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*Intisar, 2006*

# *ABSTRACT*

The main aim of this work is divided into four objects. These are summarized as follows:

The first objectives, is to study the ordinary differential equations with deviating arguments.

The second objectives, is to derive an estimate of the magnitude of the solutions for special types of linear and nonlinear ordinary differential equations with deviating arguments in order to solve them by any suitable methods.

The third objectives, is to devote the existence of a unique bounded solution for special types of the partial differential equations with deviating arguments.

The fourth objectives, is to give an estimate of the magnitude of solutions for special types of $1^{st}$ order and $2^{nd}$ order partial differential equations with deviating arguments.

# *Conclusions and Recommendations*

From the present study, we can conclude the following :

(1) The mathematical modelling for many real life applications that described as differential equations with deviating arguments is more realable than as differential equations.

(2) The estimation of the magnitude of the solutions for the linear differential equations with deviating arguments is a necessary tool for finding the solutions.

(3) The integral inequalities play an important rule for estimating the magnitude of the solutions for the linear and nonlinear differential equations with deviating arguments.

Also, for future work we can recommend the introduction of the following open problems:

(1) Discuss the existence of a unique bounded solution for the $3^{rd}$ order, $4^{th}$ order and $n^{th}$ order partial differential equations with deviating arguments.

(2) Extend the Laplace transform method to be used for solving the linear partial delay differential equations.

(3) Linearize the nonlinear ordinary differential equations with deviating arguments into ones that are linear which can be solved by the Laplace transform method.

## 1.1 Introduction:

The ordinary differential equations with deviating arguments play an important rule in many real life applications say in mixing of liquids, population growth and control systems, [Driver R.,1977].

In this chapter we give some basic concepts of the ordinary differential equations with deviating arguments.

This chapter consists of four sections:

In section two, a simple classification of the ordinary differential equations with deviating arguments is introduced with some properties of the solutions. Moreover, the initial value problems for the ordinary differential equations with deviating arguments are discussed.

In section three, some theorems that guarantee the existence of a unique solution for special types of the ordinary differential equations with deviating arguments are devoted.

In section four, some methods that can be used to solve special types of the ordinary differential equations with deviating arguments such as the method of steps and the Laplace transform method are presented.

## 1.2 The Ordinary Differential Equations with Deviating Arguments:

In this section we give some basic concepts of the ordinary differential equations with deviating arguments. These concepts include the classification of the differential equations with deviating arguments, the initial value problems for the ordinary differential equations with deviating arguments and some properties of the linear ordinary differential equations with deviating arguments.

First, recall that the differential equation with deviating arguments is a differential equation in which the unknown function enters with several different values of the argument, [El'sgol'ts L., 1964]. For example:

$$x'(t) = f(t, x(t), x(t-\tau(t)));$$

$$x'(t) = f(t, x(t), x(t+\tau_1), x'(t), x'(t+\tau_2));$$

$$x'(t) = f(t, x(t), x(t/2), x(t^3));$$

$$x'(t) = f(t, x(t-\tau), x(t), x(t+\tau));$$

$$x'(t) = f(t, x(t), x'(t), x(t-\tau), x'(t-\tau), x''(t-\tau)).$$

Sundry differential equations with deviating arguments occur as long ago in the works of Euler but the systematic study of these equations was first undertaken in the twentieth century, to meet the demands of applied science, in particular in the theory of automatic control, the theory of self-oscillating systems, the study of problems connected with combustion in rocket motion, the problem of long-range planning in economics, a series of biological problems, and in many other areas of science and technology, [El'sgol'ts L. and Norkin S., 1973].

Similarly as in the ordinary differential equations, the classification of differential equation with deviating arguments depends mainly on the linearity of the differential equation, order and degree of the differential equation whether the differential equation with deviating arguments is homogeneous or nonhomogeneous and so on, [El'sgol'ts L. and Norkin S., 1973].

The general form for the n[th] order ordinary differential equation with deviating arguments is:

$$F(x, y(x), y(x-\tau_{01}), y(x-\tau_{02}),..., y(x-\tau_{0m}), y'(x), y'(x-\tau_{11}), y'(x-\tau_{12}),...,\\ y'(x-\tau_{1m}), y''(x), y''(x-\tau_{21}), y''(x-\tau_{22}),..., y''(x-\tau_{2m}),..., y^{(n)}(x),\\ y^{(n)}(x-\tau_{n1}), y^{(n)}(x-\tau_{n2}),..., y^{(n)}(x-\tau_{nm})) = 0$$

where $F$ is a given function and $\tau_{ij} = \tau_{ij}(x, y(x), y'(x),..., y^{(n)}(x))$ is a known function for each $i = 0,1,...,n$ and $j = 1,2,...,m$.

Other literatures write the above equation in the following form:

$$F(x, y(x), y(k_{01}), y(k_{02}),..., y(k_{0m}), y'(x), y'(k_{11}), y'(k_{12}),..., y'(k_{1m}), y''(x),\\ y''(k_{21}), y''(k_{22}),..., y''(k_{2m}),..., y^{(n)}(x), y^{(n)}(k_{n1}), y^{(n)}(k_{n2}),..., y^{(n)}(k_{nm})) = 0$$

where $k_{ij} = k_{ij}(x, y(x), y'(x),..., y^{(n)}(x))$ is a known function for each $i = 0,1,...,n$ and $j = 1,2,...,m$, [El'sgol'ts L. and Norkin S., 1973].

In this work, we concerns with the special case of the above equation in which $\tau_{ij} = \tau_{ij}(x)$ for each $i = 0,1,...,n$ and $j = 1,2,...,m$ and in this case, the above equation takes the form:

$$F(x, y(x), y(x-\tau_{01}(x)), y(x-\tau_{02}(x)),..., y(x-\tau_{0m}(x)), y'(x),\\ y'(x-\tau_{11}(x)), y'(x-\tau_{12}(x)),..., y'(x-\tau_{1m}(x)), y''(x), y''(x-\tau_{21}(x)),\\ y''(x-\tau_{22}(x)),..., y''(x-\tau_{2m}(x)),..., y^{(n)}(x), y^{(n)}(x-\tau_{n1}(x)),\\ y^{(n)}(x-\tau_{n2}(x)),..., y^{(n)}(x-\tau_{nm}(x))) = 0 \qquad (1.1)$$

If $\tau_{ij}(x) = \tau(x)$ for each $i = 0,1,...,n$ and $j = 1,2,...,m$ then eq.(1.1) is said to be the n[th] order ordinary differential equation with single deviating argument, otherwise it is with multiple deviating arguments. On the other hand, if $\tau_{ij}(x)=\tau_{ij}$, where $\tau_{ij}$ is a known constant for each $i=0,1,...,n$ and $j=1,2,...,m$, then eq.(1.1) is said to be the n$^{th}$ order ordinary differential equation with constant deviating arguments, otherwise it is with variable arguments, [El'sgol'ts L. and Norkin S., 1973].

Moreover, if $\tau_{ij}(x)=\tau_{ij}\geq 0$ for each $i=0,1,...,n$ and $\mathrm{j}=1,2,...,\mathrm{m}$, where $\tau_{ij}$ is a known nonnegative real number then eq.(1.1) is said to be the n$^{th}$ order ordinary differential-difference equation (or the n$^{th}$ order ordinary delay differential equation), [El'sgol'ts L., 1964]. In this case, eq.(1.1) reduces to:

$$\begin{aligned}F(x,y(x),y(x-\tau_{01}),y(x-\tau_{02}),...,y(x-\tau_{0m}),y'(x),y'(x-\tau_{11}),y'(x-\tau_{12}),...,\\ y'(x-\tau_{1m}),y''(x),y''(x-\tau_{21}),y''(x-\tau_{22}),...,y''(x-\tau_{2m}),...,y^{(n)}(x),\\ y^{(n)}(x-\tau_{n1}),y^{(n)}(x-\tau_{n2}),...,y^{(n)}(x-\tau_{nm}))=0\end{aligned}$$

So, if $\tau_{ij}(x)=\tau>0$ for each $i=0,1,...,n$ and $\mathrm{j}=1,2,...,\mathrm{m}$ where $\tau$ is a known positive real number, then eq.(1.1) is said to be with single delay, otherwise it is with multiple delays, [El'sgol'ts L. and Norkin S., 1973].

Also, eq.(1.1) is said to be the n$^{th}$ order ordinary differential equation with retarded arguments in case the n$^{th}$ derivative of the unknown function y enter with the identical values of the argument and this value is not less than the arguments of y and its derivatives. For example:

$$y'(x)=f(x,y(x),y(x-\tau(x))),\quad \tau(\mathrm{x})\geq 0;$$

$$\mathrm{y}''(x+\tau)=f(x,y(x),y(x+\tau),y'(x),y'(x+\tau)),\quad \tau>0;$$

$$y'(x) = f(x, y(x), y(x/2), y(x - e^{-x})), \quad x \ge 0.$$

Moreover, eq.(1.1) is said to be the n^th^ order ordinary differential equation with advanced arguments in case the n^th^ derivative of the unknown function y enter with identical values of the argument and the common value is not greater than the argument in the function y or any of its derivatives that enter in this equation. For example:

$$y'(x) = f(x, y(x), y(x + \tau(x))), \quad \tau(x) \ge 0;$$

$$y''(x) = f(x, y(x), y(2x), y'(x), y'(2x), \quad x \ge 0;$$

$$y''(x) = f(x, y(x), y(x + \tau), y'(x), y'(x + \tau)), \quad \tau > 0.$$

It is possible that on some set of values of the independent variables a certain equation should appear as an equation with retarded argument while on another set it is an equation with advanced argument or perhaps belongs to neither of these types. For example $y'(x) = y(x) + y(x/2)$ is a1^st^ order differential equation with retarded argument for $x \ge 0$ and an equation with advanced argument for $x \le 0$.

All remaining types of the ordinary differential equations with deviating arguments given by eq.(1.1) are called the n^th^ order ordinary differential equations of neutral type. For example:

$$y'(x) = f(x, y(x), y(x - \tau), y(x + \tau)),$$

where $\dfrac{\partial f}{\partial y(x-\tau)} \ne 0$ and $\dfrac{\partial f}{\partial y(x+\tau)} \ne 0$;

and

$$y''(x) = f(x, y(x), y'(x), y(x-\tau), y'(x-\tau), y''(x-\tau)),$$

where $\dfrac{\partial f}{\partial y''(x-\tau)} \neq 0$, [El'sgol'ts L., 1964].

Next, eq.(1.1) is said to be the n[th] order linear ordinary differential equation with deviating arguments in case $F$ takes the form:

$$\sum_{p=0}^{n}\sum_{j=0}^{m} a_{pj}(x) y^{(p)}(x-\tau_{pj}(x)) = f(x) \tag{1.2}$$

where $\tau_{p_0}(x) = 0$, $\mathrm{a_{pj}}$ is a known function of $x$ for each $\mathrm{p} = 0,1,\ldots,\mathrm{n}$ and $\mathrm{j} = 0,1,\ldots\mathrm{m}$, $f$ is a known function of $x$, [El'sgol'ts L. and Norkin S., 1973].

Second, we described the initial value problems for special types of the ordinary differential equations with deviating arguments.

Let us first consider the simplest delay differential equation with retarded argument:

$$y'(x) = f\left(x, y(x), y(x-\tau)\right) \tag{1.3}$$

where $\tau$ is assumed to be known positive constant, the basic initial value problem consists in the determination of a continuous solution $y(x)$ of eq.(1.3) for $x > x_0$, under the condition that

$$y(x) = \varphi(x), \quad \mathrm{x}_0 - \tau \le x \le x_0,$$

where $\varphi(x)$ is a given continuous function called the initial function. The closed interval $[x_0 - \tau, x_0]$ on which the initial function is given, is called the initial set and denoted by $E_{x_0}$; the point $x_0$ is called the initial point, [El'sgol'ts L., 1964].

For the n$^{th}$ order delay differential equation with single retarded argument:

$$y^{(n)}(x) = f(x, y(x), y(x-\tau), y'(x), y'(x-\tau), ..., y^{(n-1)}(x), y^{(n-1)}(x-\tau)),$$

where $\tau$ is assumed to be known positive constant, the initial conditions in the fundamental problem are the same,

$$y(x) = \varphi(x), \quad x_0 - \tau \leq x \leq x_0,$$

where $\varphi(x)$ is continuously differential (n-1) times if the solution has continuous derivatives up to the order (n-1) inclusive, [El'sgol'ts L., 1964].

For $\tau = \tau(x)$, then for the 1$^{st}$ order ordinary delay differential equation with deviating argument:

$$y'(x) = f(x, y(x), y(x - \tau(x))),$$

the initial function $y(x) = \varphi(x)$ in the fundamental problem must be defined on a so-called initial set $E_{x_0}$ consisting of the point $x = x_0$ and of those values of the differences $x - \tau(x)$ for $x_0 \leq x \leq \beta$ which are less than $x_0$, if the solution is defined for values $x_0 \leq x \leq \beta$. For example, for the equation

$$y'(x) = f(x, y(x), y(x - \sin^2 x)), \quad 0 \leq x < \infty$$

the initial set $E_0$ consisting of all points of the interval $[-1,0]$, [El'sgol'ts L., 1964].

Also, the n^th order ordinary delay differential equation with variable delay $\tau(x) \geq 0$,

$$y^{(n)}(x) = f(x, y(x), y(x-\tau(x)), y'(x), y'(x-\tau(x)), \ldots, y^{(n-1)}(x), y^{(n-1)}(x-\tau(x)))$$

normally determines an (n-1)-fold continuously differentiable solution $y(x)$, $x_0 \leq x \leq \beta$, and the initial conditions are as described above, $y(x) = \varphi(x)$ on $E_{x_0}$ where the function $\varphi(x)$ is continuously differentiable (n-1) times except when the set $E_{x_0}$ for $x_0 \leq x \leq \beta$ consists of the single point or when the point $x_0$ is isolated in the set $E_{x_0}$. If the exceptional case occurs, then at the point $x_0$, it is necessary to define the values of the derivatives on the right-hand side up to the order (n-1) inclusive, [El'sgol'ts L., 1964].

Third, we give some properties of the solutions for the linear ordinary differential equations with deviating arguments. To do this, recall that the general form of the n^th order linear ordinary differential equation with variable deviating arguments is given by eq.(1.2), the equation

$$\sum_{p=0}^{n}\sum_{j=0}^{m} a_{pj}(x) y^{(p)}(x-\tau_{pj}(x)) = 0 \qquad (1.4)$$

is called the homogeneous linear equation corresponding to eq.(1.2). The solution of eq.(1.2) or eq.(1.4) will be denoted by $y_\varphi(x)$ such that $y^{(\ell)}(x) = \varphi^{(\ell)}(x)$, $\ell = 0,1,\ldots,\text{n}-1$.

The above homogeneous linear ordinary differential equation with deviating have the following properties:

(i) A linear combination of solutions of eq.(1.4) with arbitrary constant coefficients $\sum_{i=1}^{k} c_i y_{\varphi_i}(x) = y_{\varphi}(x)$ is also a solution, where $y_{\varphi_i}$ is solution of eq.(1.4) such that $y^{(\ell)}(x) = \varphi_i^{(\ell)}(x)$, $\ell = 0,1,...,n-1$ and $\varphi(x) = \sum_{i=1}^{k} c_i \varphi_i(x)$. This property is preserved for $k \to \infty$, if the series $\sum_{i=1}^{\infty} c_i y_{\varphi_i}(x)$ converges and allows n successive differentiation.

(ii) If all coefficients $a_{pj}(x)$ and deviations $\tau_{pj}(x)$ are real, then the real and imaginary parts of complex solution are also solutions of this equation.

The nonhomogeneous linear ordinary differential equation with deviating arguments given by eq.(1.2) satisfy the following properties:

(i) The sum $y_{\varphi}(x) + y_{\varphi_1}(x)$ of a solutions $y_{\varphi}(x)$ of eq.(1.2) and a solution $y_{\varphi_1}(x)$ of the corresponding homogeneous equation is a solution $y_{\varphi+\varphi_1}(x)$ of eq.(1.2) defined by the initial function $\varphi + \varphi_1$.

(ii) The sum $\sum_{i=1}^{k} y_{\varphi_i}(x)$ of solutions $y_{\varphi_i}(x)$ of $\sum_{p=0}^{n}\sum_{j=0}^{m} a_{pj}(x) y^{(p)}(x - \tau_{pj}(x)) = f_i(x)$, $\mathrm{i} = 1,2,...,\mathrm{k}$, is a solution $y_{\varphi}(x)$ of the equation $\sum_{p=0}^{n}\sum_{j=0}^{m} a_{pj}(x) y^{(p)}(x - \tau_{pj}(x)) = \sum_{i=1}^{k} f_i$ with the initial function $\varphi(x)=\sum_{i=1}^{k}\varphi_i(x)$. This property remains valid for $k\to\infty$ if the series $\sum_{i=1}^{\infty}y_{\varphi_i}(x)$ converges and admits n successive differentiation.

(iii) If all coefficients $a_{pj}(x)$ and derivations $\tau_j(x)$ are real, then the real part $u(x)$ and the complex part $v(x)$ of the solution $y_{\varphi+i\psi}(x)=u(x)+iv(x)$ of eq.(1.2) are solutions of the equations

$$\sum_{p=0}^{n}\sum_{j=0}^{m}a_{pj}(x)y^{(p)}(x-\tau_{pj}(x))=\operatorname{Re} f(x)$$
$$\sum_{p=0}^{n}\sum_{j=0}^{m}a_{pj}(x)y^{(p)}(x-\tau_{pj}(x))=\operatorname{Im} f(x)$$

respectively, where $u(x)=y_{\varphi}(x)$ and $v(x)=y_{\psi}(x)$, [El'sgol'ts L. and Norkin S., 1973].

## **1.3 Existence and Uniqueness Theorems of the Solution for the Ordinary Differential Equations with Deviating Arguments:**

In this section, we give some basic theorems that are necessary for establishing the existence and the uniqueness for special types of linear and nonlinear ordinary differential equations with deviating arguments.

We start this section by the following theorem. This theorem gives some necessary conditions to ensure the existence of a unique solution for the 1st order delay differential equation of retarded argument with single variable delay.

### **Theorem (1.1), [El'sgol'ts L., 1964]:**

Consider the differential equation with retarded argument

$$y'(x) = f(x, y(x), y(x - \tau(x))) \tag{1.5}$$

with the initial condition $y(x) = \varphi(x)$ on the initial set $E_{x_0}$ where the continuous function $\tau(x) \geq 0$ and $\varphi(x)$ is a given continuous function. Then the above initial value problem has a unique continuous solution for $x_0 \leq x \leq x_0 + h$ if the function $f$ is continuous in the neighborhood of the values $(x, \varphi(x))$ for $x \in E_{x_0}$ and satisfies in this neighborhood a Lipschitz condition in the second and third arguments and $h$ is sufficiently small.

## Proof:

We replace the above differential equation by the following equivalent integral equation with the same initial conditions:

$$y(x) = \varphi(x_0) + \int_{x_0}^{x} f\left(t, y(t), y(t - \tau(t))\right) dt$$

$\mathrm{y(x)} = \varphi(\mathrm{x})$ on the initial set $\mathrm{E}_{\mathrm{x}_0}$.

