ABSTRACT

This work has three objectives:

The first objective is to study fuzzy set theory including definitions, notations and examples.

The second objective is to study and proof the existence and uniqueness theorem of fuzzy boundary value problems directly without transforming the problem into fuzzy initial value problem.

The third objective is to study the numerical solution of fuzzy boundary value problems.

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Rasha Hussein Ibraheem

July, 2007

Supervisor Certification

I certify that this thesis was prepared under my supervision at the department of mathematics and computer applications, College of Science, Al-Nahrain University, in partial fulfillment of the requirements for the degree of Master of Science in mathematics

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1 FUZZY SET THEORY

In this chapter, we shall introduce the basic concepts and definitions of fuzzy set with some examples to explain these concepts. This chapter consists of five sections. In section (1.1), basic concepts related to fuzzy set theory are given including basic definitions, illustrated examples.

In section (1.2), a very strong notion which is related to fuzzy set theory will be introduced, which is the concept of α-level sets which has the utility of expressing an element that belong to the fuzzy set, The extension principle, which is used to generalize non-fuzzy concepts to fuzzy ideas is given and discussed in section (1.3).In section (1.4), we present fuzzy differential equations, In section(1.5) including solution of fuzzy differential equations, illustrated examples.

1.1 BASIC CONCEPTS OF FUZZY SETS

In this section, some of the fundamental notions and basic concepts are discussed which are related to fuzzy set theory. As mathematical notion, and throughout this thesis, X will used to denote the universal set with the generic element x and $\tilde{A}$ be fuzzy subsets of X, which is distinguished from ordinary set A by the symbol "~".

In order to distinguish between crisp (ordinary sets) and fuzzy sets, we start first with the definition of the characteristic function of ordinary sets, and then generalize this function for fuzzy sets.

<u>Definition (1.1.1), [Dubois, 1980]:</u>

The characteristic function in a classical (or ordinary or crisp) subset A of universal set X is often viewed as a function $\mu_A : X \longrightarrow \{0, 1\}$, and is denoted by:

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

where {0, 1} is called the valuation set.

If the valuation set is allowed to be the real interval [0, 1], then A is called a fuzzy set (and is denoted by $\tilde{A}$), where $\mu_{\tilde{A}}(x)$ is the grade membership of x in $\tilde{A}$ [Zadeh, 1965]. The closer value of $\mu_{\tilde{A}}(x)$ to 1, the more x belong to $\tilde{A}$, clearly $\tilde{A}$ is a subset of X that has no sharp boundary. $\tilde{A}$ which could be completely characterized by the following alterative form for representing the fuzzy set $\tilde{A}$, by:

$$\tilde{A} = \left\{(x, \mu_{\tilde{A}}(x)) : x \in X \,,\ 0 \le \mu_{\tilde{A}}(x) \le 1\right\}$$

Example (1.1.2), [Fadhel, 1998]:

We suppose a possible membership function for the fuzzy set of real numbers, which are very close to zero. This fuzzy $\tilde{A}$ set is defined using the membership function $\mu_{\tilde{A}} : \mathbb{R} \longrightarrow [0,1]$, defined by:

$$\mu_{\tilde{A}}(x) = \frac{1}{1+10x^2}, \ \forall x \in \mathbb{R}$$

Figure (1.1) illustrates the membership function for this fuzzy set.

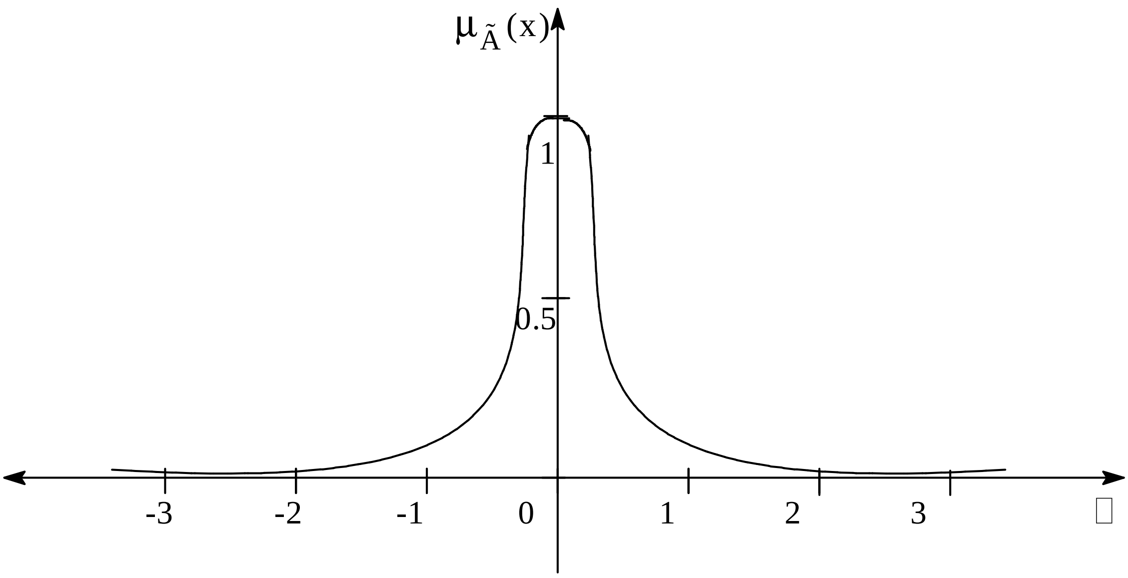

Fig. (1.1) The Membership function

Remarks (1.1.3), [Dubois, 1980], [Klir, 1997], [Zimmermann, 1988]:

1. If the universal set X is finite and given by $\{x_1, x_2, \ldots, x_n\}$, then a fuzzy set on X may be expressed as:

$$\tilde{A} = \mu_{\tilde{A}}(x_1)/x_1 + \mu_{\tilde{A}}(x_2)/x_2 + \ldots + \mu_{\tilde{A}}(x_n)/x_n$$

$$= \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)/x_i$$

where $\sum$ refers to the union over the finite index set and / refers to the restriction of the membership value $\mu_{\tilde{A}}(x_i)$, $\forall$ i = 1, 2, …, n.

2. If the universal set X is infinite, or for continuous membership functions, we express $\tilde{A}$ as:

$$\tilde{A} = \int_X \mu_{\tilde{A}}(x)/x$$

where $\int$ refers to the union over infinite index set.

3. Two fuzzy subset $\tilde{A}$ and $\tilde{B}$ of X are said to be equal (denoted by $\tilde{A} = \tilde{B}$) if:

$$\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x), \ \forall \ x \in X.$$

4. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy subsets of X, then $\tilde{A} \subseteq \tilde{B}$ if and only if:

$$\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x), \ \forall \ x \in X.$$

5. The complement of a fuzzy set $\tilde{A}$ (and is denoted by $\tilde{A}^c$) is a fuzzy set, with membership function:

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x), \ \forall \ x \in X.$$

6. The intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ (denoted by $\tilde{A} \cap \tilde{B}$) is a fuzzy set with membership function:

$$\mu_{\tilde{A} \cap \tilde{B}}(x) = \text{Min}\ \{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \ \forall \ x \in X.$$

7. The union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ denoted by $\tilde{A} \cup \tilde{B}$ is a fuzzy set with membership function:

$$\mu_{\tilde{A} \cup \tilde{B}}(x) = \text{Max}\ \{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \forall \ x \in X.$$

8. A fuzzy set $\tilde{A}$ is convex if:

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \text{Min}\left\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\right\},\ \forall\ x_1, x_2 \in X,\ \lambda \in [0, 1]$$

9. A fuzzy number $\tilde{M}$ is a convex normalized fuzzy set $\tilde{M}$ of the real line R, such that:

   i. There exists exactly one $x_0 \in$ R, with $\mu_{\tilde{M}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{M}$).

   ii. $\mu_{\tilde{M}}(x)$ is piecewise continuous.

10. The height of $\tilde{A}$ is given by:

$$\text{hgt}(\tilde{A}) = \sup_{x \in X} \mu_{\tilde{A}}(x)$$

11. $\tilde{A}$ is said to be normal if and only if there exists x $\in$ X, such that $\mu_{\tilde{A}}(x) = 1$; otherwise $\tilde{A}$ is subnormal, and if it is subnormal then we can normalized by the fuzzy set by dividing on the height the value of membership function.

12. The fuzzy singleton x_λ of a fuzzy set $\tilde{A}$ is defined by :

$$x_\lambda(w) = \begin{cases} \lambda, & \text{if } w = \lambda \\ 0, & \text{if } w \neq \lambda \end{cases}$$

for some $\lambda \in [0, 1]$.

Among the important aspects in fuzzy set theory is the law of ordinary set theory that is no longer valid here in fuzzy set theory, which is so called sometimes, the excluded middle law, because if $\tilde{A}$ is a fuzzy set and $\tilde{A}^c$ its complement then $\tilde{A} \cap \tilde{A}^c \neq \tilde{\varnothing}$ and $\tilde{A} \cup \tilde{A}^c \neq X$. Since the fuzzy set $\tilde{A}$ has no definite boundary and neither $\tilde{A}^c$, i.e., for all x $\in$ X, we have:

$$\text{Min}\,\{\mu_{\tilde{A}}(x),\ \mu_{\tilde{A}^c}(x)\} \le 0.5 \quad \text{and} \quad \text{Max}\,\{\mu_{\tilde{A}}(x),\ \mu_{\tilde{A}^c}(x)\} \ge 0.5$$

and it can be seen that $\tilde{A}$ and $\tilde{A}^c$ may overlap and then $\tilde{A} \cup \tilde{A}^c$ do not cover X exactly.

Example (1.1.4):

Let X = {1, 2, 3, 4} and define a fuzzy subset $\tilde{A}$ of X, by:

$\tilde{A}$ ={(1,0.3),(2,0.5),(3,0.7), (4,1)}

Then

$\tilde{A}^c$= {(1, 0.7), (2, 0.5), (3, 0.3), (4, 0)}

Hence

$\tilde{A} \cap \tilde{A}^c$ = {(1, 0.3), (2, 0.5), (3, 0.3), (4, 0)} $\neq \tilde{\emptyset}$

and

$\tilde{A} \cup \tilde{A}^c$ = {(1, 0.7), (2, 0.5), (3, 0.7), (4, 1)} $\neq$ X.

1.2 α-LEVEL SETS

The scope of this section is to cover some basic and most important properties of an ordinary set that can be derived from certain fuzzy set. These sets are called the α-level sets (or α-cuts), which corresponds to any fuzzy set. The α-level sets are those sets which collect between fuzzy sets and ordinary sets, which can be used to prove most of the results that are satisfied in ordinary sets are also satisfied here to fuzzy sets and rise versa, i.e., there is also another approach in which the classical sets and fuzzy sets are connected to each other [Yan, 1994].

Definition (1.2.1), [Yan, 1994]:

The α-level (or α-cut) set of a fuzzy set $\tilde{A}$, labeled by A_α, is the crisp set of all x in X such that $\mu_{\tilde{A}}(x) \geq \alpha$, i.e.,

$$A_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\}, \quad \alpha \in [0, 1]$$

One can notice that an α-level set discards those points whose membership values are less than α.Also, it is remarkable that in some literatures, if the equality is dropped in the definition of A_α then it is called a strong α-level set and is denoted by $A_{\alpha+}$ or $A_{\alpha-}$.

Remark (1.2.2), [Yan, 1994]:

If $\tilde{A}$, $\tilde{B}$ are any two fuzzy subsets of a universal set X, then it is easily checked that the following properties hold:

1. $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.
2. $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$.
3. If $\tilde{A} \subseteq \tilde{B}$ then $A_\alpha \subseteq B_\alpha$.
4. $\tilde{A} = \tilde{B}$ equivalent to $A_\alpha = B_\alpha$, $\forall\ \alpha \in [0, 1]$.
5. $A_\alpha \cap A_\beta = A_\beta$ and $A_\alpha \cup A_\beta = A_\alpha$, for all $\alpha, \beta \in [0, 1]$ and $\alpha \leq \beta$.
6. If $\alpha \leq \beta$, then $A_\alpha \supseteq A_\beta$.

Remark (1.2.3):

Sometimes for different fuzzy sets $\tilde{A}$ and $\tilde{B}$, we have $A_\alpha = B_\beta$ for different $\alpha, \beta \in [0, 1]$, as the following example illustrate:

Example (1.2.4), [Fadhel, 1998]:

Consider:

$X = \{a, b, c, e\}$

which is considered here as the universal set. Define a fuzzy subset $\tilde{A}$ of X using the following membership function:

$$\mu_{\tilde{A}}(e) = 1/2,\ \mu_{\tilde{A}}(a) = 1/3,\ \mu_{\tilde{A}}(b) = 1/4,\ \mu_{\tilde{A}}(c) = 1/4$$

Hence the image set of $\tilde{A}$ is given by:

$$\mathrm{Im}(\tilde{A}) = \{1/2, 1/3, 1/4\}$$

with the level sets:

$$\tilde{A}_{1/2} = \{e\},\ \tilde{A}_{1/3} = \{e, a\},\ \tilde{A}_{1/4} = \{e, a, b, c\}$$

Now, we define another fuzzy set $\tilde{B}$ of X with membership function defined by:

$$\mu_{\tilde{B}}(e) = 4/5,\ \mu_{\tilde{B}}(a) = 2/5,\ \mu_{\tilde{B}}(b) = 1/5,\ \mu_{\tilde{B}}(c) = 1/5$$

which has the image set:

$$\mathrm{Im}(\tilde{B}) = \{4/5, 2/5, 1/5\}$$

with the level sets:

$$\tilde{B}_{4/5} = \{e\},\ \tilde{B}_{2/5} = \{e, a\},\ \tilde{B}_{1/5} = \{e, a, b, c\}$$

Thus the fuzzy sets $\tilde{A}$ and $\tilde{B}$ have the same family of level sets, i.e.,

$$\tilde{A}_{1/2} = \tilde{B}_{4/5},\ \tilde{A}_{1/3} = \tilde{B}_{2/5},\ \tilde{A}_{1/4} = \tilde{B}_{1/5}$$

But $\tilde{A}$ and $\tilde{B}$ are different, where:

$$\tilde{A} = \{(e, 1/2), (a, 1/3), (b, 1/4), (c, 1/4)\}$$

$$\tilde{B} = \{(e, 4/5), (a, 2/5), (b, 1/5), (c, 1/5)\}.$$

1.4 THE EXTENSION PRINCIPLE OF FUZZY SETS

One of the most basic concepts of fuzzy set theory, which can be used to generalize crisp mathematical concepts to fuzzy sets, is the extension principle, which is defined as follows:

Definition (1.4.1) [Zimmermann, 1988]:

Let X be a Cartesian product of universes $X_1, X_2, \ldots, X_r$ and $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_r$ be r-fuzzy sets in $X_1, X_2, \ldots, X_r$, respectively, f is a mapping from X to a universe Y ($y = f(x_1, x_2, \ldots, x_r)$). Then a fuzzy set $\tilde{B}$ in Y is defined by:

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y = f(x_1, x_2, \ldots, x_r), (x_1, x_2, \ldots, x_r) \in X\}$$

where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{(x_1,\ldots,x_r)\in f^{-1}(y)} \operatorname{Min}\left\{\mu_{\tilde{A}_1}(x_1),\ldots,\mu_{\tilde{A}_r}(x_r)\right\}, & \text{if } f^{-1}(y) \neq \{\varnothing\} \\ 0, & \text{otherwise} \end{cases}$$

where f^{-1} is the inverse image of f.

For r = 1, the extension principle, of course, reduces to:

$$\tilde{B} = f(\tilde{A}) = f\ \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \mu_{\tilde{A}}(x), & \text{if } f^{-1}(y) \neq \{\varnothing\} \\ 0, & \text{otherwise} \end{cases}$$

which is the image of a fuzzy set, i.e., may be used to define a fuzzy function between two fuzzy sets.

Example (1.4.2), [Zimmermann, 1988]:

Let $X = \{-1, 0, 1, 2\}$, and define a fuzzy set on X by:

$\tilde{A} = \{(-1, 0.5), (0, 0.8), (1, 1), (2, 0.4)\}$

and consider $f(x) = x^2$, then by applying the extension principle we obtain a fuzzy set $\tilde{B}$, defined by:

$\tilde{B} = \{(0, 0.8), (1, 1), (4, 0.4)\}$.

1.5 FUZZY DIFFERENTIAL EQUATIONS [PEARSON, 1997]

We shall consider the fuzzy differential equation that has the property of linear differential equations of the initial state is described by a vector of fuzzy numbers. The property is related directly to the matrix defining the original non-fuzzy system by passing to the theory of complex analysis representation of the α-level sets of the fuzzy system. In modeling real systems, the fuzzy differential equation is:

$x' = f(t, x)$, $x(0) = \tilde{x}_a$, $a \le t \le b$ .... (1.1)

where $x \in \mathbb{R}^n$; $a, b \in \mathbb{R}$; f is a given vector field; and $\tilde{x}_0$ is a fuzzy number.

Methods for treating such problems depend on the use of properties of fuzzy set theory. As a first case, suppose the vector field is linear and all the parameters are assumed to be known to a certain sufficient accuracy, and the initial values of the system are fuzzy, i.e., equation (1.1) takes the form

$x'(t) = Ax$, $x(0) = \tilde{x}_a$, $a \le t \le b$

where $\tilde{x}_0$ is a fuzzy numbers, by using the properties of complex number, we can make the representation of the fuzzy dynamical system and to relate the fuzzy dynamics to the original non-fuzzy linear system.

A second case of fuzzy differential equations occurs when the coefficient of the matrix related to the vector filed is constituted to be fuzzy numbers, i.e., equation (1.1) takes the form

$$x'(t) = \tilde{A}\, x,\ x(0) \sqcup \tilde{x}_0,\ a \le t \le b$$

where $\tilde{A}$ is matrix of fuzzy numbers. Similarly, treating such systems is similar to the first case, but with more complications.

Also, it is important to notice that the conditions of the existence and uniqueness theorem for a solution of fuzzy differential equations are assumed to be imposed.

1.5 SOLUTION OF FUZZY DIFFERENTIAL EQUATIONS [PEARSON, 1997]

In this section, we shall study, as a survey, the solution of linear system of fuzzy differential equations.

Consider the problem of solving the fuzzy linear homogenous differential equation:

$$x'(t) = Ax,\ x(0) \sqcup \tilde{x}_0,\ a \le t \le b \quad \text{.......} \quad (1.2)$$

where $x \in \square^n$, A is n×n matrix and $\tilde{x}_0$ is the initial condition which is described by a vector made up of n-fuzzy numbers.

