In this chapter we discuss some basic necessary functions that will arise in our study. These are Gamma function, Beta function, Laplace transforms and Mittag-Leffler function, the fractional integral and fractional derivative has been presented.

1.1 BASIC MATHMATICAL FUNCTIONS

To understand the fractional calculus definitions and making their use to be clear, we shall quickly discuss some basic mathematical functions are addressed in the following subsections.

1.1.1 Gamma Function [Erdelyi, 1981].

The Gamma function $\Gamma(x)$ is one of basic functions which plays an important role on the theory of differentiation and generalizes the ordinary definition of factorial of an integer number n and allows n to take also any non- negative integer.

The integral transform definition for the $\Gamma(x)$ is given by

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy \ , \qquad x > 0 \qquad (1.1)$$

The following are the most important properties of gamma function:

1. $\Gamma(1) = 1$.
2. $\Gamma(x+1) = x\Gamma(x)$, X is a non- negative integer.
3. $\Gamma(n+1) = n!$, n is positive integer.

The following are some frequently encounter examples of gamma functions for different value of x.

$$\Gamma(-1)=\mp\infty,\ \Gamma(0)=\mp\infty,\ \Gamma(1)=1,\ \Gamma(2)=1,\ \Gamma(3)=2,\ \Gamma\left(\frac{-3}{2}\right)=\frac{4}{3}\sqrt{\pi},$$

$$\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\sqrt{\pi},\ \Gamma\left(\frac{-1}{2}\right)=-2\sqrt{\pi},\ \Gamma\left(\frac{5}{2}\right)=\frac{3}{4}\sqrt{\pi}.$$

These properties enable us to calculate for any positive real x the fraction $\Gamma(x)$ in terms of the fractional part of x .The following example gives a good illustration:

Γ (7/2) =5/2 Γ (5/2) = (5/2) (3/2) Γ (3/2) = (5/2) (3/2) (1/2) Γ (1/2)
where Γ (1/2) = $\sqrt{\pi}$.

1.1.2 Beta Function [Loverro, A,2004].

The Beta function is an important relationship in fractional calculus. Its solution not only defined through the use of multiple Gamma Function, but furthermore shares a form that is characteristically similar to the Fractional integral / derivative of many functions, such as, the Mittag-Leffler function (which will be discussed next). For positive values of the two parameters, p and q, the function is defined by the following integral

$$B(p,q)=\int_0^1 y^{p-1}[1-y]^{q-1}\,dy, \qquad p,q>0 \qquad (1.2\ )$$

Is called complete Beta function, also, known as Euler's integral of the second kind. If either p or q is non- positive integer, the integral diverges, The Beta function is related to the gamma function according to the following formula:

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad (1.3)$$

1.1.3 Laplace Transform [Loverro, A, 2004].

The Laplace transform is a function transformation commonly used in the solution of complicated differential equations. With the Laplace transform it is frequently possible to avoid working with equations of different differential order directly by translating the problem into a domain where the solution presents itself algebraically. The formal definition of the Laplace transform is given by

$$L\{f(t)\}=\int_0^{\infty} e^{-st} f(t)dt=\tilde{f}(s), \qquad s>0 \qquad (1.4)$$

The Laplace transform of the function $f(t)$ is said to be exist if (1.4) is a convergent integral. The requirement for this is that $f(t)$ does not grow at a rate higher than the rate at which the exponential term e^{-st} decreases

Also commonly used is the Laplace convolution, given by

$$f(t)*g(t)=\int_0^{t} f(t-\tau)g(\tau)d\tau=g(t)*f(t) \qquad (1.5)$$

The convolution of two functions in the domain of t is sometime complicated to resolve, however, in the Laplace domain(s), the convolution results in the simple function multiplication as shown in

$$L\{f(t)*g(t)\} = \tilde{f}(s)\tilde{g}(s) \quad (1.6)$$

One final important property of the Laplace transform that should be addressed is the Laplace transform of a derivative of integer order n of a function $f(t)$, given by

$$L\{f^{(n)}(t)\} = s^n \tilde{f}(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) \quad (1.7)$$

1.1.4 Mittage –Leffler function [Dzherbashyan, 1966].

The Mittage –Leffler function is an important function that finds widespread in the world of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the Mittage –Leffler function plays an analogous role in the solution of non-integer order differential equations. In fact, the exponential function itself a very specific form, one of an infinite series, of this seemingly omnipresent function.

The standard definition of the Mittage –Leffler is given by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \quad (1.8)$$

In particular, when α =1 and α=2, we have

$$E_1(z) = e^z \, and \quad E_2(z) = \cosh(z)$$

The generalized form of the Mittage –leffler function $E_{\alpha,\beta}(z)$, is

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \qquad (1.9)$$

In particular,

$$E_{\alpha,1}(z)=E_{\alpha}(z), \quad E_{1,2}(z)=\frac{e^z-1}{z} \text{ , and } \quad E_{2,2}(z)=\frac{\sinh(\sqrt{z})}{\sqrt{z}}$$

There are some special functions defined in term, of the Mittage-leffler function $E_{\alpha,\alpha}(z)$ these functions play the main role in investigating of what so- called differential equations of fractional order, such as

$$z^{\beta-1}E_{\alpha,\beta}(\lambda z^{\alpha})$$

and its Laplace transform formula

$$L\left[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\right]=\frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda} \qquad (\lambda\in C, \left|\lambda s^{-\alpha}<1\right| \qquad (1.10)$$

1.2 THE FRACTIONAL INTEGRAL **[Loverro, A, 2004].**

The formulation of the concept for fractional integrals and derivatives was a natural outgrowth of integer order integrals and derivatives in such the same way that the fractional exponent follows from the more traditional integer order exponent. For the latter, it is the notation that makes the jump seems obvious. While one can not imagine the multiplication of a quantity a fractional number of times, there seems no practical restriction to placing a non- integer into the exponential position.

The common formulation for the fractional integral can be derived directly from a traditional expression of the repeated integration of a function. This approach is commonly refereed to as the Riemann-Liouvill approach. The following formula

$$\int \dots \int_0^t f(\tau)\,d\tau = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)\,d\tau \qquad (1.11)$$

demonstrates usually attributed to Cauchy for evaluating the n^{th} integration of the function $f(t)$. For the abbreviated representation of formula (1.11), we introduce the operator I^n as shown

$$I^n f(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau \qquad (1.12)$$

Often, one will also find another operator, D^{-n}, used in place of I^n. While they represent the same formulation of the repeated integral function, and can be seen as interchangeable, one will find that use of D^{-n} may become misleading, especially when multiple operators are used in combination.

For direct use in (1.12) , n is restricted to be an integer. The primary restriction is the use of the fractional which in essence has no meaning for non- integer values. The gamma function is however an analytic expansion of the fractional for all reals, and thus can be used by replacing the fractional expression for its gamma function equivalent, we can generalize (1.13) for all $\alpha \in R_+$, as shown , below

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \qquad (1.13)$$

This formulation of the fractional integral carries with it some very important Properties that will later show importance when solving equations involving integrals and derivatives of fractional order. First, we consider integrations of order $\alpha = 0$ to be an identity operator

$$I^0 \ f(t) = f(t) \qquad (1.14)$$

Also, given the nature of the integral's definition, and based on the principle from which it came (Cauchy Repeated Integral Equation), we can see that just as

$$I^n I^m = I^{m+n} = I^m I^n \ , \qquad m,n \in N \qquad (1.15)$$

so to,

$$I^{\alpha} I^{\beta} = I^{\alpha+\beta} = I^{\beta} I^{\alpha} , \qquad \alpha , \beta \in R \qquad (1.16)$$

The one presupposed condition placed upon a function $f(t)$ that needs to be satisfied for these and other similar properties to remain true, is that $f(t)$ be a causal function, i.e. that it is vanishing for t $\leq$ 0. although this is a consequence of convention, the convenience of this condition is especially clear in the context of the property demonstrated in (1.16). The effect is such that $f(0) = f_n(0) = f_{\alpha}(0) \equiv 0$.