Define the operator $A$ by

$$A(y(x)) = \varphi(x_0) + \int_{x_0}^{x} f(t, y(t), y(t - \tau(t)))\, dt$$

on the metric space of all continuous functions given on $E_{x_0}$ and on the segment $x_0 \leq x \leq x_0 + h$ with the uniform topology and with all the functions coinciding with $\varphi(x)$ on $E_{x_0}$ in the metric

$$\rho(y(x), z(x)) = \sup_{x_0 \leq x \leq x_0 + h} \left| y(x) - z(x) \right|,$$

then

$$\rho(A(y(x)),A(z(x))) = \sup_{x_0\le x\le x_0+h}\left|\int_{x_0}^{x}\left[f(t,y(t),y(t-\tau(t))) - f(t,z(t),z(t-\tau(t)))\right]dt\right|$$

$$\le N\sup_{x_0\le x\le x_0+h}\left|\int_{x_0}^{x}\left[\left|y(t)-z(t)\right| + \left|y(t-\tau(t))-z(t-\tau(t))\right|\right]dt\right|$$

$$\le \mathrm{Nh}\left[\sup_{x_0\le x\le x_0+h}\left|y(x)-z(x)\right| + \sup_{x_0\le x\le x_0+h}\left|y(x-\tau(x))-z(x-\tau(x))\right|\right]$$

$$\le 2\mathrm{Nh}\sup_{x_0\le x\le x_0+h}\left|y(x)-z(x)\right|$$

$$= 2\mathrm{Nh}\rho(\mathrm{y(x),z(x)}).$$

For $h \le \dfrac{\alpha}{2N}$ where $0<\alpha<1$, the operator $A$ defined above is contractive. Therefore by the fixed point theorem, one can get the initial value problem for eq.(1.5) has a unique solution. ■

Next, the generalization of the theorem(1.1) to be valid for the 1st delay differential equation of retarded arguments but with multiple variable delays.

## Theorem (1.2), [El'sgol'ts L. and Norkin S., 1973]:

Consider the differential equation with retarded arguments

$$y'(x) = f(x,y(x),y(x-\tau_1(x)),\ldots,y(x-\tau_m(x))) \qquad (1.6)$$

with the initial condition

$y(x)=\varphi(x)$ on the initial set $E_{x_0}$ .

If in eq.(1.6) all $\tau_i(x)$ are continuous for $x_0 \le x \le x_0 + h,\ \mathrm{h} > 0$ and nonnegative and the function $f$ is continuous in a neighborhood of the point $(x_0, \varphi(x_0), \varphi(x_0 - \tau_1(x_0)), \ldots, \varphi(x_0 - \tau_m(x_0)))$ and satisfies a Lipschitz condition in all arguments beginning with the second, the initial function $\varphi(x)$ is continuous on $E_{x_0}$, then there exists a unique solution for the initial value problem for eq.(1.6) for $x_0 \le x \le x_0 + h$ where $h$ is sufficiently small.

## **Proof:**

We replace eq.(1.6) by the following equivalent integral equation

$$y(x) = \varphi(x_0) + \int_{x_0}^{x} f(t, y(t), y(t - \tau_1(t)), \ldots, y(t - \tau_m(t)))\, dt$$

with the initial condition

$$y(x) = \varphi(x) \quad \text{on the initial set } \mathrm{E}_{x_0}.$$

Define the operator $A$ by

$$A(y(x)) = \varphi(x_0) + \int_{x_0}^{x} f(t, y(t), y(t - \tau_1(t)), \ldots, y(t - \tau_m(t)))\, dt$$

$$\mathrm{A(y(x))} = y(x) = \varphi(x) \ \text{ on } \ \mathrm{E}_{x_0}$$

on the metric space of all continuous functions on $E_{x_0}$ such that on $E_{x_0}$ all of these functions equal $\varphi(x)$ and on the interval $[x_0, x_0 + h]$ they are sufficiently near to $\varphi(x_0)$ in the metric

$$\rho(y(x), z(x)) = \sup_{x_0 \le x \le x_0 + h} |y(x) - z(x)|$$

In fact $A(y(x))$ is continuous if $y(x)$ is continuous and defined for $x_0 \le x \le x_0 + h$, for sufficiently small $h_1$. On the other hand, in the neighborhood of the point

$$(x_0, \varphi(x_0), \varphi(x_0 - \tau_1(x_0)), \dots, \varphi(x_0 - \tau_m(x_0))),$$

$$\left|\mathrm{f}(\mathrm{x}, \mathrm{y}(\mathrm{x}), \mathrm{y}(\mathrm{x} - \tau_1(x)), \dots, y(x - \tau_m(x))\right| < M .$$

Therefore,

$$\left|A(y(x)) - \varphi(x_0)\right| < Mh_1,$$

and if $h_1$ is sufficiently small, $A(y(x))$ belongs to the above metric space.

Moreover,

$$\rho(A(y(x)), A(z(x))) = \sup_{x_0 \le x \le x_0 + h} \left| \int_{x_0}^{x} [f(x, y(x), y(x - \tau_1(x)), \dots, y(x - \tau_m(x))) - f(x, z(x), z(x - \tau_1(x)), \dots, z(x - \tau_m(x)))] dx \right|$$

$$\le \mathrm{N} \sup_{x_0 \le x \le x_0 + h} \int_{x_0}^{x} \sum_{i=0}^{m} \left| y(x - \tau_i(x)) - z(x - \tau_i(x)) \right| dx$$

$$\le \mathrm{Nh(m+1)} \sup_{x_0 \le x \le x_0 + h} \left| y(x) - z(x) \right|$$

$$= \mathrm{Nh(m+1)} \rho(\mathrm{y(x)}, \mathrm{z(x)})$$

where $\tau_0(x) = 0$. Thus $h < \frac{\alpha}{(m+1)N}$ where $0 < \alpha < 1$, $A$ is contractive. Therefore by the fixed point theorem the initial value problem for eq.(1.6) has a unique solution. ■

## Remarks (1.1):

(i) Theorem(1.2) can be easily generalized to systems of finite number of 1st order delay differential equation of retarded type with multiple variable delays, [El'sgol'ts L., 1964].

(ii) As seen before, the existence of a unique solution for special types of delay differential equations is obtained with the aid of the fixed point theorem. On the other hand, the method of successive approximation can be also used to get the same result, [Al-Kubeisy S., 2004].

Next, we discuss the continuous dependence of the solution of the differential equation with retarded arguments on the initial function. But before that we need the following lemma.

## Lemma (1.1), [Bellman R. and Cook K., 1963]:

If $w(x)$ is positive and monotone nondecreasing, if $y(x) \geq 0$, $\mathrm{v(x)} \geq 0$, if all three functions are continuous, and if

$$y(x) \leq w(x) + \int_a^x y(t)v(t)dt, \quad a \leq x \leq b \tag{1.1}$$

then

$$y(x) \leq w(x)e^{\int_a^x v(t)dt}, \quad a \leq x \leq b$$

## Proof:

From ineq.(1.1) and since $w$ is monotone nondecreasing one can get

$$\frac{y(x)}{w(x)} \le 1 + \int_a^x \frac{y(t)v(t)}{w(x)} dt \le 1 + \int_a^x \frac{y(t)v(t)}{w(t)} dt .$$

Let $g(x) = \dfrac{y(x)}{w(x)}$, then

$$\frac{g(x)v(x)}{1 + \int_a^x g(t)v(t)dt} \le v(x) .$$

By integrating the above inequality from $a$ to $x$ one can obtain

$$\ln\left[1 + \int_a^x g(t)v(t)dt\right] \le \int_a^x v(t)dt .$$

Thus

$$\frac{y(x)}{w(x)} \le 1 + \int_a^x g(t)v(t)dt \le e^{\int_a^x v(t)dt} ,$$

hence

$$y(x) \le w(x) e^{\int_a^x v(t)dt} , \quad a \le x \le b . \ \blacksquare$$

Now, we are in the position that can we give the following theorem.

## Theorem (1.3):

The solution of the differential equation with retarded arguments given by eq.(1.6) with the initial function $y(x)=\varphi(x)$ on the initial set $E_{x_0}$ satisfying the conditions of the existence theorem (1.2) is in continuously dependent upon the initial function. Moreover, if

$|\varphi_1(x)-\varphi_2(x)|<\delta, \quad \delta>0$ on the initial set $\mathrm{E}_{\mathrm{x}_0}$,

then

$$\left|y_{\varphi_1}(x)-y_{\varphi_2}(x)\right|<\delta e^{(m+1)N(x-x_0)}, \quad \mathrm{x}>\mathrm{x}_0$$

where $y_{\varphi_i}$ is the solution of eq.(1.6) with the initial function $y_{\varphi_i}(x)=\varphi_i(x)$ on the initial set $E_{x_0}$ for $\mathrm{i}=1,2$ .

## Proof:

We replace eq.(1.6) by the equivalent integral equations

$$y_{\varphi_1}(x)=\varphi_1(x_0)+\int_{x_0}^{x} f(x,y_{\varphi_1}(x),y_{\varphi_1}(x-\tau_1(x)),\dots,y_{\varphi_1}(x-\tau_m(x)))dt$$

$$y_{\varphi_2}(x)=\varphi_2(x_0)+\int_{x_0}^{x} f(x,y_{\varphi_2}(x),y_{\varphi_2}(x-\tau_1(x)),\dots,y_{\varphi_2}(x-\tau_m(x)))dt$$

then

$$\sup_{[x_0,x]}\left|y_{\varphi_1}(x)-y_{\varphi_2}(x)\right|\le\left|\varphi_1(x_0)-\varphi_2(x_0)\right|+N\int_{x_0}^{x}\sum_{i=0}^{m}\sup_{[x_0,x]}\left|y_{\varphi_1}(x-\tau_i(x))-y_{\varphi_2}(x-\tau_i(x))\right|dt$$

$$\leq \delta + N(\mathrm{m}+1)\int_{\mathrm{x}_0}^{\mathrm{x}} \sup_{[t_0,t]} \left| y_{\varphi_1}(t) - y_{\varphi_2}(t) \right| dt$$

Solving the polynomial inequality relative to $\sup\limits_{[x_0,x]} \left| y_{\varphi_1}(x) - y_{\varphi_2}(x) \right|$ by using lemma(1.1) we will have

$$\sup_{[x_0,x]} \left| y_{\varphi_1}(x) - y_{\varphi_2}(x) \right| \leq \delta\, e^{N(m+1)(x-x_0)}. \quad \blacksquare$$

## Remark (1.2):

The previous theorem can be also modified to be valid for systems of the 1st order ordinary differential equations with retarded arguments.

## 1.4 Methods of Solution of Delay Differential Equations:

Like, the ordinary differential equations, there are many methods that can be used to solve the delay differential equations, say, the method of steps,[El'sgol'ts L. and Norkin S., 1973], the Laplace transform method, [Bellman R. and Cook K., 1963], the linear multistep method, [Al-Kubeisy S.,2004], and the expansion methods, [Salih S., 2004].

In this section, we give two of such methods namely the method of steps and the Laplace transform method.

### 1.4.1 The Method of Steps, [El'sgol'ts L. and Norkin S., 1973]:

The method of steps is the best well known theoretical method for solving differential equations with deviating arguments.

This method can be described for the following cases:

## Case (1):

Consider the basic initial value problem for the simplest delay differential equation with a retarded argument

$$y'(x) = f(x, y(x), y(x-\tau)) \tag{1.7}$$

with the initial function

$$y(x) = \varphi(x) \quad \text{for} \quad x_0 - \tau \le x \le x_0 .$$

The solution of eq.(1.7) is determined from the equation without delay

$$y'(x) = f(x, y(x), \varphi(x-\tau)), \quad x_0 \le x \le x_0 + \tau$$

with the initial condition

$$y(x_0) = \varphi(x_0)$$

Next, assuming the existence of a solution $y(x) = \varphi_1(x)$ of this initial value problem on the interval $[x_0 , x_0 + \tau]$, analogously we obtain:

$$y'(x) = f(x, y(x), \varphi_1(x-\tau)), \quad x_0 + \tau \le x \le x_0 + 2\tau$$

with the initial condition

$$y(x_0 + \tau) = \varphi_1(x_0 + \tau)$$

Therefore,

$$y'(x) = f(x, y(x), \varphi_n(x-\tau)), \quad x_0 + n\tau \le x \le x_0 + (n+1)\tau$$

with the initial condition

$y(x_0 + n\tau) = \varphi_n(x_0 + n\tau)$

where $\varphi_j(x)$ is the solution of the considered initial value problem on the interval $[x_0 + (j-1)\tau, x_0 + j\tau]$.

Now, for illustration purpose, we will demonstrate the method of steps in the following example.

## Example (1.1):

Consider the 1st order differential equation with a retarded argument

$$y'(x) = 2y(x) + 6y(x-1)$$

with the initial function

$$y(x) = x + 1, \quad x \in [-1,0]$$

The solution of the initial value problem is determined from the differential equation without delay

$$y'(x) = 2y(x) + 6x, \quad 0 \le x \le 1$$

with the initial condition

$$y(0) = 1$$

The solution of the above initial value problem is

$$y(x) = \frac{5}{2}e^{2x} - 3x - \frac{3}{2}, \quad 0 \le x \le 1$$

Next, we determined the solution on the interval $[1, 2]$ from the differential equation

$$y'(x) = 2y(x) + 15e^{2(x-1)} - 18x + 9$$

with the initial condition

$$y(1) = \frac{5}{2}e^2 - \frac{9}{2}$$

The solution of this initial value problem is

$$y(x) = \frac{5}{2}e^{2x} - \frac{57}{2}e^{2(x-1)} + 15xe^{2(x-1)} + 9x$$

## Case (2):

Consider the simplest delay differential equation with a variable retarded argument:

$$y'(x) = f(x, y(x), y(x - \tau(x))) \tag{1.8}$$

with the initial function

$y(x) = \varphi(x)$ on the initial set $E_{x_0}$.

The solution of eq.(1.8) is determined from the equation without delay

$y'(x) = f(x, y(x), \varphi(x - \tau(x)))$ for $x_0 \le x \le \alpha(x_0)$

with the initial condition

$$y(x_0) = \varphi(x_0)$$

The interval $[x_0, \alpha(x_0)]$ is the largest interval beginning with the point $x_0$ for which $x - \tau(x) \le x_0$. We observe that $\alpha(x)$ is the inverse of the function $x - \tau(x)$, if this inverse exists.

Next, we determine the solution on the interval $[\alpha(x_0), \alpha(\alpha(x_0))]$ from the equation:

$$y'(x) = f(x, y(x), \varphi_1(x - \tau(x)))$$

with the initial condition

$$y(\alpha(x_0)) = \varphi_1(\alpha(x_0))$$

where $\varphi_1(x)$ is the extension of the function $\varphi(x)$ by the solution of eq.(1.8) on the interval $[x_0, \alpha(x_0)]$, and the continuation of the process may be reduced to the problem of integrating an equation without delay.

To illustrate this method in this case, see the following example.

**Example (1.2):**

Consider the differential equation with retarded argument:

$$y'(x) = 2y(x) + y(x/2)$$

with the initial function

$y(x) = x$ on the initial set $E_1 = [\frac{1}{2}, 1]$.

The solution of the above differential equation is determined from the differential equation without delay

$$y'(x) = 2y(x) + \frac{x}{2}, \quad 1 \le x \le 2$$

with the initial condition

$$y(1) = 1$$

The solution of the above initial value problem is

$$y(x) = \frac{11}{8}e^{2(x-1)} - \frac{1}{4}x - \frac{1}{8}$$

Next, we determined the solution on the interval $[2, 4]$ from the differential equation

$$y'(x) = 2y(x) + \frac{11}{8}e^{x-2} - \frac{1}{8}x - \frac{1}{8}$$

with the initial condition

$$y(2) = \frac{11}{8}e^{2} - \frac{5}{8}$$

The solution of the above initial value problem is

$$y(x) = \frac{11}{8}e^{2(x-1)} + \frac{17}{32}e^{2(x-2)} - \frac{11}{8}e^{x-2} + \frac{1}{16}x + \frac{3}{32}$$

## Case (3):

Consider the 1st order differential equation with different deviating arguments:

$$y'(x) = f(x, y(x), y(x-\tau_1(x)), \ldots, y(x-\tau_m(x))) \tag{1.9}$$

with the initial function

$y(x) = \varphi(x)$ on the initial set $E_{x_0}$.

The solution of eq.(1.9) is determined from the equation without delay

$$y'(x) = f(x, y(x), \varphi(x - \tau_1(x)), ..., \varphi(x - \tau_m(x))) \quad \text{for } x_0 \le x \le \alpha(x_0)$$

with the initial condition

$$y(x_0) = \varphi(x_0)$$

The interval $[x_0, \alpha(x_0)]$ is the largest interval with the left endpoint $x_0$ on which all of the differences $x - \tau_i(x) \le x_0, \ \ i = 1,2,...,m$.

Next, we determine the solution on the interval $[\alpha(x_0), \alpha(\alpha(x_0))]$ from the equation

$$y'(x) = f(x, y(x), \varphi_1(x - \tau_1(x)), ..., \varphi_1(x - \tau_m(x)))$$

with the initial condition

$$y(\alpha(x_0)) = \varphi_1(\alpha(x_0))$$

where $\varphi_1(x)$ is the extension of the function $\varphi(x)$ by the solution of eq.(1.9) on the interval $[x_0, \alpha(x_0)]$, and the continuation of the process may be reduced to the problem of integrating an equation without delay.

Next, we determine the solution on the interval $[\alpha(x_0), \alpha(\alpha(x_0))]$ from the equation

$$y'(x) = f(x, y(x), \varphi_1(x - \tau_1(x)), ..., \varphi_1(x - \tau_m(x)))$$

with the initial condition

$y(\alpha(x_0)) = \varphi_1(\alpha(x_0))$

where $\varphi_1(x)$ is the extension of the function $\varphi(x)$ by the solution of eq.(1.9) on the interval $[x_0, \alpha(x_0)]$, and the continuation of the process may be reduced to the problem of integrating an equation without delay.

Here, the following example is very useful to understand the above.

## Example (1.3):

Consider the differential equation with different deviating arguments

$$y'(x) = y(x) + 2y(x/2) + y(x/3)$$

with the initial function

$y(x) = x$ on the initial set $E_1 = [\frac{1}{3}, 1]$.

The solution of the above differential equation is determined from the differential equation without delay

$$y'(x) = y(x) + \frac{4}{3}x, \quad 1 \le x \le 3$$

with the initial condition

$$y(1) = 1$$

The solution of the above initial value problem is

$$y(x) = \frac{11}{3}e^{x-1} - \frac{4}{3}x - \frac{4}{3}$$

Next, we determined the solution on the interval $[3,9]$ from the differential equation

$$y'(x)=y(x)+\frac{22}{3}e^{\frac{x}{2}-1}-\frac{16}{9}x+\frac{11}{3}e^{\frac{x}{3}-1}-4$$

with the initial condition

$$y(3)=\frac{11}{3}e^{2}-\frac{16}{3}$$

The solution of the this initial value problem is

$$y(x)=\frac{11}{3}e^{x-1}-\frac{44}{3}e^{x-\frac{5}{2}}+\frac{11}{2}e^{x-3}-\frac{148}{9}e^{x-3}-\frac{44}{3}e^{\frac{x}{2}-1}-\frac{11}{2}e^{\frac{x}{3}-1}+\frac{16}{9}x+\frac{52}{9}$$

## Case (4):

Consider the 1st order neutral differential equation with deviating argument

$$y'(x)=f(x,y(x),y(x-\tau),y'(x-\tau)) \qquad (1.10)$$

with the initial function

$$y(x)=\varphi(x) \quad \text{for } x_0-\tau\le x\le x_0$$

where $\varphi$ is a continuous function that have continuous derivatives.
The solution $\varphi_1(x)$ of eq.(1.10) is determined from the equation without a deviating argument

$$y'(x)=f(x,y(x),\varphi(x-\tau),\varphi'(x-\tau)) \quad \text{for } x_0<x\le x_0+\tau$$

On the next step

$$y'(x) = f(x, y(x), \varphi_1(x-\tau), \varphi_1'(x-\tau)), \quad \text{for } x_0 + \tau \le x \le x_0 + 2\tau$$

and so on.

The contract to an equation with a deviating argument consisting of the fact that the solution is not smoothed. In fact, not only at the point $x_0$ is the left derivative $\varphi'(x_0)$, generally speaking, not equal to $\varphi_1'(x_0)$ but also at the point $x_0 + \tau$, as is obvious from eq.(1.10). Thus $y'(x)$ will be in general discontinuous at $x_0 + \tau$. Similarly $y'(x)$ will be discontinuous at $x_0 + k\tau$, $(k = 0,1,...)$.

To illustrate this approach in this case, consider the following example.

