ABSTRACT

The main aim of this work is classified into four objects, these are summarized as follows:

The first objective is to study the theory of existence and the uniqueness of the solutions for the periodic boundary value problems of the differential equations.

The second objective is to devote the existence theorems of the extremal solutions of the above periodic boundary value problems.

The third objective is to give the existence and the uniqueness theorems of the solutions for the periodic boundary value problems of the linear and nonlinear ordinary integro-differential equations. Also, the existence theorems of the extremal solutions for the above periodic boundary value problems is introduced.

The fourth objective is to solve the periodic boundary value problems for ordinary integro-differential equations by using the expansion methods.

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APPENDIX

Example (3.1):

$$\frac{d}{dt}u(t) + 3u(t) = -Ku(t) + 3\cdot t^2 - \frac{13}{12}t - 1$$

$$u(0) = u(1)$$

with the boundary condition

This example is solved by using the collocation method

The exact solution

$$u(t) := t^2 - t$$

where

$$Ku(t) = \int_0^1 t\cdot s\cdot u(s)\,ds$$

and

$$g(t) := 3\cdot t^2 - \frac{13}{12}t - 1$$

Approximate u as a polynomial of degree 2

$$u(t) := a0 + a1\cdot t + a2\cdot t^2$$

$$u(t, a0, a2) := a0 - a2t + a2\cdot t^2$$

$$Ku(t,a0,a2) := \int_0^1 t\cdot s\cdot u(s,a0,a2)\,ds \rightarrow \frac{-1}{12}\cdot t\cdot a2 + \frac{1}{2}\cdot t\cdot a0$$

$$e(t,a0,a2) := \frac{d}{dt}u(t,a0,a2) + 3u(t,a0,a2) + Ku(t,a0,a2) - g(t)$$

$$e(t,a0,a2) \rightarrow -a2 - \frac{13}{12}\cdot t\cdot a2 + 3\cdot a0 + 3\cdot a2\cdot t^2 + \frac{1}{2}\cdot t\cdot a0 - 3\cdot t^2 + \frac{13}{12}\cdot t + 1$$

Given

$$e(0,a0,a2) = 0 \rightarrow -a2 + 3\cdot a0 + 1 = 0$$

$$e\left(\frac{1}{3},a0,a2\right) = 0 \rightarrow \frac{-37}{36}\cdot a2 + \frac{19}{6}\cdot a0 + \frac{37}{36} = 0$$

$$-a2 + 3\cdot a0 + 1 = 0$$

$$\frac{-37}{36}\cdot a2 + \frac{19}{6}\cdot a0 + \frac{37}{36} = 0$$

$$Find(a0,a2) \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Another approximate for u as a polynomial of degree 3

$$u(t) := a0 + a1\cdot t + a2\cdot t^2 + a3\cdot t^3$$

$$u(t,a0,a2,a3) := a0 - (a2 + a3)\cdot t + a2\cdot t^2 + a3\cdot t^3$$

$$Ku(t,a0,a2,a3) := \int_0^1 t\cdot s\cdot u(s,a0,a2,a3)\,ds \rightarrow \frac{-2}{15}\cdot t\cdot a3 - \frac{1}{12}\cdot t\cdot a2 + \frac{1}{2}\cdot t\cdot a0$$

$$e(t,a0,a2,a3) := \frac{d}{dt}u(t,a0,a2,a3) + 3u(t,a0,a2,a3) + Ku(t,a0,a2,a3) - g(t)$$

$$e(t,a0,a2,a3) \rightarrow -a2 - a3 + \frac{23}{12}\cdot t\cdot a2 + 3\cdot a3\cdot t^2 + 3\cdot a0 - 3\cdot(a2+a3)\cdot t + 3\cdot a2\cdot t^2 + 3\cdot a3\cdot t^3 - \frac{2}{15}\cdot t\cdot a3 + \frac{1}{2}\cdot t\cdot a0 - 3\cdot t^2 + \frac{13}{12}\cdot t + 1$$

Given

$$e(0,a0,a2,a3) = 0 \rightarrow -a2 - a3 + 3\cdot a0 + 1 = 0$$

$$e\left(\frac{1}{2},a0,a2,a3\right) = 0 \rightarrow \frac{-173}{120}\cdot a3 + \frac{13}{4}\cdot a0 - \frac{19}{24}\cdot a2 + \frac{19}{24} = 0$$

$$f(1,a0,a2,a3) = 0 \rightarrow f(1,a0,a2,a3) = 0$$

$$-a2 - a3 + 3\cdot a0 + 1 = 0$$

$$\frac{-173}{120}\cdot a3 + \frac{13}{4}\cdot a0 - \frac{19}{24}\cdot a2 + \frac{19}{24} = 0$$

$$\frac{11}{12}\cdot a2 + \frac{28}{15}\cdot a3 + \frac{7}{2}\cdot a0 - \frac{11}{12} = 0$$

$$\mathrm{Find}(a0,a2,a3) \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Another approximate for u as a polynomial of degree 4

$$u(t) := a0 + a1\cdot t + a2\cdot t^2 + a3\cdot t^3 + a4\cdot t^4$$

$$u(t,a0,a2,a3,a4) := a0 - (a2 + a3 + a4)\cdot t + a2\cdot t^2 + a3\cdot t^3 + a4\cdot t^4$$

$$Ku(t,a0,a2,a3,a4) := \int_0^1 t\cdot s\cdot u(s,a0,a2,a3,a4)\,ds$$

$$g1(t,a0,a1,a2,a3,a4) := Ku(t,a0,a2,a3,a4) - g(t)$$

$$g2(t,a0,a1,a2,a3,a4) := \frac{d}{dt}u(t,a0,a2,a3,a4) + 3\cdot u(t,a0,a2,a3,a4)$$

$$e(t,a0,a2,a3,a4) := g2(t,a0,a1,a2,a3,a4) + g1(t,a0,a1,a2,a3,a4)$$

$$e(t, a0, a2, a3, a4) \rightarrow -a2 - a3 - a4 + \frac{23}{12}\cdot t\cdot a2 + 3\cdot a3\cdot t^2 + 4\cdot a4\cdot t^3 + 3\cdot a0 - 3\cdot(a2 + a3 + a4)\cdot t + 3\cdot a2\cdot t^2 + 3\cdot a3\cdot t^3 + 3\cdot a4\cdot t^4 - \frac{1}{6}\cdot t\cdot a4 - \frac{2}{15}\cdot t\cdot a3 + \frac{1}{2}\cdot t\cdot a0 - 3\cdot t^2 + \frac{13}{12}\cdot t + 1$$

Given

$$e(0, a0, a2, a3, a4) = 0 \rightarrow -a2 - a3 - a4 + 3\cdot a0 + 1 = 0$$

$$e\left(\frac{1}{2}, a0, a2, a3, a4\right) = 0 \rightarrow \frac{-173}{120}\cdot a3 - \frac{91}{48}\cdot a4 + \frac{13}{4}\cdot a0 - \frac{19}{24}\cdot a2 + \frac{19}{24} = 0$$

$$e\left(\frac{1}{3}, a0, a2, a3, a4\right) = 0 \rightarrow \frac{-37}{36}\cdot a2 - \frac{8}{5}\cdot a3 - \frac{101}{54}\cdot a4 + \frac{19}{6}\cdot a0 + \frac{37}{36} = 0$$

$$e(1, a0, a2, a3, a4) = 0 \rightarrow \frac{11}{12}\cdot a2 + \frac{28}{15}\cdot a3 + \frac{17}{6}\cdot a4 + \frac{7}{2}\cdot a0 - \frac{11}{12} = 0$$

$$-a2 - a3 - a4 + 3\cdot a0 + 1 = 0$$

$$\frac{-173}{120}\cdot a3 - \frac{91}{48}\cdot a4 + \frac{13}{4}\cdot a0 - \frac{19}{24}\cdot a2 + \frac{19}{24} = 0$$

$$\frac{-37}{36}\cdot a2 - \frac{8}{5}\cdot a3 - \frac{101}{54}\cdot a4 + \frac{19}{6}\cdot a0 + \frac{37}{36} = 0$$

$$\frac{11}{12}\cdot a2 + \frac{28}{15}\cdot a3 + \frac{17}{6}\cdot a4 + \frac{7}{2}\cdot a0 - \frac{11}{12} = 0$$

$$\mathrm{Find}(a0, a2, a3, a4) \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

<u>Example (3.2):</u>

$$\frac{d}{dt}u(t) + u(t) = \left[\int_0^1 (t + 2\cdot s)\cdot u(s)\,ds\right]^2 + \left[t - 1 + t^2 - \left(\frac{-1}{6} - \frac{1}{6}\cdot t\right)^2\right]$$

with the boundary condition

$u(0) = u(1)$

This example is solved by using the collocation method

The exact solution

$$u(t) := t^2 - t$$

where

$$Ku(t) = \int_0^1 t\cdot s\cdot u(s)\,ds$$

and

$$g(t) := t - 1 + t^2 - \left(\frac{-1}{6} - \frac{1}{6}\cdot t\right)^2$$

$$u(t) := a0 + a1\cdot t + a2\cdot t^2$$

$$u(t, a0, a2) := a0 - a2\,t + a2\cdot t^2$$

$$Ku(t, a0, a2) := \left[\int_0^1 (t + 2\cdot s)\cdot u(s, a0, a2)\,ds\right]^2 \rightarrow \left(\frac{-1}{6}\cdot a2 - \frac{1}{6}\cdot t\cdot a2 + a0 + t\cdot a0\right)^2$$

$$e(t, a0, a2) := \frac{d}{dt}u(t, a0, a2) + u(t, a0, a2) - Ku(t, a0, a2) - g(t)$$

$$e(t,a0,a2) \rightarrow -a2 + t\cdot a2 + a0 + a2\cdot t^2 - \left(\frac{-1}{6}\cdot a2 - \frac{1}{6}\cdot t\cdot a2 + a0 + t\cdot a0\right)^2 - t + 1 - t^2 + \left(\frac{-1}{6} - \frac{1}{6}\cdot t\right)^2$$

Given

$$e(0,a0,a2) = 0 \rightarrow -a2 + a0 - \left(\frac{-1}{6}\cdot a2 + a0\right)^2 + \frac{37}{36} = 0$$

$$e(1,a0,a2) = 0 \rightarrow a2 + a0 - \left(\frac{-1}{3}\cdot a2 + 2\cdot a0\right)^2 - \frac{8}{9} = 0$$

$$-a2 + a0 - \left(\frac{-1}{6}\cdot a2 + a0\right)^2 + \frac{37}{36} = 0$$

$$a2 + a0 - \left(\frac{-1}{3}\cdot a2 + 2\cdot a0\right)^2 - \frac{8}{9} = 0$$

$$\mathrm{Find}(a0,a2) \rightarrow \begin{pmatrix} 0 & \frac{70}{81} \\ 1 & \frac{41}{27} \end{pmatrix}$$

$$e(t,0,1) \rightarrow 0$$

$$e\left(t,\frac{70}{81},\frac{41}{27}\right) \rightarrow \frac{28}{81} + \frac{14}{27}\cdot t + \frac{14}{27}\cdot t^2 - \left(\frac{11}{18} + \frac{11}{18}\cdot t\right)^2 + \left(\frac{-1}{6} - \frac{1}{6}\cdot t\right)^2$$

Example (3.3):

$$\frac{d}{dt}u(t) + 3u(t) = -2Ku(t) + 2\cdot t(t-2\cdot\pi) + t^2 + 3\cdot t^2(t-2\cdot\pi) + 8\pi^2\cdot\sin(t) + 24\cdot\pi\cos(t)$$

$$u(0) = u(2\cdot\pi)$$

with the boundary condition

This example is solved by using the Galerkin's method

The exact solutions

$$u(t) := t^2\cdot(t-2\cdot\pi)$$

where

$$Ku(t) = \int_0^{2\cdot\pi} \sin(t+s)\cdot u(s)\,ds$$

and

$$g(t) := 2\cdot t(t-2\cdot\pi) + t^2 + 3\cdot t^2\cdot(t-2\cdot\pi) + 8\cdot\pi^2\cdot\sin(t) + 24\cdot\pi\cdot\cos(t)$$

$$\frac{d}{dt}u(t) + 3\cdot u(t) + 2\cdot\int_0^{2\cdot\pi} \sin(t+s)\cdot u(s)\,ds \rightarrow 2\cdot t\cdot(t-2\cdot\pi) + t^2 + 3\cdot t^2\cdot(t-2\cdot\pi) + 24\cdot\cos(t)\cdot\pi + 8\cdot\sin(t)\cdot\pi^2$$

Approximate u as a polynomial of degree 3

$$u(t) := a0 + a1\cdot t + a2\cdot t^2 + a3\cdot t^3$$

$$u(t,a0,a2,a3) := a0 - \left(2\cdot\pi a2 + 4\cdot\pi^2 a3\right)\cdot t + a2\cdot t^2 + a3\cdot t^3$$

$$p(t,a0,a2,a3) := -\left(2\cdot\pi a2 + 4\cdot\pi^2 a3\right) + 2\cdot a2\cdot t + 3\cdot a3\cdot t^2$$

$$Ku(t,a0,a2,a3) := \int_0^{2\cdot\pi} \sin(t+s)\cdot u(s,a0,a2,a3)\,ds \rightarrow 4\cdot\pi\cdot a2\cdot\sin(t) + 12\cdot a3\cdot\sin(t)\cdot\pi^2 + 12\cdot a3\cdot\cos(t)\cdot\pi$$

$$e(t,a0,a2,a3) := p(t,a0,a2,a3) + 3\cdot u(t,a0,a2,a3) + 2\cdot Ku(t,a0,a2,a3) - g(t)$$

$$e(t,a0,a2,a3) \rightarrow -2\cdot\pi\cdot a2 - 4\cdot\pi^2\cdot a3 + 2\cdot t\cdot a2 + 3\cdot a3\cdot t^2 + 3\cdot a0 - 3\cdot\left(2\cdot\pi\cdot a2 + 4\cdot\pi^2\cdot a3\right)\cdot t + 3\cdot a2\cdot t^2 + 3\cdot a3\cdot t^3 + 8\cdot\pi\cdot a2\cdot\sin(t) + 24\cdot a3\cdot\sin(t)\cdot\pi^2 + 24\cdot a3\cdot\cos(t)\cdot\pi - 2\cdot t\left(t - 2\cdot\pi\right) - t^2 - 3\cdot t^2\cdot\left(t - 2\cdot\pi\right) - 8\cdot\sin(t)\cdot\pi^2 - 24\cdot\cos(t)\cdot\pi$$

$$e\left(t,0,-2\pi,1\right) \rightarrow -4\cdot\pi\cdot t + 2\cdot t^2 - 6\cdot\pi\cdot t^2 + 3\cdot t^3 - 2\cdot t\left(t - 2\cdot\pi\right) - 3\cdot t^2\cdot\left(t - 2\cdot\pi\right)$$

Given

$$-a2 - a3 + 3\cdot a0 + 1 = 0$$

$$\frac{-173}{120}\cdot a3 + \frac{13}{4}\cdot a0 - \frac{19}{24}\cdot a2 + \frac{19}{24} = 0$$

$$\frac{11}{12}\cdot a2 + \frac{28}{15}\cdot a3 + \frac{7}{2}\cdot a0 - \frac{11}{12} = 0$$

$$\mathrm{Find}(a0,a2,a3) \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Another approximate for u as a polynomial of degree 4

$$u(t) := a0 + a1\cdot t + a2\cdot t^2 + a3\cdot t^3 + a4\cdot t^4$$

$$u(t,a0,a2,a3,a4) := a0 - (a2 + a3 + a4)\cdot t + a2\cdot t^2 + a3\cdot t^3 + a4\cdot t^4$$

$$Ku(t,a0,a2,a3,a4) := \int_0^1 t\cdot s\cdot u(s,a0,a2,a3,a4)\,ds$$

$$g1(t,a0,a1,a2,a3,a4) := Ku(t,a0,a2,a3,a4) - g(t)$$

$$g2(t,a0,a1,a2,a3,a4) := \frac{d}{dt}u(t,a0,a2,a3,a4) + 3\cdot u(t,a0,a2,a3,a4)$$

$$e(t,a0,a2,a3,a4) := g2(t,a0,a1,a2,a3,a4) + g1(t,a0,a1,a2,a3,a4)$$

$$e(t,a0,a2,a3,a4) \rightarrow -a2 - a3 - a4 + \frac{23}{12}\cdot t\cdot a2 + 3\cdot a3\cdot t^2 + 4\cdot a4\cdot t^3 + 3\cdot a0 - 3\cdot(a2 + a3 + a4)\cdot t + 3\cdot a2\cdot t^2 + 3\cdot a3\cdot t^3 + 3\cdot a4\cdot t^4 - \frac{1}{6}\cdot t\cdot a4 - \frac{2}{15}\cdot t\cdot a3 + \frac{1}{2}\cdot t\cdot a0 - 2\cdot t\left(t - 2\cdot\pi\right) - t^2 - 3\cdot t^2\cdot\left(t - 2\cdot\pi\right) - 8\cdot\sin(t)\cdot\pi^2 - 24\cdot\cos(t)\cdot\pi$$

Given

$$e(0,a0,a2,a3,a4) = 0 \rightarrow$$

$$e\left(\frac{1}{2},a0,a2,a3,a4\right) = 0 \rightarrow \frac{-173}{120}\cdot a3 - \frac{91}{48}\cdot a4 + \frac{13}{4}\cdot a0 - \frac{19}{24}\cdot a2 - 2\cdot 1/2\left(\frac{1}{2} - 2\cdot\pi\right) - \frac{5}{8} + \frac{3}{2}\cdot\pi - 8\cdot\pi^2\cdot\sin\left(\frac{1}{2}\right) - 24\cdot\pi\cdot\cos\left(\frac{1}{2}\right) = 0$$

$$e\left(\frac{1}{3},a0,a2,a3,a4\right) = 0 \rightarrow \frac{-37}{36}\cdot a2 - \frac{8}{5}\cdot a3 - \frac{101}{54}\cdot a4 + \frac{19}{6}\cdot a0 - 2\cdot 1/3\left(\frac{1}{3} - 2\cdot\pi\right) - \frac{2}{9} + \frac{2}{3}\cdot\pi - 8\cdot\pi^2\cdot\sin\left(\frac{1}{3}\right) - 24\cdot\pi\cdot\cos\left(\frac{1}{3}\right) = 0$$

$$e(1,a0,a2,a3,a4) = 0 \rightarrow$$

$$-a2 - a3 - a4 + 3\cdot a0 + 1 = 0$$

$$\frac{-173}{120}\cdot a3 - \frac{91}{48}\cdot a4 + \frac{13}{4}\cdot a0 - \frac{19}{24}\cdot a2 + \frac{19}{24} = 0$$

$$\frac{-37}{36}\cdot a2 - \frac{8}{5}\cdot a3 - \frac{101}{54}\cdot a4 + \frac{19}{6}\cdot a0 + \frac{37}{36} = 0$$

$$\frac{11}{12}\cdot a2 + \frac{28}{15}\cdot a3 + \frac{17}{6}\cdot a4 + \frac{7}{2}\cdot a0 - \frac{11}{12} = 0$$

$$\text{Find}(a0,a2,a3,a4) \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

<u>Example (3.4):</u>

$$\frac{d}{dt}u(t) + 6\cdot u(t) = -4\cdot\int_0^2 (t+s)\cdot u(s)\,ds + \left(-8\cdot t^3 - 6\cdot t^2 + 6\cdot t^4 - \frac{128}{15} - \frac{32}{5}\cdot t\right)$$

with the boundary condition

$u(0) = u(2)$

This example is solved by using the Least squares method
The exact solution

$$u(t) := t^3 \cdot (t-2)$$

where

$$Ku(t) = \int_0^2 (t+s) \cdot u(s)\, ds$$

and

$$g(t) := -8 \cdot t^3 - 6 \cdot t^2 + 6 \cdot t^4 - \frac{128}{15} - \frac{32}{5} \cdot t$$

Approximate u as a polynomial of degree 4

$$u(t) := a0 + a1 \cdot t + a2 \cdot t^2 + a3 \cdot t^3 + a4 \cdot t^4$$

$$u(t, a0, a2, a3, a4) := a0 + (-2a2 - 4a3 - 8a4)t + a2 \cdot t^2 + a3 \cdot t^3 + a4 \cdot t^4$$

$$ku(t, a0, a2, a3, a4) := \int_0^2 (t+s) \cdot u(s, a0, a2, a3, a4)\, ds$$

$$k1(t, a0, a2, a3, a4) := -2 \cdot a2 - 4 \cdot a3 - 8 \cdot a4 + 2 \cdot a2 \cdot t + 3 \cdot a3 \cdot t^2 + 4 \cdot a4 \cdot t^3$$

$$k1(t, 0, 0, -2, 1) + 6 \cdot u(t, 0, 0, -2, 1) + 4 \cdot ku(t, 0, 0, -2, 1) - g(t) \rightarrow 0$$

$$e(t, a0, a2, a3, a4) := k1(t, a0, a2, a3, a4) + 6 \cdot u(t, a0, a2, a3, a4) + 4 \cdot ku(t, a0, a2, a3, a4) - g(t)$$

$$e(t, a0, a2, a3, a4) \rightarrow \frac{-22}{3} \cdot a2 - \frac{316}{15} \cdot a3 - \frac{152}{3} \cdot a4 - \frac{10}{3} \cdot t \cdot a2 + 3 \cdot t^2 \cdot a3 + 4 \cdot a4 \cdot t^3 + 14 \cdot a0 + 6 \cdot t \cdot (-2 \cdot a2 - 4 \cdot a3 - 8 \cdot a4) + 6 \cdot a2 \cdot t^2 + 6 \cdot a3 \cdot t^3 + 6 \cdot a4 \cdot t^4 - \frac{192}{5} \cdot t \cdot a4 - 16 \cdot t \cdot a3 + 8 \cdot t \cdot a0 + 8 \cdot t^3 + 6 \cdot t^2 - 6 \cdot t^4 + \frac{128}{15} + \frac{32}{5} \cdot t$$

Here we minimize the functional:

$$f(a0, a2, a3, a4) := \int_0^2 (e(t, a0, a2, a3, a4))^2\, dt$$

$$a0 := 1$$

$a2 := 0$
$a3 := 2$
$a4 := 5$
Given
$P := \mathrm{Minimize}(f, a0, a2, a3, a4)$

$$P = \begin{pmatrix} -7.548 \times 10^{-8} \\ 1.437 \times 10^{-7} \\ -2 \\ 1 \end{pmatrix}$$

