional weakly singular non-linear integral equations, Su and Sakar in 1999, [33] used the moment method for solving the two-dimensional Fredholm integral equations of the first kind, Han and Wang in 2001, [20] gave some approximated solutions of the two-dimensional Fredholm integral equations via Galerkin's method, Kulkarni R. in 2004, [26] gave approximate solutions for the multivariable integral equations of the second kind, Xu and Zhou in 2004, [34] used the collocation method for solving the multi-dimensional integral equations, Gonzalez in 2005, [19] used the regularization method for solving the two and three-dimensional Fredholm integral equations of the first kind, Hasson in 2005, [21] gave some approximated solution for two-dimensional integral equations, Cardone and et al. in 2006, [11] used some iterative methods for solving the two-dimensional Volterra-Fredholm integral equations and Al-Niamey in 2006, [4] studied the multi-dimensional integral equations. Moreover Omran H. in 2007, [30] presented solutions of the generalized multi-dimensional Volterra integral and integro-differential equations and Majeed S. in 2007, [28] gave the solutions for the generalized multi-dimensional linear Fredholm integral equations.

The main purpose of this work is to study and classify the n-generalized integral equations that contain n integral operators. This study include the existence of a unique solution for special types of these integral equations and solving them via some methods of the quadrature methods, namely the trapezoidal rule, the modified trapezoidal rule and the Simpson's rule.

This thesis consists of three chapters

In chapter one, we give simple definitions for the one-dimensional integral equations and extend them into the generalized one-dimensional integral equations. Also, some of the existence and uniqueness theorems for the standard one-dimensional integral equations extended for the generalized ones.

In chapter two, we use special types of the numerical methods namely, the trapezoidal rule and Simpson's rule to solve special types of the integral equations namely, the one-dimensional Fredholm and Volterra linear integral equations of the second kind and shows validity to the generalized ones.

Moreover we use some of the numerical methods namely, the modified trapezoidal rule to solve some of the special types of the integral equations.

In chapter three, we classify the multi-dimensional integral equations and describe the generalized multi-dimensional integral equations with their numerical solutions via the quadrature methods. Also, we modify some quadrature methods, specially the trapezoid rule to solve special types of the multi-dimensional integral equations, namely the multi-dimensional Fredholm linear integral equation of the second kind.

For each method, some numerical examples are solved and computer programs are written in MathCAD (professional 2001i) software package, and the results are presented in tabular forms.