## Example (1.4):

Consider the differential equation with deviating argument

$$y'(x) = y(x) + 2y'(x-2) + 3y^2(x-2)$$

with the initial function

$$y(x) = x + 1, \quad -2 \le x \le 0$$

Then

$$y'(x) = y(x) + 2 + 3(x-1)^2$$

with the initial condition

$$y(0) = 1$$

The solution of the above initial value problem for $0 \le x \le 2$ is

$$y(x) = 6e^x - 5 - 3x^2$$

Note that $\varphi(x) = x + 1$ on $-2 \le x \le 0$ and $\varphi_1(x) = 6e^x - 5 - 3x^2$ on $0 \le x \le 2$. Since $\varphi'(0) = 1$ and $\varphi_1'(0) = 6$, therefore $y'(0)$ doesn't exists. On the next step, substitute the solution $\varphi_1(x)$ into the original delay differential equation to obtain

$$y'(x) = y(x) + 108e^{2(x-2)} + (-108x^2 + 432x - 600)e^{x-2} + 27x^4 - 216x^3 + 1386x^2 - 5556x + 10827$$

The solution of the above differential equation together with the initial condition $y(2) = 6e^2 - 17$ is

$$y(x) = 6e^x - 11360e^{x-2} - 108e^{2(x-2)} + (36x^3 - 216x^2 + 600x)e^{x-2} + 27x^2 - 108x^3 + 1710x^2 - 2136x + 8691$$

Note that,

$$\varphi_2(x) = 6e^x - 11360e^{x-2} - 216e^{2(x-2)} + (36x^3 - 216x^2 + 600x)e^{x-2} + (108x^2 - 432x + 600)e^{x-2} + 54x - 324x^2 + 3420x - 2136$$

on the interval $2 \le x \le 4$.

Since $\varphi_1'(2) = 6e^2 - 12$ and $\varphi_2'(2) = 6e^2 - 7268$, therefore $y'(2)$ doesn't exists.

## 1.4.2 The LaplaceTransform Method:

It is known that the Laplace transform method is one of the important methods that can be used to solve the linear ordinary differential equation with constant coefficients, [Brauer F. and Nohel J., 1973]. Here we use it to solve the same types of ordinary differential equations but with deviating arguments. This method depends mainly on applying the method of steps for the ordinary differential equations with deviating arguments to transform them to ordinary differential equations and then use the Laplace transform method to solve the resulting ordinary differential equations.

To illustrate this approach, consider the following examples.

**Example (1.5):**

Consider the 1st order differential equation with a retarded argument

$$y'(x) = 2y(x) + 4y(x-2)$$

with the initial function

$$y(x) = x, \quad x \in [-2,0]$$

To find the solution in the first step interval [0,2], we apply the method of steps, to get

$$y'(x) = 2y(x) + 4x - 8, \quad 0 \le x \le 2$$

and this is an ordinary differential equation of the 1st order.

Now, taking the Laplace transform produces

$$L(y'(x)) = 2L(y(x)) + 4L(x) - 8L(1)$$

$$sY(s) - y(0) = 2Y(s) + 4\frac{1}{s^2} - \frac{8}{s}$$

and so the Laplace transform of the solution $y(x)$ into $Y(s)$ is given by

$$Y(s) = \frac{5}{s} - \frac{2}{s^2} - \frac{5}{(s-2)}$$

Taking inverse Laplace transform, we have

$$y(x) = 5\,L^{-1}\left(\frac{1}{s}\right) - \frac{1}{1!}L^{-1}\left(\frac{1!}{s^2}\right) - 5\,L^{-1}\left(\frac{1}{s-2}\right)$$

Hence, the solution in the first step interval is given by

$$y(x) = \varphi_1(x) = 5 - 2x - 5e^{2x}, \quad 0 \le \mathrm{x} \le 2$$

In order to find the solution in the second step time interval [2,4], we proceed similarly as in the first step with initial function

$$\varphi_1(x) = 5 - 2x - 5e^{2x}, \quad 0 \le \mathrm{x} \le 2$$

and hence,

$$y'(x) = 2y(x) - 20e^{2(x-2)} - 8x + 36$$

with the initial condition

$$y(2) = 1 - 5e^4$$

By making changing of independent variable $x-2$ to move the initial time to zero. Let $w = x-2$ then $w \in [0,2]$, so that

$$y'(w+2) = 2y(w+2) - 20\,\mathrm{e}^{2\mathrm{w}} - 8(w+2) + 36$$

and by considering

$$z(w) = y(x+2)$$

implies that

$$z'(w) = 2z(w) - 20\,\mathrm{e}^{2w} - 8(w+2) + 36$$

with $z(0) = 1 - 5\,\mathrm{e}^{4}$

Taking the Laplace transform of both sides, we have

$$sZ(s) - z(0) = 2Z(s) - \frac{20}{s-2} - \frac{8}{s^2} - \frac{16}{s} + \frac{36}{s}$$

where $Z(s)$ is the Laplace transform of $Z(w)$, hence

$$Z(s) = -\frac{11}{s} + \frac{4}{s^2} + \frac{12-5e^4}{s-2} - \frac{20}{(s-2)^2}$$

Taking inverse Laplace transform, we have

$$Z(w) = -11 + 4w + (12 - 5e^4)\,e^{2w} - 20we^{2w}$$

Hence, the solution in the second step time interval [2,4] is given by

$$z(w) = y(x) = -11 + 4(x-2) + (12 - 5e^4)e^{2(x-2)} - 20(x-2)e^{2(x-2)}.$$

Similarly, we proceed to the next intervals.

## Example (1.6):

Consider the 2nd order differential equation with a retarded argument

$$y''(x) = -2y(x-1)$$

with the initial function

$y(x) = x, \quad x \in [-1,0]$

To find the solution in the first step interval [0,1], we apply the method of steps, to get

$y''(x) = -2x + 2, \quad 0 \le x \le 1$

and this is an ordinary differential equation of the 1st order.

Now, taking the Laplace transform produces

$L(y''(x)) = -2L(x) + 2L(1)$

$$s^2 Y(s) - sy(0) - y'(0) = \frac{-2}{s^2} + \frac{2}{s}$$

and so the Laplace transform of the solution $y(x)$ into $Y(s)$ is given by

$$Y(s) = \frac{-2}{s^4} + \frac{2}{s^3} + \frac{1}{s^2}$$

Taking inverse Laplace transform, we have

$$y(x) = \frac{-2}{3!} L^{-1}\left(\frac{3!}{s^4}\right) + L^{-1}\left(\frac{2!}{s^3}\right) + L^{-1}\left(\frac{1!}{s^2}\right)$$

Hence, the solution in the first step interval is given by

$$y(x) = \varphi_1(x) = -\frac{x^3}{3} + x^2 + x, \quad 0 \le x \le 1$$

In order to find the solution in the second step time interval [1,2], we proceed similarly as in the first step with initial function

$$\varphi_1(x) = -\frac{x^3}{3} + x^2 + x, \quad 0 \le \text{x} \le 1$$

and hence,

$$y''(x) = \frac{2}{3}(x-1)^3 - 2(x-1)^2 - 2(x-1)$$

with the initial conditions

$$y(1) = \frac{5}{3} \quad \text{and} \quad y'(1) = 2$$

By making changing of independent variable $x-1$ to move the initial time to zero. Let $w = x-1$ then $w \in [0,1]$, so that

$$y''(x+1) = \frac{2}{3}w^3 - 2w^2 - 2w$$

and by considering

$$z(w) = y(x+1)$$

implies that

$$z''(w) = \frac{2}{3}w^3 - 2w^2 - 2w$$

with $z(0) = \frac{5}{3}$ and $z'(0) = 2$

Taking the Laplace transform of both sides, we have

$$s^2 Z(s) - sz(0) - z'(0) = \frac{4}{s^4} - \frac{4}{s^3} - \frac{2}{s^2}$$

where $Z(s)$ is the Laplace transform of $Z(w)$, hence

$$Z(s)=\frac{4}{s^6}-\frac{4}{s^5}-\frac{2}{s^4}+\frac{5}{3}s+2$$

Taking inverse Laplace transform, we have

$$Z(w)=\frac{4}{5!}w^5-\frac{4}{4!}w^4-\frac{2}{3!}w^3+\frac{5}{3}+2w$$

Hence, the solution in the second step time interval [1,2] is given by

$$z(w)=y(x)=\frac{(t-1)^5}{3}-\frac{(t-1)^4}{6}-\frac{(t-1)^3}{3}+2(t-1)+\frac{5}{3}.$$

Similarly, we proceed to the next intervals.

## Remark(1.3):

The Laplace transform method can be directly used to solve the linear ordinary differential equations with deviating arguments with constant coefficients (i.e., without using the method of steps), [Bellman R. and Cook K., 1963].

## 3.1 Introduction:

As seen before, the ordinary differential equations with deviating arguments is more general than the ordinary differential equations.

So, the partial differential equations with deviating arguments is more general than the partial differential equations.

So, in this chapter we give some basic concepts of the partial differential equations with deviating arguments.

Also, the method of successive approximation is devoted to ensure the existence and uniqueness of a bounded solution for special types of the partial differential equations with deviating arguments.

Moreover, an estimate of the magnitude of the solutions for special types of the $1^{st}$ and $2^{nd}$ order linear partial differential equations with deviating arguments are introduced.

This chapter consists of four sections.

In section two, a simple classification of the partial differential equations with deviating arguments is given.

In section three, some existence and uniqueness theorems for special types of the $1^{st}$ and $2^{nd}$ order partial differential equations with deviating arguments are introduced.

In section four, an estimate of the magnitude of the solutions for special types of the $1^{st}$ and $2^{nd}$ order partial differential equations with deviating arguments is derived. This section consists the main part of this work.

## 3.2 The Partial Differential Equations with Deviating Arguments and Their Solutions:

It is known that the partial differential equations with deviating arguments are differential equations in which the unknown function (depends on two or more independent variables) or its partial derivatives enter with several different values of the argument, [El'sgol'ts L. and Norkin S., 1973]. For example

$$\frac{\partial u}{\partial x}(x,t)+u(x,t)=f(x,t,u(x,t-1))$$

$$\frac{\partial^2 u}{\partial x^2}(x,t)+\frac{\partial^2 u}{\partial t^2}(x,t)=2\frac{\partial^2 u}{\partial t^2}(x-1,t)$$

In this work, we restrict our discussion for the partial differential equations with deviating arguments in which the unknown function depends only on two variables.

The general form for the 1$^{st}$ order partial differential equation with deviating arguments is:

$$F\left(x,t,u(x,t),u\left(x-\tau_1\left(x,t,u(x,t),\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right),t-\tau_2\left(x,t,u(x,t),\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right)\right),\right.$$
$$\frac{\partial u}{\partial x},\frac{\partial u}{\partial t},\frac{\partial u}{\partial x}\left(x-\tau_3\left(x,t,u(x,t),\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right),t-\tau_4\left(x,t,u(x,t),\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right)\right),$$
$$\left.\frac{\partial u}{\partial t}\left(x-\tau_5\left(x,t,u(x,t),\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right),t-\tau_6\left(x,t,u(x,t),\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right)\right)\right)=0 \qquad (3.1)$$

where $\tau_i$ is known function for each $i=1,2,...,6$. If $\tau_i=0$ for each $i=1,2,...,6$, then eq.(3.1) reduces to the general form for the 1st order partial differential equation. If $\tau_i\left(x,t,u(x,t),\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right)=\tau_i$, where $\tau_i$ is a known constant for each $i=1,2,...,6$, then eq.(3.1) is said to be the general form for the 1st order partial differential equation with constant deviating arguments, otherwise it is with variable deviating arguments.

Also, the general form for the 1st order linear partial differential equation with deviating arguments can be obtained by reducing eq.(3.1) to the following form:

$$a_1(x,t)\frac{\partial u}{\partial x}+a_2(x,t)\frac{\partial u}{\partial t}+a_3(x,t)\frac{\partial u}{\partial x}(x-\tau_1(x,t),t-\tau_2(x,t))+$$

$$a_4(x,t)\frac{\partial u}{\partial t}(x-\tau_3(x,t),t-\tau_4(x,t))+a_5(x,t)u(x,t)+$$

$$a_6(x,t)u(x-\tau_5(x,t),t-\tau_6(x,t))=g(x,t) \qquad (3.2)$$

if $a_3=a_4=a_6=0$, then eq.(3.2) reduces to the general form of the 1st order linear partial differential equation.

Moreover, the general form for the 2nd order partial differential equation with deviating arguments is:

$$F\left(x,t,u(x,t),u(x-\tau_1,t-\tau_2),\frac{\partial u}{\partial x},\frac{\partial u}{\partial t},\frac{\partial u}{\partial x}(x-\tau_3,t-\tau_4),\frac{\partial u}{\partial t}(x-\tau_5,t-\tau_6),\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial x\partial t},\right.$$

$$\left.\frac{\partial^2 u}{\partial t^2},\frac{\partial^2 u}{\partial x^2}(x-\tau_7,t-\tau_8),\frac{\partial^2 u}{\partial x\partial t}(x-\tau_9,t-\tau_{10}),\frac{\partial^2 u}{\partial t^2}(x-\tau_{11},t-\tau_{12})\right)=0 \qquad (3.3)$$

where $\tau_i = \tau_i\left( x,t,u(x,t),\frac{\partial u}{\partial x},\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial x \partial t},\frac{\partial^2 u}{\partial t^2} \right)$ is a known function for each $i = 1,2,...,12$.

If $\tau_i$ is a known constant for each $i = 1,2,...,12$, then eq.(3.3) is said to be the general form for the 2nd order partial differential equation with constant deviating arguments, otherwise it is with variable deviating arguments.

So, the general form of the 2nd order linear partial differential equation with deviating arguments is:

$$a_1 \frac{\partial^2 u}{\partial x^2} + b_1 \frac{\partial^2 u}{\partial x \partial t} + c_1 \frac{\partial^2 u}{\partial t^2} + d_1 \frac{\partial u}{\partial x} + e_1 \frac{\partial u}{\partial t} + f_1 u + a_2 \frac{\partial^2 u}{\partial x^2}(x-\tau_1, t-\tau_2) +$$

$$b_2 \frac{\partial^2 u}{\partial x \partial t}(x-\tau_3, t-\tau_4) + c_2 \frac{\partial^2 u}{\partial t^2}(x-\tau_5, t-\tau_6) + d_2 \frac{\partial u}{\partial x}(x-\tau_7, t-\tau_8) +$$

$$e_2 \frac{\partial u}{\partial t}(x-\tau_9, t-\tau_{10}) + f_2 u(x-\tau_{11}, t-\tau_{12}) = g(x,t) \qquad (3.4)$$

where $a_j, b_j, c_j, d_j, e_j, f_j, \tau_i$ and g are assumed to be known functions of $x$ and $t$ only for each $i = 1,2,...,12$, $\mathrm{j} = 1,2$.

A special case eq.(3.4) is the following partial differential equation with deviating arguments is:

$$a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial t} + c \frac{\partial^2 u}{\partial t^2} + d \frac{\partial u}{\partial x} + e \frac{\partial u}{\partial t} + fu + hu(x-\tau_1, t-\tau_2) = g(x,t) \qquad (3.5)$$

where $a,b,c,d,e,f,h,g,\tau_1,\tau_2$ are known functions of $x$ and $t$. Moreover

(i) if $b^2 - 4ac > 0$, then eq.(3.5) is said to be of the hyperbolic type.

(ii) if $b^2 - 4ac = 0$, then eq.(3.5) is said to be of the parabolic type.

(iii) if $b^2 - 4ac < 0$, then eq.(3.5) is said to be of the elliptic type.

So, one can easily recognize the linear and the nonlinear $n^{th}$ order partial differential equation with deviating arguments.

Next, it is known that, many methods can be used to solve the partial differential equations, say separation of variables, Laplace transform method, [Farlow S., 1982], finite-difference method, [Farlow S., 1982] and [Smith G., 1965], etc.

One of the important methods that can be used to solve the partial differential equations with deviating arguments is the separation of variables. El'sgol'ts L. and Norkin S. in 1973 and El'sgol'ts L. in 1964 used the separation of variables to solve the following problems:

(i) the generalized diffusion equation:

$$\frac{\partial u}{\partial t}(x,t) = a^2 \frac{\partial^2 u}{\partial x^2}(x,t) + b^2 \frac{\partial^2 u}{\partial x^2}(x,t-\tau)$$

where $a$ , b, and $\tau$ are constants, $\tau > 0$ together with the initial and boundary conditions

$$u(t,x) = \varphi(t,x) \text{ for } 0 \le x \le \ell,\ \ 0 \le t \le \tau \text{ and } u(0,t) = 0,\ \ u(\ell,t) = 0.$$

(ii) the generalized wave equation:

$$\frac{\partial^2 u}{\partial t^2}(x,t) = a^2 \frac{\partial^2 u}{\partial x^2}(x,t) + b^2 \frac{\partial^2 u}{\partial x^2}(x,t-\tau)$$

where $a$ , b, and $\tau$ are constants, $\tau > 0$ together with the initial and boundary conditions

$$u(t,x) = \varphi(t,x) \quad \text{for } 0 \le \mathrm{x} \le \ell,\ \ 0 \le \mathrm{t} \le \tau \quad \text{and} \quad \mathrm{u}(0,\mathrm{t}) = 0,\ \ \mathrm{u}(\ell,\mathrm{t}) = 0 .$$

Also, Vandewalle S. and Gander M. in 2003 used this method to solve the parabolic partial delay differential equations.

## 3.3 Existence and Uniqueness of Solution for Special Types of the Partial Delay Differential Equations:

The partial differential equations with deviating arguments appear in mathematical models for many real life applications say in dynamics of gas absorption, [Poorkarimi H. and Wiener J., 1987], and arise from many biological, chemical, and physical systems which are characterized by both spatial and temporal variables and exhibit various spatio-temporal patterns. The systematic study of such equations from the dynamical systems and semigroups point of view began in the 70s, and considerable advances have been achieved since then, [Wu. J, 1996].

This section concerned with the existence and the uniqueness of the solution for special types of partial delay differential equations.

## 3.3.1 Existence and Uniqueness of Solution for 1st order Partial Delay Differential Equations:

In this section, we discuss the existence and the uniqueness of a bounded solution for special types of the 1st order linear and nonlinear partial delay differential equations. This section consists of the main part of this work.

We start this section by recalling the 1st order partial delay differential equation:

$$\frac{\partial u}{\partial t}(x,t)+a(x,t)u(x,t)=c(x,t,u(x,t),u(x,t-\tau)) \qquad (3.6)$$

together with the following initial function

$$u(x,t)=\varphi(x,t), \quad 0\le x\le 1, \quad -\tau\le t\le 0, \quad \tau>0 \qquad (3.7)$$

where $a$ and $\varphi$ are given functions of $x$ and $t$, and $c$ is a known function of $x$, t, u(x,t) and u(x,t - $\tau$).

The following theorem shows that eq.(3.6)-(3.7) has a unique bounded solution. To the best of our knowledge, this theorem seems to be new.

**Theorem(3.1):**

Consider eq.(3.6)-(3.7). Assume the following conditions:

(i) $\varphi(x,t)$ is continuous on $[0,1]\times[-\tau,0]$;

(ii) $a(x,t)$ is continuous in $\Delta=\{(x,t)\,|\,0\le x\le 1,\ t\ge 0\}$ and satisfies

$a(x,t)\ge m>0$ in $\Delta$;

(iii) $c(x,t,u,v)$ is continuous and bounded on $\Delta \times \Re \times \Re$ and satisfies the Lipschitz condition

$$|c(x,t,u,v) - c(x,t,u^*,v)| \leq L|u - u^*|; \tag{3.1}$$

(iv) $\frac{L}{m} < 1$.

Then there exists a unique continuous solution of the problem given by eq.(3.6)-(3.7) defined in $\Delta$ and bounded there.