Approximate u as a polynomial of degree 5

$$u(t) := a0 + a1 \cdot t + a2 \cdot t^2 + a3 \cdot t^3 + a4 \cdot t^4 + a5 \cdot t^5$$

$$u(t, a0, a2, a3, a4, a5) := a0 + (-2a2 - 4a3 - 8a4 - 16 \cdot a5)t + a2 \cdot t^2 + a3 \cdot t^3 + a4 \cdot t^4 + a5 \cdot t^5$$

$$ku(t, a0, a2, a3, a4, a5) := \int_0^2 (t + s) \cdot u(s, a0, a2, a3, a4, a5)\, ds$$

$$k1(t, a0, a2, a3, a4, a5) := -2 \cdot a2 - 4 \cdot a3 - 8 \cdot a4 - 16 \cdot a5 + 2 \cdot a2 \cdot t + 3 \cdot a3 \cdot t^2 + 4 \cdot a4 \cdot t^3 + 5 \cdot a5 \cdot t^4$$

$$e(t, a0, a2, a3, a4, a5) := k1(t, a0, a2, a3, a4, a5) + 6 \cdot u(t, a0, a2, a3, a4, a5) + 4 \cdot ku(t, a0, a2, a3, a4, a5) - g(t)$$

$$e(t, a0, a2, a3, a4, a5) \rightarrow \frac{-4094}{15} - \frac{2384}{21} \cdot a5 + 12 \cdot t^2 + 40 \cdot t^3 + 5 \cdot a5 \cdot t^4 + 6 \cdot t \cdot (-48 - 16 \cdot a5) + 24 \cdot t^4 + 6 \cdot a5 \cdot t^5 - \frac{256}{3} \cdot t \cdot a5 - \frac{1048}{5} \cdot t$$

Here we minimize the functional:

$$f(a0, a2, a3, a4, a5) := \int_0^2 (e(t, a0, a2, a3, a4, a5))^2\, dt$$

$a0 := 1$

$a2 := 2$
$a3 := .2$
$a4 := 0$
$a5 := 1$
Given
$P := \text{Minimize}(f, a0, a2, a3, a4, a5)$

$$P = \begin{pmatrix} -2.47 \times 10^{-4} \\ -1.087 \times 10^{-3} \\ -1.999 \\ 0.999 \\ 2.947 \times 10^{-4} \end{pmatrix}$$

<u>Example (3.5):</u>

$$\frac{d}{dt}u(t) + u(t) = \left[\int_0^1 (t + 2 \cdot s) \cdot u(s)\, ds\right]^2 + t - 1 + t^2 - \left(\frac{-1}{6} - \frac{1}{6} \cdot t\right)^2$$

with the boundary condition
$u(0) = u(1)$
This example is solved by using the Least squares method
The exact solution
$u(t) := t \cdot (t - 1)$
where

$$Ku(t) = \left[\int_0^1 (t + 2\cdot s)\cdot u(s)\,ds\right]^2$$

and

$$g(t) := t - 1 + t^2 - \left(\frac{-1}{6} - \frac{1}{6}\cdot t\right)^2$$

Approximate u as a polynomial of degree 4

$$u(t) := a0 + a1\cdot t + a2\cdot t^2 + a3\cdot t^3 + a4\cdot t^4$$

$$u(t, a0, a2, a3, a4) := a0 + (-2a2 - 4a3 - 8a4)t + a2\cdot t^2 + a3\cdot t^3 + a4\cdot t^4$$

$$ku(t, a0, a2, a3, a4) := \left[\int_0^1 (t + 2\cdot s)\cdot u(s, a0, a2, a3, a4)\,ds\right]^2$$

$$k1(t, a0, a2, a3, a4) := -a2 - a3 - a4 + 2\cdot a2\cdot t + 3\cdot a3\cdot t^2 + 4\cdot a4\cdot t^3$$

$$k1(t, 0, 1, 0, 0) + u(t, 0, 1, 0, 0) + -ku(t, 0, 1, 0, 0) - g(t) \rightarrow -\left(\frac{-5}{6} - \frac{2}{3}\cdot t\right)^2 - t + \left(\frac{-1}{6} - \frac{1}{6}\cdot t\right)^2$$

$$e(t, a0, a2, a3, a4) := k1(t, a0, a2, a3, a4) + u(t, a0, a2, a3, a4) + -ku(t, a0, a2, a3, a4) - g(t)$$

$$e(t, a0, a2, a3, a4) \rightarrow -a2 - a3 - a4 + 2\cdot t\cdot a2 + 3\cdot t^2\cdot a3 + 4\cdot a4\cdot t^3 + a0 + (-2\cdot a2 - 4\cdot a3 - 8\cdot a4)\cdot t + a2\cdot t^2 + a3\cdot t^3 + a4\cdot t^4 - \left(-5\cdot a4 - \frac{19}{5}\cdot t\cdot a4 - \frac{34}{15}\cdot a3 - \frac{7}{4}\cdot t\cdot a3 - \frac{5}{6}\cdot a2 - \frac{2}{3}\cdot t\cdot a2 + a0 + t\cdot a0\right)^2 - t + 1 - t^2 + \left(\frac{-1}{6} - \frac{1}{6}\cdot t\right)^2$$

Here we minimize the functional:

$$f(a0, a2, a3, a4) := \int_0^1 (e(t, a0, a2, a3, a4))^2\,dt$$

$$a0 := 1$$
$$a2 := 2$$

a3 := 0
a4 := 0
Given
P := Minimize(f , a0 , a2 , a3 , a4)

$$P = \begin{pmatrix} 0.961 \\ 1.922 \\ -0.066 \\ -0.031 \end{pmatrix}$$

Example (3.6):

$$\frac{d^2}{dt^2}u(t) + u(t) = -\int_0^1 (3\cdot t + s)\cdot u(s)\,ds + 8\cdot t\cdot(t-1) + 2\cdot t^2 + 2\cdot(t-1)^2 + t^2\cdot(t-1)^2 + \frac{1}{60} + \frac{1}{10}\cdot t$$

with the boundary condition

$$u(0) = u(1)$$

$$\frac{d}{dt}u(0) = \frac{d}{dt}u(1)$$

This example is solved by using the collocation method
The exact solution

$$u(t) := t^2\cdot(t-1)^2$$

where

$$Ku(t) = \int_0^1 (3\cdot t + s)\cdot s^2 (s-1)^2\,ds \rightarrow Ku(t) = \frac{1}{60} + \frac{1}{10}\cdot t$$

and

$$g(t) := 6\cdot t\cdot(t-1)^2 + 12\cdot t^2\cdot(t-1) + 2\cdot t^3 + 4\cdot t^3\cdot(t-1)^2 + \frac{32}{21} + \frac{256}{105}\cdot t + \frac{16}{15}\cdot t^2$$

Approximate u as a polynomial of degree 4

$$u(t) := a0 + a1\cdot t + a2\cdot t^2 + a3\cdot t^3 + a4\cdot t^4$$

$$u(t, a0, a3, a4) := a0 + \left(-a3 - a4 + \frac{3}{2}a3 + 2a4\right)\cdot t + \left(-\frac{3}{2}a3 - 2a4\right)\cdot t^2 + \left(a3\cdot t^3 + a4\cdot t^4\right)$$

$$Ku(t, a0, a3, a4) := \int_0^1 (3\cdot t + s)\cdot u(s, a0, a3, a4)\, ds$$

$$e(t, a0, a3, a4) := \frac{d^2}{dt^2}u(t, a0, a3, a4) + u(t, a0, a3, a4) + Ku(t, a0, a3, a4) - g(t)$$

$$e(t, a0, a3, a4) \rightarrow \frac{-361}{120}\cdot a3 - 4\cdot a4 + 6\cdot t\cdot a3 + 12\cdot a4\cdot t^2 + \frac{3}{2}\cdot a0 + t\cdot\left(\frac{1}{2}\cdot a3 + a4\right) + \left(\frac{-3}{2}\cdot a3 - 2\cdot a4\right)\cdot t^2 + a3\cdot t^3 + a4\cdot t^4 + \frac{1}{10}\cdot t\cdot a4 + 3\cdot t\cdot a0 - 6\cdot t\cdot(t-1)^2 - 12\cdot t^2\cdot(t-1) - 2\cdot t^3 - 4\cdot t^3\cdot(t-1)^2 - \frac{32}{21} - \frac{256}{105}\cdot t - \frac{16}{15}\cdot t^2$$

Given

$$e(0, a0, a3, a4) = 0 \rightarrow \frac{-361}{120}\cdot a3 - 4\cdot a4 + \frac{3}{2}\cdot a0 - \frac{32}{21} = 0$$

$$e\left(\frac{1}{2}, a0, a3, a4\right) = 0 \rightarrow \frac{-71}{80}\cdot a4 + 3\cdot a0 - \frac{1}{120}\cdot a3 - \frac{2213}{840} = 0$$

$$e(1, a0, a3, a4) = 0 \rightarrow \frac{359}{120}\cdot a3 + \frac{81}{10}\cdot a4 + \frac{9}{2}\cdot a0 - \frac{246}{35} = 0$$

$$\frac{-361}{120}\cdot a3 - 4\cdot a4 + \frac{3}{2}\cdot a0 - \frac{32}{21} = 0$$

$$\frac{-71}{80}\cdot a4 + 3\cdot a0 - \frac{1}{120}\cdot a3 - \frac{2213}{840} = 0$$

$$\frac{359}{120}\cdot a3 + \frac{81}{10}\cdot a4 + \frac{9}{2}\cdot a0 - \frac{246}{35} = 0$$

$$\mathrm{Find}(a0, a3, a4) \rightarrow \begin{pmatrix} \frac{18528577}{17790675} \\ \frac{-866196}{1186045} \\ \frac{394}{705} \end{pmatrix}$$

Example (3.7):

$$\frac{d^2}{dt^2}u(t) + 4u(t) = -\int_0^2 (t+s)^2\cdot u(s)\,ds + 6\cdot t\cdot(t-2)^2 + 12\cdot t^2\cdot(t-2) + 2\cdot t^3 + 4\cdot t^3\cdot(t-2)^2 + \frac{32}{21} + \frac{256}{105}\cdot t + \frac{16}{15}\cdot t^2$$

$$\frac{d}{dt}u(0) = \frac{d}{dt}u(2)$$

with the boundary condition

$$u(0) = u(2)$$

This example is solved by using the Galerkin's method

The exact solution

$$u(t) := t^3\cdot(t-2)^2$$

where

$$Ku(t) = \int_0^2 (t+s)^2 \cdot s^3 (s-2)^2 \, ds \rightarrow Ku(t) = \frac{32}{21} + \frac{256}{105} \cdot t + \frac{16}{15} \cdot t^2$$

and

$$g(t) := 6 \cdot t \cdot (t-2)^2 + 12 \cdot t^2 \cdot (t-2) + 2 \cdot t^3 + 4 \cdot t^3 \cdot (t-2)^2 + \frac{32}{21} + \frac{256}{105} \cdot t + \frac{16}{15} \cdot t^2$$

Another approximation for u as a polynomial of degree 5

$$u(t) := a0 + a1 \cdot t + a2 \cdot t^2 + a3 \cdot t^3 + a4 \cdot t^4 + a5 \cdot t^5$$

$$u(t, a0, a3, a4, a5) := a0 + (2a3 + 8a4 + 24a5) \cdot t - (3 \cdot a3 + 8 \cdot a4 + 20 \cdot a5) \cdot t^2 + a3 \cdot t^3 + a4 \cdot t^4 + a5 \cdot t^5$$

$$Ku(t, a0, a3, a4, a5) := \int_0^2 (t+s)^2 \cdot u(s, a0, a3, a4, a5) \, ds$$

$$e(t, a0, a3, a4, a5) := \frac{d^2}{dt^2} u(t, a0, a3, a4, a5) + 4 u(t, a0, a3, a4, a5) + Ku(t, a0, a3, a4, a5) - g(t)$$

Given

$$\int_0^2 e(t, a0, a3, a4, a5) \, dt = 0 \rightarrow \frac{80}{3} \cdot a0 - \frac{32}{15} \cdot a3 + \frac{1664}{315} \cdot a4 + \frac{2816}{63} \cdot a5 - \frac{4736}{315} = 0$$

$$\int_0^2 e(t, a0, a3, a4, a5) \cdot t^2 \, dt = 0 \rightarrow \frac{-6496}{225} + \frac{5568}{35} \cdot a5 + \frac{18304}{525} \cdot a4 + \frac{104}{45} \cdot a3 + \frac{2096}{45} \cdot a0 = 0$$

$$\int_0^2 e(t, a0, a3, a4, a5) \cdot t^3 \, dt = 0 \rightarrow \frac{-25472}{525} + \frac{88352}{315} \cdot a5 + \frac{19904}{315} \cdot a4 + \frac{2728}{525} \cdot a3 + \frac{368}{5} \cdot a0 = 0$$

$$\int_0^2 e(t,a0,a3,a4,a5)\cdot t^5\,dt = 0 \rightarrow \frac{-3690496}{24255} + \frac{4392704}{4851}\cdot a5 + \frac{7296}{35}\cdot a4 + \frac{704}{35}\cdot a3 + \frac{13120}{63}\cdot a0 = 0$$

$$\frac{-4736}{315} - \frac{15712}{21}\cdot a5 + \frac{80}{3}\cdot a0 - \frac{56}{3}\cdot a3 - \frac{65984}{315}\cdot a4 = 0$$

$$\frac{-6496}{225} - \frac{137408}{105}\cdot a5 - \frac{62592}{175}\cdot a4 - \frac{1048}{45}\cdot a3 + \frac{2096}{45}\cdot a0 = 0$$

$$\frac{-25472}{525} - \frac{637856}{315}\cdot a5 - \frac{172736}{315}\cdot a4 - \frac{15752}{525}\cdot a3 + \frac{368}{5}\cdot a0 = 0$$

$$\frac{-3690496}{24255} - \frac{26831104}{4851}\cdot a5 - \frac{155264}{105}\cdot a4 - \frac{1216}{21}\cdot a3 + \frac{13120}{63}\cdot a0 = 0$$

$$\text{Find}(a0,a3,a4,a5) \rightarrow \begin{pmatrix} 0 \\ 4 \\ -4 \\ 1 \end{pmatrix}$$

<u>Example (3.8):</u>

$$\frac{d^2}{dt^2}u(t) + 4u(t) = -\int_0^2 (t+s)^2\cdot u(s)\,ds + 6\cdot t\cdot(t-2)^2 + 12\cdot t^2\cdot(t-2) + 2\cdot t^3 + 4\cdot t^3\cdot(t-2)^2 + \frac{32}{21} + \frac{256}{105}\cdot t + \frac{16}{15}\cdot t^2$$

with the boundary condition

$$u(0) = u(2)$$

$$\frac{d}{dt}u(0) = \frac{d}{dt}u(2)$$

This example is solved by using the least squares method

The exact solution

$$u(t) := t^3\cdot(t-2)^2$$

where

$$Ku(t) = \int_0^2 (t+s)^2\cdot s^3(s-2)^2\,ds \rightarrow Ku(t) = \frac{32}{21} + \frac{256}{105}\cdot t + \frac{16}{15}\cdot t^2$$

and

$$g(t) := 6\cdot t\cdot(t-2)^2 + 12\cdot t^2\cdot(t-2) + 2\cdot t^3 + 4\cdot t^3\cdot(t-2)^2 + \frac{32}{21} + \frac{256}{105}\cdot t + \frac{16}{15}\cdot t^2$$

Another approximation for u as a polynomial of degree 5

$$u(t) := a0 + a1\cdot t + a2\cdot t^2 + a3\cdot t^3 + a4\cdot t^4 + a5\cdot t^5$$

$$u(t, a0, a3, a4, a5) := a0 + (2a3 + 8a4 + 24a5)\cdot t - (3a3 + 8a4 + 20a5)\cdot t^2 + a3\cdot t^3 + a4\cdot t^4 + a5\cdot t^5$$

$$Ku(t, a0, a3, a4, a5) := \int_0^2 (t+s)^2\cdot u(s, a0, a3, a4, a5)\,ds$$

$$e(t, a0, a3, a4, a5) := \frac{d^2}{dt^2}u(t, a0, a3, a4, a5) + 4u(t, a0, a3, a4, a5) + Ku(t, a0, a3, a4, a5) - g(t)$$

$$f(a0, a3, a4, a5) := \int_0^2 (e(t, a0, a3, a4, a5))^2\,dt$$

$$a0 := 1$$

$a3 := 2$

$a4 := 5$

$a5 := 0$

Given

$P := \text{Minimize}(f, a0, a3, a4, a5)$

$$P = \begin{pmatrix} -5.399 \times 10^{-8} \\ 4 \\ -4 \\ 1 \end{pmatrix}$$

CHAPTER 1

EXISTENCE OF THE EXTREMAL SOLUTIONS FOR THE PERIODIC BOUNDARY VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

Introduction

The periodic boundary value problems of the ordinary differential equations has been widely studied in the last years and has many real life applications in physics, engineering and mathematical biology, [Nieto J., 1991]. There are many authors who study the periodic boundary value problem for the ordinary differential equations, for example,. [Lakshmikantham V. and Leela S , 1983], studied the periodic boundary value problems for the first order ordinary differential equations, [Leela S. and Nieto J., 1988] devoted the second order linear and nonlinear periodic boundary value problems, [Aftabizadeh A., et al., 1990]concerned with the periodic boundary value problems for the third order ordinary differential equations and [Aggarwal R., 1986], studied the periodic boundary value problems for higher order ordinary differential equations

In this chapter, we study periodic boundary value problems which consists of the first, second and third order ordinary differential equations together with periodic boundary conditions. This study include the existence of the extremal solutions of equations.

This chapter consists of three sections.

In section one; we introduce some necessary condition for the existence of the extremal solutions for the first order periodic boundary value problems.

In section two, we devote the existence theorem of the extremal solutions for the second order periodic boundary value problems.

In section three, we discuss the existence of the extremal solutions for the third order periodic boundary value problems.

1.1 Existence of the Extremal Solutions for the Periodic Boundary Value Problems of the First Order Ordinary Differential Equations:

In this section, we give the existence theorem of the extremal solutions for the periodic boundary value problems for the first order ordinary differential equations. For this purpose, we start with some basic mathematical concepts that will be needed later.

Definition (1.1), [Rama M., 1980]:

Let r(t) be any solution of the differential equation defined by:

$u'(t) = f(t, u(t))$

on the interval I, r(t) is said to be the maximal(minimal) solution of the above differential equation if for every solution u(t) of it existing on I, the inequality $u(t) \leq r(t)$($u(t) \geq r(t)$)holds for all $t \in$ I.

Remark (1.1), [Rama M., 1980]:

It easy to check that, the maximal and minimal solutions of the above differential equation are unique.

Next, the following proposition gives some necessary conditions for the existence of solutions for some special types of boundary value problems of the first order linear ordinary differential equations. It appeared in [Nieto J., et al., 2000] without proof. Here we give its proof.

Proposition (1.1):

Consider the boundary value problem for the first order linear ordinary differential equation which consists of the differential equation:

$u'(t) + Mu(t) = \sigma(t),\ t \in J = [0, T]$ ....(1.1a)

together with the following boundary conditions:

$u(0) = u(T) + \lambda$ .... (1.1b)

Then any solution of eq.(1.1) can be written as:

$$u(t) = \int_0^T G(t, s)\sigma(s)\,ds + h_\lambda(t),\ t \in J$$

where, $\lambda \in \mathbb{R}$, $M \in \mathbb{R}\backslash\{0\}$, $\sigma \in C(J)$,

$$G(t, s) = \frac{1}{1-e^{-MT}}\begin{cases} e^{-M(t-s)}, & 0 \le s \le t \le T \\ e^{-M(T+t-s)}, & 0 \le t < s \le T \end{cases}$$

and

$$h_\lambda(t) = \frac{\lambda e^{-Mt}}{1-e^{-MT}}$$

Proof:

Multiply eq.(1.1) by G(t, s) and integrating the resulting differential equation from 0 to T to get:

$$\int_0^T [u'(s) + Mu(s)]G(t,s)\,ds = \int_0^T G(t, s)\sigma(s)\,ds$$

$$\frac{1}{1-e^{-MT}}\Big[\int_0^t u'(s)e^{-M(t-s)}\,ds + \int_0^t Mu(s)e^{-M(t-s)}\,ds +$$

$$\int_t^T u'(s)\, e^{-M(T+t-s)}\,ds + M\int_t^T u(s)\, e^{-M(T+t-s)}\,ds\Big] = \int_0^T G(t, s)\sigma(s)\,ds$$

Thus:

$$\frac{1}{1-e^{-MT}}\left[\int_0^t \frac{d}{ds}\left(ue^{-M(t-s)}\right)ds + \int_t^T \frac{d}{ds}\left(ue^{-M(T+t-s)}\right)ds\right] = \int_0^T G(t, s)\sigma(s)\, ds$$

Therefore:

$$u(t) + \frac{e^{-Mt}}{1-e^{-MT}}[-u(0) + u(T)] = \int_0^T G(t, s)\sigma(s)\, ds$$

But this solution must satisfy the boundary condition given by eq.(1.1b). Thus

$$u(t) - \lambda \frac{e^{-Mt}}{1-e^{-MT}} = \int_0^T G(t, s)\sigma(s)\, ds$$

Hence

$$u(t) = \int_0^T G(t, s)\sigma(s)\, ds + \frac{\lambda e^{-Mt}}{1-e^{-MT}} = \int_0^T G(t, s)\sigma(s)\, ds + h_\lambda(t)$$

is a solution of eq.(1.1). ■

Next, the following proposition shows that the under converse of the previous proposition is true. This proposition appeared in [Nieto J., et al., 2000] without proof; here we give its proof.

<u>Proposition (1.2):</u>

If $u \in C(J)$ satisfies the equation

$$u(t) = \int_0^T G(t, s)\sigma(s)\, ds + h_\lambda(t),\ t \in J,$$

then it is a solution of eq.(1.1).

Proof:

From the definition of G(t, s) and $h_\lambda(t)$, the above equation can be written as:

$$u(t) = \frac{1}{1-e^{-MT}}\left[\int_0^t e^{-M(t-s)}\sigma(s)\,ds + \int_t^T e^{-M(T+t-s)}\sigma(s)\,ds\right] + \frac{\lambda e^{-Mt}}{1-e^{-MT}}$$

then

$$u'(t)=\frac{1}{1-e^{-MT}}\left[-M\int_0^t e^{-M(t-s)}\sigma(s)ds + \sigma(t) - M\int_t^T e^{-M(T+t-s)}\sigma(t)ds - e^{-MT}\sigma(t)\right]$$

$$-\frac{\lambda M e^{-Mt}}{1-e^{-MT}}$$

$$= \sigma(t) - \frac{M}{1-e^{-MT}}\left[\int_0^t e^{-M(t-s)}\sigma(s)\,ds + \int_t^T e^{-M(T+t-s)}\sigma(s)\,ds\right] - \frac{\lambda M e^{-Mt}}{1-e^{-MT}}$$

$$= \sigma(t) - M\left[\int_0^T G(t,s)\sigma(s)\,ds\right] - \frac{\lambda M e^{-Mt}}{1-e^{-MT}}$$

Thus

$$u'(t) + Mu(t) = \sigma(t) - M\int_0^T G(t,s)\sigma(s)\,ds + M\int_0^T G(t,s)\sigma(s)\,ds$$

$$= \sigma(t)$$

which means that

$$u(t) = \int_0^T G(t, s)\sigma(s)\,ds + h_\lambda(t)$$

is a solution of eq.(1.1a).