$$
A=\begin{pmatrix}
1-\frac{\lambda h}{2}k_{0,0} & -\lambda hk_{0,1} & -\lambda hk_{0,2} & \cdots & -\lambda hk_{0,p-1} & -\lambda hk_{0,p}-\frac{\mu h}{2}\ell_{0,p} & -\lambda hk_{0,p+1}-\mu h\ell_{0,p+1} & \cdots & -\lambda hk_{0,q-1}-\mu h\ell_{0,q-1} & -\lambda hk_{0,q}-\frac{\mu h}{2}\ell_{0,q} & -\lambda hk_{0,q+1} & \cdots & -\lambda hk_{0,n-1} & -\frac{\lambda h}{2}k_{0,n} \\
-\frac{\lambda h}{2}k_{1,0} & 1-\lambda hk_{1,1} & -\lambda hk_{1,2} & \cdots & -\lambda hk_{1,p-1} & -\lambda hk_{1,p}-\frac{\mu h}{2}\ell_{1,p} & -\lambda hk_{1,p+1}-\mu h\ell_{1,p+1} & \cdots & -\lambda hk_{1,q-1}-\mu h\ell_{1,q-1} & -\lambda hk_{1,q}-\frac{\mu h}{2}\ell_{1,q} & -\lambda hk_{1,q+1} & \cdots & -\lambda hk_{1,n-1} & -\frac{\lambda h}{2}k_{1,n} \\
-\frac{\lambda h}{2}k_{2,0} & -\lambda hk_{2,1} & 1-\lambda hk_{2,2} & \cdots & -\lambda hk_{2,p-1} & -\lambda hk_{2,p}-\frac{\mu h}{2}\ell_{2,p} & -\lambda hk_{2,p+1}-\mu h\ell_{2,p+1} & \cdots & -\lambda hk_{2,q-1}-\mu h\ell_{2,q-1} & -\lambda hk_{2,q}-\frac{\mu h}{2}\ell_{2,q} & -\lambda hk_{2,q+1} & \cdots & -\lambda hk_{2,n-1} & -\frac{\lambda h}{2}k_{2,n} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
-\frac{\lambda h}{2}k_{p-1,0} & -\lambda hk_{p-1,1} & -\lambda hk_{p-1,2} & \cdots & 1-\lambda hk_{p-1,p-1} & -\lambda hk_{p-1,p}-\frac{\mu h}{2}\ell_{p-1,p} & -\lambda hk_{p-1,p+1}-\mu h\ell_{p-1,p+1} & \cdots & -\lambda hk_{p-1,q-1}-\mu h\ell_{p-1,q-1} & -\lambda hk_{p-1,q}-\frac{\mu h}{2}\ell_{p-1,q} & -\lambda hk_{p-1,q+1} & \cdots & -\lambda hk_{p-1,n-1} & -\frac{\lambda h}{2}k_{p-1,n} \\
-\frac{\lambda h}{2}k_{p,0} & -\lambda hk_{p,1} & -\lambda hk_{p,2} & \cdots & -\lambda hk_{p,p-1} & 1-\lambda hk_{p,p}-\frac{\mu h}{2}\ell_{p,p} & -\lambda hk_{p,p+1}-\mu h\ell_{p,p+1} & \cdots & -\lambda hk_{p,q-1}-\mu h\ell_{p,q-1} & -\lambda hk_{p,q}-\frac{\mu h}{2}\ell_{p,q} & -\lambda hk_{p,q+1} & \cdots & -\lambda hk_{p,n-1} & -\frac{\lambda h}{2}k_{p,n} \\
-\frac{\lambda h}{2}k_{p+1,0} & -\lambda hk_{p+1,1} & -\lambda hk_{p+1,2} & \cdots & -\lambda hk_{p+1,p-1} & -\lambda hk_{p+1,p}-\frac{\mu h}{2}\ell_{p+1,p} & 1-\lambda hk_{p+1,p+1}-\mu h\ell_{p+1,p+1} & \cdots & -\lambda hk_{p+1,q-1}-\mu h\ell_{p+1,q-1} & -\lambda hk_{p+1,q}-\frac{\mu h}{2}\ell_{p+1,q} & -\lambda hk_{p+1,q+1} & \cdots & -\lambda hk_{p+1,n-1} & -\frac{\lambda h}{2}k_{p+1,n} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
-\frac{\lambda h}{2}k_{q-1,0} & -\lambda hk_{q-1,1} & -\lambda hk_{q-1,2} & \cdots & -\lambda hk_{q-1,p-1} & -\lambda hk_{q-1,p}-\frac{\mu h}{2}\ell_{q-1,p} & -\lambda hk_{q-1,p+1}-\mu h\ell_{q-1,p+1} & \cdots & 1-\lambda hk_{q-1,q-1}-\mu h\ell_{q-1,q-1} & -\lambda hk_{q-1,q}-\frac{\mu h}{2}\ell_{q-1,q} & -\lambda hk_{q-1,q+1} & \cdots & -\lambda hk_{q-1,n-1} & -\frac{\lambda h}{2}k_{q-1,n} \\
-\frac{\lambda h}{2}k_{q,0} & -\lambda hk_{q,1} & -\lambda hk_{q,2} & \cdots & -\lambda hk_{q,p-1} & -\lambda hk_{q,p}-\frac{\mu h}{2}\ell_{q,p} & -\lambda hk_{q,p+1}-\mu h\ell_{q,p+1} & \cdots & -\lambda hk_{q,q-1}-\mu h\ell_{q,q-1} & 1-\lambda hk_{q,q}-\frac{\mu h}{2}\ell_{q,q} & -\lambda hk_{q,q+1} & \cdots & -\lambda hk_{q,n-1} & -\frac{\lambda h}{2}k_{q,n} \\
-\frac{\lambda h}{2}k_{q+1,0} & -\lambda hk_{q+1,1} & -\lambda hk_{q+1,2} & \cdots & -\lambda hk_{q+1,p-1} & -\lambda hk_{q+1,p}-\frac{\mu h}{2}\ell_{q+1,p} & -\lambda hk_{q+1,p+1}-\mu h\ell_{q+1,p+1} & \cdots & -\lambda hk_{q+1,q-1}-\mu h\ell_{q+1,q-1} & -\lambda hk_{q+1,q}-\frac{\mu h}{2}\ell_{q+1,q} & 1-\lambda hk_{q+1,q+1} & \cdots & -\lambda hk_{q+1,n-1} & -\frac{\lambda h}{2}k_{q+1,n} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
-\frac{\lambda h}{2}k_{n-1,0} & -\lambda hk_{n-1,1} & -\lambda hk_{n-1,2} & \cdots & -\lambda hk_{n-1,p-1} & -\lambda hk_{n-1,p}-\frac{\mu h}{2}\ell_{n-1,p} & -\lambda hk_{n-1,p+1}-\mu h\ell_{n-1,p+1} & \cdots & -\lambda hk_{n-1,q-1}-\mu h\ell_{n-1,q-1} & -\lambda hk_{n-1,q}-\frac{\mu h}{2}\ell_{n-1,q} & -\lambda hk_{n-1,q+1} & \cdots & 1-\lambda hk_{n-1,n-1} & -\frac{\lambda h}{2}k_{n-1,n} \\
-\frac{\lambda h}{2}k_{n,0} & -\lambda hk_{n,1} & -\lambda hk_{n,2} & \cdots & -\lambda hk_{n,p-1} & -\lambda hk_{n,p}-\frac{\mu h}{2}\ell_{n,p} & -\lambda hk_{n,p+1}-\mu h\ell_{n,p+1} & \cdots & -\lambda hk_{n,q-1}-\mu h\ell_{n,q-1} & -\lambda hk_{n,q}-\frac{\mu h}{2}\ell_{n,q} & -\lambda hk_{n,q+1} & \cdots & -\lambda hk_{n,n-1} & 1-\frac{\lambda h}{2}k_{n,n}
\end{pmatrix}
$$

