Abstract

The main objective of thesis is oriented toward two objectives. The first objective is to introduce and study new type of differential equations, which are the so called fuzzy fractional order differential equations. This type of equations is the collection between two different theories in mathematics which are fuzzy set theory and theory of fractional calculus, where the study include some illustrative examples and theoretical aspects. The second objective is the statement and proof of the existence and uniqueness theorem of fuzzy fractional order differential equations using Sadoviskii's fixed point theorem.

Acknowledgements

Thanks to Allah for all his blessing during the pursuit of my academic and career gools.

I would like to express my sincere appreciation to my supervisor, Dr. Fadhel S. Fadhel, for giving me the major steps to go on to explore the subject, shearing with me the ideas in my research " Existence and Uniqueness Theorem of Fuzzy Fractional Order Differential Equations" and perform the points that I felt were important.

Special thanks to the staff members of Mathematics Department at Al-Nahrain University for their help and during my study.

Finally, I would like to say "Thank you" to my family, all of them, for their help.

Russol

October, 2006 ✍

Appendix A
Computer Programs

Following the computer programs used in this thesis:

Prog.1: { Computer programming for solving

$$\tilde{y}^{(1/4)}(x)=\tilde{y}^{-3/4}\exp(-1/\tilde{y}),\quad \tilde{y}(0;\alpha)=[0.1+0.9\alpha,1.5-0.5\alpha]$$

using explicit Euler's method and explicit Runge-Kutta method}.

h:=0.1

α:=0,0.2..1

$\underline{\tilde{y}}_0(\alpha)$:=0.1+0.9.α

$\overline{\tilde{y}}_0(\alpha)$:=1.5-0.5.α

$$\underline{\tilde{y}}(\alpha):=\left(\underline{\tilde{y}}_0(\alpha)\right)^{-5/4}.\exp\left(\frac{-1}{\underline{\tilde{y}}_0(\alpha)}\right)$$

$$\overline{\tilde{y}}(\alpha):=\left(\overline{\tilde{y}}_0(\alpha)\right)^{-5/4}.\exp\left(\frac{-1}{\overline{\tilde{y}}_0(\alpha)}\right)$$

$$\underline{\tilde{k}}1(\alpha):=\underline{\tilde{y}}(\alpha)$$

$$\overline{\tilde{k}}1(\alpha):=\overline{\tilde{y}}(\alpha)$$

$$\underline{\tilde{k}}2(\alpha):=\left(\underline{\tilde{y}}_0(\alpha)+h\left(\underline{\tilde{y}}_0^{-5/4}(\alpha).\exp\left(-1/\underline{\tilde{y}}_0(\alpha)\right)\right)\right)^{-5/4}.$$
$$\exp\left(-1/\underline{\tilde{y}}_0(\alpha)+h\left(\underline{\tilde{y}}_0^{-5/4}(\alpha)\exp\left(-1/\underline{\tilde{y}}_0(\alpha)\right)\right)\right)$$

$$\overline{\tilde{k}}2(\alpha):=\left(\overline{\tilde{y}}_0(\alpha)+h\left(\overline{\tilde{y}}_0^{-5/4}(\alpha)\exp\left(-1/\overline{\tilde{y}}_0(\alpha)\right)\right)\right)^{-5/4}.$$
$$\exp\left(-1/\overline{\tilde{y}}_0(\alpha)+h\left(\overline{\tilde{y}}_0^{-5/4}(\alpha).\exp\left(-1/\overline{\tilde{y}}_0(\alpha)\right)\right)\right)$$

$$\underline{\tilde{y}}_1(\alpha):=\underline{\tilde{y}}_0(\alpha)+h.\ \underline{\tilde{y}}(\alpha)$$

$\bar{\tilde{y}}_1(\alpha) := \bar{\tilde{y}}_0(\alpha) + h.\ \tilde{y}(\alpha)$

$\underline{\tilde{Y}}1(\alpha) := \underline{\tilde{y}}_0(\alpha) + \frac{h}{2}.\left(\underline{\tilde{k}}1(\alpha) + \underline{\tilde{k}}2(\alpha)\right)$

$\bar{\tilde{Y}}1(\alpha) := \bar{\tilde{y}}_0(\alpha) + \frac{h}{2}.\left(\bar{\tilde{k}}1(\alpha) + \bar{\tilde{k}}2(\alpha)\right)$

Prog.2: { Computer programming for solving

$$D^{3/2}\tilde{y}(x) = \tilde{y}, \quad \tilde{y}(0;\alpha) = [0.5 + 0.5\alpha, 1.25 - 0.25\alpha]$$

using Adem Bashforth method}.

h:=0.1

α:=0,0.2..1

$\underline{\tilde{y}}_0(\alpha)$:=0.5+0.5.α

$\bar{\tilde{y}}_0(\alpha)$:=1.25-0.25.α

$\underline{\tilde{y}}_1(\alpha) := \underline{\tilde{y}}_0(\alpha) + h$

$\bar{\tilde{y}}_1(\alpha) := \bar{\tilde{y}}_0(\alpha) + h$

$\underline{\tilde{y}}_2(\alpha) := \underline{\tilde{y}}_1(\alpha) + \frac{h}{\sqrt{\pi}}.\left(3.\sqrt{\underline{\tilde{y}}_1(\alpha)} - \sqrt{\underline{\tilde{y}}_0}\right)$

$\bar{\tilde{y}}_2(\alpha) := \bar{\tilde{y}}_1(\alpha) + \frac{h}{\sqrt{\pi}}.\left(3.\sqrt{\bar{\tilde{y}}_1(\alpha)} - \sqrt{\bar{\tilde{y}}_0}\right)$

Supervisor Certification

I certify that this thesis was prepared under my supervision at the Al-Nahrain University, College of Science, in partial fulfillment of the requirements for the degree of master of science in mathematics

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Examining Committee Certification

We certify that we have read this thesis entitled *"Existence and Uniqueness Theorem of Some Fuzzy Fractional Order Differential Equation"* and as examining committee examined the student *(Russol Najem Abdullah Al-Husseiny)* in its contents and in what it connected with, and that is in our opinion it meets the standards of a thesis for the degree of Master of Science in Mathematics.

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1 General Concepts

This chapter we present some general concepts related to this work, including fuzzy set theory and fractional calculus. This chapter consist of three sections, in section 1.1 general introduction to fuzzy set theory is given including basic definitions, general properties, algebraic operation, the membership function and the extension principle.

In section 1.2 primitive concept and definitions related to fractional calculus are given, including gamma function, beta function, Riemann-Liouville formula of fractional differentiation and integration as well as the fractional integration and the fractional derivatives of the some well known functions.

Finally in section 1.3 an introduction and statement to the problem of a fuzzy fractional order differential equations (FFODE's) is given.

1.1 Fuzzy Set Theory

In every day of real life, we are using so many properties, which cannot be dealt with satisfactory simple “Yes” or “No” basis. Assigning each individual in a population by “Yes” or “No”, i.e., “1” or “0” values, as is done in ordinary set theory, is not an adequate way for dealing with properties of this type, [Zadeh, 1965].

In 1965, Zadeh suggested a modified approach where an individual can have a degree of membership value ranged over a continuum of values rather than being either 0 or 1. He showed how set operations such as union and intersection can be define for these “fuzzy” sets, and developed a consistent framework for dealing with such type of problems. This system allows fuzzy sets to be manipulated in a consistent and reasonably intuitive way, [Yan, 1994].

Lack of crispness is an aspect of many real world properties, and one must be catered in defining the linguistic terms used to name these properties. The framework provided by fuzzy sets is perhaps the most natural and accurate currently available for dong this, [Yan, 1994].

The fuzzy set theory was initiated by Zadeh in the early 1960's 1964's. Since 1965, fuzzy theoretical approach had developed by Zadeh himself and some other researchers as a tool for modeling human centered systems applied in a wide range of scientific and engineering areas.

The use of fuzzy sets in pattern recognition and classification may spot some light on the general problem of decision making and fuzzy processes in general. Although a great amount of literatures had been published dealing with fuzzy techniques in pattern recognition, cluster analysis, and related topics, [Kandel, 1982].

It is frequently stated that the process of recognition and classification is one of the most fundamental of human activities. As a matter of fact, one of the most primitive and common activities of animals (human beings included) consists of sorting like items into groups. These groups are described by patterns and what we perform is the act recognition of certain pattern and then classification of them into groups, [Al-Doury, 2002].

It has been claimed that the concept of vagueness underlying fuzzy theory is more appropriate of such systems than the probabilistic concepts of randomness.

1.1.1 Basic Definitions and General Properties of Fuzzy Sets:

This subsection consists of some basic definitions and concepts related to fuzzy set theory. These concepts and definitions has an analogy from in some cases, in non fuzzy set theory. As a classification between fuzzy and non-fuzzy sets, each fuzzy set is assigned with the symbol "~" in the rest of this work. We start first with the definition of fuzzy sets:

Definition (1.1), [Zadeh, 1965]:

Let X be any set of elements. A ***fuzzy set*** $\tilde{A}$ is characterized by a membership function $\mu_{\tilde{A}}(x)$: X⟶ I, where I is the closed unit interval [0, 1]. Then we can write a fuzzy set $\tilde{A}$ by the set of points:

$$\tilde{A} = \left\{ \left(x, \mu_{\tilde{A}}(x) \right) \middle| \, x \in X, \; 0 \le \mu_{\tilde{A}}(x) \le 1 \right\}$$

The collection of all fuzzy sets in X will be denoted by I^X, i.e.,

$$I^X = \{ \tilde{A} : \tilde{A} \text{ is a fuzzy subset of } X \}.$$

Following, some fundamental concepts related to the basic algebraic operations and relations of fuzzy sets ([Zadeh, 1965], [Zimmerman, 1985] and [Kandel, 1986]).

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy subsets of a universal set X with membership functions $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ respectively, then:

1. $\tilde{A} = \varnothing$ if and only if $\mu_{\tilde{A}}(x) = 0$, $\forall x \in X$, where $\varnothing$ is the ***empty*** fuzzy set.
2. $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x)$, $\forall x \in X$.
3. $\tilde{A} = \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$, $\forall x \in X$.

4. The ***complement*** of $\tilde{A}$ (denoted by $\tilde{A}^c$) is a fuzzy set with membership function

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x).$$

5. The ***support*** of $\tilde{A}$ (denoted by $S(\tilde{A})$), is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}}(x) > 0$.

6. The ***height*** of $\tilde{A}$ (denoted by hgt $(\tilde{A})$) is the supremum value of $\mu_{\tilde{A}}(x)$ over all $x \in X$. If hgt $(\tilde{A})$ = 1, then $\tilde{A}$ is ***normal,*** otherwise it is ***subnormal***.

7. A point $x \in X$ is said to be ***crossover point*** of $\tilde{A}$ if $\mu_{\tilde{A}}(x) = 0.5$.

8. $\tilde{C} = \tilde{A} \cap \tilde{B}$ is a fuzzy set with membership function

$$\mu_{\tilde{C}}(x) = \text{Min}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \ \forall x \in X$$

9. $\tilde{D} = \tilde{A} \cup \tilde{B}$ is a fuzzy set with membership function

$$\mu_{\tilde{D}}(x) = \text{Max}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \ \forall x \in X$$

10. The ***m-th power*** of $\tilde{A}$ is a fuzzy set with the membership function

$$\mu_{\tilde{A}^m}(x) = \left[\mu_{\tilde{A}}(x)\right]^m, \ \forall \ x \in X.$$

11. The ***algebraic sum*** of $\tilde{A}$ and $\tilde{B}$ is a fuzzy set $\tilde{C}$ (termed by $\tilde{C} = \tilde{A} + \tilde{B}$) which is defined as:

$$\tilde{C} = \left\{\left(x, \mu_{\tilde{A}+\tilde{B}}(x)\right) \middle| \ \forall x \in X\right\}$$

where**:**

$$\mu_{\tilde{A}+\tilde{B}}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x).\mu_{\tilde{B}}(x)$$

12. The ***algebraic product*** of $\tilde{A}$ and $\tilde{B}$ is a fuzzy set $\tilde{C}$ (termed by $\tilde{C} = \tilde{A} \cdot \tilde{B}$) which is defined as:

$$\tilde{C} = \left\{\left(x, \mu_{\tilde{A}.\tilde{B}}(x)\right) \middle| \ \forall x \in X\right\}$$

where**:**

$$\mu_{\tilde{A}.\tilde{B}}(x)=\mu_{\tilde{A}}(x).\mu_{\tilde{B}}(x)$$

13. The ***absolute difference*** of $\tilde{A}$ and $\tilde{B}$ (denoted by $|\tilde{A}-\tilde{B}|$) and is defined by:

$$\mu_{|\tilde{A}-\tilde{B}|}(x)=\left|\mu_{\tilde{A}}(x)-\mu_{\tilde{B}}(x)\right|.$$

14. If $\mu_{\tilde{A}\cap\tilde{B}}=0$, $\forall\, x\in X$, then $\tilde{A}$ and $\tilde{B}$ are said to be ***separated*** sets.

Example (1.1), [Kandel, 1986]:

Let the universal set be the interval [0, 120], with x interpreted as the age. A fuzzy subset $\tilde{A}$ of X labeled old may be defined by a grade of membership function, such as:

$$\mu_{\tilde{A}}(x)=\begin{cases} 0 & , \quad \text{for } 0\le x\le 40 \\ \left(1+\left(\dfrac{x-40}{5}\right)^{-2}\right)^{-1} & , \quad \text{for } 40< x\le 120 \end{cases}$$

Then $S(\tilde{A})$ is the interval (40, 120].

The $hgt(\tilde{A})$ is equal to 1.

The crossover point of $\tilde{A}$ is 45.

Definition (1.2), [Zimmermann, 1985]:

The (crisp) set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α is called the ***weak α-level set,*** (see Fig. (1.1)), is defined by:

$$A_\alpha=\{x\in X:\mu_{\tilde{A}}(x)\ge\alpha\}$$

while the ***"strong α-level set"*** or ***"strong α-cut",*** is defined by:

$$A'_\alpha=\{x\in X:\mu_{\tilde{A}}(x)>\alpha\}$$

Remarks (1.1), [Klir, 1997]:

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy subset of a universal set X, then it is easily checked that the following properties are satisfied for all $\alpha \in (0, 1]$:

i. $\left(\tilde{A} \cup \tilde{B}\right)_\alpha = A_\alpha \cup B_\alpha$.

ii. $\left(\tilde{A} \cap \tilde{B}\right)_\alpha = A_\alpha \cap B_\alpha$.

iii. If $\tilde{A} \subseteq \tilde{B}$ then $A_\alpha \subseteq B_\alpha$.

iv. $\tilde{A} = \tilde{B}$ equivalent to $A_\alpha = B_\alpha$, $\forall\ \alpha \in (0, 1]$.

v. $A_\alpha \cap A_\beta = A_\beta$ and $A_\alpha \cup A_\beta = A_\alpha$, if $\alpha \le \beta$.

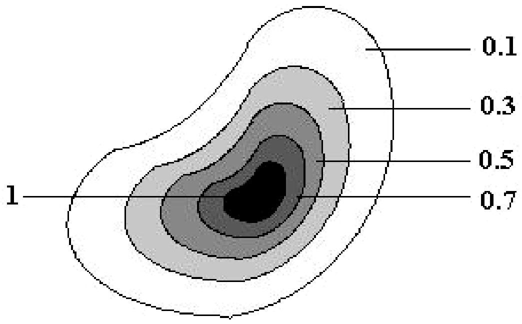

Fig.(1.1) Nested α-level sets.

Definition (1.3), [Zadeh, 1965]:

A fuzzy subset $\tilde{A}$ of a universal vector space X is ***convex*** if and only if the sets A_α defined by:

$$A_\alpha = \left\{x \middle| \mu_{\tilde{A}}(x) \ge \alpha\right\}$$

are convex for all α in the interval (0, 1].

Or equivalently, we can define a convex fuzzy set using directly its membership function to satisfy:

$$\mu_{\tilde{A}}\left[\lambda x_1 + (1-\lambda)x_2\right] \ge \mathrm{Min}\left\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\right\}$$

for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$.

Among the definitions of fuzzy number, is the next definition given by [Zimmermann, 1985].

Definition (1.4), [Zimmermann, 1985]:

A ***fuzzy number*** $\tilde{M}$ is convex normalized fuzzy set of the real line $\mathbb{R}$ such that

i. It exists exactly one $x_0 \in \mathbb{R}$, with $\mu_{\tilde{M}}(x_0) = 1$, (x_0 is called the mean value of $\tilde{M}$).

ii. $\mu_{\tilde{M}}(x)$ is a piecewise continuous function.