Moreover,

$$u(0) = \frac{1}{1-e^{-MT}} \int_0^T e^{-M(T-s)} \sigma(s)\, ds + \frac{\lambda}{1-e^{-MT}}$$

and

$$u(T) = \frac{1}{1-e^{-MT}} \int_0^T e^{-M(T-s)} \sigma(s)\, ds + \frac{\lambda e^{-MT}}{1-e^{-MT}}$$

Thus

$$u(T) + \lambda = \frac{1}{1-e^{-MT}} \int_0^T e^{-M(T-s)} \sigma(s)\, ds + \frac{\lambda e^{-MT}}{1-e^{-MT}} + \lambda$$

$$= \frac{1}{1-e^{-MT}} \int_0^T e^{-M(T-s)} \sigma(s)\, ds + \frac{\lambda}{1-e^{-MT}} = u(0)$$

which means that the function u defined by

$$u(t) = \int_0^T G(t, s)\sigma(s)\, ds + h_{\lambda}(t),$$

is a solution of eq.(1.1b). Thus the function u defined above is a solution of eq.(1.1). ■

Next, the following proposition shows that under certain conditions, the solution of eq.(1.1) is nonnegative .

<u>Proposition (1.3):</u>

Consider the boundary value problem given by eq.(1.1), if $M > 0$, $\lambda \geq 0$ and $\sigma(t) \geq 0$, for each $t \in J$. Then eq.(1.1) has a nonnegative unique solution.

<u>Proof:</u>

As seen before, eq.(1.1) has the unique solution

$$u(t) = \int_{0}^{T} G(t, s)\sigma(s)\, ds + h_\lambda(t),\ t \in J$$

where G and h_λ are defined previously.

Since M > 0, then $e^{-MT} < 1$ and hence G(t, s) > 0 for each (t, s) $\in$ J$\times$J. Thus

$$\int_{0}^{T} G(t, s)\sigma(s)\, ds \geq 0$$

On the other hand, since $\lambda \geq 0$ and M > 0, then $h_\lambda(t) \geq 0$. Therefore, $u(t) \geq 0$ for each $t \in$ J. ■

Before we give the existence theorem of the extremal solutions for the periodic boundary value problem of the first order ordinary differential equation, we need the following lemma.

<u>Lemma (1.1), [Lakshmikantham V. and Leela S., 1983]:</u>

Let $m \in C^1[[0, 2\pi], \mathbb{R}]$ and $m'(t) \leq -Mm(t)$, $0 \leq t \leq 2\pi$, where M > 0. Then $m(2\pi) \geq m(0)$, implies that $m(t) \leq 0$, on $[0, 2\pi]$.

Now the following theorem gives necessary conditions to ensure the existence of the extremal solutions for the boundary value problem which consists of the first order ordinary differential equation:

$$u'(t) = f(t, u(t))\ ,\ t \in [0, 2\pi]. \qquad \text{(1.2a)}$$

together with the following periodic boundary condition:

$$u(0) = u(2\pi) \qquad \text{(1.2b)}$$

where $f \in C[[0, 2\pi]\times\mathbb{R}, \mathbb{R}]$.

This theorem appeared in [Lakshmikantham V. and Leela S., 1989]. Here we give the details of its proof

Theorem (1.1), [Lakshmikantham V. and Leela S., 1983]:

Consider the periodic boundary value problem given by eq.(1.2). If the following conditions hold:

(1) There exist $\alpha, \beta \in C^1[[0, 2\pi], \mathbb{R}]$, such that $\alpha(t) \le \beta(t)$ on $[0, 2\pi]$ and

(i) $\alpha' \le f(t, \alpha)$, $t \in (0, 2\pi]$ and

$\alpha(0) \le \alpha(2\pi)$.

(ii) $\beta' \ge f(t, \beta)$, $t \in (0, 2\pi]$ and

$\beta(0) \ge \beta(2\pi)$.

(2) There exists M>0 such that $f(t, u_1) - f(t, u_2) \ge -M(u_1 - u_2)$, $t \in [0, 2\pi]$ for any u_1, u_2 such that $\alpha(t) \le u_2 \le u_1 \le \beta(t)$.

Then there exist monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$, with $\alpha_0 = \alpha$, $\beta_0=\beta$, such that $\lim_{n\to\infty} \alpha_n(t) = p(t)$, $\lim_{n\to\infty} \beta_n(t) = r(t)$ uniformly and monotonically on $[0,2\pi]$ and p, r are the minimal and the maximal solutions of the periodic boundary value problem given by eq.(1.2).

Proof:

For any $\eta \in [\alpha, \beta] = \{\eta \in C[[0, 2\pi], \mathbb{R}], \alpha(t) \le \eta(t) \le \beta(t), t \in [0, 2\pi]\}$, consider the linear periodic boundary value problem which consists of the first order ordinary differential equation

$$u' = G(t,u) \quad \text{.......} (1.3a)$$

together with the following periodic boundary condition

$$u(0) = u(2\pi) \quad \text{.......} (1.3b)$$

where $G(t, u) = f(t, \eta(t)) - M(u(t) - \eta(t))$

By rewriting eq.(1.3) in the form

$$u'(t) + Mu(t) = \sigma(t),\ t \in [0, 2\pi]$$

together with the following periodic boundary condition:

$$u(0) = u(2\pi)$$

where $\sigma(t) = f(t, \eta(t)) + M\eta(t)$, it is easy to see that

$$u(t) = u(0)e^{-Mt} + \int_0^t \sigma(s)e^{-M(t-s)}ds$$

is the solution of eq.(1.3a)

On the other hand,

$$u(0) = u(2\pi)e^{-2\pi M} + \int_0^{2\pi} \sigma(s)e^{-M(2\pi-s)}ds$$

Thus

$$u(0) = \frac{1}{e^{2M\pi} - 1}\int_0^{2\pi} \sigma(s)e^{Ms}ds$$

Therefore

$$u(t) = \frac{1}{e^{2M\pi} - 1}\int_0^{2\pi} \sigma(s)e^{-M(t-s)}ds + \int_0^t \sigma(s)e^{-M(t-s)}ds$$

is the solution of eq. (1.3)

Now, suppose that there exist two solutions u_1, u_2 of eq.(1.3). Then, setting $v(t) = u_1(t) - u_2(t)$, we get:

$$v'(t) = f(t,\eta(t)) - M(u_1(t) - \eta(t)) - f(t,\eta(t)) + M(u_2(t) - \eta(t)) = -Mv(t)$$

and

$$v(2\pi) = u_1(2\pi) - u_2(2\pi) = u_1(0) - u_2(0) = v(0)$$

Therefore, $v(t) = v(0)e^{-Mt}$ is a solution of the above differential equation. But

$$v(0) = v(2\pi),$$

thus

$$v(0) = v(2\pi) = v(0)e^{-2M\pi}$$

Thus

$$v(0)[1 - e^{-2M\pi}] = 0. \text{ But } e^{-2M\pi} \neq 1.$$

Hence $v(0) = 0$ and therefore $v(t) = 0$ is the solution of the above initial value problem. This shows the uniqueness of the solution of eq.(1.3).

For any $\eta \in [\alpha, \beta]$, we define a mapping A by $A\eta = u$, where u is the unique solution of eq.(1.3). We shall show that:

(a) If $\eta \in [\alpha, \beta]$, then $A\eta \in [\alpha, \beta]$.

(b) A is a monotone non-decreasing on $[\alpha, \beta]$.

To show (a), first we prove $\alpha \leq A\eta$. To do this we consider $v_1 = \alpha - A\eta$. Thus

$$v_1 = \alpha - u_1,$$

where u_1 is the unique solution of eq.(1.3). Hence

$$v_1'(t) = \alpha'(t) - u_1'(t) = \alpha'(t) - f(t, \eta(t)) + M(u_1(t) - \eta(t))$$

But from the hypothesis, $\alpha'(t) \leq f(t,\alpha(t))$

hence

$$v'_1(t) \leq f(t, \alpha(t)) - f(t, \eta(t)) + M(u_1(t) - \eta(t))$$

from the condition (2) and since $\alpha(t)\le \eta(t) \le\beta(t)$, one can get

$$v_1'(t) \le M(\eta(t)-\alpha(t)) + M(u_1(t)-\eta(t)) = -M(\alpha(t)-u_1(t)) = -Mv_1(t)$$

Also $v_1(0) = \alpha(0)-u_1(0) = \alpha(0)-u_1(2\pi)$.

Since $\alpha(0)\le\alpha(2\pi)$, thus

$$v_1(0) \le \alpha(2\pi) - u_1(2\pi) = v_1(2\pi)$$

by using lemma (1.1), one can get $v_1(t) \le 0$ on $[0, 2\pi]$. Therefore $\alpha \le A\eta$.

Second we prove $\beta \ge A\eta$. To do this we consider $v_2 = A\eta - \beta$. Thus

$$v_2 = u_2 - \beta$$

where u_2 is the unique solution of eq.(1.3).Thus

$$v_2'(t) = u_2'(t) - \beta'(t) = f(t,\eta(t)) - M(u_2(t)-\eta(t)) - \beta'(t)$$

But from the hypothesis, $\beta'(t) \ge f(t,\beta(t))$

hence

$$v_2'(t) \le f(t,\eta(t)) - f(t,\beta(t)) + M(u_2(t)-\eta(t)).$$

from the condition (2) and since $\alpha(t) \le \eta(t) \le \beta(t)$, one can get

$$v_2'(t) \le M(\beta(t)-\eta(t)) + M(u_2(t)-\eta(t)) = -M(u_2(t)-\beta(t)) = -M(v_2(t))$$

Also $v_2(0) \le u_2(0) - \beta(0) = u_2(2\pi) - \beta(0)$.

Since $\beta(0) \ge \beta(2\pi)$, thus

$$v_2(0) = u_2(2\pi) - \beta(2\pi) = v_2(2\pi)$$

and by using lemma (1.1), one can get $v_2(t) \le 0$ on $[0,2\pi]$. Therefore $\beta \ge A\eta$.

In order to prove (b), let $\eta_1, \eta_2 \in [\alpha, \beta]$, such that $\eta_1 \le \eta_2$, consider

$$v_3 = A\eta_1 - A\eta_2.$$

Thus

$$v_3(t) = u_1(t) - u_2(t)$$

where u_1 and u_2 are the unique solutions of eq.(1.3) with respect to η_1 and η_2 respectively.

Hence

$$v_3'(t) = u_1'(t) - u_2'(t)$$
$$= f(t, \eta_1(t)) - M(u_1(t) - \eta_1(t)) - f(t, \eta_2(t)) + M(u_2(t) - \eta_2(t))$$

from the condition (2) and since $\alpha(t) \leq \eta(t) \leq \beta(t)$, one can get

$$v_3'(t) \leq M(\eta_2(t) - \eta_1(t)) - M(u_1(t) - u_2(t)) - M(\eta_2(t) - \eta_1(t)) = -Mv_3(t)$$

Also

$$v_3(0) = u_1(0) - u_2(0) = u_1(2\pi) - u_2(2\pi) = v_3(2\pi)$$

Hence, by using lemma (1.1), one can get $v_1(t) \leq 0$ on $[0, 2\pi]$. Therefore

$A\eta_1 \leq A\eta_2$.

It therefore follows that we can define the sequence $\{\alpha_n\}, \{\beta_n\}$ with $\alpha_0 = \alpha$ $\beta_0 = \beta$ such that

$$\alpha_n = A\alpha_{n-1}, \quad n \in N$$

$$\beta_n = A\beta_{n-1}, \quad n \in N$$

we shall show that:

$$\alpha \leq \alpha_n \leq \alpha_{n+1} \leq \beta, \ n \in \mathbb{N} \quad \text{...(1.4a)}$$

To do this, we use the mathematical induction.

For n = 1, we must prove

$$\alpha \leq \alpha_1 \leq \alpha_2 \leq \beta.$$

Since $\alpha \leq \alpha_0 \leq \beta$, then from the part (a), one can get $\alpha \leq A\alpha_0 \leq \beta$, that is $\alpha \leq \alpha_1 \leq \beta$. Also from part (a) one can have $\alpha \leq A\alpha_1 \leq \beta$, that is $\alpha \leq \alpha_2 \leq \beta$.

Since, $\alpha \le \alpha \le \alpha_1 \le \beta$, then from the part (b) one can obtain $\alpha \le A\alpha \le A\alpha_1 \le \beta$, that is $\alpha \le \alpha_1 \le \alpha_2 \le \beta$. Therefore, ineq. (1.4a) holds for $n = 1$.

Assume ineq.(1.4a), holds for n = k, we must prove this inequality holds for n = k + 1. Since $\alpha_{k+1} = A\alpha_k$ and $\alpha_k \in [\alpha, \beta]$, then by using the property (a), we obtain that $\alpha_{k+1} \in [\alpha, \beta]$. Again, by using the property (a), one can get $\alpha_{k+2} = A\alpha_{k+1} \in [\alpha, \beta]$. But $\alpha_k \le \alpha_{k+1}$, hence, by using the property (b), one can have $\alpha_{k+1} = A\alpha_k \le A\alpha_{k+1} = \alpha_{k+1}$. Therefore, $\alpha \le \alpha_{k+1} \le \alpha_{k+2} \le \beta$ and hence ineq.(1.4) holds for all $n \in \mathbb{N}$.

Moreover, we shall show that:

$\alpha \le \beta_{n+1} \le \beta_n \le \beta$, $n \in \mathbb{N}$ .... (1.4b)

To do this, we use the mathematical induction.

For n = 1, we can prove

$\alpha \le \beta_2 \le \beta_1 \le \beta$.

Since $\alpha \le \beta_0 \le \beta$, then from the part (a), one can get $\alpha \le A\beta_0 \le \beta$, that is $\alpha \le \beta_1 \le \beta$. Again by using part (a) one can have $\alpha \le A\beta_1 \le \beta$, that is $\alpha \le \beta_2 \le \beta$. Since, then $\alpha \le \beta_1 \le \beta \le \beta$, then from the part (b) one can obtain $\alpha \le A\beta_1 \le A\beta \le \beta$, that is $\alpha \le \beta_2 \le \beta_1 \le \beta$. Therefore, ineq.(1.4b) holds for $n = 1$.

Assume ineq.(1.4b), holds for n = k, we must prove this inequality holds for n = k + 1. Since, $\beta_{k+1} = A\beta_k$ and $\beta_k \in [\alpha, \beta]$, then by using the property (a), we obtain $\beta_{k+1} \in [\alpha, \beta]$. Again, by using the property (a), one can get $\beta_{k+2} = A\beta_{k+1} \in [\alpha, \beta]$. But $\beta_{k+1} \le \beta_k$, hence, by using the property (b), one can have $\beta_{k+2} = A\beta_{k+1} \leq A\beta_k = \beta_{k+1}$. Therefore, $\alpha \leq \beta_{k+2} \leq \beta_{k+1} \leq \beta$ and hence ineq.(1.4b) holds for all $n \in \mathbb{N}$.

Now we shall show that, $\alpha_n \leq \beta_n$ for $n \in \mathbb{N}$. To do this, we also use the mathematical induction.

For n=1, since $\alpha \leq \alpha_0 \leq \beta_0 \leq \beta$, then from the part (b), one can have

$\alpha \leq A\alpha_0 \leq A\beta_0 \leq \beta$, that is $\alpha \leq \alpha_1 \leq \beta_1 \leq \beta$.

Assume $\alpha_k \leq \beta_k$, we must prove $\alpha_{k+1} \leq \beta_{k+1}$ since $\alpha \leq \alpha_k \leq \beta_k \leq \beta$, then from the part (b), one can obtain $\alpha \leq A\alpha_k \leq A\beta_k \leq \beta$, that is $\alpha \leq \alpha_{k+1} \leq \beta_{k+1} \leq \beta$. Therefore $\alpha_n \leq \beta_n$ for all $n \in N$.

Therefore $\lim_{n\to\infty} \alpha_n(t) = p(t)$ and $\lim_{n\to\infty} \beta_n(t) = r(t)$ uniformly and monotonically on $[0, 2\pi]$. We will show that p and r are solutions of eq.(1.2)

Since $\alpha_n = A\alpha_{n-1}$, then

$$\begin{aligned}\alpha'_n(t) &= G(t,\alpha_n(t)) \\ &= f(t,\alpha_{n-1}(t)) - M(\alpha_n(t) - \alpha_{n-1}(t))\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$ of the both sides of the above equation, we have:

$$\lim_{n\to\infty} \alpha'_n(t) = \lim_{n\to\infty} f(t,\alpha_{n-1}(t)) - \lim_{n\to\infty} M(\alpha_n(t) - \alpha_{n-1}(t))$$

Thus

$$p'(t) = f(t, \lim_{n\to\infty} \alpha_{n-1}(t)) - M\left[\lim_{n\to\infty} \alpha_n(t) - \lim_{n\to\infty} \alpha_{n-1}(t)\right]$$

$$= f(t,p(t)) - M\left[p(t) - p(t)\right] = f(t,p(t))$$

and hence p is a solution of eq.(1.2a).

Also, since

$$\alpha_n(0) = \alpha_n(2\pi)$$

and by taking the limit as $n \longrightarrow \infty$, one can get

$$p(0) = p(2\pi)$$

Thus p is a solution of eq.(1.2).

Moreover, since $\beta_n = A\beta_{n-1}$, then

$$\begin{aligned}\beta_n'(t) &= G(t, \beta_n(t)) \\ &= f(t, \beta_{n-1}(t)) - M(\beta_n(t) - \beta_{n-1}(t))\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$ of the both sides of the above equation, we have

$$\lim_{n\to\infty} \beta_n'(t) = f(t, \lim_{n\to\infty} \beta_{n-1}(t)) - M(\lim_{n\to\infty} \beta_n(t) - \lim_{n\to\infty} \beta_{n-1}(t))$$

Thus

$$r'(t) = f(t, r(t))$$

Also $r(0) = r(2\pi)$, hence r is a solution of eq.(1.2).

To prove that p, r are the minimal and maximal solutions of eq.(1.2), we have to show that if u is any solution of eq.(1.2), such that $u \in [\alpha, \beta]$ on $[0, 2\pi]$, then $\alpha \le p \le u \le r \le \beta$ on $[0, 2\pi]$.

To do this, we shall show that, for any solution with $\alpha \le u \le \beta$, we have $\alpha_n \le u \le \beta_n$ on $[0, 2\pi]$. This can be proved by using the mathematical induction.

For n = 1, since $\alpha \le \alpha \le u \le \beta$, then from the part (b), one can have $\alpha \le A\alpha \le Au \le \beta$. Also, since $\alpha \le u \le \beta \le \beta$ then from the part (b), one can have $\alpha \le Au \le A\beta \le \beta$. Thus $\alpha_1 \le Au \le \beta_1$. But u is a solution of eq.(1.2), thus

$$u'(t) = f(t, u(t)) - M(u(t) - u(t))$$

and hence $Au = u$. Therefore $\alpha_1 \le u \le \beta_1$. Assume $\alpha_k \le u \le \beta_k$, we must prove $\alpha_{k+1} \le u \le \beta_{k+1}$.

Since $\alpha_k \le u \le \beta_k$, then $A\alpha_k \le Au \le A\beta_k$. But $Au = u$, $A\alpha_k = A\alpha_{k+1}$ and $A\beta_k = \beta_{k+1}$, hence $\alpha_{k+1} \le u \le \beta_{k+1}$. Therefore $\alpha_n \le u \le \beta_n$ on $[0,2\pi]$ hence $\alpha \le p \le u \le r \le \beta$ on $[0,2\pi]$. ■

Remark (1.2):

From theorem(1.1), one can deduce that eq.(1.2) has at least two solutions .

Corollary (1.1):

Consider the periodic boundary value problem which consists of the first order linear ordinary differential equation

$$u'(t) + a(t)u(t) = b(t), \quad t \in [0,2\pi]$$

together with the periodic boundary condition:

$$u(0) = u(2\pi)$$

If the condition (1) in the previous theorem holds and $a(t) \ge -M$ for some $M > 0$, then the same previous result holds.

1.2 Existence of the Extremal Solutions for the Periodic Boundary Value Problems of the Second Order Ordinary Differential Equations:

In this section, we give the existence theorem of the extremal solutions for the periodic boundary value problems of the second order ordinary differential equations.

For this purpose, we need the following lemma.

Lemma (1.2), [Lakshminkantham V. and Leela, S., 1984]:

Let $m \in C^2[[0, 2\pi], \mathbb{R}]$, and $-m''(t) \le -M^2 m(t)$, $t \in [0, 2\pi]$ for some M. Then $m(0) = m(2\pi)$ and $m'(0) \ge m'(2\pi)$ implies that $m(t) \le 0$ on $[0, 2\pi]$.

Now the following theorem gives necessary conditions to ensure the existence of the extremal solutions for the periodic boundary value problem which consists of the second order ordinary differential equation:

$$-u''(t) = f(t, u(t)) \qquad \text{.......(1.5a)}$$

together with the following periodic boundary conditions:

$$\left.\begin{aligned} u(0) &= u(2\pi) \\ u'(0) &= u'(2\pi) \end{aligned}\right\} \qquad \text{.......(1.5b)}$$

where $f \in C[[0, 2\pi]\times\mathbb{R}, \mathbb{R}]$.

Theorem (1.2), [Laskshminkantham V. and Leela, S., 1984]:

Consider the periodic boundary value problem given by eq.(1.5). If the following conditions hold:

(1) There exist $\alpha, \beta \in C^2[[0, 2\pi], \mathbb{R}]$, such that $\alpha(t) \le \beta(t)$ on $[0, 2\pi]$ and

(i) $-\alpha'' \le f(t, \alpha)$, $t \in [0, 2\pi]$,

$\alpha(0) = \alpha(2\pi)$ and $\alpha'(0) \ge \alpha'(2\pi)$.

(ii) $-\beta'' \ge f(t, \beta)$, $t \in [0, 2\pi]$,

$\beta(0) = \beta(2\pi)$ and $\beta'(0) \le \beta'(2\pi)$.

(2) There exists $M > 0$ such that $f(t, u_1) - f(t, u_2) \ge -M^2 (u_1 - u_2)$, $t \in [0, 2\pi]$

for any u_1, u_2 such that $\alpha(t) \le u_2 \le u_1 \le \beta(t)$.

Then there exist monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$, with $\alpha_0 = \alpha$, $\beta_0=\beta$, such that $\lim\limits_{n\to\infty} \alpha_n(t) = p(t)$, $\lim\limits_{n\to\infty} \beta_n(t) = r(t)$ uniformly and monotonically on $[0,2\pi]$ and p, r are the minimal and the maximal solutions of the periodic boundary value problem given by eq.(1.5).