Table (2.6) represents the numerical and the exact solutions of example (2.5) for different values of n

Nodes	Numerical Solution						Exact Solution
	n=10	n=20	n=30	n=40	n=50	n=100	
x=0	-9.10527×10^{-3}	-2.29747×10^{-3}	-1.02286×10^{-3}	-5.75706×10^{-4}	-3.68555×10^{-4}	-9.21732×10^{-5}	0
x=0.1	-8.52746×10^{-4}	7.26094×10^{-3}	8.78049×10^{-3}	9.31360×10^{-3}	9.56058×10^{-3}	9.89010×10^{-3}	0.01
x=0.2	0.027336	0.03680	0.03858	0.03920	0.03949	0.03987	0.04
x=0.3	0.07545	0.08633	0.08836	0.08908	0.08941	0.08985	0.09
x=0.4	0.1435	0.15583	0.15814	0.15896	0.15933	0.15983	0.16
x=0.5	0.23148	0.24532	0.24792	0.24883	0.24925	0.24981	0.25
x=0.6	0.3394	0.35480	0.35768	0.35870	0.35917	0.35979	0.36
x=0.7	0.46725	0.48425	0.48744	0.48856	0.48908	0.48977	0.49
x=0.8	0.61503	0.63369	0.63719	0.63842	0.63899	0.63975	0.64
x=0.9	0.78274	0.80311	0.80693	0.80827	0.80890	0.80972	0.81
x=1	0.97038	0.99252	0.99667	0.99813	0.99880	0.99970	1