Remarks (1.2), [Nguyen, 2000]:

In fact, fuzzy number is fuzzy interval; the only difference is that fuzzy number contain the value 1 at only one place while a fuzzy interval can have several value of 1 on many places, (see Fig.(1.2) and Fig.(1.3)).

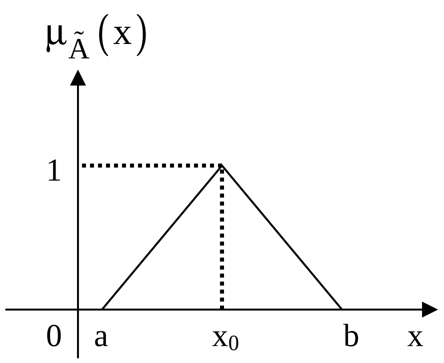

Fig.(1.2) Triangular Fuzzy Number

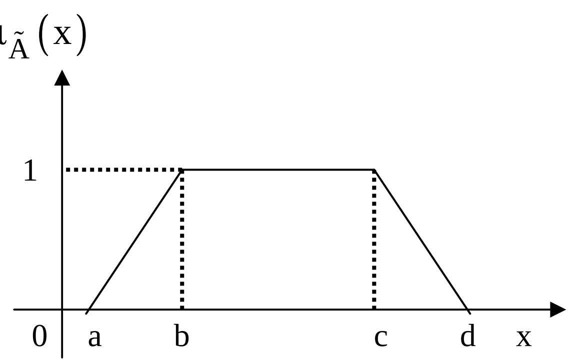

Fig.(1.3) Triangular Fuzzy Interval

The distance D between two fuzzy sets $\tilde{U}$ and $\tilde{V}$ to be D: $E^n \times E^n \longrightarrow [0,\infty)$ given by:

$$D(\tilde{U},\tilde{V}) = \sup_{0<\alpha\leq 1} d(U_\alpha, V_\alpha)$$

where E^n is the set of fuzzy subsets from $\mathbb{R}^n$ to [0, 1] and d is the Hausdorff metric defined in:

$$d(A, B)=\text{Max}\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}$$

where A, B are any two non-empty closed and bounded subsets of $\mathbb{R}^n$ and:

$$\|a-b\|=\left[\sum_{i=1}^{n}\left(a_i-b_i\right)^n\right]^{1/n}.$$

Then D is a metric on E^n.

1. Also, D satisfies the following (see[Song, 2000]):
2. $\left(E^n, D\right)$ is a complete metric space.
3. $D\left(\tilde{U}+\tilde{W}, \tilde{V}+\tilde{W}\right)=D\left(\tilde{U}, \tilde{V}\right)$, for all $\tilde{U}, \tilde{V}, \tilde{W}\in E^n$.
4. $D\left(k\tilde{U}, k\tilde{V}\right)=|k|D\left(\tilde{U}, \tilde{V}\right)$, for all $\tilde{U}, \tilde{V}\in E^n$ and $k\in\mathbb{R}$.

1.1.2 The Membership Function, [Kandel, 1986]:

An important task of the theory of fuzzy sets is the definition and construction of membership functions, which admits certain properties of fuzzy sets. The characteristic function assigns to each element x of X a number, $\mu_{\tilde{A}}(x)$, in the closed unit interval [0, 1] that characterizes the degree of membership of x in $\tilde{A}$, membership functions are functions of the form:

$$\mu_{\tilde{A}}: X \longrightarrow [0, 1].$$

In defining the membership function, the universal set X always assumed to be classical set.

The membership function falls into three categories to be defined either numerically or analytically or by inspection of the reader (see [Zadeh , 1965], [Al-Hamaiwand, 2001] and [Al-Doury, 2002]).

1.1.3 The Extension Principle, [Zimmermann, 1985]:

One of the most basic concepts of fuzzy set theory that can be used to generalize crisp mathematical concepts to fuzzy set theory is the extension principle. In its elementary from it was already implied in Zadeh's first contribution in 1965. Following Zadeh we define the extension principle as follows:

Definition (1.6), [Zimmermann, 1985]:

Let X be the cartesion product of universes $X_1,\ldots,X_s$ and $\tilde{A}_1,...,\tilde{A}_s$ be s fuzzy sets in $X_1,\ldots,X_s$, respectively. f is a mapping from X to a universe Y, $y=f(x_1,\ldots,x_s)$. Then the ***extension principle*** allows us to define a fuzzy set $\tilde{B}$ in Y by:

$$\tilde{B}=f(\tilde{A})=\left\{\left(y,\mu_{\tilde{B}}(y)\right)\middle|\ y=f(x_1,\ldots,x_s),(x_1,\ldots,x_s)\in X\right\}$$

where

$$\mu_{\tilde{B}}(y)=\begin{cases}\sup\limits_{(x_1,\ldots,x_s)\in f^{-1}(y)}\min\left\{\mu_{\tilde{A}_1}(x_1),...,\mu_{\tilde{A}_s}(x_s)\right\}, & \text{if } f^{-1}(y)\neq\varnothing\\ 0 & ,\ \text{Otherwise}\end{cases}$$

where f^{-1} is the inverse image of f.

For s=1, the extension principle, of course, reduces to:

$$\tilde{B}=f(\tilde{A})=\left\{\left(y,\mu_{\tilde{B}}(y)\right)\middle|\ y=f(x),x\in X\right\}$$

where

$$\mu_{\tilde{B}}(y)=\begin{cases}\sup\limits_{x\in f^{-1}(y)}\mu_{\tilde{A}}(x), & \text{if } f^{-1}(y)\neq\varnothing\\ 0 & ,\ \text{Otherwise}\end{cases}$$

1.2 Fractional Calculus

Fractional calculus, is an important branch of applied mathematics, which seems first to have many vague notations and ill defined concepts to the readers which are interest in this subject. This type of differentiation and integration could be considered as a generalization to the usual definition of differentiation and integration, [Oldham, 1974].

Recently, fractional derivatives have been used in building models of physical processes, leading to the formulation with fractional differential equations. The fractional calculus may be considered as an old and yet a new topic. Since, it is an old topic since, starting from some speculations of G. W. Leibniz (1695, 1697) and L. Euler (1730), and new since it had been developed up to now days.

A list of mathematicians, who have provided contributions up to the middle of the 19^{th} century, includes P. S. Laplace (1812), J. B. J. Fourier (1822), N. H. Abel (1823- 1826), J. Liouville (1823- 1873), B. Riemann (1847), H. Holmgren (1865-1867), A. K. Grunwald (1867- 1872), A. V. Letnikov (1868-1872), H. Laurent (1884), P. A. Nekrassov (1888), A. Krug (1890), J. Hudamard (1892), O. Heariside (1892- 1912), S. Pincherle (1902), G. H. Hordy and J. E. Little Wood (1917-1928), H. Weyl (1917), P. Levy (1923), H. T. D. Avis (1924-1936), H. Kobor (1940), D. V. Widder (1941), M. Riesz (1949), [Oldham, 1974].

As well, since only from a little more than to the later fifty years it has been an object of specialized conferences and treatises. For the first conference the merit is a scribed to B. Ross who organized the first conference on fractional calculus and its applications at the University of New Haven in June 1974. For the first monograph the merit is a scribed to K. B. Oldham and J. Spanier, (1974), who after a joint collaboration started in

1968, published a book devoted to fractional calculus in 1974, [Al-Saltani, 2003].

In recent years, considerable interest in fractional calculus have been stimulated by the applications that this subject finds in numerical analysis, differential equations and different areas of applied sciences, especially in physics and engineering, possibly including fractal phenomena [Kalil, 2006].

1.2.1 Fundamental Notions:

It is important to notice that fractional calculus is so difficult to understand and because of this difficulty we shall present in this section the most important notions and definitions that are necessary for understanding this subject.

1.2.1.1 Gamma and Beta Functions, [Oldham, 1974]:

Gamma function $\Gamma(x)$ plays an important role in the theory of differentegration, since in fractional calculus, the gamma function generalizes the concepts of a factorial of a given natural number n to any real number and it is defined by:

$$\Gamma(x) \equiv \int_0^{\infty} y^{x-1} e^{-y} dy, \; x > 0 \quad \ldots\ldots (1.1)$$

The following are the most important properties of gamma function:

1. $\Gamma(1) = 1$.
2. $\Gamma(x+1) = x\Gamma(x)$, x<0.
3. $\Gamma(n+1) = n!, \; n \in \mathbb{N}$.
4. $\Gamma(x-1) = \dfrac{\Gamma(x)}{x-1}$, x≠1.

5. $\Gamma\left(\frac{1}{2}-n\right)=\frac{[-4]^n\, n!\,\sqrt{\pi}}{(2n)!}$.

6. $\Gamma(-x)=\frac{-\pi\csc(\pi x)}{\Gamma(x+1)}$.

7. $\Gamma(nx)=\sqrt{\frac{2\pi}{n}}\left[\frac{n^x}{\sqrt{2\pi}}\right]^n\prod_{k=0}^{n-1}\Gamma\left(n+\frac{k}{n}\right)$, and in particular:

$$\Gamma(2x)=\frac{4^x\Gamma(x)\Gamma(x+1/2)}{2\sqrt{\pi}}.$$

The following are some frequently encounter examples of gamma functions for different value of x.

$\Gamma(-1)=\infty$, $\Gamma(0)=\infty$, $\Gamma(1)=1$, $\Gamma(2)=1$, $\Gamma(3)=2$, $\Gamma\left(\frac{-3}{2}\right)=\frac{4}{3}\sqrt{\pi}$, $\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\sqrt{\pi}$, $\Gamma\left(\frac{-1}{2}\right)=-2\sqrt{\pi}$, $\Gamma\left(\frac{5}{2}\right)=\frac{3}{4}\sqrt{\pi}$, $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

Another type of functions is called the beta function defined by:

$$B(p,q)=\int_0^1 y^{p-1}(1-y)^{q-1}dy,\ \ p,q>0$$

If either p or q is non-positive, the integral diverges otherwise $B(p,q)$ is defined by the relationship:

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

which valid for all p and q.

Both the beta and gamma functions have “incomplete” analogs. The incomplete beta function of argument x is defined by the integral:

$$B_x(p,q)=\int_0^x y^{p-1}(1-y)^{q-1}dy$$

1.2.1.2 Riemann- Liouville Formula of Fractional Differentiation and Integration, [Oldham, 1974], [Nishimoto, 1997]:

Fractional differentiation and integration may involve Riemann-Liouville formula of fractional order q>0, which takes the form:

$$D_{x_0}^{q} y(x) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_{x_0}^{x} \frac{y(u)}{(x-u)^{q-m+1}} du \qquad (1.2)$$

where $D_{x_0}^{0} = I$ (identity operator), and m is a positive integer number defined by $m-1 < q \leq m$, and x_0 is an initial condition.

Such equations have recently proved to be valuable tools in modeling many physical phenomena. The case of 0<q<1 seems to be particularly important, but there are also some applications for q>1. It is well known that D^q has an m-dimensional kernel, and therefore we certainly need to specify m initial conditions in order to obtain a unique solution of the straightforward form of the fractional differential equation:

$$D^q y(x) = f(x, y(x)), \quad x \in [a,b] \qquad (1.3)$$

where f is some given continuous function and a,b are any real numbers. The initial conditions of eq. (1.3) must takes the form:

$$\frac{d^{q-k}}{dx^{q-k}} y(x_0) = b_k, \; k=1, 2, \dots, m$$

where b_k's are given constants and m is positive integer.

1.2.1.3 The Fractional Integral, [Oldham, 1974]:

The most frequently encountered definition of an integral of fractional order is via an integral transform, which is called the Riemann-Liouville integral. So, the generalization to non-integer q is:

$$\frac{d^q}{dx^q} f = \frac{1}{\Gamma(-q)} \int_0^x (x-y)^{-q-1} f(y)\,dy, \; q<0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.4)$$

For the function $f(x) = x^{q-1}\exp(-1/x)$, using the formula (1.4) on substituting $y = \frac{x}{(xz+1)}$, we find:

$$\frac{d^q}{dx^q}\left\{\frac{\exp(-1/x)}{x^{1-q}}\right\} = \frac{1}{\Gamma(-q)}\int_0^x \frac{y^{q-1}\exp(-1/y)}{[x-y]^{q+1}}dy$$

$$= \frac{\exp(-1/x)}{\Gamma(-q)x^{q+1}}\int_0^\infty \frac{\exp(-z)}{z^{q+1}}dz$$

From eq. (1.1), the integral is evaluated simply as $\Gamma(-q)$ so that the final result is given by:

$$\frac{d^q}{dx^q}\left\{\frac{\exp(-1/x)}{x^{1-q}}\right\} = \frac{\exp(-1/x)}{x^{q+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.5)$$

We shall omit the proof of the more general results, such as:

$$\frac{d^q}{dx^q}\left\{x^{q-n}\exp\left(\frac{-1}{x}\right)\right\} = \exp\left(\frac{-1}{x}\right)\sum_{j=0}^{n-1}\frac{\Gamma(j-q)}{\Gamma(-q)}\binom{n-1}{j}x^{j-n-q}, \; n=1,\,2,...,$$

where eq. (1.5) is a special case when n=1.

1.2.1.4 The Fractional Derivatives [Oldham, 1974], [Bertram, 1974]:

As it is given in literatures, fractional differentiation is given by:

$$\frac{d^q}{dt^q} f(t) = \lim_{N\to\infty}\left[\left(\frac{t}{N}\right)^{-q}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} f\left(t - j\frac{t}{N}\right)\right], \; q>0 \ldots. (1.6)$$

Similarly, as in natural differentiation, we can give the following examples for fractional differentiations:

1. $\frac{d^q[1]}{dx^q} = \frac{x^{-q}}{\Gamma(1-q)}$, x>0.

2. $\dfrac{d^q}{dx^q}[c]=c\dfrac{d^q}{dx^q}(1)=\dfrac{c\,x^{-q}}{\Gamma(1-q)}$, c is a constant.

3. $\dfrac{d^q}{dx^q}(0)=0$, for all q.

4. $\dfrac{d^q}{dx^q}x^p=\dfrac{x^{p-q}}{\Gamma(-q)}B(p+1,-q)$

$$=\frac{\Gamma(p+1)x^{p-q}}{\Gamma(p-q+1)}\text{, p>-1, q<1.}$$

5. $\dfrac{d^q\exp(k-cx)}{d(x-a)^q}=\dfrac{\exp(k-cx)}{(x-a)^q}\gamma^*(-q,-c(x-a))$

Since $\gamma^*(c,x)$ is the incomplete gamma function which is defined by:

$$\gamma^*(c,x)=\frac{c^{-x}}{\Gamma(x)}\int_0^c y^{x-1}\exp(-y)\,dy$$

$$=\exp(-x)\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+c+1)}.$$

6. $\dfrac{d^q}{dx^q}\left[\dfrac{x^q}{1-x}\right]=\dfrac{\Gamma(q+1)}{[1-x]^{q+1}}$, q>-1.

7. $\dfrac{d^q}{dx^q}\left[\dfrac{x^p}{1-x}\right]=x^{p-q}\displaystyle\sum_{j=0}^{\infty}\frac{\Gamma(j+p+1)}{\Gamma(j+p-q+1)}x^j$

$$=\frac{\Gamma(p+1).B_x(p-q,q+1)}{\Gamma(p-q).[1-x]^{q+1}}\text{, 0<x<1 and p>-1.}$$

8. $\dfrac{d^q}{dx^q}[1-x]^{q-1}=\dfrac{x^{-q}}{\Gamma(1-q).[1-x]}$, with $|q|<1$.

9. $\dfrac{d^q}{dx^q}[1-x]^p=\dfrac{[1-x]^{p-q}.B_x(-q,q-p)}{\Gamma(-q)}$, p>0.

1.2.2 Some Properties of Fractional Differential Operator D_x^q*, [Beteram, 1974], [Nishimoto, 1997]:*

In this subsection, some important properties of the fractional differential operator D_x^q are presented for completeness perpouse:.

1. The operator D_x^q is linear, i.e.,

   $D_x^q\{c_1 f(x) + c_2 g(x)\} = c_1 D_x^q\{f(x)\} + c_2 D_x^q\{g(x)\}$, where c_1 and c_2 are constants.

2. $D_x^q D_x^\beta f(x) = D_x^{q+\beta} f(x)$.

1.3 Fuzzy Fractional Order Differential Equations

In this section, a new type of differential equations is formulated by mixing two well known types of differential equations which are the fractional order differential equations and fuzzy order differential equations. This type of equations will be called fuzzy fractional order differential equations and has the following form:

$$\begin{aligned} &\tilde{y}^{(q)}(x) = f(x, \tilde{y}(x)) \\ &\tilde{y}^{(q-k)}(x_0) = \tilde{y}_0, \quad k = 1, 2, \ldots, n+1, \; n < q < n+1 \end{aligned} \qquad \text{.............. (1.7)}$$

where n is an integer number and $\tilde{y}_0 \in E^n$.