Proof:

For any $\eta\in[\alpha, \beta] = \{\eta\in C[[0, 2\pi], \square\ ], \alpha(t) \le \eta(t) \le \beta(t), t\in[0, 2\pi]\}$, consider the linear periodic boundary value problem which consists of the second order ordinary differential equation

$$-u'' = G(t,u) \quad \text{.......(1.6a)}$$

together with the following periodic boundary conditions:

$$\left.\begin{array}{l} u(0)=u(2\pi) \\ u'(0)=u'(2\pi) \end{array}\right\} \quad \text{.......(1.6b)}$$

where $G(t, u) = f(t, \eta(t)) - M^2(u(t) - \eta(t))$

By rewriting eq.(1.6a) in the form

$$-u''+M^2u = \sigma(t),\ t \in [0, 2\pi]$$

where

$$\sigma(t) = f(t, \eta(t)) + M^2\eta(t),$$

the solution u(t) of eq.(1.6) is given by:

$$u(t) = c_1e^{Mt} + c_2e^{-Mt} - \frac{e^{Mt}}{2M}\int_0^t \sigma(s)e^{-Ms}ds + \frac{e^{-Mt}}{2M}\int_0^t \sigma(s)e^{-Ms}ds$$

where

$$c_1 = \frac{1}{2M(e^{2M\pi}-1)}\int_0^{2\pi} \sigma(s)e^{(2M\pi-s)}ds$$

and

$$c_2 = \frac{1}{2M(e^{2M\pi}-1)} \int_0^{2\pi} \sigma(s)e^{Ms}ds$$

Now, suppose that there exist two solutions u_1, u_2 of eq.(1.6). Then, setting $v(t) = u_1(t) - u_2(t)$, we get:

$$-v''(t) = f(t,\eta(t)) - M^2(u_1-\eta(t)) - f(t,\eta(t)) + M^2(u_2-\eta(t)) = -M^2v(t)$$

and

$$v(2\pi) = u_1(2\pi) - u_2(2\pi) = u_1(0) - u_2(0) = v_1(0)$$

$$v'(2\pi) = u_1'(2\pi) - u_2'(2\pi) = u_1'(0) - u_2'(0) = v_1'(0)$$

The solution of the differential equation $-v''(t) = -M^2v(t)$ is

$$v(t) = c_1e^{Mt} + c_2e^{-Mt}$$

But $v(0) = v(2\pi)$, thus

$$c_1 + c_2 = c_1e^{2M\pi} + c_2e^{-2M\pi}$$

and since $v'(0) = v'(2\pi)$, then

$$Mc_1 - Mc_2 = c_1Me^{2M\pi} - c_2Me^{-2M\pi}$$

Therefore

$$2Mc_1 = 2Mc_1e^{2M\pi}$$

But $e^{2M\pi} \neq 1$, thus $c_1 = 0$. Hence $c_2 = c_2e^{-2M\pi}$ and since $e^{-2M\pi} \neq 1$, then $c_2 = 0$. Therefore $v(t) = 0$ and hence $u_1(t) = u_2(t)$. This shows the uniqueness of the solution of eq.(1.6).

For any $\eta \in [\alpha, \beta]$, we define a mapping A by $A\eta = u$, where u is the unique solution of eq.(1.6). We shall show that

(a) If $\eta \in [\alpha, \beta]$, then $A\eta \in [\alpha, \beta]$.

(b) A is a monotone non-decreasing on $[\alpha, \beta]$.

To show (a), first we prove $\alpha \le A\eta$. To do this we consider $v_1 = \alpha - A\eta$. Thus

$$v_1 = \alpha - u_1$$

where u_1 is the unique solution of eq.(1.6). Hence

$$-v_1''(t) = -\alpha''(t) + u_1''(t) = -\alpha''(t) - f(t,\eta(t)) + M^2(u_1(t) - \eta(t))$$

But from the hypothesis, $-\alpha''(t) \le f(t,\alpha(t))$, hence

$$-v_1''(t) \le f(t,\alpha(t)) - f(t,\eta(t)) + M^2(u_1(t) - \eta(t))$$

from the condition (2) and since $\alpha(t) \le \eta(t) \le \beta(t)$, one can get

$$-v_1''(t) \le M^2(\eta(t) - \alpha(t)) + M^2(u_1(t) - \eta(t)) = -M^2(\alpha(t) - u_1(t)) = -M^2 v_1(t).$$

Also

$$v_1(0) = \alpha(0) - u_1(2\pi) = \alpha(2\pi) - u_1(2\pi) = v_1(2\pi), \text{ and}$$

$$v_1'(0) = \alpha'(0) - u_1'(0) \ge \alpha'(2\pi) - u_1'(2\pi) = v_1'(2\pi)$$

by using lemma (1.2), one can get $v_1(t) \le 0$ on $[0, 2\pi]$. Therefore $\alpha \le A\eta$.

Second we prove $\beta \ge A\eta$. To do this we consider $v_2 = A\eta - \beta$.Thus

$$v_2 = u_2 - \beta$$

where u_2 is the unique solution of eq.(1.6).Thus

$$-v_2''(t) = -u_2''(t) + \beta''(t) = f(t,\eta(t)) - M^2(u_2(t) - \eta(t)) + \beta''(t)$$

But from the hypothesis, $-\beta''(t) \ge f(t,\beta(t))$, hence

$$-v_2''(t) \le f(t,\eta(t)) - f(t,\beta(t)) - M^2(u_2(t) - \eta(t))$$

from the condition (2) and since $\alpha(t) \leq \eta(t) \leq \beta(t)$,one can get

$$-v_2''(t) \leq M^2(\beta(t)-\eta(t)) - M^2(u_2(t)-\eta(t)) = -M^2(u_2(t)-\beta(t)) = -Mv_2(t)$$

Also

$$v_2(0) = u_2(0) - \beta(0) = u_2(2\pi) - \beta(2\pi) = v_2(2\pi) \text{, and}$$

$$v_2'(0) = u_2'(0) - \beta'(0) \geq u_2'(2\pi) - \beta'(2\pi) = v_2'(2\pi).$$

and by using lemma (1.2), one can get $v_2(t) \leq 0$ on $[0,2\pi]$. Therefore $\beta \geq A\eta$. In order to prove (b), let $\eta_1, \eta_2 \in [\alpha, \beta]$, such that $\eta_1 \leq \eta_2$, consider

$$v_3 = A\eta_1 - A\eta_2$$

Thus

$$v_3(t) = u_1(t) - u_2(t)$$

where u_1 and u_2 are the unique solutions of eq.(1.6) with respect to η_1 and η_2 respectively. Hence

$$-v_3''(t) = -u_1''(t) + u_2''(t) = f(t,\eta_1(t)) - M^2(u_1(t)-\eta_1(t)) - f(t,\eta_2(t)) + M^2(u_2(t)-\eta_2(t))$$

from the condition (2)and since $\alpha(t) \leq \eta(t) \leq \beta(t)$, one can get

$$-v_3''(t) \leq M^2(\eta_2(t)-\eta_1(t)) - M^2(u_1(t)-u_2(t)) - M^2(\eta_2(t)-\eta_1(t)) = -M^2 v_3(t)$$

Also

$$v_3(0) = u_1(0) - u_2(0) = u_1(2\pi) - u_2(2\pi) = v_3(2\pi)$$

$$v_3'(0) = u_1'(0) - u_2'(0) = u_1'(2\pi) - u_2'(2\pi) = v_3'(2\pi)$$

Hence, by using lemma (1.2), one can get $v_3(t) \leq 0$ on $[0, 2\pi]$. Therefore

$$A\eta_1 \leq A\eta_2.$$

It therefore follows that we can define the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ as in theorem (1.1) and obtain $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq \beta_n \leq \ldots \leq \beta_1 \leq \beta_0$ on $[0, 2\pi]$.

Therefore $\lim_{n\to\infty} \alpha_n(t) = p(t)$ and $\lim_{n\to\infty} \beta_n(t) = r(t)$, uniformly and monotonically on $[0, 2\pi]$

We will show that p and r are solutions of eq.(1.5) . Since $\alpha_n = A\alpha_{n-1}$, then $-\alpha''_n = f(t, \alpha_{n-1}) - M^2(\alpha_n - \alpha_{n-1})$,

Taking the limit as $n \longrightarrow \infty$ of the both sides of the above equation, we have:

$$-p''(t)= f(t, p(t)) - M^2(p(t) - p(t)) = f(t, p(t))$$

Also

$$p(0) = \lim_{n\to\infty} \alpha_n(0) = \lim_{n\to\infty} \alpha_n(2\pi) = p(2\pi) \text{ ,and}$$

$$p'(0) = \lim_{n\to\infty} \alpha'_n(0) = \lim_{n\to\infty} \alpha'_n(2\pi) = p'(2\pi)$$

Similarly,

$$-\beta''_n = f(t, \beta_{n-1}) - M^2(\beta_n - \beta_{n-1}),$$

Taking the limit as $n \longrightarrow \infty$ of the both sides of the above equation, we have:

$$\begin{aligned} -r''(t) &= f(t, r(t)) - M^2(r(t) - r(t)) \\ &= f(t, r(t)) \end{aligned}$$

Also:

$$r(0) = \lim_{n\to\infty} \beta_n(0) = \lim_{n\to\infty} \beta_n(2\pi) = r(2\pi), \text{ and}$$

$$r'(0) = \lim_{n\to\infty} \beta'_n(0) = \lim_{n\to\infty} \beta'_n(2\pi) = r'(2\pi)$$

Hence p and r are solutions of eq.(1.5).

The proof that p, r are minimal and maximal solutions of eq.(1.5) is similar to that in theorem (1.1). ■

1.3 Existence of the Extremal Solutions for the Periodic Boundary Value Problems of the Third Order Ordinary Differential Equations:

In this section, we give some basic theorems that are necessary for establishing the existence of the extremal solutions for the periodic boundary value problem of the third order ordinary differential equations. For this purpose, we need the following lemma.

Lemma (1.3), [Nieto J., 1991]:

Let $q \in C^3[[0, 2\pi], \mathbb{R}]$ and $q'''(t) - Mq''(t) - Mq'(t) + M^2 q(t) \geq 0$ where $M > 0$. Then $q(0) = q(2\pi)$, $q'(0) \leq q'(2\pi)$ and $q''(0) \geq q''(2\pi)$. Implies that $q(t) \geq 0$ on $[0,2\pi]$.

Next, the following theorem gives the necessary conditions to ensure the existence of the extremal solutions for periodic boundary value problem which consists of the third order ordinary differential equation:

$u'''(t) - mu''(t) - mu'(t) = f(t, u(t)),\ t \in [0, 2\pi]$ ....(1.7a)

together with the following periodic boundary conditions

$$\left.\begin{aligned} u(0) &= u(2\pi) \\ u'(0) &= u'(2\pi) \\ u''(0) &= u''(2\pi) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1.7b)}$$

where $m > 0$ and $f \in C[[0, 2\pi] \times \mathbb{R}, \mathbb{R}]$

This theorem is a modification of the theorem that appeared in [Nieto J., 1991]. To the best of our knowledge this theorem seems to be new.

Theorem (1.3):

Consider the periodic boundary value problem given by eq.(1.7). If the following conditions hold:

(1) There exists $\alpha, \beta \in C^3[[0, 2\pi], \mathbb{R}]$, such that $\alpha(t) \leq \beta(t)$ on $[0, 2\pi]$ and

(i) $\alpha'''(t) - m\alpha''(t) - m\alpha'(t) \leq f(t, \alpha(t)),\ t \in (0, 2\pi]$

$\alpha(0) = \alpha(2\pi)$, $\alpha'(0) \geq \alpha'(2\pi)$ and $\alpha''(0) \leq \alpha''(2\pi)$.

(ii) $\beta'''(t) - m\beta''(t) - m\beta'(t) \geq f(t, \beta(t)),\ t \in (0, 2\pi]$,

$\beta(0) = \beta(2\pi)$, $\beta'(0) \leq \beta'(2\pi)$ and $\beta''(0) \geq \beta''(2\pi)$

(2) f satisfy the following inequality

$$f(t,u_1) - f(t,u_2) \geq -m^2(u_1 - u_2), \quad t \in [0,2\pi]$$

for any u_1, u_2 such that $\alpha(t) \leq u_2 \leq u_1 \leq \beta(t)$.

Then there exist monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ with $\alpha_0=\alpha$, $\beta_0=\beta$ such that $\lim\limits_{n\to\infty} \alpha_n(t) = p(t)$, $\lim\limits_{n\to\infty} \beta_n(t) = r(t)$ uniformly and monotonically on $[0, 2\pi]$ and that p, r are the minimal and the maximal solutions of the periodic boundary value problem given by eq.(1.7).

Proof:

For any $\eta \in [\alpha, \beta] = \{\eta \in C[[0, 2\pi], \mathbb{R}];\ \alpha(t) \leq \eta \leq \beta(t),\ t \in [0, 2\pi]\}$, consider the linear periodic boundary value problem which consists of the third order ordinary differential equation:

$$u'''(t) - mu''(t) - mu'(t) + m^2u(t) = f(t,\eta(t)) + m^2\eta(t) \quad \text{.............(1.8a)}$$

together with the following periodic boundary conditions:

$$\left.\begin{aligned} u(0) &= u(2\pi) \\ u'(0) &= u'(2\pi) \\ u''(0) &= u''(2\pi) \end{aligned}\right\} \quad \text{.. (1.8b)}$$

since $m > 0$, then the above periodic boundary value problem has a unique solution, [Nieto J., 1991].

For any $\eta \in [\alpha, \beta]$ define a mapping A by $A\eta = u$, where u is the unique solution of the above periodic boundary value problem. We shall show that:

(a) If $\eta \in [\alpha, \beta]$, then $A\eta \in [\alpha, \beta]$.

(b) A is a monotone non-decreasing on $[\alpha, \beta]$.

To prove (a), first we prove $\alpha \le A\eta$. To do this we consider $v_1 = A\eta - \alpha$. Thus

$$v_1 = u - \alpha$$

where u is the unique solution of eq.(1.8). Hence

$$\begin{aligned} v_1'''(t) - mv_1''(t) - mv_1'(t) + m^2 v_1(t) &= u_1'''(t) - \alpha'''(t) - mu_1''(t) + m\alpha''(t) - mu_1'(t) \\ &\quad + m\alpha'(t) + m^2 u_1(t) - m^2\alpha(t) \\ &= f(t,\eta(t)) + m^2\eta(t) - (\alpha'''(t) - m\alpha''(t) - m\alpha'(t)) - m^2\alpha(t) \\ &\ge f(t,\eta(t)) + m^2\eta(t) - m^2\alpha(t) - f(t,\alpha(t)) \\ &\ge -m^2(\eta(t) - \alpha(t)) + m^2(\eta(t) - \alpha(t)) = 0 \end{aligned}$$

Also

$$v_1(0) = u_1(0) - \alpha(0) = u_1(2\pi) - \alpha(2\pi) = v_1(2\pi),$$

$$v_1'(0) = u_1'(0) - \alpha'(0) = u_1'(2\pi) - \alpha'(0) \le u_1'(2\pi) - \alpha'(2\pi) = v_1'(2\pi),$$

and

$$v_1''(0) = u_1''(0) - \alpha''(0) = u_1''(2\pi) - \alpha''(0) \ge u_1''(2\pi) - \alpha''(2\pi) = v_1''(2\pi).$$

Then by using lemma (1.3), one can get $v_1(t) \ge 0$ on $[0, 2\pi]$. Therefore $\alpha \le A\eta$.

Second we prove $\beta \ge A\eta$. To do this we consider $v_2 = \beta - A\eta$. Thus

$$v_2 = \beta - u_2$$

where u_2 is the unique solution of eq.(1.8). Thus

$$v_2'''(t) - mv_2''(t) - mv_1'(t) + m^2 v_1(t) = \beta'''(t) - u_2'''(t) - m\beta''(t) + mu_2''(t) - m\beta'(t) + mu_2'(t) + m^2\beta(t) - m^2 u_2(t)$$

$$= \beta'''(t) - m\beta''(t) - m\beta'(t) + m^2\beta(t) - f(t,\eta(t)) - m^2\eta(t)$$

$$\geq f(t,\beta(t)) + m^2\beta(t) - f(t,\eta(t)) - m^2\eta(t)$$

$$\geq -m^2(\beta(t) - \eta(t)) + m^2(\beta(t) - \eta(t)) = 0$$

Also

$$v_2(0) = \beta(0) - u_2(0) = \beta(2\pi) - u_2(2\pi) = v_2(2\pi),$$

$$v_2'(0) = \beta'(0) - u_2'(0) = \beta'(0) - u_2'(2\pi) \leq \beta'(2\pi) - u_2'(2\pi) = v_2'(2\pi),$$

and

$$v_2''(0) = \beta''(0) - u_2''(0) \geq \beta''(2\pi) - u_2''(2\pi) = v_2''(2\pi).$$

Then by using lemma (1.3), one can get $v_2(t) \geq 0$ on [0, 2π]. Therefore $\beta \geq A\eta s$.

In order to prove (b), let $\eta_1,\ \eta_2 \in [\alpha,\beta]$, such that $\eta_1 \leq \eta_2$, consider

$$v_3 = A\eta_2 - A\eta_1$$

Thus

$$v_3(t) = u_2(t) - u_1(t)$$

where u_1 and u_2 are the unique solutions of eq.(1.8) with respect to η_1 and η_2, respectively. Hence

$$v_3'''(t) - mv_3''(t) - mv_3'(t) + m^2 v_3(t) = u_2'''(t) - u_1'''(t) - mu_2''(t) + mu_1''(t) - mu_2'(t) +$$

$$mu_1'(t) + m^2 u_2(t) - m^2 u_1(t)$$

$$\geq f(t,\eta_2(t)) + m^2\eta_2(t) - f(t,\eta_1(t)) - m^2\eta_1(t)$$

$$\geq -m^2(\eta_2(t) - \eta_1(t)) + m^2(\eta_2(t) - \eta_1(t)) = 0$$

Also,

$$v_3(0) = u_2(0) - u_1(0) = u_2(2\pi) - u_1(2\pi) = v_3(2\pi),$$

$$v_3'(0) = u_2'(0) - u_1'(0) = u_2'(s2\pi) - u_1'(2\pi) = v_3'(2\pi),$$

and

$$v_3''(0) = u_2''(0) - u_1''(0) = u_2''(2\pi) - u_1''(2\pi) = v_3''(2\pi).$$

Hence, by using lemma (1.3), one can have $A\eta_2 \geq A\eta_1$.

It therefore follows that we can define the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha,\ \beta_0 = \beta$ such that

$$\alpha_n = A\alpha_{n-1},\ \ n \in N$$

$$\beta_n = A\beta_{n-1},\ \ n \in N$$

In the same manner as in theorem(1.1), one can deduce that

$$\alpha \leq \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n \leq \beta_n \leq \ldots \leq \beta_2 \leq \beta_1 \leq \beta_0 = \beta$$

Therefore $\lim\limits_{n\to\infty} \alpha_n(t) = p(t)$ and $\lim\limits_{n\to\infty} \beta_n(t) = r(t)$. We will show that p and r are solutions of eq.(1.7).

Since $\alpha_n = A\alpha_{n-1}$, then

$$\alpha_n'''(t) - m\alpha_n''(t) - m\alpha_n'(t) + m^2\alpha_n(t) = f(t,\alpha_{n-1}(t)) + m^2\alpha_{n-1}(t)$$

Taking limit as $n \to \infty$, we get

$$p'''(t) - mp''(t) - mp'(t) + m^2 p(t) = f(t, p(t)) + m^2 p(t)$$

Thus

$$p'''(t) - mp''(t) - mp'(t) = f(t, p(t))$$

and hence p is a solution of eq.(1.7a). Moreover

$$\alpha_n(0) = \alpha_n(2\pi) \Rightarrow p(0) = p(2\pi)$$

$$\alpha'_n(0) = \alpha'_n(2\pi) \Rightarrow p'(0) = p'(2\pi)$$

$$\alpha''_n(0) = \alpha''_n(2\pi) \Rightarrow p''(0) = p''(2\pi)$$

Therefore p is a solution of eq.(1.7). Similarly, one can deduce that r is another solution of eq.(1.7). The proof that p and r are the minimal and the maximal solutions of eq.(1.7) is similar to that in theorem(1.1). ■

Remarks (1.4):

(1) Nieto J. in 1991 gives necessary conditions for the existence of the extremal solutions of the periodic boundary value problem which consists of the third order ordinary differential equation:

$$u'''(t) = f(t, u(t)), \quad t \in [0, 2\pi]$$

together with the periodic boundary conditions:

$$u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0,1,2$$

(2) To the best of our knowledge, the problem for finding the extremal solutions for the periodic boundary value problem which consists of the n-th order ordinary differential equation:

$$u^{(n)}(t) = f(t, u(t)), \quad t \in [0, 2\pi]$$

together with the periodic boundary conditions:

$$u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0,1,...,n\text{-}1$$

is still open. However, this problem under certain conditions has at least one positive solution, [Yongxiang L., 2002].

CHAPTER 3

EXPANSION METHODS FOR SOLVING PERIODIC BOUNDARY VALUE PROBLEMS OF THE ORDINARY INTEGRO-DIFFERENTIAL EQUATIONS

Introduction:

In many real life problems it is so difficult sometimes to find the exact solution, especially for problems of nonlinear type. Therefore, more attention had been paid to the approximation methods. So, in this chapter the treatment for the periodic boundary value problems of the ordinary integro-differential equations centered mainly about finding the approximate solutions by using expansion methods

This chapter consists of two sections.

In section one, we give the expansion methods to solve the periodic boundary value problems of first-order linear integro-differential equations with some illustrative examples.

In section two, we use the same above methods to solve the periodic boundary value problems of the second-order linear integro-differential equations with some illustrative examples.

3.1 Methods of Solution for the Periodic Boundary Value Problem of the First Order Ordinary Integro-Differential Equations:

In this section, we give some approximation methods, namely the expansion methods to solve the periodic boundary value problem for the first order integro-differential equations which consists of the integro-differential equation.

$$u'(t) + Mu(t) = -N[Ku](t) + \sigma(t) \quad \text{.....(3.1a)}$$

together with the periodic boundary conditions

$$u(0) = u(T) \quad \text{.....(3.1b)}$$

where $t \in J = [0, T]$, $M, N \in \mathbb{R}$, $\sigma \in C(J)$ and $K : C(J) \longrightarrow C(J)$ is the integral operator

$$[Ku](t) = \int_0^T k(t, s)u(s)\, ds$$

Here, we use three methods of expansion methods to solve the periodic boundary value problems given by eq.(3.1).

3.1.1 The Collocation Methods:

The collocation method is one of the most common methods used to approximate the solution of the differential and integral equations, [Doyce D., 2001] and [Delves L., Mohamed J., 1985].

Here, we use this method to solve the periodic boundary value problem of the first order ordinary integro-differential equation given by eq.(3.1).