A fuzzy number $\tilde{x}_0$, can be prescribed easily by its α-level sets, as:

$$[\tilde{x}_0]_\alpha = \{ s \in x : \tilde{x}_0(s) \ge \alpha,\ 0 \le \alpha \le 1 \}$$

Due to the properties of the so defined fuzzy numbers, there corresponds to $\tilde{x}_0$ an interval for each given value of $\alpha \in [0, 1]$, given by:

$$[\tilde{x}_0]_\alpha = [\, \underline{\tilde{x}}_0, \overline{\tilde{x}}_0 \,]$$

Where $\underline{\tilde{x}}_0$ and $\overline{\tilde{x}}_0$ represents the lower and upper bounds of the α-level set of the fuzzy number $\tilde{x}_0$.

Now, rewrite x as a fuzzy solution $\tilde{x}$ and suppose that each element of the vector x in (1.2) may be represented in its α-level set, as:

$$x^k(t) = \left[\underline{x}^k_\alpha(t), \overline{x}^k_\alpha(t)\right], \ k = 1, 2, \cdots n, \ \alpha \in [0,1] \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.3)$$

it is shown that the evolution of the system (1.1) can be described by 2n-differential equations for the end points of the intervals, this is for each given time instant t and value of α. These equations for the end points of the intervals are:

$$\left.\begin{aligned} \underline{x}'^k_\alpha(t) &= \text{Min}\ \Big\{\sum_{j=1}^{n} a_{kj} u^j \ : u^i \in [\underline{x}^i_\alpha(t), \overline{x}^i_\alpha(t)]\Big\} \\ \overline{x}'^k_\alpha(t) &= \text{Max}\ \Big\{\sum_{j=1}^{n} a_{kj} u^j \ : u^i \in [\underline{x}^i_\alpha(t), \overline{x}^i_\alpha(t)]\Big\} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1.4)$$

with initial conditions $\underline{x}^k_\alpha(0) = \underline{x}^k_{\alpha_0}$ and $\overline{x}^k_\alpha(0) = \overline{x}^k_{\alpha_0}$

The vector in (1.2) is linear, and then equation (1.4) may be rewritten as:

$$\underline{x}'^k_\alpha(t) = \sum_{j=1}^{n} a_{kj} u^j \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1.5)$$

where

$$u^j = \begin{cases} \underline{x}^j_\alpha(t), & \text{if} \quad a_{kj} \ge 0 \\ \overline{x}^j_\alpha(t), & \text{if} \quad a_{kj} < 0 \end{cases}$$

and

$$\overline{x}'^{k}_{\alpha}(t) = \sum_{j=1}^{n} a_{kj} u^{j} \quad \ldots\ldots (1.6)$$

where:

$$u^{j} = \begin{cases} \overline{x}^{j}_{\alpha}(t), & \text{if } a_{kj} \geq 0 \\ \underline{x}^{j}_{\alpha}(t), & \text{if } a_{kj} < 0 \end{cases}$$

The method for solving directly the linear fuzzy system is meaningless; therefore an introduction of the representation of the fuzzy system using complex numbers is necessary.

Recall that, there are two equations of type (1.5) and (1.6) which can easily be written out explicitly.

Now, define new complex variables as follows:

$$z^{k}_{\alpha} = \underline{x}^{k}_{\alpha}(t) + i\overline{x}^{k}_{\alpha}(t) \quad \ldots\ldots (1.7)$$

and the two operations carried on the complex numbers are:

(a) The identity operation e, defined by:

$$e\, z^{k}_{\alpha} = z^{k}_{\alpha} \quad \ldots\ldots (1.8)$$

(b) The flip operation g, defined by

$$g(z^{k}_{\alpha}) = g(\underline{x}^{k}_{\alpha}(t) + i\,\overline{x}^{k}_{\alpha}(t)) = \overline{x}^{k}_{\alpha}(t) + i\,\underline{x}^{k}_{\alpha}(t) \quad \ldots\ldots (1.9)$$

where $g^{k} = e$ if k is even and $g^{k} = g$ if k is odd, and therefore:

$$(ug)\; z^{k}_{\alpha} = (gu) z^{k}_{\alpha} \;\text{ for } u \in \square \quad \ldots\ldots (1.10)$$

Using (1.7), (1.8) and (1.9), it is fairly easy to se that equations (1.5) and (1.6) can be written as:

$$z'^{k}_{\alpha} = B\, z^{k}_{\alpha},\; z_{\alpha}(0) = z_{\alpha_0} \quad \ldots\ldots (1.11)$$

where the elements of the matrix B are determined from those of A as follows:

$$b_{ij} = \begin{cases} ea_{ij}, & a_{ij} \geq 0 \\ ga_{ij}, & a_{ij} < 0 \end{cases} \qquad \ldots\ldots(1.12)$$

Now, $x' = Ax$, has the solution $x = ce^{At}$ and since $x(0) = x_0$. Then:

$x(t) = x_0\, e^{At}$

Similarly:

$$z_\alpha(t) = z_{\alpha_0}\, e^{Bt} \qquad \ldots\ldots(1.13)$$

But since the problem is to evaluate the exponential of the matrix B, then certain elements are multiplied by the operators e and g, where $b_{ij} = ea_{ij}$ if $a_{ij} \geq 0$ and $b_{ij} = ga_{ij}$ if $a_{ij} < 0$. This can be achieved for small values of t and represent the matrix B as the sum of two matrices C and D, one of which is multiplied by the operator e and the other by g, i.e.,

$B = eC + gD$

and for small t, we have:

$$\exp(tB)\, z_{\alpha_0} = \exp(\, t(eC + gD))\, z_{\alpha_0}$$

$$= \exp(teC)\, \exp(tgD)\, z_{\alpha_0} + O(t)$$

where O (t) is a function of t, such that $O(t)/t \longrightarrow 0$ as $t \longrightarrow 0$.

The first part exp(teC) is simply the standard matrix exponential, because e is the identity operator. For the second part exp(tgD), noting that $g^k = e$ if k is even and $g^k = g$ if it is odd and then proceed to calculate the formal power series of exp (tgD) as follows :

$$\exp(tgD)z_0 = \left(I + tgD + \frac{t^2}{2!}g^2D^2 + \frac{t^3}{3!}g^3D^3 + \ldots \right) z_0$$

$$= \left(I + \frac{t^2}{2!}g^2D^2 + \ldots \right) z_0 + \left(tgD + + \frac{t^3}{3!}g^3D^3 + \ldots \right) z_0$$

$$= \left(I + \frac{t^2}{2!}D^2 + \ldots \right) z_0 + \left(tD + + \frac{t^3}{3!}D^3 + \ldots \right) gz_0$$

$$= \cosh(tD)z_0 + \sinh(tD)gz_0$$

Hence:

$$z_{\alpha 0}(t) = \exp(tC)(\cosh(tD)z_{\alpha 0} + \sinh(tD)gz_{\alpha 0})$$

Let φ (t) =exp(tC)cosh(tD) and ψ (t) = exp(tC)sinh(tD). Then:

$$z_\alpha^k = \varphi_{kj}(t)\, z_{\alpha 0}^{j} + \psi_{kj}(t)g\, z_{\alpha 0}^{j}$$

but $z_\alpha^k = \underline{x}_\alpha^k(t) + i\, \overline{x}_\alpha^k(t)$, one get:

$$\underline{x}_\alpha^k(t) + i\, \overline{x}_\alpha^k(t) = \varphi_{kj}(t)\, z_{\alpha 0}^{j} + \psi_{kj}(t)\, z_{\alpha 0}^{j}$$

$$= \varphi_{kj}(t)(\underline{x}_{\alpha 0}^{j}(t) + i\,\overline{x}_{\alpha 0}^{j}(t)) + \psi_{kj}(t)(\overline{x}_{\alpha 0}^{j}(t) + i\,\underline{x}_{\alpha 0}^{j}(t))$$

Therefore:

$$\left.\begin{aligned} \underline{x}_\alpha^k(t) &= \sum_{j=1}^{n} \varphi_{kj}(t)\underline{x}_{\alpha_0}^{j}(t) + \psi_{kj}(t)\overline{x}_{\alpha_0}^{j}(t) \\ \overline{x}_\alpha^k(t) &= \sum_{j=1}^{n} \varphi_{kj}(t)\overline{x}_{\alpha_0}^{j}(t) + \psi_{kj}(t)\underline{x}_{\alpha_0}^{j}(t) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.14)$$

and hence the fuzzy solution is given in terms of its α-level sets as:

$$[x(t)]_\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)].$$

Example (1.6.1) [Pearson, 1997]:

Consider the linear system $\tilde{x}' = A\tilde{x}$, where $A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$ with initial values to be $x^1(0)$ about 1 and $x^2(0)$ about −1, which are fuzzy numbers and using the membership functions defined by setting, for example:

$$x_0^1(s) = \begin{cases} 0, & s < 0 \\ 2s - s^2, & 0 \le s < 2 \\ 0, & s > 2 \end{cases}$$

and

$$x_0^2(s) = \begin{cases} 0, & s < -2 \\ -2s - s^2, & -2 \le s < 0 \\ 0, & s > 0 \end{cases}$$

Thus, for $\alpha \in [0, 1]$, we can represent the initial conditions using its α-levels as:

$$[x_0^1]_\alpha = [\underline{x}_{0_\alpha}^1, \overline{x}_{0_\alpha}^1] = [1 - \sqrt{1-\alpha}, 1 + \sqrt{1-\alpha}]$$

$$[x_0^2]_\alpha = [\underline{x}_{0_\alpha}^2, \overline{x}_{0_\alpha}^2] = [-1 - \sqrt{1-\alpha}, -1 + \sqrt{1-\alpha}]$$

The fuzzy solution $\tilde{x}(t)$ is given by eq. (1.13) and if we let for simplicity purpose:

$$a = 1 - \sqrt{1-\alpha},\ b = 1 + \sqrt{1-\alpha},\ c = -1 - \sqrt{1-\alpha},\ d = -1 + \sqrt{1-\alpha}$$

Then the approximate solution may be evaluated as follows:

To find B, recall that $b_{ij} = ea_{ij}$ if $a_{ij} \ge 0$ and $b_{ij} = ga_{ij}$ if $a_{ij} < 0$, then $a_{11} = -1$ implies that $b_{11} = g(-1) = -i$, $a_{12} = 1 \ge 0$ implies $b_{12} = e(1) = 1$ and so on. Hence:

$$B = \begin{bmatrix} ga_{11} & ea_{12} \\ ea_{21} & ga_{22} \end{bmatrix} = \begin{bmatrix} -i & 1 \\ 0 & -2i \end{bmatrix}$$

and we can rewrite the matrix B as the sum of two matrices, the first matrix is multiplied by the operator e and the other is multiplied by g, as follows:

$$B = \begin{bmatrix} -i & 1 \\ 0 & -2i \end{bmatrix}$$

$$= e\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + g\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= eC + gD$$

It is easy to find e^{Ct}, which is $e^{Ct} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, and

$$\cosh(tD) = \begin{bmatrix} 1+\frac{t^2}{2!}+\frac{t^4}{4!}+\ldots & 0 \\ 0 & 1+2t^2+\frac{2}{3}t^4+\ldots \end{bmatrix}$$

Therefore:

$$\varphi(t) = e^{Ct}\cosh(tD)$$

$$= \begin{bmatrix} 1+\frac{t^2}{2!}+\frac{t^4}{4!}+\ldots & t+2t^3+\frac{2}{3}t^5+\ldots \\ 0 & 1+2t^2+\frac{2}{3}t^4+\ldots \end{bmatrix}$$

Similarly, one can find sinh(tD), which takes the form:

$$\sinh(tD) = \begin{bmatrix} -t-\frac{t^3}{3!}-\ldots & 0 \\ 0 & -2t-\frac{4}{3}t^3-\ldots \end{bmatrix}$$

Hence:

$$\psi(t) = e^{Ct}\sinh(tD)$$

$$= \begin{bmatrix} -t - \frac{t^3}{3!} - \ldots & -2t^2 - \frac{4}{3}t^4 - \ldots \\ 0 & -2t - \frac{4}{3}t^3 - \ldots \end{bmatrix}$$

therefore the approximate solution of equation (1.13) is given by:

$$\underline{x}^1_\alpha(t) = (1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots)\, a + (t + 2t^3 + \frac{2}{3}t^5 + \cdots)\, c + (-t - \frac{t^3}{3!} - \cdots)\, b + (-2t^2 - \frac{4}{3}t^4 - \cdots)\, d$$

$$\overline{x}^1_\alpha(t) = (1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots)\, b + (t + 2t^3 + \frac{2}{3}t^5 + \cdots)\, d + (-t - \frac{t^3}{3!} - \cdots)\, a + (-2t^2 - \frac{4}{3}t^4 - \cdots)\, c$$

$$\underline{x}^2_\alpha(t) = (1 + 2t^2 + \frac{2}{3}t^4 + \cdots)\, c + (-2t^2 - \frac{4}{3}t^4 - \cdots)\, d$$

$$\overline{x}^2_\alpha(t) = (-2t^2 - \frac{4}{3}t^4 - \cdots)\, c + (1 + 2t^2 + \frac{2}{3}t^4 + \cdots)\, d$$

Similar approach followed in solving linear system of ordinary differential equations with fuzzy initial conditions can be used to solve linear system of ordinary differential equations with fuzzy coefficients and non fuzzy initial conditions, as the following example illustrates:-

<u>Example (1.6.2), [Wuhaib, 2005]:</u>

Consider the linear system of fuzzy differential equations:

$$x' = \tilde{A}\, x \;, x(0) = x_0$$

where A is 2×2 fuzzy matrix, with entries:

(1) a_{11} approximately equals to –1, with membership function:

$$\mu_{a_{11}}(x) = \left|\frac{-1}{x}\right|, x \neq 0$$

(2) a_{12} approximately equals to 1, with membership function:

$$\mu_{a_{12}}(x) = \frac{1}{x}, x \neq 0$$

(3) a_{21} approximately equals to 0, with membership function:

$$\mu_{a_{21}}(x) = \frac{1}{1+x}, x \neq -1$$

(4) a_{22} approximately equals to –2, with membership function:

$$\mu_{a_{22}}(x) = \left|\frac{-2}{x}\right|, x \neq 0$$

and initial conditions $x_1(0) = 1$ and $x_2(0) = 2$.

Hence the lower and upper levels of A are given by:

$$\underline{A}_\alpha = \begin{bmatrix} -1-\sqrt{1-\alpha} & 1-\sqrt{1-\alpha} \\ -\sqrt{1-\alpha} & -2-\sqrt{1-\alpha} \end{bmatrix}$$

and

$$\overline{A}_\alpha = \begin{bmatrix} -1+\sqrt{1-\alpha} & 1+\sqrt{1-\alpha} \\ \sqrt{1-\alpha} & -2+\sqrt{1-\alpha} \end{bmatrix}$$

and if $a_{ij} \geq 0$, then $b_{ij} = ea_{ij}$, and if $a_{ij} < 0$ then $b_{ij} = ga_{ij}$. This yields the calculation of $\underline{B}_\alpha$ and $\overline{B}_\alpha$ as:

$$\underline{B}_\alpha = \begin{bmatrix} g\left(-1-\sqrt{1-\alpha}\right) & e\left(1-\sqrt{1-\alpha}\right) \\ g\left(-\sqrt{1-\alpha}\right) & g\left(-2-\sqrt{1-\alpha}\right) \end{bmatrix}$$

and

$$\overline{B}_\alpha = \begin{bmatrix} g\left(-1+\sqrt{1-\alpha}\right) & e\left(1+\sqrt{1-\alpha}\right) \\ e\left(\sqrt{1-\alpha}\right) & g\left(-2+\sqrt{1-\alpha}\right) \end{bmatrix}$$

For simplicity, if we let $r = \sqrt{1-\alpha}$, the matrix $\underline{B}_\alpha$ takes the form:

$$\underline{B}_\alpha = \begin{bmatrix} g(-1-r) & e(1-r) \\ g(-r) & g(-2-r) \end{bmatrix}$$

$$= e\begin{bmatrix} 0 & 1-r \\ 0 & 0 \end{bmatrix} + g\begin{bmatrix} -1-r & 0 \\ -r & -2-r \end{bmatrix} = e\underline{C}_\alpha + g\underline{D}_\alpha$$

Hence:

$$e^{\underline{C}_\alpha t} = e^{t\begin{bmatrix} 0 & 1-r \\ 0 & 0 \end{bmatrix}} = e^{\begin{bmatrix} 0 & t(1-r) \\ 0 & 0 \end{bmatrix}}$$

$$= I + \underline{C}_\alpha t = \begin{bmatrix} 1 & t(1-r) \\ 0 & 1 \end{bmatrix}$$

and:

$$\underline{\cosh}\,(t\underline{D}_\alpha) = I + \frac{t^2}{2!}\underline{D}_\alpha{}^2 + \frac{t^4}{4!}\underline{D}_\alpha{}^4 + O\,(t^6)$$

$$= \begin{bmatrix} 1+\frac{t^2}{2}+t^2 r+\frac{t^2 r^2}{2} & 0 \\ \frac{3}{2}t^2 r + t^2 r^2 & 1+2t^2+2t^2 r+\frac{t^2 r^2}{2} \end{bmatrix}$$

Therefore:

$$\underline{\theta}_\alpha(t) = e^{\underline{C}_\alpha t}\,\underline{\cosh}\,(t\underline{D}_\alpha)$$

$$= \begin{bmatrix} 1+t^2\left(\frac{1}{2}+r+\frac{r^2}{2}\right)+t^3\left(\frac{3}{2}r-\frac{1}{2}r^2-r^3\right) & t(1-r)+t^3\left(2-\frac{3}{2}r^2-\frac{r^3}{2}\right) \\ t^2\left(\frac{3}{2}r+r^2\right) & 1+t^2\left(2+2r+\frac{r^2}{2}\right) \end{bmatrix}$$

and similarly:

$$\underline{\psi}_\alpha(t) = \begin{bmatrix} \begin{array}{l} t(-1-r)+t^2(r^2-r)+ \\ t^3\left(-\frac{1}{6}-\frac{r}{2}-\frac{r^2}{2}-\frac{r^3}{6}\right)- \\ t^4\left(\frac{7}{6}r+\frac{1}{3}r^2-r^3-\frac{r^4}{2}\right) \end{array} & \begin{array}{l} t^2\left(r+r^2-2\right)+ \\ t^4\left(-\frac{4}{3}-\frac{2}{3}r+r^2-\frac{5}{6}r^3-\frac{r^4}{6}\right) \end{array} \\ -tr+t^3\left(-\frac{7}{6}r-\frac{3}{2}r^2-\frac{r^3}{2}\right) & t(-2-r)+t^3\left(-\frac{4}{3}-2r-r^2-\frac{r^3}{6}\right) \end{bmatrix}$$

Also, in a similar manner we can evaluate:

$$\overline{\theta}_\alpha(t) = \begin{bmatrix} 1+t^2\left(\frac{1}{2}-r+\frac{r^2}{2}\right) & t(1+r)+t^3\left(2-\frac{3}{2}r^2+\frac{r^3}{2}\right) \\ tr+t^3\left(\frac{1}{2}r-r^2+\frac{r^3}{2}\right) & 1+t^2\left(2-2r+\frac{r^2}{2}\right) \end{bmatrix}$$

and

$$\overline{\psi}_\alpha(t) = \begin{bmatrix} t(r-1)+t^3\left(-\frac{1}{6}+\frac{r}{2}-\frac{r^2}{2}+\frac{r^3}{6}\right) & t^2(-2-r+r^2)+t^4\left(\frac{r^4}{6}-\frac{5}{6}r^3+r^2-\frac{2}{3}r-\frac{4}{3}\right) \\ t^2(r-r^2)+t^4\left(-\frac{r}{6}-\frac{r^2}{2}-\frac{r^3}{2}+\frac{r^4}{6}\right) & t(r-2)+t^3\left(-\frac{4}{3}+2r-r^2+\frac{r^3}{6}\right) \end{bmatrix}$$

Now, letting t = 0.2 and $\alpha = 1$, we have:

$$\underline{\theta}(0.2) = \begin{bmatrix} 1.02 & 0.216 \\ 0 & 1.08 \end{bmatrix}$$

$$\underline{\psi}(0.2) = \begin{bmatrix} -0.201333 & -0.0821333 \\ 0 & -0.410666 \end{bmatrix}$$

$$\overline{\theta}(0.2) = \begin{bmatrix} 1.02 & 0.216 \\ 0 & 1.08 \end{bmatrix}$$

$$\overline{\psi}(0.2) = \begin{bmatrix} -0.201333 & -0.0821333 \\ 0 & -0410666 \end{bmatrix}$$

and hence $\underline{x}_1^1(0.2) = 1.0863944$, $\overline{x}_1^1(0.2) = 1.0863994$, $\underline{x}_1^2(0.2) = 1.33868$ and $\overline{x}_1^2(0.2) = 1.33868$

It is clear that for $\alpha = 1$, we have $\underline{x}_1^1(t) = \overline{x}_1^1(t)$ and $\underline{x}_1^2(t) = \overline{x}_1^2(t)$, which is the crisp value of the solution vector.