Now, for this type of equations, the same basic concepts related to such type of equations is considered, such as:

1. The statement and proof of the existence and uniqueness theorem of the solution of such type of equations.
2. Studying and introducing some method for solving such type of equations analytically and numerically.

The above two aspects will be considered in the next two chapters.

2 Existence and Uniqueness Theorem of Fuzzy Fractional Order Differential Equations

Among the important tasks in fuzzy order differential equations and in fractional order differential equations is the study and proof of the existence and uniqueness theorem. Therefore, several researchers had been studied this theorem using either Brower fixed points theorem (see [Leipzig, 1986]) or using Schauder fixed point theorem (see [Al-Ani, 2005]). This chapter we introduce a new approach in the study and proof of this theorem using Sadoviskii fixed point theorem, as well as, an introduction to some additional concepts in non-linear functional analysis, such as non-compactness measure, condensing mapping, etc. Also, in this chapter, $\tilde{y}$ will stands to denote the solution of the fuzzy fractional order differential equation.

Finally, this chapter consists of two sections. In section 2.1 we give some preliminary concepts of mixing between fuzzy order differential equations and fractional order differential equations. While in section 2.2 we state and prove the existence and uniqueness theorem of fuzzy fractional order differential equations using Sadovislii's fixed-point theorem for condensing mapping (this theorem seems to be new to the best of our knowledge).

2.1 Preliminaries

Assume that $f\left(x,\tilde{y}(x)\right): I^{*}\times E^{n} \longrightarrow E^{n}$ is a levelwise continuous function, where the interval $I^{*}=\left\{x:\left|x-x_{0}\right|\leq\delta\right\}$ and $E^{n}=\left\{\tilde{y}:R^{n}\to[0,1]\right\}$.

Consider the fuzzy fractional order differential equation (FFODE):

$$\begin{aligned}&\tilde{y}^{(q)}(x)=f\left(x,\tilde{y}(x)\right)\\&\tilde{y}^{(q-k)}(x_{0})=\tilde{y}_{0},\quad k=1,2,\ldots,n+1,\ n<q<n+1\end{aligned} \qquad \text{......................... (2.1)}$$

where n is an positive integer number and $\tilde{y}_{0}\in E^{n}$.

The next definition collects between FFODE, and Volterra integral equations through their solution.

Definition (2.1):

A mapping $\tilde{y}: I^{*}\longrightarrow E^{n}$ is a solution to the FFODE, given by eq. (2.1) and it is levelwise continuous function and satisfies the Volterra singular integral equation:

$$\tilde{y}(x)=\tilde{y}_{0}+\frac{1}{\Gamma(q)}\int_{x_{0}}^{x}(x-z)^{q-1}f\left(z,\tilde{y}(z)\right)dz \qquad \text{.................................. (2.2)}$$

for 0<q<1 and $x\in I^{*}$.

The eq. (2.2) can be written in operator form as:

$$T\tilde{y}=I\tilde{y}-\frac{1}{\Gamma(q)}\int_{x_{0}}^{x}(x-z)^{q-1}f\left(z,\tilde{y}(z)\right)dz\text{, where } T\tilde{y}=\tilde{y}_{0} \qquad \text{............... (2.3)}$$

Returning to the main question of proving the existence and uniqueness of solution of eq. (2.2), we outline next a plausible method of attacking this problem. We start by using the constant function $\tilde{y}_{0}(x)=\tilde{y}_{0}$ as an approximation to the solution and substitute this approximation into the right-hand side of eq. (2.2), and use the result:

$$\tilde{y}_1(x) = \tilde{y}_0 + \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x-z)^{q-1} f\left(z, \tilde{y}_0(z)\right) dz$$

as a next approximation to the solution. Then after substituting this approximation $\tilde{y}_1(x)$ again into the right-hand side of eq. (2.2) to obtain what we hope is a still better approximation $\tilde{y}_2(x)$, given by:

$$\tilde{y}_2(x) = \tilde{y}_0 + \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x-z)^{q-1} f\left(z, \tilde{y}_1(z)\right) dz$$

and so on continuing in this process. The final goal is to find a mapping $\tilde{y}$ with the property that when it is substituted in the right-hand side of eq. (2.2), the result is the same mapping $\tilde{y}$ (i.e., $\tilde{y}$ is a Sadoviskii's fixed point). If we continue in our approximation procedure, we may hope that the sequence of functions $\left\{\tilde{y}_k(x)\right\}$, called successive approximation, converges to a limit function with this property. Under suitable hypotheses this is the case, and precisely this approach is used to prove the existence of the solution of eq. (2.2).

We shall consider problem (2.1), with f as a continuous function on the rectangle:

$$R^* = \left\{(x, \tilde{y}) \middle| \ |x - x_0| \le a,\ D(\tilde{y}, \tilde{y}_0) \le b\right\},$$

centered at $(x_0, \tilde{y}_0)$. We assume that f and $\frac{\partial f}{\partial \tilde{y}}$ are bounded on R*, that is, there exist constants M>0 and L>0, such that:

$$\left|\left[f(x, \tilde{y})\right]_\alpha\right| \le M, \quad \left|\left[\frac{\partial}{\partial \tilde{y}} f(x, \tilde{y})\right]_\alpha\right| \le L \quad \text{...} \ (2.4)$$

for all $\alpha \in [0,1]$, and for all points $(x,\tilde{y})$ in R*. If $(x,\tilde{y}_1)$ and $(x,\tilde{y}_2)$ are any two points in R*, then by the mean-value theorem, there exists a number $\tilde{\Psi}$ between $\tilde{y}_1$ and $\tilde{y}_2$, such that:

$$\left|\left[f(x,\tilde{y}_2)\right]_\alpha - \left[f(x,\tilde{y}_1)\right]_\alpha\right| = \left|\left[\frac{\partial}{\partial \tilde{y}} f\left(x,\tilde{\Psi}\right)\right]_\alpha \left([\tilde{y}_2]_\alpha - [\tilde{y}_1]_\alpha\right)\right|$$

$$= \left|\left[\frac{\partial}{\partial \tilde{y}} f\left(x,\tilde{\Psi}\right)\right]_\alpha\right| \cdot \left|[\tilde{y}_2]_\alpha - [\tilde{y}_1]_\alpha\right|$$

Since the point $\left(x,\tilde{\Psi}\right)$ is also in R*, then $\left|\left[\frac{\partial}{\partial \tilde{y}} f\left(x,\tilde{\Psi}\right)\right]_\alpha\right| \leq L$, and we obtain that:

$$d\left(\left[f(x,\tilde{y}_2)\right]_\alpha, \left[f(x,\tilde{y}_1)\right]_\alpha\right) \leq L\, d\left([\tilde{y}_2]_\alpha, [\tilde{y}_1]_\alpha\right) \qquad (2.5)$$

valid whenever $(x,\tilde{y}_1)$ and $(x,\tilde{y}_2)$ are in R*.

Definition (2.2), [Al-Ani, 2005]:

A function $\left[f\left(x,\tilde{y}(x)\right)\right]_\alpha$ which satisfies inequality (2.5), for all $(x,\tilde{y}_1)$, $(x,\tilde{y}_2)$ in the region R* is said to satisfy a Lipschitz condition in R*.

We have already indicated that we shall use an approximation procedure to establish the existence of solutions. Now, let us define the successive approximations in general case by the equations:

$$\tilde{y}_0(x) = \tilde{y}_0$$

and

$$\tilde{y}_j(x) = \tilde{y}_0 + \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x-z)^{q-1} f\left(z,\tilde{y}_{j-1}(z)\right) dz, \quad j = 1,2,\ldots \qquad (2.6)$$

Before we can do anything with these successive approximations, we must show that they are defined properly. This means that in order to define $\tilde{y}_j$ on some interval I*, we must first show that the point $\left(x, \tilde{y}_j(x)\right)$ remains in the rectangle R* for every x in I*.

Lemma (2.1):

Define δ to be the smaller of the two positive number a and b/M. Then the successive approximations $\tilde{y}_j(x), \forall j = 0,1,...$ given by eq. (2.6) are defined on the interval I* given by $|x - x_0| \le \delta$. On this interval, we have:

$$D\left(\tilde{y}_j(x), \tilde{y}_0\right) \le M|x - x_0| \le M\delta \le b, \ j = 0,1,2,\ldots \quad (2.7)$$

where $M = D\left(f(x, \tilde{y}), \hat{0}\right), \ \hat{0} \in E^n$, such that:

$$\hat{0}(x) = \begin{cases} 1, & x = 0 \\ 0, & x \ne 0 \end{cases},$$

and for any $(x, \tilde{y}) \in J_0$, where $J_0 = I^* \times B\left(\tilde{y}_0, b\right)$, where a>0, b>0, $\tilde{y}_0 \in E^n$ by:

$$B\left(\tilde{y}_0, b\right) = \left\{\tilde{y} \in E^n \,\middle|\, D\left(\tilde{y}, \tilde{y}_0\right) \le b\right\}.$$

Proof:

We will prove this lemma by induction. It is obvious for $j = 0$. Let $x \in I^*$, then for eq. (2.6), it follows however that, for j=1:

$$\tilde{y}_1(x) = \tilde{y}_0 + \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x - z)^{q-1} f\left(z, \tilde{y}_0(z)\right) dz \quad (2.8)$$

which prove $\tilde{y}_1(x)$ is a levelwise continuous on I*, since $\tilde{y}_0$ and f are levelwise continuous. Moreover, for any $\alpha \in [0,1]$, we have:

$$d\left(\left[\tilde{y}_1(x)\right]_\alpha,[\tilde{y}_0]_\alpha\right)=d\left(\left[\tilde{y}_1(x)\right]_\alpha-[\tilde{y}_0]_\alpha,\{0\}\right)$$

$$=d\left(\left[\frac{1}{\Gamma(q)}\int_{x_0}^{x}(x-z)^{q-1}f\left(z,\tilde{y}_0(z)\right)dz\right]_\alpha,\{0\}\right)$$

$$\leq\int_{x_0}^{x}d\left(\left[\frac{1}{\Gamma(q)}(x-z)^{q-1}f\left(z,\tilde{y}_0(z)\right)\right]_\alpha,\{0\}\right)dz$$

$$\leq\left|\int_{x_0}^{x}d\left(\left[\frac{1}{\Gamma(q)}(x-z)^{q-1}f\left(z,\tilde{y}_0(z)\right)\right]_\alpha,\{0\}\right)dz\right| \qquad \text{........................ (2.9)}$$

Taking the supremum over all $\alpha\in[0,1]$ of inequality (2.9), gives:

$$\sup_{0\leq\alpha\leq1}d\left([\tilde{y}_1(x)]_\alpha,[\tilde{y}_0]_\alpha\right)\leq\sup_{0\leq\alpha\leq1}\left(\left|\int_{x_0}^{x}d\left(\left[\frac{1}{\Gamma(q)}(x-z)^{q-1}f\left(z,\tilde{y}_0(z)\right)\right]_\alpha,\{0\}\right)dz\right|\right)$$

$$\leq\left|\int_{x_0}^{x}\sup_{0\leq\alpha\leq1}d\left(\left[\frac{1}{\Gamma(q)}(x-z)^{q-1}f\left(z,\tilde{y}_0(z)\right)\right]_\alpha,\{0\}\right)dz\right|$$

Then:

$$D\left(\tilde{y}_1(x),\tilde{y}_0\right)\leq\left|\int_{x_0}^{x}D\left(\frac{1}{\Gamma(q)}(x-z)^{q-1}f\left(z,\tilde{y}_0(z)\right),\hat{0}\right)dz\right|$$

$$\leq\left|\int_{x_0}^{x}M\,dz\right|\leq M\left|x-x_0\right|$$

$$\leq M\delta\leq b \qquad \text{.. (2.10)}$$

If $\left|x-x_0\right|\leq\delta$, where $M=D\left(f\left(x,\tilde{y}(x)\right),\hat{0}\right)$, $\hat{0}\in E^n$ and for any $(x,\tilde{y})\in J_0$.

Now, assume that for 1<j-1<j, $\tilde{y}_j(x)$ is levelwise continuous on $|x-x_0|\leq\delta$, and that:

$$D(\tilde{y}_{n-1}(x),\tilde{y}_0)\leq M|x-x_0|\leq M\delta\leq b \qquad (2.11)$$

From eq. (2.6), one can deduce that $\tilde{y}_j(x)$ is levelwise continuous on $|x-x_0|\leq\delta$, and since $\tilde{y}_0$ and f are levelwise continuous. Then in a similar manner as in inequality (2.10), we have:

$$D(\tilde{y}_n(x),\tilde{y}_0)\leq M|x-x_0|\leq M\delta\leq b \qquad (2.12)$$

If $|x-x_0|\leq\delta$, where $M=D(f(x,\tilde{y}(x)),\hat{0})$, $\hat{0}\in E^n$ and for any $(x,\tilde{y})\in J_0$.

This establishes the lemma. ■

2.2 Existence and Uniqueness Theorem Using Sadoviskii's Fixed-Point Theorem for Condensing Mapping of FFODE's:

In this section we shall prove the existence and uniqueness theorem of FFODE, using Sadoviskii's fixed-point theorem.

Before, introducing this theorem, some fundamental concepts related to this theorem are given for completeness purpose.

Definition (2.3), [Leipzig, 1986]:

Let N be a bounded set in a metric space (X, d). The ***Kuratowski noncompactness measure*** $\chi(N)$ is defined to be the infimum of the set of all numbers ε>0 with the property that N can be covered by finitely many sets, each of whose diameter is less than or equal to ε, i.e.,

$$\chi(N)=\inf\{\varepsilon>0:\text{ N be a finite cover sets of diameter }\leq\varepsilon\}.$$

The measure of noncompactness determines the deviation from relative compactness of a set, i.e., $\chi(N)=0$ is equivalent to relative compactness for N. As $\chi(N)$ increases, N deviates more strongly, from relative compactness.

The following properties present some of the fundamental property of noncompactness measure.

Proposition (2.1), [Leipzig, 1986]:

Let (X, d) be a Metric-space over $\mathbb{K}$=□,□. Then for all bounded subsets $N, N_1, N_2, \dots, N_n$ and M of X, we have the following results:

1. $\chi(\varnothing)=0$.
2. $\chi(N)=0$ if and only if N is relatively compact.
3. $N \subseteq M$ implies that $\chi(N) \le \chi(M)$.
4. $0 \le \chi(N) \le \operatorname{diam}(N)$.
5. $\chi(N+M) \le \chi(N)+\chi(M)$.
6. $\chi(\beta N)=|\beta|\chi(N)$, for all $\beta \in \mathbb{K}$.
7. $\chi(N)=\chi(\overline{N})$, where $\overline{N}$ stands for the closure of N .
8. $\chi\left(\bigcup_{i=1}^{n} N_i\right)=\max\{\chi(N_1),\chi(N_2),\dots,\chi(N_n)\}$.

Definition (2.4), [Leipzig, 1986]:

Let $T: D(T) \subseteq X \longrightarrow X$ be an operator on a Banach-space X. T is called a ***k-set contraction*** if and only if T is bounded and continuous and there is a number k ≥0, such that:

$\chi(T(N)) \le k\chi(N)$, for all bounded sets N in $D(T)$.

In addition, T is called ***condensing*** if and only if T is bounded and continuous, and $\chi(T(N)) \le \chi(N)$, for all bounded sets N in $D(T)$ with $\chi(N) > 0$.

Lemma (2.2), [Leipzig, 1986]:

Let K, C: P $\subseteq$ X$\longrightarrow$ X are operators on Banach-space X, then K+C is also an operator which is k-contraction with $0 \le k < 1$, and also condensing, if:

i. K is k-contraction, i.e.,

$$\|Kx - Ky\|_\infty \le k\|x - y\|_\infty \quad \text{...} \quad (2.13)$$

for all x,y$\in$ P and fixed k$\in$ [0,1).

ii. C is compact.

***Proof*:**

Let N$\subseteq$P be a bounded set.

By definition (2.3), if follows easily from (2.13), that $\chi(K(N)) \le k\chi(N)$

By proposition (2.1) (2), $\chi(C(N)) = 0$ set T=K+C.

Now:

$$\chi(T(N)) = \chi((K + C)(N))$$
$$= \chi(K(N) + C(N))$$
$$\le \chi(K(N)) + \chi(C(N)), \text{ (By proposition (2.1) (5))}$$
$$\le k\chi(N) \quad \blacksquare$$

Then statement of the Sadovskii fixed-point theorem is given in the next theorem.