An additional property of the Reimann-Louivill integral appears after the introduction of the function Φ_{α} as

$$\Phi_{\alpha} = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \Rightarrow \Phi_{\alpha}(t) * f(t) = \int_0^t \frac{(t-\tau)_+^{\alpha-1}}{\Gamma(\alpha)} f(\tau) \qquad (1$$

where $(t-\tau)_+$ denotes the function vanishes for $t \leq 0$ and hence, (1.17) is a causal function. From the definition of the Laplace Convolution given in (1.5), it follows that

$$I^{\alpha} f(t) = \Phi_{\alpha}(t) * f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \qquad (1.18)$$

Here we shall also find the Laplace transform of the Reimann-Louivill fractional integral. In (1.17) we show that the fractional integral could be expressed as the convolution of two terms, Φ_{α} and $f(t)$.The Laplace Transform of $t^{\alpha-1}$ is given by

$$L\{t^{\alpha-1}\} = \Gamma(\alpha) s^{-\alpha}, \qquad \alpha \in R \qquad (1.19)$$

Thus, given the convolution relation to the fractional integral through Φ_{α} demonstrated in (1.19), and the Laplace Transform of the convolution shown in (1.5), lead to The Laplace Transform of the fractional integral to be

$$L\{I^{\alpha}\} = s^{-\alpha} \tilde{f}(s) \qquad (1.20)$$

1.3 FRACTIONAL DERIVATIVE [J.D.Munkhammar, 2005].

Because the Reimann-Louivill approach to the fractional derivative began with an expression for the repeated integration of a function, one's first instinct may be to imitate a similar approach for the derivative.It is also possible to formulate a definition for the fractional order derivative using the definition already obtained for the analogous integral.

In the same fashion, as in the definition of fractional integral we let

$$ {}_aD^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{\alpha}} dt \qquad (1.21)$$

which is called the Riemann-Liouville fractional derivative of f of order α ,where x.> a .The connection between the Riemann-Liouville fractional integral and derivative can, as Riemann realized, be traced back to the solvability of Abel's integral equation for any $\alpha \in (0,1)$

$$ f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\Phi(t)}{(x-t)^{1-\alpha}} dt , \qquad x > 0 \qquad (1.22)$$

Equation (1.22) can be solved by changing x *to* t and t *to* s respectively, further by multiplying both sides of the equation by $(x-t)^{-\alpha}$ and integrating we get:

$$ \int_a^x \frac{dt}{(x-t)^{\alpha}} \int_a^t \frac{\Phi(s)\ ds}{(t-s)^{1-\alpha}} = \Gamma(\alpha) \int_a^x \frac{f(t)\ dt}{(x-t)^{\alpha}} \qquad (1.23)$$

Interchanging the order of integration in the left hand side we obtain

$$\int_a^x \Phi(s)\,ds \int_s^x \frac{dt}{(x-t)^{\alpha}(t-s)^{1-\alpha}} = \Gamma(\alpha)\int_a^x \frac{f(t)\ dt}{(x-t)^{\alpha}} \qquad (1.24)$$

The inner integral is easily evaluated after the change of variable $t = s + \tau(x-s)$ and use of the formulae of the Beta-function

$$\int_a^x (x-t)^{-\alpha}(t-s)^{\alpha-1}dt = \int_0^1 \tau^{\alpha-1}(1-\tau)^{-\alpha}d\tau = B(\alpha,1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha) \qquad (1.25)$$

Therefore we get

$$\int_a^x \Phi(s)\,ds \ = \frac{1}{\Gamma(1-\alpha)}\int_a^x \frac{f(t)\ dt}{(x-t)^{\alpha}} \qquad (1.26)$$

Hence after differentiation we have

$$\Phi(x) \ = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)\ dt}{(x-t)^{\alpha}} \qquad (1.27)$$

Thus if (1.22) has a solution it is necessarily given by (1.27) for any $\alpha \in (0,1)$. One observe that (1.22) is in a sense the α- order integral and the inversion (1.27) is the α- order derivative.

This chapter is devoted to derive the solutions of Cauchy type problems of partial fractional order differential equations on a finite intervals and proving their existence and uniqueness.

Many authers (**[Gerasimov, A. N, 1948]** **[Zhemukhov,K. K,1986]** **[Conlan, J, 1983]**) derived and solved partial fractional order differential equations for special applied problems.

Gerasimov **[Gerasimov, A. N, 1948]** derived and solved fractional –order partial differential equations for special applied problems. He studied two problems of viscoelasticity describing the motion of a viscous fluid between moving surfaces, and reduced these problems to the partial differential equations of fractional order. Zhemukhov **[Zhemukhov,K. K,1986]** studied the second order degenerate loaded hyperbolic equations involving the partial Riemann-Lioville fractional derivatives. He reduced these problems to equivalent non-linear Voltera integral equations of the second kind and proved their solvability by the method of the contraction mapping principle, and then solved Cauchy problems for these Voltera equations in order to obtain the solutions

Conlan **[Conlan, J, 1983]** considered the nonlinear partial differential equation and used the Banach Contraction Mapping Principle, he proved local theorems for the existence and the uniqueness of the solution of the corresponding integral equation.

Atanackovic and Stankovic (**[Atanackovic, 2002] [Stankovic, 2002]**) used the Laplace transform to solve a system of partial differential equations with fractional derivatives. Kilbas and Repin **[Kilbas, 2004]** studied the mixed type Riemann-Lioville fractional derivative.

In this work we consider a partial fractional order differential equation of the form

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} u(x,t) = \lambda^2 \Delta_x u(x,t) \qquad (x \in R^n,\ \ t > 0,\ \ 0 < \alpha \le 1,\ ) \qquad (3.1)$$

involving the partial RL-fractional derivative of order α w.r.t t >0 defined by (2.4) and the Laplacin $\Delta_x u(x,t)$ w.r.t $x \in R^n$

$$\Delta_x u(x,t) = \frac{\partial^2 u(x,t)}{\partial x_1^2} + ... + \frac{\partial^2 u(x,t)}{\partial x_n^2} \qquad (n \in N) \qquad (3.2)$$

In particular, when n=1, the equation (3.1) takes the form

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} u(x,t) = \lambda^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad (3.3)$$

known as the fractional diffusion heat equation, with the Cauchy type initial condition

$$\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}} u(x,t)\Big|_{t=0} = f(x) \qquad (3.4)$$

where $x \in R^n$, $0 < \alpha \le 1$. Also, $u(0,t) = u_x(0,t) = 0$.

Theorem (3.1): [Oldham, 1974]

1- $L_t\left\{\frac{\partial^\alpha f(x,t)}{\partial t^\alpha}\right\}=s^\alpha L_t\{f(x,t)\}-\sum_{k=0}^{n-1}s^k\frac{\partial^{\alpha-1-k}f(x,t)}{\partial t^{\alpha-1-k}}\Big|_{t=0}$

$2-L_t\left\{\frac{\partial^\alpha}{\partial x^\alpha}f(x,t)\right\}=\frac{\partial^\alpha}{\partial x^\alpha}F(x,s)$

3- $L_t\left\{\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial t^\beta}f(x,t)\right\}=s^\beta\frac{\partial^\beta}{\partial x^\beta}F(x,s)-\sum_{k=0}^{n-1}s^k\frac{\partial^{\beta-1-k}}{\partial t^{\beta-1-k}}(\frac{\partial^\alpha}{\partial x^\alpha}f(x,t))\Big|_{t=0}$

Where n is an integer such that n-1< α < n .the term in the sum is vanishing where (α -1- k) < 0.