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THEORETICAL RESULTS IN FUZZY BOUNDARY VALUE PROBLEMS

This chapter consists of three sections. In section (2.1), we introduce some basic and fundamental concepts in theoretical fuzzy spaces and then discuss fuzzy inner and normed spaces. In Section (2.2), we introduce some basic and fundamental concepts of fuzzy boundary value problems. In Section (2.3), we state and prove the existence and uniqueness theorem of fuzzy boundary value problems, in which the proof is given directly without transforming the problem into a fuzzy initial value problem.

2.1 FUZZY NORMED SPACES

Fuzzy normed spaces are not studied previously by other researchers, therefore in this section; we give a brief and new introduction as a construction to the fuzzy normed spaces (to the best of our knowledge).

We start first with the following definition of ordinary normed linear spaces, that will be extended later to fuzzy set theory using the extension principle (given in chapter one).

Definition (2.1.1), [Erwin, 1978]:

A vector space X is said to be ***normed space*** if to every $x \in X$ there is an associated non-negative real number $\|x\|$ (called norm of x), such that:

1- $\|x\| > 0$ if $x \neq 0$, $\forall\, x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

2- $\|\lambda x\| = |\lambda|\, \|x\|$, $\forall\, x \in X$, λ is a scalar.

3- $\|x + y\| \leq \|x\| + \|y\|$, $\forall\, x, y \in X$.

The pair $(X, \|.\|)$ is called normed linear space.

As an examples of a normed spaces are the following:

i- The Euclidean space $\mathbb{R}^n$, where n is a fixed natural number which is the space of all $x = (x_1, x_2, \ldots, x_n)$, with the norm

$$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$$

ii- The space ℓ^p, where $p \geq 1$ is a fixed real number, which is the space of all sequences $x = \langle x_n \rangle$ with norm:

$$\|x\| = \sum_{n=1}^{\infty} |x_n|^p < \infty$$

iii- The space C[a, b] of all continuous real valued functions on the closed interval [a, b], with the following norm defined on it

$$\|x\| = \sup_{t \in [a, b]} |x(t)|$$

For more details and examples, see [Erwin, 1978].

Depending on the extension principle, we can extend the Euclidian normed space $(X, \|.\|)$ to a fuzzy normed space $(X^{\sim}, \|.\|^{\sim})$, as follows:

Definition (2.1.2):

Let $(X,\|.\|)$ be a non fuzzy normed space, and let $\tilde{A}$ be a fuzzy subset of X, and let I^x be the set of all fuzzy subsets of X, and then a fuzzy norm $\|\,.\,\|^{\sim} : I^X \longrightarrow \mathbb{R}^+$ is a function with membership function defined by:

$$\mu_{\|\tilde{A}\|^{\sim}}(\delta) = \begin{cases} \sup\limits_{\delta=\|x\|} \quad \min\left\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2), \ldots, \mu_{\tilde{A}}(x_n)\right\}, & \text{if } \delta = \|x\| \\ 0, & \text{,if } \delta \neq \|x\| \end{cases}$$

.... (2.1)

Definition (2.1.3), [Al-Yassiri, 2000]:

The normed space $(X^{\sim}, \|.\|^{\sim})$ is called ***Banach space*** if the normed space is complete.

Definition (2.1.4), [Al-Saeed, 2004]:

Let $(X, <.,.>)$ be a scalar product space and let $\tilde{A}$ be a fuzzy vector subspace of X, a mapping denoted by $<.,,>^{\sim}$ from $\tilde{A} \times \tilde{A}$ into a field F ($\mathbb{R}$ or $\mathbb{C}$) is called a ***fuzzy scalar product*** if the following conditions hold:

1- $< a_t, b_s >^{\sim} = \overline{< b_s, a_t >^{\sim}}$.

2- $< \lambda_1 a_t + \lambda_2 b_s, c_k >^{\sim} = \lambda_1 < a_t, c_k >^{\sim} + \lambda_2 < b_s, c_k >^{\sim}$, $\lambda_1, \lambda_2 \in F$.

3- $< a_t, a_t >^{\sim} \geq 0$, where $t \neq 0$ for all $t \in [0, 1]$.

4- $< a_t, a_t >^{\sim} = 0$ if and only if $t = 0$.

where $a_t, b_s, c_k \in \tilde{A}$; $a, b \in X$; and $t, s, k \in [0, 1]$.

Definition (2.1.5), [Al-Saeed, 2004]:

Let $(X, <.,.>)$ be a scalar product space, a pair $(X, <.,.>^{\sim})$ is said to be a ***fuzzy scalar product space*** if the mapping denoted by $<.,.>^{\sim}$ from $A \times A$ into a field F ($\mathbb{R}$ or $\mathbb{C}$) is a fuzzy scalar product on a fuzzy vector subspace A of X.

Remark (2.1.6):

We can define a fuzzy norm on a fuzzy subset of I^X, to be as follows:

$$\| \tilde{A} \|^{\sim} = \sqrt{< a_t, b_s >^{\sim}}$$

where a_t, b_s are fuzzy singleton of A.

2.2 FUNDAMENTAL CONCEPTS OF FUZZY BOUNDARY VALUE PROPLEMS

In order to study the existence and uniqueness theorem of the solution of a first-order fuzzy boundary value problem given by:

$$x' = f(t, x),\ t \in [a, c] \ \ldots\ldots \ (2.2)$$

subject to the boundary condition:

$$Mx(a) + R\,x(c) = \tilde{\beta}, \quad \tilde{\beta} \in E^n \ \ldots\ldots \ (2.3)$$

where $f : [a, c] \times E^n \longrightarrow E^n$ is continuous, non-linear function, a and c are given constants, M and R are given constant in $\mathbb{R}$, $\tilde{\beta}$ is a fuzzy number, and we define E^n by:

$$E^n = \{\tilde{A} / \mu_{\tilde{A}} : \mathbb{R}^n \longrightarrow [0,1]\}$$

and each $\tilde{A} \in E^n$ satisfies the following conditions:

i- $\tilde{A}$ is a fuzzy convex set.

ii- $\tilde{A}$ is semicontinuous, i.e., its α-cuts are closed, $\forall\ \alpha$.

iii- $\tilde{A}$ is normal.

iv- $[\tilde{A}]^0 = \mathrm{cl}\left\{ y \in \mathbb{R}^n : \mu_{\tilde{A}}(y) > 0 \right\}$ is compact.

It is remarkable that the solution of problem (2.2) is denoted by $\tilde{x}(t)$ rather than x(t), since it is fuzzy set.

Eq.(2.2) subject to eq.(2.3) is known as a fuzzy boundary value problem (FBVP).

An important formula that collects between fuzzy boundary value problems and fuzzy integral equations is given in the next lemma:

<u>Lemma (2.2.1):</u>

Suppose M + R ≠ 0 holds, and if $\tilde{x} \in C([a,c];E^n)$ satisfies the boundary value problem given by eqs (2.2) and (2.3), then:

$$\tilde{x}(t) = \tilde{A} + \int_a^t f(s,\tilde{x})\, ds,\quad t \in [a,c]$$

where:

$$\tilde{A} = \frac{1}{(M+R)}\left[\tilde{\beta} - R\int_a^c f(s,\tilde{x})ds\right]$$

Proof:

Integrating the boundary value problem given by equation (2.2) from a to t, give:

$$\tilde{x}(t) = \tilde{x}(a) + \int_a^t f(s, \tilde{x})\, ds, \quad t \in [a, c] \quad \text{...} \quad (2.4)$$

and hence substituting t = c, yields

$$\tilde{x}(c) = \tilde{x}(a) + \int_a^c f(s, \tilde{x})\, ds \quad \text{..} \quad (2.5)$$

So substituting (2.5) in (2.3) gives:

$$\tilde{\beta} = M\, \tilde{x}(a) + R\left(\tilde{x}(a) + \int_a^c f(s, \tilde{x})\, ds \right) \quad \text{..} \quad (2.6)$$

and rearranging eq.(2.6), we obtain that:

$$\tilde{x}(a) = \frac{1}{(M+R)}\left[\tilde{\beta} - R\int_a^c f(s, \tilde{x})\, ds \right] \quad \text{...} \quad (2.7)$$

So substituting eq.(2.7) in eq.(2.4) gives for $t \in [a, c]$

$$\tilde{x}(t) = \frac{1}{(M+R)}\left[\tilde{\beta} - R\int_a^c f(s, \tilde{x})\, ds \right] + \int_a^t f(s, \tilde{x})\, ds \quad \text{............................} \quad (2.8)$$

and hence the proof is complete. ■

As it is known obviously in the proof of the existence and uniqueness theorem, the proof depends on certain fixed point theorem such as Brouwer fixed point theorem, Schauder fixed point theorem, or Sadoveskii fixed point theorem , etc., depending on the nature of the problem under consideration.

Next, we state Schauder fixed point theorem which will be used in the proof of the existence and uniqueness theorem of fuzzy boundary value problems:

Theorem (2.2.2), (Schauder Fuzzy Fixed Point), [Al-Hamaiwand, 2001]:

Let I^X be a nonempty, closed, bounded and convex subset of a fuzzy Banach space B, and suppose that $\tilde{T} : I^X \longrightarrow I^X$ is a compact fuzzy operator, then $\tilde{T}$ has a fuzzy fixed point.

Also, the next theorem is of great importance in the proof of the existence and uniqueness theorem of fuzzy boundary value problems:

Theorem (2.2.3), (Finite Dimensional Rang), [Erwin, 1978]:

Let X and Y be two normed space and $T : X \longrightarrow Y$, a Linear operator. If T is bounded and dim $T(x) < \infty$, then the operator T is compact.

Definition (2.2.4), [Al-Hamaiwand, 2001]:

Given a fuzzy point $\tilde{x}_0 \in E^n$ and a number $r > 0$, we define on open ball B of radius r and center $\tilde{x}_0$, by:

$$B(\tilde{x}_0, r) = \left\{ \tilde{x} \in E^n : \left\| \tilde{x} - \tilde{x}_0 \right\|^{\sim} < r \right\}$$

Definition (2.2.5), [Al-Hamaiwand, 2001]:

For every mapping $T : X \longrightarrow Y$, then the fuzzy mapping $\tilde{T} : I^X \longrightarrow I^Y$, is defined by:

$$\tilde{T}(\tilde{A}) = \sup \mu_{\tilde{A}}(w)$$

where $\tilde{A}$ is a fuzzy subset of X and $w \in T^{-1}(x)$. If I^X and I^Y are fuzzy Banach spaces, then $\tilde{T}$ is called ***fuzzy operator***. Also, if:

$$\tilde{T}(c_1\tilde{A} + c_2\tilde{B}) = c_1\tilde{T}(\tilde{A}) + c_2\tilde{T}(\tilde{B}),\ \forall\ \tilde{A}, \tilde{B} \in I^X \text{ and } c_1, c_2 \in \mathbb{R} \text{ or } \mathbb{C}$$

Then $\tilde{T}$ is called ***linear operator***.

Definition (2.2.6), [Erwin, 1978]:

Let X and Y be two ordinary normed space and $\tilde{T}: I^X \longrightarrow I^Y$ a linear fuzzy operator and let B_{L+1} be a fuzzy ball, then the fuzzy operator $\tilde{T}$ is said to be ***bounded*** if there is a real number k such that:

$$\|\tilde{T}\,\tilde{x}\|^{\sim} \leq k\, \|\tilde{x}\|^{\sim}, \text{ for all } \tilde{x} \in B_{L+1}.$$

Definition (2.2.7), [Al-Hamaiwand, 2001]:

A fuzzy set $\tilde{A}$ is ***bounded fuzzy set*** if there exists a real number L > 0, such that:

$$\|\tilde{x} - \tilde{y}\|^{\sim} \leq L, \quad \forall\ \tilde{x}, \tilde{y} \in \tilde{A}.$$

Definition (2.2.8), [Park, 1999]:

Suppose $T = [c, d] \subset \mathbb{R}$ be a compact interval, then a mapping $F : T \longrightarrow E^n$ is called ***levelwise continuous*** at $x_0 \in T$ if the set-valued mapping $F_\alpha(x) = [F(x)]^\alpha$ is continuous at $x = x_0$ with respect to the Hausdorff metric d for all $\alpha \in [0, 1]$.

Now, define the fuzzy operator related to the fuzzy boundary value problem given by eqs. (2.2) and (2.3), by:

$$\tilde{T}. = I. + \frac{R}{(M+R)}\int_a^c f(s,.)\,ds - \int_a^t f(s,.)\,ds, \quad t \in [a, c] \text{ .... } (2.9)$$

and if f is taken to be linear then (2.9), takes the form:

$$\tilde{T}. = I. + \frac{R}{(M+R)}\int_a^c K(s)\,.\,ds - \int_a^t K(s)\,.\,ds, \quad t \in [a, c]$$

where K(s) is an ordinary function of s, I is the identity operator.

<u>Lemma (2.2.9):</u>

The fuzzy operator $\tilde{T}$ is linear.

Proof:

Since from the definition of the fuzzy operator related to the boundary value problems, we have:

$$\tilde{T}(\tilde{x}(t)) = \frac{\tilde{\beta}}{(M+R)} = g$$

i.e.,

$$\tilde{x}(t) + \frac{R}{M+R}\int_a^c K(s)\,\tilde{x}(t)\,ds - \int_a^t K(s)\,\tilde{x}(t)\,ds = g$$

Hence:

$$\tilde{x}(t) = g - \frac{R}{M+R}\int_a^c K(s)\,\tilde{x}(t)\,ds + \int_a^t K(s)\,\tilde{x}(t)\,ds$$

Now, to prove that $\tilde{T}$ is linear, we must prove that:

$$\tilde{T}(c_1\tilde{x}_1 + c_2\tilde{x}_2) = c_1\tilde{T}\,\tilde{x}_1 + c_2\tilde{T}\,\tilde{x}_2$$

Now:

$$\tilde{T}(c_1\tilde{x}_1+c_2\tilde{x}_2)=(c_1\tilde{x}_1+c_2\tilde{x}_2)+\frac{R}{(M+R)}\int_a^c K(s)\,(c_1\tilde{x}_1+c_2\tilde{x}_2)\,ds-\int_a^t K(s)(c_1\tilde{x}_1+c_2\tilde{x}_2)\,ds$$

$$=c_1\tilde{x}_1+c_2\tilde{x}_2+c_1\frac{R}{(M+R)}\int_a^c K(s)\,\tilde{x}_1\,ds+c_2\frac{R}{(M+R)}\int_a^c K(s)\,\tilde{x}_2\,ds-c_1\int_a^t K(s)\tilde{x}_1\,ds-c_2\int_a^t K(s)\,\tilde{x}_2\,ds$$

$$=c_1\left[\tilde{x}_1+\frac{R}{(M+R)}\int_a^c K(s)\,\tilde{x}_1\,ds-\int_a^t K(s)\,\tilde{x}_1\,ds\right]+c_2\left[\tilde{x}_2+\frac{R}{(M+R)}\int_a^c K(s)\,\tilde{x}_2\,ds-\int_a^t K(s)\,\tilde{x}_2\,ds\right]$$

$$=c_1\left[I.+\frac{R}{(M+R)}\int_a^c K(s)\,.\,ds-\int_a^t K(s)\,.\,ds\right]\tilde{x}_1+c_2\left[I.+\frac{R}{(M+R)}\int_a^c K(s)\,.\,ds-\int_a^t K(s)\,.\,ds\right]\tilde{x}_2$$

$$=c_1\tilde{T}\,\tilde{x}_1+c_2\tilde{T}\,\tilde{x}_2$$

Hence $\tilde{T}$ is linear. ■

2.3 THE EXISTENCE AND UNIQUENESS THEOREM

The study of the existence and uniqueness theorem plays an important roll in the theory of fuzzy differential equations in general, and in fuzzy boundary value problems in particular, therefore in this section we shall study and prove the existence and the uniqueness theorem of fuzzy boundary value problems directly without transforming the problem as in some literatures into fuzzy initial value problem.

Theorem (2.3.1), (The Existence Theorem):

Suppose M + R $\neq$ 0 holds and f$\in C([a, c]\times E^n; E^n)$. If there exist function k $\in$ $C([a, c]; \mathbb{R}^+)$, such that:

$$\| f(t, \tilde{q}) \|^{\sim} \leq k(t) \| \tilde{q} \|^{\sim}, \text{ for all } t \in [a, c], \quad \tilde{q} \in E^n \text{ .... } (2.10)$$

and if:

$$\left[1 + \left\| \frac{1}{(M+R)} R \right\|^{\sim} \right] \int_a^c k(s)\, ds < 1 \text{ .... } (2.11)$$

Then the fuzzy boundary value problem (2.2)-(2.3) has at least one solution in $C([a, c]; E^n)$.

Proof:

The existence of at least one solution to the fuzzy boundary value problem (2.2)-(2.3) is equivalent to the proof that the fuzzy integral equation given by eq. (2.8) has a fixed point, therefore the proof will be dependent on Schauder fixed point theorem.