$$A=\begin{pmatrix}
1-\frac{\lambda h}{3}k_{0,0} & -\frac{4\lambda h}{3}k_{0,1} & -\frac{2\lambda h}{3}k_{0,2} & \cdots & -\frac{2\lambda h}{3}k_{0,p-1} & -\frac{4\lambda h}{3}k_{0,p}-\frac{\mu h}{2}\ell_{0,p} & -\frac{2\lambda h}{3}k_{0,p+1}-\mu h\ell_{0,p+1} & -\frac{4\lambda h}{3}k_{0,q-1}-\mu h\ell_{0,q-1} & -\frac{2\lambda h}{3}k_{0,q}-\frac{\mu h}{2}\ell_{0,q} & -\frac{4\lambda h}{3}k_{0,q+1} & \cdots & -\frac{4\lambda h}{3}k_{0,n-1} & -\frac{\lambda h}{3}k_{0,n} \\
-\frac{\lambda h}{3}k_{1,0} & 1-\frac{4\lambda h}{3}k_{1,1} & -\frac{2\lambda h}{3}k_{1,2} & \cdots & -\frac{2\lambda h}{3}k_{1,p-1} & -\frac{4\lambda h}{3}k_{1,p}-\frac{\mu h}{2}\ell_{1,p} & -\frac{2\lambda h}{3}k_{1,p+1}-\mu h\ell_{1,p+1} & -\frac{4\lambda h}{3}k_{1,q-1}-\mu h\ell_{1,q-1} & -\frac{2\lambda h}{3}k_{1,q}-\frac{\mu h}{2}\ell_{1,q} & -\frac{4\lambda h}{3}k_{1,q+1} & \cdots & -\frac{4\lambda h}{3}k_{1,n-1} & -\frac{\lambda h}{3}k_{1,n} \\
-\frac{\lambda h}{3}k_{2,0} & -\frac{4\lambda h}{3}k_{2,1} & 1-\frac{2\lambda h}{3}k_{2,2} & \cdots & -\frac{2\lambda h}{3}k_{2,p-1} & -\frac{4\lambda h}{3}k_{2,p}-\frac{\mu h}{2}\ell_{2,p} & -\frac{2\lambda h}{3}k_{2,p+1}-\mu h\ell_{2,p+1} & -\frac{4\lambda h}{3}k_{2,q-1}-\mu h\ell_{2,q-1} & -\frac{2\lambda h}{3}k_{2,q}-\frac{\mu h}{2}\ell_{2,q} & -\frac{4\lambda h}{3}k_{2,q+1} & \cdots & -\frac{4\lambda h}{3}k_{2,n-1} & -\frac{\lambda h}{3}k_{2,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-\frac{\lambda h}{3}k_{p-1,0} & -\frac{4\lambda h}{3}k_{p-1,1} & -\frac{2\lambda h}{3}k_{p-1,2} & \cdots & 1-\frac{2\lambda h}{3}k_{p-1,p-1} & -\frac{4\lambda h}{3}k_{p-1,p}-\frac{\mu h}{2}\ell_{p-1,p} & -\frac{2\lambda h}{3}k_{p-1,p+1}-\mu h\ell_{p-1,p+1} & -\frac{4\lambda h}{3}k_{p-1,q-1}-\mu h\ell_{p-1,q-1} & -\frac{2\lambda h}{3}k_{p-1,q}-\frac{\mu h}{2}\ell_{p-1,q} & -\frac{4\lambda h}{3}k_{p-1,q+1} & \cdots & -\frac{4\lambda h}{3}k_{p-1,n-1} & -\frac{\lambda h}{3}k_{p-1,n} \\
-\frac{\lambda h}{3}k_{p,0} & -\frac{4\lambda h}{3}k_{p,1} & -\frac{2\lambda h}{3}k_{p,2} & \cdots & -\frac{2\lambda h}{3}k_{p,p-1} & 1-\frac{4\lambda h}{3}k_{p,p}-\frac{\mu h}{2}\ell_{p,p} & -\frac{2\lambda h}{3}k_{p,p+1}-\mu h\ell_{p,p+1} & -\frac{4\lambda h}{3}k_{p,q-1}-\mu h\ell_{p,q-1} & -\frac{2\lambda h}{3}k_{p,q}-\frac{\mu h}{2}\ell_{p,q} & -\frac{4\lambda h}{3}k_{p,q+1} & \cdots & -\frac{4\lambda h}{3}k_{p,n-1} & -\frac{\lambda h}{3}k_{p,n} \\
-\frac{\lambda h}{3}k_{p+1,0} & -\frac{4\lambda h}{3}k_{p+1,1} & -\frac{2\lambda h}{3}k_{p+1,2} & \cdots & -\frac{2\lambda h}{3}k_{p+1,p-1} & -\frac{4\lambda h}{3}k_{p+1,p}-\frac{\mu h}{2}\ell_{p+1,p} & 1-\frac{2\lambda h}{3}k_{p+1,p+1}-\mu h\ell_{p+1,p+1} & -\frac{4\lambda h}{3}k_{p+1,q-1}-\mu h\ell_{p+1,q-1} & -\frac{2\lambda h}{3}k_{p+1,q}-\frac{\mu h}{2}\ell_{p+1,q} & -\frac{4\lambda h}{3}k_{p+1,q+1} & \cdots & -\frac{4\lambda h}{3}k_{p+1,n-1} & -\frac{\lambda h}{3}k_{p+1,n} \\