Proof:

1- for α <0, so that the R-L definition

$$\frac{\partial^\alpha f}{\partial t^\alpha}=\frac{1}{\Gamma(-\alpha)}\int_0^x\frac{f(x,y)}{(t-y)^{\alpha+1}}dy \quad , \alpha<0$$

may be used in direct application of convolution theorem which gives:

$$L_t\left\{\frac{\partial^\alpha f(x,t)}{\partial t^\alpha}\right\}=\frac{1}{\Gamma(-\alpha)}L_t\{t^{-1-\alpha}\}L_t\{f(x,t)\} \quad (3.5)$$

$$=s^\alpha L_t\{f(x,t)\}$$

For α > 0, we are using the property that:

$$\left(\frac{\partial^{\alpha} f(x,t)}{\partial t^{\alpha}}\right) = \frac{\partial^{n}}{\partial t^{n}}\left[\frac{\partial^{\alpha-n}}{(\partial(t)^{\alpha-n})}\right] f(x,t)$$

$$= \frac{\partial^{n}}{\partial t^{n}}\frac{\partial^{\alpha-n}}{\partial t^{\alpha-n}} f(x,t)$$

Where n-1< α <n .Now we are making use of the Laplace transform formula of integer derivative, we find that

$$L_t\left\{\frac{\partial^{\alpha} f(x,t)}{\partial t^{\alpha}}\right\} = L_t\left\{\frac{\partial^{n}}{\partial t^{n}}\left\{\frac{\partial^{\alpha-n} f(x,t)}{\partial t^{\alpha-n}}\right\}\right\}$$

$$= s^{n} L_t\left\{\frac{\partial^{\alpha-n} f(x,t)}{\partial t^{\alpha-n}}\right\} - \sum_{k=0}^{n-1} s^{k} \frac{\partial^{n-1-k}}{\partial t^{n-1-k}}\left(\frac{\partial^{\alpha-n}}{\partial t^{\alpha-n}}\right) f(x,t)$$

The first right-hand term may be evaluated by using (3.5), since α-n<0, the composition rule may be applied to the terms within the summation.

$$L_t\left\{\frac{\partial^{\alpha} f(x,t)}{\partial t^{\alpha}}\right\} = s^{\alpha} L_t\{f(x,t)\} - \sum_{k=0}^{n-1} s^{k} \frac{\partial^{\alpha-1-k} f(x,t)}{\partial t^{\alpha-1-k}} \Big|_{t=0}$$

2 - Since $\frac{\partial^{\alpha}}{\partial x^{\alpha}} f(x,t)$ is not derivative w.r.t. t, it a function of x, and the laplace transform w.r.t. t is $\frac{\partial^{\alpha}}{\partial x^{\alpha}} F(x,s)$.

(3) - $L_t\left\{\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{\beta}}{\partial t^{\beta}} f(x,t)\right\} = L_t\left\{\frac{\partial^{\beta}}{\partial t^{\beta}}\left(\frac{\partial^{\alpha}}{\partial x^{\alpha}} f(x,t)\right)\right\}$, using the definition (2.5)

$$= s^{\beta} L_t \left\{ \frac{\partial^{\alpha}}{\partial x^{\alpha}} f(x,t) \right\} - \sum_{k=0}^{n-1} s^k \frac{\partial^{\beta-1-k}}{\partial t^{\beta-1-k}} \left(\frac{\partial^{\alpha}}{\partial x^{\alpha}} f(x,t) \right) \Big|_{t=0}$$

$$= s^{\beta} \frac{\partial^{\alpha}}{\partial x^{\alpha}} F(x,s) - \sum_{k=0}^{n-1} s^k \frac{\partial^{\beta-1-k}}{\partial t^{\beta-1-k}} \left(\frac{\partial^{\alpha}}{\partial x^{\alpha}} f(x,t) \right) \Big|_{t=0}$$

■

Theorem (3.2):

The function

$$u(x,t) = D_t^{\alpha-1} u(x,0)[t^{\alpha-1} E_{\alpha,\alpha}((\lambda^2 / x) t^{\alpha})]$$

is the solution of equation (3.3) with initial condition (3.4), $0 < \alpha < 1$.

Proof:

Applying laplace transform formula to equation (3.3) w.r.t x

$$L_x \{ \frac{\partial^{\alpha}}{\partial t^{\alpha}} u(x,t) \} = L_x \left\{ \lambda^2 \frac{\partial^2 u(x,t)}{\partial x^2} \right\}$$

$$D_t^{\alpha} U(s,t) = \lambda^2 (s^2 U(s,t) - sU(0,t) - U_x(0,t))$$

$$D_t^{\alpha} U(s,t) = \lambda^2 s^2 U(s,t)$$

$$U(s,t) = \frac{D_t^{\alpha} U(s,t)}{\lambda^2} \frac{1}{s^2}$$

$$L_x^{-1} \{ U(s,t) \} = \frac{D_t^{\alpha} u(x,t)}{\lambda^2} L^{-1} \left\{ \frac{1}{s^2} \right\}$$

$$u(x,t) = \frac{x}{\lambda^2} D_t^{\alpha} u(x,t)$$

which can be written as

$$D_t^{\alpha} u(x,t) - (\lambda^2 / x) u(x,t) = 0 \tag{3.6}$$

This is Cauchy type problem of ordinary fractional order differential equation, in which the variable x is considered as a constant parameter.

Now, applying laplace transform w.r.t t

$$L_t\left\{D_t^{\alpha} u(x,t) - (\lambda^2 / x) u(x,t)\right\} = 0$$

$$s^{\alpha} L_t\{u(x,t)\} - s^0 D_t^{\alpha-1} u(x,t)\Big|_{t=0} - (\lambda^2 / x) L_t\{u(x,t)\} = 0$$

$$L_t\{u(x,t)\} = \frac{D_t^{\alpha-1} u(x,.t)\Big|_{t=0}}{s^{\alpha} - (\lambda^2 / x)}$$

and from (1.10)

$$L\left\{t^{\gamma_2 - 1} E_{\gamma_1,\gamma_2}((\lambda^2 / x) t^{\gamma_1})\right\} = \frac{s^{\gamma_1 - \gamma_2}}{s^{\gamma_1} - (\lambda^2 / x)}, \qquad \left| (\lambda^2 / x) s^{-\gamma_1} \right| < 1$$

with $\gamma_2 = \gamma_1 = \alpha$

$$L\left\{t^{\alpha-1} E_{\alpha,\alpha}((\lambda^2 / x) t^{\alpha})\right\} = \frac{1}{s^{\alpha} - (\lambda^2 / x)}$$

we can derive the following solution

$$u(x,t) = D_t^{\alpha-1} u(x,0)[t^{\alpha-1} E_{\alpha,\alpha}((\lambda^2 / x) t^{\alpha})]$$

■

Now, we could state the existence and uniqueness theorem:

Theorem (3.3):

Let $0<\alpha<1$, and $0\le\gamma<1$ be such that $\gamma\ge 1-\alpha$. Also, let $\lambda\in R$, if the Lipschitz condition

$\left|D_t^{\alpha}u(x,t_1)-D_t^{\alpha}u(x,t_2)\right| \le L\left|u(x,t_1)-u(x,t_2)\right|$,

$\left(L(t_1-a)^{\alpha}\Gamma(\alpha)/\Gamma(2\alpha)\right)<1$ and

$\left\|I_t^{\alpha}u(x,t)\right\|_{C_{1-\alpha}[a,b]} \le (b-a)^{\alpha}\dfrac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}\left\|u(x,t)\right\|_{C_{1-\alpha}[a,b]}$ are satisfied,

Then the Cauchy type problem (3.6) with initial condition (3.4) has a unique solution $u(x,t)\in C_{1-\alpha}[a,b]$ and this solution is given by

$$u(x,t)=f(x)[t^{\alpha-1}E_{\alpha,\alpha}((\lambda^2/x)t^{\alpha})]$$

Proof:

First we are proving that the solution of Cauchy problem (3.6) with initial condition (3.4) is equivalent to the solution of the following linear Volterra integral equation of the second kind

$$u(x,t)=\frac{f(x)t^{\alpha-1}}{\Gamma(\alpha)}+\frac{(\lambda^2/x)}{\Gamma(\alpha)}\int_0^t\frac{u(x,r)}{(t-r)^{1-\alpha}}\,dr \qquad (t>0) \qquad (3.7)$$

By applying the method of successive approximations to solve (3.6), we set

$$u_0(x,t)=\frac{f(x)}{\Gamma(\alpha)}t^{\alpha-1} \qquad (3.8)$$

$$u_m(x,t) = u_0(x,t) + \frac{(\lambda^2/x)}{\Gamma(\alpha)} \int_0^t \frac{u_{m-1}(x,r)}{(t-r)^{1-\alpha}} dr \tag{3.9}$$

Using the definition (1), equation (3.8) and taking equation (2.3) into account, we find:

$$u_m(x,t) = u_0(x,t) + (\lambda^2/x) I_t^{\alpha} u_0(x,t)$$

$$u_1(x,t) = \frac{f(x)}{\Gamma(\alpha)} t^{\alpha-1} + (\lambda^2/x) \frac{f(x)}{\Gamma(2\alpha)} t^{2\alpha-1} \tag{3.10}$$

$$= f(x)\sum_{k=1}^{2} \frac{(\lambda^2/x)^{k-1}}{\Gamma(k\alpha)} t^{k\alpha-1}$$

Similarly,

$$u_2(x,t) = \frac{f(x)}{\Gamma(\alpha)} t^{\alpha-1} + f(x) \sum_{k=2}^{3} \frac{(\lambda^2/x)^{k-1}}{\Gamma(\alpha k)} t^{\alpha k-1}$$

$$= f(x)\sum_{k=1}^{3} \frac{(\lambda^2/x)^{k-1}}{\Gamma(k\alpha)} t^{k\alpha-1} \tag{3.11}$$

Continuing this process, we derive the following

$$u_m(x,t) = f(x)\sum_{k=1}^{m+1} \frac{(\lambda^2/x)^{k-1}}{\Gamma(k\alpha)} t^{k\alpha-1} \tag{3.12}$$

By replacing the index of summation k by k+1, and taking $\lim m \to \infty$

$$u(x,t) = f(x)\sum_{k=0}^{\infty} \frac{(\lambda^2/x)^{k}}{\Gamma(k\alpha+\alpha)} t^{k\alpha+\alpha-1} \tag{3.13}$$

Taking into account the relation (1.9), we can rewrite this solution in terms of Mittag-Leffler Function $E_{\alpha,\beta}(z)$:

$$u(x,t) = f(x)\, t^{\alpha-1} E_{\alpha,\alpha}((\lambda^2 / x)\, t^{\alpha}) \qquad (3.14)$$

This yields an explicit solution to the linear Volterra integral equation of the second kind (3.7) and hence from theorem (3.2) it's also a solution to the Cauchy type problem (3.6)

Now, if (3.6) satisfy the Lipschitz condition

$$\left| D_t^{\alpha} u(x,t_1) - D_t^{\alpha} u(x,t_2) \right| \le L \; \left| u(x,t_1) - u(x,t_2) \right| \qquad (3.15)$$

Where L > 0, does not depend on t, and defining the operator T by

$$T(u(x,t)) = u_0(x,t) + \frac{(\lambda^2 / x)}{\Gamma(\alpha)} \int_0^t \frac{u(x,r)}{(t-r)^{1-\alpha}}\, dr \qquad (3.16)$$

as a map from the space $C_{1-\alpha}[a,b]$ onto $C_{1-\alpha}[a,b]$ where $C_{1-\alpha}[a,b]$ is the Banach space of functions $u(x,t)$ which are continuously differentiable on [a,b] up to order α **[Banach, S., 1932]** and have the derivative $\frac{\partial}{\partial t} u(x,t)$ on [a,b].

Now, we choose $t_1 \in (a,b)$ such that the condition

$$\left(L(t_1 - a)^{\alpha} \Gamma(\alpha) / \Gamma(2\alpha) \right) < 1 \qquad (3.17)$$

is satisfied, and rewrite the integral equation (3.7) in the form

$$u(x,t) = T\,u(x,t)$$

$$\| Tu_1 - Tu_2 \| \le K \; \| u_1 - u_2 \| \qquad (3.18)$$

It follows from (3.8) that $u_0 \in C_{1-\alpha}[a,t_1]$. Since $D_t^{\alpha} u(x,t) \in C_{1-\alpha}[a,t_1]$, then the integral in the right-hand side of (3.16) also belongs to $C_{1-\alpha}[a,t_1]$, and hence $T\,u(x,t) \in C_{1-\alpha}[a,t_1]$. Now by (3.16), (3.8) and the definition of $I_t^{\alpha} u(x,t)$, using the lipschitzian condition (3.15) and applying the relation

$$\| I_t^{\alpha} u(x,t) \|_{C_{\gamma[a,b]}} \le (b-a)^{\alpha} \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} \| u(x,t) \|_{C_{\gamma[a,b]}} \qquad (3.19)$$

with $\gamma = 1-\alpha, \quad b = t_1$

We have

$$\| Tu_1 - Tu_2 \|_{C_{1-\alpha}[a,t_1]} \le \left\| I_t^{\alpha} \left(\left| D_t^{\alpha} u(x,t_1) - D_t^{\alpha} u(x,t_2) \right| \right) \right\|_{C_{1-\alpha}[a,t_1]}$$

$$\le L \left\| I_t^{\alpha} \left(|u_1 - u_2| \right) \right\|_{C_{1-\alpha}[a,t_1]}$$

$$\le L\,(t_1 - a)^{\alpha} \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} \| u_1 - u_2 \|_{C_{1-\alpha}[a,t_1]}$$

which yields (3.18) with $K = L(t_1 - a)^{\alpha} \dfrac{\Gamma(\alpha)}{\Gamma(2\alpha)}$. In accordance with (3.17), 0 < k < 1, and by the classical “Banach fixed point Theorem”

[Banach, S., 1932] in a complete metric space, there exists a unique solution $u^*(x,t)=u_0^*(x,t)\in C_{1-\alpha}[a,t_1]$ to the integral equation (3.7) on the interval $[a,t_1]$. Also, this solution $u^*(x,t)$ is a limit of a convergent sequence $T^m\, u_0^*(x,t)$

$$\lim_{m\to\infty}\left\| T^m u_0^* - u^* \right\|_{C_{1-\alpha}[a,t_1]} = 0 \tag{3.20}$$

If $f(x)\neq 0$ in the initial condition (3.4), we can take $u^*(x,t)=u_0^*(x,t)$ defined by (3.8). the equation (3.20) can be rewritten in the form

$$\lim_{m\to\infty}\left\| u_m(x,t) - u\ (x,t) \right\|_{C_{1-\alpha}[a,t_1]} = 0 \tag{3.21}$$

where

$$u_m(x,t) = T^m u_0^*(x,t) = u_0(x,t) + \frac{(\lambda^2/x)}{\Gamma(\alpha)}\int_0^t \frac{T^{m-1}u_0^*(x,r)}{(t-r)^{1-\alpha}}\,dr \tag{3.22}$$

Next, we consider the interval $[t_1,t_2]$, where $t_2 = t_1 + h_1$, $h_1 > 0$ are such that $t_2 < b$. Rewrite (3.7) in the form

$$u(x,t) = u_{01}(x,t) + \frac{(\lambda^2/x)}{\Gamma(\alpha)}\int_{t_1}^t \frac{u(x,r)}{(t-r)^{1-\alpha}}\,dr \tag{3.23}$$

where $$u_{01}(x,t) = \frac{f(x)}{\Gamma(\alpha)}t^{\alpha-1} + \frac{(\lambda^2/x)}{\Gamma(\alpha)}\int_0^{t_1} \frac{u(x,r)}{(t-r)^{1-\alpha}}\,dr \tag{3.24}$$

is known function.