Consider the mapping $\tilde{T} : C([a, c]; E^n) \longrightarrow C([a, c]; E^n)$, defined by:

$$\tilde{T}(\tilde{x}(t)) = \frac{1}{(M+R)} \left[\tilde{\beta} - R \int_a^c f\left(s, \tilde{x}(s)\right) ds \right] + \int_a^t f\left(s, \tilde{x}(s)\right) ds \text{ .... } (2.12)$$

for all t $\in$ [a, c].

Thus our problem is reduced to prove the existence of at least one $\tilde{x}$, such that:

$$\tilde{T}(\tilde{x}) = \tilde{x} \text{ .... } (2.13)$$

Now, from (2.13), we see that for all $t \in [a, c]$

$$\|\tilde{x}(t)\|^{\sim} \le \|\tilde{T}\,\tilde{x}(t)\|^{\sim}$$

$$= \left\| \frac{1}{(M+R)}\left[\tilde{\beta} - R\int_a^c f\left(s,\tilde{x}(s)\right)ds\right] + \int_a^t f\left(s,\tilde{x}(s)\right)ds \right\|^{\sim}$$

and since t ∈ [a, c], i.e., t ≤ c, hence:

$$\int_a^t f\left(s,\tilde{x}(s)\right)ds \le \int_a^c f\left(s,\tilde{x}(s)\right)ds$$

$$\|\tilde{x}(t)\|^{\sim} \le \left(1+\left|\frac{1}{(M+R)}R\right|\right)\int_a^c \|f\left(s,\tilde{x}(s)\right)\|^{\sim}\,ds + \left|\frac{1}{(M+R)}\right|\|\tilde{\beta}\|^{\sim}$$

$$\le \left(1+\left|\frac{1}{(M+R)}R\right|\right)\int_a^c k(s)\|\tilde{x}(s)\|^{\sim}\,ds + \left|\frac{1}{(M+R)}\right|\|\tilde{\beta}\|^{\sim}$$

Therefore:

$$\|\tilde{x}(t)\|^{\sim} \le \left(1+\left|\frac{1}{(M+R)}\right|R\right)\left[\max_{t\in[a,c]}\|\tilde{x}(t)\|^{\sim}\int_a^c k(s)\,ds\right] + \left|\frac{1}{(M+R)}\right|\|\tilde{\beta}\|^{\sim}$$

So rearranging the last inequality and taking the supremum over all t ∈ [a, c] we obtain that:

$$\sup_{t\in[a,c]}\|\tilde{x}(t)\|^{\sim} \le \frac{\left|\frac{1}{M+R}\right|\|\tilde{\beta}\|^{\sim}}{1-\left(1+\left|\frac{1}{(M+R)}R\right|\right)\int_a^c k(s)ds} = L \quad \text{.....................} \ (2.14)$$

Now, define the open ball with center 0 and radius L+1 by:

$$B_{L+1} = \left\{\tilde{x}\in C([a,c];E^n) : \|\tilde{x}(t)\|^{\sim} < L+1, \forall\, t\in[a,c]\right\}$$

We can see from inequality (2.14) that all possible solution of eq.(2.13) satisfies:

$$\|\tilde{x}(t)\|^{\sim} < L+1, \text{ for all } t \in [a, c]$$

Now, we will defined a fuzzy operator $\tilde{T}$, as follow

$$\tilde{T} : B_{L+1} \longrightarrow C([a, c]; E^n) \subseteq B_{L+1}$$

To prove, B_{L+1} is closed, bounded and convex fuzzy subset of $C([a,c];E^n)$

First, it is clear that, B_{L+1} is closed and bounded fuzzy set (by construction)

Now to Prove, B_{L+1} is convex set of fuzzy solutions.

Let $\tilde{x}_1(t),\ \tilde{x}_2(t) \in B_{L+1}$, hence we have:

$$\tilde{x}_1(t) \in C\big([a, c]; E^n\big), \text{ where } \|\tilde{x}_1(t)\|^{\sim} < L+1, \forall\, t \in [a, c]$$

and

$$\tilde{x}_2(t) \in C\big([a, c]; E^n\big) \text{ ,where } \|\tilde{x}_2(t)\|^{\sim} < L+1, \ \forall\, t \in [a, c]$$

To prove:

$$\tilde{x}(t) = \lambda\, \tilde{x}_1(t) + (1-\lambda)\, \tilde{x}_2(t) \in B_{L+1}$$

i.e., to prove that $\tilde{x}(t) \in C([a, c]; E^n\big)$ and $\|\tilde{x}(t)\|^{\sim} < L+1$

Now since, $\tilde{x}_1(t), \tilde{x}_2(t) \in C([a,c];E^n)$, and since the linear combination of levelwise continuous functions is also a levelwise continuous function, hence $\tilde{x}(t) \in C([a, c]; E^n\big)$. Also:

$$\begin{aligned}
\|\tilde{x}(t)\|^{\sim} &= \|\lambda\, \tilde{x}_1(t) + (1-\lambda)\, \tilde{x}_2(t)\|^{\sim} \\
&\leq \|\lambda\, \tilde{x}_1(t)\|^{\sim} + \|(1-\lambda)\, \tilde{x}_2(t)\|^{\sim} \\
&= |\lambda|\, \|\tilde{x}_1(t)\|^{\sim} + |1-\lambda|\, \|\tilde{x}_2(t)\|^{\sim} \\
&\leq \lambda\, L + (1-\lambda)\, L \\
&= L < L+1
\end{aligned}$$

Hence, $\| \tilde{x}(t) \|^{\sim} < L+1$, i.e., $\tilde{x}(t) \in B_{L+1}$

Hence, B_{L+1} is convex set

Now, to prove that $\tilde{T}$ is bounded, i.e., to prove $\| \tilde{T}\,\tilde{x} \|^{\sim} \le 2 \| \tilde{x} \|^{\sim}$, for any $\tilde{x}(t) \in C([a,c];E^n)$, we have:

$$\tilde{T}\tilde{x}(t) = \tilde{x}(t) + \frac{R}{(M+R)} \int_a^c f\left(s, \tilde{x}(s)\right) ds - \int_a^t f\left(s, \tilde{x}(s)\right) ds$$

and therefore

$$\left\|\tilde{T}\tilde{x}(t)\right\|^{\sim} = \left\| \tilde{x}(t) + \frac{R}{(M+R)} \int_a^c f\left(s, \tilde{x}(s)\right) ds - \int_a^t f\left(s, \tilde{x}(s)\right) ds \right\|^{\sim}$$

$$\le \| \tilde{x}(t) \|^{\sim} + \left(1 + \left| \frac{1}{(M+R)} R \right| \right) \int_a^c \left\| f\left(s, \tilde{x}(s)\right) \right\|^{\sim} ds$$

$$\le \| \tilde{x}(t) \|^{\sim} + \left(1 + \left| \frac{1}{(M+R)} R \right| \right) \int_a^c |k(s)| \, \| \tilde{x}(s) \|^{\sim} ds$$

$$\le \sup_{t \in [a,c]} |\tilde{x}(t)| + \left(1 + \left| \frac{1}{(M+R)} R \right| \right) \sup_{t \in [a,c]} |\tilde{x}(t)| \int_a^c k(s)\, ds$$

$$\le \left[1 + \left(1 + \left| \frac{1}{(M+R)} R \right| \right) \int_a^c k(s)\, ds \right] \sup_{t \in [a,c]} |\tilde{x}(t)|$$

$$\le \left[1 + \left(1 + \left| \frac{1}{(M+R)} R \right| \right) \int_a^c k(s)\, ds \right] \| \tilde{x} \|^{\sim}$$

$$\le 2 \| \tilde{x} \|^{\sim}$$

Hence, the fuzzy operator $\tilde{T}$ is bounded.

Then using theorem (2.2.3), we have $\tilde{T}$ is compact fuzzy operator,

Finally, by using Schauder fixed point theorem we get that $\tilde{T}$ has a fuzzy fixed point which shows that the existence of at least one solution in B_{L+1} and hence to (2.2), (2.3). ■

<u>Theorem (2.3.2), (The Uniqueness Theorem):</u>

Suppose M + R ≠ 0 holds and $f \in C([a, c]\times E^n; E^n)$ If there exists a function $k \in C([a, c];[0,\infty])$, such that:

$$\| f(t, \tilde{u}) - f(t, \tilde{v}) \|^{\sim} \leq k(t) \| \tilde{u} - \tilde{v} \|^{\sim}, \text{ for all } t \in [a, c], \tilde{u}, \tilde{v} \in E^n$$

.... (2.15)

and eq.(2.11) holds, then the fuzzy boundary value problem (2.2)-(2.3) has a unique solution in $C([a, c]; E^n)$

Proof:

Suppose that there exist two fuzzy solutions $\tilde{u}_1$ and $\tilde{u}_2$ to equations (2.2) and (2.3), and let $\tilde{z} = \tilde{u}_1 - \tilde{u}_2$. Now consider the fizzy boundary value problem:

$$\tilde{z}' = f(t, \tilde{u}_1) - f(t, \tilde{u}_2), \quad t \in [a, c] \quad \text{...} (2.16)$$

$$M\,\tilde{z}(a) + R\,\tilde{z}(c) \;\square\; \tilde{0} \quad \text{...} (2.17)$$

Arguing as in the proofs of lemma (2.2.1) and theorem (2.3.1) for t ∈ [a, c]. Integrating the boundary value problem given by equation (2.16) from a to t, gives:

$$\tilde{z}(t) = \tilde{z}(a) + \int_a^t \left[f(s, \tilde{u}_1) - f(s, \tilde{u}_2) \right] ds, \quad t \in [a, c] \quad \text{..........................} (2.18)$$

and at t = c, we have:

$$\tilde{z}(c) = \tilde{z}(a) + \int_a^c \left[f(s, \tilde{u}_1) - f(s, \tilde{u}_2) \right] ds \quad \ldots\ldots (2.19)$$

So substituting (2.19) in (2.17) gives:

$$\tilde{0} \,\square\, M\tilde{z}(a) + R\left[\tilde{z}(a) + \int_a^c \left[f(s, \tilde{u}_1) - f(s, \tilde{u}_2) \right] ds \right] \quad \ldots\ldots (2.20)$$

and rearranging (2.20) we obtain that:

$$\tilde{z}(a) = \frac{1}{(M+R)}\left[-R\int_a^c \left[f(s, \tilde{u}_1) - f(s, \tilde{u}_2) \right] ds \right] \quad \ldots\ldots (2.21)$$

So substituting eq. (2.21) in eq. (2.18), to get:

$$\tilde{z}(t) = \frac{1}{(M+R)}\left[-R\int_a^c \left[f(s, \tilde{u}_1) - f(s, \tilde{u}_2) \right] ds \right] + \int_a^t \left[f(s, \tilde{u}_1) - f(s, \tilde{u}_2) \right] ds$$

taking the norm on both sides of the last eq., yields:

$$\| \tilde{z}(t) \|^{\sim} = \left\| \frac{1}{(M+R)}\left[-R\int_a^c \left[f(s, \tilde{u}_1) - f(s, \tilde{u}_2) \right] ds \right] + \int_a^t \left[f(s, \tilde{u}_1) - f(s, \tilde{u}_2) \right] ds \right\|^{\sim}$$

$$\leq \left(1 + \left| \frac{1}{(M+R)} R \right| \right) \int_a^c \| \left[f(s, u_1) - f(s, u_2) \right] \|^{\sim} \, ds$$

$$\leq \left(1 + \left| \frac{1}{(M+R)} R \right| \right) \int_a^c k(s) \, \| \tilde{u}_1 - \tilde{u}_2 \|^{\sim} \, ds$$

$$\leq \left(1 + \left| \frac{1}{(M+R)} R \right| \right) \left[\max_{t \in [a,c]} \| \tilde{z}(t) \|^{\sim} \int_a^c k(s) ds \right]$$

and rearranging we obtain

$$\max_{t\in[a,\,c]} \|\tilde{z}(t)\|^{\sim} \left(1-\left(1+\left|\frac{1}{(M+R)}R\right|\right)\int_a^c k(s)\,ds\right) \le 0$$

So we have $\|\tilde{z}(t)\|^{\sim} = 0$, for all $t \in [a, c]$ but $\left(1-\left(1+\left|\frac{1}{(M+R)}R\right|\right)\int_a^c k(s)\,ds\right) \neq 0$

since $\left[1+\left\|\frac{1}{(M+R)}R\right\|^{\sim}\right]\int_a^c k(s)\,ds < 1$ and from the properties of the norm, we have $\tilde{z}(t) = 0$, i.e., $\tilde{u}_1(t) = \tilde{u}_2(t),\ \forall\, t \in [a, c]$, which the uniqueness of the solution. ■

3 SOLUTION OF FUZZY BOUNDARY VALUE PROBLEMS

In chapter one, we studied methods for evaluating solutions o fuzzy initial value problems. In an initial value problem the values of the solution and lower order derivative up to order one less than the order of the differential equation are specified at the initial time. Another type of problems that arises frequently in applications is a fuzzy boundary value problem in which in addition to the differential equation information about the solution and perhaps some derivatives is specified at two different values of the independent variable.

This chapter consists of the numerical solution of linear and nonlinear fuzzy boundary value problems using the shooting method and finite difference method, as well as some well known results in ordinary boundary value problems and its generalization to the undertaken problem of fuzzy boundary value problems.

3.1 SOME THEORETICAL RESULTS

3.1.1 Sturm-Liouville Boundary value Problem [Sagan, 1961]***:***

Consider the second order ODE:

$$\frac{d}{dx}[P(x)y'] + [Q(x) + \lambda R(x)]y = 0, \ x \in [a, b] \quad \text{................................} \ (3.1)$$

with boundary conditions:

$$\left.\begin{array}{l} a_{11}y(a) + a_{12}y'(a) = 0 \\ a_{21}y(b) + a_{22}y'(b) = 0 \end{array}\right\} \quad \text{..} \ (3.2)$$

where $a_{ij}, \forall i, j = 1, 2$ are prescribed constants and P, P′, Q and R are continuous functions on the interval $a \le x \le b$. Furthermore, $P(x) > 0$ and $R(x) > 0$, for all $x \in [a, b]$ and λ is a constant called the eigenvalue.

The problem of solving eq.(3.1) under the boundary conditions (3.2) is called the Sturm-Liouville problem's, in particular (self adjoint BVP's, in general).

A parameter λ for which eq.(3.1) has a nontrivial solution is called an eigenvalue (characteristic value) and the corresponding solution is an eigenfunction (characteristic function) of Sturm-Liouville problem.

The problem of evaluating such values of λ and y(x) is called eigenvalue problem (characteristic value problem).

3.1.2 Eigenvalues and Eigenfunctions of Sturm-Liouville Problem:

The asymptotic properties of the eigenvalues and eigenfunctions related to Sturm-Liouville have been studied previously. These results can be summarized in the following theorems that will be given here without proof (see [William and Richard, 1986], [Mustafa, 1996]).

Theorem (3.1.2.1):

All the eigenvalues of Sturm-Liouville problem (3.1) and (3.2) are real.

.Theorem (3.1.2.2):

If Q(x) is non-positive, then the eigenvalues of Sturm-Liouville problem (3.1)-(3.2) are positive.

Remark (3.1.2.3):

If P(x), Q(x) and R(x) are continuous functions, p(x) is differentiable and P(x) > 0, R(x) > 0, for $x \in [a, b]$. Then, the eigenvalues λ_1, λ_2, … of Sturm-Liouville problem satisfies $\lambda_1 < \lambda_2 < \ldots$, for which the Sturm-Liouville problem (3.1)-(3.2) has a nontrivial solution.

Theorem (3.1.2.4):

If Q_i and Q_j, $i \neq j$ are two eigenfunctions of the Sturm-Liouville problem (3.1)-(3.2) corresponding to the eigenvalues λ_i and λ_j, respectively, then:

$$\int_a^b R(x)Q_i(x)Q_j(x)\,dx = 0, \text{ provided that } Q_i \neq Q_j \text{ and } \lambda_i \neq \lambda_j.$$

where R(x) is the weighted function.

Remark (3.1.2.5):

From the above theorems, one can see that the analysis depends mainly of the differential equation (not on the boundary conditions). Hence, the generalization to fuzzy BVP's is quite similar since the fuzziness appears in the boundary conditions.

Example (3.1.2.6):

Consider the fuzzy BVP of Sturm-Liouville type:

$$y'' + \lambda y = 0,\ 0 \le x \le \pi \quad \text{............} (3.3)$$

with fuzzy boundary conditions:

$$y(0) \sqcup \tilde{0} \text{ and } y(\pi) \sqcup \tilde{0}$$

in this case the parametric equations of eq.(3.3), are given by:

$$\underline{y}''_{\alpha}(x) + \underline{\lambda}\,\underline{y}_{\alpha}(x) = 0,\ x \in [0, \pi] \quad \text{............} (3.4)$$

with:

$$\underline{y}_{\alpha}(0) = 0 - \sqrt{1-\alpha} \quad \text{and} \quad \underline{y}_{\alpha}(\pi) = 0 - \sqrt{1-\alpha}$$

and

$$\overline{y}''_{\alpha}(x) + \overline{\lambda}\,\overline{y}_{\alpha}(x) = 0,\ x \in [0, \pi] \quad \text{............} (3.5)$$

with:

$$\overline{y}_{\alpha}(0) = 0 + \sqrt{1-\alpha} \quad \text{and} \quad \overline{y}_{\alpha}(\pi) = 0 + \sqrt{1-\alpha}$$

which are both ordinary BVP with different boundary condition.

It is easy to see that associated with $\underline{\lambda} > 0$.The general solution corresponding to this problem is given by:

$$\underline{y}(x) = c_1 \sin\sqrt{\underline{\lambda}}\,x + c_2 \cos\sqrt{\underline{\lambda}}\,x$$

Using the first boundary conditions, the general solution reduced to:

$$\underline{y}(x) = c_1 \sin\sqrt{\underline{\lambda}}\,x$$

and by using the second boundary condition, we have $\sqrt{\underline{\lambda}} = n$, for some positive integer, therefore $\underline{\lambda} = n^2$.

Hence, the corresponding nontrivial solutions could be written in the form:

$\underline{y}_n(x) = a_n \sin(nx),\ n = 1, 2, \ldots$

This shows that all the eigenvalues are positive and $\underline{\lambda}_1 < \underline{\lambda}_2 < \underline{\lambda}_3 < \ldots$

The orthogonally of eigenfunctions also can be proved easily, since for any two eigenfunctions $\underline{y}_n(x)$ and $\underline{y}_m(x)$ with $n \neq m$, we have:

$$\int_0^{\pi} \underline{y}_n(x)\,\underline{y}_m(x)\,dx = \int_0^{\pi} a_n \sin(nx) a_m \sin(mx)\,dx$$

$$= a_n a_m \int_0^{\pi} \frac{\cos(nx - mx) - \cos(nx + mx)}{2}\,dx$$

$$= \frac{a_n a_m}{2}\left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m}\right]_0^{\pi} = 0$$

Hence $\underline{y}_n$ is orthogonal to $\underline{y}_m$, $\forall\ n \neq m$.