<u>Theorem (2.1), (Sadovskii's Fixed-Point Theorem), [Leipzig, 1986]:</u>

Suppose that:

i. The operator $T: N \subseteq X \longrightarrow N$ is condensing.

ii. N is a nonempty, closed, bounded and convex subset of a Banach-space X.

Then T has a fixed point.

<u>Theorem (2.2), (The Existence Theorem):</u>

Consider the FFODE, (2.1) and suppose $J_0 = I^* \times B(\tilde{y}_0, b)$, where $B(\tilde{y}_0, b) = \left\{\tilde{y} \in E^n \,\middle|\, D(\tilde{y}(x), \tilde{y}_0) \le b\right\}$, $I^* = \{x : |x - x_0| \le \delta\}$ and $f(x, \tilde{y}(x)) : I^* \times E^n \longrightarrow E^n$ be levelwise continuous and bounded function for any $(x_0, \tilde{y}) \in J_0$, then there exist a solution of eq. (2.1) which passes through $(x_0, \tilde{y})$.

Proof:

In order to prove the existence of a solution to the FFODE, a use to the Sadoviskii fixed-point theorem will be used, i.e., we must prove that the two conditions of the theorem are satisfied. This is shown as follows:

i. Consider defn. (2.1) and the eq. (2.3):

$$T\tilde{y} = I\tilde{y} - \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x - z)^{q-1} f(z, \tilde{y}(z)) dz, \text{ where } T\tilde{y} = \tilde{y}_0$$

Then, we have to prove that T is condensing.

i.e., to prove T is bounded, continuous and $\chi(T\tilde{y}(x)) \le \chi(\tilde{y}(x))$.

Since T is linear operator.

Then it is either to prove that T is bounded or continuous, (for simplicity, we prove T is bounded).

Now,

$$\|T\tilde{y}\|_\infty = \left\| I\tilde{y} - \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x-z)^{q-1} f(z,\tilde{y}(z))dz \right\|_\infty$$

$$\leq \| I\tilde{y}\|_\infty + \left\| \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x-z)^{q-1} f(z,\tilde{y}(z))dz \right\|_\infty$$

$$\leq \| I\tilde{y}\|_\infty + \frac{1}{\Gamma(q)} \sup_{x\in I^*} \left| \int_{x_0}^{x} (x-z)^{q-1} f(z,\tilde{y}(z))dz \right|$$

$$\leq \| I\tilde{y}\|_\infty + \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x-z)^{q-1} \sup_{x\in I^*} |f(z,\tilde{y}(z))| dz$$

$$\leq \| I\tilde{y}\|_\infty + \frac{\|f\|_\infty}{\Gamma(q)} \int_{x_0}^{x} (x-z)^{q-1} dz$$

Since f is bounded function, then $\| f\|_\infty \leq c^*$. Thus:

$$\|T\tilde{y}\|_\infty \leq \| I\tilde{y}\|_\infty + \frac{c^*}{\Gamma(q)} \int_{x_0}^{x} (x-z)^{q-1} dz$$

$$\leq \| I\tilde{y}\|_\infty + \frac{c^*}{q\Gamma(q)} (x-x_0)^q$$

and hence if $|x-x_0| \leq \delta$, then:

$$\|T\tilde{y}\|_\infty \leq \|I\tilde{y}\|_\infty + \frac{c^*}{\Gamma(q+1)} \delta^q \leq M^*$$

where $q\Gamma(q)=\Gamma(q+1)$, $\|f\|_\infty = \sup_{x\in I^*} |f(x,\tilde{y}(x))|$ and $M^* = \|I\tilde{y}\|_\infty + \frac{c^*}{\Gamma(q+1)}\delta^q$

Therefore, T is bounded operator and also continuous.

Also, to prove that $\chi(T\tilde{y}(x)) \leq \chi(\tilde{y}(x))$.

By lemma (2.2), let $\tilde{y}(x) \in N$ which is a bounded set.

Set $T(\tilde{y}(x)) = K(\tilde{y}(x)) + C(\tilde{y}(x))$, for $\tilde{y}(x) \in N$.

Now, from proposition (2.1) (5), imply that:

$$\begin{aligned}\chi(T(\tilde{y}(x))) &= \chi(K(\tilde{y}(x)) + C(\tilde{y}(x))) \\ &\leq \chi(K(\tilde{y}(x))) + \chi(C(\tilde{y}(x))) \\ &\leq \chi(K(\tilde{y}(x))) \\ &\leq k\chi(\tilde{y}(x))\end{aligned}$$

Since $\chi(C(\tilde{y}(x))) = 0$, by proposition (2.1) (2), and hence C is relatively compact
and since C is closed, then C is compact.

Thus, if k=1, then: $\chi(T(\tilde{y}(x))) \leq \chi(\tilde{y}(x))$

Therefore, the operator $T : N \subseteq X \longrightarrow N$ is condensing.

ii. Suppose that:

$$N = \{\tilde{y}(x) \in B(\tilde{y}_0, b) : \tilde{y}(0) \cong 0, \tilde{y}(x) \in B_\beta, x \in I^*\}$$

where

$$B_\beta = \{\tilde{\eta}(x) \in B(\tilde{y}_0, b) : \|\tilde{\eta}(x)\|_\infty \leq 1 + \beta\} \quad .$$

Now, to prove that N is a nonempty, closed, bounded and convex FFODE. subset of Banach-space X.

It is clear that N is nonempty, closed and bounded, (by construction).

To prove that N is a convex set.

Let $\tilde{y}_1(x), \tilde{y}_2(x) \in N$, then:

$$\tilde{y}_1(x) \in B(\tilde{y}_0, b),\ \tilde{y}_1(0) \cong 0,\ \tilde{y}_1(x) \in B_\beta$$
$$\tilde{y}_2(x) \in B(\tilde{y}_0, b),\ \tilde{y}_2(0) \cong 0,\ \tilde{y}_2(x) \in B_\beta$$

To prove that: $\tilde{z}(x)=\lambda\tilde{y}_1(x)+(1-\lambda)\tilde{y}_2(x)\in N$

i.e., to prove $\tilde{z}(x)\in B(\tilde{y}_0,b),\ \tilde{z}(0)\cong 0,\ \tilde{z}(x)\in B_\beta$

Now, since $\tilde{y}_1(x),\tilde{y}_2(x)\in B(\tilde{y}_0,b)$ and since the linear combination of levelwise continuous functions is also a levelwise continuous function. Hence $\tilde{z}(x)$ is a levelwise continuous function, and

$$\begin{aligned}\tilde{z}(0)&=\lambda\tilde{y}_1(0)+(1-\lambda)\tilde{y}_2(0)\\&=\lambda.0+(1-\lambda).0=0\end{aligned}$$

therefore, $\tilde{z}(x)\in B(\tilde{y}_0,b)$

Moreover, to prove $\|\tilde{z}(x)\|_\infty\in N$, i.e., to prove $\|\tilde{z}(x)\|_\infty\le 1+\beta$

$$\begin{aligned}\|\tilde{z}(x)\|_\infty&=\|\lambda\tilde{y}_1(x)+(1-\lambda)\tilde{y}_2(x)\|_\infty\\&\le\|\lambda\tilde{y}_1(x)\|_\infty+\|(1-\lambda)\tilde{y}_2(x)\|_\infty\\&\le|\lambda|.\|\tilde{y}_1(x)\|_\infty+|(1-\lambda)|.\|\tilde{y}_2(x)\|_\infty\\&\le\lambda.(1+\beta)+(1-\lambda).(1+\beta)\\&\le 1+\beta\end{aligned}$$

So $\tilde{z}(x)=\lambda\tilde{y}_1(x)+(1-\lambda)\tilde{y}_2(x)\in N$

Hence, N is a convex set of FFODE's of Banach-space X.

From, (i) and (ii), T has a fixed point.■

Let $\tilde{y}(x)$ and $\tilde{y}^*(x)$ be two solution of eq. (2.1).

Now, went to prove that this solution is unique, that is from:

$$T\tilde{y}^*=I\tilde{y}^*-\frac{1}{\Gamma(q)}\int_{x_0}^{x}(x-z)^{q-1}f\left(z,\tilde{y}^*(z)\right)dz \qquad\text{.........................}\ (2.14)$$

on $|x-x_0|\le\delta$, it follows that $D(T\tilde{y},\ T\tilde{y}^*)\equiv 0$.

Indeed, from eq. (2.3) and eq. (2.14)

$$d\left([T\tilde{y}]_\alpha,\left[T\tilde{y}^*\right]_\alpha\right)=d\left(\left[I\tilde{y}-\frac{1}{\Gamma(q)}\int_{x_0}^{x}(x-z)^{q-1}f\left(z,\tilde{y}(z)\right)dz\right]_\alpha,\right.$$

$$\left.\left[I\tilde{y}^*-\frac{1}{\Gamma(q)}\int_{x_0}^{x}(x-z)^{q-1}f\left(z,\tilde{y}^*(z)\right)dz\right]_\alpha\right)$$

$$\leq\int_{x_0}^{x}d\left(\left[I\tilde{y}-\frac{1}{\Gamma(q)}(x-z)^{q-1}f\left(z,\tilde{y}(z)\right)\right]_\alpha,\right.$$

$$\left.\left[I\tilde{y}^*-\frac{1}{\Gamma(q)}(x-z)^{q-1}f\left(z,\tilde{y}^*(z)\right)\right]_\alpha\right)dz$$

By inequality (2.5), we have:

$$d\left(\left[T\tilde{y}^*\right]_\alpha,[T\tilde{y}]_\alpha\right)\leq\int_{x_0}^{x}L\,d\left([\tilde{y}(z)]_\alpha,\left[\tilde{y}^*(z)\right]_\alpha\right)dz$$

Taking the supremum over $\alpha\in[0,1]$ to the both sides, give:

$$\sup_{0\leq\alpha\leq 1}d\left([T\tilde{y}]_\alpha,\left[T\tilde{y}^*\right]_\alpha\right)\leq\int_{x_0}^{x}L\sup_{0\leq\alpha\leq 1}d\left([\tilde{y}(z)]_\alpha,\left[\tilde{y}^*(z)\right]_\alpha\right)dz$$

Hence

$$D\left(T\tilde{y},T\tilde{y}^*\right)\leq L\int_{x_0}^{x}D\left(\tilde{y}(z),\tilde{y}^*(z)\right)dz$$

$$\leq L\,b\int_{x_0}^{x}dz=L\,b\left|x-x_0\right| \quad\text{.......................................} \quad (2.15)$$

where $\left|x-x_0\right|\leq\delta$, therefore, repeating (2.15) n-times, yields:

$$D\left(T\tilde{y},T\tilde{y}_n^*\right)\leq bL^n\frac{\left|x-x_0\right|^n}{n!} \quad\text{...} \quad (2.16)$$

Consequently, eq. (2.16) holds for any n, which leads to the conclusion that:

$$D\left(T\tilde{y}, T\tilde{y}^{*}\right) = D\left(T\tilde{y}, T\tilde{y}_{n}^{*}\right) \longrightarrow 0 \text{ as } n \to \infty$$

$$\Rightarrow D\left(T\tilde{y}, T\tilde{y}^{*}\right) \equiv 0, \text{ on } \left|x - x_{0}\right| \leq \delta \text{ as } n \to \infty.$$

$$\Rightarrow T\tilde{y} = T\tilde{y}^{*}$$

i.e., $\tilde{y}(x) = \tilde{y}^{*}(x), \quad \forall x$ .... (2.17)

From (2.17), we have (2.1) has a unique solution. ■

3 Solution of Fuzzy Fractional Order Differential Equations

Pearson in 1997, introduced the analytical method for solving linear system of fuzzy differential equations with the cooperation of complex numbers, while the solution of fractional differential equations was introduced by several researchers either analytically or numerically. Analytic solution introduced by Oldham in 1974 using inverse operator method and Laplace transformation method, while numerical and approximate methods introduced by Al-Saltani in 2003 using variational approach, Kalil in 2006 using supline interpolation function and by Al-Authab in 2005 using numerical methods.

This chapter consists of four sections. In section 3.1 we introduce the analytic solution of fuzzy differential equations. In section 3.2 we introduce methods for solving fractional differential equations analytically and numerically. In section 3.3 the solution of fuzzy fractional order differential equations have been introduced using numerical methods (linear multistep methods and Runge-Kutta). Finally, section 3.4 introduces some numerical illustrative examples of fuzzy fractional order differential equation.

3.1 Solution of Linear Fuzzy Differential Equations:

This section consists of three cases for solving a linear system of fuzzy differential equation. As a first case, suppose the vector field is linear and all the parameters are assumed to be known to a certain sufficient accuracy, and the initial values of the system are fuzzy.

A second case of fuzzy differential equations occurs when the coefficient matrix related to the vector field constituted a fuzzy numbers.

Also, the third case of fuzzy differential equations occurs when the vector is fuzzy and the matrix is approximately fuzzy.

3.1.1 Analytic Solution of Linear Fuzzy Differential Equations:

Each of the above cases will be considered in details with an illustrative example.

Case (1), [Pearson, 1997]:

Consider the system:

$$\tilde{y}' = A\tilde{y}, \quad \tilde{y}(0) = \tilde{y}_0, \quad x \in [a,b] \quad \text{.......} \quad (3.1)$$

where $A: \mathbb{R}^n \to \mathbb{R}^n$, $\tilde{y}$ is a fuzzy mapping $\tilde{y} \in \mathbb{R}^n \longrightarrow [0,1]$, where $\tilde{y}$ is a vector made up of n-fuzzy mapping.

Suppose that each element of the vector $\tilde{y}$ in (3.1) is a fuzzy number, which is similarly represented as the following α-level set:

$$\tilde{y}^k_\alpha(x) = \left[\underline{\tilde{y}}^k_\alpha(x), \overline{\tilde{y}}^k_\alpha(x)\right], k = 1,2,...,n \quad \text{.......} \quad (3.2)$$

it is shown that the evaluation of the system (3.1) can be described by 2n-differential equations for the end points of the intervals (3.2), this is for each given x and each value of α. These equations for the end points of the intervals are:

$$\begin{aligned}\underline{\tilde{y}}'^{k}_{\alpha}(x) &= \text{Min}\left\{(A\tilde{u})_k : \tilde{u}^i \in \left[\underline{\tilde{y}}^i_{\alpha}(x), \overline{\tilde{y}}^i_{\alpha}(x)\right]\right\} \\ \overline{\tilde{y}}'^{k}_{\alpha}(x) &= \text{Max}\left\{(A\tilde{u})_k : \tilde{u}^i \in \left[\underline{\tilde{y}}^i_{\alpha}(x), \overline{\tilde{y}}^i_{\alpha}(x)\right]\right\}\end{aligned} \quad \ldots\ldots (3.3)$$

with the initial conditions $\underline{\tilde{y}}^k_{\alpha}(0) = \underline{\tilde{y}}^k_{\alpha_0}$, $\overline{\tilde{y}}^k_{\alpha}(0) = \overline{\tilde{y}}^k_{\alpha_0}$, where:

$$(A\tilde{u})_k = a_{kj}\tilde{u}^j$$

is the k^{th} row of $A\tilde{u}$. The vector field in eq. (3.1) is linear, and so the following rule applies in eq. (3.3).

$$\underline{\tilde{y}}'^{k}_{\alpha}(x) = a_{kj}\tilde{u}^j \quad \ldots\ldots (3.4)$$

where: $\tilde{u}^j = \begin{cases} \underline{\tilde{y}}^j_{\alpha}(x), & \text{if } a_{kj} \geq 0 \\ \overline{\tilde{y}}^j_{\alpha}(x), & \text{if } a_{kj} < 0 \end{cases}$

and

$$\overline{\tilde{y}}'^{k}_{\alpha}(x) = a_{kj}\tilde{u}^j \quad \ldots\ldots (3.5)$$

where: $\tilde{u}^j = \begin{cases} \overline{\tilde{y}}^j_{\alpha}(x), & \text{if } a_{kj} \geq 0 \\ \underline{\tilde{y}}^j_{\alpha}(x), & \text{if } a_{kj} < 0 \end{cases}$

Where $a_{kj}\tilde{u}^j = \sum_{j=1}^{n} a_{kj}\tilde{u}^j$. Equations (3.4) and (3.5) are called parametric equations.

Now, in order to solve the fuzzy system of differential equations, we let $\tilde{y}$ to be a vector of fuzzy numbers, and hence:

$$\tilde{y}'(x) = A\tilde{y}(x), \quad \tilde{y}(0) \approx \tilde{y}_0, \quad x \in [a,b]$$

Recall that, there are two equations of the type of equations (3.4) and (3.5) which can easily be written out explicitly.