Using the same arguments as above, we derive that there exist a unique solution $u_1^*(x,t)\in C_{1-\alpha}[t_1,t_2]$ to the equation (3.7) on the interval $[t_1,t_2]$. taking the next interval $[t_2,t_3]$, where $t_3 = t_2 + h_2,\ h_2 > 0$ are such that $t_3 < b$, and repeating this process, we find that there exist a unique solution $u(x,t)$ to the equation (3.7) such that $u(x,t) = u_k^*(x,t)$ and $u_i^*(x,t)\in C_{1-\alpha}[t_{i-1},t_i]$, $(i = 1,...,n),\ where\ \ a = t_0 < t_1 < ... < t_n = b$.

Then there exists a unique solution $u(x,t)\in C_{1-\alpha}[a,b]$ on the whole interval [a,b]. Thus there exists a unique solution $u(x,t) = u^*(x,t)\in C_{1-\alpha}[a,b]$ to the linear Voltera integral equation of the second kind (3.7), and hence to the Cauchy type problem (3.4) & (3.6).

Now, we show that $D_t^\alpha u(x,t)\in C_\gamma[a,b]$. since $\gamma \geq 1-\alpha$, then

$$\left\| D_t^\alpha u_m - D_t^\alpha u \right\|_{c_\gamma[a,b]} \leq L \left\| u_m - u \right\|_{c_\gamma[a,b]}$$

$$\leq L\,(a,b)^{\gamma-1+\alpha} \left\| u_m - u \right\|_{C_{1-\alpha}[a,b]}$$

Thus, by the above estimates and the Lipschitzion condition (3.15), we find that

$$\lim_{m\to 0} \left\| D_t^\alpha u_m - D_t^\alpha u \right\|_{c_\gamma[a,b]} = 0$$

By the definition of the space

$$C_{1-\alpha}^\alpha[a,b] = \left\{ u(x,t)\in C^\alpha{}_{1-\alpha}[a,b] : D_t^\alpha u(x,t)\in C_\gamma[a,b] \right\}$$

Hence there exists a unique solution $u(x,t) \in C_{1-\alpha}[a,b]$ which has the property $D_t^{\alpha} u(x,t) \in C_{\gamma}[a,b]$.

Now, It is sufficient to prove the existence of a unique solution $u(x,t)$ to linear Voltera integral equation of the second kind (3.7). First we are proving the necessity, by letting $u(x,t) \in C^{\alpha}{}_{1-\alpha}[a,b]$ satisfy (3.4) & (3.6). according to the definition (1)

$$D_t^{\alpha} u(x,t) = \frac{d}{dt} I_t^{1-\alpha} u(x,t)$$

In which $I_t^{1-\alpha} u(x,t) \in AC_{1-\alpha}[a,b]$, from the lemma in chapter (2)

$$I_t^{\alpha} D_t^{\alpha} u(x,t) = u(x,t) - \frac{D_t^{\alpha-1} u(x,0)}{\Gamma(\alpha)} t^{\alpha-1}$$

Since the integral $\left(I_t^{\alpha} D_t^{\alpha} u(x,t)\right) \in L(a,b)$, exists almost every where on [a,b]. Applying the operator I_t^{α} to the both side of (3.6) and the definition of fractional integral I_t^{α} we obtain the following

$$u(x,t) = \frac{f(x)}{\Gamma(\alpha)} t^{\alpha-1} + \frac{(\lambda^2 / x)}{\Gamma(\alpha)} \int_0^t \frac{u(x,r)}{(t-r)^{1-\alpha}}\, dr \qquad (3.7)$$

and hence the necessity is proved.

Now, applying the operator D_t^{α} to both sides of (3.7), we have

$$D_t^{\alpha} u(x,t) = \frac{f(x)}{\Gamma(\alpha)} (D_t^{\alpha} t^{\alpha-1}) + D_t^{\alpha} I_t^{\alpha} ((\lambda^2 / x)\, u(x,t))$$

from property (3), we have (3.6)

Now, to show that (3.4) also hold, we apply the operator $D_t^{\alpha-k}$ (k=1,…,n) to both sides of (3.7). By property (3) and property (4) we have:

$$D_t^{\alpha-k}\, u(x,t) = \frac{f(x)}{\Gamma(\alpha)}\left(D_t^{\alpha-k}\, t^{\alpha-1}\right) + D_t^{\alpha-k} I_t^{\alpha}\left((\lambda^2/x)\,u(x,t)\right)$$

$$= \frac{f(x)}{\Gamma(\alpha)}\, t^{k-1} + I^{k}\left((\lambda^2/x)\,u(x,t)\right)$$

$$D_t^{\alpha-k}\, u(x,t) = \frac{f(x)}{(k-1)!}\, t^{k-1} + \frac{1}{(k-1)!}\int_0^t \frac{u(x,r)}{(x-r)^{1-k}}\, dr \qquad (3.25)$$

If k=n, then , in accordance with property (5) and property (3) we obtain

$$D_t^{\alpha-n}\, u(x,t) = \frac{f(x)}{(n-1)!}\, t^{n-1} + \frac{1}{(n-1)!}\int_0^t \frac{u(x,r)}{(x-r)^{1-n}}\, dr \qquad (3.26)$$

Taking in (3.25) & (3.26) a limit as $t \to 0$ almost everywhere, we obtain (3.4) as n=1, and the sufficiency is proved.

■

In this chapter we give the definitions and some properties of multidimensional partial fractional integrals and fractional derivatives presented by **[S.Samko, 1993]**. Such operations of fractional integration and fractional differentiation in the n- dimensional Euclidean space R^n $(n \in N)$ are natural generalizations of the corresponding one – dimensional fractional integrals and fractional derivatives, being taken with respect to one or several variables.

2.1 BASIC DEFINITIONS

It is known

$$(I_a^{\alpha} u(\overline{x}) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{u(t)}{(x-t)^{1-\alpha}}\, dt, \qquad (x > a)$$

and

$$(I_b^{\alpha} u(\overline{x}) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{u(t)}{(t-x)^{1-\alpha}}\, dt, \qquad (x < b)$$

Therefore the partial Riemann-Liouville fractional integrals of order $\alpha_k > 0$ with respect to the k th variable x_k are defined by

$$(I_{a_k, x_k}^{\alpha_k} u(\overline{x}) = \frac{1}{\Gamma(\alpha)} \int_{a_k}^{x_k} \frac{u(x_1, \ldots, x_{k-1}, t_k, x_{k+1}, \ldots, x_n)}{(x_k - t_k)^{1-\alpha_k}}\, dt_k \qquad (x_k > a_k) \qquad (2.1)$$

and

$$(I_{b_k, x_k}^{\alpha_k} u(\overline{x}) = \frac{1}{\Gamma(\alpha)} \int_{x_k}^{b_k} \frac{u(x_1, \ldots, x_{k-1}, t_k, x_{k+1}, \ldots, x_n)}{(t_k - x_k)^{1-\alpha_k}}\, dt_k \qquad (x_k < b_k) \qquad (2.2)$$

These definitions are valid for functions $u(\overline{x}) = u(x_1,\ldots,x_n)$ defined for $x_k > a_k$ and $x_k < b_k$, respectively. By analogy with the one-dimensionsl case, the fractional integrals (2.1) and (2.2) are called left- and right- sided partial Riemann-Liouville fractional integrals.