Similarly, we can find the upper solution $\overline{y}_\alpha$.

Therefore, $\tilde{y}_\alpha = [\underline{y}_\alpha, \overline{y}_\alpha]$ is the solution of the original fuzzy BVP.

3.2 BOUNDARY VALUE PROBLEMS OF FUZZY DIFFERENTIAL EQUATIONS

Solution of fuzzy boundary value problems have not been discussed previously in details, either analytically or numerically. Therefore, in the next two subsections, we will discuss the numerical solution of fuzzy boundary value problems of linear and nonlinear case. We considered the general form of a fuzzy BVP is :

$$y'' = p(x)y' + q(x)y + r(x),\ a \leq x \leq b$$

$$y(a) \square\ \tilde{\alpha},\ y(b) \square\ \tilde{\beta}$$

3.2.1 The Shooting Method for Solving Fuzzy Linear BVP's:

The shooting method for solving fuzzy linear equation is based on the replacement of the fuzzy boundary value problem by its two related fuzzy initial value problems, as it is usual case in solving non-fuzzy boundary value problems.

Now, consider the linear second order fuzzy boundary value problem:

$$y'' = p(x)y' + q(x)y + r(x),\ a \le x \le b \quad (3.6)$$

$$y(a) \sqcup \tilde{\alpha},\ y(b) \sqcup \tilde{\beta} \quad (3.7)$$

hence, the related two fuzzy initial value problems are given by:

$$u'' = p(x)u' + q(x)u,\ a \le x \le b,\ u(a) \sqcup \tilde{0},\ u'(a) \sqcup \tilde{1} \quad (3.8)$$

and

$$v'' = p(x)v' + q(x)v + r(x),\ a \le x \le b,\ v(a) \sqcup \tilde{\alpha},\ v'(a) \sqcup \tilde{0} \quad (3.9)$$

To find the solutions of the fuzzy initial value problems (3.8) and (3.9), respectively, then the α-level equations these fuzzy differential equations are:

$$[u'']_\alpha = p(x)[u']_\alpha + q(x)[u]_\alpha,\ [u(a)]_\alpha \sqcup \tilde{0}_\alpha,\ [u'(a)]_\alpha \sqcup \tilde{1}_\alpha \quad (3.10)$$

and

$$[v'']_\alpha = p(x)[v']_\alpha + q(x)[v]_\alpha + r(x),\ [v(a)]_\alpha \sqcup \tilde{\alpha}_\alpha,\ [v'(a)]_\alpha \sqcup \tilde{0}_\alpha. \quad (3.11)$$

Hence, for solution in the range $a \le x \le b$, where a and b are finite, suppose that the differential equations satisfies the existence and uniqueness theorem (as illustrated in chapter two). The solution of eqs.(3.10) and (3.11) are denoted, respectively, by:

$$[u(x)]_\alpha = [\underline{u}(x;\alpha),\ \overline{u}(x;\alpha)],\ \alpha \in [0, 1]$$

and

$$[v(x)]_\alpha = [\underline{v}(x;\alpha),\ \overline{v}(x;\alpha)],\ \alpha \in [0, 1]$$

where $\underline{u}$, $\overline{u}$, $\underline{v}$ and $\overline{v}$ are the lower ad upper ordinary solutions related to the fuzzy initial value problems at certain level. Also, the initial conditions can be rewritten as:

$$[u(a)]_\alpha = [\tilde{0}]_\alpha = [\underline{\tilde{0}}(\alpha), \overline{\tilde{0}}(\alpha)],\ [u'(a)]_\alpha = [\tilde{1}]_\alpha = [\underline{\tilde{1}}(\alpha), \overline{\tilde{1}}(\alpha)] \quad \text{.... (3.12)}$$

and

$$[v(a)]_\alpha = [\tilde{\alpha}]_\alpha = [\underline{\tilde{\alpha}}(\alpha), \overline{\tilde{\alpha}}(\alpha)],\ [v'(a)]_\alpha = [\tilde{0}]_\alpha = [\underline{\tilde{0}}(\alpha), \overline{\tilde{0}}(\alpha)] \quad \text{.. (3.13)}$$

and hence the final solution of the fuzzy BVP can be obtained using previous discussed methods which are given for lower and upper cases by:

$$\left.\begin{aligned} \underline{y}(x) &= \underline{v}(x) + \underline{\lambda}\,\underline{u}(x) \\ \overline{y}(x) &= \overline{v}(x) + \overline{\lambda}\,\overline{u}(x) \end{aligned}\right\} \quad \text{.. (3.14)}$$

where:

$$\underline{\lambda} = \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)},\ \underline{u}(b) \neq 0 \quad \text{and} \quad \overline{\lambda} = \frac{\tilde{\beta} - \overline{v}(b)}{\overline{u}(b)},\ \overline{u}(b) \neq 0$$

Eq. (3.14) is derived for the lower case, by letting

$$\underline{y}(x) = c_1\underline{v}(x) + c_2\underline{u}(x) \quad \text{.. (3.15)}$$

and it is easily found that $c_1 = 1$ and $c_2 = \dfrac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}$ by using the boundary conditions at a and b. Also, we can check that $\underline{y}(x)$ is really the solution of the original fuzzy BVP, since:

$$\underline{y}'(x) = \underline{v}'(x) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}\,\underline{u}'(x)$$

and

$$\underline{y}''(x) = \underline{v}''(x) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}\,\underline{u}''(x)$$

So:

$$\underline{y}''(x) = p(x)\underline{v}'(x) + q(x)\underline{v}(x) + r(x) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}(p(x)\underline{u}'(x) + q(x)\underline{u}(x))$$

$$= p(x)\{\underline{v}'(x) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}\,\underline{u}'(x)\} + q(x)\{\underline{v}(x) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}\,\underline{u}(x)\} + r(x)$$

$$= p(x)\underline{y}'(x) + q(x)\underline{y}(x) + r(x)$$

Moreover:

$$\underline{y}(a) = \underline{v}(a) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}\,\underline{u}(a)$$

$$= \tilde{\alpha} + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)} \times 0 = \tilde{\alpha}$$

and

$$\underline{y}(b) = \underline{v}(b) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}\,\underline{u}(b)$$

$$= \underline{v}(b) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}\,\underline{u}(b) = \underline{v}(b) + \tilde{\beta} - \underline{v}(b) = \tilde{\beta}$$

Hence, $\underline{y}(x)$ is the unique solution to the linear BVP, provided, of course, that $\underline{u}(b) \neq 0$.

Similarly, as in upper case, we have:

$$\overline{y}(x) = \overline{v}(x) + \frac{\tilde{\beta} - \overline{v}(b)}{\overline{u}(b)}\,\overline{u}(x)$$

The following example illustrates the above approach:

Example (3.2.1.1):

To solve the non-homogeneous fuzzy BVP:

$$y'' = \frac{-2}{x}y' + \frac{2}{x^2}y + x^2,\ 1 \le x \le 2$$

$$y(1) \square\ \tilde{1},\ y(2) \square\ \tilde{2}$$

by using the liner shooting method. This problem has the exact crisp solution:

$$y(x) = -\frac{1}{8}x + \frac{1}{x^2} + \frac{1}{8}x^4$$

Now, to solve the homogeneous problem:

$$u'' = \frac{-2}{x}u' + \frac{2}{x^2}u,\ u(1) \square\ \tilde{0},\ u'(1) \square\ \tilde{1}$$

Let $u_1 = u$, then $u' = u_2$ and so $u'_2 = \frac{-2}{x}u_2 + \frac{2}{x^2}u_1$. Therefore in matrix form:

$$\begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{2}{x^2} & \frac{-2}{x} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

So, the desired system of homogeneous system of initial value problem, and carrying out similar whenever $x = 2$, we have for all $\alpha \in [0, 1]$:

$$\underline{u}^{(1)}_{\alpha}(2) = 2a + \frac{22}{3}c - \frac{20}{3}d,\ \overline{u}^{(1)}_{\alpha}(2) = \frac{-20}{3}c + 2b + \frac{22}{3}d$$

and

$$\underline{u}^{(2)}_{\alpha}(2) = a + \frac{22}{3}c - \frac{20}{3}d,\ \overline{u}^{(2)}_{\alpha}(2) = \frac{-20}{3}c + b + \frac{22}{3}d$$

where $a = -\sqrt{1-\alpha}$, $b = \sqrt{1-\alpha}$, $c = 1 - \sqrt{1-\alpha}$, $d = 1 + \sqrt{1-\alpha}$.

To check the validity of the results, if $\alpha = 1$, then $a = 0$, $b = 0$, $c = 1$, $d = 1$, and hence:

$$\underline{u}_{\alpha}^{(1)}(2) = \overline{u}_{\alpha}^{(1)}(2) = 0.666666 \quad \text{and} \quad \underline{u}_{\alpha}^{(2)}(2) = \overline{u}_{\alpha}^{(2)}(2) = 0.666666$$

Also, for the non-homogeneous problem:

$$v'' = \frac{-2}{x}v' + \frac{2}{x^2}v + x^2,\ v(1) \sqcup \tilde{1},\ v'(1) \sqcup \tilde{0}$$

which has an equivalent matrix form:

$$\begin{bmatrix} v' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{2}{x^2} & \dfrac{-2}{x} \end{bmatrix} \begin{bmatrix} v \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ x^2 \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix}$$

So, the desire non-homogeneous system of differential equations with fuzzy initial conditions could also be solved. Solving the non-homogeneous system using Euler's method given in parametric form for v(x) and y(x). In this case is given as follows:

$$\left.\begin{aligned} &\underline{v}'(x,\alpha) = \underline{y}(x,\alpha),\ \overline{v}'(x,\alpha) = \overline{y}(x,\alpha) \\ &\underline{y}'(x,\alpha) = \frac{2}{x^2}\underline{v} - \frac{2}{x}\underline{y} + x^2,\ \overline{y}'(x,\alpha) = \frac{2}{x^2}\overline{v} - \frac{2}{x}\overline{y} + x^2 \end{aligned}\right\} \quad .. (3.16)$$

With initial conditions are given for all $\alpha \in [0, 1]$, by:

$$\underline{v}(1,\alpha) \sqcup \tilde{1},\ \overline{v}(1,\alpha) \sqcup \tilde{1}$$

$$\underline{y}(1,\alpha) \sqcup \tilde{0},\ \overline{y}(1,\alpha) \sqcup \tilde{0}$$

Using fist order explicit Euler method:

$$\left.\begin{aligned} &\underline{v}_{n+1}(\alpha) = \underline{v}_n(\alpha) + hF_1(x_n,\ \underline{v}_n(\alpha), \underline{y}_n(\alpha)) \\ &\overline{v}_{n+1}(\alpha) = \overline{v}_n(\alpha) + hF_2(x_n,\ \overline{v}_n(\alpha), \overline{y}_n(\alpha)) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.17)$$

and

$$\left.\begin{aligned} \underline{y}_{n+1}(\alpha) &= \underline{y}_n(\alpha) + hG_1(x_n, \underline{v}_n(\alpha), \underline{y}_n(\alpha)) \\ \overline{y}_{n+1}(\alpha) &= \overline{y}_n(\alpha) + hG_2(x_n, \overline{v}_n(\alpha), \overline{y}_n(\alpha)) \end{aligned}\right\} \quad \text{............................ (3.18)}$$

Using eq.(3.16) with eqs.(3.17) and (3.18), we get:

$$\left.\begin{aligned} \underline{v}_{n+1}(\alpha) &= \underline{v}_n(\alpha) + h\,\underline{y}_n(\alpha) \\ \overline{v}_{n+1}(\alpha) &= \overline{v}_n(\alpha) + h\,\overline{y}_n(\alpha) \end{aligned}\right\} \quad \text{.. (3.19)}$$

and

$$\left.\begin{aligned} \underline{y}_{n+1}(\alpha) &= \underline{y}_n(\alpha) + h\left(\frac{2}{x_n^2}\,\underline{v}_n - \frac{2}{x_n}\,\underline{y}_n + x^2_n\right) \\ \overline{y}_{n+1}(\alpha) &= \overline{y}_n(\alpha) + h\left(\frac{2}{x_n^2}\,\overline{v}_n - \frac{2}{x_n}\,\overline{y}_n + x^2_n\right) \end{aligned}\right\} \quad \text{......................... (3.20)}$$

Hence, by using the first problem of eqs.(3.19) and (3.20), we get the following results which are given in table (3.1), for all $x \in [1, 2]$, $h = 0.1$ and $\alpha = 1$.

Table (3.1).

The lower level of fuzzy solution of eqs.(3.19) and (3.20)

x_n	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$\underline{v}_n$		1	1.03	1.08	1.15	1.24	1.34	1.46	1.60	1.76	1.95
$\underline{y}_n$	0	0.3	0.53	0.72	0.90	1.08	1.27	1.47	1.68	1.91	2.16

And by using the second problem of eqs.(3.19) and (3.20) we get the following results which are given in table (3.2) for all $x \in [1, 2]$ and $h = 0.1$, $\alpha = 1$.

Table (3.2).

The upper level of fuzzy solution of eqs.(3.19) and (3.20)

x_n	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$\overline{v}_n$	1	1	1.03	1.08	1.15	1.24	1.34	1.46	1.60	1.76	1.95
$\overline{y}_n$	0	0.3	0.53	0.72	0.90	1.08	1.27	1.47	1.68	1.91	2.16

where:

$$\underline{\lambda} = \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)} = \frac{2 - 1.95}{0.666666} = 0.075$$

and

$$\overline{\lambda} = \frac{\tilde{\beta} - \overline{v}(b)}{\overline{u}(b)} = \frac{2 - 1.95}{0.666666} = 0.075$$

As a result, the general solution of the fuzzy BVP using the linear shooting method is given by:

$$\underline{y}(x_i) = \underline{v}(x_i) + \underline{\lambda}\, \underline{u}(x_i)$$

$$\overline{y}(x_i) = \overline{v}(x_i) + \overline{\lambda}\, \overline{u}(x_i)$$

The results may be checked by comparing the solution with the crisp solution at $\alpha = 1$, and for $x = b = 2$, we have:

$$\underline{y}(2) = \underline{v}(2) + \underline{\lambda}\, \underline{u}(2)$$

$$= 1.95 + 0.075 \times 0.666666 = 2$$

$$\overline{y}(2) = \overline{v}(2) + \overline{\lambda}\, \overline{u}(2)$$

$$= 1.95 + 0.075 \times 0.666666 = 2$$

Clearly $\underline{y}(x)$ and $\overline{y}(x)$ are equal only when $\alpha = 1$.

The results of the calculations using N = 10 and h = 0.1, are given in table (3.3).

Table (3.3).

The lower and upper levels of fuzzy solution for example (3.2.1.1)

x_i	$\underline{y}(x_i)$	$\bar{y}(x_i)$	***Exact solution***
1	1	1	1.000
1.1	1.012	1.012	1.013
1.2	1.121	1.121	1.125
1.3	1.240	1.240	1.248
1.4	1.353	1.353	1.359
1.5	1.458	1.458	1.455
1.6	1.567	1.567	1.563
1.7	1.671	1.671	1.677
1.8	1.785	1.785	1.788
1.9	1.891	1.891	1.893
2	2	2	2.000

3.2.2 The Shooting Method for Solving Non Linear BVP:

The idea behind the shooting technique for solving the nonlinear second order fuzzy boundary value problem:

$$y'' = f(x, y, y'),\ a , x \le b,\ y(a) \sqcup \tilde{\alpha},\ y(b) \sqcup \tilde{\beta} \quad \text{.............................} (3.21)$$

Is similar to that in linear case, except that the solution to a nonlinear problem cannot be simply expressed as a linear combination of the solutions of the two initial value problems. Instead, we will need to utilize the solutions to a sequence of fuzzy initial value problem of the form:

$$y'' = f(x, y, y'),\ a , x \le b,\ y(a) \sqcup \tilde{\alpha},\ y'(a) \sqcup \tilde{t}_k \quad \text{............................} (3.22)$$

where $\underline{y}(x)$, $\overline{y}(x)$ denotes the solution to the BVP (3.21) in lower and upper case and $\underline{y}(x, \underline{\tilde{t}}_k)$, $\overline{y}(x, \overline{\tilde{t}}_k)$ denotes the solution to the IVP (3.22) in lower and upper cases, which involves a parameter, to approximate the solution to our BVP. We do this by choosing the parameters $\tilde{t}_k$ in lower and upper cases as $\underline{\tilde{t}}_k$ and $\overline{\tilde{t}}_k$ in a manner which will ensure that:

$$\lim_{k\to\infty} \underline{y}(b, \underline{\tilde{t}}_k) = \underline{y}(b) = \tilde{\beta} \quad \text{and} \quad \lim_{k\to\infty} \overline{y}(x, \overline{\tilde{t}}_k) = \overline{y}(b) = \tilde{\beta}$$

This method is called the "shooting method" by analogy with the procedure of firing objects at a stationary target.

We start with parameters $\underline{\tilde{t}}_0$ and $\overline{\tilde{t}}_0$ which determines the initial elevation at which the object is fired from the point (a, $\tilde{\alpha}$) and along the curve described by the solution to the IVP:

$$y'' = f(x, y, y'),\ a \le x \le b,\ y(a) \square\ \tilde{\alpha},\ y'(a) \square\ \tilde{t}_0$$

If $\underline{y}(b, \underline{\tilde{t}}_0)$ and $\overline{y}(b, \overline{\tilde{t}}_0)$ is not sufficiently close to $\tilde{\beta}$, we attempt to correct our approximation by choosing another elevations $\underline{\tilde{t}}_1$, $\overline{\tilde{t}}_1$ and so on until $\underline{y}(b, \underline{\tilde{t}}_k)$ and $\overline{y}(b, \overline{\tilde{t}}_k)$ are sufficiently close to the "hitting" $\tilde{\beta}$.

The next theorem may be considered as consequence result to the non-fuzzy case of BVP's:

<u>Theorem (3.2.2.1):</u>

The boundary value problem:

$$y'' = f(x, y, y'),\ a\ ,\ x \le b,\ y(a) \square\ \tilde{\alpha},\ y(b) \square\ \tilde{\beta} \quad \text{.............................} \quad (3.23)$$

and the initial value problem:

$$y'' = f(x, y, y'),\ a\ ,\ x \le b,\ y(a) \square\ \tilde{\alpha},\ y'(a) \square\ \tilde{t} \quad \text{.............................} \quad (3.24)$$

In lower and upper cases and for an arbitrary parameters $\underline{\tilde{t}}$ and $\overline{\tilde{t}}$, will have a unique solutions $\underline{y}$ and $\overline{y}$ on [a, b] provided that for $\underline{y}$ and $\overline{y}$ on:

$$D = \{(x, y, y') \mid a \le x \le b, -\infty < y < \infty, -\infty < y' < \infty\}$$

(i) $f, \dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial y'}$ are continuous.