## Proof:

On the interval $0 \leq t \leq \tau$, eq.(3.6) becomes

$$\frac{\partial u}{\partial t}(x,t) + a(x,t)u(x,t) = c(x,t,u(x,t),\varphi(x,t-\tau)). \tag{3.8}$$

Integrating eq.(3.8) from $0$ to $t$, we obtain

$$u(x,t) = u(x,0)e^{-\int_0^t a(x,s)ds} + \int_0^t e^{-\int_s^t a(x,\theta)d\theta} c(x,s,u(x,s),\varphi(x,s-\tau))ds \tag{3.9}$$

To prove existence-uniqueness, we apply to eq.(3.9) the method of successive approximations. Put

$$u_1(x,t) = u(x,0)e^{-\int_0^t a(x,s)ds},$$

then, since

$$|u(x,0)| = |\varphi(x,0)| \leq N,$$

and $a(x,t) \geq m > 0$, we have

$$|u_1(x,t)| \le Ne^{-\int_0^t mds} = Ne^{-mt} \le N .$$

for some constant $N$ .

Furthermore,

$$u_2(x,t) = u(x,0)e^{-\int_0^t a(x,\theta)d\theta} + \int_0^t e^{-\int_s^t a(x,\theta)d\theta} c(x,s,u_1(x,s),\varphi(x,s-\tau))ds ,$$

and

$$|u_2(x,t) - u_1(x,t)| \le \int_0^t e^{-m(t-s)} |c(x,s,u_1(x,s),\varphi(x,s-\tau))| ds$$

$$\le \frac{M}{m} e^{-m(t-s)} \Big|_0^t = \frac{M}{m}(1-e^{-mt}) < \frac{M}{m} ,$$

where $|c(x,t,u,v)| \le M$ , for each $(x,t,u,v) \in \Delta \times \Re \times \Re$.

Also, by virtue of ineq.(3.1),

$$|u_3(x,t) - u_2(x,t)| \le \int_0^t e^{-\int_s^t a(x,\theta)d\theta} |c(x,s,u_2(x,s),\varphi(x,s-\tau)) -$$

$$c(x,s,u_1(x,s),\varphi(x,s-\tau))| ds$$

$$\le \int_0^t \mathrm{L} \mathrm{e}^{-\int_s^t \mathrm{a}(\mathrm{x},\theta)\mathrm{d}\theta} |u_2(x,s) - u_1(x,s)| ds$$

$$\le \mathrm{L} \int_0^t \mathrm{e}^{-\int_s^t \mathrm{m}\mathrm{d}\theta} |u_2(x,s) - u_1(x,s)| ds$$

$$\le \mathrm{L} \frac{\mathrm{M}}{\mathrm{m}} \int_0^t e^{-m(t-s)} ds \le \mathrm{L} \frac{\mathrm{M}}{\mathrm{m}} \frac{1}{m} (1 - \mathrm{e}^{-\mathrm{mt}}) < \frac{M}{L} \left( \frac{\mathrm{L}}{\mathrm{m}} \right)^2 ,$$

and similarly,

$$\left|u_4(x,t)-u_3(x,t)\right|<\frac{M}{L}\left(\frac{L}{m}\right)^3 .$$

Continuing this procedure gives the estimate

$$\left|u_{n+1}(x,t)-u_n(x,t)\right|<\frac{M}{L}\left(\frac{L}{m}\right)^n , \quad \mathrm{n}=1,2,...$$

and since

$$u(x,t)=u_1(x,t)+\sum_{n=1}^{\infty}(u_{n+1}(x,t)-u_n(x,t)),$$

then

$$\left|u(x,t)\right|\le N+\sum_{n=1}^{\infty}\frac{M}{L}\left(\frac{L}{m}\right)^n=\mathrm{N}+\frac{M}{L}\frac{\frac{\mathrm{L}}{\mathrm{m}}}{1-\frac{\mathrm{L}}{\mathrm{m}}}$$

For the solution of eq.(3.6)-(3.7) on the interval $\tau\le t\le 2\tau$, we have

$$u(x,t)=u(x,0)e^{-\int_0^t a(x,s)ds}+\int_0^t e^{-\int_s^t a(x,\theta)d\theta}c(x,s,u(x,s),u^{(o)}(x,s-\tau))ds,$$

where $u^{(o)}(x,t)$ is the solution of eq.(3.6)-(3.7) for $0\le t\le\tau$.

Repeating the above calculations yields

$$\left|u(x,t)\right|<N+\frac{M}{L}\frac{\frac{L}{m}}{1-\frac{L}{m}}, \quad \tau\le \mathrm{t}\le 2\tau, \quad 0\le \mathrm{x}\le 1.$$

Therefore,

$$\left|u(x,t)\right|<N+\frac{M}{m-L}, \quad \mathrm{t}\ge 0, \quad 0\le \mathrm{x}\le 1. \quad \blacksquare$$

Next, recall that the 1st order partial delay differential equation:

$$\frac{\partial u}{\partial t}(x,t)+a(x,t)u(x,t)=c(x,t,u(x,t),u(x,g(t))), \qquad (3.10)$$

together with the following initial function

$$u(x,0)=\varphi(x),\ \ 0\le x\le 1, \qquad (3.11)$$

where $a$ and $\varphi$ are given functions of $x$ and $t$ and $c$ is a known function of $x$, t, u(x,t) and $u(x,g(t))$.

The following theorem shows that eq.(3.10)-(3.11) has a unique bounded solution. To the best of our knowledge, this theorem seems to be new.

## Theorem(3.2):

Consider eq.(3.10)-(3.11). Assume the following conditions:

(i) $\varphi(x)$ is continuous on $[0,1]$;

(ii) $a(x,t)$ is continuous in $\Delta=\{(x,t)\mid 0\le x\le 1,\ t\ge 0\}$ and satisfies $a(x,t)\ge m>0$ in $\Delta$;

(iii) $c(x,t,u,v)$ is continuous on $\Delta\times\Re\times\Re$ with $c(x,t,0,0)$ bounded on $\Delta$, and satisfies a Lipschitz condition in $u$ and v.

(iv) $g(t)$ is continuous on $[0,\infty)$ and $0\le g(t)\le t$.

(v) $\frac{2L}{m}<1$.

Then there exists a unique continuous solution of the problem given by eq.(3.10)-(3.11) defined in $\Delta$ and bounded there.

## Proof:

Integrating eq.(3.10) from $0$ to $t$, we obtain

$$u(x,t)=\varphi(x)e^{-\int_0^t a(x,s)ds}+\int_0^t e^{-\int_s^t a(x,\theta)d\theta}c(x,s,u(x,s),u(x,g(s)))ds \qquad (3.12)$$

To prove existence-uniqueness, we apply to eq.(3.12) the method of successive approximations. Put

$$u_1(x,t)=\varphi(x)e^{-\int_0^t a(x,s)ds},$$

then, since

$$|u(x,0)|=|\varphi(x)|\le N,$$

and $a(x,t)\ge m>0$, we have

$$|u_1(x,t)|\le Ne^{-\int_0^t mds}=Ne^{-mt}\le N,$$

with some constant N.

Furthermore,

$$u_2(x,t)=\varphi(x)e^{-\int_0^t a(x,s)ds}+\int_0^t e^{-\int_s^t a(x,\theta)d\theta}c(x,s,u_1(x,s),u_1(x,g(s)))ds,$$

and

$$|u_2(x,t)-u_1(x,t)|\le\int_0^t e^{-m(t-s)}\,|c(x,s,u_1(x,s),u_1(x,g(s)))|ds.$$

Now,

$$|c(x,s,u_1(x,s),u_1(x,g(s)))| \le |c(x,s,u_1(x,s),u_1(x,g(s)))-c(x,s,0,0)| + |c(x,s,0,0)|$$

$$\le L\left(\sup|u_1(x,t)| + \sup|u_1(x,g(t))|\right) + \sup|c(x,t,0,0)| = M$$

Therefore,

$$|u_2(x,t)-u_1(x,t)| \le \frac{M}{m}(1-e^{-mt}) < \frac{M}{m}.$$

Also,

$$u_3(x,t) = \varphi(x)e^{-\int_0^t a(x,s)ds} + \int_0^t e^{-\int_s^t a(x,\theta)d\theta} \mathrm{c}(\mathrm{x},\mathrm{s},\mathrm{u}_2(x,s),u_2(x,g(s)))ds,$$

and

$$|u_3(x,t)-u_2(x,t)| \le \int_0^t e^{-\int_s^t a(x,\theta)d\theta} |c(x,s,u_2(x,s),u_2(x,g(s)))-c(x,s,u_1(x,s),u_1(x,g(s)))|\, ds$$

$$\le \int_0^t \mathrm{e}^{-\int_s^t \mathrm{a}(\mathrm{x},\theta)\mathrm{d}\theta} \mathrm{L}\left(|u_2(x,s)-u_1(x,s)| + |u_2(x,g(s))-u_1(x,g(s))|\right)ds$$

$$\le \frac{1}{\mathrm{m}} L\left(\sup|u_2(x,t)-u_1(x,t)| + \sup|u_2(x,g(t))-u_1(x,g(t))|\right)$$

By virtue of $0 \le g(t) \le t$, we have

$$\sup|u_2(x,g(t))-u_1(x,g(t))| \le \sup|u_2(x,t)-u_1(x,t)|.$$

Hence,

$$|u_3(x,t)-u_2(x,t)| \le \frac{1}{m}\left(2L\sup|u_2(x,t)-u_1(x,t)|\right) < \frac{2LM}{m^2} = \frac{M}{2L}\left(\frac{2L}{m}\right)^2.$$

Similarly,

$$\left|u_4(x,t)-u_3(x,t)\right|<\frac{M}{2L}\left(\frac{2L}{m}\right)^3,$$

and

$$\left|u_{n+1}(x,t)-u_n(x,t)\right|<\frac{M}{2L}\left(\frac{2L}{m}\right)^n,\quad \mathrm{n}=1,2,...$$

Since

$$u(x,t)=u_1(x,t)+\sum_{n=1}^{\infty}(u_{n+1}(x,t)-u_n(x,t)),$$

then

$$\left|u(x,t)\right|<N+\frac{M}{2L}\left(\frac{\frac{2L}{m}}{1-\frac{2L}{m}}\right),\quad 0\le \mathrm{x}\le 1,\ \mathrm{t}\ge 0$$

which proves the boundedness of $u(x,t)$ in the domain $\Delta$. ■

## 3.3.2 Existence and Uniqueness of Solution for 2nd Order Partial Delay Differential Equations:

In this section, we use the method of successive approximations to guarantee the existence of a unique bounded solution for special types of linear and nonlinear 2nd order partial delay differential equations, namely the hyperbolic type.

We start this section by recalling the 2nd order partial delay differential equation:

$$\frac{\partial^2 u}{\partial x\partial t}(x,t)+a(x,t)\frac{\partial u}{\partial x}(x,t)=c(x,t,u(x,t),u(x,t-\tau)) \qquad (3.13)$$

together with the initial and boundary functions

$$u(x,t) = \varphi(x,t),\ 0 \le \mathrm{x} \le 1,\ -\tau \le \mathrm{t} \le 0 \qquad (3.14)$$

$$u(0,t) = u_0(t),\ \mathrm{t} \ge 0 \qquad (3.15)$$

where $\varphi$ is a given function of $x$ and $t$ and $u_0$ is a known function of $t$.

The following theorem gives necessary conditions for the existence and uniqueness of a bounded solution to the problem given by eq.(3.13)-(3.15).

## **Theorem(3.3), [Poorkarimi H. and Wiener J., 1987]:**

Consider eq.(3.13)-(3.15). Assume the following conditions:

(i) $u_0(t)$ is bounded and continuously differentiable on $t \ge 0$;

(ii) $\varphi(x,t)$ and $\frac{\partial \varphi}{\partial x}(x,t)$ are continuous on $[0,1]\times[-\tau,0]$ and $\varphi(0,0) = u_0(0)$;

(iii) $a(x,t)$ is continuous in $\Delta = \{(x,t) \mid 0 \le x \le 1,\ \mathrm{t} \ge 0\}$ and satisfies $a(x,t) \ge m > 0$ in $\Delta$;

(iv) $c(x,t,u,v)$ is continuous and bounded on $\Delta \times \Re \times \Re$ and satisfies the Lipschitz condition

$$\left| c(x,t,u,v) - c(x,t,u^*,v) \right| \le L \left| u - u^* \right| \qquad (3.3)$$

Then there exists a unique continuous solution of the problem given by eq.(3.13)-(3.15), defined in $\Delta$ and bounded there.

**Proof:**

Let $w(x,t)=\frac{\partial u}{\partial x}(x,t)$, then

$$\frac{\partial w}{\partial t}(x,t)+a(x,t)w(x,t)=c(x,t,u(x,t),u(x,t-\tau)) \quad (3.16)$$

To solve this equation, we use the method of successive integration. On the interval $0\le t\le\tau$, eq.(3.16) becomes

$$\frac{\partial w}{\partial t}(x,t)+a(x,t)w(x,t)=c(x,t,u(x,t),\varphi(x,t-\tau)).$$

Integrating the above equation from $0$ to $t$, we obtain

$$w(x,t)=\frac{\partial\varphi}{\partial x}(x,0)e^{-\int_0^t a(x,s)ds}+\int_0^t e^{-\int_s^t a(x,\theta)d\theta}c(x,s,u(x,s),\varphi(x,s-\tau))ds \quad (3.17$$

and by integrating eq.(3.17) from $0$ to $x$, one can get

$$u(x,t)=u_0(t)+\int_0^x\frac{\partial\varphi}{\partial\xi}(\xi,0)e^{-\int_0^t a(\xi,s)ds}d\xi+$$

$$\int_0^x\int_0^t e^{-\int_s^t a(\xi,\theta)d\theta}c(\xi,s,u(\xi,s),\varphi(\xi,s-\tau))dsd\xi. \quad (3.18)$$

To prove existence-uniqueness, we apply to eq.(3.18) the method of successive approximations. Put

$$u_1(x,t)=u_0(t)+\int_0^x\frac{\partial\varphi}{\partial\xi}(\xi,0)e^{-\int_0^t a(\xi,s)ds}d\xi,$$

then, since

$$|u_0(t)| \le \mathrm{N}\,, \ \left|\frac{\partial \varphi}{\partial x}(x,0)\right| \le K\,,$$

and $a(x,t) \ge m > 0$, we have

$$|u_1(x,t)| \le |u_0(t)| + \int_0^x \left|\frac{\partial \varphi}{\partial \xi}(\xi,0)\right| e^{-\int_0^t a(\xi,s)ds} d\xi$$

$$\le N + \int_0^x K d\xi \le N + K\,.$$

Let $A = N + K$ , then $|u_1(x,t)| \le A$.

Furthermore,

$$u_2(x,t) = u_0(t) + \int_0^x \frac{\partial \varphi}{\partial \xi}(\xi,0) e^{-\int_0^t a(\xi,s)ds} d\xi +$$

$$\int_0^x \int_0^t e^{-\int_s^t a(\xi,\theta)d\theta} c(\xi, s, u_1(\xi,s), \varphi(\xi, s-\tau)) ds d\xi\,,$$

and

$$|u_2(x,t) - u_1(x,t)| \le \int_0^x \int_0^t e^{-\int_s^t a(\xi,\theta)d\theta} |\mathrm{c}(\xi, \mathrm{s}, \mathrm{u}_1(\xi,s), \varphi(\xi, s-\tau))| ds d\xi < \frac{M}{m} x\,,$$

where $|c(x,t,u,v)| \le M$ , for each $(x,t,u,v) \in \Delta \times \Re \times \Re$.

Also, by using the Lipschitz condition given by ineq.(3.3), one can get

$$|u_3(x,t)-u_2(x,t)| \le \int_0^x \int_0^t e^{-\int_s^t a(x,\theta)d\theta} |c(\xi,s,u_2(\xi,s),\varphi(\xi,s-\tau))-$$

$$c(\xi,s,u_1(\xi,s),\varphi(\xi,s-\tau))|dsd\xi$$

$$< \int_0^x \int_0^t e^{-\int_s^t a(x,\theta)d\theta} L|u_2(\xi,s)-u_1(\xi,s)|dsd\xi$$

$$< \int_0^x \int_0^t e^{-\int_s^t a(x,\theta)d\theta} \frac{LM}{m}\xi dsd\xi$$

$$= \int_0^x \frac{1}{m}(1-e^{-mt})\frac{LM}{m}\xi d\xi$$

$$< \frac{LM}{m^2}\frac{x^2}{2!} = \frac{M}{L}\left(\frac{Lx}{m}\right)^2 \frac{1}{2!},$$

and similarly,

$$|u_4(x,t)-u_3(x,t)| < \frac{M}{L}\left(\frac{Lx}{m}\right)^3 \frac{1}{3!}.$$

Continuing this process gives the estimate

$$|u_{n+1}(x,t)-u_n(x,t)| < \frac{M}{L}\left(\frac{Lx}{m}\right)^n \frac{1}{n!}, \ \text{n} = 1,2,...$$

and since

$$u(x,t) = u_1(x,t) + \sum_{i=1}^{\infty}(u_{n+1}(x,t)-u_n(x,t)),$$

then

$$|u(x,t)| < A + \sum_{n=1}^{\infty}\frac{M}{L}\left(\frac{Lx}{m}\right)^n \frac{1}{n!} = A + \frac{M}{L}e^{\frac{Lx}{m}}, \quad 0 \le \text{x} \le 1, \ 0 \le \text{t} \le \tau.$$

For the solution of eq.(3.13)-(3.15) on the interval $\tau \le t \le 2\tau$, we have

$$u(x,t)=u_0(t)+\int_0^x \frac{\partial u^{(o)}}{\partial \xi}(\xi,\tau)e^{-\int_\tau^t a(\xi,s)ds}d\xi+\int_0^x\int_0^t e^{-\int_s^t a(\xi,\theta)d\theta}c(\xi,s,u(\xi,s),u^{(o)}(\xi,s-\tau))dsd\xi$$

where $u^{(o)}(x,t)$ is the solution of eq.(3.13)-(3.15) on $0\le t\le\tau$.

Repeating the above calculations yields

$$|u(x,t)|\le A_1+\frac{M}{L}e^{\frac{Lx}{m}}\ ,\quad 0\le \mathrm{x}\le 1,\ \tau\le \mathrm{t}\le 2\tau,$$

where

$$A_1=\sup_\Delta\left(|u_0(t)|+\int_0^x\left|\frac{\partial u^{(o)}}{\partial \xi}(\xi,\tau)\right|e^{-\int_\tau^t a(\xi,s)ds}d\xi\right).$$

To evaluate $\frac{\partial u^{(o)}}{\partial x}(x,\tau)$, we use eq.(3.17) from which

$$\frac{\partial \mathrm{u}^{(o)}}{\partial x}(x,\tau)=\frac{\partial u}{\partial x}(x,0)e^{-\int_0^\tau a(x,s)ds}+\int_0^\tau e^{-\int_s^\tau a(x,\theta)d\theta}c(x,s,u^{(o)}(x,s),\varphi(x,s-\tau))ds\ ,$$

thus

$$\left|\frac{\partial u^{(o)}}{\partial x}(x,\tau)\right|\le Ke^{-m\tau}+M\int_0^\tau e^{-m(\tau-s)}ds$$

$$=Ke^{-m\tau}+\frac{M}{m}e^{-m(\tau-s)}\Big|_0^\tau$$

$$=Ke^{-m\tau}+\frac{M}{m}\left(1-e^{-m\tau}\right)<Ke^{-m\tau}+\frac{M}{m},$$

and

$$A_1\le N+Ke^{-m\tau}+\frac{M}{m}.$$

In the region $0 \le x \le 1,\ 2\tau \le t \le 3\tau$, we have

$$|u(x,t)| \le A_2 + \frac{M}{L} e^{\frac{Lx}{m}},$$

where

$$A_2 \le N + Ke^{-2m\tau} + \frac{M}{m} e^{-m\tau} + \frac{M}{m}.$$

Finally, it is easy to see that

$$|u(x,t)| \le A_n + \frac{M}{L} e^{\frac{Lx}{m}}, \quad 0 \le x \le 1,\ n\tau \le t \le (n+1)\tau,$$

with

$$A_n \le N + Ke^{-nm\tau} + \frac{M}{m}\sum_{i=0}^{n-1} e^{-im\tau} < N + K + \frac{M}{m}\sum_{i=0}^{\infty} e^{-im\tau} = N + K + \frac{M}{m}\frac{1}{1-e^{-m\tau}}$$

Thus all values $A_n$ are uniformly bounded, and the proof is complete. ■

Next, recall that the hyperbolic partial delay differential equation:

$$\frac{\partial^2 u}{\partial x \partial t}(x,t) + a(x,t)\frac{\partial u}{\partial x}(x,t) = c(x,t,u(x,t),u(x,g(t))) \qquad (3.19)$$

together with the initial and boundary conditions:

$$u(x,0) = \varphi(x), \quad 0 \le x \le 1 \qquad (3.20)$$

$$u(0,t) = u_0(t), \quad t \ge 0 \qquad (3.21)$$

where $a$ is a known function of $x$ and $t$, $\varphi$ and $u_0$ are known functions of $x$ and $t$ respectively and $c$ is a known function of $x$, t, u(x,t), u(x,g(t)).