The left- and right- sided partial Riemann-Liouville fractional derivatives of order $\alpha_k \in C(\ R(\alpha) \geq 0)$ with respect to the k th variable x_k are defined by

$$(\frac{\partial^{\alpha_k}}{\partial a_k^{\alpha_k}} u(\overline{x})) = \frac{\partial^{L_k}}{\partial x_k^{L_k}} (I_{a_k}^{L_k-\alpha_k}\ u(\overline{x}))$$

$$= \frac{1}{\Gamma(L_k - \alpha_k)} \frac{\partial^{L_k}}{\partial x_k^{L_k}} \int_{a_k}^{x_k} \frac{u(x_1,\ldots,x_{k-1},t_k,x_{k+1},\ldots,x_n)}{(x_k - t_k)^{\alpha_k - L_k + 1}}\ dt_k \qquad (x_k > a_k) \qquad (2.3)$$

and

$$(\frac{\partial^{\alpha_k}}{\partial b_k^{\alpha_k}} u(\overline{x})) = -\frac{\partial^{L_k}}{\partial x_k^{L_k}} (I_{b_k}^{L_k-\alpha_k}\ u(\overline{x}))$$

$$= \frac{1}{\Gamma(L_k - \alpha_k)} - \frac{\partial^{L_k}}{\partial x_k^{L_k}} \int_{x_k}^{b_k} \frac{u(x_1,\ldots,x_{k-1},t_k,x_{k+1},\ldots,x_n)}{(t_k - x_k)^{\alpha_k - L_k + 1}}\ dt_k \qquad (x_k < b_k) \ (2.4)$$

where $L_k = [\alpha_k] + 1.$

take the following forms

$$(\frac{\partial^{\alpha_k}}{\partial a_k^{\alpha_k}} u(\overline{x})) = \frac{\partial}{\partial x_k} (I_{a_k}^{1-\alpha_k}\ u(\overline{x}))$$

$$= \frac{1}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int_{a_k}^{x_k} \frac{u(x_1,\dots,x_{k-1},t_k,x_{k+1},\dots,x_n)}{(x_k-t_k)^{\alpha_k}}\, dt_k \qquad (x_k > a_k)$$

$$\left(\frac{\partial^{\alpha_k}}{\partial b_k^{\alpha_k}} u(\overline{x})\right) = -\frac{\partial}{\partial x_k}\left(I_{b_k}^{1-\alpha_k} u(\overline{x})\right)$$

$$= -\frac{1}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int_{x_k}^{b_k} \frac{u(x_1,\dots,x_{k-1},t_k,x_{k+1},\dots,x_n)}{(t_k-x_k)^{\alpha_k}}\, dt_k \qquad (x_k < b_k)$$

If $\alpha_k = L_k \in N_0$, we have the usual partial derivatives as follows:

$$\left(\frac{\partial^{L_k}}{\partial a_k^{L_k}} u(\overline{x})\right) = \frac{\partial^{L_k}}{\partial x_k^{L_k}} u(\overline{x}) \quad \text{and} \quad \left(\frac{\partial^{L_k}}{\partial b_k^{L_k}} u(\overline{x})\right) = (-1)^{L_K} \frac{\partial^{L_k}}{\partial x_k^{L_k}} u(\overline{x})$$

If we denote the following notations:

$$AC[a,b] = \{ f : [a,b] \to C, \quad f(x) = c + \int_a^b f'(x)\,dt \quad and \quad (f'(x) \in L(a,b)) \}$$

$$AC_{1-\alpha}[a,b] = \{ f : [a,b] \to C \quad and \quad f'(x) \in AC[a,b) \}$$

where AC[a,b] is the space of all absolutely continuous function.

Based on (2.1)-(2.3), we could state the following definition:

Definition (1): The partial fractional derivative $\frac{\partial^\alpha}{\partial x_i^\alpha}$ of order α > 0 of

absolutely continuous function u ($\bar{x}$)is defined as

$$\frac{\partial^\alpha}{\partial x_i^\alpha} u(\bar{x}) = \frac{1}{\Gamma(n-1)} \frac{\partial^n}{\partial x_i^n} \int_o^{x_i} (x_t - t)^{-\alpha} \; u(x_1, x_2, \ldots.. x_{i-1}, t, \ldots, x_n) dt$$

$$= I_{x_i}^{n-\alpha} \frac{\partial^n}{\partial x_i^n} u(\bar{x})$$

$$= \frac{x_i^{n-\alpha-1}}{\Gamma(n-\alpha)} u_{x_i}(\bar{x}) \;, \quad \text{for } x \geq 0 \qquad (2.5)$$

$$= 0 \;, \qquad \text{for } x \leq 0$$

Where n=[α]+1.

2.2 PROPERTIES AND THEORIES OF PARTIAL FRACTIONAL DERIVATIVE $\frac{\partial^\alpha}{\partial x_i^\alpha}$

Property (1):

The operator $\frac{\partial^\alpha}{\partial x_i^\alpha}$ **,** α > 0, is linear.

Proof:

Let u, v $\in$ AC [a,b] , $r_1, r_2 \in R$, n=[α]+1

$$\frac{\partial^\alpha}{\partial x_i^\alpha} \quad (r_1 \; u(\bar{x}) + r_2 v(\bar{x})) = {}_0I_{x_i}^{n-\alpha} \; \frac{\partial^n}{\partial x_i^n} \; (r_1 u(\bar{x}) + r_2 v(\bar{x}))$$

$$={}_0I_{x_i}^{n-\alpha}(r_1 u_{x_i}(\overline{x})+r_2 v_{x_i}(\overline{x}))$$

$$=r_1\ {}_0I_{x_i}^{n-\alpha}u_{x_i}(\overline{x})\ +\ r_2\ {}_0I_{x_i}^{n-\alpha}v_{x_i}(\overline{x})$$

$$=r_1\ \frac{\partial^\alpha}{\partial x_i^\alpha}u(\overline{x})\ +\ r_2\ \frac{\partial^\alpha}{\partial x_i^\alpha}v(\overline{x})$$

Property (2):

if u ∈ AC [a,b] , then

$$\frac{\partial^\alpha}{\partial x_i^\alpha}\frac{\partial^\beta}{\partial x_i^\beta}u(\overline{x})=\frac{\partial^\beta}{\partial x_i^\beta}\frac{\partial^\alpha}{\partial x_i^\alpha}u(\overline{x})=\frac{\partial^{\alpha+\beta}}{\partial x_i^{\alpha+\beta}}u(\overline{x})$$

Proof:

$$\frac{\partial^\alpha}{\partial x_i^\alpha}\frac{\partial^\beta}{\partial x_i^\beta}u(\overline{x})=\frac{\partial^\alpha}{\partial x_i^\alpha}({}_0I_{x_i}^{1-\beta}u_{x_i})={}_0I_{x_i}^{1-\alpha}(\frac{\partial}{\partial x_i}({}_0I_{x_i}^{1-\beta}u_{x_i}))$$

$$=I_{x_i}^{1-\alpha}(I_{x_i}^{-\beta}u_{x_i})=\ I_{x_i}^{1-\alpha-\beta}u_{x_i}$$

$$\frac{\partial^\beta}{\partial x_i^\beta}\frac{\partial^\alpha}{\partial x_i^\alpha}u(\overline{x})=\frac{\partial^\beta}{\partial x_i^\beta}({}_0I_{x_i}^{1-\alpha}u_{x_i})={}_0I_{x_i}^{1-\beta}(\frac{\partial}{\partial x_i}({}_0I_{x_i}^{1-\alpha}u_{x_i}))$$

$$=I_{x_i}^{1-\beta}(I_{x_i}^{-\alpha}u_{x_i})=I_{x_i}^{1-\beta-\alpha}u_{x_i}$$

Property (3): [Loverro, A, 2004]

$$(I_a^\alpha(t-a)^{\beta-1})(x)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\ (x-a)^{\beta+\alpha-1}$$

$$(\frac{\partial^\alpha}{\partial a^\alpha}(t-a)^{\beta-1})(x)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\ (x-a)^{\beta-\alpha-1}$$

and

$$(I_b^{\alpha}(b-t)^{\beta-1})(x)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\;(b-x)^{\beta+\alpha-1}$$

$$(\frac{\partial^{\alpha}}{\partial b^{\alpha}}(b-t)^{\beta-1})(x)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\;(b-x)^{\beta-\alpha-1}$$

In particular**,** $if\ \beta=1\ and\ \alpha\geq 0$, then the Riemann-Lioville fractional derivative of a constant are, in general, not equal to zero.