-\frac{\lambda h}{3}k_{q-1,0} & -\frac{4\lambda h}{3}k_{q-1,1} & -\frac{2\lambda h}{3}k_{q-1,2} & \cdots & -\frac{2\lambda h}{3}k_{q-1,p-1} & -\frac{4\lambda h}{3}k_{q-1,p}-\frac{\mu h}{2}\ell_{q-1,p} & -\frac{2\lambda h}{3}k_{q-1,p+1}-\mu h\ell_{q-1,p+1} & 1-\frac{4\lambda h}{3}k_{q-1,q-1}-\mu h\ell_{q-1,q-1} & -\frac{2\lambda h}{3}k_{q-1,q}-\frac{\mu h}{2}\ell_{q-1,q} & -\frac{4\lambda h}{3}k_{q-1,q+1} & \cdots & -\frac{4\lambda h}{3}k_{q-1,n-1} & -\frac{\lambda h}{3}k_{q-1,n} \\
-\frac{\lambda h}{3}k_{q,0} & -\frac{4\lambda h}{3}k_{q,1} & -\frac{2\lambda h}{3}k_{q,2} & \cdots & -\frac{2\lambda h}{3}k_{q,p-1} & -\frac{4\lambda h}{3}k_{q,p}-\frac{\mu h}{2}\ell_{q,p} & -\frac{2\lambda h}{3}k_{q,p+1}-\mu h\ell_{q,p+1} & -\frac{4\lambda h}{3}k_{q,q-1}-\mu h\ell_{q,q-1} & 1-\frac{2\lambda h}{3}k_{q,q}-\frac{\mu h}{2}\ell_{q,q} & -\frac{4\lambda h}{3}k_{q,q+1} & \cdots & -\frac{4\lambda h}{3}k_{q,n-1} & -\frac{\lambda h}{3}k_{q,n} \\
-\frac{\lambda h}{3}k_{q+1,0} & -\frac{4\lambda h}{3}k_{q+1,1} & -\frac{2\lambda h}{3}k_{q+1,2} & \cdots & -\frac{2\lambda h}{3}k_{q+1,p-1} & -\frac{4\lambda h}{3}k_{q+1,p}-\frac{\mu h}{2}\ell_{q+1,p} & -\frac{2\lambda h}{3}k_{q+1,p+1}-\mu h\ell_{q+1,p+1} & -\frac{4\lambda h}{3}k_{q+1,q-1}-\mu h\ell_{q+1,q-1} & -\frac{2\lambda h}{3}k_{q+1,q}-\frac{\mu h}{2}\ell_{q+1,q} & 1-\frac{4\lambda h}{3}k_{q+1,q+1} & \cdots & -\frac{4\lambda h}{3}k_{q+1,n-1} & -\frac{\lambda h}{3}k_{q+1,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-\frac{\lambda h}{3}k_{n-1,0} & -\frac{4\lambda h}{3}k_{n-1,1} & -\frac{2\lambda h}{3}k_{n-1,2} & \cdots & -\frac{2\lambda h}{3}k_{n-1,p-1} & -\frac{4\lambda h}{3}k_{n-1,p}-\frac{\mu h}{2}\ell_{n-1,p} & -\frac{2\lambda h}{3}k_{n-1,p+1}-\mu h\ell_{n-1,p+1} & -\frac{4\lambda h}{3}k_{n-1,q-1}-\mu h\ell_{n-1,q-1} & -\frac{2\lambda h}{3}k_{n-1,q}-\frac{\mu h}{2}\ell_{n-1,q} & -\frac{4\lambda h}{3}k_{n-1,q+1} & \cdots & 1-\frac{4\lambda h}{3}k_{n-1,n-1} & -\frac{\lambda h}{3}k_{n-1,n} \\
-\frac{\lambda h}{3}k_{n,0} & -\frac{4\lambda h}{3}k_{n,1} & -\frac{2\lambda h}{3}k_{n,2} & \cdots & -\frac{2\lambda h}{3}k_{n,p-1} & -\frac{4\lambda h}{3}k_{n,p}-\frac{\mu h}{2}\ell_{n,p} & -\frac{2\lambda h}{3}k_{n,p+1}-\mu h\ell_{n,p+1} & -\frac{4\lambda h}{3}k_{n,q-1}-\mu h\ell_{n,q-1} & -\frac{2\lambda h}{3}k_{n,q}-\frac{\mu h}{2}\ell_{n,q} & -\frac{4\lambda h}{3}k_{n,q+1} & \cdots & -\frac{4\lambda h}{3}k_{n,n-1} & 1-\frac{\lambda h}{3}k_{n,n}
\end{pmatrix}$$