(ii) A constant M exists with $\left|\dfrac{\partial f}{\partial y'}\right| \le M$.

(iii) A constant L exists with $\dfrac{\partial f}{\partial y} > 0$.

To determine how the parameters $\underline{\tilde{t}}_k$ and $\overline{\tilde{t}}_k$ can be chosen, suppose that we have a boundary value problem of the form (3.23), which satisfies the hypotheses of theorem (3.2.2.1).

If $\underline{y}(x, \underline{\tilde{t}})$ and $\overline{y}(x, \overline{\tilde{t}})$ are used to denote the solution to the IVP (3.24) in lower and upper cases, the question is how to choose $\underline{\tilde{t}}$ and $\overline{\tilde{t}}$, so that:

$$\underline{y}(b, \underline{\tilde{t}}) - \underline{\tilde{\beta}} = 0 \quad \text{and} \quad \overline{y}(b, \overline{\tilde{t}}) - \overline{\tilde{\beta}} = 0 \qquad (3.25)$$

We will carry for simplicity the approach of lower case only, as follows:

If we wish to employ the secant method to solve the problem, we need to choose initial approximations $\underline{\tilde{t}}_0$ and $\underline{\tilde{t}}_1$ and then generate the remaining terms of the sequence by:

$$\underline{\tilde{t}}_k = \underline{\tilde{t}}_{k-1} - \frac{(\underline{y}(b, \underline{\tilde{t}}_{k-1}) - \underline{\tilde{\beta}})(\underline{\tilde{t}}_{k-1} - \underline{\tilde{t}}_{k-2})}{\underline{y}(b, \underline{\tilde{t}}_{k-1}) - \underline{y}(b, \underline{\tilde{t}}_{k-2})}, \; k = 2, 3, \ldots$$

In order to use the more powerful Newton's method to generate the sequence $\{\underline{\tilde{t}}_k\}$, only one initial value, $\underline{\tilde{t}}_0$ is needed. However, the iterations would have the form:

$$\underline{\tilde{t}}_k = \underline{\tilde{t}}_{k-1} - \frac{(\underline{y}(b, \underline{\tilde{t}}_{k-1}) - \tilde{\beta})}{\frac{d\underline{y}}{d\underline{t}}(b, \underline{\tilde{t}}_{k-1})}, \text{ where } \frac{d\underline{y}}{d\underline{t}}(b, \underline{\tilde{t}}_{k-1}) \equiv \frac{d\underline{y}(b, \underline{\tilde{t}}_{k-1})}{d\underline{t}} \quad \ldots (3.26)$$

This present a difficulty, since $\underline{y}(b, \underline{\tilde{t}}_k)$ is known only for $\underline{\tilde{t}} = \underline{\tilde{t}}_0, \underline{\tilde{t}}_1, \ldots, \underline{\tilde{t}}_{k-1}$.

Suppose we rewrite the IVP (3.24), emphasizing that the solution depends on both x and $\underline{\tilde{t}}$:

$$\underline{y}''(x, \underline{\tilde{t}}) = f(x, \underline{y}(x, \underline{\tilde{t}}), \underline{y}'(x, \underline{\tilde{t}})), \ a \le x \le b \quad \ldots (3.27)$$

with boundary conditions:

$$\underline{y}(a, \underline{\tilde{t}}) \cong \tilde{\alpha}, \ \underline{y}'(a, \underline{\tilde{t}}) \cong \underline{\tilde{t}}$$

Since we are interested in determining $\frac{d\underline{y}}{d\underline{\tilde{t}}}(b, \underline{\tilde{t}})$, when $\underline{\tilde{t}} = \underline{\tilde{t}}_{k-1}$, we will first take the partial derivative of eq.(3.27) with respect to $\underline{\tilde{t}}$. This implies that:

$$\begin{aligned}
\frac{\partial \underline{y}''}{\partial \underline{\tilde{t}}}(x, \underline{\tilde{t}}) &= \frac{\partial}{\partial t} f(x, \underline{y}(x, \underline{\tilde{t}}), \underline{y}'(x, \underline{\tilde{t}})) \\
&= \frac{\partial}{\partial x} f(x, \underline{y}(x, \underline{\tilde{t}}), \underline{y}'(x, \underline{\tilde{t}})) \frac{\partial x}{\partial \underline{\tilde{t}}} + \frac{\partial}{\partial \underline{y}} f(x, \underline{y}(x, \underline{\tilde{t}}), \underline{y}'(x, \\
&\quad \underline{\tilde{t}})) \frac{\partial \underline{y}}{\partial \underline{\tilde{t}}}(x, \underline{\tilde{t}}) + \frac{\partial}{\partial \underline{y}'} f(x, \underline{y}(x, \underline{\tilde{t}}), \underline{y}'(x, \underline{\tilde{t}})) \frac{\partial \underline{y}'}{\partial \underline{\tilde{t}}}(x, \underline{\tilde{t}}) \\
&= \frac{\partial}{\partial \underline{y}} f(x, \underline{y}(x, \underline{\tilde{t}}), \underline{y}'(x, \underline{\tilde{t}})) \frac{\partial \underline{y}}{\partial \underline{\tilde{t}}}(x, \underline{\tilde{t}}) + \frac{\partial}{\partial \underline{y}'} f(x, \underline{y}(x, \underline{\tilde{t}}), \\
&\quad \underline{y}'(x, \underline{\tilde{t}})) \frac{\partial \underline{y}'}{\partial \underline{\tilde{t}}}(x, \underline{\tilde{t}})
\end{aligned}$$

for $a \le x \le b$, and the initial conditions give:

$$\frac{\partial \underline{y}}{\partial \underline{\tilde{t}}}(a,\ \underline{\tilde{t}}) = \tilde{0} \quad \text{and} \quad \frac{\partial \underline{y}'}{\partial \underline{\tilde{t}}}(a,\ \underline{\tilde{t}}) = \tilde{1}$$

If we simplify the notation by using $\underline{z}(x,\ \underline{\tilde{t}})$ to denote $\frac{\partial \underline{y}}{\partial \underline{\tilde{t}}}(x,\ \underline{\tilde{t}})$ and assume that the order of differentiation with respect to x and $\underline{\tilde{t}}$ can be reversed, we have the IVP:

$$\underline{z}'' = \frac{\partial f}{\partial \underline{y}}(x,\ \underline{y},\ \underline{y}')\underline{z} + \frac{\partial f}{\partial \underline{y}'}(x,\ \underline{y},\ \underline{y}')\underline{z}',\ a \le x \le b,\ \underline{z}(a) = \tilde{0},\ \underline{z}'(a) = \tilde{1}$$

.... (3.28)

Newton's method would therefore require that two initial value problems be solved for each iteration, since eqs.(3.26) would become:

$$\underline{\tilde{t}}_k = \underline{\tilde{t}}_{k-1} - \frac{\underline{y}(b, \underline{\tilde{t}}_{k-1}) - \tilde{\beta}}{\underline{z}(b, \underline{\tilde{t}}_{k-1})} \quad \text{.. (3.29)}$$

Similar approach may be carried for evaluating $\underline{\tilde{t}}_k$ as in lower case of solution, we have:

$$\overline{\tilde{t}}_k = \overline{\tilde{t}}_{k-1} - \frac{\overline{y}(b, \overline{\tilde{t}}_{k-1}) - \tilde{\beta}}{\overline{z}(b, \overline{\tilde{t}}_{k-1})} \quad \text{.. (3.30)}$$

The following example illustrates the above discussion:

Example (3.2.2.2):

Consider the homogeneous non-linear fuzzy boundary value problem:

$$y'' = 2y^3,\ 1 \le x \le 1.3,\ y(1) = \tilde{1},\ y(1.3) = \widetilde{0.77}$$

and to solve this problem using the shooting method for fuzzy nonlinear BVP's.

Now, the problem requires the following equivalent initial value problems:

$$y'' = 2y^3,\ y(1) = \tilde{1},\ y'(1) = \tilde{t}_0 = \frac{\beta - \alpha}{b - a} = -0.77 \quad \text{.........................} (3.31)$$

$$z'' = 6y^2 z,\ z(1) = \tilde{0},\ z'(1) = \tilde{0} \quad \text{..} (3.32)$$

eq.(3.31) can be written in system form, as:

$$y' = v = F(x, y, v)$$

$$v' = 2y^3 = G(x, y, v)$$

the parametric form of y(x) and v(x) in this case is given by:

$$\underline{y}'(x, \alpha) = \underline{v}(x, \alpha),\ \overline{y}'(x, \alpha) = \overline{v}(x, \alpha)$$

and

$$\underline{v}'(x, \alpha) = 2\,\underline{y}^3(x, \alpha),\ \overline{v}'(x, \alpha) = 2\,\overline{y}^3(x, \alpha)$$

With initial conditions given for all $\alpha \in [0, 1]$ and $\underline{\tilde{t}}_0 = -0.77$ and $\overline{\tilde{t}}_0 = -0.77$.

$$\underline{y}(1, \alpha) = \tilde{1},\ \overline{y}(1, \alpha) = \tilde{1}$$

and

$$\underline{v}(1, \alpha) = -0.\tilde{7}7,\ \overline{v}(1, \alpha) = -0.\tilde{7}7$$

Using Euler's method, we have:

$$\left.\begin{aligned} \underline{y}_{n+1}(\alpha) &= \underline{y}_n(\alpha) + hF_1(x_n, \underline{y}_n(\alpha), \underline{v}_n(\alpha)) \\ \overline{y}_{n+1}(\alpha) &= \overline{y}_n(\alpha) + hF_2(x_n, \overline{y}_n(\alpha), \overline{v}_n(\alpha)) \end{aligned}\right\} \quad \text{............................} (3.33)$$

and

$$\left.\begin{aligned} \underline{v}_{n+1}(\alpha) &= \underline{v}_n(\alpha) + hG_1(x_n, \underline{y}_n(\alpha), \underline{v}_n(\alpha)) \\ \overline{v}_{n+1}(\alpha) &= \overline{v}_n(\alpha) + hG_2(x_n, \overline{y}_n(\alpha), \overline{v}_n(\alpha)) \end{aligned}\right\} \quad \text{............................} (3.34)$$

Applying F_1, F_2, G_1 and G_2 in eqs.(3.33) and (3.34) yields:

$$\left.\begin{aligned} \underline{y}_{n+1}(\alpha) &= \underline{y}_n(\alpha) + h\,\underline{v}_n(\alpha) \\ \overline{y}_{n+1}(\alpha) &= \overline{y}_n(\alpha) + h\,\overline{v}_n(\alpha) \end{aligned}\right\} \quad \ldots\ldots\ldots (3.35)$$

and

$$\left.\begin{aligned} \underline{v}_{n+1}(\alpha) &= \underline{v}_n(\alpha) + 2h\,\underline{y}_n^3(\alpha) \\ \overline{v}_{n+1}(\alpha) &= \overline{v}_n(\alpha) + 2h\,\overline{y}_n^3(\alpha) \end{aligned}\right\} \quad \ldots\ldots\ldots (3.36)$$

Hence, by using the first problem of eq.(3.35) and the first problem of eq.(3.36), we get the following results which are given in table (3.4), for all $x \in [1, 1.3]$ and $h = 0.1$, $\alpha = 1$ with $\underline{\tilde{t}}_0 = -0.\tilde{7}7$.

Table (3.4).

The lower level of fuzzy solution of eqs.(3.35) and (3.36)

x_n	$\underline{y}_n(a)$	$\underline{v}_n(a)$
1	1	–0.77
1.1	0.923	–0.57
1.2	0.866	–0.41
1.3	0.825	–0.28

Since $\underline{y}(1.3, \underline{\tilde{t}}_0) = 0.825$, and by using the second problem of eq.(3.35) and the second problem of eq.(3.36), we get the following results which are given in table (3.5) for all $x \in [1, 1.3]$ and for $h = 0,1$, $\alpha = 1$ with $\overline{\tilde{t}}_0 = -0.\tilde{7}7$.

Table (3.5)

The upper level of fuzzy solution of eqs.(3.35) and (3.36)

x_n	$\overline{y}_n(a)$	$\overline{v}_n(a)$
1	1	–0.77
1.1	0.923	–0.57
1.2	0.866	–0.41
1.3	0.825	–0.28

Since $\overline{y}(1.3, \tilde{0}) = 0.825$. Also, eq.(3.32) can be written as:

$$z' = v = H(x, z, v)$$

$$v' = 6y^2 z = B(x, z, v)$$

The parametric form z(x) and v(x) in this case is given by:

$$\underline{z}'(x, \alpha) = \underline{v}(x, \alpha),\ \overline{z}'(x, \alpha) = \overline{v}(x, \alpha)$$

and

$$\underline{v}(x, \alpha) = 6\,\underline{y}^2(x, \alpha)\,\underline{z}(x, \alpha),\ \overline{v}'(x, \alpha) = 6\,\overline{y}^2(x, \alpha)\,\overline{z}(x, \alpha)$$

with initial conditions are given for all $\alpha \in [0, 1]$, by:

$$\underline{z}(1, \alpha) = \tilde{0},\ \overline{z}(1, \alpha) = \tilde{0}$$

and

$$\underline{v}(1, \alpha) = \tilde{1},\ \overline{v}(1, \alpha) = \tilde{1}$$

And similarly, using Euler's method for solving the linear system, we have:

$$\left.\begin{aligned} \underline{z}_{n+1}(\alpha) &= \underline{z}_n(\alpha) + h\,\underline{v}_n(\alpha) \\ \overline{z}_{n+1}(\alpha) &= \overline{z}_n(\alpha) + h\,\overline{v}_n(\alpha) \end{aligned}\right\} \qquad \text{.. (3.37)}$$

and

$$\left.\begin{aligned}\underline{v}_{n+1}(\alpha) &= \underline{v}_n(\alpha) + 6h\,\underline{y}_n^2\,\underline{z}_n\\ \overline{v}_{n+1}(\alpha) &= \overline{v}_n(\alpha) + 6h\,\overline{y}_n^2\,\overline{z}_n\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.38)$$

Hence, by using the first problem of eq.(3.37) and the first problem of eq.(3.38), we get the following results which are given in table (3.6) for all $x \in [1, 1.3]$ and for $h = 0.1$, $\alpha = 1$ with $\underline{\tilde{t}}_0 = -0.\tilde{7}7$.

Table (3.6)

The lower level of fuzzy solution of eqs.(3.37) and (3.38)

x_n	$\underline{z}_n(a)$	$\underline{v}_n(a)$
1	0	1
1.1	0.1	1
1.2	0.2	1.05
1.3	0.3	1.13

Since $\underline{z}(1.3, \underline{\tilde{t}}_0) = 0.3$, and by using the second problem of eq.(3.37) and the second problem of eq.(3.38), we get the following table of results which are given in table (3.7) for all $x \in [1, 1.3]$ and for $h = 0.1$, $\alpha = 1$ with $\overline{\tilde{t}}_0 = -0.\tilde{7}7$

Table (3.7)

The upper level of fuzzy solution of eqs.(3.37) and (3.38)

x_n	$\overline{z}_n(a)$	$\overline{v}_n(a)$
1	0	1
1.1	0.1	1
1.2	0.2	1.05
1.3	0.3	1.13

Since $\overline{z}(1.3, \overline{\tilde{t}}_0) = 0.3$.

Now, we can evaluate $\underline{\tilde{t}}_1$ and $\overline{\tilde{t}}_1$ by using eqs.(3.29) and (3.30), we get:

$$\underline{\tilde{t}}_1 = \underline{\tilde{t}}_0 - \frac{\underline{y}(1.3, \underline{\tilde{t}}_0) - \tilde{\beta}}{\underline{z}(1.3, \underline{\tilde{t}}_0)} = -\tilde{0.9}$$

$$\overline{\tilde{t}}_1 = \overline{\tilde{t}}_0 - \frac{\overline{y}(1.3, \overline{\tilde{t}}_0) - \tilde{\beta}}{\overline{z}(1.3, \overline{\tilde{t}}_0)} = -\tilde{0.9}$$

Now, the initial condition of $\underline{\tilde{t}}_1 = -0.9$ and $\overline{\tilde{t}}_1 = -0.9$ are given for all $\alpha \in [0, 1]$, by:

$$\underline{y}(1, \alpha) = \tilde{1}, \quad \overline{y}(1, \alpha) = \tilde{1}$$

and

$$\underline{v}(1, \alpha) = -\tilde{0.9}, \quad \overline{v}(1, \alpha) = -\tilde{0.9}$$

Hence, using the first problem of eq.(3.35) and the first problem of eq.(3.36), we get the following results which are given in table (3.8) for all $x \in [1, 1.3]$ and for $h = 0.1$, $\alpha = 1$ with $\underline{\tilde{t}}_1 = -\tilde{0.9}$.

Table (3.8).

The lower level of fuzzy solution of eqs.(3.35) and (3.36)

x_n	$\underline{y}_n(\alpha)$	$\underline{v}_n(\alpha)$
1	1	–0.95
1.1	0.905	–0.75
1.2	0.830	–0.60
1.3	0.770	–0.48

Since $\underline{y}(1.3,\ \tilde{\underline{t}}_1) = 0.77 = \tilde{\beta}$, and by using the second problem of eq.(3.35) and second problem of eq. (3.36), we get the following result which are given in table (3.9) for all $x \in [1, 1.3]$ and for $\alpha = 1$ with $\bar{\tilde{t}}_1 = -0.9$.

Table (3.9)

The upper level of fuzzy solution of eqs.(3.35) and (3.36)

x_n	$\bar{y}_n(a)$	$\bar{v}_n(a)$
1	1	–0.95
1.1	0.905	–0.75
1.2	0.830	–0.60
1.3	0.770	–0.48

Since $\bar{y}(1.3,\ \bar{\tilde{t}}_1) = 0.77 = \tilde{\beta}$.

The solution requires one iteration and $t_1 = -0.9$. The results obtained for this value of t are shown in table (3.10).

Table (3.10).

The lower and upper levels of fuzzy solution for example (3.2.2.2)

x_n	$\underline{y}(a)$	$\bar{y}(a)$
1	1	1
1.1	0.905	0.905
1.2	0.830	0.830
1.3	0.770	0.770

3.2.3 Finite Difference Method for Solving Fuzzy Linear Boundary Value Problems:

The methods we will present here have better stability characteristics, but generally require more computations to obtain a pre-specified accuracy.

Methods involving finite differences for solving boundary value problems consist of replacing each of the derivatives in the differential equation by an appropriate difference approximation. The difference equation is generally chosen so that a certain order of truncation error is maintained.

The finite difference method for the linear second order fuzzy BVP:

$$y'' = p(x)y' + q(x)y + r(x),\ a \le x \le b \qquad (3.39)$$

$$y(a) \square \tilde{\alpha},\ y(b) \square \tilde{\beta}$$

requires that the difference approximations be used for approximating both y' and y''. To accomplish this, we select an integer $N > 0$ and divide the interval $[a, b]$ into $N + 1$ equal subintervals, whose end points are the mesh points $x_i = a + ih$, for $i = 0, 1, \ldots, N + 1$; where $h = \dfrac{b - a}{N + 1}$. Choosing the constant h in this manner will facilitate the application of a matrix algorithm, which in this form will require of solving a linear system that involves $N \times N$ matrix.