Now, define new complex variable as follows:

$$\tilde{z}^k_{\alpha} = \underline{\tilde{y}}^k_{\alpha}(x) + i\overline{\tilde{y}}^k_{\alpha}(x) \quad \ldots\ldots (3.6)$$

and the two operations carried on the complex numbers are:

a) The identity operation, e, such that:

$$e\tilde{z}_{\alpha}^{k} = \tilde{z}_{\alpha}^{k} \quad \text{(3.7)}$$

b) The flip operation g, about the diagonal in the complex plane, i.e.

$$g\tilde{z}_{\alpha}^{k} = g\left(\underline{\tilde{y}}_{\alpha}^{k}(x) + i\overline{\tilde{y}}_{\alpha}^{k}(x)\right)$$

$$= \overline{\tilde{y}}_{\alpha}^{k}(x) + i\underline{\tilde{y}}_{\alpha}^{k}(x) \quad \text{(3.8)}$$

where $g^2 = e$ and $g^k = e$ if k is even and $g^k = g$ if k is odd, and:

$$(\nu g)\tilde{z}_{\alpha}^{k} = (g\nu)\tilde{z}_{\alpha}^{k}, \ \text{for } \nu \in \mathbb{R} \quad \text{(3.9)}$$

From eq. (3.6), we have:

$$\tilde{z}_{\alpha}^{\prime k} = \underline{\tilde{y}}_{\alpha}^{\prime k}(x) + i\overline{\tilde{y}}_{\alpha}^{\prime k}(x)$$

but

$$\underline{\tilde{y}}_{\alpha}^{\prime k}(x) = a_{kj}\tilde{u}^{j} \ \text{and} \ i\overline{\tilde{y}}_{\alpha}^{\prime k}(x) = ia_{kj}\tilde{u}^{j}.$$

Then:

$$\overline{\tilde{y}}_{\alpha}^{\prime k}(x) + i\underline{\tilde{y}}_{\alpha}^{\prime k}(x) = a_{kj}\tilde{u}^{j} + ia_{kj}\tilde{u}^{j}$$

Hence:

$$\tilde{z}_{\alpha}^{\prime k} = a_{kj}\left(\tilde{u}^{j} + i\tilde{u}^{j}\right)$$

$$= \begin{cases} a_{kj}\left(\underline{\tilde{y}}_{\alpha}^{k} + i\overline{\tilde{y}}_{\alpha}^{k}\right), & \text{if } a_{kj} \geq 0 \\ a_{kj}\left(\overline{\tilde{y}}_{\alpha}^{k} + i\underline{\tilde{y}}_{\alpha}^{k}\right), & \text{if } a_{kj} < 0 \end{cases}$$

$$= \begin{cases} a_{kj}\,\tilde{z}_{\alpha}^{k}, & \text{if } a_{kj} \geq 0 \\ a_{kj}\left(g\tilde{z}_{\alpha}^{k}\right), & \text{if } a_{kj} < 0 \end{cases}$$

Now, by using eq. (3.9), we have:

$$\tilde{z}_{\alpha}^{\prime k} = \begin{cases} a_{kj}\,\tilde{z}_{\alpha}^{k}, & \text{if } a_{kj} \geq 0 \\ g\,a_{kj}\,\tilde{z}_{\alpha}^{k}, & \text{if } a_{kj} < 0 \end{cases}$$

In order to simplify the last formula, let:

$$b_{ij} = \begin{cases} ea_{ij}, & \text{if } a_{ij} \geq 0 \\ ga_{ij}, & \text{if } a_{ij} < 0 \end{cases} \quad \text{...} \quad (3.10)$$

then:

$$\tilde{z}'^{k}_{\alpha} = \begin{cases} b_{ij}\,\tilde{z}^{k}_{\alpha} & , \text{ if } a_{kj} \geq 0 \\ b_{ij}\,\tilde{z}^{k}_{\alpha} & , \text{ if } a_{kj} < 0 \end{cases}$$

or in matrix from:

$$\tilde{z}'^{k}_{\alpha} = B\tilde{z}^{k}_{\alpha}, \quad \tilde{z}_{\alpha}(0) = \tilde{z}_{\alpha_0}$$

Now, $\tilde{y}' = A\tilde{y}$, which has the solution $\tilde{y} = ce^{Ax}$ and since $\tilde{y}(0) = \tilde{y}_0$, then $\tilde{y}(x) = \tilde{y}_0 e^{Ax}$. Similarly:

$$\tilde{z}_{\alpha}(x) = \tilde{z}_{\alpha_0} e^{Bx} \quad \text{...} \quad (3.11)$$

but since the problem is to calculate the exponential of the matrix B, and upon carrying some little calculations to solve eq. (3.11), we get:

$$\tilde{z}^{k}_{\alpha} = \varphi_{kj}(x)\tilde{z}^{j}_{\alpha_0} + \psi_{kj}(x)g\,\tilde{z}^{j}_{\alpha_0}$$

where $\varphi(x) = \exp(xC)\cosh(xD)$, $\psi(x) = \exp(xC)\sinh(xD)$, and B=eC+gD and since $\tilde{z}^{k}_{\alpha}(x) = \underline{\tilde{y}}^{k}_{\alpha}(x) + i\,\overline{\tilde{y}}^{k}_{\alpha}(x)$, hence:

$$\left.\begin{aligned} \underline{\tilde{y}}^{k}_{\alpha}(x) &= \sum_{j=1}^{n}\Big(\varphi_{kj}(x)\,\underline{\tilde{y}}^{j}_{\alpha_0}(x) + \psi_{kj}(x)\,\overline{\tilde{y}}^{j}_{\alpha_0}(x)\Big) \\ \overline{\tilde{y}}^{k}_{\alpha}(x) &= \sum_{j=1}^{n}\Big(\varphi_{kj}(x)\,\overline{\tilde{y}}^{j}_{\alpha_0}(x) + \psi_{kj}(x)\,\underline{\tilde{y}}^{j}_{\alpha_0}(x)\Big) \end{aligned}\right\} \quad \text{...........................} \quad (3.12)$$

As an illustration, we consider the next example:

Example (3.1), [Pearson, 1997]:

Consider the linear system $\tilde{y}' = A\tilde{y}$, where $A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$ with initial values to be $\tilde{y}^1(0)$ about 1 and $\tilde{y}^2(0)$ about -1, which are fuzzy numbers and using the membership function defined by setting for example:

$$\tilde{y}_0^1(x) = \begin{cases} 0 & , x < 0 \\ 2x - x^2 & , 0 \le x < 2 \\ 0 & , x > 2 \end{cases}$$

and

$$\tilde{y}_0^2(x) = \begin{cases} 0 & , x < -2 \\ -2x - x^2 & , -2 \le x < 0 \\ 0 & , x > 0 \end{cases}$$

Thus, for $\alpha \in [0, 1]$, we can represent the initial condition in terms of its α-levels:

$$\tilde{y}_{0_\alpha}^1 = \left[\underline{\tilde{y}}_{0_\alpha}^1, \overline{\tilde{y}}_{0_\alpha}^1\right] = \left[1 - \sqrt{1-\alpha}, 1 + \sqrt{1-\alpha}\right]$$
$$\tilde{y}_{0_\alpha}^2 = \left[\underline{\tilde{y}}_{0_\alpha}^2, \overline{\tilde{y}}_{0_\alpha}^2\right] = \left[-1 - \sqrt{1-\alpha}, -1 + \sqrt{1-\alpha}\right]$$

Hence, carrying the above procedure, we can find the final solution in terms of the α-levels, as:

$$\underline{\tilde{y}}_\alpha^1(x) = \left(1 + \frac{x^2}{2!} + \cdots\right)a - \left(x + \frac{x^3}{3!} + \cdots\right)b + \left(x + 2x^3 + \cdots\right)c - \left(2x^2 + \frac{4}{3}x^4 + \cdots\right)d$$

$$\bar{\tilde{y}}_\alpha^1(x)=\left(1+\frac{x^2}{2!}+\cdots\right)b-\left(x+\frac{x^3}{3!}+\cdots\right)a+\left(x+2x^3+\cdots\right)d-\left(2x^2+\frac{4}{3}x^4+\cdots\right)c$$

$$\underline{\tilde{y}}_\alpha^2(x)=\left(1+2x^2+\cdots\right)c-\left(2x+\frac{4}{3}x^3+\cdots\right)d$$

$$\bar{\tilde{y}}_\alpha^2(x)=\left(1+2x^2+\cdots\right)d-\left(2x+\frac{4}{3}x^3+\cdots\right)c$$

where $a=1-\sqrt{1-\alpha}$, $b=1+\sqrt{1-\alpha}$, $c=-1-\sqrt{1-\alpha}$ and $d=-1+\sqrt{1-\alpha}$.

For example, if $\alpha=0.1$ and x=0.2, then $a=0.051317$, $b=1.94868$, $c=-1.94868$ and $d=-0.051317$:

$$\underline{\tilde{y}}_{0.1}^1(0.2)=-0.7571089$$
$$\bar{\tilde{y}}_{0.1}^1(0.2)=2.1264348$$
$$\underline{\tilde{y}}_{0.1}^2(0.2)=-2.085581$$
$$\bar{\tilde{y}}_{0.1}^2(0.2)=0.744947$$

Case (2), [Wuhaib, 2004]:

In this case, we discuss the solution of linear differential equations when the elements of the coefficient matrix are fuzzy numbers. We can solve such type of linear differential equations by using a modified approach of case (1). The linear differential equation is:

$$\tilde{y}'=\tilde{A}y, \quad y(0)=y_0$$

where $y\in\mathbb{R}$, $\tilde{A}$ is n×n fuzzy matrix, i.e., all elements of $\tilde{A}$ are approximate fuzzy numbers. Each elements of $\tilde{A}$ could be written in its α-level as $\tilde{a}_\alpha=\left[\underline{\tilde{a}},\bar{\tilde{a}}\right]$, $\alpha\in[0,1]$. Then each entry of the matrix $\tilde{A}$, could be give as: $\tilde{a}_{ij\alpha}=\left[\underline{\tilde{a}}_{ij},\bar{\tilde{a}}_{ij}\right]$ and hence, at any x, we have:

$$\tilde{a}_{ij_\alpha}(x) = \left[\underline{\tilde{a}}_{ij_\alpha}(x), \overline{\tilde{a}}_{ij_\alpha}(x)\right] \quad \text{.......} \quad (3.13)$$

It is shown that (as in case (1)) the evaluation of the system $y' = \tilde{A}y$, $y(0) = y_0$ can be described by 2n-differential equations for the points of the intervals in (3.13). This is for each x and each value of α, since $\tilde{a}_{ij_\alpha}(x) = \left[\underline{\tilde{a}}_{ij}(x;\alpha), \overline{\tilde{a}}_{ij}(x;\alpha)\right]$, which implies that:

$$\tilde{A} = \begin{bmatrix} [\underline{\tilde{a}}_{11}, \overline{\tilde{a}}_{11}] & [\underline{\tilde{a}}_{12}, \overline{\tilde{a}}_{12}] & \cdots & [\underline{\tilde{a}}_{1n}, \overline{\tilde{a}}_{1n}] \\ [\underline{\tilde{a}}_{21}, \overline{\tilde{a}}_{21}] & [\underline{\tilde{a}}_{22}, \overline{\tilde{a}}_{22}] & \cdots & [\underline{\tilde{a}}_{2n}, \overline{\tilde{a}}_{2n}] \\ \vdots & \vdots & \ddots & \vdots \\ [\underline{\tilde{a}}_{n1}, \overline{\tilde{a}}_{n1}] & [\underline{\tilde{a}}_{n2}, \overline{\tilde{a}}_{n2}] & \cdots & [\underline{\tilde{a}}_{nn}, \overline{\tilde{a}}_{nn}] \end{bmatrix}$$

Let $\underline{\tilde{A}}$ be the matrix of all $\underline{\tilde{a}}_{ij}$ and $\overline{\tilde{A}}$ be the matrix of all $\overline{\tilde{a}}_{ij}$, i.e.

$$\underline{\tilde{A}} = \begin{bmatrix} \underline{\tilde{a}}_{11} & \underline{\tilde{a}}_{12} & \cdots & \underline{\tilde{a}}_{1n} \\ \underline{\tilde{a}}_{21} & \underline{\tilde{a}}_{22} & \cdots & \underline{\tilde{a}}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\tilde{a}}_{n1} & \underline{\tilde{a}}_{n2} & \cdots & \underline{\tilde{a}}_{nn} \end{bmatrix}$$

and

$$\overline{\tilde{A}} = \begin{bmatrix} \overline{\tilde{a}}_{11} & \overline{\tilde{a}}_{12} & \cdots & \overline{\tilde{a}}_{1n} \\ \overline{\tilde{a}}_{21} & \overline{\tilde{a}}_{22} & \cdots & \overline{\tilde{a}}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\tilde{a}}_{n1} & \overline{\tilde{a}}_{n2} & \cdots & \overline{\tilde{a}}_{nn} \end{bmatrix}$$

Then, we can obtain two matrices $\underline{\tilde{B}}$ and $\overline{\tilde{B}}$ from $\underline{\tilde{A}}$ and $\overline{\tilde{A}}$ respectively, to be defined as follows:

$$\underline{\tilde{B}} = \begin{cases} e\underline{\tilde{a}}_{ij}, & \text{if } \underline{\tilde{a}}_{ij} \geq 0 \\ g\underline{\tilde{a}}_{ij}, & \text{if } \underline{\tilde{a}}_{ij} < 0 \end{cases}$$

and

$$\overline{\tilde{B}} = \begin{cases} e\overline{\tilde{a}}_{ij}, & \text{if } \overline{\tilde{a}}_{ij} \geq 0 \\ g\overline{\tilde{a}}_{ij}, & \text{if } \overline{\tilde{a}}_{ij} < 0 \end{cases}$$

Similarly as in case (1), we have:

$$\underline{\tilde{\theta}}(x) = e^{\underline{\tilde{C}}x}\cosh\left(x\underline{\tilde{D}}\right), \quad \overline{\tilde{\theta}}(x) = e^{\overline{\tilde{C}}x}\cosh\left(x\overline{\tilde{D}}\right)$$

and

$$\underline{\tilde{\psi}}(x) = e^{\underline{\tilde{C}}x}\sinh\left(x\underline{\tilde{D}}\right), \quad \overline{\tilde{\psi}}(x) = e^{\overline{\tilde{C}}x}\sinh\left(x\overline{\tilde{D}}\right)$$

where $\underline{\tilde{C}}, \underline{\tilde{D}}, \overline{\tilde{C}}$ and $\overline{\tilde{D}}$ belongs to $\underline{\tilde{B}}$ and $\overline{\tilde{B}}$ respectively.

For illustration purpose, we will consider a system of 2×2 differential equations with fuzzy coefficients.

Example (3.2), [Wuhaib, 2004]:

Consider the linear system of fuzzy differential equations:

$$y' = \tilde{A}y, \quad y(0) = y_0$$

where $\tilde{A}$ is 2×2 fuzzy matrix, such that:

$\tilde{a}_{11}$ approximately equals to -1, with membership function:

$$\mu_{\tilde{a}_{11}}(y) = \left|\frac{-1}{y}\right|, \; y \neq 0$$

$\tilde{a}_{12}$ approximately equals to 1, with membership function:

$$\mu_{\tilde{a}_{12}}(y) = \frac{1}{y}, \; y \neq 0$$

$\tilde{a}_{21}$ approximately equals to 0, with membership function:

$$\mu_{\tilde{a}_{21}}(y) = \frac{1}{1+y}, \; y \neq -1$$

$\tilde{a}_{22}$ approximately equals to -2, with membership function:

$$\mu_{\tilde{a}_{22}}(y) = \left|\frac{-2}{y}\right|, \; y \neq 0$$

and with initial conditions $y_1(0) = 1, \; y_2(0) = 2$.