This method is based on approximating the unknown function u(t) as a linear combination of n+1 linearly independent functions $\{\varphi_i(t)\}_{i=0}^{n}$, that is write

$$u(t) = \sum_{i=0}^{n} a_i \varphi_i(t) \quad \text{.......................} (3.2)$$

and by substituting this approximated solution into eq.(3.1b), one can obtain:

$$\sum_{i=0}^{n} a_i \varphi_i(0) = \sum_{i=0}^{n} a_i \varphi_i(T) \quad \text{.......................} (3.3)$$

Hence, we illustrate the following three cases:

Case(1): If $\varphi_k(0) \neq 0$, for some $k \in \{0, 1, \ldots, n\}$, then

$$a_k = \frac{\sum\limits_{i=0}^{n} a_i \varphi_i(T) - \sum\limits_{\substack{i=0 \\ i\neq k}}^{n} a_i \varphi_i(0)}{\varphi_k(0)}$$

and hence the approximated solution given by eq.(3.2) can be written as

$$u(t) = \sum_{\substack{i=0 \\ i\neq k}}^{n} a_i \varphi_i(t) + a_k \varphi_k(t)$$

By substituting this approximated solution into eq.(3.1a), one can obtain:

$$\sum_{\substack{i=0 \\ i\neq k}}^{n} a_i \varphi_i'(t) + a_k \varphi'_k(t) + M\left[\sum_{\substack{i=0 \\ i\neq k}}^{n} a_i \varphi_i(t) + a_k \varphi_k(t)\right] = \varepsilon(t, \vec{a})$$

$$-N\int_0^T k(t, s)\left[\sum_{\substack{i-0 \\ i\neq k}}^{n} a_i \varphi_i(s) + a_k \varphi_k(s)\right] ds + \sigma(t)$$

that is

$$\varepsilon(t,\vec{a}) = \sum_{\substack{i=0\\ i\neq k}}^{n} a_i\varphi'_i(t) + a_k\varphi'_k(t) + M\sum_{\substack{i=0\\ i\neq k}}^{n} a_i\varphi_i(t) + M\, a_k\varphi_k(t) +$$

$$N\sum_{\substack{i-0\\ i\neq k}}^{n} a_i\int_0^T k(t,s)\varphi_i(s)ds + Na_k\int_0^T k(t,s)\varphi_k(s)\; ds - \sigma(t) \;\ldots\ldots\; (3.4)$$

where $\varepsilon(t,\vec{a})$ is said to be the error in the approximation of eq.(3.1), where $\vec{a}$ is a vector of n of a_i's that must be determined .

Next to find $\vec{a}$, one must choose n points say t_ℓ , ℓ =1,2,..,n in which the error function given by eq.(3.4) vanishes at these points. That is

$$\sum_{\substack{i=0\\ i\neq k}}^{n} a_i\varphi'_i(t_\ell) + a_k\varphi'_k(t_\ell) + M\sum_{\substack{i=0\\ i\neq k}}^{n} a_i\varphi_i(t_\ell) + Ma_k\varphi_k(t_\ell) +$$

$$N\sum_{\substack{i0\\ i\neq k}}^{n} a_i\int_0^T k(t_\ell,s)\varphi_i(s) + Na_k\int_0^T k(t_\ell,s)\varphi_k(s)\; ds - \sigma(t_\ell) = 0 \;\ldots\ldots\ldots\; (3.5)$$

By evaluating eq.(3.5) at each ℓ = 1,2,...,n. one can get a system of n linear equations which can be solved by any suitable method to find the values of n of a_i's that appeared in eq.(3.4).

Case (2): If $\varphi_j(T) \neq 0$, for some $j \in \{0, 1, \ldots, n\}$, then:

$$a_j = \frac{\sum_{i=0}^{n} a_i\varphi_i(0) - \sum_{\substack{i=0\\ i\neq j}}^{n} a_i\varphi_i(T)}{\varphi_j(T)} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (3.6)$$

and hence the approximated solution given by eq.(3.2) can be written as

$$u(t) = \sum_{\substack{i=0\\ i\neq j}}^{n} a_i\varphi_i(t) + a_j\varphi_j(t)$$

where a_j is defined by eq.(3.6).

By substituting this approximated solution into eq.(3.1a), one can obtain

$$\sum_{\substack{i=0\\i\neq j}}^{n} a_i\varphi_i'(t) + a_j\varphi'_j(t) + M\left[\sum_{\substack{i=0\\i\neq j}}^{n} a_i\varphi_i(t) + a_j\varphi_j(t)\right] = \varepsilon(t, \vec{a})$$

$$-N\int_0^T k(t, s)\left[\sum_{\substack{i=0\\i\neq j}}^{n} a_i\varphi_i(s) + a_j\varphi_j(s)\right] ds + \sigma(t)$$

that is

$$\varepsilon(t, \vec{a}) = \sum_{\substack{i=0\\i\neq j}}^{n} a_i\varphi_i'(t) + a_j\varphi'_j(t) + M\sum_{\substack{i=0\\i\neq j}}^{n} a_i\varphi_i(t) + M\, a_j\, \varphi_j(t) +$$

$$N\sum_{\substack{i-0\\i\neq j}}^{n} a_i\int_0^T k(t,s)\varphi_i(s) + Na_j\int_0^T k(t,s)\varphi_j(s)\; ds - \sigma(t) \;............\; (3.7)$$

In this case $\vec{a}$ is also a vector of n of a_i's that must be determined by choosing n points say t_ℓ, $\ell = 1,2,...,n$ in which the error function given by eq.(3.7) vanishes at these points . That is

$$\sum_{\substack{i=0\\i\neq j}}^{n} a_i\varphi_i'(t_\ell) + a_j\varphi'_j(t_\ell) + M\sum_{\substack{i=0\\i\neq j}}^{n} a_i\varphi_i(t_\ell) + Ma_j\varphi_j(t_\ell) +$$

$$N\sum_{\substack{i=0\\i\neq j}}^{n} a_i\int_0^T k(t_\ell,s)\varphi_i(s) + Na_j\int_0^T k(t_\ell,s)\varphi_j(s)ds - \sigma(t_\ell) = 0, \ell=1,2,\ldots,n ... \; (3.8)$$

By evaluating eq.(3.8) at each $\ell = 1, 2, \ldots, n$ one can get a system of n linear equations which can be solved by any suitable method to find the values of n of a_i's that appeared in eq.(3.7).

Case (3): If $\varphi_i(0) = \varphi_i(T)$ for each i = 0,1,..,n.

That is, if the approximated solution given by eq.(3.2) is automatically satisfy the periodic boundary condition given by eq.(3.1b), then

$$\varepsilon(t,\vec{a}) = \sum_{i=0}^{n} a_i \varphi_i'(t) + M\sum_{i=0}^{n} a_i \varphi_i(t) + N\sum_{i=0}^{n} a_i \int_0^T k(t,s)\varphi_i(s)\,ds - \sigma(t)$$

where $\vec{a} = \{a_i\}_{i=0}^{n}$ is a vector that must be determined.

Next, to find $\vec{a}$, one must choose n+1 points , say, t_ℓ, $\ell = 0, 1, \ldots, n$; in which the error given by eq.(3.9) vanishes at these points. That is

$$\sum_{i=0}^{n} a_i \varphi_i'(t_\ell) + M\sum_{i=0}^{n} a_i \varphi_i(t_\ell) + N\sum_{i=0}^{n} a_i \int_0^T k(t_\ell,s)\varphi_i(s)\,ds - \sigma(t_\ell) = 0 \;..\; (3.9)$$

By evaluating eq.(3.9) at each $\ell = 0,1,..,n$, one can get a system of n+1 linear equations with n+1 unknowns $\{a_i\}_{i=0}^{n}$ which can be solved by any suitable method.

To illustrate this approximated method, consider the following example.

Example (3.1):

Consider the periodic boundary value problem which consists of the first order linear integro-differential equation:-

$$\frac{d}{dt}u(t) + 3u(t) = -Ku(t) + 3t^2 - \frac{13}{12}t - 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.10a)$$

$t \in [0, 1]$,together with the periodic boundary condition

$$u(0) = u(1) \qquad (3.10b)$$

where

$$Ku(t) = \int_0^1 tsu(s)\, ds$$

We solve this periodic boundary value problem by using the collocation method, to do this, first, approximate the unknown function u(t) by a polynomial of degree two, that is, write:-

$$u(t) = a_0 + a_1 t + a_2 t^2$$

This solution must satisfy the periodic boundary conditions given by eq.(3.10b), thus u(t) becomes:

$$u(t) = a_0 - a_2 t + a_2 t^2$$

By substituting this solution into eq.(3.10a), one can get:

$$\varepsilon(t, a_0, a_2) = -a_2 - \frac{13}{12} a_2 t + 3a_0 + 3a_2 t^2 + \frac{1}{2} a_0 t - 3t^2 + \frac{13}{12} t + 1 \,. \qquad (3.11)$$

where $\varepsilon(t, a_0, a_2)$ is the error function and a_0 and a_2 are the unknown parameters that must be determined.

To find a_0 and a_2, we choose two points in the interval [0, 1] in which the error given by eq.(3.11) vanishes at them.

Here we take $t_0 = 0$, $t_1 = 1/2$, such that $\varepsilon(t_0, a_0, a_2) = \varepsilon(t_1, a_0, a_2) = 0$ to get the following system of equations, which has the solution $a_0 = 0$ and $a_2 = 1$. Thus:

$$u(t) = t^2 - t$$

is the solution of the periodic boundary value problem given by eq.(3.10).

Second, if we approximate the solution of eq.(3.10) by a polynomial of degree three, that is

$$u(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

This solution must satisfy the periodic boundary conditions given by eq.(3.10b), thus u(t) becomes:

$$u(t) = a_0 - \left(a_2 + a_3\right)\cdot t + a_2\cdot t^2 + a_3\cdot t^3$$

Hence

$$\varepsilon(t, a_0, a_2, a_3) = -a_2 - a_3 + \frac{23}{12} t a_2 + 3a_3 t^2 + 3a_0 - 3(a_2 + a_3)t + 3a_2 t^2 + 3a_3 t^3$$
$$- \frac{2}{15} t a_3 + \frac{1}{2} t a_0 - 3t^2 + \frac{13}{12} t + 1$$

and by taking $t_0 = 0$, $t_1 = 1/2$ and $t_2 = 1/3$; in which

$\varepsilon(t_0, a_0, a_2, a_3) = \varepsilon(t_1, a_0, a_2, a_3) = \varepsilon(t_2, a_0, a_2, a_3) = 0$, to get the following system of equations:

$$-a_2 - a_3 + 3a_0 + 1 = 0$$
$$\frac{-173}{120} a_3 + \frac{13}{4} a_0 - \frac{19}{24} a_2 + \frac{19}{24} = 0$$
$$\frac{11}{12} a_2 + \frac{28}{15} a_3 + \frac{7}{2} a_0 - \frac{11}{12} = 0$$

which has the solution $a_0 = 0$, $a_2 = 1$, and $a_3 = 0$. Therefore, $u(t) = t^2 - t$ is the solution of eq.(3.10).

Third, if we approximate the solution of eq.(3.10) as a polynomial of degree four. That is:

$$u(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$$

This solution must satisfy the periodic boundary conditions given by eq.(3.10b), thus u(t) becomes

$$u(t) = a_0 - (a_2+a_3+ a_4)t + a_2t^2+ a_3t^3+ a_4t^4$$

By substituting this solution into eq.(3.10a) in this case $\varepsilon(t,a_0,a_2,a_3,a_4)$ takes in the appendix, example(3.1)), where $\varepsilon(t,a_0,a_2,a_3,a_4)$ is the error function and a_0, a_2, a_3, a_4 are the unknown parameters that must be determined. To find a_0, a_2, a_3 and a_4 choose four points in the interval [0,1] in which the error function vanishes at them.

Here we take t_0=0, t_2=1/2, 3,t_3=1/3, t_4=1 such that $\varepsilon(t_i, a_0, a_2, a_3, a_4) = 0$, i = 1,2,3,4to get the following system of equations:

$$-a_2 - a_3 - a_4 + 3a_5 + 1 = 0$$

$$\frac{-173}{120}a_3 - \frac{91}{48}a_4 + \frac{13}{4}a_o - \frac{19}{24}a_2 + \frac{19}{24} = 0$$

$$\frac{-37}{36}a_2 - \frac{8}{5}a_3 - \frac{101}{54}a_4 + \frac{19}{6}a_o + \frac{37}{36} = 0$$

$$\frac{11}{12}a_2 + \frac{28}{15}a_3 + \frac{17}{6}a_4 + \frac{7}{2}a_o - \frac{11}{12} = 0$$

.

which has the solution $a_0 = 0$, $a_2 = 1$, and $a_3 = 0$ a_4=0. Therefore, $u(t) = t^2 - t$, is the solution of eq.(3.10).

For more details, see the appendix, example(3.1))

Remark (3.1):

The collocation method can be used to solve the periodic boundary value problems of the first order nonlinear integro-differential equations, given by eq.(2.23).

To see this, consider the following example:

Example (3.2):

Consider the periodic boundary value problem which consists of the first order nonlinear integro-differential equation:

$$\frac{d}{dt}u(t) + u(t) = \left[\int_0^1 (t+2s)u(s)\,ds \right]^2 + \left[t - 1 + t^2 - \left(\frac{-1}{6} - \frac{1}{6}t \right)^2 \right]$$

.........................(3.12a)

$t \in [0, 1]$, together with the periodic boundary condition

$$u(0) = u(1) \qquad \text{(3.12b)}$$

We solve this periodic boundary value problem by using the collocation method, to do this, first, approximate the unknown function u(t) by a polynomial of degree two, that is, write:

$$u(t) = a_0 + a_1 t + a_2 t^2$$

This solution must satisfy the periodic boundary conditions given by eq.(3.12b), thus u(t) becomes:

$$u(t) = a_0 - a_2 t + a_2 t^2$$

By substituting this solution into eq.(3.12a), one can get

$$e(t, a_0, a_2) = -a_2 + t a_2 + a_0 + a_2 t^2 - \left(\frac{-1}{6} a_2 - \frac{1}{6} a_2 t + a_0 + a_0 t \right)^2$$
$$-t + 1 - t^2 + \left(\frac{-1}{6} - \frac{1}{6} t \right)^2$$

.... (3.13)

where $\varepsilon(t, a_0, a_2)$ is the error function and a_0 and a_2 are the unknown parameters that must be determined.

To find a_0 and a_2, we choose two points in the interval [0,1] in which the error given by eq.(3.13) vanishes at them.

Here we take $t_0 = 0$, $t_2 = 1/2$, such that $\varepsilon(t_0, a_0, a_2) = \varepsilon(t_1, a_0, a_2) = 0$ to get the following nonlinear system of equations:

$$-a_2 + a_o - \left(\frac{-1}{6} a_2 + a_o\right)^2 + \frac{37}{36} = 0$$

$$a_2 + a_o - \left(\frac{-1}{3} a_2 + 2 a_o\right)^2 - \frac{8}{9} = 0$$

which has the solutions $a_0 = 0$ and $a_2 = 1$. Therefore, $u(t) = t^2 - t$ is the approximated solution of eq.(3.12).

For more details, see the appendix,(example(3.2)).

3.1.2 The Galerkin's Method:

The Galerkin's method is also one of the important methods that can be used to approximate the solution of the differential and integral equations [Chambers L., 1976] and [Doyce D., 2001].

Here we use this method to solve the periodic boundary value problems of the first order ordinary integro-differential equation by eq.(3.1).

Like the collocation method, this method is based on approximating the unknown function u(t) as given in eq.(3.2).

Next we consider the same previous cases:

Case(1):- The Galerkin's method here establishes n conditions necessary for determination of n of a_i 's that appeared in eq.(3.4) by making the error function defined by eq.(3.4) orthogonal to n given linearly independent functions $\{\psi_\ell(t)\}_{\ell=1}^{n}$ on the interval [0,T]. That is $\int_0^T \Psi_\ell(t)\varepsilon(t,\vec{a})dt = 0$, ℓ = 1,2,…,n. In other word write

$$\sum_{\substack{i=0\\i\neq k}}^{n} a_i \int_0^T \Psi_\ell(t)\varphi_i'(t)dt + a_k \int_0^T \Psi_\ell(t)\varphi_k'(t)dt + M\sum_{\substack{i=0\\i\neq k}}^{n} a_i \int_0^T \Psi_\ell(t)\varphi_i(t)dt +$$

$$Ma_k \int_0^T \Psi_\ell(t)\varphi_k(t)dt + N\sum_{\substack{i=0\\i\neq k}}^{n} a_i \int_0^T \Psi_\ell(t)\left[\int_0^T k(t,s)\varphi_i(s)ds\right]dt +$$

$$Na_k \int_0^T \Psi_\ell(t)\left[\int_0^T k(t,s)\varphi_k(s)ds\right]dt - \int_0^T \Psi_\ell(t)\sigma(t)dt = 0,\ \ell=1,2,\dots,n$$

.... (3.14)

By evaluating eq.(3.14) at each ℓ=1,2,...,n, one can get a system of n linear equations which can be solved by any suitable method to find the values of n of a_i's that appeared in eq.(3.14).

Case(2):-The Galerkin's method here establishes n conditions necessary for determination of n of a_i's appeared in eq.(3.7) by making the error function defined by eq.(3.7) orthogonal to n given linearly independent functions $\{\psi_\ell(t)\}_{\ell=1}^{n}$ on the interval [0, T]. That is,

$$\sum_{\substack{i=0\\i\neq j}}^{n} a_i \int_0^T \Psi_\ell(t)\varphi_i'(t)dt + a_j \int_0^T \Psi_\ell(t)\varphi_j'(t)dt + M\sum_{\substack{i=0\\i\neq j}}^{n} a_i \int_0^T \Psi_\ell(t)\varphi_i(t)dt + Ma_j \int_0^T \Psi_\ell(t)\varphi_j(t)dt +$$

$$N\sum_{\substack{i=0\\i\neq j}}^{n} a_i \int_0^T \Psi_\ell(t)\left[\int_0^T k(t,s)\varphi_i(s)ds\right]dt + Na_j \int_0^T \Psi_\ell(t)\left[\int_0^T k(t,s)\varphi_j(s)ds\right]dt - \int_0^T \Psi_\ell(t)\sigma(t)dt = 0$$

.... (3.15)

By evaluating eq.(3.15) at each ℓ =1,2,...,n. one can get a system of n linear equations which can be solved by any suitable method to find the values of n of a_i's that appeared in eq.(3.15).

Case(3):- The Galerkin's method here establishes n+1 conditions necessary for determination of n+1 of a_i's that appeared in eq.(3.9) by making the error function defined by eq.(3.9) orthogonal to n+1 given linearly independent functions $\{\Psi_\ell(t)\}_{\ell=0}^{n}$ on the interval [0,T]. That is

$$\sum_{i=0}^{n} a_i \int_0^T \Psi_\ell(t)\varphi_i'(t)dt + M\sum_{i=0}^{n} a_i \int_0^T \Psi_\ell(t)\varphi_i(t)dt +$$

$$N\sum_{i=0}^{n} a_i \int_0^T \Psi_\ell(t)\left[\int_0^T k(t,s)\varphi_i(s)ds\right]dt - \int_0^T \Psi_\ell(t)\sigma(t)dt = 0,\ \ell = 0,1,\ldots,n$$

.... (3.16)

By evaluating eq.(3.16) at each ℓ = 0, 1, ..., n. one can get a system of n+1 linear equations can be solved to find the values of n+1 of a_i's that appeared in eq.(3.16).

To illustrate this method, consider the following example:

Example (3.3):

Consider the periodic boundary value problem which consists of the first order linear integro-differential equation:

$$\frac{d}{dt}u(t) + 3u(t) = -2Ku(t) + 2t(t - 2\cdot\pi) + t^2 + 3t^2(t - 2\pi) + 8\pi^2\sin(t) + 24\pi\cos(t)$$

.........................(3.17a)

$t \in [0,2\pi]$, together with the periodic boundary condition

$$u(0) = u(2\pi) \quad \text{...} \quad (3.17b)$$

where

$$Ku(t) = \int_0^{2\pi} \sin(t+s)u(s)\,ds$$

We solve this periodic boundary value problem by using the Galerkin's method, to do this, first, approximate the unknown function u(t) by a polynomial of degree three, that is, write:

$$u(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

This solution must satisfy the periodic boundary conditions given by eq.(3.17), thus u(t) becomes

$$u(t) = a_0 - (a_2 + a_3)t + a_2 t^2 + a_3 t^3$$

By substituting this solution into eq.(3.17a), we get $\varepsilon(t, a_0, a_2, a_3)$ takes in the appendix ,example(3.3)

we choose three linearly independent functions say; 1, t^2, t^3 to be orthogonal to the error function defined by appendix to get the following system of equations:

$$-a_2 - a_3 + 3a_o + 1 = 0$$

$$\frac{-173}{120}a_3 + \frac{13}{4}a_o - \frac{19}{24}a_2 + \frac{19}{24} = 0$$

$$\frac{11}{12}a_2 + \frac{28}{15}a_3 + \frac{7}{2}a_o - \frac{11}{12} = 0$$

which has the solutions $a_0 = 0$, $a_2 = 1$ and $a_3 = 0$, Therefore, u(t) is a solution of eq.(3.17).

For more details, see the appendix.

Remark (3.2):

The Galerkian's method can be used to solve the periodic boundary value problems of the first order nonlinear integro-differential equations, given by eq.(2.23):

Remark (3.3):

The linear independent functions $\Psi_i(t)$, i = 0,1,2, used in the Galerkin's method are in general different from $\phi_i(t)$, i = 0,1,2, that used in the approximated solution given by eq.(3.2). If $\varphi_i(t) = \Psi_i(t)$, i = 0, 1, 2, then the Galerkin's method is said to be the moment method.

3.1.3 The Least Square Method:

This method is also one of the approximated methods used to solve the integral and integro-differential equations [Mohammed S., 2002], [Salih A., 2003], [Kareem R., 2003].

Here, we use it to solve periodic boundary value problems for the first order ordinary integro-differential equation given by eq.(3.1).

Like the collocation method, this method is based on approximating the unknown function u(t) as given in eq.(3.2). Next, we consider the same previous cases.

Case (1): The least square method requires minimizing the functional:

$$L(\vec{a}) = \int_0^T [\varepsilon(t, \vec{a})]^2 w(t)\, dt$$

$$L(\vec{a}) = \int_0^T \left[\sum_{\substack{i=0 \\ i\neq k}}^{n} a_i \varphi_i'(t) + a_k \varphi_k'(t) + M \left[\sum_{\substack{i=0 \\ i\neq k}}^{n} a_i \varphi_i(t) + a_k \varphi_k(t) \right] + \right.$$

$$\left. N \int_0^T k(t,s) \left[\sum_{\substack{i=0 \\ i\neq k}}^{n} a_i \varphi_i(s) + a_k \varphi_k(s) \right] ds - \sigma(t) \right]^2 w(t)dt \quad \ldots (3.18)$$

To do this differentiate $L(\vec{a})$ with respect to n of a_i's that appeared in eq.(3.18), and equating to zero, to get a system of n linear equations which can be solved to find the values of n of a_i's that appeared in eq.(3.18)where w(t) is any positive function defined in the region D and it is called the weight function.

Case (2): The least square method requires minimizing the functional:

$$L(\vec{a}) = \int_0^T [\varepsilon(t, \vec{a})]^2 w(t)\, dt$$

$$L(\vec{a}) = \int_0^T \left[\sum_{\substack{i=0 \\ i\neq j}}^{n} a_i \varphi_i'(t) + a_j \varphi_j'(t) + M \left[\sum_{\substack{i=0 \\ i\neq j}}^{n} a_i \varphi_i(t) + a_j \varphi_j(t) \right] + \right.$$

$$\left. N \int_0^T k(t,s) \left[\sum_{\substack{i=0 \\ i\neq j}}^{n} a_i \varphi_i(s) + a_j \varphi_j(s) \right] ds - \sigma(t) \right]^2 w(t)dt \quad \ldots. (3.19)$$

The values of $\vec{a}$ can be obtained by the following the same previous steps as in case (1).