Table (2.8) represents the numerical and the exact solutions of example (2.6) for different values of n

Nodes	Numerical Solution						Exact Solution
	n=10	n=20	n=30	n=40	n=50	n=100	
x=0	-9.505×10^{-3}	-2.29849×10^{-3}	-1.02272×10^{-3}	-5.75601×10^{-4}	-3.68455×10^{-4}	-9.21722×10^{-5}	0
x=0.1	-9.52×10^{-4}	7.26122×10^{-3}	8.78056×10^{-3}	9.31370×10^{-3}	9.56065×10^{-3}	9.89019×10^{-3}	0.01
x=0.2	0.03733	0.03701	0.03870	0.03935	0.03969	0.03999	0.04
x=0.3	0.08545	0.08424	0.08847	0.08918	0.08952	0.08988	0.09
x=0.4	0.1535	0.15692	0.15825	0.15904	0.15945	0.15991	0.16
x=0.5	0.24148	0.24576	0.24801	0.24895	0.24930	0.24989	0.25
x=0.6	0.3594	0.35505	0.35767	0.35887	0.35921	0.35987	0.36
x=0.7	0.47725	0.48450	0.48755	0.48865	0.48912	0.48990	0.49
x=0.8	0.62503	0.63375	0.63730	0.63847	0.63957	0.63982	0.64
x=0.9	0.79274	0.80401	0.80699	0.80839	0.80898	0.80977	0.81
x=1	0.98038	0.99260	0.99672	0.99823	0.99960	0.99980	1