In this case, the parametric equations related to eq.(3.39) are given by:

$$\underline{y}'' = p(x)\underline{y}' + q(x)\underline{y} + r(x),\ a \le x \le b$$

With boundary conditions:

$$\underline{y}_{\alpha}(a) = \tilde{a} - \sqrt{1-\alpha} \quad \text{and} \quad \underline{y}_{\alpha}(b) = \tilde{b} - \sqrt{1-\alpha}$$

and

$$\overline{y}'' = p(x)\overline{y}' + q(x)\overline{y} + r(x),\ a \le x \le b$$

with boundary conditions:

$$\overline{y}_{\alpha}(a) = \tilde{a} + \sqrt{1-\alpha} \quad \text{and} \quad \overline{y}_{\alpha}(b) = \tilde{b} + \sqrt{1-\alpha}$$

As in the previous work, we will carry the discussion for lower case problem and we notify that similar approach is carried in upper case.

$$\underline{y}''(x_i) = p(x_i)\,\underline{y}'(x_i) + q(x_i)\,\underline{y}(x_i) + r(x_i) \quad \text{.......................................} \quad (3.40)$$

Expanding y in a third-degree Taylor polynomial about x_i evaluated at x_{i+1} and x_{i-1}, and assuming $\underline{y} \in C^4[x_{i-1}, x_{i+1}]$, we have:

$$\underline{y}(x_{i+1}) = \underline{y}(x_i + h)$$

$$= \underline{y}(x_i) + h\,\underline{y}'(x_i) + \frac{h^2}{2}\,\underline{y}''(x_i) + \frac{h^3}{6}\,\underline{y}'''(x_i) + \frac{h^4}{24}\,\underline{y}''''(\xi_i^{+})$$

for some point ξ_i^{+}, $x_i < \xi_i^{+} < x_{i+1}$, and:

$$\underline{y}(x_{i-1}) = \underline{y}(x_i - h)$$

$$= \underline{y}(x_i) - h\,\underline{y}'(x_i) + \frac{h^2}{2}\,\underline{y}''(x_i) - \frac{h^3}{6}\,\underline{y}'''(x_i) + \frac{h^4}{24}\,\underline{y}''''(\xi_i^{-})$$

for some point ξ_i^{-}, $x_{i-1} < \xi_i^{-} < x_i$.

If these equations are added together, the terms involving $\underline{y}'(x_i)$ and $\underline{y}'''(x_i)$ are eliminated, and a simple algebraic manipulation gives:

$$\underline{y}''(x_i) = \frac{1}{h^2}[\underline{y}(x_{i+1}) - 2\,\underline{y}(x_i) + \underline{y}(x_{i-1})] - \frac{h^2}{24}[\underline{y}^{(4)}(\xi_i^{+}) + \underline{y}^{(4)}(\xi_i^{-})]$$

The intermediate value theorem can be used to simplify the last equation even further:

$$\underline{y}''(x_i) = \frac{1}{h^2}[\underline{y}(x_{i+1}) - 2\,\underline{y}(x_i) + \underline{y}(x_{i-1})] - \frac{h^2}{12}\,\underline{y}^{(4)}(\xi_i) \quad \text{................} \quad (3.41)$$

for some point ξ_i, $x_{i-1} < \xi_i < x_{i+1}$.

Equation (3.41) is called the central difference formula for $\underline{y}''(x_i)$.

A central-difference formula for $\underline{y}'(x_i)$ can be obtained in a similar manner, resulting in:

$$\underline{y}'(x_i) = \frac{1}{2h}[\underline{y}(x_{i+1}) - \underline{y}(x_{i-1})] - \frac{h^2}{6}\underline{y}'''(\eta_i)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.42)$$

for some η_i, where $x_{i-1} < \eta_i < x_{i+1}$.

The use of these central-difference formulas in eq.(3.40) results in the equation:

$$\frac{1}{h^2}[\underline{y}(x_{i+1}) - 2\underline{y}(x_i) + \underline{y}(x_{i-1})] = p(x_i)\frac{1}{2h}[\underline{y}(x_{i+1}) - \underline{y}(x_{i-1})] + q(x_i)$$

$$\underline{y}(x_i) + r(x_i) - \frac{h^2}{12}[2p(x_i)\ \underline{y}'''(\eta_i) - \underline{y}^{(4)}(\xi_i)]$$

A finite difference method wit truncation error of order $o(h^2)$ results by using this equation together with the boundary conditions $\underline{y}(a) = \tilde{\alpha} - \sqrt{1-\alpha}$ and $\underline{y}(b) = \tilde{\beta} - \sqrt{1-\alpha}$, to define:

$$\underline{y}_0 = \tilde{\alpha} - \sqrt{1-\alpha} \text{ and } \underline{y}_{N+1} = \tilde{\beta} - \sqrt{1-\alpha}$$

and

$$\frac{1}{h^2}[2\underline{y}_i - \underline{y}_{i+1} - \underline{y}_{i-1}] + \frac{1}{2h}p(x_i)[\underline{y}_{i+1} - \underline{y}_{i-1}] + q(x_i)\ \underline{y}_i = -r(x_i)\ ..\ (3.43)$$

for each i = 1, 2, …, N.

In the form we will consider eq.(3.43) is rewritten as:

$$-\left(1 + \frac{h}{2}p(x_i)\right)\underline{y}_{i-1} + \left(2 + h^2 q(x_i)\right)\underline{y}_i - \left(1 - \frac{h}{2}p(x_i)\right)\underline{y}_{i+1} = -h^2 r(x_i)$$

and the resulting system of algebraic equations is expressed in the tridiagonal $N \times N$ matrix form:

$$A\,\underline{y} = b \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.44)$$

where:

$$A = \begin{bmatrix} 2+h^2q(x_1) & -1+\frac{h}{2}p(x_1) & 0 & \cdots & \cdots & 0 \\ -1-\frac{h}{2}p(x_2) & 2+h^2q(x_2) & -1+\frac{h}{2}p(x_2) & \ddots & & \vdots \\ 0 & -1-\frac{h}{2}p(x_i) & 2+h^2q(x_i) & -1+\frac{h}{2}p(xi) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -1+\frac{h}{2}p(x_{n-1}) \\ 0 & \cdots & \cdots & 0 & -1-\frac{h}{2}p(x_n) & 2+h^2q(x_n) \end{bmatrix}$$

$$\underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_i \\ \vdots \\ \underline{y}_{N-1} \\ \underline{y}_N \end{bmatrix}, \quad b = \begin{bmatrix} -h^2r(x_1) + \left(1+\frac{h}{2}p(x_1)\right)\underline{y}_0 \\ -h^2r(x_2) \\ \vdots \\ -h^2r(x_i) \\ \vdots \\ -h^2r(x_{N-1}) \\ -h^2r(x_N) + \left(1-\frac{h}{2}p(x_N)\right)\underline{y}_{N+1} \end{bmatrix}$$

Example (3.2.3.1):

Consider the fuzzy BVP:

$$y'' = \frac{-2}{x}y' + \frac{2}{x^2}y + x^2, \; 1 \le x \le 2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.45)$$

$$y(1) = \tilde{1}, \; y(2) = \tilde{2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.46)$$

For this example, we will use N = 9, so that h = 0.1. In this case, the parametric equations related to eqs.(3.45) and (3.46), are:

$$\underline{y}''_\alpha(x) = \frac{-2}{x}\underline{y}'_\alpha(x) + \frac{2}{x^2}\underline{y}_\alpha(x) + x^2,\ x \in [1, 2]$$

With boundary conditions:

$$\underline{y}_\alpha(1) = 1 - \sqrt{1-\alpha} \quad \text{and} \quad \underline{y}_\alpha(2) = 2 - \sqrt{1-\alpha}$$

which is a non fuzzy BVP.

By using eq.(3.42), we get:

$$\frac{\underline{y}(x_{i+1}) - 2\underline{y}(x_i) + \underline{y}(x_{i-1})}{h^2} = \frac{-2}{x_i}\left[\frac{\underline{y}(x_{i+1}) - \underline{y}(x_{i-1})}{2h}\right] + \frac{2}{x_i^2}\underline{y}(x_i) + x_i^2$$

Now, define:

$$\underline{y}_0(1) = 1 - \sqrt{1-\alpha} \quad \text{and} \quad \underline{y}_{10}(2) = 2 - \sqrt{1-\alpha}$$

Hence:

$$\frac{2\underline{y}_i - \underline{y}_{i+1} - \underline{y}_{i-1}}{h^2} - \frac{2}{x_i}\left[\frac{\underline{y}_{i+1} - \underline{y}_{i-1}}{2h}\right] + \frac{2}{x_i^2}\underline{y}_i = -x_i^2 \quad \text{..................} \ (3.47)$$

For each i = 1, 2, …, 9.

In this form, we will consider eq.(3.47) which can be written as:

$$-\left(1 - \frac{h}{x_i}\right)\underline{y}_{i-1} + \left(2 + \frac{2}{x_i^2}h^2\right)\underline{y}_i - \left(1 + \frac{h}{x_i}\right)\underline{y}_{i+1} = -x_i^2 h^2$$

and the resulting system of algebraic equations may be expressed in the triadiagonal 9×9 matrix form:

$$A\,\underline{y} = b \quad \text{...} \ (3.48)$$

Where:

$$A=\begin{bmatrix}
2+\frac{2}{x_1^2}h^2 & -1-\frac{h}{x_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1+\frac{h}{x_2} & 2+\frac{2}{x_2^2}h^2 & -1-\frac{h}{x_2} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1+\frac{h}{x_3} & 2+\frac{2}{x_3^2}h^2 & -1-\frac{h}{x_3} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1+\frac{h}{x_4} & 2+\frac{2}{x_4^2}h^2 & -1-\frac{h}{x_4} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1+\frac{h}{x_5} & 2+\frac{2}{x_5^2}h^2 & -1-\frac{h}{x_5} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1+\frac{h}{x_6} & 2+\frac{2}{x_6^2}h^2 & -1-\frac{h}{x_6} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1+\frac{h}{x_7} & 2+\frac{2}{x_7^2}h^2 & -1-\frac{h}{x_7} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1+\frac{h}{x_8} & 2+\frac{2}{x_8^2}h^2 & -1-\frac{h}{x_8}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+\frac{h}{x_9} & 2+\frac{2}{x_9^2}h^2
\end{bmatrix}$$

$$\underline{y}=\begin{bmatrix}\underline{y}_1\\ \underline{y}_2\\ \underline{y}_3\\ \underline{y}_4\\ \underline{y}_5\\ \underline{y}_6\\ \underline{y}_7\\ \underline{y}_8\\ \underline{y}_9\end{bmatrix},\ b=\begin{bmatrix}-h^2x_1^2+\left(1-\frac{h}{x_1}\right)\left(1-\sqrt{1-\alpha}\right)\\ -h^2x_2^2\\ -h^2x_3^2\\ -h^2x_4^2\\ -h^2x_5^2\\ -h^2x_6^2\\ -h^2x_7^2\\ -h^2x_8^2\\ -h^2x_9^2+\left(1+\frac{h}{x_9}\right)\left(2-\sqrt{1-\alpha}\right)\end{bmatrix}$$

We can carry similar calculations as it is followed for lower case of solution and find the upper case of solution from the following system of algebraic equations:

$$A\,\overline{y}=b \qquad (3.49)$$

Where

$$A=\begin{bmatrix}
2+\frac{2}{x_1^2}h^2 & -1-\frac{h}{x_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1+\frac{h}{x_2} & 2+\frac{2}{x_2^2}h^2 & -1-\frac{h}{x_2} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1+\frac{h}{x_3} & 2+\frac{2}{x_3^2}h^2 & -1-\frac{h}{x_3} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1+\frac{h}{x_4} & 2+\frac{2}{x_4^2}h^2 & -1-\frac{h}{x_4} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1+\frac{h}{x_5} & 2+\frac{2}{x_5^2}h^2 & -1-\frac{h}{x_5} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1+\frac{h}{x_6} & 2+\frac{2}{x_6^2}h^2 & -1-\frac{h}{x_6} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1+\frac{h}{x_7} & 2+\frac{2}{x_7^2}h^2 & -1-\frac{h}{x_7} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1+\frac{h}{x_8} & 2+\frac{2}{x_8^2}h^2 & -1-\frac{h}{x_8}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+\frac{h}{x_9} & 2+\frac{2}{x_9^2}h^2
\end{bmatrix}$$

$$\underline{y}=\begin{bmatrix}\overline{y}_1\\ \overline{y}_2\\ \overline{y}_3\\ \overline{y}_4\\ \overline{y}_5\\ \overline{y}_6\\ \overline{y}_7\\ \overline{y}_8\\ \overline{y}_9\end{bmatrix},\ b=\begin{bmatrix}-h^2x_1^2+\left(1-\frac{h}{x_1}\right)\left(1+\sqrt{1-\alpha}\right)\\ -h^2x_2^2\\ -h^2x_3^2\\ -h^2x_4^2\\ -h^2x_5^2\\ -h^2x_6^2\\ -h^2x_7^2\\ -h^2x_8^2\\ -h^2x_9^2+\left(1+\frac{h}{x_9}\right)\left(2+\sqrt{1-\alpha}\right)\end{bmatrix}$$

Where α is a parameter between 0 and 1, and when $\alpha = 1$, the results are given in table (3.11) which represented the crisp solution

Table (3.11)

The lower and upper levels of fuzzy solution for example (3.2.3.1)

x_i	$\underline{y}(x_i)$	$\bar{y}(x_i)$
1	1	1
1.1	1.012	1.012
1.2	1.127	1.127
1.3	1.244	1.244
1.4	1.3747	1.3747
1.5	1.455	1.455
1.6	1.561	1.561
1.7	1.673	1.673
1.8	1.781	1.781
1.9	1.895	1.895
2	2	2

Hence $y_\alpha = [\underline{y}_\alpha, \bar{y}_\alpha]$ is the solution of the original fuzzy BVP.

3.2.4 Finite Difference Method for Solving Fuzzy Non Linear Boundary Value Problem:

For the general non linear fuzzy BVP:

$y'' = f(x, y, y'), \; a \le x \le b$ .... (3.50)

$y(a) \square \tilde{\alpha}, \; y(b) \square \tilde{\beta}$

the difference method is similar to the method applied to linear problem. Here, however, the system of equations which are derived will not be linear, so an iteration process is required to solve the problem.

For the development of the procedure, we will assume throughout that f satisfies the following conditions:

(i) f and the partial derivatives f_y and $f_{y'}$ are all continuous on:

$$D = \{(x, y, y') \mid a , x , b, -\infty < y < \infty, -\infty < y' < \infty\}$$

(ii) $f_y(x, y, y') \geq \delta > 0$ on D for some $\delta > 0$.

(iii) constants K and L exist, with:

$$K = \max_{(x,y,y')\in D}\left|\frac{\partial f}{\partial y}(x,y,y')\right|,\ L = \max_{(x,y,y')\in D}\left|\frac{\partial f}{\partial y'}(x,y,y')\right|$$

As in the linear case, we dived [a, b] into N + 1 equal subintervals se endpoints are at the mesh points $x_i = a + ih$, for $i = 0, 1, \ldots, N + 1$. Assuming that the exact solution has a bounded fourth derivative allows us to replace $y''(x_i)$ and $y'(x_i)$ in each of the equations. In this case, the parametric equations related to eq.(3.50) are given by:

$$\underline{y}'' = f(x, \underline{y}, \underline{y}'),\ a \leq x \leq b$$

with boundary conditions:

$$\underline{y}_{\alpha}(a) = \alpha - \sqrt{1-\alpha} \quad \text{and} \quad \underline{y}_{\alpha}(b) = \beta - \sqrt{1-\alpha}$$

and

$$\overline{y}'' = f(x, \overline{y}, \overline{y}'),\ a \leq x \leq b$$

with boundary conditions:

$$\overline{y}_{\alpha}(a) = \alpha + \sqrt{1-\alpha} \quad \text{and} \quad \overline{y}_{\alpha}(b) = \beta + \sqrt{1-\alpha}$$

As in the previous work, we will carry the discussion for lower case problem an we notify that similar approach i carried in upper case.

$$\underline{y}''(x_i) = f(x_i, \underline{y}(x_i), \underline{y}'(x_i))$$

By the appropriate centered-difference formula given in eqs.(3.41) and (3.42), to obtain, for each i = 1, 2, …, N.

$$\frac{\underline{y}(x_{i+1}) - 2\underline{y}(x_i) + \underline{y}(x_{i-1})}{h^2} = f\left(x_i, \underline{y}(x_i), \frac{\underline{y}(x_{i+1}) - \underline{y}(x_{i-1})}{2h} - \frac{h^2}{6}\underline{y}'''(\eta_i)\right)$$

$$+ \frac{h^2}{12} y^{(4)}(\xi i) \quad \ldots\ldots (3.51)$$

for some points ξ_i, η_i in the interval (x_{i-1}, x_{i+1}).

As in the linear case, the difference method results when the error terms are deleted and the boundary conditions employed

$$\underline{y}_0 = \tilde{\alpha}, \ \underline{y}_{N+1} = \tilde{\beta}$$

and

$$-\frac{\underline{y}_{i+1} - 2\underline{y}_i + \underline{y}_{i-1}}{h^2} + f\left(x_i, \underline{y}_i, \frac{\underline{y}_{i+1} - \underline{y}_{i-1}}{2h}\right) = 0$$

for each i = 1, 2, …, N.

The N×N non linear system obtained from this method

$$\left.\begin{array}{l}
2\underline{y}_1 - \underline{y}_2 + h^2 f\left(x_1, \underline{y}_1, \dfrac{\underline{y}_2 - \tilde{\alpha}}{2h}\right) - \tilde{\alpha} = 0 \\[2ex]
-\underline{y}_1 + 2\underline{y}_2 - \underline{y}_3 + h^2 f\left(x_2, \underline{y}_2, \dfrac{\underline{y}_3 - \underline{y}_1}{2h}\right) = 0 \\[2ex]
\quad \vdots \\[2ex]
-\underline{y}_{N-2} + 2\underline{y}_{N-1} - \underline{y}_N + h^2 f\left(x_{N-1}, \underline{y}_{N-1}, \dfrac{\underline{y}_N - \underline{y}_{N-2}}{2h}\right) = 0 \\[2ex]
-\underline{y}_{N-2} + 2\underline{y}_N + h^2 f\left(x_N, \underline{y}_N, \dfrac{\tilde{\beta} - \underline{y}_{N-1}}{2h}\right) - \tilde{\beta} = 0
\end{array}\right\} \ldots (3.\ 52)$$

will have a unique solution, provided that h ≤ 2/L.