On the other hand, $for\ \ j=1,2,...,[\alpha]+1,$

$$(\frac{\partial^{\alpha}}{\partial a^{\alpha}}(t-a)^{\alpha-j})(x)=0\ \ ,\qquad (\frac{\partial^{\alpha}}{\partial b^{\alpha}}(b-t)^{\beta-j})(x)=0$$

<u>Property (4):</u> [Loverro, A, 2004]

If $u(x)\in L_p(a,b)$ the relations

$$(\frac{\partial^{\beta}}{\partial a^{\alpha}}I_a^{\alpha}\,u(\overline{x}))=I_a^{\alpha-\beta}\,u(\overline{x})\quad and\quad (\frac{\partial^{\beta}}{\partial b^{\beta}}I_b^{\alpha}\,u(\overline{x})=I_b^{\alpha-\beta}u(\overline{x})$$

holds almost everywhere on [a,b]

In particular, when $\beta = k\in N\ and\ R(\alpha)>K,\ then$

$$(\frac{\partial^{k}}{\partial a^{\alpha}}I_a^{\alpha}\,u(\overline{x}))=I_a^{\alpha-k}\,u(\overline{x})\quad and\quad (\frac{\partial^{k}}{\partial b^{\beta}}I_b^{\alpha}\,u(\overline{x})=I_b^{\alpha-k}u(\overline{x})$$

<u>Property (5):</u> [Loverro, A, 2004]

$$\frac{\partial^{\alpha-k}}{\partial t^{\alpha-k}} u(x,t) = \lim_{t\to 0} \frac{\partial^{\alpha-k}}{\partial t^{\alpha-k}} u(x,t), \quad (1 \le k \le n-1),\ n = [\alpha]+1$$

$\equiv$ It means the limit is taken at almost all points of the right-sided neighborhood $(0,\in)$ $(\in > 0)$.

$$\frac{\partial^{\alpha-n}}{\partial t^{\alpha-n}} u(x,t) = \lim_{x\to 0} I^{n-\alpha} u(x,t) \qquad (n \ne \alpha)$$

$$= u(x,t)\ \Big|_{t=0} \qquad (n = \alpha)$$

<u>Lemma:</u> [Loverro, A, 2004]

Let $\alpha > 0, n = [\alpha]+1$ *and let* $u_{n-\alpha}(\overline{x}) = (I_a^{n-\alpha} u(\overline{x}))$ be the fractional integral of order $n-\alpha$.

(*a*) *If* $1 \le p \le \infty$ *and* $u(\overline{x}) \in I_a^{\alpha}(L_p)$, *then*

$$(I_a^{\alpha} \frac{\partial^{\alpha}}{\partial a^{\alpha}} u(\overline{x})) = u(\overline{x}).$$

(*b*) *If* $u(\overline{x}) \in L_1(a,b)$ *and* $u_{n-\alpha}(\overline{x}) \in AC\ [a,b]$, then the equality

$$(I_a^{\alpha} \frac{\partial^{\alpha}}{\partial a^{\alpha}} u(\overline{x})) = u(\overline{x}) - \sum_{j=1}^{n} \frac{u_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha - j + 1)}\ (x-a)^{\alpha-j}$$

holds almost everywhere on [a,b].

In particular, *if* $0 < \alpha < 1$, *then*

$$(I_a^{\alpha} \frac{\partial^{\alpha}}{\partial a^{\alpha}} u(\overline{x})) = u(\overline{x}) - \frac{u_{1-\alpha}(a)}{\Gamma(\alpha)}\ (x-a)^{\alpha-1}$$

where $u_{1-\alpha}(\overline{x}) = (I_a^{1-\alpha}u(\overline{x}))$, *while for* $\alpha = n \in N$, then the following equality holds:

$$(I_a^n \frac{\partial^n}{\partial a^n} u(\overline{x})) = u(\overline{x}) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!} \ (x-a)^k$$

<u>Theorem (2.1):</u>

Let $u(\overline{x})$ & $v(\overline{x}) \in AC[a,b]$, $\frac{\partial^\alpha}{\partial x_i^\alpha}$ be the partial fractional derivative of order $\alpha \in (0,1]$, *and* λ is any real number. Then

(i) $\frac{\partial^\alpha}{\partial x_i^\alpha}(u+v)(\overline{x}) = \frac{\partial^\alpha}{\partial x_i^\alpha}u(\overline{x}) + \frac{\partial^\alpha}{\partial x_i^\alpha}v(\overline{x})$

(ii) $\frac{\partial^\alpha}{\partial x_i^\alpha}(\lambda u)(\overline{x}) = \lambda \frac{\partial^\alpha}{\partial x_i^\alpha}u(\overline{x})$

(iii) $\frac{\partial^\alpha}{\partial x_i^\alpha}(uv)(\overline{x}) = v(\overline{x})\frac{\partial^\alpha}{\partial x_i^\alpha}u(\overline{x}) + u(\overline{x})\frac{\partial^\alpha}{\partial x_i^\alpha}v(\overline{x})$

Proofs of (i) and (ii) are trivial from property (1)

Proof (iii): using equation (2.5)

$$\frac{\partial^\alpha}{\partial x_i^\alpha}(uv)(\overline{x}) = \frac{x_i^{-\alpha}}{\Gamma(1-\alpha)}\frac{\partial}{\partial x_i}(uv)(\overline{x})$$

$$= \frac{x_i^{-\alpha}}{\Gamma(1-\alpha)}[v(\overline{x})u_{x_i}(\overline{x}) + u(\overline{x})v_{x_i}(\overline{x})]$$

$$= v(\overline{x})\frac{x_i^{\alpha}}{\Gamma(1-\alpha)}u_{x_i}(\overline{x}) + u(\overline{x})\frac{x_i^{-\alpha}}{\Gamma(1-\alpha)}v_{x_i}(\overline{x})$$

$$= v(\overline{x})D_{x_i}^\alpha u(\overline{x}) + u(\overline{x})D_{x_i}^\alpha v(\overline{x})$$

■

From the theorem, one can be calculate $\frac{\partial^{\alpha}}{\partial x^{2\alpha}}$, $\frac{\partial^{\beta}}{\partial x^{2\beta}}$ and $\frac{\partial^{\alpha,\beta}}{\partial x^{\alpha}\partial y^{\beta}}$ as follows:

Let U ≡ U(x,y), α >0, β >0

$$\frac{\partial^{\alpha}}{\partial x^{2\alpha}}u(x,y)=I^{n-2\alpha}(\frac{\partial^{n}}{\partial x^{n}}u(x,y)), \quad n=[\alpha]+1$$

$$=\frac{x^{n-2\alpha-1}}{\Gamma(n-2\alpha)}\frac{\partial^{n}}{\partial x^{n}}u(x,y)$$

Similarly

$$\frac{\partial^{\beta}}{\partial y^{2\beta}}u(x,y)=I^{m-2\beta}\frac{\partial^{m}}{\partial y^{m}}u(x,y), \quad m=[\beta]+1$$

$$=\frac{y^{m-2\beta-1}}{\Gamma(m-2\beta)}\frac{\partial^{m}}{\partial y^{m}}u(x,y)$$

Also,

$$\frac{\partial^{\alpha,\beta}}{\partial x^{\alpha}\partial y^{\beta}}=\frac{\partial^{\alpha}}{\partial x^{\alpha}}(\frac{y^{-\beta}u_{y}}{\Gamma(1-\beta)})=I^{1-\alpha}(\frac{\partial}{\partial x}(\frac{1}{\Gamma(1-\beta)}y^{-\beta}u_{y}))$$

$$=I^{1-\alpha}(\frac{1}{\Gamma(1-\beta)}y^{-\beta}u_{xy)})=\frac{x^{-\alpha}}{\Gamma(1-\alpha)}(\frac{1}{\Gamma(1-\beta)}y^{-\beta}u_{xy})$$

$$=\frac{x^{-\alpha}y^{-\beta}}{\Gamma(1-\alpha)\Gamma(1-\beta)}u_{xy}$$

Abstract

In this work, we obtain an analytical solution for Cauchy type problem of partial fractional order differential equation in terms of Mittage – Leffler function using Laplace transformation. The existence and uniqueness of the analytical solution also, is reviewed by reducing the Cauchy type problem of partial fractional order differential equation into linear Volterra integral equation of the second kind and showing that the solution of our Cauchy type problem is equivalent to the solution of linear Volterra integral equation of the second kind.