Case (3): The least square method requires minimizing the functional:

$$L(\vec{a}) = \int_0^T [\varepsilon(t, \vec{a})]^2 w(t)\, dt$$

$$L(\vec{a}) = \int_0^T \left[\sum_{i=0}^{n} a_i \varphi_i'(t) + M\left[\sum_{i=0}^{n} a_i \varphi_i(t)\right] + N \sum_{i=0}^{n} \int_0^T k(t,s) a_i \varphi_i(s) ds - \sigma(t) \right]^2 w(t)dt$$

.... (3.20)

The values of $\vec{a}$ can be obtained by following the same previous steps as in case (1).

To illustrate this method, consider the following example:

Example (3.4):

Consider the periodic boundary value problem which consists of the first order linear integro-differential equation:

$$\frac{d}{dt}u(t) + 6u(t) = -4\cdot\int_0^2 (t+s)u(s)\, ds + \left(-8t^3 - 6t^2 + 6t^4 - \frac{128}{15} - \frac{32}{5}t\right)$$

.........................(3.21a)

$t \in [0, 2]$, together with the periodic boundary condition

$u(0) = u(2)$.. (3.21b)

We solve this periodic boundary value problem by using the least square method, to do this, first, approximate the unknown function u(t) by a polynomial of degree four, that is, write

$$u(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$$

This solution must satisfy the periodic boundary conditions given by eq.(3.21b), thus u(t) becomes

$u(t) = a_0 - (2a_2+4a_3+8a_4)t + a_2t^2 + a_3t^3 + a_4t^4$

By substituting this solution into eq.(3.21a), one can get the error term defined in the appendix ,then by minimizing the functional:

$$L(a_0, a_2, a_3, a_4) = \int_0^2 (\varepsilon(a_0, a_2, a_3, a_4))^2\, dt$$

one can get $a_0 = 7.548\times10^{-8}$, $a_2 = 1.437\times10^{-7}$, $a_3 = -2$ and $a_4 = 1$. Therefore, $u(t) \cong t^4 - 2t^3$ is the approximated solution of eq.(3.21).

For more details, see the appendix.

<u>Remark (3.4):</u>

The least square method can be used to solve the periodic boundary value problems of the first order nonlinear integro-differential equations, given by eq.(2.23).

To see this, consider the following example:

<u>Example (3.5):</u>

Consider the periodic boundary value problem which consists of the first order nonlinear integro-differential equation:

$$\frac{d}{dt}u(t) + u(t) = \left[\int_0^1 (t + 2s)u(s)\, ds\right]^2 + t - 1 + t^2 - \left(\frac{-1}{6} - \frac{1}{6}t\right)^2$$

.........................(3.22a)

together with the periodic boundary condition

$u(0) = u(2)$ .... (3.22b)

This solution must satisfy the periodic boundary conditions given by eq.(3.22), thus u(t) becomes

$u(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4$

This solution must satisfy the periodic boundary conditions given by eq.(3.22b), thus u(t) becomes

$u(t) = a_0 -(2a_2+4a_3+8a_4)t+ a_2t^2+ a_3t^3 + a_4t^4$

By substituting this solution into eq.(3.22), one can get the error function defined by

$$\varepsilon(t,a_2,a_3,a_4)=-a_2-a_3-a_4+2a_2t+3a_3t^2+4a_4t^3+a_0+(-a_2-a_3-a_4)t+a_2t^2+a_3t^3+a_4t^4-\left(-\frac{1}{3}a_4-\frac{3}{10}ta_4-\frac{4}{15}a_3-\frac{1}{4}ta_3-\frac{1}{6}ta_2+a_o+ta_o\right)^2-t+1-t^2-\left(-\frac{1}{6}-\frac{1}{6}t\right)^2$$

then we minimize the functional:

$$L(a_0, a_2, a_3, a_4) = \int_0^1 (\varepsilon(a_0, a_2, a_3, a_4))^2\, dt$$

to get $a_0=1.556\times10^{-6}$, $a_2=1.001$, $a_3=-1.348\times10^{-3}$, and $a_4=6.535\times10^{-4}$. Therefore, u(t) $\cong$ t(t – 1) is the approximated solution of eq.(3.22).

For more details, see the appendix (example(3.5)).

3.2 Method of Solution for Periodic Boundary Value Problems of the Second Order Ordinary Integro-Differential Equations:

In this section, we use the same previous methods to solve periodic boundary value problems of the second order ordinary integro-differential equation.

3.2.1 The Collocation Method:

As seen before the collocation method is used to solve the periodic boundary value problems of the first order integro-differential equations. Here we used to solve the periodic boundary value problem which consists of the second order integro- differential equation:

$$u''(t) + Mu(t) = -N[K\,u](t) + \sigma(t) \quad \text{.........(3.23a)}$$

together with the periodic boundary conditions

$$\left.\begin{aligned} u(0) &= u(T) \\ u'(0) &= u'(T) \end{aligned}\right\} \quad \text{.........(3.23b)}$$

where K , M, N and σ are defined similar to the previous.

This method is based on approximating the unknown function u(t) given ineq.(3.2)and by substituting this approximated solution into eq.(3.23b), one can obtain:

$$\left.\begin{aligned} \sum_{i=0}^{n} a_i \varphi_i(0) &= \sum_{i=0}^{n} a_i \varphi_i(T) \\ \sum_{i=0}^{n} a_i \varphi_i{}'(0) &= \sum_{i=0}^{n} a_i \varphi_i'(T) \end{aligned}\right\} \quad \text{.........(3.24)}$$

Hence , the error function $\varepsilon(t, \vec{a})$,is given by :

$$\varepsilon(t,\vec{a}) = \left[\sum_{i=0}^{n} a_i \varphi_i''(t) + M \sum_{i=0}^{n} a_i \varphi_i(t) + N\left[K \sum_{i=0}^{n} a_i \int_0^T k(t,s)\varphi_i(s)ds \right] - \sigma(t) \right] \quad \text{.........(3.25)}$$

where $\vec{a}$ is a vector of j of a_i's where $n-1 \le j \le n+1$. Similar to the previous to find $\vec{a}$ one must choose j points say t_ℓ, ℓ = 1, 2, …, j; in which $\varepsilon(t_\ell, \vec{a})=0, \ell =$ 1,2,…,j to get system of linear equations which can be solved to find $\vec{a}$.

To illustrate this method, consider the following example:

Example (3.6):

Consider the periodic boundary value problem which consists of the second order linear integro-differential equation:

$$\frac{d^2}{dt^2}u(t) + u(t) = -\int_0^1 (3t + s)u(s)\,ds + 8t(t-1) + 2t^2 + 2(t-1)^2 + t^2(t-1)^2 + \frac{1}{60} + \frac{1}{10}t$$

.........................(3.26a)

$t \in [0, 1]$, together with the periodic boundary conditions

$$\begin{aligned} u(0) &= u(1) \\ u'(0) &= u'(1) \end{aligned} \quad \text{.. (3.26b)}$$

Approximate the solution of the periodic boundary value problem given by eq.(3.26) by a polynomial of degree four, i.e.,

$$u(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$$

By substituting this solution into eq.(3.26b), one can get

$$u(t) = a_0 + (-a_3 - a_4 + \frac{3}{2}a_3 + 2\,a_4)t + (-\frac{3}{2}a_3 - 2\,a_4)t^2 + a_3 t^3 + a_4 t^4$$

where ε (t, a_0, a_3, a_4) is the error function defined in the appendix(example(3.6))and a_0 ,a_3 and a_4 are the unknown parameters that must be determined.

To find a_0, a_3 and a_4, choose three points in the interval [0, 1] in which the error function given in the appendix vanishes at them.

Here we take $t_0 = 0$, $t_1 = 1/2$, $t_3 = 1$, we get:

$$\frac{-361}{120}\cdot a_3 - 4\cdot a_4 + \frac{3}{2}\cdot a_0 - \frac{32}{21} = 0$$

$$\frac{-71}{80}\cdot a_4 + 3\cdot a_0 - \frac{1}{120}\cdot a_3 - \frac{2213}{840} = 0$$

$$\frac{359}{120}\cdot a_3 + \frac{81}{10}\cdot a_4 + \frac{9}{2}\cdot a_0 - \frac{246}{35} = 0$$

which has the solutions a_0 =1.041, a_3 = -0.73, a_4 =0.559Therefore,

$u(t)\Box\ t^{2}\,(t-1)^{2}$ is the approximated solution of eq.(3.26).

For more details, see the appendix.

3.2.2 The Galerkin's Method:

As seen before, the Galerkin's method can be used to solve the periodic boundary value problems for the first order linear and non-linear integro-differential equations.

Here we use it to solve the periodic boundary value problems given by equation (3.23).

The Galerkin's method is based on choosing j linearly independent functions $\{\psi_\ell(t)\}_{\ell=1}^{j}$ that are orthogonal on the error function $\varepsilon(t, \vec{a})$ given by eq.(3.25), where $n-1 \le j \le n+1$ to get a system of linear equations that can be solved to obtain $\vec{a}$.

To illustrate method, consider the following example:

Example (3.7):

Consider the periodic boundary value problem which consists of the first order non linear integro-differential equation:

$$\frac{d^2}{dt^2}u(t) + 4u(t) = -\int_0^2 (t+s)^2 u(s)\,ds + 6t(t-2)^2 + 12t^2(t-2) + 2t^3 + 4t^3(t-2)^2 + \frac{32}{21} + \frac{256}{105}\cdot t + \frac{16}{15}t^2$$

.... (3.27a)

$t \in [0, 2]$, together with the periodic boundary conditions

$$u(0) = u(2)$$

$$u'(0) = u'(2) \quad \text{...} (3.27b)$$

We solve this periodic boundary value problem by using the Galerkin's method method, to do this, approximate the unknown function u(t) by a polynomial of degree five, that is, write

$$u(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

This solution must satisfy the periodic boundary conditions given by

eq.(3.٢7b), thus u(t) becomes

$$u(t) = a_0 + (2a_3 + 8a_4 + 24a_5)t - (3a_3 + 8a_4 + 20a_5)t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

By substituting this solution into eq.(3.27a), one can get ε (t, a_0, a_3, a_4, a_5) where ε (t, a_0, a_3, a_4, a_5) is error function defined in the appendix (example(3.7)) and a_0, a_2, a_3 and a_4 are the unknown parameters that must be determined.

To find a_0, a_3, a_4 and a_5, choose fou points in the interval [0, 1] in which the error given in the appendix vanishes at them

Here we take $t_0 = 0$, $t_1 = 1/2$, $t_2 = 1/3$ and $t_3 = 1$, we get:

$$\frac{2816}{63}a_5 + \frac{80}{3}\cdot a_o - \frac{32}{15}a_3 + \frac{1664}{315}a_4 - \frac{4736}{315} = 0$$

$$\frac{-6496}{225} + \frac{5568}{35}a_5 + \frac{18304}{525}a_4 + \frac{104}{45}a_3 + \frac{2096}{45}\cdot a_o = 0$$

$$\frac{-25472}{525} + \frac{88352}{315}a_5 + \frac{19904}{315}a_4 + \frac{2728}{525}a_3 + \frac{368}{5}\cdot a_o = 0$$

$$\frac{-3690496}{24255} + \frac{4392704}{4851}a_5 + \frac{7296}{35}a_4 + \frac{704}{35}a_3 + \frac{13120}{63}a_o = 0$$

which has the solutions $a_0 = 0$, $a_3 = 4$, $a_4 = -4$ and $a_5 = 1$. Therefore $u(t) = t^3(t-2)^2$ is the approximated solution of eq.(3.27).

For more details, see the appendix.

3.2.3 The Least Square Method:

This method is based on minimizing the functional:

$$L(\vec{a}) = \int_0^T [\varepsilon(t,\vec{a})]^2\, w(t)\, dt\,,$$

where $\vec{a}$ is a vector of j of a_i's and $\varepsilon(t, \vec{a})$ is the error function defined by eq.(3.25)

To illustrate this method, consider the following example:

Example (3.8):

Consider the periodic boundary value problem which consists of the second order linear integro-differential equation:

$$\frac{d^2}{dt^2}u(t) + 4u(t) = -\int_0^2 (t+s)^2 u(s)\, ds + 6t(t-2)^2 + 12t^2(t-2) + 2t^3 + 4t^3(t-2)^2 + \frac{32}{21} + \frac{256}{105}\cdot t + \frac{16}{15}t^2$$

.........................(3.28a)

$t \in [0, 2]$, together with the periodic boundary conditions

$$\begin{aligned} u(0) &= u(2) \\ u'(0) &= u'(2) \end{aligned} \qquad \text{...} (3.28b)$$

We solve this periodic boundary value problem by using the least square method, to do this, first, approximate the unknown function u(t) by a polynomial of degree five, that is, write

$$u(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

By substituting this solution into eq.(3.28 b), one can get

$$u(t) = a_0 + (2a_3+8a_4+24a_5)t - (3a_3+8a_4+20a_5)t^2 + a_3 t^3 + a_4 t^4$$

By substituting this solution into eq.(3.28a), one can get the error function defined in the appendix(example(3.8)), then we minimize the functional:

$$L(a_0, a_3, a_4, a_5)) = \int_0^2 (\varepsilon(a_0, a_3, a_4, a_5))^2\, dt$$

one can get $a_0 = -5.399\times10^{-8}$, $a_3 = 4, a_4 = -4$ and $a_5=1$.

Therefore, $u(t) \cong t^3(t-2)^2$ is the approximated solution of eq.(3.28).

For more details, see the appendix.

<u>Remark (3.5):</u>

The other methods of expansion methods can be also used to solve the periodic boundary value problems of the ordinary integro-differential equations, say the moment method and the partition method.

CHAPTER

2

PERIODIC BOUNDARY VALUE PROBLEMS FOR ORDINARY INTEGRO-DIFFERENTIAL EQUATIONS

Introduction:

Periodic boundary value problems of ordinary integro-differential equations has been widely studied in the last years and has many real life applications in various mathematical problems, [Lakshmikantham V. & Hu S., 1986] studied the periodic boundary value problems for the integro-differential equations of Volterra types. [Nieto J. & Liz E., 1994] devoted the periodic boundary value problems for the Fredholm integro-differential equations with general kernel. [Hong Xu H. & Nieto J., 1997] gave the extremal solutions of a class of nonlinear integro-differential equations in Banach spaces.

In this chapter, we study periodic boundary value problems which consist of first order linear and nonlinear ordinary integro-differential equation together with periodic boundary conditions. This study includes the existence and the uniqueness of the solutions and the existence of the extremal solutions for such type of equations..

This chapter consists of three sections. In sections one, we give some basic concepts of the ordinary integro-differential equations.

In section two, we devote the existence and the uniqueness of the periodic boundary value problems for the first order linear ordinary integro-differential equations. Also the existence of the extremal solutions for them is discussed.

In section three, the same above study for the linear case is extended to include the periodic boundary value for the first order nonlinear ordinary integro-differential equations.

2.1 Integro-Differential Equations (Definition and Types):

In this section we present the definition and types of the first, second and n-th order linear and nonlinear ordinary integro-differential equations,

An integro differential equation is an equation involving unknown function u, together with both differential and integral operation on u. This means that it is an equation contains unknown function u, which appears inside the differential and integral signs,[Chambers L., 1976].

If the derivative is always taken with respect to one variable, the integro-differential equation is called ordinary. Other integro-differential equations, on the contrary, which often occur in mathematical physics, contain derivatives with respect to different variables, are called partial integro differential equations, [Vito Volterra., 1959].

In this work we restrict ourselves to study periodic boundary value problems of the one dimensional ordinary integro-differential equations.

Recall that the general form for the first order ordinary integro-differential equation is given by:

$$F(t, u(t), u'(t), [Ku](t)) = 0, \; t \in [a, b] \quad \text{...} \; (2.1)$$

$$[Ku](t) = \int_{a}^{\beta(t)} k(t, s, u(t), u(s), u'(t), u'(s))\, ds$$

where β is the known functions of t and a is the known constants. If $\beta(t) = t$ then it is called the general form for the first order Volterra ordinary integro-differential equation. If $\beta(t) = b$ then it is called the general form for first order Fredholm ordinary integro-differential equation. This equation is said to be linear if it takes the form:

$$a_1(t)u'(t) + a_2(t)u(t) + a_3(t)[Ku](t) + a_4(t)[Lu](t) = \sigma(t) \quad \text{............... (2.2)}$$

where

$$[Ku](t) = \int_a^{\beta(t)} k(t, s)u(s)\, ds,$$

$$[Lu](t) = \int_a^{\gamma(t)} \ell(t, s)\, u'(s)\, ds,$$

$\{a_i\}_{i=1}^{4}, \beta, \gamma, \sigma$ are known functions of t. Otherwise, it is said to be nonlinear. The initial value problem for the first order ordinary integro-differential equation consists of the ordinary integro-differential equation given by eq.(2.1) together with the initial condition

$$u(a) = \alpha_1$$

The periodic boundary value problem for the first order ordinary integro-differential equation consists of the ordinary integro- differential equation given by eq.(2.1) together with the periodic boundary condition

$$u(a) = u(b) \quad \text{.. (2.3)}$$

On the other hand the general form for the second order integro-differential equation is:

$$F(t, u(t), u'(t), u''(t), [Ku](t)) = 0,\ t \in [a, b] \quad \text{.............................. (2.4)}$$

where

$$[Ku](t) = \int_a^{\beta(t)} k(t, s, u(t), u(s), u'(t), u'(s), u''(t), u''(s))\, ds$$

If $\beta(t) = t$ then it is called the general form for second order Volterra ordinary integro-differential equation. If $\beta(t) = b$ then it is called the general form for the second order Fredholm ordinary integro-differential equation. This equation is said to be linear if it takes the form:

$$a_1(t)u''(t) + a_2(t)u'(t) + a_3(t)u(t) + a_4(t)[Ku](t) + a_5(t)[L_1 u](t) + a_6(t)[L_2u](t) = \sigma(t),\ t \in [a, b] \quad \text{.....} (2.5)$$

where

$$[Ku](t) = \int_a^{\beta(t)} k(t, s)u(s)\, ds,$$

$$[L_1u](t) = \int_a^{\gamma_1(t)} \ell_1(t, s)\, u'(s)\, ds,$$

and

$$[L_2u](t) = \int_a^{\gamma_2(t)} \ell_2(t, s)\, u''(s)\, ds$$

$\{a_i\}_{i=1}^{6}$, $\beta, \sigma, \{\gamma_i\}_{i=1}^{2}$ are known functions of t. Otherwise, it is said to be nonlinear

The initial value problem for the second order ordinary integro-differential equation consists of the ordinary integro-differential equation given by eq.(2.4) together with the initial conditions:

$$u(a) = \alpha_1$$

$$u'(a) = \alpha_2$$

The boundary value problem for the second order ordinary integro-differential equation consists of the ordinary integro- differential equation given by eq.(2.4) together with the boundary conditions:

$\alpha_1 u(a) + \beta_1 u'(a) = \gamma_1$

$\alpha_2 u(b) + \beta_2 u'(b) = \gamma_2$

The periodic boundary value problem for the second order ordinary integro-differential equation consists of the ordinary integro-differential equation given by eq.(2.4) together with the periodic boundary conditions:

$u(a) = u(b)$

$u'(a) = u'(b)$

So, the general form for the n-th order ordinary integro-differential equation is

$F(t, u(t), u'(t), \ldots, u^{(n)}(t), [Ku](t)) = 0, t \in [a, b]$.......................... (2.6)

where

$$[Ku](t) = \int_a^{\beta(t)} k(t, s, u(t), u(s), u'(t), u'(s), u^{(n)}(t), u^{(n)}(s))\, ds$$

Similar to the pervious, one can easily recognize the Fredholm and Volterra types for the above n-th order ordinary integro-differential equation.

The above integro-differential equation is said to be linear if it takes the form:

$$\sum_{i=0}^{n} a_i(t)u^{(i)}(t) + \sum_{i=0}^{n} b_i(t)[K_i u(t)] = \sigma(t)$$

where

$$[K_i u](t) = \int_{\alpha}^{\beta_i(t)} k_i(t, s)u^{(i)}(s)\, ds,\ i = 0,1,2,\ldots n,$$

and $\{a_i\}_{i=0}^{n}, \{b_i\}_{i=0}^{n}, \{\beta_i\}_{i=0}^{n}, \sigma$ are known functions of t.

Otherwise, it is said to be nonlinear.

The initial value problem for the n-th order ordinary integro-differential equation consists of the ordinary integro-differential equation given by eq.(2.6) together with the initial conditions:

$$u^{(i)}(a) = \alpha_i,\ i = 0, 1, \ldots, n - 1$$

The boundary value problem for the n-th order ordinary integro-differential equation consists of the ordinary integro- differential equation given by eq.(2.6) together with the boundary conditions:

$$\sum_{i=0}^{n-1} \alpha_{ij} u^{(i)}(a_j) = \gamma_j,\ j = 0, 1, \ldots, n-1$$

where $a_j \in [a, b]$ for each $j = 0,1,2,\ldots n-1$.

The periodic boundary value problem for the n-th order ordinary integro-differential equation consists of the ordinary integro- differential equation given by eq.(2.6) together with periodic boundary condition

2.2 Existence of the Extremal Solutions for the Periodic Boundary Value Problem for the Linear Integro-Differential Equations:

In this section we present some theorems that are necessary to establish the existence and the uniqueness of the solutions for special types of the periodic boundary value problems which consists of the first order linear integro-differential equation together with the periodic boundary condition.

Also, some theorems are introduced to ensure the existence of the extremal solutions for the periodic boundary value problems of the first order linear integro-differential equations. To the best of our knowledge, these theorems seem to be new.

We start this section by the following theorem. This theorem gives some necessary conditions for the existence of the solutions for special types of the boundary value problems of the first order linear integro-differential equations. It appeared in [Nieto J., et al., 2000] without proof. Here we give its proof.

Theorem (2.1):

Consider the boundary value problem for the first order linear Fredholm ordinary integro-differential equations which consists of the integro-differential equation

$$u'(t) + Mu(t) = -N[Ku](t) + \sigma(t),\ t \in J = [0, T] \quad \text{.........................(2.7a)}$$

together with the boundary conditions

$$u(0) = u(T) + \lambda \quad \text{.. (2.7b)}$$

where $N, \lambda \in \mathbb{R}$, $M \in \mathbb{R}\backslash\{0\}$, $\sigma \in C(J)$,and $K:C(J) \longrightarrow C(J)$ is an integral operator defined by

$$[Ku](t) = \int_0^T k(t, s)u(s)\, ds$$

If $u \in C^1(J)$ is a solution of eq.(2.7) then

$$u(t) = \int_0^T G(t, s)\{-N[Ku](s) + \sigma(s)\}\, ds + h_\lambda(t) \quad \text{............................ (2.8)}$$

where

$$G(t, s) = \frac{1}{1-e^{-MT}}\begin{cases} e^{-M(t-s)}, & 0 \le s \le t \le T \\ e^{-M(T+t-s)}, & 0 \le t < s \le T \end{cases}$$

and

$$h_\lambda(t) = \frac{\lambda e^{-Mt}}{1-e^{-MT}}$$

Moreover if $u \in C(J)$ satisfies eq.(2.8), then it is a solution of eq.(2.7).