$$
A=\begin{pmatrix}
1-\lambda z_{0,0}k_{0,0}^{0,0} & -\lambda z_{1,0}k_{0,0}^{1,0} & \cdots & -\lambda z_{n,0}k_{0,0}^{n,0} & -\lambda z_{0,1}k_{0,0}^{0,1} & -\lambda z_{1,1}k_{0,0}^{1,1} & \cdots & -\lambda z_{n,1}k_{0,0}^{n,1} & \cdots & -\lambda z_{0,m}k_{0,0}^{0,m} & -\lambda z_{1,m}k_{0,0}^{1,m} & \cdots & -\lambda z_{n,m}k_{0,0}^{n,m} \\
-\lambda z_{0,0}k_{1,0}^{0,0} & 1-\lambda z_{1,0}k_{1,0}^{1,0} & \cdots & -\lambda z_{n,0}k_{1,0}^{n,0} & -\lambda z_{0,1}k_{1,0}^{0,1} & -\lambda z_{1,1}k_{1,0}^{1,1} & \cdots & -\lambda z_{n,1}k_{1,0}^{n,1} & \cdots & -\lambda z_{0,m}k_{1,0}^{0,m} & -\lambda z_{1,m}k_{1,0}^{1,m} & \cdots & -\lambda z_{n,m}k_{1,0}^{n,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-\lambda z_{0,0}k_{n,0}^{0,0} & -\lambda z_{1,0}k_{n,0}^{1,0} & \cdots & 1-\lambda z_{n,0}k_{n,0}^{n,0} & -\lambda z_{0,1}k_{n,0}^{0,1} & -\lambda z_{1,1}k_{n,0}^{1,1} & \cdots & -\lambda z_{n,1}k_{n,0}^{n,1} & \cdots & -\lambda z_{0,m}k_{n,0}^{0,m} & -\lambda z_{1,m}k_{n,0}^{1,m} & \cdots & -\lambda z_{n,m}k_{n,0}^{n,m} \\
-\lambda z_{0,0}k_{0,1}^{0,0} & -\lambda z_{1,0}k_{0,1}^{1,0} & \cdots & -\lambda z_{n,0}k_{0,1}^{n,0} & 1-\lambda z_{0,1}k_{0,1}^{0,1} & -\lambda z_{1,1}k_{0,1}^{1,1} & \cdots & -\lambda z_{n,1}k_{0,1}^{n,1} & \cdots & -\lambda z_{0,m}k_{0,1}^{0,m} & -\lambda z_{1,m}k_{0,1}^{1,m} & \cdots & -\lambda z_{n,m}k_{0,1}^{n,m} \\
-\lambda z_{0,0}k_{1,1}^{0,0} & -\lambda z_{1,0}k_{1,1}^{1,0} & \cdots & -\lambda z_{n,0}k_{1,1}^{n,0} & -\lambda z_{0,1}k_{1,1}^{0,1} & 1-\lambda z_{1,1}k_{1,1}^{1,1} & \cdots & -\lambda z_{n,1}k_{1,1}^{n,1} & \cdots & -\lambda z_{0,m}k_{1,1}^{0,m} & -\lambda z_{1,m}k_{1,1}^{1,m} & \cdots & -\lambda z_{n,m}k_{1,1}^{n,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-\lambda z_{0,0}k_{n,1}^{0,0} & -\lambda z_{1,0}k_{n,1}^{1,0} & \cdots & -\lambda z_{n,0}k_{n,1}^{n,0} & -\lambda z_{0,1}k_{n,1}^{0,1} & -\lambda z_{1,1}k_{n,1}^{1,1} & \cdots & 1-\lambda z_{n,1}k_{n,1}^{n,1} & \cdots & -\lambda z_{0,m}k_{n,1}^{0,m} & -\lambda z_{1,m}k_{n,1}^{1,m} & \cdots & -\lambda z_{n,m}k_{n,1}^{n,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-\lambda z_{0,0}k_{0,m}^{0,0} & -\lambda z_{1,0}k_{0,m}^{1,0} & \cdots & -\lambda z_{n,0}k_{0,m}^{n,0} & -\lambda z_{0,1}k_{0,m}^{0,1} & -\lambda z_{1,1}k_{0,m}^{1,1} & \cdots & -\lambda z_{n,1}k_{0,m}^{n,1} & \cdots & 1-\lambda z_{0,m}k_{0,m}^{0,m} & -\lambda z_{1,m}k_{0,m}^{1,m} & \cdots & -\lambda z_{n,m}k_{0,m}^{n,m} \\
-z_{0,0}k_{1,m}^{0,0} & -\lambda z_{1,0}k_{1,m}^{1,0} & \cdots & -z_{n,0}k_{1,m}^{n,0} & -\lambda z_{0,1}k_{1,m}^{0,1} & -\lambda z_{1,1}k_{1,m}^{1,1} & \cdots & -\lambda z_{n,1}k_{1,m}^{n,1} & \cdots & -\lambda z_{0,m}k_{1,m}^{0,m} & 1-\lambda z_{1,m}k_{1,m}^{1,m} & \cdots & -\lambda z_{n,m}k_{1,m}^{n,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-\lambda z_{0,0}k_{n,m}^{0,0} & -\lambda z_{1,0}k_{n,m}^{1,0} & \cdots & -z_{n,0}k_{n,m}^{n,0} & -\lambda z_{0,1}k_{n,m}^{0,1} & -\lambda z_{1,1}k_{n,m}^{1,1} & \cdots & -\lambda z_{n,1}k_{n,m}^{n,1} & \cdots & -\lambda z_{0,m}k_{n,m}^{0,m} & -\lambda z_{1,m}k_{n,m}^{1,m} & \cdots & 1-\lambda z_{n,m}k_{n,m}^{n,m}
\end{pmatrix}
$$

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Republic of Iraq
Ministry of Higher Education
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Al-Nahrain University
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MODIFIED NUMERICAL METHODS FOR SOLVING THE MULTI-DIMENSIONAL INTEGRAL EQUATIONS