To approximate the solution to this system, we will use Newton's method for nonlinear system. A sequence of iterates $\{(\underline{y}_1^{(k)}, \underline{y}_2^{(k)}, \ldots, \underline{y}_N^{(k)})\}$ is generated which will converge to the solution of (3.52), provided that the initial approximation $\{(\underline{y}_1^{(0)}, \underline{y}_2^{(0)}, \ldots, \underline{y}_N^{(0)})\}$ is sufficiently close to the solution $\{(\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_N)\}$ and that the Jacobian matrix for the system is nonsingular defined by:

$$J(x) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}$$

However, for the system (3.52), the Jacobian matrix $J(\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_n)$ is triadiagonal.

$$J(\underline{y}_1,\ldots,\underline{y}_N) = \begin{bmatrix} 2+h^2 f_{\underline{y}}\left(x_1, \underline{y}_1, \dfrac{\underline{y}_2-\tilde{\alpha}}{2h}\right) & -1+\dfrac{h}{2} f_{\underline{y}'}\left(x_1, \underline{y}_1, \dfrac{\underline{y}_2-\tilde{\alpha}}{2h}\right) & 0 & \cdots & 0 \\ -1-\dfrac{h}{2} f_{\underline{y}'}\left(x_2, \underline{y}_2, \dfrac{\underline{y}_3-\underline{y}_1}{2h}\right) & 2+h^2 f_{\underline{y}}\left(x_2, \underline{y}_2, \dfrac{\underline{y}_3-\underline{y}_1}{2h}\right) & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1+\dfrac{h}{2} f_{\underline{y}'}\left(x_{N-1}, \underline{y}_{N-1}, \dfrac{\underline{y}_N-\underline{y}_{N-2}}{2h}\right) \\ 0 & \cdots & 0 & -1-\dfrac{h}{2} f_{\underline{y}'}\left(x_N, \underline{y}_N, \dfrac{\tilde{\beta}-\underline{y}_{N-1}}{2h}\right) & 2+h^2 f_{\underline{y}}\left(x_N, \underline{y}_N, \dfrac{\tilde{\beta}-\underline{y}_{N-1}}{2h}\right) \end{bmatrix}$$

Newton's method for non linear system requires that at each iteration, the N×N linear system:

$$J(\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_N)(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_N)^t = -\Bigg(2\underline{y}_1 - \underline{y}_2 - \tilde{\alpha} + h^2 f\left(x_1, \underline{y}_1, \frac{\underline{y}_2-\tilde{\alpha}}{2h}\right), -\underline{y}_1 + 2\underline{y}_2 - \underline{y}_3 + h^2 f\left(x_2, \underline{y}_2, \frac{\underline{y}_3-\underline{y}_1}{2h}\right), \ldots, -\underline{y}_{N-2} + 2\underline{y}_{N-1} - \underline{y}_N + h^2 f\left(x_{N-1}, \underline{y}_{N-1}, \frac{\underline{y}_N-\underline{y}_{N-2}}{2h}\right), -\underline{y}_{N-1} + 2\underline{y}_N + h^2 f\left(x_N, \underline{y}_N, \frac{\tilde{\beta}-\underline{y}_{N-1}}{2h}\right) - \tilde{\beta}\Bigg)^t$$

Solving for $\underline{v}_1$, $\underline{v}_2$, …, $\underline{v}_N$, since:

$$\underline{y}_i^{(k)} = \underline{y}_i^{(k-1)} + \underline{v}_i^{(k-1)}$$

for each i = 1, 2, …, N.

<u>Example (3.2.4.1):</u>

Consider the fuzzy BVP:

$$y'' = 2y^3,\ 1 \le x \le 3 \qquad (3.53)$$

$$y(1) = \tilde{1},\ y(1.3) = \widetilde{0.77} \qquad (3.54)$$

For this example we will use N = 4, so that h = 0.1. In this case, the parametric equations related to eqs.(3.53) and (3.54).

$$\underline{y}''_\alpha(x) = 2\,\underline{y}^3_\alpha(x),\ x \in [1, 1.3]$$

with boundary conditions:

$$\underline{y}_\alpha(1) = 1 - \sqrt{1-\alpha} \quad \text{and} \quad \underline{y}_\alpha(1.3) = 0.77 - \sqrt{1-\alpha}$$

which is a non fuzzy BVP.

By using eq.(3.51), we get:

$$\frac{\underline{y}_{i+1} - 2\underline{y}_i + \underline{y}_{i-1}}{h^2} = 2\,\underline{y}^3_i$$

We define:

$$\underline{y}_0(1) = 1 - \sqrt{1-\alpha} \quad \text{and} \quad \underline{y}_5(1.3) = 0.77 - \sqrt{1-\alpha}$$

Hence:

$$-\frac{\underline{y}_{i+1} - 2\underline{y}_i + \underline{y}_{i-1}}{h^2} + 2\,\underline{y}^3_i = 0,\ i = 1, 2, 3, 4 \qquad (3.55)$$

for each i = 1, 2, 3, 4.

In this form, we will consider eqs.(3.55) which can be rewritten as:

$$-\underline{y}_{i+1} + 2\underline{y}_i - \underline{y}_{i-1} + 2h^2 \underline{y}^3_i = 0$$

The 4×4 non linear system obtained from this method:

$$\begin{aligned} &2\underline{y}_1 - \underline{y}_2 + 2h^2 \underline{y}^3_1 - (1 - \sqrt{1-\alpha}) = 0 \\ &-\underline{y}_1 + 2\underline{y}_2 - \underline{y}_3 + 2h^2 \underline{y}^3_2 = 0 \\ &-\underline{y}_2 + 2\underline{y}_3 - \underline{y}_4 + 2h^2 \underline{y}^3_3 = 0 \\ &-\underline{y}_3 + 2\underline{y}_4 + 2h^2 \underline{y}^3_4 - (0.77 - \sqrt{1-\alpha}) = 0 \end{aligned} \qquad \text{...............................} \ (3.56)$$

Newton's method has been used to approximate the solution when the initial approximation is:

$$\underline{y}^{(0)} = (1, 1, 1, 1)$$

The Jacobin matrix $J(\underline{y})$ for the system (3.56) is given by:

$$J(\underline{y}_1, \underline{y}_2, \underline{y}_3, \underline{y}_4) = \begin{bmatrix} 2+6h^2\underline{y}_1^2 & -1 & 0 & 0 \\ -1 & 2+6h^2\underline{y}_2^2 & -1 & 0 \\ 0 & -1 & 2+6h^2\underline{y}_3^2 & -1 \\ 0 & 0 & -1 & 2+6h^2\underline{y}_4^2 \end{bmatrix}$$

Thus, at the k-th step, the 4×4 linear system:

$$\begin{bmatrix} 2+6h^2\left(\underline{y}_1^{(k-1)}\right)^2 & -1 & 0 & 0 \\ -1 & 2+6h^2\left(\underline{y}_2^{(k-1)}\right)^2 & -1 & 0 \\ 0 & -1 & 2+6h^2\left(\underline{y}_3^{(k-1)}\right)^2 & -1 \\ 0 & 0 & -1 & 2+6h^2\left(\underline{y}_4^{(k-1)}\right)^2 \end{bmatrix}$$

$$\begin{bmatrix} \underline{v}_1^{(k-1)} \\ \underline{v}_2^{(k-1)} \\ \underline{v}_3^{(k-1)} \\ \underline{v}_4^{(k-1)} \end{bmatrix} = -\begin{bmatrix} 2\underline{y}_1^{(k-1)} - \underline{y}_2^{(k-1)} + 2h^2(\underline{y}_1^{(k-1)})^3 - (1-\sqrt{1-\alpha}) \\ -\underline{y}_1^{(k-1)} + 2\underline{y}_2^{(k-1)} - \underline{y}_3^{(k-1)} + 2h^2(\underline{y}_2^{(k-1)})^3 \\ -\underline{y}_2^{(k-1)} + 2\underline{y}_3^{(k-1)} - \underline{y}_4^{(k-1)} + 2h^2(\underline{y}_3^{(k-1)})^3 \\ -\underline{y}_3^{(k-1)} + 2\underline{y}_4^{(k-1)} + 2h^2(\underline{y}_4^{(k-1)})^3 - (0.77-\sqrt{1-\alpha}) \end{bmatrix}$$

Be solved or each k = 1, 2, …; since:

$$\underline{y}_i^{(k)} = \underline{y}_i^{(k-1)} + \underline{v}_i^{(k-1)}$$

for each i = 1, 2, 3, 4; where α is a parameter between 0 and 1, when $\alpha = 1$; the results are presented in table (3.12).

Table (3.12)

The lower level of fuzzy solution for example (3.2.4.1)

x_i	$\underline{y}_1$	$\underline{y}_2$	$\underline{y}_3$	$\underline{y}_4$
1	1	1	1	1
1.1	0.971	0.968	0.968	0.982
1.2	0.894	0.803	0.821	0.844
1.3	0.770	0.780	0.730	0.710

We can carry similar calculations as it is followed for lower case of solution an fin the upper solution from the following system:

$$\begin{bmatrix} 2+6h^2\left(\overline{y}_1^{(k-1)}\right)^2 & -1 & 0 & 0 \\ -1 & 2+6h^2\left(\overline{y}_2^{(k-1)}\right)^2 & -1 & 0 \\ 0 & -1 & 2+6h^2\left(\overline{y}_3^{(k-1)}\right)^2 & -1 \\ 0 & 0 & -1 & 2+6h^2\left(\overline{y}_4^{(k-1)}\right)^2 \end{bmatrix}$$

$$\begin{bmatrix} \overline{v}_1^{(k-1)} \\ \overline{v}_2^{(k-1)} \\ \overline{v}_3^{(k-1)} \\ \overline{v}_4^{(k-1)} \end{bmatrix} = -\begin{bmatrix} 2\overline{y}_1^{(k-1)} - \overline{y}_2^{(k-1)} + 2h^2(\overline{y}_1^{(k-1)})^3 - (1+\sqrt{1-\alpha}) \\ -\overline{y}_1^{(k-1)} + 2\overline{y}_2^{(k-1)} - \overline{y}_3^{(k-1)} + 2h^2(\overline{y}_2^{(k-1)})^3 \\ -\overline{y}_2^{(k-1)} + 2\overline{y}_3^{(k-1)} - \overline{y}_4^{(k-1)} + 2h^2(\overline{y}_3^{(k-1)})^3 \\ -\overline{y}_3^{(k-1)} + 2\overline{y}_4^{(k-1)} + 2h^2(\overline{y}_4^{(k-1)})^3 - (0.77+\sqrt{1-\alpha}) \end{bmatrix}$$

be solved for each k = 1, 2, 3; since:

$$\overline{y}_i^{(k)} = \overline{y}_i^{(k-1)} + \overline{v}_i^{(k-1)}$$

for each i = 1, 2, 3, 4, Where α is a parameter between 0 and 1, when $\alpha = 1$, the results are presented in table (3.13)

Table (3.13).

The upper level of fuzzy solution for example (3.2.4.1)

x_i	$\overline{y}_1$	$\overline{y}_2$	$\overline{y}_3$	$\overline{y}_4$
1	1	1	1	1
1.1	0.971	0.968	0.968	0.982
1.2	0.894	0.803	0.821	0.844
1.3	0.770	0.780	0.730	0.710

CONCLUSIONS AND RECOMMENDATIONS

From the present study, the following conclusions are drowning:

1. In some cases, it is so difficult to generalize the ordinary concepts of set theory to fuzzy set theory, unless using the extension principle.
2. Some literatures state and prove the existence and uniqueness theorem of boundary value problems in terms of initial value problems.
3. The approximate solution of the shooting method for linear problems seems to be more accurate than the other numerical method in comparison with the exact solution.

Also, we can recommend the following statements for the future work as open problems:

1. Studying fuzzy boundary value problems with generalized boundary conditions, including derivatives in the boundary conditions.
2. Studying fuzzy periodic boundary value problems.
3. Studying new methods of solution for solving fuzzy boundary value problems, such as the collocation method.

CONTENTS

الإهداء

إلى ...

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مثلي الأعلى وبكل فخر ... والدي الحبيب

من أحتوت الدنيا بحنانها وبكل إعتزاز ... والدتي الحبيبة (رَحِمها الله)

رفاق دربي ومصابيح حياتي ... أخوتي وأخواتي الاعزاء

أهدي ثمرة جهدي المتواضع

رشا

INTRODUCTION

Fuzzy set theory has been studied extensively over the past 40 years. Most of the early interest in fuzzy sets pertained to representing uncertainty in human cognitive processes. Fuzzy set theory is now applied to problems in engineering, business, model and related health science, and the natural sciences, in this basic sciences, we construct exact mathematical model of empirical phenomena and then w use these models to make predictions. While some aspects of the real world problems always escape from such precise mathematical models and usually there is an elusive inexactness as a part of the original model, [Kandel, 1986].

One of the aims of fuzzy set theory is to develop the methodology of the formulations and solutions of problems that are too complicated or ill-defined to be acceptable to analysis by conventional techniques.

Since 1965, fuzzy theoretical approach had been developed by Zadeh himself and some other researchers in which they apply this theory in a wide range of scientific and engineering areas, in which Zadeh's original definition of fuzzy set is as follow "a fuzzy set a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assign to each object a grade of membership ranging between zero and one", [Pal, 1986].

Such a type of objects most encountered in real life problems, which do not have precisely well defined criteria of membership function. For example, the class of animals clearly includes dog, horses, birds, etc., and clearly excludes such objects as rock, plants, etc. However, such objects as starfish, bacteria, etc., have an ambiguous status with respect to the class of animal, such type of elements may be classified easily using fuzzy set with the cooperation of the membership function, [Al-Yassiri, 2000].

Kaufmann and Gupta (1988) reported that over 7000reearch papers, reports, monographs, and books on fuzzy set theory and applications have been published since 1965.

Katsara A. K. (1984) defined the fuzzy noremd spaces, but the concept of fuzzy normed spaces which were introduced by Wu Congxin and Fang Ginxuon (1984).

Kandel and Byatt (1978 and 1980) applied the concept of fuzzy differential equations to the analysis of fuzzy dynamical problems, but the boundary value problems was treated, rigorously by Lakshmikantham, V., Murty, K. N. and Turner, J. (2001).

Henderon, J. and Peterson, A. (2004) obtained a theorem of the existence and uniqueness of solutions for the boundary value problems of fuzzy differential equations. Al-Saedy, A. J. in 2006 studied the solution of boundary fuzzy differential equations.

This thesis consists of three chapters. Chapter one, entitled "Fuzzy set Theory" consists of five sections. Section one consists of basic concepts and definitions related to fuzzy set theory which are necessary for the completeness of this work. Section two and because of the importance in solving fuzzy differential equations, we study α-level sets, as well as, some of its properties. Section three was devoted to define the membership function in general without details. Section four presents the extension principle which is necessary to generalize crisp (nonfuzzy) mathematical concepts of mathematical logic. Finally, in section five, we discuss linear fuzzy differential equation, as well as their solution whenever the fuzziness occurs in the initial conditions or in the coefficient of the system.

Chapter two, entitled "Theoretical Results in Boundary Value Problems" consists of three sections. In section one; we introduce some basic and fundamental concepts of fuzzy normed spaces. In section two, we introduce the fundamental concepts of fuzzy boundary value problems with some related definitions. In section three, we study in details the statement and the proof of the existence n uniqueness theorem of fuzzy boundary value problems using Schauder fuzzy fixed point theorem.

Finally, chapter three, entitled "Solution of Fuzzy Boundary Value problems" consists of three sections. In section one; we introduce some theoretical results of boundary value problems, such as Sturm-Liouville equation. In section two, we discuss boundary value problems for fuzzy differential equations in details, using the shooting method for linear and non linear problems and finite difference method for liner and non linear problems to solve numerically boundary value problems

LIST OF SYMBOLS

A	Ordinary or nonfuzzy set.
$\tilde{A}$	Fuzzy set.
$\mu_{\tilde{A}}$	The membership function of a fuzzy set $\tilde{A}$.
$\tilde{A}^c$	The complement of a fuzzy set $\tilde{A}$.
A_α	α-level set of a fuzzy set $\tilde{A}$.
$\mathrm{Im}(\tilde{A})$	The image of a fuzzy set $\tilde{A}$.
$\lvert . \rvert$	Absolute value.
$\lVert . \rVert$	Norm.
$(X, \lVert . \rVert)$	Ordinary normed space.
$C[a, b]$	The set of all continuous functions on [a, b].
$\mathbb{R}^n$	Euclidean space.
sup	Least upper bound.
B-space	Banach space.
F	Field.
$<., .>$	Scalar (inner) product mapping.
$\overline{<x, y>}$	Conjugate of <x, y>.
$(X, <., .>)$	Ordinary scalar (inner) product space.
$<., .>^{\sim}$	Fuzzy scalar mapping.
$(\tilde{A}, <., .>^{\sim})$	Fuzzy scalar product space.

I^X	The collection of all fuzzy subsets of X.
$\dim T(x) < \infty$	The dimension of the operator T is finite.
∞	Infinity.
$\tilde{T}$	Fuzzy operator.
$\lVert \tilde{T} \rVert$	The norm of a fuzzy operator.
$X^{\sim}$	The set of all fuzzy subsets of X.
$(X^{\sim}, \lVert . \rVert^{\sim})$	Fuzzy normed space.
I	The closed unit interval [0, a].
max.	The maximum.
min.	The minimum.
E^n	The set of all fuzzy sets from X onto $\mathbb{R}^n$.

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Numerical Solution of Fuzzy Boundary Value Problems

A Thesis

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جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم

الحل العددي لمسائل القيم الحدودية الضبابية

رسالة
مقدمة إلى كلية العلوم – جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
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(نرْفَع دَرَجَاتٍ مَن نشاءُ وَفَوْقَ كُلِ ذِي عِلْمٍ عَلِيم)

صدق الله العظيم

(سورة يوسف، الآية ٧٦)

المستخلص

لقد أصبحت نظرية المجموعات الضبابية في السنوات الاخيرة على قدر كبير من الأهمية، وقد استخدمت في العديد من حقول الرياضيات. حيث ان عدد من الدراسات قد ظهرت في مجال الجبر، التحليل والرياضات التطبيقية، والتي اعتمدت على هذه النظرية.

لهذه الاطروحة ثلاث أهداف رئيسية. الهدف الاول هو لدراسة المجموعات الضبابية حيث تضمنت الدراسة التعاريف الاساسية، المصطلحات الرئيسية وبعض الامثلة التوضيحية.

الهدف الثاني هو لاعطاء صياغة وبرهان لمبرهنة وجود ووحدانية حلول المعادلات التفاضلية الحدودية الضبابية بصورة مباشرة وبدون تحويل المعادلة الى مهعادلة تفاضلية ابتدائية ضبابية.

الهدف الثالث هو لدراسة الحلول العددية للمعادلات التفاضلية الحدودية الضبابية.