<u>Proof:</u>

Multiply eq.(2.7a) by G(t, s) and integrating the resulting integro-differential equation from 0 to T, to get:

$$\int_0^T [u'(s) + Mu(s)]G(t, s)\, ds = \int_0^T G(t, s)\{-N[Ku](s) + \sigma(s)\}\, ds$$

Then from the definition of G(t,s), the above equation reduces to:

$$\frac{1}{1-e^{-MT}}\left[\int_0^t u'(s)e^{-M(t-s)}ds + \int_0^t Mu(s)e^{-M(t-s)}ds + \int_t^T u'(s)e^{-M(T+t-s)}ds + \right.$$
$$\left.\int_t^T Mu(s)e^{-M(T+t-s)}ds\right] = \int_0^T G(t, s)\{-N[Ku](s) + \sigma(s)\}\, ds$$

Thus

$$u(t) + \frac{e^{-Mt}}{1-e^{-MT}}[-u(0) + u(T)] = \int_0^T G(t, s)\{-N[Ku](s) + \sigma(s)\}\, ds$$

But $u(0) = u(T) + \lambda$, then

$$u\,(t) = \int_0^T G(t, s)\{-N[Ku](s) + \sigma(s)\}\, ds + \frac{\lambda e^{-Mt}}{1-e^{-MT}}$$

$$= \int_0^T G(t, s)\{-N[Ku](s) + \sigma(s)\}\, ds + h_\lambda(t)$$

is a solution of eq.(2.7).

Conversely, if $u \in C^1(J)$ satisfies eq.(2.8), then from the definition of $G(t, s)$ and $h_\lambda(t)$, eq.(2.8) can be written as:

$$u(t) = \frac{1}{1-e^{-MT}}\left[\int_0^t e^{-M(t-s)}\{-N[Ku](s)+\sigma(s)\}ds + \int_t^T e^{-M(T+t-s)}\{-N[Ku](s)+\sigma(s)\}ds\right] + \frac{\lambda e^{-Mt}}{1-e^{-MT}}$$

Then

$$u'(t)=\frac{1}{1-e^{-MT}}\left[-M\int_0^t e^{-M(t-s)}\{-N[Ku](s)+\sigma(s)\}ds + \{-N[Ku](t)+\sigma(t)\} - M\int_t^T e^{-M(T+t-s)}\{-N[Ku](s)+\sigma(s)\}ds - e^{-MT}\{-N[Ku](t)+\sigma(t)\}\right] - \frac{M\lambda e^{-Mt}}{1-e^{-MT}}$$

$$= -N[Ku](t) + \sigma(t) - \frac{M}{1-e^{-MT}}\left[\int_0^t e^{-M(t-s)}\{-N[Ku](s)+\sigma(s)\}ds + \int_t^T e^{-M(T+t-s)}\{-N[Ku](s)+\sigma(s)\}ds\right] - \frac{M\lambda e^{-Mt}}{1-e^{-MT}}$$

$$= -N[Ku](t) + \sigma(t) - M\int_0^T G(t,s)\{-N[Ku](s)+\sigma(s)\}ds - \frac{M\lambda e^{-Mt}}{1-e^{-MT}}$$

Thus

$$u'(t) + Mu(t) = -N[Ku](t) + \sigma(t) - M\int_0^T G(t,s)\{-N[Ku](s)+\sigma(s)\}ds + M\int_0^T G(t,s)\{-N[Ku](s)+\sigma(s)\}ds + \frac{M\lambda e^{-Mt}}{1-e^{-MT}}$$

$$= -N[Ku](t) + \sigma(t)$$

which means that eq.(2.8) is a solution of eq.(2.7a). Moreover

$$u(0) = \frac{1}{1-e^{-MT}} \int_0^T e^{-M(T-s)} \{-N[Ku](s) + \sigma(s)\} ds \ + \ \frac{\lambda}{1-e^{-MT}}$$

and

$$u(T) = \frac{1}{1-e^{-MT}} \int_0^T e^{-M(T-s)} \{-N[Ku](s) + \sigma(s)\} ds \ + \ \frac{\lambda e^{-MT}}{1-e^{-MT}}$$

Thus

$$u(T) + \lambda = \frac{1}{1-e^{-MT}} \int_0^T e^{-M(T-s)} \{-N[Ku](s) + \sigma(s)\} ds \ + \ \frac{\lambda e^{-MT}}{1-e^{-MT}} \ + \lambda$$

$$= \frac{1}{1-e^{-MT}} \int_0^T e^{-M(T-s)} \{-N[Ku](s) + \sigma(s)\} ds \ + \ \frac{\lambda}{1-e^{-MT}} \ = u(0)$$

which means that the function defined by eq.(2.8) is a solution of eq.(2.7b). Thus, the function u defined by eq.(2.8) is a solution of eq.(2.7). ■

The proof of the following corollary is clear, thus we omitted it.

Corollary (2.1):

Consider the periodic boundary value problem for the first order linear Fredholm ordinary integro-differential equation which consists of the integro-differential equation:

$u'(t) + Mu(t) = -N[Ku](t) + \sigma(t)$, $t \in J = [0,T]$ ....(2.9a)

together with the periodic boundary conditions

$u(0) = u(T)$.. (2.9b)

where $N \in \mathbb{R}$, $M \in \mathbb{R} \backslash \{0\}$, $\sigma \in C(J)$, and K is the integral operator defined previously. Then if $u \in C^1(J)$ is a solution of eq.(2.9), then:

$$u(t) = \int_0^T G(t, s)\{-N[Ku](s) + \sigma(s)\}\, ds \quad \text{......................................} \quad (2.10)$$

where G defined previously. Moreover if $u \in C(J)$ satisfies eq.(2.10), then it is a solution of eq.(2.9).

Remark (2.1):

It is easy to check that the fixed points of the operators $A_1: C(J) \longrightarrow C(J)$ and $A_2: C(J) \longrightarrow C(J)$, which are defined by:

$$[A_1 u](t) = \int_0^T G(t, s)\{-N[Ku](s) + \sigma(s)\}\, ds + h_\lambda(t),\ t \in J = [0, T]$$

and

$$[A_2 u](t) = \int_0^T G(t, s)\{-N[Ku](s) + \sigma(s)\}\, ds,\ t \in J = [0, T]$$

are precisely the solutions of the boundary value problem given by eq.(2.7) and eq.(2.9) respectively.

Next in the following theorem we find sufficient conditions to ensure the existence and the uniqueness of solution for the boundary value problem given by eq.(2.7). This theorem appeared in [Nieto J., et al., 2000]. Hence, we give the details of this proof.

Theorem (2.2):

Consider the boundary value problem given by eq.(2.7). If

$$\|k\|_\infty < \frac{|M|}{|N|\,T}$$

where $N, M \in \mathbb{R}\setminus\{0\}$, $\sigma \in C(J)$, and K is the integral operator defined previously Then eq.(2.7) has a unique solution.

Proof:

We show that the operator $A : C(J) \longrightarrow C(J)$, defined by

$$[Au](t) = \int_0^T G(t, s)\{-N[Ku](s) + \sigma(s)\}\, ds + h_\lambda(t)$$

is a contraction operator. To do this, consider:

$$\|Au - Av\| = \left\| \int_0^T G(t,s)\{-N[Ku](s) + \sigma(s)\} ds + h_\lambda(t) - \right.$$

$$\left. \int_0^T G(t,s)\{-N[Kv](s) + \sigma(s)\} ds - h_\lambda(t) \right\|$$

$$\leq |N| \max_{t\in[0,T]} \left| \int_0^T G(t,s)\{[Ku](s) - [Kv](s)\}\, ds \right|$$

$$\leq T|N| \max_{t\in[0,T]} \|k\|_\infty \|u - v\| \int_0^T | G(t,s) | ds$$

$$= \frac{T|N|}{|M|} \|k\|_\infty \|u - v\|_\infty .$$

But $\|k\|_\infty < \frac{|M|}{|N|T}$, thus $\frac{T|N|}{|M|}\|k\|_\infty < 1$. Therefore one can conclude that A is a contraction operator and hence A has a unique fixed point, which is the solution of eq.(2.7). ■

The proof of the following corollary is easy, thus we omitted it.

Corollary (2.2):

Consider the periodic boundary value problem given by eq.(2.9). If

$$\|k\|_\infty < \frac{|M|}{|N|T}$$

where N ,M $\in \mathbb{R}\backslash\{0\}$, $\sigma \in C(J)$, and K is the integral operator defined previously .Then eq.(2.9) has a unique solution.

Next we find a kind of Green's function to represent the solution for the boundary value problem given by eq.(2.7). This theorem appeared in [Nieto J., et al., 2000]. Here, we give the details of its proof.

Theorem (2.3):

Consider the boundary value problem given by eq.(2.7). Assume

$$\|k\|_\infty < \frac{M}{|N|T}$$

where N ,M $\leq \in \mathbb{R}\backslash\{0\}$, $\sigma \in C(J)$, and K is the integral operator defined previously. Then there exist H, Q $\in C(J\times J)$, such that the solution of eq.(2.7) is given by

$$u(t) = \int_0^T H(t, s)\sigma(s)\,ds + \int_0^T Q(t, s)h_\lambda(s)\,ds + h_\lambda(t),\ t \in J$$

where

$$H = G + F,$$

$$F(t, s) = \int_0^T Q(t, r)G(r, s)\,dr,$$

$$Q(t, s) = \sum_{i=1}^{\infty} R^{(i)}(t, s) \ldots\ldots (2.11)$$

and $R^{(i)}$ are the iterated kernels of R, i.e., $R^{(1)} = R$ and for $(t, s)\times J\times J$.

$$R^{(i)}(t, s) = \int_0^T R^{(i-1)}(t, s)R(r, s)\,dr,\ i \geq 2 \ldots\ldots (2.12)$$

Proof:

Since A is a contraction operator, then the sequence of iterates $\{A^n u_0\}$ converges to the solution u, for any $u_0 \in C(J)$. Let:

$$u_0(t) = \int_0^T G(t, s)\sigma(s)\, ds + h_\lambda(t),\ t \in J$$

By using Fubini's theorem and the mathematical induction one can have:

$$[A^n u_0](t) = u_0(t) + \int_0^T Q_n(t, s)h_\lambda(s)\, ds + \int_0^T F_n(t, s)\sigma(s)\, ds,\ n \geq 1$$

where

$$F_n(t, s) = \int_0^T Q_n(t, r)G(r, s)\, dr,$$

and

$$Q_n(t, s) = \sum_{i=1}^{\infty} R^{(i)}(t, s)$$

Since

$$\|k\|_\infty < \frac{M}{|N|T}$$

Then one can have the following estimate:

$$\|R\|_\infty \leq \frac{|N|\,\|k\|_\infty}{M}$$

$$= d < \frac{1}{T}$$

In consequence, we show that:

$$\|R^{(n)}\|_\infty \leq (dT)^{n-1} d,\ n \geq 1$$

To do this, we use the mathematical induction.

Let n = 1, then:

$$\|R^{(1)}\|_\infty = \|R\|_\infty \le d$$

$$= (dT)^{1-1}d$$

Let n = 2, then:

$$\|R^{(2)}\|_\infty = \left\|\int_0^T R(t,r)R(r,s)\,dr\right\|$$

$$= \max_{t,s\in J}\left|\int_0^T R(t,r)R(r,s)\,dr\right|$$

$$\le \max_{t,s\in J}\int_0^T |R(t,r)||R(r,s)|dr$$

$$\le \int_0^T (d^2)\,dr$$

$$= d^2T = (dT)^{2-1}d$$

Assume

$$\|R^{(k)}\|_\infty \le (dT)^{k-1}d$$

Then:

$$\|R^{(k+1)}\|_\infty = \left\|\int_0^T R^{(k)}(t,r)R(r,s)\,dr\right\|$$

$$= \max_{t,s\in J}\left|\int_0^T R^{(k)}(t,r)R(r,s)\,dr\right|$$

$$\leq \max_{t,s\in J} \int_0^T |R^{(k)}(t,r)||R(r,s)|dr$$

$$\leq (dT)^{k-1} d(dT) = (dT)^k d$$

Therefore

$$\|R^{(n)}\|_\infty \leq (dT)^{n-1} d, \; n \geq 1$$

Since T > 0, then 0< dT < 1, thus the series $\sum_{i=1}^{\infty} R^{(i)}(t,s)$ is absolutely and uniformly convergent in C(J×J) and $\sum_{i=1}^{\infty} R^{(i)}(t,s) = Q \in$ C(J×J). Then the sequence F_n is convergent in C(J×J) to a function F. ■

Remark (2.2):

The unique solution of the periodic boundary value problem given by eq.(2.9) is given by

$$u(t) = \int_0^T H(t, s)\sigma(s)\, ds$$

where H is defined previously. Thus we say H is the Green's function of the integro- differential problem given by eq.(2.9).

Next, the following theorem gives necessary conditions for the periodic boundary value problem given by eq.(2.9) to ensure the existence of a non-negative unique solution. This theorem appeared in [Nieto J., et al., 2000]. Hence, we give the details of this proof.

Theorem (2.4), [Nieto J., et al., 2000]:

Consider the periodic boundary value problem given by eq.(2.9). If

$$\|k\|_\infty \le \frac{M^2 e^{-|M|T}}{|N|[1+(|M|T-1)e^{-|M|T}]}$$

where $N, M \in \mathbb{R}\setminus\{0\}$, $\sigma \in C(J)$, and K is the integral operator defined previously .Then eq.(2.9) has a unique solution u and if $M > 0$, $\sigma \ge 0$ on J, and hence

$u(t) \ge 0$ on J.

Proof:

First of all we use corollary (2.2) to ensure that eq.(2.9) has a unique solution. Second if $M > 0$ and $\sigma \ge 0$ on J, then we must prove $u(t) \ge 0$ on J .To do this we obtain

$$\|Q\|_\infty = \left\| \sum_{i=1}^{\infty} R^{(i)}(t,s) \right\|_\infty$$

$$\le \sum_{i=1}^{\infty} \| R^{(i)} \|_\infty$$

$$\le \sum_{i=1}^{\infty} (dT)^{i-1} d = \frac{d}{1-dT}$$

This implies that

$$\|Q\|_\infty \le \frac{|N|c_1}{|M-|N|c_1T}, \text{where } c_1 = \frac{M^2 e^{-|M|T}}{|N|[1+(|M|T-1)e^{-|M|T}]}$$

Hence

$$\|F\|_\infty \le \frac{|N|c_1}{M(M-|N|c_1T)}$$

Recall that, $F \le |F|$ and $-F < |F|$ and $|F| \le ||F||$, which implies that:

$$F \ge -|F| \ge -||F||$$

Hence, for every $(t, s) \in J \times J$, we get:

$$H(t, s) \ge \frac{e^{-MT}}{1-e^{-MT}} - ||F||_\infty \ge 0$$

Hence

$$u(t) \ge 0 \text{ on } J. \quad \blacksquare$$

Next the following theorem gives another necessary conditions to guarantee the existence of non-negative unique solution of the boundary value problem given by eq.(2.7). This theorem appeared in [Nito J., et al., 2000]. Here, we give the details of its proof.

Theorem (2.5):

Consider the boundary value problem given by eq.(2.7). Assume that $M > 0$, $N > 0$, $\lambda \ge 0$, $\sigma \in C(J)$ and $\sigma(t) \ge 0$, for each $t \in J = [0, T]$. Assume also that $k \in C(J \times J)$, $k \ge 0$, and

$$||k||_\infty \le \frac{M}{2NT}\left(\sqrt{\left(\frac{e^{MT}-1}{MT}\right)^2 + 4} - \frac{e^{MT}-1}{MT}\right)$$

Then eq.(2.7) has a unique solution, with $u(t) > 0$, for each $t \in J$.

Proof:

Let

$$c = \frac{M}{2NT}\left(\sqrt{\left(\frac{e^{MT}-1}{MT}\right)^2 + 4} - \frac{e^{MT}-1}{MT}\right)$$

then it is easy to check that:

$$\|k\|_\infty \leq c \leq \frac{M}{NT}$$

Thus, by using theorem (2.2), eq.(2.7) has a unique solution.

Next, we prove $u(t) \geq 0$, for each $t \in J$.

Since $N > 0$ and $k \geq 0$ then $Nk \geq 0$ on $J\times J$ and hence:

$$R(t, s) = -N\int_0^T G(t,r)k(r,s)\, ds \leq 0, \text{ on } J\times J$$

In consequence:

$$(-1)^i R^{(i)}(t, s) \geq 0 \text{ on } J\times J \text{ for } i = 1, 2, \ldots$$

where $R^{(i)}$ is the function defined by eq.(2.12).

If we consider the series formed by odd terms of the series defined by eq.(2.11), we have

$$Q(t, s) \geq Q^*(t, s) = \sum_{k=1}^{\infty} R^{(2k-1)}(t, s),\ t, s \in J$$

Then

$$\|Q^*\|_\infty \leq \|\sum_{k=1}^{\infty} R^{(2k-1)}\|_\infty$$

$$\leq \sum_{k=1}^{\infty} (dT)^{2k-2} d = \frac{d}{1-(dT)^2}$$

where $d = \frac{N}{M}\|k\|_\infty$

Thus:

$$d \le \frac{N}{M} c$$

and hence

$$1 - (dT)^2 \ge 1 - \left(\frac{N}{M} Tc\right)^2$$

therefore

$$\|Q^*\|_\infty \le \frac{d}{1-(dT)^2} \le \frac{NcM}{M^2-(NcT)^2}$$

Then for every $(t, s) \in J\times J$

$$H(t, s) = G(t, s) + F(t, s)$$

$$= G(t, s) + \int_0^T Q(t,r)G(r,s)\, dr$$

Since $Q(t, s) \ge Q^*(t, s)$, $t, s \in J$, and $Q^*(t, s) \le \|Q^*\|$, $t, s \in J$, Hence:

$$Q(t, s) \ge Q^*(t, s) \ge -\|Q^*\|,\ t, s \in J$$

On the other hand, $G(t, s) \ge \min\limits_{t,s\in J} G(t, s)$. Thus:

$$H(t, s) \ge -\|Q^*\|_\infty \int_0^T G(r, s)\, dr + \min_{t,s\in J} G(t, s)$$

It is easy to check that:

$$\int_0^T G(r, s)\, dr = \frac{1}{M}$$

and

$$\min_{t,s\in J} G(t, s) = \frac{1}{e^{MT}-1}$$

So

$$H(t, s) \geq -\frac{\|Q^*\|_\infty}{M} + \frac{1}{e^{MT}-1}$$

$$\geq \frac{-NcM}{M^2-(NcT)^2} + \frac{1}{e^{MT}-1}$$

Now

$$(NcT)^2 = \frac{M^2}{4}\left[2\left(\frac{e^{MT}-1}{MT}\right)^2 + 4 - 2\sqrt{\left(\frac{e^{MT}-1}{MT}\right)^2 + 4}\,\frac{e^{MT}-1}{MT}\right]$$

$$\geq \frac{M^2}{4}\left[2\left(\frac{e^{MT}-1}{MT}\right)^2 + 4 - 2\left[\frac{e^{MT}-1}{MT}+2\right]\left(\frac{e^{MT}-1}{MT}\right)\right]$$

$$= \frac{M^2}{4}\left[4 - 4\frac{e^{MT}-1}{MT}\right] = M^2\left[1-\frac{e^{MT}-1}{MT}\right]$$

Thus

$$M^2-(NcT)^2 \leq \frac{M\left(e^{MT}-1\right)}{T}$$

$$\frac{1}{M^2-(NcT)^2} \geq \frac{T}{M\left(e^{MT}-1\right)} \quad \text{...} \quad (2.13)$$

Also:

$$Nc = \frac{M}{2T}\left[\sqrt{\left(\frac{e^{MT}-1}{MT}\right)^2 + 4}\,\frac{e^{MT}-1}{MT}\right]$$

$$\leq \frac{M}{2T}[2] = \frac{M}{T}$$

$$-Nc \geq \frac{-M}{T} \quad \ldots \quad (2.14)$$

From inequality (2.13) and inequality (2.14), one can get:

$$\frac{-Nc}{M^2-(NcT)^2} \geq \frac{-M}{T}\,\frac{T}{M\left(e^{MT}-1\right)}$$

$$\geq \frac{-1}{e^{MT}-1}$$

Therefore:

$$H(t, s) \geq \frac{-Nc}{M^2-(NcT)^2} + \frac{1}{e^{MT}-1}$$

$$\geq \frac{-1}{e^{MT}-1} + \frac{1}{e^{MT}-1} = 0$$

Since $\sigma(t) \geq 0$, for each $t \in J$. Therefore:

$$\int_0^T H(t, s)\sigma(s)\, ds \geq 0$$

on the other hand

$$\int_0^T Q(t, s)h_\lambda(s)\, ds + h_\lambda(t) \geq -\|Q^*\|_\infty \int_0^T h_\lambda(s)\, ds + \min_{t\in J} h_\lambda(t)$$

$$= -\|Q^*\|_\infty \int_0^T \frac{\lambda e^{-Ms}}{1-e^{-MT}}\, ds + \frac{\lambda}{1-e^{-MT}} \min_{t\in J} e^{-MT}$$

$$= \frac{-\|Q^*\|_\infty \lambda}{-M\left(1-e^{-MT}\right)} e^{-Ms}\Big|_0^T + \frac{\lambda}{1-e^{-MT}} e^{-Mt}$$

$$= \frac{-\|Q^*\|_\infty \lambda}{-M\left(1-e^{-MT}\right)}(e^{-MT} - 1) + \frac{\lambda}{e^{MT}-1}$$

$$= \frac{-\lambda \|Q^*\|_\infty}{M} + \frac{\lambda}{e^{MT}-1}$$

$$\geq \lambda\left(\frac{-Nc}{M^2-(NcT)^2} + \frac{1}{e^{MT}-1}\right)$$

$$\geq \lambda(0) = 0$$

From the above results, one can conclude that

$$u(t) = \int_0^T H(t,s)\sigma(s)\,ds + \int_0^T Q(t,s)h_\lambda(s)\,ds + h_\lambda(t) \geq 0, \text{ for each } t \in J. \ \blacksquare$$

The proof of the following corollary is straightforward.

Corollary (2.3):

Consider the boundary value problem given by eq.(2.9). Assume that $M > 0$, $N > 0$, $\sigma \in C(J)$ and $\sigma(t) \geq 0$, for each $t \in J = [0, T]$. Assume also that $k \in C(J \times J)$, $k \geq 0$

$$\|k\|_\infty \leq \frac{M}{2NT}\left(\sqrt{\left(\frac{e^{MT}-1}{MT}\right)^2 + 4} - \frac{e^{MT}-1}{MT}\right)$$

Then eq.(2.9) has a unique solution u, with $u(t) \geq 0$, for each $t \in J$.

Now, we are in a position that we can give the following theorem. This theorem shows that the existence of the extremal solutions for the periodic boundary value problem which consists of the first order linear integro-differential equation

$u'(t) = -N[Ku](t), t \in J = [0, T]$ ....(215a)

together with the following periodic boundary condition:

$u(0) = u(T)$... (2.15b)

to the best of our knowledge, this theorem seems to be new.