The following theorem gives necessary conditions to ensure the existence of a unique bounded solution of eq.(3.19)-(3.21).

## Theorem (3.4), [Poorkarimi H. and Wiener J., 1987]:

Consider eq.(3.19)-(3.21). Assume the following conditions:

(i) $u_o(t)$ is bounded and continuously differentiable on $t \geq 0$;

(ii) $\varphi(x)$ is continuously differentiable on [0,1];

(iii) $a(x,t)$ is continuous in $\Delta = \{(x,t) | \, 0 \leq x \leq 1, \mathrm{t} \geq 0\}$ and satisfies $a(x,t) \geq m > 0$ in $\Delta$;

(iv) $c(x,t,u,v)$ is continuous on $\Delta \times \Re \times \Re$, with $c(x,t,0,0)$ bounded on $\Delta$, and satisfies a Lipschitz condition in $u$ and $v$;

(v) $g(t)$ is continuous on $[0,\infty)$ and $0 \leq \mathrm{g(t)} \leq \mathrm{t}$.

Then there exists a unique continuous solution of the problem given by eq.(3.19)-(3.21) defined in $\Delta$ and bounded there.

## Proof:

Let $w(x,t) = \dfrac{\partial u}{\partial x}(x,t)$, then eq.(3.19) becomes

$$\frac{\partial w}{\partial t}(x,t) + a(x,t)w(x,t) = c(x,t,u(x,t),u(x,g(t))) \qquad (3.22)$$

Integrating eq.(3.22) from $0$ to $t$ to obtain

$$w(x,t) = w(x,0)e^{-\int_0^t a(x,s)ds} + \int_0^t e^{-\int_s^t a(x,\theta)d\theta} c(x,s,u(x,s),u(x,g(s)))ds \qquad (3.23)$$

Integrating the above equation from $0$ to $x$ gives

$$u(x,t) = u_o(t) + \int_0^x \varphi'(\xi) e^{-\int_0^t a(\xi,s)ds} d\xi +$$

$$\int_0^x \int_0^t e^{-\int_s^t a(\xi,\theta)d\theta} c(\xi, s, u(\xi,s), u(\xi, g(s))) ds d\xi . \qquad (3.24)$$

To apply to eq.(3.24) the method of successive approximations, we construct a sequence $\{u_i(x,t)\}$, for which the following estimates are satisfied in $\Delta$:

$$u_1(x,t) = u_o(t) + \int_0^x \varphi'(\xi) e^{-\int_0^t a(\xi,s)ds} d\xi ,$$

$$|u_1(x,t)| \le |u_o(t)| + \int_0^x |\varphi'(\xi)| e^{-\int_0^t a(\xi,s)ds} d\xi \le N + K_1 = A ,$$

where $|u_o(t)| \le N$ and $|\varphi'(x)| \le K_1$.

Also,

$$u_2(x,t) = u_1(x,t) + \int_0^x \int_0^t e^{-\int_s^t a(\xi,\theta)d\theta} c(\xi, s, u_1(\xi,s), u_1(\xi, g(s))) ds d\xi ,$$

and

$$|u_2(x,t) - u_1(x,t)| \le \int_0^x \int_0^t e^{-\int_s^t a(\xi,\theta)d\theta} |c(\xi, s, u_1(\xi,s), u_1(\xi, g(s)))| ds d\xi .$$

Now,

$$\left|c(\xi,s,u_1(\xi,s),u_1(\xi,g(s)))\right| \le \left|c(\xi,s,u_1(\xi,s),u_1(\xi,g(s))) - c(x,s,0,0)\right| + \left|c(x,s,0,0)\right|$$

$$\le \mathrm{L}(\sup\left|\mathrm{u}_1(x,t)\right| + \sup\left|u_1(x,g(t))\right|) + \sup\left|c(x,t,0,0)\right| = M$$

Hence

$$\left|u_2(x,t) - u_1(x,t)\right| \le \int_0^x \frac{1}{m} \sup\left|c(\xi,s,u_1(\xi,s),u_1(x,g(s)))\right| \le \frac{\mathrm{M}}{\mathrm{m}} x .$$

Also,

$$u_3(x,t) = u_1(x,t) + \int_0^x \int_0^t e^{-\int_s^t a(\xi,\theta)d\theta} c(\xi,s,u_2(\xi,s),u_2(\xi,g(s)))d\xi ds ,$$

and

$$\left|\mathrm{u}_3(x,t) - u_2(x,t)\right| \le \int_0^{\mathrm{x}} \int_0^{\mathrm{t}} e^{-\int_s^t a(\xi,\theta)d\theta} \left|\mathrm{c}(\xi,\mathrm{s},\mathrm{u}_2(\xi,s),u_2(\xi,g(s))) - \right.$$

$$\left. c(\xi,s,u_1(\xi,s),u_1(\xi,g(s)))\right| ds d\xi$$

$$\le \int_0^x \frac{L}{m} \left(\left|u_2(\xi,t) - u_1(\xi,t)\right| + \left|u_2(\xi,g(t)) - u_1(\xi,g(t))\right|\right) d\xi$$

Since $0 \le g(t) \le t$, we have

$$\sup\left|u_2(x,g(t)) - u_1(x,g(t))\right| \le \sup\left|u_2(x,t) - u_1(x,t)\right| .$$

Hence,

$$\left|u_3(x,t) - u_2(x,t)\right| \le \int_0^x \frac{1}{m} \left(2L \sup\left|u_2(\xi,t) - u_1(\xi,t)\right|\right) d\xi$$

$$\le \frac{2\mathrm{ML}}{\mathrm{m}^2} \frac{x^2}{2} = \frac{2M}{L} \left(\frac{Lx}{m}\right)^2 \frac{1}{2!} .$$

Similarly,

$$|u_4(x,t)-u_3(x,t)| \le \int_0^x \frac{2L}{m}\left(\frac{LM}{m^2}\xi^2\right)d\xi$$

$$= 2ML^2\frac{x^3}{m^3}\frac{1}{3} = \frac{2M}{L}\left(\frac{Lx}{m}\right)^3\frac{1}{3!},$$

and

$$|u_{n+1}(x,t)-u_n(x,t)| \le \frac{2M}{L}\left(\frac{Lx}{m}\right)^n\frac{1}{n!}.$$

Since

$$u(x,t)=u_1(x,t)+\sum_{n=1}^{\infty}(u_{n+1}(x,t)-u_n(x,t)),$$

then

$$|u(x,t)| \le A+\frac{2M}{L}e^{\frac{Lx}{m}}\ ,\ \ 0\le x\le 1,\ t\ge 0.$$

which proves the boundedness of $u(x,t)$ in the domain $\Delta$. ■

## Remark (3.1):

The method of successive approximations can be also used to ensure the unique bounded solution for special types of nonlinear delay parabolic partial differential equations, [Poorkarimi H. and Wiener J., 1999].

## 3.4 An Estimate of the magnitude of Solutions for Partial Delay Differential Equations:

In this section, we generalize the estimation of the magnitude of the solutions for linear ordinary differential equations with deviating arguments to include the linear partial differential equations with deviating arguments. This section consists the main part of this work.

We start this section by deriving an estimate of the magnitude of the solution for special types of the 1st order linear partial delay differential equations. To the best of our knowledge, this theorem seems to be new.

## Theorem(3.5):

Let $u(x,t)$ be a solution of the partial delay differential equation

$$a_0 \frac{\partial u}{\partial t} + b_0 u(x,t) + b_1 u(x,t-\tau) = f(x,t) \qquad (3.25)$$

which is of class $C^1$ on $[0,1]\times[-\tau,\infty)$. Suppose that $f$ is of class $C^0$ on $[0,1]\times[-\tau,\infty)$ and that

$$|f(x,t)| \le c_1 e^{c_2 t}\ , \quad 0 \le x \le 1,\ t \ge 0,$$

where $c_1$ and $c_2$ are positive constants. Let

$$m = \max_{\substack{0\le x\le 1 \\ -\tau\le t\le 0}} |u(x,t)|,$$

Then there are positive constant $c_3$ and $c_4$ depending only on $c_2$ and the coefficients in eq.(3.25) such that

$$|u(x,t)| \le c_3 (m + c_1) e^{c_5 \tau} e^{c_4 t}\ , \quad 0 \le x \le 1,\ t \ge -\tau$$

## Proof:

Integrating eq.(3.25) from $0$ to $t$, we obtain

$$a_0 u(x,t) = a_0 u(x,0) - b_0 \int_0^t u(x,s)ds - b_1 \int_0^t u(x,s-\tau)ds + \int_0^t f(x,s)ds$$

Therefore,

$$|u(x,t)| \le m + \frac{|b_0|}{|a_0|} \int_{-\tau}^{t} |u(x,s)| ds + \frac{|b_1|}{|a_0|} \int_{-\tau}^{t} |u(x,s)| ds + \frac{c_1}{c_2 |a_0|} e^{c_2 t} .$$

Let $c_3 = \max\left\{1, \frac{1}{c_2 |a_0|}\right\}$, and $c_5 = \frac{|b_0| + |b_1|}{|a_0|}$. Then

$$|u(x,t)| \le c_3 (m + c_1) e^{c_2 t} + c_5 \int_{-\tau}^{t} |u(x,s)| ds, \quad t \ge 0 \qquad (3.4)$$

Since $|u(x,t)| \le m \le m c_3 e^{c_2 t}$ for $0 \le x \le 1$, $-\tau \le t \le 0$ ineq.(3.4) holds for all $0 \le x \le 1$ and $t \ge -\tau$. It therefore follows from lemma (1.1) that

$$|u(x,t)| \le c_3 (m + c_1) e^{c_5 \tau} e^{(c_2 + c_5) t} , \ 0 \le x \le 1, \ t \ge -\tau .$$

which proves the theorem. ■

## Corollary(3.1):

Let $u(x,t)$ be a solution of the partial delay differential equation

$$a_0 \frac{\partial u}{\partial t} + b_0 u(x,t) + b_1 u(x,t-\tau) = f(x,t) \qquad (3.26)$$

which is of class $C^1$ on $[0,1]\times[0,\infty)$. Suppose that $f$ is of class $C^0$ on $[0,1]\times[0,\infty)$ and that

$$\left| f(x,t) \right| \le c_1 e^{c_2 t}\ , \quad 0 \le x \le 1,\ t \ge 0,$$

where $c_1$ and $c_2$ are positive constants. Let

$$m = \max_{\substack{0\le x\le 1\\ 0\le t\le \tau}} \left| u(x,t) \right|,$$

Then there are positive constant $c_3$ and $c_4$ depending only on $c_2$ and the coefficients in eq.(3.26) such that

$$\left| u(x,t) \right| \le c_3 (m + c_1) e^{c_4 t}\ , \quad 0 \le x \le 1,\ t \ge 0. \ \blacksquare$$

Next, an estimate of the magnitude of solutions for special types of the 2nd order linear partial delay differential equations is discussed below. But before that we need the following lemma.

## **Lemma (3.1), [Bainov D. and Simeonov P., 1992]:**

Let $u(x,t)$ and b(x,t) be nonnegative continuous functions defined for $x \ge \alpha_1,\ t \ge \alpha_2$.

Assume that $a(x,t)$ is positive nondecreasing continuous function in each of the variables $x \ge \alpha_1,\ t \ge \alpha_2$.

If

$$u(x,t) \le a(x,t) + \int_{\alpha_1}^{x} \int_{\alpha_2}^{t} b(s,\theta) u(s,\theta)\, d\theta\, ds, \quad \text{for } x \ge \alpha_1,\ t \ge \alpha_2$$

then

$$u(x,t) \le a(x,t) e^{\int_{\alpha_1}^{x}\int_{\alpha_2}^{t} b(s,\theta)d\theta ds} \quad \text{for } x \ge \alpha_1,\ t \ge \alpha_2 .$$

## Proof:

Since $a(x,t)$ is a positive nondecreasing in each of the variables $x \ge \alpha_1,\ t \ge \alpha_2$ then

$$\frac{u(x,t)}{a(x,t)} \le 1 + \int_{\alpha_1}^{x}\int_{\alpha_2}^{t} b(s,\theta)\frac{u(s,\theta)}{a(s,\theta)} d\theta ds \quad \text{for } x \ge \alpha_1,\ t \ge \alpha_2 .$$

Let $v(x,t) = \dfrac{u(x,t)}{a(x,t)}$, then

$$v(x,t) \le 1 + \int_{\alpha_1}^{x}\int_{\alpha_2}^{t} b(s,\theta)v(s,\theta) d\theta ds \quad \text{for } x \ge \alpha_1,\ t \ge \alpha_2$$

Let $w(x,t) = 1 + \int_{\alpha_1}^{x}\int_{\alpha_2}^{t} b(s,\theta)v(s,\theta) d\theta ds$,

then $v(x,t) \le w(x,t)$. Moreover

$$w_x(x,t) \le \int_{\alpha_2}^{t} b(x,\theta)v(x,\theta)d\theta \le \int_{\alpha_2}^{t} b(x,\theta)w(x,\theta)d\theta .$$

the function $w(x,t)$ is nondecreasing in $t$ and

$$w_x(x,t) \le w(x,t)\int_{\alpha_2}^{t} b(x,\theta)d\theta ,$$

thus

$$\frac{w_x(x,t)}{w(x,t)} \le \int_{\alpha_2}^{t} b(x,\theta)d\theta .$$

Integrating both sides of the above inequality from $\alpha_1$ to x yields:

$$w(x,t) \le e^{\int_{\alpha_1}^{x}\int_{\alpha_2}^{t} b(s,\theta)d\theta ds} .$$

Since $v(x,t) \le w(x,t)$, this implies that

$$u(x,t) \le a(x,t)e^{\int_{\alpha_1}^{x}\int_{\alpha_2}^{t} b(s,\theta)d\theta ds} \quad \text{for } x \ge \alpha_1,\ t \ge \alpha_2 . \quad \blacksquare$$

Next, we are in the position that we can give the following theorem. To the best of our knowledge, this theorem seems to be new.

## Theorem(3.6):

let $u(x,t)$ be a solution of the partial delay differential equation

$$a_0 \frac{\partial^2 u}{\partial x \partial t} + a_1 u(x,t-\tau) = f(x,t) \qquad (3.27)$$

which is of class $C^1$ on $[0,1]\times[0,\infty)$. Suppose that $f$ is of class $C^0$ on $[0,1]\times[0,\infty)$ and that

$$|f(x,t)| \le c_1 e^{c_2 t} , \quad 0 \le x \le 1,\ t \ge 0,$$

where $c_1$ and $c_2$ are positive constants. Let

$$m_1 = \max_{\substack{0\le x\le 1\\ 0\le t\le \tau}} |u(x,t)|$$

$$m_2 = \max_{t\ge 0} |u(0,t)|$$

Then there are positive constant $c_3$ and $c_4$ depending only on $c_2$ and the coefficients in eq.(3.27) such that

$$|u(x,t)| \le c_3(m_1 + m_2 + c_1)e^{c_4 t}\ , \quad 0 \le \mathrm{x} \le 1,\ \mathrm{t} \ge 0$$

## Proof:

Integrating eq.(3.27) from $0$ to $\mathrm{x}$, we obtain

$$a_0 \frac{\partial u}{\partial t}(x,t) = a_0 \frac{\partial u}{\partial t}(0,t) - a_1\int_0^x u(s,t-\tau)ds + \int_0^x f(s,t)ds$$

Again, integrating the above equation from $\tau$ to $\mathrm{t}$, we get

$$a_0 u(x,t) = a_0 u(x,\tau) + a_0 u(0,t) - a_0 u(0,\tau) - a_1\int_\tau^t\int_0^x u(s,\theta-\tau)dsd\theta +$$

$$\int_\tau^t\int_0^x f(s,\theta)dsd\theta,\ \mathrm{t} \ge \tau$$

Therefore,

$$|a_0 u(x,t)| \le |a_0|\,|\mathrm{u}(\mathrm{x},\tau)| + |a_0|\,|\mathrm{u}(0,\mathrm{t})| + |a_0|\,|\mathrm{u}(0,\tau)| + |a_1|\int_\tau^t\int_0^x |\mathrm{u}(\mathrm{s},\theta-\tau)|dsd\theta +$$

$$\int_\tau^t\int_0^x |f(s,\theta)|dsd\theta$$

$$|a_0 u(x,t)| \le |a_0| m_1 + |a_0| m_2 + |a_0| m_1 + |a_1| \int_0^t \int_0^x |u(s,\theta)| ds d\theta + \int_0^t \int_0^x |f(s,\theta)| ds d\theta$$

$$\le |a_0| m_1 + |a_0| m_2 + |a_0| m_1 + |a_1| \int_0^t \int_0^x |u(s,\theta)| ds d\theta + \int_0^t \int_0^x c_1 e^{c_2 \theta} d\theta$$

$$\le |a_0| m_1 + |a_0| m_2 + |a_0| m_1 + |a_1| \int_0^t \int_0^x |u(s,\theta)| ds d\theta + \frac{c_1}{c_2} e^{c_2 t}, \quad t \ge \tau .$$

Thus

$$|u(x,t)| \le 2m_1 + m_2 + \frac{c_1}{c_2 |a_0|} e^{c_2 t} + \frac{|a_1|}{|a_0|} \int_0^t \int_0^x |u(s,\theta)| ds d\theta .$$

Let $c_3 = \max\left\{2 , \frac{1}{c_2 |a_0|}\right\}$, and $c_5 = \frac{|a_1|}{|a_0|}$.

Then

$$|u(x,t)| \le c_3 (m_1 + m_2 + c_1) e^{c_2 t} + c_5 \int_0^t \int_0^x |u(s,\theta)| ds d\theta \qquad (3.5)$$

Since $|u(x,t)| \le m_1 \le c_3 m_1 e^{c_2 t}$ for $0 \le x \le 1$, $0 \le t \le \tau$ ineq.(3.5) holds for all $0 \le x \le 1$, $t \ge 0$.

Also, since $|u(0,t)| \le m_2 \le c_3 m_2 e^{c_2 t}$ for $t \ge 0$ ineq.(3.5) holds for all $t \ge 0$. It therefore follows from lemma (3.1) that

$$|u(x,t)| \le c_3 (m_1 + m_2 + c_1) e^{(c_2 + c_5)t}, \quad 0 \le x \le 1, \ t \ge 0 . \ \blacksquare$$

## 2.1 Introduction:

As seen before in chapter one, the Laplace transform technique is one of the important methods that can be used to solve the linear ordinary delay differential equations.

Also, it is known that, the Laplace transform of a function $f$, denoted by $\mathrm{L}\{f\}$ or $F(s)$ is defined by

$$F(s)=\int_0^\infty e^{-sx}f(x)dx,$$

this improper integral converges absolutely for $\mathrm{Re}(s)>a$ in case the function $f$ possess bounds of the form

$$|f(x)|\leq ce^{ax}$$

Therefore, the Laplace transform technique can be used to solve the linear ordinary delay differential equations in case the magnitude of the solutions for such equations satisfy the above inequality.