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Hala
June, 2007✍

Contents

Discussion

In this work, based on Riemann – Liouville definition , we state the partial Riemann – Liouville fractional integral and derivative , and we give the definition and some properties of partial fractional derivative $\frac{\partial^{\alpha}}{\partial x_i^{\alpha}}$ of order α > 0.

Also, we derive the solution of partial fractional order differential equation which is known as the fractional diffusion heat equation , with the caushy type initial condition , the existence and uniqueness is proved by reducing our caushy type problem into linear Volterra integral equation of the second kind and showing that its solution is equvelant to the solution of caushy type problem .

Examining Committee's Certification

We certify that we read this thesis entitled ***"SOLUTION OF CAUCHY TYPE PROBLEM OF PARTIAL FRACTIONAL ORDER DIFFERENTIAL EQUATIONS"***and as examining committee examined the student, ***Hala Abed AL-Ameer Naji*** in its contents and in what it connected with, and that is in our opinion it meet the standard of thesis for the degree of Master of Science in Mathematics.

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Future work

The future work may be considered are the following:

- Different types of partial fractional order differential equations using other transformations.
- Using the above mentioned approaches on a system of partial fractional order differential equations.

Introduction

Fractional calculus is a field of mathematical study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. Any one can verify that $x^3 = x.x.x$, how might one describe the physical meaning of $x^{3.4}$, or moreover the transcendental exponent x^{π}.One cannot conceive what it might be like to multiply a number or quantity by it self 3.4 times, or π times, and yet these expressions have a definite value for any value x, verifiable by infinite series expansion, or more practically, by calculator. Now, in the same way consider the integral and derivative. Fractional calculus has its origins in the generalizations of the differential and integral calculus **[Lubich, 1986]**. It deals with the investigation and applications of integrals and derivative of arbitrary order. Fractional derivative have been used in building models of physical processes, leading to the formulation with fractional differential equation. The fractional calculus may be considered as an old and yet a new topic which is still under development and investigation**[Gorenflo et al., 2000]** .

Fractional calculus considered as a novel topic as well, since only from a little more than to the later fifty years it has been an object of specialized conferences and treatises. Most authors on this topic will cite a particular date as the birthday of so called "Fractional Calculus ". In a letter dated September 30th , 1695 L'Hopital wrote to Leibniz asking him about a particular notation he had used in his publications for the nth –

derivative $\frac{D^n x}{Dx^n}$ of the linear function $f(x) = x$, L'Hopital's posed the question to Leibniz , what would the result be if n=1/2 . Leibniz's response:" An apparent paradox, from which one day useful consequences will be drawn." In these words fractional calculus was born. Following L'Hopital's and Leibniz's first inquisition, fractional calculus was primarily a study reserved for the best minds in mathematics. Fourier,Euler, Laplace were among the many that dabbled with fractional calculus and the mathematical consequences **[K.Nishimoto, 1991]**. Many found, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative .The most famous these definitions that have been popularized in the word of fractional calculus are the Riemann – Liouville and Grunwald-Letnikove definitions.

The pioneers in the early of last century, notable contributions have been made to both theory and applications of fractional calculus, Weyl (1917), Hardy (1917), Hardy and little wood (1925), and Post (1930) used difference quotients to define generalized differentiation for operator $f(D)$, where D denotes differentiation and f is a suitable restricted function. Furthermore Kober (1940) and Kauthner (1953) examine some rather special, but natural properties of differe integrals of functions belonging to Lebesgue class.

In the recent years, considerable interest in fractional calculus has been stimulated by the applications that it find in different fields of science, including numerical analysis, physics, engineering, biology, economics and finance **[Gorenflo et al., 2000]** For more details in historical development of the fractional calculus we refer to “the fractional calculus” **[Oldham, 1974]**.

Nowadays, many researchers are interested in the field of fractional calculus. **[Al-Shather, H. A., 2003]**, solved a fractional order multiple delay integro- differential equations using the collection method and a varitional approach. Also, **[Al-Azawi, S. N., 2004]**, studied some results in fractional calculus, **[Aziz, 2006]**, discussed analytical study of partial fractional order differential equations, & **[Al-Saltani B. K., 2003]** ,studied the solution of some fractional differential equations. Moreover, **[Salih H. M., 2005]** used an approximate method to solve fractional differential equations.

This thesis concern with Partial Fractional Order Differential Equation. It consists of three chapters: The first chapter presented the basic concepts of Fractional calculus. The second chapter presented extensions definitions and theories for Partial Fractional Order Differential Equations. In chapter three we are presented the solutions of Cauchy type problems for Partial Fractional Order Differential Equations and their existence and uniqueness is proved.

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Supervisor Certification

I certify that this thesis was prepared under my supervision at the Al-Nahrain University, College of Science, in partial fulfillment of the requirements for the degree of Master of Science in mathematics

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In view of the available recommendations I forward this thesis for debate by the examining committee.

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الاهداء

الى

من اسأل ربي ان يرحمهما كما ربياني صغيرا

(امي و ابي) اكثر الناس تضحية من اجلي

الى

اقرب الناس لي وخيرهم نصحا وعونا

اختي (نغم)

الى

مكمن قوتي في هذه الحياة اخوتي الاعزاء

(احمد ،علي، مصطفى)

Republic of Iraq
Ministry of Higher Education
and Scientific Research
Al-Nahrain University
College of Science

Solution of Cauchy Type Problem Of Partial Fractional Order Differential Equations

A Thesis

Submitted to the College of Science, AL-Nahrain University
as a Partial Fulfillment of the Requirements for the Degree of
Master of Science in Mathematics

By
Hala Abed AL- Ameer Al -Azzawi
(B.Sc., Al-Nahrain University, 2004)

Supervised by
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June 2007

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم

حل مسألة كوشي التفاضلية الجزئية ذات الرتب الكسرية

رسالة
مقدمة إلى كلية العلوم – جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل
هالة عبد الامير العزاوي
(بكلوريوس علوم ، جامعة النهرين، ٢٠٠٤)

بأشراف
د. علاء الدين نوري احمد

حزيران ٢٠٠٧

المستخلص

في هذه الرساله تم ايجاد الحل التحليلي لمسألة كوشي التفاضلية الجزئية ذات الرتب الكسرية بدلالة دالة ميتج- لفلر (Mittag-Leffler Function) وبأستخدام تحويل لابلاس.

و تم برهنة وجود ووحدانية الحل التحليلي وذلك بتحويل مسألة كوشي التفاضلية الجزئية ذات الرتب الكسرية الى معادلة فولتيرا التكاملية الخطية من النوع الثاني واثبتنا ان حل مسألة كوشي يناظر حل معادلة فولتيرا التكاملية الخطية ومن النوع الثاني.