Hence the lower and upper α-levels of $\tilde{A}$ are given by:

$$\underline{\tilde{A}} = \begin{bmatrix} -1-\sqrt{1-\alpha} & 1+\sqrt{1-\alpha} \\ -\sqrt{1-\alpha} & -2-\sqrt{1-\alpha} \end{bmatrix}$$

and

$$\overline{\tilde{A}} = \begin{bmatrix} -1+\sqrt{1-\alpha} & 1+\sqrt{1-\alpha} \\ \sqrt{1-\alpha} & -2+\sqrt{1-\alpha} \end{bmatrix}$$

and if $\tilde{a}_{ij} \geq 0$, then $\tilde{b}_{ij} = e\tilde{a}_{ij}$ and if $\tilde{a}_{ij} < 0$, then $\tilde{b}_{ij} = g\tilde{a}_{ij}$. This yields the calculation of $\underline{\tilde{B}}$ and $\overline{\tilde{B}}$ as:

$$\underline{\tilde{B}} = \begin{bmatrix} g\left(-1-\sqrt{1-\alpha}\right) & e\left(1+\sqrt{1-\alpha}\right) \\ g\left(-\sqrt{1-\alpha}\right) & g\left(-2-\sqrt{1-\alpha}\right) \end{bmatrix}$$

and

$$\overline{\tilde{B}} = \begin{bmatrix} g\left(-1+\sqrt{1-\alpha}\right) & e\left(1+\sqrt{1-\alpha}\right) \\ e\left(\sqrt{1-\alpha}\right) & g\left(-2+\sqrt{1-\alpha}\right) \end{bmatrix}$$

For simplicity, if we let $s = \sqrt{1-\alpha}$, then the matrix $\underline{\tilde{B}}$ could be written as:

$$\begin{aligned} \underline{\tilde{B}} &= \begin{bmatrix} i(-1-s) & e(1+s) \\ i(-s) & i(-2-s) \end{bmatrix} \\ &= e\begin{bmatrix} 0 & 1+s \\ 0 & 0 \end{bmatrix} + g\begin{bmatrix} -1-s & 0 \\ -s & -2-s \end{bmatrix} \\ &= e\underline{\tilde{C}} + g\underline{\tilde{D}} \end{aligned}$$

It easy to find $e^{\underline{\tilde{C}}x}$, to be:

$$e^{\underline{\tilde{C}}x} = \begin{bmatrix} 1 & x(1+s) \\ 0 & 1 \end{bmatrix}$$

and

$$\cosh\left(x\underline{\tilde{D}}\right)=\begin{bmatrix}1+\frac{x^2}{2}\left(1+2s+s^2\right) & 0\\ 1+\frac{x^2}{2}s^2 & 1+\frac{x^2}{2}\left(4+4s+s^2\right)\end{bmatrix}$$

Therefore:

$$\underline{\tilde{\theta}}(x)=e^{\underline{\tilde{C}}x}\cosh\left(x\underline{\tilde{D}}\right)$$
$$=\begin{bmatrix}\underline{\tilde{\theta}}_{11}(x) & \underline{\tilde{\theta}}_{12}(x)\\ \underline{\tilde{\theta}}_{21}(x) & \underline{\tilde{\theta}}_{22}(x)\end{bmatrix}$$

where $\underline{\tilde{\theta}}_{11}(x)=1+\frac{x^2}{2}\left(1+2s+s^2\right)+x(1+s)+\frac{x^3}{2}\left(s^2+s^3\right)$,

$\underline{\tilde{\theta}}_{12}(x)=x(1+s)+\frac{x^3}{2}\left(s^3+5s^2+8s+4\right)$, $\underline{\tilde{\theta}}_{21}(x)=xs+\frac{x^2}{2}s^2$ and

$\underline{\tilde{\theta}}_{22}(x)=1+\frac{x^2}{2}\left(4+4s+s^2\right)$

and similarly:

$$\underline{\tilde{\psi}}(x)=\begin{bmatrix}\underline{\tilde{\psi}}_{11}(x) & \underline{\tilde{\psi}}_{12}(x)\\ \underline{\tilde{\psi}}_{21}(x) & \underline{\tilde{\psi}}_{22}(x)\end{bmatrix}$$

where $\underline{\tilde{\psi}}_{11}(x)=-x(1+s)-x^2\left(s+s^2\right)-\frac{x^3}{6}\left(s^3+3s^2+3s+1\right)-\frac{x^4}{6}\left(s^3+s^4\right)$,

$\underline{\tilde{\psi}}_{12}(x)=-x^2\left(s^2+3s+2\right)-\frac{x^4}{6}\left(s^4+5s^3+6s^2+4s+8\right)$,

$\underline{\tilde{\psi}}_{21}(x)=-xs-\frac{x^3s^3}{6}$ and $\underline{\tilde{\psi}}_{22}(x)=-x(2+s)-\frac{x^3}{6}\left(s^3+6s^2+12s+8\right)$

Also, in a similar manner we can evaluate $\overline{\tilde{\theta}}(x)$ and $\overline{\tilde{\psi}}(x)$ as:

$$\overline{\tilde{\theta}}(x)=\begin{bmatrix}\overline{\tilde{\theta}}_{11}(x) & \overline{\tilde{\theta}}_{12}(x)\\ \overline{\tilde{\theta}}_{21}(x) & \overline{\tilde{\theta}}_{22}(x)\end{bmatrix}$$

where $\bar{\tilde{\theta}}_{11}(x)=1+\frac{x^2}{2}\left(s^2-2s+1\right)$, $\bar{\tilde{\theta}}_{12}(x)=x(1+s)+\frac{x^3}{2}\left(s^3-3s^2+4\right)$,

$\bar{\tilde{\theta}}_{21}(x)=xs+\frac{x^3}{2}\left(s^3-2s^2+s\right)$ and $\bar{\tilde{\theta}}_{22}(x)=1+\frac{x^2}{2}\left(s^2-4s+4\right)$.

and

$$\bar{\tilde{\psi}}(x)=\begin{bmatrix}\bar{\tilde{\psi}}_{11}(x) & \bar{\tilde{\psi}}_{12}(x)\\ \bar{\tilde{\psi}}_{21}(x) & \bar{\tilde{\psi}}_{22}(x)\end{bmatrix}$$

where $\bar{\tilde{\psi}}_{11}(x)=x(s-1)+\frac{x^3}{6}\left(s^3-3s^2+3s-1\right)$,

$\bar{\tilde{\psi}}_{12}(x)=x^2\left(s^2-s-2\right)+\frac{x^4}{6}\left(s^4-5s^3+6s^2+4s-8\right)$,

$\bar{\tilde{\psi}}_{21}(x)=x^2\left(s^2-s\right)+\frac{x^4}{6}\left(s^4-3s^3+3s^2-s\right)$ and

$\bar{\tilde{\psi}}_{22}(x)=x(s-2)+\frac{x^3}{6}\left(s^3-6s^2+12s-8\right)$

Therefore:

$$\underline{y}_{\alpha}^{k}(x)=\sum_{j=1}^{n}\left(\underline{\tilde{\theta}}_{kj}(x)\,y_j(0)+\underline{\tilde{\psi}}_{kj}(x)\,y_j(0)\right)$$

$$\bar{y}_{\alpha}^{k}(x)=\sum_{j=1}^{n}\left(\bar{\tilde{\theta}}_{kj}(x)\,y_j(0)+\bar{\tilde{\psi}}_{kj}(x)\,y_j(0)\right)$$

Now, letting x=0.2 and α=1, we have:

$$\underline{\tilde{\theta}}(0.2)=\begin{bmatrix}1.02 & 0.216\\ 0 & 1.08\end{bmatrix},\ \underline{\tilde{\psi}}(0.2)=\begin{bmatrix}-0.201333 & -0.0821333\\ 0 & -0.410666\end{bmatrix}$$

$$\bar{\tilde{\theta}}(0.2)=\begin{bmatrix}1.02 & 0.216\\ 0 & 1.08\end{bmatrix},\ \bar{\tilde{\psi}}(0.2)=\begin{bmatrix}-0.201333 & -0.0821333\\ 0 & -0.410666\end{bmatrix}$$

and hence $\underline{y}_1^1(0.2)=1.0864004$, $\bar{y}_1^1(0.2)=1.0864004$, $\underline{y}_1^2(0.2)=1.338668$

and $\bar{y}_1^2(0.2)=1.338668$.

It is clear that for α=1, we have:

$$\underline{y}_1^1(x)=\overline{y}_1^1(x)=y^1(x) \text{ and } \underline{y}_1^2(x)=\overline{y}_1^2(x)=y^2(x)$$

which is the same as the crisp value of the solution vector.

Case (3), [Wuhaib, 2004]:

In this case, we discuss the solution of linear differential equations when the vector of initial condition is fuzzy and the matrix is approximately fuzzy. We can solve this kind of problems by using a mix of case (1) and case (2) together, the formula to the lower and upper bounds of solutions are:

$$\underline{\tilde{y}}_\alpha^k(x)=\sum_{j=1}^{n}\left(\underline{\tilde{\theta}}_{kj}(x)\underline{\tilde{y}}_{\alpha_0}^j+\overline{\tilde{\psi}}_{kj}(x)\overline{\tilde{y}}_{\alpha_0}^j\right)$$

and

$$\overline{\tilde{y}}_\alpha^k(x)=\sum_{j=1}^{n}\left(\overline{\tilde{\theta}}_{kj}(x)\overline{\tilde{y}}_{\alpha_0}^j+\underline{\tilde{\psi}}_{kj}(x)\underline{\tilde{y}}_{\alpha_0}^j\right)$$

where $\underline{\tilde{\theta}}_{kj}(x)$, $\overline{\tilde{\theta}}_{kj}(x)$, $\underline{\tilde{\psi}}_{kj}(x)$ and $\overline{\tilde{\psi}}_{kj}(x)$ are obtained from the lower and upper bounds of the coefficient matrices and $\underline{\tilde{y}}_{\alpha_0}^j$, $\overline{\tilde{y}}_{\alpha_0}^j$ from the initial conditions which are also fuzzy.

3.2 Solution of Fractional Differential Equations

In opposite to differential equations of integer order in which derivatives depends only on the local behaviour of the function. An important type of differential equations, which is called fractional differential equations where the differentegration is of fractional order. Such type of problems may be considered to have the form:

$$y^{(q)}=f(x,y),\ y^{(q-k)}(x_0)=y_0,\ k=1, 2,..., n+1,\ n<q<n+1.$$

where n is an integer number $q\in\mathbb{R}^+$.

This section consists of two approaches, the first approach is the analytic method, while the second are some of the numerical and approximation methods.

3.2.1 Analytic Methods for Solving Fractional Differential Equations, [Oldham, 1974]:

Several analytical methods are proposed for solving fractional differential equations, and among such methods:

1. Inverse Operator Method:

Let f be an unknown function and let q be an arbitrary real number, F is known function, then we can construct the simplest of all fractional differential equations by:

$$\frac{d^q f}{dx^q} = F \qquad (3.14)$$

hence upon taking the inverse operator $\frac{d^{-q}}{dx^{-q}}$, gives:

$$f = \frac{d^{-q}F}{dx^{-q}}$$

where it is clear that it is not always the case that they are equal, but this is not the most general solution:

$$f - \frac{d^{-q}}{dx^{-q}}\frac{d^q}{dx^q} f = 0 \qquad (3.15)$$

additional terms must be added to eq. (3.15), which are $c_1 x^{q-1}, c_2 x^{q-2}$, $\ldots, c_m x^{q-m}$ and hence:

$$f - \frac{d^{-q}}{dx^{-q}}\frac{d^q}{dx^q} f = c_1 x^{q-1} + c_2 x^{q-2} + \cdots + c_m x^{q-m}$$

where $c_1, c_2, ..., c_m$ are an arbitrary constants to be determined from the initial conditions and $q \le m < q+1$.

Thus:

$$f - c_1 x^{q-1} - c_2 x^{q-2} - \cdots - c_m x^{q-m} = \frac{d^{-q}}{dx^{-q}} \frac{d^q}{dx^q} f = \frac{d^{-q}}{dx^{-q}} F$$

Hence, the most general solution of eq. (3.14) is given by:

$$f = \frac{d^{-q}}{dx^{-q}} F + c_1 x^{q-1} + c_2 x^{q-2} + \cdots + c_m x^{q-m}$$

where $0 < q \le m < q+1$ or m=0 for q≤0.

As an illustration example, consider the fractional differential equation:

$$D^{1/2} y = x^{1/2},\ y^{-1/2}(0) = 0.1$$

Now, since $q = \frac{1}{2}$ and $F = x^{1/2}$, hence $\frac{1}{2} < m < \frac{1}{2} + 1$. Therefore, m=1. So with the cooperation of the initial condition:

$$\begin{aligned} y &= D^{-1/2} x^{1/2} + c_1 x^{-1/2} \\ &= \frac{\sqrt{\pi}}{2} x + \frac{0.1}{\sqrt{\pi}} x^{-1/2} \end{aligned}$$

2. Laplace Transformation Method:

Laplace transformation method can be used to solve fractional differential equations; but first of all, we start with Reimann-Liouville formula**:**

$$D_x^q f(x) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_{x_0}^{x} (x-t)^{n-q-1} f(t) dt$$

and letting:

$$g(x) = \int_{x_0}^{x} (x-t)^{n-q-1} f(t) dt$$

so that:

$$D_x^q f(x) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} g(x)$$

or equivalently $g^{(n)}(x) = \Gamma(n-q)h(x)$, where $h(x) = D_x^q f(x)$, and upon integrating the above equation n-times, leads to:

$$\int_{x_0}^{x} \cdots \int_{x_0}^{x} g^n(x)dx\ldots dx = \int_{x_0}^{x} \cdots \int_{x_0}^{x} \left(\Gamma(n-q)h(x)\right)dx\ldots dx$$

$$g(x) = \frac{\Gamma(n-q)}{\Gamma(n)} \int_{x_0}^{x} (x-t)^{n-1} h(t)dt$$

Since:

$$g(x) = \int_{x_0}^{x} (x-t)^{n-q-1} f(t)dt$$

Then:

$$\frac{\Gamma(n-q)}{\Gamma(n)} \int_{x_0}^{x} (x-t)^{n-1} h(t)dt = \int_{x_0}^{x} (x-t)^{n-q-1} f(t)dt$$

Taking the Laplace transformation to the both sides, yields:

$$\frac{\Gamma(n-q)}{\Gamma(n)} \mathscr{L}\left(\int_{x_0}^{x} (x-t)^{n-1} h(t)dt \right) = \mathscr{L}\left(\int_{x_0}^{x} (x-t)^{n-q-1} f(t)dt \right)$$

solving this equation for $\mathscr{L}$(f) and evaluating the inverse Laplace transform to get the desired solution f.

As an illustrative example, consider:

$$D^{1/2}y = x,\ y^{-1/2}(0) = 0$$

$$D_x^{1/2}y(x) = \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_{x_0=0}^{x} (x-t)^{-1/2} f(t)dt$$

and let

$$g(x) = \int_0^x (x - t)^{-1/2} f(t)\,dt$$, then:

$$D_x^{1/2} y = \frac{1}{\sqrt{\pi}} g'(x)$$

or equivalently:

$$g'(x) = \sqrt{\pi}\, x$$

Therefore:

$$g(x) = \sqrt{\pi}\left(\int_0^x t\,dt\right)$$

Hence

$$\sqrt{\pi}\left(\int_0^x t\,dt\right) = \int_0^x (x - t)^{-1/2} f(t)\,dt$$

Taking the Laplace transform and using the convolution theorem, one can find that:

$$\sqrt{\pi}\,\mathscr{L}\left(\int_0^x t\,dt\right) = \mathscr{L}\left(\int_0^x (x - t)^{-1/2} f(t)dt\right)$$

$$\sqrt{\pi}\frac{1}{s^3} = F(s)\frac{\sqrt{\pi}}{s^{1/2}}$$

which implies that:

$$F(s) = \frac{1}{s^{5/2}}$$

and upon taking the inverse Laplace transform, we have:

$$f(x) = \frac{1}{\Gamma(5/2)} x^{3/2}$$

as the desired solution to the fractional differential equation.

3.2.2 Numerical and Approximate Methods for Solving Fractional Differential Equations:

Fractional differential equations can be solved using numerical and approximate methods. Therefore, in this section, some of the well known methods for solving such type of equations will be discussed.

1. The Collocation Method, [Al-Saltani, 2003]:

One of the approximate methods for solving differential equations "in general" and fractional differential equations "in particular" is the so called collocation method, which has also other application in solving integral equations, partial differential equations, etc. This method has its basis on approximating the solution of the problem under consideration by a complete sequence of functions $\{\phi_i\}$ and certain function which satisfying the non-homogenous initial and boundary conditions $\zeta(x)$, such that:

$$y(x) = \zeta(x) + \sum_{i=1}^{n} a_i \phi_i(x)$$

where $\phi_i(x)$ satisfy the homogenous conditions and a_i's are constants to be determined. Evaluating the last equation, at some point of the region of definition to get a linear system of algebraic equations.