The main purpose of this chapter is to devote some necessary conditions to estimate the magnitude of the solutions for the linear ordinary delay differential equations in order to find them. Also, an estimate of the magnitude of the solutions for special types of nonlinear ordinary delay differential equations is given.

Moreover, an estimate of the magnitude of the solutions for systems of the $1^{st}$ order linear delay differential equations is introduced.

This chapter consists of five sections.

In section two, an estimate of the magnitude of only one solution for the 1st order linear delay differential equations is derived.

In section three, an estimate of the magnitude of only one solution for the 1st order nonlinear delay differential equation is introduced.

In section four, an estimate of the magnitude of two solutions for the 1st order delay differential equation is obtained.

In section five, an estimate of the magnitude of solutions for a system of the 1st order delay differential equations.

## 2.2 An Estimate of the Magnitude of One Solution of Linear Ordinary Delay Differential Equations:

In this section, we give an estimate of the magnitude of only one solution for the linear ordinary delay differential equations of single and multiple delays with constant coefficients.

We start this section by deriving an estimate of the magnitude of only one solution for special types of the 1st order linear ordinary differential equations with single delay with the aid of the lemma (1.1).

**Proposition (2.1), [Bellman R. and Cooke K., 1963]:**

Let $y(x)$ be a solution of the delay differential equation

$$a_0 y'(x) + b_1 y(x-\tau) = f(x) \tag{2.1}$$

which is of class $C^1$ on $[0,\infty)$. Suppose that $f$ is of class $C^o$ on $[0,\infty)$ and that

$$|f(x)| \leq c_1 \, e^{c_2 x}, \quad x \geq 0$$

where $c_1$ and $c_2$ are positive constants. Let

$$m = \max_{0 \leq x \leq \tau} |y(x)|,$$

then there are positive constants $c_3$ and $c_4$ depending mainly on $c_2$ and the coefficients in eq.(2.1) such that

$$|y(x)| \leq c_3 (c_1 + m) e^{c_4 x}, \quad x \geq 0$$

## Proof:

From eq.(2.1), we find that

$$a_0 y(x) = a_0 y(\tau) + \int_\tau^x f(t)dt - b_1 \int_\tau^x y(t-\tau)dt, \quad x \geq \tau .$$

Therefore,

$$|a_0 y(x)| \leq |a_0| m + c_1 \int_\tau^x e^{c_2 t} dt + |b_1| \int_0^{x-\tau} |y(t)| dt, \quad x \geq \tau ,$$

hence

$$|y(x)| \leq m + \frac{c_1}{c_2 |a_0|} e^{c_2 x} + \frac{|b_1|}{|a_0|} \int_0^x |y(t)| dt, \quad x \geq \tau .$$

Let $c_3 = \max\left\{1, \ \frac{1}{c_2 |a_0|}\right\}$, and $c_5 = \frac{|b_1|}{|a_0|}$.

Then

$$|y(x)| \le c_3(c_1 + m)e^{c_2 x} + c_5 \int_0^x |y(t)| dt, \quad x \ge \tau . \qquad (2.1)$$

Since $|y(x)| \le m \le c_3 m e^{c_2 x}$ for $0 \le x \le \tau$, ineq.(2.1) holds for all $x \ge 0$. It therefore follows from lemma (1.1) that

$$|y(x)| \le c_3(c_1 + m)e^{(c_2 + c_5)x}, \quad x \ge 0. \ \blacksquare$$

Next, using this proposition, we can prove this theorem.

## Theorem (2.1):

Let $y(x)$ be the continuous solution of

$$a_0 y'(x) + b_1 y(x - \tau) = f(x)$$

which satisfies the initial condition $y(x) = g(x), \ 0 \le x \le \tau$. Assume that $g$ is $C^o[0,\tau]$, that $f$ is $C^o[0,\infty)$ and that

$$|f(x)| \le c_1 e^{c_2 x}, \quad x \ge 0$$

where $c_1$ and $c_2$ are positive constants. Then for any sufficiently large constant $c$,

$$y(x) = \int_{(c)} e^{xs} h^{-1}(s)[p_0(s) + q(s)]ds, \quad x > \tau,$$

where

$$p_0(s) = a_0 g(\tau)e^{-\tau s} - b_1 e^{-\tau s}\int_0^{\tau} g(x_1)e^{-sx_1}dx_1,$$

$$q(s) = \int_{\tau}^{\infty} f(x_1)e^{-sx_1}dx_1 .$$

Also,

$$y(x) = \int_{(c)} e^{xs}h^{-1}(s)[p(s) + q(s)]ds, \quad x > 0,$$

where

$$p(s) = a_0 g(\tau)e^{-\tau s} + a_0 s\int_0^{\tau} g(x_1)e^{-sx_1}dx_1,$$

provided $g$ is $C^1[0,\tau]$.

## Proof:

See [Bellman R. and Cooke K., 1963]. ■

The following theorem is a generalization of the previous proposition which gives an estimate of only one solution for special types of the 1st order linear ordinary delay differential equation.

## Theorem (2.2), [Bellman R. and Cooke K., 1963]:

Let $y(x)$ be a solution of the delay differential equation

$$a_0 y'(x) + b_0 y(x) + b_1 y(x-\tau) = f(x) \tag{2.2}$$

which is of class $C^1$ on $[0,\infty)$. Suppose that $f$ is of class $C^o$ on $[0,\infty)$ and that

$$|f(x)| \le c_1 e^{c_2 x}, \quad x \ge 0,$$

where $c_1$ and $c_2$ are positive constants. Let

$$m = \max_{0 \le x \le \tau} |y(x)|,$$

then there are positive constants $c_3$ and $c_4$ depending only on $c_2$ and the coefficients in eq.(2.2) such that

$$|y(x)| \le c_3 (c_1 + m) e^{c_4 x}, \quad x \ge 0.$$

## Proof:

From eq.(2.2), we find that

$$a_0 y(x) = a_0 y(\tau) + \int_\tau^x f(t)dt - b_0 \int_\tau^x y(t)dt - b_1 \int_\tau^x y(t-\tau)dt, \quad x \ge \tau.$$

Therefore,

$$|y(x)| \le m + \frac{c_1}{c_2 |a_0|} e^{c_2 x} + \frac{|b_0| + |b_1|}{|a_0|} \int_0^x |y(t)| dt, \quad x \ge \tau.$$

Let $c_3 = \max\left\{ 1, \frac{1}{c_2 |a_0|} \right\}$, and $c_5 = \frac{|b_0| + |b_1|}{|a_0|}$.

Then

$$|y(x)| \le c_3 (c_1 + m) e^{c_2 x} + c_5 \int_0^x |y(t)| dt, \quad x \ge \tau. \qquad (2.2)$$

Since $|y(x)| \le m \le c_3 m e^{c_2 x}$ for $0 \le x \le \tau$, ineq.(2.2) holds for all $x \ge 0$. It therefore follows from lemma (1.1) that

$$|y(x)| \le c_3 (c_1 + m) e^{(c_2 + c_5)x} , \quad \mathrm{x} \ge 0 ,$$

which proves the theorem. ■

Next, using this theorem, we can prove the following theorem.

## **Theorem (2.3):**

Let $y(x)$ be the continuous solution of

$$a_0 y'(x) + +b_0 y(x) + b_1 y(x-\tau) = f(x)$$

which satisfies the initial condition $y(x) = g(x), \quad 0 \le \mathrm{x} \le \tau$. Assume that $g$ is $C^o[0,\tau]$, that $f$ is $C^o[0,\infty)$ and that

$$|f(x)| \le c_1 \, \mathrm{e}^{c_2 x}, \quad \mathrm{x} \ge 0$$

where $c_1$ and $c_2$ are positive constants. Then for any sufficiently large constant $c$,

$$y(x) = \int_{(c)} e^{xs} h^{-1}(s)[p_0(s) + q(s)]ds, \quad \mathrm{x} > \tau ,$$

where

$$p_0(s) = a_0 g(\tau) e^{-\tau s} - b_1 e^{-\tau s} \int_0^{\tau} g(x_1) e^{-s x_1} dx_1 ,$$

$$q(s) = \int_{\tau}^{\infty} f(x_1) e^{-s x_1} dx_1 .$$

Also,

$$y(x) = \int_{(c)} e^{xs} h^{-1}(s)[p(s) + q(s)]ds, \quad x > 0,$$

where

$$p(s) = a_0 g(\tau)e^{-\tau s} + (a_0 s + b_0)\int_0^{\tau} g(x_1)e^{-sx_1} dx_1,$$

provided $g$ is $C^1[0,\tau]$.

## Proof:

See [Bellman R. and Cooke K., 1963]. ■

Now, the following theorem is an extended theorem of the previous facts which gives the same result. To the best of our knowledge, this theorem seems to be new.

## Theorem (2.4):

Let $y(x)$ be a solution of the delay differential equation

$$a_0 y'(x) + a_1 y'(x-\tau) + b_0 y(x) + b_1 y(x-\tau) = f(x) \tag{2.3}$$

which is of class $C^1$ on $[0,\infty)$. Suppose that $f$ is of class $C^o$ on $[0,\infty)$ and that

$$|f(x)| \le c_1 e^{c_2 x}, \quad x \ge 0,$$

where $c_1$ and $c_2$ are positive constants. Let

$$m = \max_{0 \le x \le j\tau} |y(x)|,$$

where $j \in N$, then there are positive constants $c_3$ and $c_4$ depending only on $c_2$ and the coefficients in eq.(2.3) such that

$$|y(x)| \le c_3(c_1 + m)e^{c_4 x}, \quad 0 \le \text{x} \le (\text{j}+1)\tau .$$

## Proof:

From eq.(2.3), we find that

$$a_0 y(x) = a_0 y(j\tau) + \int_{j\tau}^{x} f(t)dt - a_1 \int_{j\tau}^{x} y'(t-\tau)dt - b_0 \int_{j\tau}^{x} y(t)dt - b_1 \int_{\tau}^{x} y(t-\tau)dt, \quad \text{x} \ge \text{j}\tau$$

$$= \text{a}_0 y(j\tau) + \int_{j\tau}^{x} f(t)dt - a_1 \int_{(j-1)\tau}^{x-\tau} y'(t)dt - b_0 \int_{j\tau}^{x} y(t)dt - b_1 \int_{(j-1)\tau}^{x-\tau} y(t)dt ,$$

Therefore,

$$|y(x)| \le m + \frac{c_1}{c_2|a_0|} e^{c_2 x} + 2\frac{|a_1|}{|a_0|} m + \frac{|b_0| + |b_1|}{|a_0|} \int_0^x |y(t)|dt, \quad \text{j}\tau \le \text{x} \le (\text{j}+1)\tau$$

Let $c_3 = \max\left\{ 1 + 2\frac{|a_1|}{|a_0|}, \frac{1}{c_2|a_0|} \right\}$, and $\text{c}_5 = \frac{|b_0| + |b_1|}{|a_0|}$.

Then

$$|y(x)| \le c_3(c_1 + m)e^{c_2 x} + c_5 \int_0^x |y(t)|dt . \qquad (2.3)$$

Since $|y(x)| \le m \le c_3 m e^{c_2 x}$ for $0 \le x \le j\tau$, ineq.(2.3) holds for all $0 \le x \le (j+1)\tau$. It therefore follows from lemma (1.1) that

$$|y(x)| \le c_3(c_1 + m)e^{(c_2 + c_5)x}, \quad 0 \le \text{x} \le (\text{j}+1)\tau . \blacksquare$$

Next, an estimate for the magnitude of only one solution for special types of the 1st order linear ordinary delay differential equations with multiple delays is discussed below. To the best of our knowledge, this theorem seems to be new.

## Theorem (2.5):

Let $y(x)$ be a solution of the equation

$$a_0 y'(x) + b_0 y(x) + b_1 y(x-\tau_1) + b_2 y(x-\tau_2) = f(x), \tag{2.4}$$

which is of class $C^1$ on $[0,\infty)$. Suppose that $f$ is of class $C^o$ on $[0,\infty)$ and that

$$|f(x)| \le c_1 e^{c_2 x}, \quad \mathrm{x} \ge 0,$$

where $c_1$ and $c_2$ are positive constants. Let

$$m = \max_{0 \le x \le \tau_2} |y(x)|,$$

then there are positive constants $c_3$ and $c_4$ depending only on $c_2$ and the coefficients in eq.(2.4), such that

$$|y(x)| \le c_3(c_1 + m)e^{c_4 x}, \quad \mathrm{x} \ge 0.$$

## Proof:

From eq.(2.4), we find that

$$a_0 y(x) = a_0 y(\tau_2) + \int_{\tau_2}^{x} f(t)dt - b_0 \int_{\tau_2}^{x} y(t)dt - b_1 \int_{\tau_2}^{x} y(t-\tau_1)dt - b_2 \int_{\tau_2}^{x} y(t-\tau_2)dt.$$

Therefore,

$$|a_0 y(x)| \le |a_0| m + c_1 \int_{\tau_2}^{x} e^{c_2 t} dt + |b_0| \int_0^x |y(t)| dt + |b_1| \int_{\tau_2-\tau_1}^{x-\tau_2} |y(t)| dt + |b_2| \int_0^{x-\tau_2} |y(t)| dt, \quad \mathrm{x} \ge \tau_2$$

thus

$$|y(x)| \le m + \frac{c_1}{c_2 |a_0|} e^{c_2 x} + \frac{|b_0| + |b_1| + |b_2|}{|a_0|} \int_0^x |y(t)| dt, \quad \mathrm{x} \ge \tau_2 \,.$$

Let $c_3 = \max\left\{1, \frac{1}{c_2 |a_0|}\right\}$, and $\mathrm{c}_5 = \frac{|b_0| + |b_1| + |b_2|}{|a_0|}$.

Then

$$|y(x)| \le c_3 (c_1 + m) e^{c_2 x} + c_5 \int_0^x |y(x)| dx, \quad \mathrm{x} \ge \tau_2 \,. \tag{2.4}$$

Since $|y(x)| \le m \le c_3 m e^{c_2 x}$ for $0 \le x \le \tau_2$, ineq.(2.4) holds for all $x \ge 0$. It therefore follows from lemma (1.1) that

$$|y(x)| = \mathrm{c}_3 (c_1 + m) e^{(c_2 + c_5) x}, \quad \mathrm{x} \ge 0 \,.$$

which proves the theorem. ■

## 2.3 An Estimate of the Magnitude of One Solution of Nonlinear Ordinary Delay Differential Equations:

As seen before, [Bellman R. and Cooke K., 1963] discussed an estimate of the magnitude of only one solution for linear ordinary delay differential equations.

In this section, we give an estimate of the magnitude of only one solution for special types of the nonlinear ordinary delay differential equations with constant coefficients. To the best of our knowledge, this section seems to be new and consists of the main part of this work.

We start this section by deriving an estimate of the magnitude of only one solution for special types of the 1st order nonlinear ordinary delay differential equations with constant coefficients, but before that we need the following lemma.

## Lemma (2.1), [Bainov D. and Simeonov P., 1992]:

Let $y(x)$, $a(x)$ and $k(x)$ be nonnegative continuous functions in $J=[\alpha,\beta]$, and suppose

$$y(x) \le a(x) + \int_{\alpha}^{x} k(t)y^{p}(t)dt, \quad x\in J, \tag{2.5}$$

where $0 < p < 1$. Then

$$y(x) \le a(x) + {z_0}^{p}\left[\int_{\alpha}^{x} k^{\frac{1}{q}}(t)dt\right]^{q}, \quad q = 1 - p, \tag{2.6}$$

where $z_0$ is the unique positive root of the equation

$$z - a - bz^{p} = 0, \quad a = \int_{\alpha}^{\beta} a(t)dt, \quad b = \int_{\alpha}^{\beta}\left[\int_{\alpha}^{x} k^{\frac{1}{q}}(t)dt\right]^{q} dx.$$

## Proof:

From Holder's inequality, we obtain

$$\int_{\alpha}^{x} k(t) y^{p}(t)dt \le \left[\int_{\alpha}^{x} k^{\frac{1}{q}}(t)dt\right]^{q} \left[\int_{\alpha}^{x} y(t)dt\right]^{p} .$$

Then ineq.(2.5) implies

$$y(x) \le a(x) + \left[\int_{\alpha}^{x} y(t)dt\right]^{p} \left[\int_{\alpha}^{x} k^{\frac{1}{q}}(t)dt\right]^{q} , \qquad (2.7)$$

and integration of ineq.(2.7) from $\alpha$ to $\beta$ gives

$$\int_{\alpha}^{\beta} y(t)dt \le \int_{\alpha}^{\beta} a(t)dt + \left[\int_{\alpha}^{\beta} y(t)dt\right]^{p} \int_{\alpha}^{\beta}\left[\int_{\alpha}^{x} k^{\frac{1}{q}}(t)dt\right]^{q} dx$$

If $z = \int_{\alpha}^{\beta} y(t)dt,\ \mathrm{a} = \int_{\alpha}^{\beta} \mathrm{a(t)dt},\ \mathrm{b} = \int_{\alpha}^{\beta}\left[\int_{\alpha}^{\mathrm{x}} \mathrm{k}^{\frac{1}{q}}(t)dt\right]^{q} dx$, then the last inequality shows that the nonnegative number $z$ satisfies the inequality $z \le a + bz^{p}$. Analyzing this, we conclude that $z \le z_0$ where $z_0$ is the unique positive root of the equation $z = a + bz^{p}$. Hence ineq.(2.7) implies ineq.(2.6). ■

Now, we can prove the following proposition. This proposition gives an estimate of the magnitude of only one solution for a special type of nonlinear ordinary delay differential equations.

## Proposition (2.2):

Let $y(x)$ be a solution of the delay differential equation

$$a_0 y'(x) + b_1 y^p(x-\tau) = f(x), \quad 0 < \mathrm{p} < 1, \tag{2.5}$$

which is of class $C^1$ on $[0,\infty)$. Suppose that $f$ is of class $C^o$ on $[0,\infty)$ and that

$$|f(x)| \le c_1 e^{c_2 x}, \quad \mathrm{x} \ge 0,$$

where $c_1$ and $c_2$ are positive constants. Let

$$m = \max_{0\le x\le \tau} |y(x)|,$$

then there are positive constants $c_3$ and $c_4$, depending only on $c_2$ and the coefficients in eq.(2.5) such that

$$|y(x)| \le c_3(c_1+m)e^{c_2 x} + c_4 {z_0}^p x^{1-p}, \quad 0 \le \mathrm{x} \le \beta,$$

where $z_0$ is the unique positive root of the equation

$$z - \frac{c_3(c_1+m)}{c_2}(e^{c_2\beta} - 1) - c_4 \frac{\beta^{2-p}}{2-p} z^p = 0, \quad \beta \ge 0.$$

## Proof:

From eq.(2.5), we find that

$$a_0 y(x) = a_0 y(\tau) + \int_\tau^x f(t)dt - b_1 \int_\tau^x y^p(t-\tau)dt, \quad \mathrm{x} \ge \tau.$$

Thus

$$|a_0 y(x)| \le m|a_0| + c_1\int_{\tau}^{x} e^{c_2 t}dt + |b_1|\int_{\tau}^{x} |y^p(t-\tau)|dt$$

$$\le \mathrm{m}|\mathrm{a}_0| + c_1\int_{\tau}^{x} e^{c_2 t}dt + |b_1|\int_{0}^{x-\tau} |y^p(t)|dt,$$

hence

$$|y(x)| \le m + \frac{c_1}{c_2|a_0|}e^{c_2 x} + \frac{|b_1|}{|a_0|}\int_{0}^{x} |y^p(t)|dt.$$

Let $c_3 = \max\left\{1, \frac{1}{c_2|a_0|}\right\}$, and $\mathrm{c}_4 = \frac{|b_1|}{|a_0|}$.

Then

$$|y(x)| \le c_3(\mathrm{m}+\mathrm{c}_1)e^{c_2 x} + c_4\int_{0}^{x} |y(t)|^p dt, \quad \mathrm{x} \ge \tau. \tag{2.8}$$

Since $|y(x)| \le m \le c_3 m e^{c_2 x}$ for $0 \le x \le \tau$, ineq.(2.8) holds for all $x \ge 0$.