A Thesis

Submitted to the College of Science, AL-Nahrain University
as a Partial Fulfillment of the Requirements for the Degree of
Master of Science in Mathematics

By
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Supervised by
Ast. Prof. Dr. Ahlam Jameel Khalil

October 2007 — Ramadhan 1428

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم

الطرق العددية المطورة لحل المعادلات التكامليه
الموسعة المتعددة الابعاد

رسالة
مقدمة إلى كلية العلوم – جامعة النهرين
كجزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل

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بأشراف
د.احلام جميل خليل

تشرين الاول ٢٠٠٧

رمضان ١٤٢٨

الأهداء

الى من لم يبخلا علي بسنين عمرهم ويستنير دربي بدعائهم
الى أمي وأبي

الى من شجعني وغمرني بحنانه وهم اعز الناس الى قلبي
الى أختاي و أخي

الى من شد ازري وشاركني همومي
الى خالتي

الى كل الاصدقاء الذين وقفوا بجانبي

الى مشرفتي الفاضلة واساتذتي الاعزاء

اهدي لكم هذا الجهد عرفانا ووفاءا

يسر سهيل علي

بَســَم اللهِ الـــَرحَمنِ الـــَرحِيمِ

وَ عَلمَ ءَادَمَ الاسمَآءَ كُلهَا ثُمَ عَرضَهُم عَلى

المَلائِكَة ِفقَاَل أنبئُونِـي بِأسمَاءَ هَؤُلاءِ ان كُنتُم

صَادِقِينَ ✵ قَاَلُوا سبحانك لاعِلمَ لنَا الا مَا

عَلمتَنَا انكَ انتَ العَلِيمُ الحَكِيمُ ✵

صَدَقَ اللهُ العَظيمَ
سورة البقرة ٣٠-٣١

المستخلص

هذا العمل يتمحور باتجاه هدفين:

الهدف الاول هو تصنيف و دراسة المعادلات التكاملية الموسعة ذات البعد الواحد والتي تحتوي على n من المؤثرات التكاملية ذات البعد الواحد. هذه الدراسة تتضمن وجود و وحدانية الحل لانواع خاصة من هذه المعادلات التكاملية وحل هذه المعادلات باستخدام بعض الطرق من الطرق التربيعية والتي هي قاعدة شبه المنحرف وقاعدة شبه المنحرف مع التصحيح وقاعدة سمسون.

الهدف الثاني هو تصنيف و دراسة المعادلات التكاملية الموسعة المتعددة الابعاد والتي تحتوي على n من المؤثرات التكاملية المتعددة الابعاد. هذه الدراسة شملت حل هذه المعادلات باستخدام بعض الطرق من الطرق التربيعية والتي هي قاعدة شبه المنحرف وقاعدة شبه المنحرف مع التصحيح وقاعدة سمسون.

معلـــومـــات عامـــــــــه

وزارة التعليم العالي والبحث العلمي

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المفاتيح الاستدلالية :- المعادلات التكاملية، الطرق العددية لحل المعادلات التكاملية، معادلات تكاملية متعددة الابعاد

المستخلص

هذا العمل يتمحور باتجاه هدفين:

الهدف الاول هو تصنيف و دراسة المعادلات التكاملية الموسعة ذات البعد الواحد والتي تحتوي على n من المؤثرات التكاملية ذات البعد الواحد. هذه الدراسة تتضمن وجود و وحدانية الحل لانواع خاصة من هذه المعادلات التكاملية وحل هذه المعادلات باستخدام بعض الطرق من الطرق التربيعية والتي هي قاعدة شبه المنحرف وقاعدة شبه المنحرف مع التصحيح وقاعدة سمسون.

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Key words:- Integral equations, numerical solution of integral equations, Multidimensional integral equations.

Abstract

This work is oriented towards two objectives:

The first objective is to classify and study the generalized one-dimensional integral equations that contain n one-dimensional integral operators. This study includes the existence of a unique solution for special types of these integral equations and their solutions by using some quadrature methods, namely the trapezoidal rule, the modified trapezoidal rule and Simpson's rule.

The second objective is to classify and study the generalized multi-dimensional integral equations that contain n multi-dimensional integral operators. This study includes their solutions by using some quadrature methods, namely the trapezoidal rule, the modified trapezoidal rule and Simpson's rule.