As an illustrative example, consider the fractional differential equation:

$$D^{1/2} y = x^{1/2},\ y^{(-1/2)}(0) = 0.1$$

and in order to solve this problem approximately using the collocation method, we let:

$$y(x) = \frac{0.1}{\Gamma(1/2)} x^{-1/2} + a_1 x + a_2 x^2 + a_3 x^3$$

Then:

$$D^{1/2}\left\{\frac{0.1}{\Gamma(1/2)}x^{-1/2}+a_1x+a_2x^2+a_3x^3\right\}=x^{1/2}$$

Hence

$$\frac{0.1}{\Gamma(1/2)}D^{1/2}x^{-1/2}+a_1D^{1/2}x+a_2D^{1/2}x^2+a_3D^{1/2}\ x^3=x^{1/2}$$

hence carrying out some simplifications, gives:

$$\frac{0.1}{\Gamma(1/2)}(0)+a_1\frac{x^{1/2}}{\Gamma(3/2)}+a_2\frac{2}{\Gamma(5/2)}x^{3/2}+a_3\frac{6}{\Gamma(7/2)}x^{5/2}=x^{1/2}$$

$$a_1\frac{x^{1/2}}{\Gamma(3/2)}-x^{1/2}+a_2\frac{2}{\Gamma(5/2)}x^{3/2}+a_3\frac{6}{\Gamma(7/2)}x^{5/2}=0$$

$$\left(\frac{a_1}{\Gamma(3/2)}-1\right)x^{1/2}+a_2\frac{2}{\Gamma(5/2)}x^{3/2}+a_3\frac{6}{\Gamma(7/2)}x^{5/2}=0$$

we get $\frac{a_1}{\Gamma(3/2)}-1=0\Rightarrow\frac{a_1}{\Gamma(3/2)}=1\Rightarrow a_1=\Gamma\left(\frac{3}{2}\right)=0.886$, $a_2=$ $a_3=0$, and

therefore the approximation solution, is given by:

$$y(x)=\frac{0.1}{\sqrt{\pi}}x^{-1/2}+0.886\,x$$

As a comparison, this problem has the exact solution:

$$y(x)=\frac{0.1}{\sqrt{\pi}}x^{-1/2}+0.886\,x$$

2. *The Least-Squares Method :*

Among the popular methods used to approximate the solution of fractional differential equations is the so called least-square method. To illustrate this method, consider the following fractional differential equation:

$$D^q y=g(x)$$

$g\in C[0,1]$, q>0 and approximate the solution by:

$$y_n(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$$

Hence substituting in the differential equation yields:

$$D_x^q y_n(x) = g(x)$$

The polynomial y_n is of degree at most n is required to minimize the error:

$$\int_a^b \left(g(x) - D_x^q y_n(x)\right)^2 dx, \quad 0 < q < 1 \text{ .... } (3.16)$$

To determine such least- square approximating polynomial, that is, a polynomial that minimizes the expression (3.16), we must evaluate its coefficient $a_0, a_1, \ldots, a_n$. For this purpose define:

$$E(a_0, a_1, \ldots, a_n) = \int_a^b \left(g(x) - D_x^q \sum_{k=0}^{n} a_k x^k \right)^2 dx$$

$$= \int_a^b \left(g(x) - \sum_{k=0}^{n} a_k D_x^q x^k \right)^2 dx \text{ .... } (3.17)$$

Hence, the problem now is reduced to find real coefficients $a_0, a_1, \ldots, a_n$ that minimizes E. From the calculus of functions of several variables, a necessary condition for the coefficients $a_0, a_1, \ldots, a_n$ which minimizes E is that $\frac{\partial E}{\partial a_j} = 0$ for each j=0, 1, ... , n. Since from eq. (3.17), we have:

$$E = \int_a^b (g(x))^2 dx - 2\sum_{k=0}^{n} a_k \int_a^b g(x)\, D_x^q x^k dx + \int_a^b \left(\sum_{k=0}^{n} a_k D_x^q x^k \right)^2 dx$$

Hence

$$\frac{\partial E}{\partial a_j} = -2\int_a^b g(x).D_x^q x^j dx + 2\sum_{k=0}^{n} a_k \int_a^b D_x^q x^{j+k} dx, \ \forall j=0, 1, \ldots, n$$

Hence to find y_n, the following (n+1) linear system:

$$\sum_{k=0}^{n} a_k \int_a^b D_x^q x^{j+k} dx = \int_a^b g(x) D_x^q x^j dx, \ j = 0,1,\ldots,n \text{ .... } (3.18)$$

must be solved for the (n+1) unknowns a_j , j=0,1,...,n.

As an illustration to this method, consider the following example of solving the fractional differential equation:

$$D^{1/2}y_n(x) = x^5,\ x \in [0,1]$$

Suppose that $y_n(x) = a_0 + a_1x + a_2x^2$, hence the related linear system obtained from eq. (3.18), is given by:

$$a_0\frac{2}{\sqrt{\pi}} + a_1\frac{4}{3\sqrt{\pi}} + a_2\frac{16}{15\sqrt{\pi}} = \frac{2}{11\sqrt{\pi}}$$

$$a_0\frac{4}{3\sqrt{\pi}} + a_1\frac{16}{15\sqrt{\pi}} + a_2\frac{12}{7\cdot\Gamma(7/2)} = \frac{4}{13\sqrt{\pi}}$$

$$a_0\frac{16}{15\sqrt{\pi}} + a_1\frac{12}{7\cdot\Gamma(7/2)} + a_2\frac{48}{9\cdot\Gamma(9/2)} = \frac{16}{45\sqrt{\pi}}$$

which is a linear system of three unknowns $a_0, a_1$and a_2 which is solved using any numerical method for solving linear systems of algebraic equations, to give:

$$a_0 = 0.221,\ a_1 = -3.129 \text{ and } a_2 = 3.666.$$

Consequently the least-squares polynomial approximation of degree two for $D^{1/2}y(x) = x^5$ on [0, 1] is:

$$y_2(x) = 0.221 - 3.129x + 3.666x^2 .$$

3.3 Solution of Fuzzy Fractional Order Differential Equations

A fuzzy fractional order differential equation of order q, 0<q<1, is an equation of the form:

$$\tilde{y}^{(q)}(x) = f(x, \tilde{y}(x)),\quad x \in I = [x_0, b] \qquad (3.19)$$

where $f(x, \tilde{y}(x)) : I \times E^n \longrightarrow E^n$ is a levelwise continuous function, with an initial condition given as a fuzzy number $\tilde{y}(x_0) = \tilde{y}_0$, where $\tilde{y}_0 \in E^n$ and $x_0 \in I$, (see [Add, 2005]).

We start with the numerical solution of fuzzy fractional order differential equations using LMM, which seems to be new:

3.3.1 Linear Multistep Methods:

Consider the fuzzy fractional order differential equation given by eq. (3.19), then the α-level set related to this equation is given by:

$$\tilde{y}_\alpha^{(q)} = f_\alpha(x, \tilde{y}),\ \tilde{y}_\alpha(x_0) = \tilde{y}_{0_\alpha} \qquad (3.20)$$

Then we seek for a solution in the range $x \in [x_0, b]$, where x_0 and b are finite, and we assume that f satisfies the conditions of the existence and uniqueness theorem, which we shall indicate by :

$$\tilde{y}_\alpha(x) = \left[\underline{y}(x;\alpha), \overline{y}(x;\alpha)\right],\ \alpha \in [0, 1].$$

Consider the sequence of points $\{x_n\}$ defined by $x_n = x_0 + nh$, n=0,1,...The parameter h, which will always be regarded as constant, except where otherwise indicated, is called the step length. An essential property of the majority computational methods for the solution of eq. (3.20) is that of discretization; that is, we seek for a numerical solution not on the continuous interval $x_0 \le x \le b$, but on the discrete points of the set $\{x_n \,|\, n = 0,1,...,$ $(b - x_0)/h\}$.

Let $\tilde{y}_n(\alpha)$ be an approximation to the theoretical solution at x_n that is, to $\tilde{y}_\alpha(x_n)$ and let $G_n(\alpha) \equiv G_\alpha(x_n, \tilde{y}_n)$. If a computational method for determining the sequence $\{\tilde{y}_n(\alpha)\}$ takes the form of a linear relationship between $\tilde{y}_{n+j}(\alpha), G_{n+j}(\alpha)$, $j = 0,1,...,k$, or a linear k-step method.

It is we known that, the general form of linear multistep method may thus be written as:

$$\sum_{j=0}^{k} \Omega_j \, \tilde{y}_{n+j}(\alpha) = h \sum_{j=0}^{k} \beta_j G_{n+j}(\alpha) \quad \text{...} \quad (3.21)$$

In the case of lower and upper solutions, eq. (3.21) can be decomposed into the following two equations:

$$\sum_{j=0}^{k} \Omega_j^* \, \underline{\tilde{y}}_{n+j}(\alpha) = h \sum_{j=0}^{k} \beta_j^* G_{n+j}^*(\alpha)$$

and .... (3.22)

$$\sum_{j=0}^{k} \Omega_j \, \overline{\tilde{y}}_{n+j}(\alpha) = h \sum_{j=0}^{k} \beta_j F_{n+j}(\alpha)$$

where $\Omega_j^*, \Omega_j, \beta_j^*$ and β_j are constants to be determined. The arbitrariness will be removed by assuming throughout this subsection by letting $\Omega_k^* = 1$ and $\Omega_k = 1$. Hence, eq. (3.22) can be rewritten equivalently as:

$$\underline{\tilde{y}}_{n+k}(\alpha) = h \sum_{j=0}^{k} \beta_j^* G_{n+j}^*(\alpha) - \sum_{j=0}^{k-1} \Omega_j^* \, \underline{\tilde{y}}_{n+j}(\alpha)$$

and .... (3.23)

$$\overline{\tilde{y}}_{n+k}(\alpha) = h \sum_{j=0}^{k} \beta_j F_{n+j}(\alpha) - \sum_{j=0}^{k-1} \Omega_j \overline{\tilde{y}}_{n+j}(\alpha)$$

Remark (3.1):

1. Such equations are so difficult to handle theoretically than are non-linear FFODE.'s, but they have practical advantage of permitting us to compute the sequence $\{\tilde{y}_n(\alpha)\}$ numerically. In order to do this, one must supply a set of starting values, $\tilde{y}_0(\alpha), \tilde{y}_1(\alpha), \ldots, \tilde{y}_{k-1}(\alpha)$ (supply by using any explicit one step method), (see [Al-Ani, 2005]).

2. The two eq.'s in (3.23) are explicit if $\beta_k^* = 0$ and $\beta_k = 0$, and implicit if $\beta_k^* \neq 0$ and $\beta_k \neq 0$.

3. In eq.'s (3.23) each equation is a k-step method and each one contain 2k-1 unknowns, (see [Lambert, 1973]).

3.3.2 Euler's Method for Solving Fuzzy Fractional Order Differential Equations:

To use the Euler's method to solve fuzzy fractional order differential equation, the following approach is followed:

Consider the FFODE's:

$$\tilde{y}^{(q)} = f\left(x, \tilde{y}(x)\right), \quad \tilde{y}(x_0) = \tilde{y}_0$$

and since Euler's method with α-level reads as follows:

$$\tilde{y}_{n+1}(\alpha) = \tilde{y}_n(\alpha) + h\tilde{y}'(\alpha) + O\left(h^2\right)$$

Hence:

$$\begin{aligned}\tilde{y}_{n+1}(\alpha) &= \tilde{y}_n(\alpha) + hD^{1-q}D^q\tilde{y} + O\left(h^2\right) \\ &= \tilde{y}_n(\alpha) + hD^{1-q}f\left(x, \tilde{y}(x); \alpha\right) + O\left(h^2\right) \\ &= \tilde{y}_n(\alpha) + hf^*\left(x, \tilde{y}(x); \alpha\right) + O\left(h^2\right) \quad \text{.................................} (3.24)\end{aligned}$$

where $f^*\left(x, \tilde{y}(x); \alpha\right) = D^{1-q}f\left(x, \tilde{y}(x); \alpha\right)$ could be evaluated easily by using fractional calculus.

<u>Remark (3.2):</u>

Consider the first order of FFODE, given by:

$$\left.\begin{aligned}\tilde{y}'(x) &= f^*\left(x, \tilde{y}(x)\right) \\ \tilde{y}(x_0) &= \tilde{y}_0\end{aligned}\right\} \quad \text{..} (3.25)$$

where $f^*\left(x,\tilde{y}(x)\right)=D^{1-q}f\left(x,\tilde{y}(x);\alpha\right)$ and $\tilde{y}(x_0)=\tilde{y}_0$ is a fuzzy number.

The α-level set of $\tilde{y}(x)$ for $x\in[x_0,b]$ is $\tilde{y}_\alpha(x)=\left[\underline{\tilde{y}}(x;\alpha),\overline{\tilde{y}}(x;\alpha)\right]$.

Also $\tilde{y}'_\alpha(x)=\left[\underline{\tilde{y}}'(x;\alpha),\overline{\tilde{y}}'(x;\alpha)\right]$, and

$$f^*_\alpha\left(x,\tilde{y}(x)\right)=\left[\underline{f}^*\left(x,\tilde{y}(x);\alpha\right),\overline{f}^*\left(x,\tilde{y}(x);\alpha\right)\right]$$
$$=\left[G^*\left(x,\underline{\tilde{y}}(x;\alpha),\overline{\tilde{y}}(x;\alpha)\right),F\left(x,\underline{\tilde{y}}(x;\alpha),\overline{\tilde{y}}(x;\alpha)\right)\right]$$

Because of $\tilde{y}'=f^*\left(x,\tilde{y}(x)\right)$, we have:

$$\underline{\tilde{y}}'(x;\alpha)=\underline{f}^*\left(x,\tilde{y}(x);\alpha\right)=G^*\left(x,\underline{\tilde{y}}(x;\alpha),\overline{\tilde{y}}(x;\alpha)\right) \quad (3.26)$$

$$\overline{\tilde{y}}'(x;\alpha)=\overline{f}^*\left(x,\tilde{y}(x);\alpha\right)=F\left(x,\underline{\tilde{y}}(x;\alpha),\overline{\tilde{y}}(x;\alpha)\right) \quad (3.27)$$

Also:

$$\tilde{y}_\alpha(x_0)=\tilde{y}_{0_\alpha}=\left[\underline{\tilde{y}}_0(\alpha),\overline{\tilde{y}}_0(\alpha)\right]$$

3.3.3 General Concepts of Runge-Kutta Methods:

There are two approaches for evaluating the solution of an ordinary differential equation, “analytically” or “numerically”. The analytic solution is usually obtained directly from the mathematical representation of the model formulation, while the numerical solution is generally an approximate solution obtained at certain node points.

The idea of extending the Euler method by allowing for a multiplicity of evolutions of the function f^* within each step was originally proposed by Runge (1895). Further contributions were made by Heun (1900) and by Kutta (1901). The letter completely characterized the set of Runge-Kutta method of order 4 and proposed the first methods of order 5. Special methods for second-order differential equations were proposed by Nystrom (1925) who also contributed to the development of methods for first-order equations.

Since from advent of digital computers, a fresh interest had been focused on Runge-Kutta methods, and a large number of research workers have contributed to recent extensions to the theory and the development of particular methods. Although, early studies were devoted entirely to explicit Runge-Kutta methods, interest has now extended to implicit methods, which are now recognized as appropriate for stiff differential equations. (see [Butcher, 1964]).

3.3.3.1 Formulation of Runge-Kutta Methods:

The general form of an R-stages Runge-Kutta method for solving fuzzy fractional ordinary differential equation in its α-level sets (for simplicity the α-level sets is termed by $\tilde{y}(\alpha)$ instead of $\tilde{y}_\alpha$) is given by:

$$\tilde{y}_{n+1}(\alpha) = \tilde{y}_n(\alpha) + h\sum_{i=1}^{R} c_i\, \tilde{k}_i(\alpha)$$

where $\alpha \in [0,1]$ and,

$$\tilde{k}_i(\alpha) = f^*\left(x_n + h a_i, \tilde{y}_n(\alpha) + h\sum_{s=1}^{R} b_{is}\, \tilde{k}_s \right)$$

and

$$a_i = \sum_{s=1}^{R} b_{is}$$

where c_i, a_i and b_{is}, for all i, s=1,2,...,R; are constants to be determined.

Then the general form of an upper R-stages Runge-Kutta methods is given by:

$$\overline{\tilde{y}}_{n+1}(\alpha) = \overline{\tilde{y}}_n(\alpha) + h\sum_{i=1}^{R} c_i\, \overline{\tilde{k}}_i(\alpha)$$

where

$$\overline{\tilde{k}}_i(\alpha)=\overline{f}^*\left(x_n+ha_i,\overline{\tilde{y}}_n(\alpha)+h\sum_{s=1}^{R}b_{is}\,\overline{\tilde{k}}_s(\alpha)\right)$$

Similarly, the general form of lower R-stages Runge-Kutta method is given by:

$$\underline{\tilde{y}}_{n+1}(\alpha)=\underline{\tilde{y}}_n(\alpha)+h\sum_{i=1}^{R}c_i\,\underline{\tilde{k}}_i(\alpha)$$

where

$$\underline{\tilde{k}}_i(\alpha)=\underline{f}^*\left(x_n+ha_i,\underline{\tilde{y}}_n(\alpha)+h\sum_{s=1}^{R}b_{is}\,\underline{\tilde{k}}_s(\alpha)\right)$$

For convenience, we design the process by an array of constants, as follows:

$$\begin{array}{cccc|c} b_{11} & b_{12} & \cdots & b_{1R} & a_1 \\ b_{21} & b_{22} & \cdots & b_{2R} & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{1R} & b_{2R} & \cdots & b_{nR} & a_R \\ \hline c_1 & c_2 & \cdots & c_s & \end{array}$$

And it is easy to classify Runge-Kutta methods, as follows:

- If $b_{ij}=0$, $\forall i<j$, then the method is called semi-explicit.
- If $b_{ij}=0$, $\forall i\leq j$, then the method is called explicit.
- Otherwise it is called implicit.