Thus

$$|y(x)| \le c_3(m+c_1)e^{c_2 x} + c_4\int_{0}^{x} |y(t)|^p dt, \quad \mathrm{x} \ge 0.$$

By using lemma (2.1), one can get

$$|y(x)| \le c_3(m+c_1)e^{c_2 x} + c_4 {z_0}^p x^q, \quad \mathrm{q} = 1 - \mathrm{p},$$

where $z_0$ is the unique positive root of the equation $z - a - bz^p = 0$,

$$\mathrm{a} = \int_{0}^{\beta} c_3(\mathrm{m}+\mathrm{c}_1)\, e^{c_2 x}dx = \frac{c_3(m+c_1)}{c_2}(\mathrm{e}^{\mathrm{c}_2\beta} - 1),$$ and

$$b=\int_0^{\beta}\left[\int_0^{x}c_4^{\frac{1}{q}}\right]^q dx=c_4\left.\frac{x^{q+1}}{q+1}\right|_0^{\beta}=c_4\frac{\beta^{q+1}}{q+1},\quad \mathrm{q}=1-\mathrm{p}.\quad \blacksquare$$

Next, the following theorem is an extension of the above proposition.

## Theorem (2.6):

Let $y(x)$ be a solution of the delay differential equation

$$a_0y'(x)+b_0y^p(x)+b_1y^p(x-\tau)=f(x),\quad 0<\mathrm{p}<1, \qquad (2.6)$$

which is of class $C^1$ on $[0,\infty)$. Suppose that $f$ is of class $C^o$ on $[0,\infty)$ and that

$$|f(x)|\le c_1e^{c_2x},\quad \mathrm{x}\ge 0,$$

where $c_1$ and $c_2$ are positive constants. Let

$$m=\max_{0\le x\le\tau}|y(x)|,$$

then there are positive constants $c_3$ and $c_4$ depending only on $c_2$ and the coefficients in eq.(2.6) such that

$$|y(x)|\le c_3(c_1+m)e^{c_2x}+c_4{z_0}^p x^{1-p},\quad 0\le \mathrm{x}\le\beta,$$

where $z_0$ is the unique positive root of the equation

$$z-\frac{c_3(c_1+m)}{c_2}(e^{c_2\beta}-1)-c_4\frac{\beta^{q+1}}{q+1}z^p=0,\quad \beta\ge 0.$$

## Proof:

From eq.(2.6), we find that

$$a_0 y(x) = a_0 y(\tau) + \int_\tau^x f(t)dt - b_0 \int_\tau^x y^p(t)dt - b_1 \int_\tau^x y^p(t-\tau)dt, \quad x \geq \tau .$$

Thus

$$|a_0 y(x)| \leq m|a_0| + c_1 \int_\tau^x e^{c_2 t} dt + |b_0| \int_\tau^x |y^p(t)| dt + |b_1| \int_\tau^x |y^p(t-\tau)| dt$$

$$\leq m|a_0| + c_1 \int_\tau^x e^{c_2 t} dt + |b_0| \int_0^x |y^p(t)| dt + |b_1| \int_0^{x-\tau} |y^p(t)| dt ,$$

hence

$$|y(x)| \leq m + \frac{c_1}{c_2 |a_0|} e^{c_2 x} + \frac{|b_0| + |b_1|}{|a_0|} \int_0^x |y^p(t)| dt .$$

Let $c_3 = \max\left\{ 1, \ \frac{1}{c_2 |a_0|} \right\}$, and $c_4 = \frac{|b_0| + |b_1|}{|a_0|}$.

Then by following the same previous steps as in proposition (2.2), one can get the same result. ■

Next, we generalize the previous theorem to be valid for $p > 1$, but before that we need the following lemma.

## Lemma (2.2), [Bainov D. and Simeonov P., 1992]:

Let $y(x)$, a(x), b(x) and k(x) be nonnegative continuous functions in $J=[\alpha,\beta]$ and let $p>1$ be a constant. Suppose $\frac{a}{b}$ is nondecreasing in $J$ and

$$y(x)\le a(x)+b(x)\int_{\alpha}^{x}k(t)y^{p}(t)dt,\quad x\in J. \tag{2.9}$$

Then

$$y(x)\le a(x)\left[1-(p-1)\int_{\alpha}^{x}k(t)b(t)a^{p-1}(t)dt\right]^{\frac{1}{1-p}},\quad \alpha\le x\le\beta_{p}, \tag{2.10}$$

where $\beta_p=\sup\left\{x\in J:(p-1)\int_{\alpha}^{x}k(t)b(t)a^{p-1}(t)dt<1\right\}$.

## Proof:

Set $v(x)=\int_{\alpha}^{x}k(t)y^{p}(t)dt$. Then for $\alpha\le x\le\tau<\beta_p$, ineq.(2.9) implies

$$y(x)\le a(x)+b(x)v(x), \tag{2.11}$$

and

$$v'(x)=k(x)y^{p}(x)\le k(x)[a(x)+b(x)v(x)]^{p-1}[a(x)+b(x)v(x)]$$

$$\le k(x)b(x)[a(x)+b(x)v(x)]^{p-1}\left[\frac{a(\tau)}{b(\tau)}+v(x)\right],$$

that is,

$$v'(x) \le R(x)\left[\frac{a(\tau)}{b(\tau)} + v(x)\right], \qquad (2.12)$$

where $R(x) = k(x)b(x)[a(x) + b(x)v(x)]^{p-1}$.

From ineq.(2.12), one can get

$$[v'(t) - R(t)v(t)]e^{\int_t^x R(t_1)dt_1} \le R(t)\frac{a(\tau)}{b(\tau)}e^{\int_t^x R(t_1)dt_1}, \quad \mathrm{t} \ge \alpha .$$

Thus

$$\frac{d}{dt}\left[v(t)e^{\int_t^x R(t_1)dt_1}\right] \le R(t)\frac{a(\tau)}{b(\tau)}e^{\int_t^x R(t_1)dt_1}$$

Integrating over $t$ from $\alpha$ to $x$ gives

$$v(x) - v(\alpha)e^{\int_\alpha^x R(t_1)dt_1} \le \frac{a(\tau)}{b(\tau)}\int_\alpha^x R(t)e^{\int_t^x R(t_1)dt_1}dt ,$$

which implies

$$v(x) \le \frac{a(\tau)}{b(\tau)}\int_\alpha^x R(t)e^{\int_t^x R(t_1)dt_1}dt .$$

Therefore,

$$v(x) + \frac{a(\tau)}{b(\tau)} \le \frac{a(\tau)}{b(\tau)}e^{\int_\alpha^x R(t)dt}, \quad \alpha \le \mathrm{x} \le \tau .$$

Hence, for $x = \tau$

$$a(x)+b(x)v(x)\le a(x)e^{\int_{\alpha}^{x}R(t)dt} \tag{2.13}$$

From ineq.(2.13), we obtain

$$[a(x)+b(x)v(x)]^{p-1}\le a^{p-1}(x)e^{\int_{\alpha}^{x}(p-1)R(t)dt},$$

that is,

$$R(x)\le k(x)b(x)a^{p-1}(x)e^{\int_{\alpha}^{x}(p-1)R(t)dt}.$$

Let $z(x)=(p-1)R(x)$, then

$$\left[-e^{-\int_{\alpha}^{x}z(t)dt}\right]'\le (p-1)k(x)b(x)a^{p-1}(x).$$

Integrating the above inequality from $\alpha$ to $x$ yields

$$1-e^{-\int_{\alpha}^{x}z(t)dt}\le\int_{\alpha}^{x}(p-1)k(t)b(t)a^{p-1}(t)dt,$$

from which we conclude that

$$e^{\int_{\alpha}^{x}R(t)dt}\le\left[1-(p-1)\int_{\alpha}^{x}k(t)b(t)a^{p-1}(t)dt\right]^{\frac{1}{1-p}}$$

The above inequality, together with ineq.(2.11) and ineq.(2.13), implies ineq.(2.10). ■

Now, we are in the position that we can state the following proposition. This proposition gives an estimate of the magnitude of only one solution for another types of nonlinear ordinary delay differential equations.

## Proposition (2.3):

Let $y(x)$ be a solution of the delay differential equation

$$a_0 y'(x) + b_1 y^p(x-\tau) = f(x), \quad \mathrm{p} > 1 \tag{2.7}$$

which is of class $C^1$ on $[0,\infty)$. Suppose that $f$ is of class $C^o$ on $[0,\infty)$ and that

$$|f(x)| \le c_1 e^{c_2 x}, \quad \mathrm{x} \ge 0,$$

where $c_1$ and $c_2$ are positive constants. Let

$$m = \max_{0 \le x \le \tau} |y(x)|,$$

then there are positive constants $c_3$ and $c_4$ depending only on $c_2$ and the coefficients in eq.(2.7) such that

$$|y(x)| \le c_3(c_1+m)e^{c_2 x}\left[1 - \frac{c_4 {c_3}^{p-1}(c_1+m)^{p-1}}{c_2}\left(e^{(p-1)c_2 x} - 1\right)\right]^{\frac{1}{p-1}}, \quad 0 \le \mathrm{x} \le \beta_{\mathrm{p}}$$

where $\beta_p = \sup\left\{ x \in J : (p-1)\int_0^x c_4\left[c_3(c_1+m)\right]^{p-1} e^{(p-1)c_2 t} dt < 1 \right\}, \quad \mathrm{J} = [0,\beta]$ and

$\beta$ is any positive real number.

## Proof:

By following the same previous steps as in proposition (2.2), one can get

$$|y(x)| \le c_3(c_1+m)e^{c_2 x} + c_4\int_0^x |y(t)|^p dt, \quad x \ge 0,$$

and by using lemma (2.2) with $a(x) = c_3(c_1+m)e^{c_2 x}$, $\mathrm{b(x)=1,\ k(x)=c_4}$ and $\alpha = 0$, one can get

$$|y(x)| \le c_3(c_1+m)e^{c_2 x}\left[1-(p-1)\int_0^x c_4 {c_3}^{p-1}(c_1+m)^{p-1}e^{(p-1)c_2 t}dt\right]^{\frac{1}{1-p}}$$

$$= \mathrm{c}_3(c_1+m)e^{c_2 x}\left[1-\frac{c_4 {c_3}^{p-1}(c_1+m)^{p-1}}{c_2}(e^{(p-1)c_2 x}-1)\right]^{\frac{1}{1-p}}, \quad 0 \le \mathrm{x} \le \beta_{\mathrm{p}},$$

where $\tau_p = \sup\left\{x \in J : (p-1)\int_0^x c_4[c_3(c_1+m)]^{p-1}e^{(p-1)c_2 t}dt < 1\right\}$, $\mathrm{J} = [0, \beta]$ and

$\beta$ is any positive real number. ■

The following theorem is a generalization of the previous proposition which gives the same result.

## Theorem (2.7):

Let $y(x)$ be the solution of the delay differential equation

$$a_0 y'(x) + b_0 y^p(x) + b_1 y^p(x-\tau) = f(x), \quad \mathrm{p} > 1 \qquad (2.8)$$

which is of class $C^1$ on $[0,\infty)$. Suppose that $f$ is of class $C^o$ on $[0,\infty)$ and that

$$|f(x)| \le c_1 e^{c_2 x}, \quad x \ge 0,$$

where $c_1$ and $c_2$ are positive constants. Let

$$m = \max_{0 \le x \le \tau} |y(x)|,$$

then there are positive constants $c_3$ and $c_4$ depending only on $c_2$ and the coefficients in eq.(2.8) such that

$$|y(x)| \le c_3(c_1+m)e^{c_2 x}\left[1 - \frac{c_4 {c_3}^{p-1}(c_1+m)}{c_2}(e^{(p-1)c_2 x}-1)\right]^{\frac{1}{1-p}}, \quad 0 \le x \le \beta_p,$$

where $\tau_p = \sup\left\{x \in J : \frac{c_4 {c_3}^{p-1}(c_1+m)^{p-1}}{c_2}(e^{(p-1)c_2 x}-1) < 1\right\}, \quad J=[0,\beta]$

and $\beta$ is any positive real number.

## Proof:

By following the same previous steps as in theorem (2.6), one can get

$$|y(x)| \le c_3(c_1+m)e^{c_2 x} + c_4 \int_0^x |y(t)|^p\, dt, \quad t \ge 0,$$

and by using lemma (2.2) with $a(x) = c_3(c_1+m)e^{c_2 x}$, $b(x)=1$, $k(x)=c_4$ and $\alpha = 0$, one can get

$$|y(x)| \le c_3(c_1+m)e^{c_2 x}\left[1-(p-1)c_4\int_0^x c_3{}^{p-1}(c_1+m)^{p-1}e^{(p-1)c_2 t}dt\right]^{\frac{1}{1-p}}.$$

Thus

$$|y(x)| \le c_3(c_1+m)e^{c_2 x}\left[1-\frac{c_4 c_3{}^{p-1}(c_1+m)^{p-1}}{c_2}(e^{(p-1)c_2 x}-1)\right]^{\frac{1}{1-p}}, \quad 0 \le \mathrm{x} \le \beta_p$$

where $\beta_p = \sup\left\{x \in J : \frac{c_4 c_3{}^{p-1}(c_1+m)^{p-1}}{c_2}(e^{(p-1)c_2 x}-1) < 1\right\}, \quad J=[0,\beta]$

and $\beta$ is any positive real number.

## 2.4 An Estimate of the Magnitude of Two Solutions of Linear Ordinary Delay Differential Equations:

In this section, we give an estimate of the magnitude of two solutions for linear ordinary delay differential equations with constant coefficients.

We start this section by the following proposition. This proposition appeared in [Bellmen R. and Cook K., 1963] without proof. Here we give it's proof.

## Proposition(2.4):

Let $y_1(x)$ and $y_2(x)$ be two solutions of the delay differential equation

$$a_0 y'(x) + b_1 y(x-\tau) = f(x) \tag{2.9}$$

which are of class $C^1$ on $[0,\infty)$. Suppose that $f$ is of class $C^o$ on $[0,\infty)$. Let

$$m = \max_{0 \le x \le \tau} |y_1(x) - y_2(x)|,$$

then there is a positive constant $c$, depending only on the coefficients in eq.(2.9) such that

$$|y_1(x) - y_2(x)| \le m e^{cx}, \quad \mathrm{x} \ge 0.$$

## Proof:

Since $y_1(x)$ and $y_2(x)$ are solutions of eq.(2.9), then

$$a_0 y_1(x) = a_0 y_1(\tau) + \int_\tau^x f(t)dt - b_1 \int_\tau^x y_1(t-\tau)dt, \quad \mathrm{x} \ge \tau,$$

and

$$a_0 y_2(x) = a_0 y_2(\tau) + \int_\tau^x f(t)dt - b_1 \int_\tau^x y_2(t-\tau)dt, \quad \mathrm{x} \ge \tau.$$

Thus

$$|a_0||y_1(x) - y_2(x)| \le |a_0||y_1(\tau) - y_2(\tau)| + |b_1| \int_0^{x-\tau} |y_1(t) - y_2(t)| dt,$$

hence

$$|y_1(x) - y_2(x)| \le m + \frac{|b_1|}{|a_0|} \int_0^x |y_1(t) - y_2(t)| dt.$$

Let $c = \frac{|b_1|}{|a_0|}$, then

$$|y_1(x) - y_2(x)| \le m + c\int_0^x |y_1(t) - y_2(t)|dt, \quad \mathrm{x} \ge \tau . \qquad (2.14)$$

Since $|y_1(x) - y_2(x)| \le m$ for $0 \le x \le \tau$, ineq.(2.14) holds for all $x \ge 0$. It therefore follows from lemma (1.1) that

$$|y_1(x) - y_2(x)| \le me^{\int_0^x cdt} = me^{cx}, \quad \mathrm{x} \ge 0. \ \blacksquare$$

Next, this proposition can be also extended to give the following theorem. This theorem appeared in [ Bellman R. and Cooke K., 1963] without proof. Here we give it's proof.

## Theorem (2.8):

Let $y_1(x)$ and $y_2(x)$ be two solutions of the delay differential equation

$$a_0 y'(x) + b_0 y(x) + b_1 y(x - \tau) = f(x) \qquad (2.10)$$

which are of class $C^1$ on $[0, \infty)$. Suppose that $f$ is of class $C^o$ on $[0, \infty)$. Let

$$m = \max_{0 \le x \le \tau} |y_1(x) - y_2(x)|,$$

then there is a positive constant $c$ depending only on the coefficients in eq.(2.10) such that

$$|y_1(x) - y_2(x)| \le me^{cx}, \quad \mathrm{x} \ge 0.$$

## Proof:

By following the same previous steps as in proposition (2.4), one can get

$$|y_1(x) - y_2(x)| \le m + c\int_0^x |y_1(t) - y_2(t)|dt, \quad x \ge \tau,$$

where $c = \dfrac{|b_0| + |b_1|}{|a_0|}$.

The solution of the above inequality is

$$|y_1(x) - y_2(x)| \le me^{cx}, \quad x \ge 0. \quad ■$$

Next, the following theorem is a generalization of the previous theorem which discussed an estimate of the magnitude of two solutions for two 1st order delay differential equations with same differential operator. This theorem appeared in [Bellman R. and Cooke K., 1963] without proof. Here, we give it's proof.

## Theorem (2.9):

Let $y_1(x)$ and $y_2(x)$ be solutions of the delay differential equations

$$a_0 y'(x) + b_0 y(x) + b_1 y(x-\tau) = f_1(x) \tag{2.11}$$

$$a_0 y'(x) + b_0 y(x) + b_1 y(x-\tau) = f_2(x) \tag{2.12}$$

respectively which are of class $C^1$ on $[0,\infty)$. Suppose that $f_1$ and $f_2$ are of class $C^o$ on $[0,\infty)$. Let

$$m = \max_{0 \le x \le \tau} \left| y_1(x) - y_2(x) \right|,$$

then there is a positive constant $c$ depending only on the coefficients in eq.(2.11)-(2.12) such that

$$\left| y_1(x) - y_2(x) \right| \le \left[ m + |a|^{-1} \int_0^x \left| f_1(t) - f_2(t) \right| dt \right] e^{cx}, \quad \text{x} \ge 0.$$

## Proof:

From eq.(2.11)-(2.12), we find that

$$a_0 y_1(x) = a_0 y_1(\tau) + \int_\tau^x f_1(t)dt - b_0 \int_\tau^x y_1(t)dt - b_1 \int_\tau^x y_1(t-\tau)dt, \quad \text{x} \ge \tau$$

and

$$a_0 y_2(x) = a_0 y_2(\tau) + \int_\tau^x f_2(t)dt - b_0 \int_\tau^x y_2(t)dt - b_1 \int_\tau^x y_2(t-\tau)dt, \quad \text{x} \ge \tau.$$

Thus

$$\left| y_1(x) - y_2(x) \right| \le \left| y_1(\tau) - y_2(\tau) \right| + \frac{1}{|a_0|} \int_0^x \left| f_1(t) - f_2(t) \right| dt + \frac{|b_0| + |b_1|}{|a_0|} \int_0^x \left| y_1(t) - y_2(t) \right| dt$$

$$\left| y_1(x) - y_2(x) \right| \le m + |a_0|^{-1} \int_0^x \left| f_1(t) - f_2(t) \right| dt + c \int_0^x \left| y_1(t) - y_2(t) \right| dt, \quad \text{x} \ge \tau \qquad (2.15)$$

where $c = \dfrac{|b_0| + |b_1|}{|a_0|}$.

Since $|y_1(x)-y_2(x)| \le m$ for $0 \le x \le \tau$, ineq.(2.15) holds for all $x \ge 0$. It therefore follows from lemma (1.1) that

$$|y_1(x)-y_2(x)| \le \left[ m + |a_0|^{-1} \int_0^x |f_1(t)-f_2(t)| dt \right] e^{cx}, \quad x \ge 0. \quad \blacksquare$$

## Remark (2.1):

In theorem (2.9), if $f_1(x) = f_2(x)$ for each $x \ge 0$, then this theorem reduces to theorem (2.8).