3.4 Numerical Examples:

In this section, some illustrative examples are given as a comparison between the numerical methods and focuse on the powerfull approaches used in solving this new field in differential equations, which are fuzzy fractional order differential equations.

Example (3.3):

Consider the non-linear FFODE:

$$\tilde{y}^{(1/4)}(x) = \tilde{y}^{-3/4}\exp(-1/\tilde{y}), \quad \tilde{y}(0;\alpha) = [0.1 + 0.9\alpha, 1.5 - 0.5\alpha]$$

over the interval [0, 1].

In order to solve this equation, we given the following alterative form:

$$\tilde{y}' = D^{1-1/4}\left(\tilde{y}^{-3/4}\exp(-1/\tilde{y})\right)$$
$$= \tilde{y}^{-5/4}\exp(-1/\tilde{y})$$

and since:

$$\tilde{y}' = \tilde{y}^{-5/4}\exp(-1/\tilde{y}), \ \tilde{y}(0;\alpha) = [0.1 + 0.9\alpha, 1.5 - 0.5\alpha] \quad \text{.............} (3.28)$$

Now, then from eq. (3.28), we have:

$$\begin{aligned} \underline{\tilde{y}}'(x;\alpha) &= \underline{f}^*(x, \tilde{y}(x);\alpha) = G^*\left(x, \underline{\tilde{y}}(x;\alpha), \overline{\tilde{y}}(x;\alpha)\right) \\ &= \underline{\tilde{y}}^{-5/4}(x;\alpha)\exp\left(-1/\underline{\tilde{y}}(x;\alpha)\right) \\ \overline{\tilde{y}}'(x;\alpha) &= \overline{f}^*(x, \tilde{y}(x);\alpha) = F\left(x, \underline{\tilde{y}}(x;\alpha), \overline{\tilde{y}}(x;\alpha)\right) \\ &= \overline{\tilde{y}}^{-5/4}(x;\alpha)\exp\left(-1/\overline{\tilde{y}}(x;\alpha)\right) \end{aligned} \quad \text{......................} (3.29)$$

This example has no analytic solution therefore numerical methods will be used with step size h=0.1.

Using Euler's method:

$$\begin{aligned} \underline{\tilde{y}}_{n+1}(\alpha) &= \underline{\tilde{y}}_n(\alpha) + hG^*\left(x, \underline{\tilde{y}}(x;\alpha), \overline{\tilde{y}}(x;\alpha)\right) \\ \overline{\tilde{y}}_{n+1}(\alpha) &= \overline{\tilde{y}}_n(\alpha) + hF\left(x, \underline{\tilde{y}}(x;\alpha), \overline{\tilde{y}}(x;\alpha)\right) \end{aligned} \quad \text{...............................} (3.30)$$

Then from eq. (3.29) and eq. (3.30):

$$\begin{aligned} \underline{\tilde{y}}_{n+1}(\alpha) &= \underline{\tilde{y}}_n(\alpha) + h\left(\underline{\tilde{y}}_n^{-5/4}(\alpha)\exp\left(-1/\underline{\tilde{y}}_n(\alpha)\right)\right) \\ \overline{\tilde{y}}_{n+1}(\alpha) &= \overline{\tilde{y}}_n(\alpha) + h\left(\overline{\tilde{y}}_n^{-5/4}(\alpha)\exp\left(-1/\overline{\tilde{y}}_n(\alpha)\right)\right) \end{aligned} \quad \text{........................} (3.31)$$

Similarly, ***Runge-Kutta method*** can be applied which has the form:

$$\tilde{y}_{n+1}(\alpha) = \tilde{y}_n(\alpha) + h\left(c_1\tilde{k}_1(\alpha) + c_2\tilde{k}_2(\alpha)\right)$$

such that $c_1, c_2 = 1/2$, $a_2 = b_{21} = 1$ and

$$\underline{\tilde{k}}_1(\alpha) = \underline{f}^*\left(x, \underline{\tilde{y}}_n(\alpha)\right) = G^*\left(x, \underline{\tilde{y}}(x;\alpha), \overline{\tilde{y}}(x;\alpha)\right)$$
$$= \underline{\tilde{y}}_n^{-5/4}(\alpha)\exp\left(-1/\underline{\tilde{y}}_n(\alpha)\right)$$

$$\overline{\tilde{k}}_1(\alpha) = \overline{f}^*\left(x, \overline{\tilde{y}}_n(\alpha)\right) = F\left(x, \underline{\tilde{y}}(x;\alpha), \overline{\tilde{y}}(x;\alpha)\right)$$
$$= \overline{\tilde{y}}_n^{-5/4}(\alpha)\exp\left(-1/\overline{\tilde{y}}_n(\alpha)\right)$$

and

$$\underline{\tilde{k}}_2(\alpha) = \underline{f}^*\left(x_n + a_2 h, \underline{\tilde{y}}_n(\alpha) + h b_{21}\underline{\tilde{k}}_1\right)$$
$$= \left(\underline{\tilde{y}}_n(\alpha) + h\left(\underline{\tilde{y}}_n^{-5/4}(\alpha)\exp\left(-1/\underline{\tilde{y}}_n(\alpha)\right)\right)\right)^{-5/4}.$$
$$\exp\left(-1/\underline{\tilde{y}}_n(\alpha) + h\left(\underline{\tilde{y}}_n^{-5/4}(\alpha)\exp\left(-1/\underline{\tilde{y}}_n(\alpha)\right)\right)\right)$$

$$\overline{\tilde{k}}_2(\alpha) = \overline{f}^*\left(x_n + a_2 h, \overline{\tilde{y}}_n(\alpha) + h b_{21}\overline{\tilde{k}}_1\right)$$
$$= \left(\overline{\tilde{y}}_n(\alpha) + h\left(\overline{\tilde{y}}_n^{-5/4}(\alpha)\exp\left(-1/\overline{\tilde{y}}_n(\alpha)\right)\right)\right)^{-5/4}.$$
$$\exp\left(-1/\overline{\tilde{y}}_n(\alpha) + h\left(\overline{\tilde{y}}_n^{-5/4}(\alpha)\exp\left(-1/\overline{\tilde{y}}_n(\alpha)\right)\right)\right)$$

Hence

$$\begin{aligned} \underline{\tilde{y}}_{n+1}(\alpha) &= \underline{\tilde{y}}_n(\alpha) + \frac{h}{2}\left(\underline{\tilde{k}}_1(\alpha) + \underline{\tilde{k}}_2(\alpha)\right) \\ \overline{\tilde{y}}_{n+1}(\alpha) &= \overline{\tilde{y}}_n(\alpha) + \frac{h}{2}\left(\overline{\tilde{k}}_1(\alpha) + \overline{\tilde{k}}_2(\alpha)\right) \end{aligned} \quad \text{.. (3.32)}$$

Also, carrying these two eq.'s (3.31) and (3.32) for n=0, 1, ..., we get the following results in table (3.3) using program (Prog. 1).

Table (3.3)
Numerical results of example (3.3)

h=0.1 α	Euler's Method $\tilde{y}_{n+1} = \tilde{y}_n + h\tilde{y}'_n$		Runge-Kutta $\tilde{y}_{n+1} = \tilde{y}_n + \frac{h}{2}\left[\tilde{k}_1 + \tilde{k}_2\right]$	
	$\underline{\tilde{y}}(\alpha)$	$\overline{\tilde{y}}(\alpha)$	$\underline{\tilde{y}}(\alpha)$	$\overline{\tilde{y}}(\alpha)$
0	0.1	1.519	0.1	1.51
0.2	0.294	1.421	0.287	1.412
0.4	0.49	1.324	0.479	1.314
0.6	0.677	1.228	0.664	1.216
0.8	0.858	1.132	0.8433	1.118
1.0	1.037	1.037	1.022	1.022

For comparison perpouse, the crisp solution could be evaluated numerically, using Euler method which is given by: $y_1 = 1.036788$, while using Runge-Kutta method we have: $y_1 = 1.02115$.

Example (3.4):

Consider the FFODE:

$$D^{3/2}\tilde{y}(x) = \tilde{y}, \quad \tilde{y}(0;\alpha) = [0.5 + 0.5\alpha, 1.25 - 0.25\alpha]$$

over the interval [0,1].

In order to solve the above FFODE, we multiply both sides by $D^{1-3/2}$, to get:

$$\tilde{y}'(x) = D^{1-3/2}\tilde{y}$$

$$= \frac{2}{\sqrt{\pi}}\tilde{y}^{1/2} \qquad (3.33)$$

Then from eq. (3.33), we have:

$$\left.\begin{aligned}\underline{\tilde{y}}'(x;\alpha) &= \underline{f}^{*}\left(x,\tilde{y}(x);\alpha\right) = G^{*}\left(x,\underline{\tilde{y}}(x;\alpha),\overline{\tilde{y}}(x;\alpha)\right) = \frac{2}{\sqrt{\pi}}\underline{\tilde{y}}^{1/2}(x;\alpha)\\ \overline{\tilde{y}}'(x;\alpha) &= \overline{f}^{*}\left(x,\tilde{y}(x);\alpha\right) = F\left(x,\underline{\tilde{y}}(x;\alpha),\overline{\tilde{y}}(x;\alpha)\right) = \frac{2}{\sqrt{\pi}}\overline{\tilde{y}}^{1/2}(x;\alpha)\end{aligned}\right\} \qquad \text{........................ (3.34)}$$

Using Adam-Bashforth method:

$$\begin{aligned}\underline{\tilde{y}}_{n+2}(\alpha) &= \underline{\tilde{y}}_{n+1}(\alpha) + \frac{h}{2}\Big(3G^{*}\left(x_{n+1},\underline{\tilde{y}}_{n+1}(\alpha),\overline{\tilde{y}}_{n+1}(\alpha)\right) - \\ &\qquad G^{*}\left(x_n,\underline{\tilde{y}}_n(r),\overline{\tilde{y}}_n(r)\right)\Big)\\ \overline{\tilde{y}}_{n+2}(\alpha) &= \overline{\tilde{y}}_{n+1}(\alpha) + \frac{h}{2}\Big(3F\left(x_{n+1},\underline{\tilde{y}}_{n+1}(\alpha),\overline{\tilde{y}}_{n+1}(\alpha)\right) - \\ &\qquad F\left(x_n,\underline{\tilde{y}}_n(\alpha),\overline{\tilde{y}}_n(\alpha)\right)\Big)\end{aligned} \qquad \text{........ (3.35)}$$

From eq. (3.34) and eq. (3.35), we have:

$$\underline{\tilde{y}}_{n+2}(\alpha) = \underline{\tilde{y}}_{n+1}(\alpha) + \frac{h}{\sqrt{\pi}}\left(3\underline{\tilde{y}}_{n+1}^{1/2}(\alpha) - \underline{\tilde{y}}_{n}^{1/2}(\alpha)\right)$$

$$\overline{\tilde{y}}_{n+2}(\alpha) = \overline{\tilde{y}}_{n+1}(\alpha) + \frac{h}{\sqrt{\pi}}\left(3\overline{\tilde{y}}_{n+1}^{1/2}(\alpha) - \overline{\tilde{y}}_{n}^{1/2}(\alpha)\right)$$

For n=0 and taking h=0.1, 0.05, then we have:

$$\begin{aligned}\underline{\tilde{y}}_{2}(\alpha) &= \underline{\tilde{y}}_{1}(\alpha) + \frac{h}{\sqrt{\pi}}\left(3\underline{\tilde{y}}_{1}^{1/2}(\alpha) - \underline{\tilde{y}}_{0}^{1/2}(\alpha)\right)\\ \overline{\tilde{y}}_{2}(\alpha) &= \overline{\tilde{y}}_{1}(\alpha) + \frac{h}{\sqrt{\pi}}\left(3\overline{\tilde{y}}_{1}^{1/2}(\alpha) - \overline{\tilde{y}}_{0}^{1/2}(\alpha)\right)\end{aligned} \qquad \text{.................................... (3.36)}$$

The results of eq. (3.36) are presented in table (3.4) using program (Prog. 2).

Table (3.4)
Numerical results of example (3.4) using Adam's method

α	Adam-Bashforth with h=0.1		Adam-Bashforth with h=0.05	
	$\underline{y}(\alpha)$	$\overline{y}(\alpha)$	$\underline{y}(\alpha)$	$\overline{y}(\alpha)$
0	0.691	1.484	0.593	1.365
0.2	0.798	1.431	0.696	1.314
0.4	0.904	1.379	0.8	1.262
0.6	1.01	1.326	0.903	1.211
0.8	1.116	1.274	1.006	1.16
1	1.221	1.221	1.109	1.109

Comparing the fuzzy solution with level α =1 with the crisp solution is given by: $y_2 = 1.221$ at h=0.1 and $y_2 = 1.109$ at h=0.05.

Conclusions and Recommendations

Form the present study, we can conclude the following:

1. Exact solution of fuzzy fractional differential equations, may be in sometimes so difficult to be evaluated, especially in non-linear cases.

2. The method of successive approximations for solving integral equations can be used to solve fuzzy fractional ordinary differential equations.

Also, we can recommend the following problems for future work:

1. Studying the existence and uniqueness theorem of fuzzy fractional differential equations using, such as, Bourbaki-Kneser fixed point theorem, Amann and Tarski fixed point theorem, etc.

2. Extending the work of this thesis to study the solution of fuzzy fractional partial differential equations, numerically and analytically.

3. Studying fuzzy fractional differential equations with boundary conditions.

Contents

Introduction

Since its inception of 40 years ago, the theory of fuzzy sets has advanced in a variety of ways and in many disciplines. Applications of this theory can be found, for example, in artificial intelligence, computer science, control engineering, decision theory, expert systems, logic, management, science operation research and robotics. Further more, it has developed theoretically and it has been applied to new areas of real life problems. Moreover, in every day life, we are used too properties which can not be dealt with satisfactorily on a simple “belong” or “not belong” basis. Whether these properties perhaps best indicated by a shade of gray, rather than by the black or white, assigning each individual in a population on a “belong” or “not belong” value, as is done in ordinary set theory is not an adequate way of dealing with properties of this type.

Zadeh had introduced fuzzy set theory in 1965, in which, Zadeh’s original definition of fuzzy sets could be given as follows “a fuzzy set is a class of objects with a continuum grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership value ranging between zero and one”, [Zadeh , 1965].

Since 1965, fuzzy theoretical approach had developed by Zadeh himself and about 300 other researchers as a tool of modeling human centered systems, in which they applied this theory in a wide range of scientific and engineering areas, [Yan, 1994].

Also, the theory of fuzzy sets have been applied in 1965 by Zadeh for solving fuzzy differential equations. In addition in this thesis new area to the subject is devoted to solve fuzzy fractional order differential equations, in which fractional differential equation could be considered as an important type of differential equations, where the differeintegration that appears in the equation is of non-integer order. Such type of problems may be considered to have the form:

$$y^{(q)} = f(x, y), \quad y^{(q-k)}(x_0) = y_0$$

where k=1, 2,..., n+1; n<q<n+1, and n is an integer number.

Real life problems with fractional differential equations are of great importance, since fractional differential equations accumulate the whole information of the function in a weighted form. This has many applications in physics, chemistry, engineering, etc. For that reason, we need a method to solve such equations, effectively, easy to use and applied in different problems.

However, the well known methods used for solving fractional differential equations have some difficulties rather than that of usual methods for solving ordinary differential equations, and therefore this thesis is oriented towards introducing a new type of equations called the fuzzy fractional order differential equation and to introduce some methods for solving such type of equations by using sutible numerical methods.

This thesis consists of three chapters, the first chapter devoted to discuss the general concepts of fuzzy differential equation and fractional differential equation, which is necessary for understanding and solving the fuzzy fractional order differential equation, as well as some of the basic definitions in mathematics.

Chapter two discuss the existence and uniqueness theorem of fuzzy fractional order differential equations (FFODE's) using Sadoviskii's fixed-point theorem for condensing mapping.

In chapter three, we discuss some types of fuzzy fractional order differential equations and they are solution using some well known analytical methods and numerical methods.

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