Now, the following theorem is more general than the pervious theorem.

## Theorem (2.10):

Let $y_1(x)$ and $y_2(x)$ be solutions of the delay differential equation

$$a_0 y'(x) + b_0 y(x) + b_1 y(x-\tau) = f_1(x) \tag{2.13}$$

and

$$a_0 y'(x) + (b_0 + \varepsilon_0 ) y(x) + (b_1 + \varepsilon_1)_1 y(x-\tau) = f_2(x) \tag{2.14}$$

respectively which are of class $C^1$ on $[0,\infty)$. Suppose that $f_1$ and $f_2$ are of class $C^o$ on $[0,\infty)$. Let

$$m_1 = \max_{0 \le x \le \tau} |y_1(x) - y_2(x)|$$

$$m_2 = \max_{0 \le x \le \tau} |y_2(x)|$$

then there are positive constant $c$ and $\varepsilon$ depending only on the coefficients in eq.(2.13)-(2.14) such that

$$\left|y_1(x)-y_2(x)\right| \le \left[ m_1 + |a|^{-1}\int_0^x \left| f_1(t)-f_2(t)\right| dt + \frac{\varepsilon\, m_2 \tau}{|a_0|}\right] e^{cx}, \quad \mathrm{x} \ge 0 .$$

## Proof:

From eq.(2.13)-(2.14), we find that

$$a_0 y_1(x) = a_0 y_1(\tau) + \int_\tau^x f_1(t)dt - b_0\int_\tau^x y_1(t)dt - b_1\int_\tau^x y_1(t-\tau)dt, \quad \mathrm{x} \ge \tau .$$

and

$$a_0 y_2(x) = a_0 y_2(\tau) + \int_\tau^x f_2(t)dt - (b_0+\varepsilon_0)\int_\tau^x y_2(t)dt - (b_1+\varepsilon_1)\int_\tau^x y_2(t-\tau)dt, \quad \mathrm{x} \ge \tau$$

Thus

$$\left|y_1(x)-y_2(x)\right| \le \left|y_1(\tau)-y_2(\tau)\right| + \frac{1}{|a_0|}\int_0^x \left| f_1(t)-f_2(t)\right| dt + \frac{|b_0|+|b_1|}{|a_0|}\int_0^x \left|y_1(t)-y_2(t)\right| dt +$$

$$\frac{|\varepsilon|}{|\mathrm{a}_0|}\left[\int_0^x |y_2(t)|dt - \int_{\mathrm{x}}^\tau |y_2(t)|dt\right], \quad \mathrm{x} \ge \tau$$

$$\le m_1 + |a_0|^{-1}\int_0^x \left| f_1(t)-f_2(t)\right| dt + c\int_0^x \left|y_1(t)-y_2(t)\right| dt +$$

$$\frac{\varepsilon\, \mathrm{m}_2 \tau}{|\mathrm{a}_0|}, \quad \mathrm{x} \ge \tau \tag{2.16}$$

where $c = \dfrac{|b_0| + |b_1|}{|a_0|}$ and $\varepsilon = \max\{|\varepsilon_0|, |\varepsilon_1|\}$.

Since $|y_1(x) - y_2(x)| \le m_1$ for $0 \le x \le \tau$, ineq.(2.16) holds for all $x \ge 0$. It therefore follows from lemma (1.1) that

$$|y_1(x) - y_2(x)| \le \left[ m_1 + |a_0|^{-1} \int_0^x |f_1(t) - f_2(t)| dt + \frac{\varepsilon\, m_2 \tau}{|a_0|} \right] e^{cx}, \quad x \ge 0. \quad \blacksquare$$

## Remarks (2.2):

(i) If $\varepsilon = 0$ then $\varepsilon_0 = \varepsilon_1 = 0$ and hence theorem (2.10) reduces to theorem (2.9).

(ii) If $\varepsilon = 0$ and $f_1(x) = f_2(x)$ for each $x \ge 0$ then theorem (2.10) reduces to theorem (2.8).

## 2.5 An Estimate of the Magnitude of One Solution for Systems Ordinary Delay Differential Equations:

In this section we derive an estimate of the solutions for special types of system of the1$^{st}$ order ordinary delay differential equations with constant coefficients.

We start this section by generalizing theorem (2.2) to system of the 1$^{st}$ order linear ordinary delay differential equations.

## Theorem (2.11):

Let $y(x)$ be a solution of the system of $1^{st}$ order delay differential equations

$$y'(x)+B_0 y(x)+\sum_{i=1}^{m} B_i y(x-\tau_i)=f(x), \tag{2.15}$$

which is of class $C^1$ on $[0,\infty)$ where $B_i$ is $n\times n$ matrix for each $i=0,1,...,m$. Suppose that $f$ is $n\times 1$ vector of class $C^o$ on $[0,\infty)$ and that

$$\|f(x)\|\le c_1 e^{c_2 x}, \quad x\ge 0,$$

where $c_1$ and $c_2$ are positive constants. Let

$$m=\max_{0\le x\le \tau_m}\|y(x)\|,$$

where $0<\tau_1<\tau_2<...<\tau_m$ then there are positive constants $c_3$ and $c_4$ depending only on $c_2$ and the coefficients in eq.(2.15) such that

$$\|y(x)\|\le c_3(c_1+m)e^{c_4 x}, \quad x\ge 0.$$

## Proof:

From eq.(2.15), we find that

$$y(x)=y(\tau_m)-B_0\int_{\tau_m}^{x} y(t)dt-\int_{\tau_m}^{x}\sum_{i=1}^{m} B_i y(t-\tau_i)dt+\int_{\tau_m}^{x} f(t)dt.$$

Therefore,

$$\|y(x)\| \le \|y(\tau_m)\| + + \int_{\tau_m}^{x} \|f(t)\|\, dt\, \|B_0\| \int_{\tau_m}^{x} \|y(t)\| dt + \int_{\tau_m}^{x} \sum_{i=1}^{m} \|B_i\| \, \|y(t-\tau_i\| dt$$

$$\le \mathrm{m} + c_1 \int_{\tau_m}^{x} e^{c_2 t} dt + \|\mathrm{B}_0\| \int_0^{\mathrm{x}} \|\mathrm{y(t)}\|\, dt + \int_{\tau_m}^{x} \sum_{i=1}^{m} \|B_i\| \, \|y(t-\tau_i)\|\, dt$$

$$\le m + \frac{c_1}{c_2} e^{c_2 x} + \|B_0\| \int_0^x \|y(t)\| dt + \sum_{i=1}^{m} \|B_i\| \int_0^x \|y(t)\| dt\,.$$

Thus

$$\|y(x)\| \le m + \frac{c_1}{c_2} e^{c_2 x} + \sum_{i=0}^{m} \|B_i\| \int_0^x \|y(t)\| dt, \quad \mathrm{x} \ge \tau_{\mathrm{m}}\,.$$

Let $c_3 = \max\left\{1\ ,\ \frac{1}{c_2}\right\}$, and $c_5 = \sum_{i=0}^{m} \|B_i\|$.

Then

$$\|y(x)\| \le c_3 (c_1 + m) e^{c_2 x} + c_5 \int_0^x \|y(t)\| dt, \quad \mathrm{x} \ge \tau_{\mathrm{m}} \tag{2.17}$$

Since $\|y(x)\| \le m \le m c_3 e^{c_2 x}$ for $0 \le \mathrm{x} \le \tau_{\mathrm{m}}$, hence ineq.(2.17) holds for all $x \ge 0$. It therefore follows from lemma(1.1) that

$$\|y(x)\| = \mathrm{c}_3 (c_1 + m) e^{(c_2 + c_5) x}, \quad \mathrm{x} \ge 0\,.$$

Next, the following theorem is an extension of theorem (2.3) to system include of the 1st order ordinary delay differential equations.

## Theorem (2.12):

Let $y(x)$ be a solution of the linear system of the 1st order ordinary delay differential equations

$$\sum_{i=0}^{m}[A_i y'(x-\tau_i)+B_i y(x-\tau_i)]=f(x) \tag{2.16}$$

which is of class $C^1$ on $[0,\infty)$ where $A_i$ and $B_i$, $i=0,1,...,m$ are $n\times n$ matrices such that $A_0=I$. Suppose that $f$ is $n\times 1$ vector of class $C^o$ on $[0,\infty)$ and that

$$\|f(x)\|\leq c_1 e^{c_2 x}, \quad x\geq 0,$$

where $c_1$ and $c_2$ are positive constants. Let

$$m=\max_{0\leq x\leq \tau_m}\|y(x)\|,$$

where $0<\tau_1<\tau_2<...<\tau_m$ then there are positive constants $c_3$ and $c_4$ depending only on $c_2$ and the coefficients in eq.(2.16) such that

$$\|y(x)\|\leq c_3(c_1+m)e^{c_4 x}, \quad 0\leq x\leq \tau_m+\tau_1.$$

## Proof:

Since $A_0=I$, then eq.(2.16) can be rewritten as

$$y'(x)+B_0 y(x)+\sum_{i=1}^{m}\left[A_i y'(x-\tau_i)+B_i y(x-\tau_i)\right]=f(x).$$

Therefore,

$$y(x) = y(\tau_m) + \int_{\tau_m}^{x} f(t)dt - B_0 \int_{\tau_m}^{x} y(t)dt - \int_{\tau_m}^{x} \sum_{i=1}^{m} \left[A_i y'(t-\tau_i) + B_i y(t-\tau_i)\right]dt$$

$$= \mathrm{y}(\tau_m) + \int_{\tau_m}^{x} f(t)dt - \mathrm{B}_0 \int_{\tau_m}^{x} \mathrm{y(t)dt} - \sum_{i=1}^{\mathrm{m}} \mathrm{A_i}\left[\mathrm{y(x} - \tau_i) - \mathrm{y}(\tau_m - \tau_i)\right] - \int_{\tau_m}^{x} \sum_{i=1}^{m} B_i y(t-\tau_i)\, dt$$

Therefore,

$$\|y(x)\| \le \|y(\tau_m)\| + \|B_0\| \int_{\tau_m}^{x} \|y(t)\| dt + \sum_{i=1}^{m} \|A_i\| \, \|y(x-\tau_i) - y(\tau_m - \tau_i)\| +$$

$$\int_{\tau_m}^{x} \sum_{i=1}^{m} \|B_i\| \, \|y(t-\tau_i)\| dt + \int_{\tau_m}^{x} \|f(t)\|\, dt$$

$$\le \mathrm{m} + \|\mathrm{B}_0\| \int_{0}^{\mathrm{x}} \|\mathrm{y(t)}\|\, dt + \sum_{\mathrm{i=1}}^{\mathrm{m}} \|\mathrm{A_i}\| \, \|\mathrm{y(x} - \tau_{\mathrm{i}}) - y(\tau_m - \tau_i)\| +$$

$$\int_{0}^{x-\tau_i} \left[\sum_{i=1}^{m} \|B_i\|\right] \|y(t)\|\, dt + \mathrm{c}_1 \int_{\tau_m}^{x} e^{c_2 t} dt$$

$$\le \mathrm{m} + \frac{c_1}{c_2} e^{c_2 x} + \sum_{i=0}^{m} \|B_i\| \int_{0}^{x} \|y(t)\| dt + \sum_{i=1}^{m} \|A_i\| \left[\, \|y(x-\tau_i)\| + \|y(\tau_m - \tau_i)\| \,\right]$$

$$\le \left[1 + 2\sum_{\mathrm{i=1}}^{\mathrm{m}} \|\mathrm{A_i}\|\right] m + \frac{c_1}{c_2} e^{c_2 x} + \sum_{i=0}^{m} \|B_i\| \int_{0}^{x} \|y(t)\| dt, \quad \tau_{\mathrm{m}} \le x \le \tau_m + \tau_1 .$$

Thus

$$\|y(x)\| \le \left[1+2\sum_{i=1}^{m}\|A_i\|\right]m + \frac{c_1}{c_2}e^{c_2x} + \sum_{i=0}^{m}\|B_i\|\int_0^x \|y(t)\|dt, \quad \tau_m \le x \le \tau_m + \tau_1$$

Let $c_3 = \max\left\{1+2\sum_{i=1}^{m}\|A_i\|, \frac{1}{c_2}\right\}$, and $c_5 = \sum_{i=0}^{m}\|B_i\|$.

Then

$$\|y(x)\| \le c_3(c_1+m)e^{c_2x} + c_5\int_0^x \|y(x)\|dx, \quad \tau_{\mathrm{m}} \le x \le \tau_m + \tau_1. \tag{2.18}$$

Since $\|y(x)\| \le m \le mc_3e^{c_2x}$ for $0 \le \mathrm{x} \le \tau_{\mathrm{m}}$, hence ineq.(2.18) holds for all $0 \le x \le \tau_m + \tau_1$. It therefore follows from lemma(1.1) that

$$\|y(x)\| \le c_3(c_1+m)e^{(c_2+c_5)x}, \quad 0 \le x \le \tau_m + \tau_1. \quad ■$$

## **Corollary(2.1):**

Let the hypotheses of theorem (2.12) be satisfied except the condition $A_0 = I$. Assume that $|A_0| \ne 0$, then there are positive constants $c_3$ and $c_4$ depending only on $c_2$ and the coefficients in eq.(2.16) such that

$$|y(x)| \le c_3(c_1+m)e^{c_4x}, \quad 0 \le x \le \tau_m + \tau_1.$$

## **Proof:**

Since $|A_0| \ne 0$, then eq.(2.16) can be rewritten as

$$y'(x) + A_0^{-1}B_0y(x) + \sum_{i=1}^{m}[(A^{-1}A_i)y'(x-\tau_i) + (A_0^{-1}B_i)y(x-\tau_i)] = f(x),$$

and hence the proof follows directly from theorem (2.12). ■

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# *INTRODUCTION*

The differential equations with a deviating argument are differential equations in which the unknown function and its derivative enter, generally speaking, under different values of the argument, [El'sgol'ts L. and Norkin S., 1973].

These equations appeared in the literature in the second half of the eighteenth century by Kondorse in 1771, but a systematic study of equations with a deviating argument began only in the twentieth century (especially in the last forty years by Myshkis A. in the Soviet Union, Wright E. and Bellman R. in other countries) in connection with the requirements of applied science, [El'sgol'ts L. and Norkin S., 1973].

A topic of differential equations with deviating arguments which is in a rapid state of development. It was the Russian mathematician Krasovskii who found an accommodation for differential equations with deviating arguments as operators in function spaces. It is worth noting that the theory of differential equations with deviating arguments is not just a simple extension of the theory of ordinary differential equations, [Saaty T., 1967].

The differential equations with deviating arguments are integrable in closed form only under very specialized circumstances, and therefore qualitative and approximate methods are of the utmost importance in studying them, [El'sgol'ts L., 1964].

Many researchers study the delay differential equations:

Al-Saady A., 2000, gave a new approach for solving the delay differential equations. This approach depends mainly on the Gaussian quadrature numerical integration method and cubic spline interpolator functions for the unknown exact solution,

Narie N., 2001, introduced the variational formulations of the delay differential equations and solved them by using the direct Ritz method,

Al. Daynee K., 2002, evaluated the variational formulation of the delay BVPs, using two approaches (variational problem with constraint and variational problem using Rayleigh quotient formula), as well as, the equivalence between the solution of the original problem in operator form $Ly = f$ and the variational problem have been proved.

Salih S., 2004, studied and modified some numerical and approximate methods for solving the $n^{th}$ order linear delay differential equations with constant coefficients, and

Al-Kubeisy S., 2004, solved the delay differential equations numerically by using the linear multistep methods.

The main purpose of this work is to derive an estimate of the magnitude of the solutions for special types of ordinary and partial delay differential equations in order to find them by any suitable methods, say the Laplace transform method.

This thesis consists of three chapters.

In chapter one, we give some basic concepts of the ordinary differential equations with deviating arguments. These concepts include, classification, existence and uniqueness of solutions and methods of solutions of them.

In chapter two, an estimate of the magnitude of solutions (one and two) of the special types of linear and nonlinear ordinary delay differential equations is presented.

In chapter three, we devote the partial differential equations with deviating arguments and give some existence and uniqueness theorems for the solutions of them. Also an estimate of the $1^{st}$ and $2^{nd}$ order partial delay differential equations is derived.

*Ministry of Higher Education*
*and Scientific Research*
*Al-Nahrain University*
*College of Science*

# *MAGNITUDE ESTIMATION OF SOLUTIONS FOR SPECIAL TYPES OF DELAY DIFFERENTIAL EQUATIONS*

***A Thesis***

***Submitted to the College of Scienc in Al-Nahrain University as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics***

***By***

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***(B.Sc. Al-Nahrain University, 2003)***

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## ***Supervisor Certification***

I certify that this thesis was prepared under my supervision at the Department of Mathematics / College of Science / Al-Nahrain University as a partial fulfillment of the requirements for the degree of master in mathematics.

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*Date: / / 2006*

*In view of the available recommendations, I forward this thesis for debate by the examining committee.*

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*Head of the Department of Mathematics*

*Date: / / 2006*

بِسمِ الله الـرَّحمن الـرَّحِيم

وَعِنـدَهُ مَفَـاتِـحُ الـغَـيبِ لا يَـعـلَـمُـهَـا إِلَّا هُـوَ ويَـعـلَـمُ مَـا فِـي الـبَـرِّ وَالـبَـحرِ وَمَـا تـسقُـطُ مِـن وَرَقَـةٍ إِلَّا يَـعـلَـمُـهَـا وَلا حَـبَّـةٍ فِـي ظُـلُـمَـاتِ الأَرضِ وَلا رَطبٍ وَلا يَـابِـسٍ إِلا فِـي كِـتَـابٍ مُّـبِـينٍ [٥٩]

{سورة الأنـعـام}

# Chapter One

## Some Basic Concepts of the Ordinary Differential Equations with Deviating Arguments

# Chapter Two

## Estimation of the Magnitude of the Solutions for Ordinary Delay Differential Equations

# Chapter Three

## The Partial Differential Equations with Deviating Arguments

ا الإ هـد ا ء

إلى وردةٍ جميلةٍ

طالعةٍ من جُرحٍ ... بلادي

إلى واحةٍ ظليلةٍ

خضراء ... عائلتي

إلى نبع ماءٍ باردٍ

يروي الصحراء ... أساتذتي

إلى من أكنُ لهم الحب

والوفاء ... زملائي

انــتـ

صـا ر

# المستخلص

الهدف الرئيسي من هذا العمل يمكن تقسيمه إلى أربع محاور والتي يمكن تلخيصها كالآتي:

**الهدف الأول:** هو دراسة المعادلات التفاضليه الاعتيادية ذات الأزاحات الزواية المنحرفة.

**الهدف الثاني:** هو تخمين قيم الحلول لأنواع خاصة من المعادلات التفاضلية الخطية واللاخطية ذات الأزاحات الزوايـة المنحرفة حتى نتمكن من حلها بأي طريقه مناسبة.

**الهدف الثالث:** هو تبني دراسة وجود حل وحيد مقيد لأنواع خاصة من المعادلات التفاضلية الجزئية ذات الأزاحات الزواية المنحرفة.

**الهدف الرابع:** هو تخمين قيم الحلول لأنواع خاصة من المعادلات التفاضلية الجزئية ذات الأزاحات الزواية المنحرفة من ذوات الرتبة الأولى والثانية.

وزارة التعليم العالي والبحث العلمي
جامعة النهرين
كلية العلوم

# تخمين قيمي لحلول انواع خاصه من المعادلات التفاضلية التباطؤية

رسالة مقدمه إلى
كلية العلوم في جامعة النهرين كجزء من متطلبات نيل درجة ماجستير علوم في الرياضيات

من قبل

**انتصار سويدان علي العيساوي**
(بكالوريوس جامعة النهرين، ٢٠٠٣)

بإشراف

**د. أحلام جميل خليل**

أيلول ٢٠٠٦
شعبان ١٤٢٧

**تمت المناقشة يوم الاثنين المصادف ١٣\ ١١ \ ٢٠٠٦**