<u>Theorem (2.6):</u>

Consider the periodic boundary value problem given by eq.(2.15). Suppose that there exist α, $\beta \in C^1(J)$, such that $\alpha \leq \beta$ on J. Assume in addition that:

(1) $\alpha'(t) \leq -N[K\alpha](t)$, and

$\alpha(0) \leq \alpha(T)$

(2) $\beta'(t) \geq -N[K\beta]$

$\beta(0) \geq \beta(T)$

(3) There exists a constant M > 0, such that:

$N[Kx](t) - N[Ky](t) \geq -M(y - x)(t)$

for each $t \in J$ and $\alpha(t) \leq x(t) \leq y(t) \leq \beta(t)$.

Then there exist monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ with $\alpha_0 = \alpha, \beta_0 = \beta$ such that $\lim\limits_{n\to\infty} \alpha_n(t) = p(t)$, $\lim\limits_{n\to\infty} \beta_n(t) = r(t)$ uniformly and monotonically on $[0, 2\pi]$ and that p, r are the minimal and the maximal solutions of the periodic boundary value problem given by eq.(2.15).

<u>Proof:</u>

For any $\eta \in [\alpha, \beta]=\{\eta \in C[[0, 2\pi], \square], \alpha(t)\leq\eta(t)\leq\beta(t), t\in[0, 2\pi]\}$, consider the linear periodic boundary value problem which consists of the first order ordinary differential equation

$u'(t) + Mu(t) = \sigma_\eta(t)$,..(2.16a)

together with the following periodic boundary conditions:

$u(0) = u(T)$ .... (2.16b)

where $\sigma_\eta(t) = M\eta(t) - N[K\eta](t)$

From proposition (1.1)-(1.2), eq.(2.16) has a unique solution.

We consider the solution operator $A : [\alpha, \beta] \longrightarrow C(J)$, which is defined by $A\eta = v$, where v is the unique solution of eq.(2.16). We will show that A satisfies the following properties:

(a) If $\alpha \le \eta \le \beta$, then $\alpha \le A\eta \le \beta$.

(b) If $\alpha \le \eta_1 \le \eta_2 \le \beta$, then $\alpha \le A\eta_1 \le A\eta_2 \le \beta$.

To show (a), first we prove $A\eta \ge \alpha$

To do this, we consider $v_1 = A\eta - \alpha$. Thus

$$v_1 = u_1 - \alpha$$

where u_1 is the unique solution of eq.(2.16). Thus

$$\begin{aligned} v'_1(t) + Mv_1(t) &= u_1'(t) - \alpha'(t) + Mu_1(t) - M\alpha(t) \\ &= u_1'(t) + Mu_1(t) - \alpha'(t) - M\alpha(t) \\ &\ge M\eta(t) - N[K\eta](t) + N[K\alpha](t) - M\alpha(t) \\ &= M[\eta(t) - \alpha(t)] - N[K\eta](t) + N[K\alpha](t) \\ &\ge M[\eta(t) - \alpha(t)] - M[\eta(t) - \alpha(t)] = 0 \end{aligned}$$

Also

$$\begin{aligned} v_1(0) &= u_1(0) - \alpha(0) \\ &\ge u_1(T) - \alpha(T) \\ &= v_1(T) \end{aligned}$$

thus by using proposition (1.3) one can get $v_1(t) \geq 0$, for each $t \in J$. Thus $A\eta \geq \alpha$.

Second, to prove $A\eta \leq \beta$. Consider $v_2 = \beta - A\eta$.

Thus

$$v_2 = \beta - u_2.$$

where u_2 is the unique solution of eq.(2.16). Then

$$\begin{aligned} v'_2(t) + Mv_2(t) &= \beta'(t) + M\beta(t) - [u'_2(t) + Mu_2(t)] \\ &\geq -N[K\beta](t) + MB(t) - M\eta(t) + N[K\eta](t) \\ &\geq -M[\beta(t) - \eta(t)] + M[\beta(t) - \eta(t)] = 0 \end{aligned}$$

Also:

$$\begin{aligned} v_2(0) &= \beta(0) - u_2(0) \\ &\geq \beta(T) - u_2(T) = v_2(T) \end{aligned}$$

Thus by using proposition (1.3) one can get $v_2(t) \geq 0$ for each $t \in J$, hence $A\eta \leq \beta$.

To prove (b), assume that $\alpha \leq \eta_1 \leq \eta_2 \leq \beta$. From the part (a), one can get $\alpha \leq A\eta_1 \leq \beta$, and $\alpha \leq A\eta_2 \leq \beta$. Thus, it is sufficient to prove that $A\eta_1 \leq A\eta_2$. To do this, consider:

$$\begin{aligned} v_3 &= A\eta_2 - A\eta_1 \\ &= u_2 - u_1 \end{aligned}$$

where u_1, u_2 are the unique solutions of eq.(2.13) with respect to η_1 and η_2, respectively. Then

$$\begin{aligned} v'_3(t) + Mv_3(t) &= u'_2(t) + Mu_2(t) - u'_1(t) - Mv_1(t) \\ &= M\eta_2(t) - N[K\eta_2](t) - M\eta_1(t) + N[K\eta_1](t) \\ &\geq M[\eta_2(t) - \eta_1(t)] - M[\eta_2(t) - \eta_1(t)] = 0 \end{aligned}$$

Also

$v_3(0) = u_2(0) - u_1(0) = u_2(T) - v_1(T) = v_3(T)$

Thus by using proposition (1.3), one can get $v_3(t) \geq 0$, for each $t \in J$. Hence $A\eta_1 \leq A\eta_2$.

It therefore follows that we can define the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$ such that $\alpha_n = A\alpha_{n-1}$, $\beta_n = A\beta_{n-1}$, $n = 1, 2, \ldots$

Moreover, as seen before in theorem (1.1) the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following inequality:

$\alpha = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n \leq \beta_n \leq \ldots \leq \beta_1 \leq \beta_0 = \beta$.

Therefore $\lim\limits_{n\to\infty} \alpha_n(t) = p(t)$, $\lim\limits_{n\to\infty} \beta_n(t) = r(t)$ uniformly and monotonically on [0, T]. Since

$\alpha_n = A\alpha_{n-1}$, then

$\alpha'_n(t) + M\alpha_n(t) = M\alpha_{n-1}(t) - N[K\alpha_{n-1}](t)$.

Therefore

$$\lim_{n\to\infty} \alpha'_n(t) + M \lim_{n\to\infty} \alpha_n(t) = M \lim_{n\to\infty} \alpha_{n-1}(t) - N[K \lim_{n\to\infty} \alpha_{n-1}](t).$$

and hence

$p'(t)+Mp(t) = Mp(t) - N[Kp](t)$.

That is $p'(t) = -N[Kp](t)$.

In other words p is a solution of eq.(2.15a). Also

$\alpha_n(0) = \alpha_n(T)$

thus $p(0) = p(T)$. That is p is a solution of eq.(2. 15b).

Similarly, since

$\beta_n = A\beta_{n-1}$, then

$$\beta'_n(t) + M\beta_n(t) = M\beta_{n-1}(t) - N[K\beta_{n-1}](t).$$

Therefore

$$\lim_{n\to\infty} \beta'_n(t) + M \lim_{n\to\infty} \beta_n(t) = M \lim_{n\to\infty} \beta_{n-1}(t) - N[K \lim_{n\to\infty} \beta_{n-1}](t).$$

and hence

$$r'(t) + Mr(t) = Mr(t) - N[Kr](t).$$

That is

$$r'(t) = -N[Kr](t).$$

In other words r is a solution of eq.(2.15a).Also

$$\beta_n(0) = \beta_n(T)$$

thus $r(0) = r(T)$. That is p is a solution of eq.(2.15b).

Hence p and r are solutions of eq.(2.15). To prove that p and r are the minimal and maximal solutions of eq.(2.15) we have to show that if u is any solution of eq.(2.15) such that $u \in [\alpha, \beta]$ on [0,T], then $\alpha \le p \le u \le r \le \beta$ on [0, T].

To do this it is easy to check that for any solution u of eq.(2.15) with $\alpha \le u \le \beta$, we have $\alpha \le \alpha_n \le u \le \beta_n \le \beta$ on [0, T], taking the limits as $n \longrightarrow \infty$, we can conclude that $\alpha \le p \le u \le r \le \beta$ on [0, T]. ■

Next, the following theorem shows that the existence of the minimal and maximal solutions of the periodic boundary value problem:

$$u'(t) = -N[Ku](t) + \sigma(t),\ t \in J = [0, T] \qquad \text{(2.17a)}$$

together with the following periodic boundary condition:

$$u(0) = u(T) \qquad \text{(2.17b)}$$

To the best of our knowledge, this theorem seems to be new and it is a generalization of the previous theorem.

<u>Theorem (2.7):</u>

Consider the periodic boundary value problem given by eq.(2.17). Suppose that there exist α, $\beta \in C^1(J)$, such that $\alpha \leq \beta$ on J. Assume in addition that:

(1) $\alpha'(t) \leq -N[K\alpha] + \sigma(t)$, and

$\alpha(0) \leq \alpha(T)$

(2) $\beta'(t) \geq -N[K\beta] + \sigma(t)$, and

$\beta(0) \geq \beta(T)$

(3) There exists a constant $M > 0$, such that:

$N[Kx](t) - N[Ky](t) \geq -M(y - x)(t)$

for each $t \in J$ and $\alpha(t) \leq x(t) \leq y(t) \leq \beta(t)$.

(4) $\sigma(t) \geq 0$ for each $t \in J$.

Then there exist monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ with $\alpha_0=\alpha$, $\beta_0=\beta$ such that $\lim\limits_{n\to\infty} \alpha_n(t) = p(t)$, $\lim\limits_{n\to\infty} \beta_n(t) = r(t)$ uniformly and monotonically on $[0, 2\pi]$ and that p, r are the minimal and the maximal solutions of the periodic boundary value problem given by eq.(2.17).

<u>Proof:</u>

For any $\eta\in[\alpha, \beta] = \{\eta\in C[[0, 2\pi], \square], \alpha(t)\leq\eta(t)\leq\beta(t), t\in[0, 2\pi]\}$, consider the linear periodic boundary value problem which consists of the first order ordinary differential equation

$u'(t) + Mu(t) = \sigma_\eta(t)$, ...(2.18a) together with the following periodic boundary condition:

$$u(0) = u(T) \qquad (2.18b)$$

where $\sigma_\eta(t) = M\eta(t) - N[K\eta](t) + \sigma(t)$

From proposition (1.1)-(1.2), eq.(2.18) has a unique solution.

We consider the solution operator $A : [\alpha, \beta] \longrightarrow C(J)$, which is defined by $A\eta = u$, where u is the unique solution of eq.(2.18). We will show that A satisfy the following properties:

(a) If $\alpha \le \eta \le \beta$, then $\alpha \le A\eta \le \beta$.

(b) If $\alpha \le \eta_1 \le \eta_2 \le \beta$, then $\alpha \le A\eta_1 \le A\eta_2 \le \beta$.

To show (a), first we prove $A\eta \ge \alpha$

To do this, we consider $v_1 = A\eta - \alpha$. Thus

$$v_1 = u_1 - \alpha$$

where u_1 is the unique solution of eq.(2.18). Thus

$$\begin{aligned} v'_1(t) + Mv_1(t) &= u_1'(t) - \alpha'(t) + Mu_1(t) - M\alpha(t) \\ &= M\eta(t) - N[K\eta](t) + \sigma(t) - \alpha'(t) - M\alpha(t) \\ &\ge M\eta(t) - N[K\eta](t) + \sigma(t) + N[K\alpha](t) - \sigma(t) + M\alpha(t) \\ &\ge M[\eta(t) - \alpha(t)] - M[\eta(t) - \alpha(t)] = 0 \end{aligned}$$

Also

$$\begin{aligned} v_1(0) &= u_1(0) - \alpha(0) \\ &\ge u_1(T) - \alpha(T) \\ &= v_1(T) \end{aligned}$$

thus by using proposition (1.3) one can get $v_1(t) \ge 0$, for each $t \in J$. Thus $A\eta \ge \alpha$.

Second, to prove $A\eta \le \beta$. Consider $v_2 = \beta - A\eta$.

Thus

$$v_2 = \beta - u_2.$$

Where

u_2 is the unique solution of eq.(2.18). Then

$$\begin{aligned} v'_2(t) + Mv_2(t) &= \beta'(t) + M\beta(t) - [u'_2(t) + Mu_2(t)] \\ &\ge -N[K\beta](t)+\sigma(t)+ M\beta(t) - M\eta(t) + N[K\eta](t))- \sigma(t) \\ &\ge -M[\beta(t) - \eta(t)] + M[\beta(t) - \eta(t)] = 0 \end{aligned}$$

Also

$$\begin{aligned} v_2(0) &= \beta(0) - u_2(0) \\ &\ge \beta(T) - u_2(T) = v_2(T) \end{aligned}$$

Thus by using proposition (1.3) one can get $v_2(t) \ge 0$ for each $t \in J$, hence $A\eta \le \beta$.

To prove (b), assume that $\alpha \le \eta_1 \le \eta_2 \le \beta$. From the part (a), one can get $\alpha \le A\eta_1 \le \beta$, and $\alpha \le A\eta_2 \le \beta$. Thus, it is sufficient to prove that $A\eta_1 \le A\eta_2$. To do this, consider:

$$v_3 = u_2 - u_1$$

where u_1, u_2 are the unique solutions of eq.(2.18) with respect to η_1 and η_2, respectively. Then

$$\begin{aligned} v'_3(t) + Mv_3(t) &= u'_2(t) + Mu_2(t) - u'_1(t) - Mu_1(t) \\ &= M\eta_2(t) - N[K\eta_2](t)+\sigma(t)-M\eta_1(t) - \sigma(t) + N[K\eta_1](t) \\ &\ge M[\eta_2(t) - \eta_1(t)] - M[\eta_2(t) - \eta_1(t)] = 0 \end{aligned}$$

Also

$v_3(0) = u_2(0) - u_1(0)$

$= u_2(T) - u_1(T) = v_3(T)$

Thus by using proposition (1.3), one can get $v_3(t) \geq 0$, for each $t \in J$. Hence $A\eta_1 \leq A\eta_2$.

It therefore follows that we can define the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$ such that $\alpha_n = A\alpha_{n-1}$, $\beta_n = A\beta_{n-1}$, $n = 1,2,\ldots$

Moreover, as seen before in theorem (1.1) the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following inequality:

$$\alpha = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n \leq \beta_n \leq \ldots \leq \beta_1 \leq \beta_0 = \beta.$$

Therefore

$$\lim_{n\to\infty} \alpha_n(t) = p(t),\ \lim_{n\to\infty} \beta_n(t) = r(t)$$

uniformly and monotonically on [0,T]. Since

$$\alpha_n = A\alpha_{n-1}$$

then

$$\alpha'_n(t) + M\alpha_n(t) = M\alpha_{n-1}(t) - N[K\alpha_{n-1}](t) + \sigma(t).$$

Therefore

$$\lim_{n\to\infty} \alpha'_n(t) + M \lim_{n\to\infty} \alpha_n(t) = M \lim_{n\to\infty} \alpha_{n-1}(t) - N[K \lim_{n\to\infty} \alpha_{n-1}](t) + \sigma(t)$$

and hence

$$p'(t) + Mp(t) = Mp(t) - N[Kp](t) + \sigma(t)$$

That is $p'(t) = -N[Kp](t) + \sigma(t)$.

In other words p is a solution of eq.(2.17a). Also

$$\alpha_n(0) = \alpha_n(T)$$

thus p(0)=p(T). That is p is a solution of eq.(2.17b).

Therefore, p is a solution of eq.(2.17).

In the same manner, one can easily prove that r is a solution of eq.(2.17). Moreover, as seen before in theorem (2.6), one can easily have $\alpha \leq p \leq u \leq r \leq \beta$ on [0, T]. ■

Now, the following theorem is a generalization to the previous theorems which gives the extremal solutions for the linear periodic boundary value problem, which consists of the first order linear integro differential equation:

$$u'(t) + Mu(t) = -N[Ku](t) + \sigma(t),\ t \in J = [0, T] \qquad \text{(2.19a)}$$

together with the following boundary conditions:

$$u(0) = u(T) \qquad \text{(2.19b)}$$

To the best of our knowledge, this theorem seems to be new.

Theorem (2.8):

Consider the periodic boundary value problem given by eq.(2.19). Suppose that there exist $\alpha, \beta \in C^1(J)$, such that $\alpha \leq \beta$ on J = [0, T]. Assume in addition that:

(1) $\alpha'(t) + M\alpha(t) \leq -N[K\alpha] + \sigma(t)$

$\alpha(0) \leq \alpha(T)$

(2) $\beta'(t) + M\beta(t) \geq -N[K\beta] + \sigma(t)$, and

$\beta(0) \geq \beta(T)$

(3) There exists a constant $M_1 > 0$, such that:

$N[Kx](t) - N[Ky](t) \geq -M_1(y - x)(t)$

for each $t \in J$ and $\alpha(t) \leq x(t) \leq y(t) \leq \beta(t)$.

(4) $M > 0$ and $\sigma(t) \geq 0$ for each $t \in J$.

Then there exist monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\lim\limits_{n\to\infty} \alpha_n(t) = p(t)$, $\lim\limits_{n\to\infty} \beta_n(t) = r(t)$ uniformly and monotonically on $[0, 2\pi]$ and that p, r are the minimal and the maximal solutions of the periodic boundary value problem given by eq.(2.19).

<u>Proof:</u>

For any $\eta \in [\alpha, \beta] = \{\eta \in C[[0, 2\pi], \mathbb{R}], \alpha(t) \le \eta(t) \le \beta(t), t \in [0, 2\pi]\}$, consider the linear periodic boundary value problem which consists of the first order ordinary differential equation

$u'(t) + (M + M_1)u(t) = \sigma_\eta(t)$,..(2.20a)

together with the following periodic boundary conditions:

$u(0) = u(T)$ .... (2.20b)

where $\sigma_\eta(t) = M\eta(t) - N[K\eta](t)$

From proposition (1.1)-(1.2), eq.(2.20) has a unique solution.

We consider the solution operator $A : [\alpha, \beta] \longrightarrow C(J)$, which is defined by $A\eta = u$, where u is the unique solution of eq.(2.20). We will show that A satisfies the following properties:

(a) If $\alpha \le \eta \le \beta$, then $\alpha \le A\eta \le \beta$.

(b) If $\alpha \le \eta_1 \le \eta_2 \le \beta$, then $\alpha \le A\eta_1 \le A\eta_2 \le \beta$.

To show (a), first we prove $A\eta \ge \alpha$

To do this, we consider $v_1 = A\eta - \alpha$. Thus

$v_1 = u_1 - \alpha$

where u_1 is the unique solution of eq.(2.20). Thus

$$v'_1(t)+(M+M_1)v_1(t) = u_1'(t) - \alpha'(t) + Mu_1(t) - M\alpha(t) + M_1u_1(t) - M_1\alpha(t)$$

$$= u_1'(t) + (M+M_1)u_1(t) - \alpha'(t) - M\alpha(t) - M_1\alpha(t)$$

$$\geq M_1\eta(t) - N[K\eta](t) + \sigma(t) + N[K\alpha](t) - \sigma(t) - M_1\alpha(t)$$

$$= M_1 [\eta(t) - \alpha(t)] - N[K\eta](t) + N[K\alpha](t)$$

$$\geq M_1[\eta(t) - \alpha(t)] - M_1 [\eta(t) - \alpha(t)] = 0$$

Also

$$v_1(0) = u_1(0) - \alpha(0)$$

$$\geq u_1(T) - \alpha(T)$$

$$= v_1(T)$$

thus by using proposition (1.3) one can get $v_1(t) \geq 0$, for each $t \in J$. Thus $A\eta \geq \alpha$.

Second, to prove $A\eta \leq \beta$. Consider $v_2 = \beta - A\eta$.

Thus

$$v_2 = \beta - u_2.$$

Where u_2 is the unique solution of eq.(2.20). Then

$$v'_2(t) + (M+M_1)v_2(t) = \beta'(t) + M_1\beta(t) - [u'_2(t) + Mu_2(t) + M_1u_2(t)] + M_1\beta(t)$$

$$\geq -N[K\beta](t) + \sigma(t) + M\beta(t) - M_1\eta(t) + N[K\eta](t) - \sigma(t)$$

$$\geq -M_1[\beta(t) - \eta(t)] + M [\beta(t) - \eta(t)] = 0$$

Also:

$$v_2(0) = \beta(0) - u_2(0)$$

$$\geq \beta(T) - u_2(T) = v_2(T)$$

Thus by using proposition (1.3) one can get $v_2(t) \geq 0$, for each $t \in J$, hence $A\eta \leq \beta$.

To prove (b), assume that $\alpha \leq \eta_1 \leq \eta_2 \leq \beta$. From the part (a), one can get $\alpha \leq A\eta_1 \leq \beta$, and $\alpha \leq A\eta_2 \leq \beta$. Thus, it is sufficient to prove that $A\eta_1 \leq A\eta_2$. To do this, consider:

$$v_3 = u_2 - u_1$$

where u_1, u_2 are the unique solutions of eq.(2.20) with respect to η_1 and η_2, respectively. Then

$$\begin{aligned} v'_3(t) + (M+M_1)v_3(t) &= u'_2(t) + Mu_2(t) + M_1u_2(t) - u'_1(t) - Mu_1(t) - M_1u_1(t) \\ &= M_1\eta_2(t) - N[K\eta_2](t) - M_1\eta_1(t) + N[K\eta_1](t) \\ &\geq M_1[\eta_2(t) - \eta_1(t)] - M_1[\eta_2(t) - \eta_1(t)] = 0 \end{aligned}$$

Also

$$v_3(0) = u_2(0) - u_1(0) = u_2(T) - v_1(T) = v_3(T)$$

Thus by using proposition (1.3), one can get $v_3(t) \geq 0$, for each $t \in J$. Hence $A\eta_1 \leq A\eta_2$.

It therefore follows that we can define the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$ such that $\alpha_n = A\alpha_{n-1}$, $\beta_n = A\beta_{n-1}$, $n = 1,2,\ldots$.

Moreover, as seen before in theorem (1.1) the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following inequality:

$$\alpha = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n \leq \beta_n \leq \ldots \leq \beta_1 \leq \beta_0 = \beta.$$

Therefore $\lim\limits_{n\to\infty} \alpha_n(t) = p(t)$, $\lim\limits_{n\to\infty} \beta_n(t) = r(t)$ uniformly and monotonically on [0,T]. Since

$$\alpha_n = A\alpha_{n-1},$$

then

$$\alpha'_n(t) + (M + M_1)\alpha_n(t) = M_1\alpha_{n-1}(t) - N[K\alpha_{n-1}](t) + \sigma(t)$$

Therefore

$$\lim_{n\to\infty} \alpha'_n(t)+(M + M_1) \lim_{n\to\infty} \alpha_n(t) = M_1 \lim_{n\to\infty} \alpha_{n-1}(t)-N[K \lim_{n\to\infty} \alpha_{n-1}](t)+\sigma(t)$$

and hence

$$p'(t) + (M + M_1)p(t) = M_1p(t) - N[Kp](t) + \sigma(t)$$

That is $p'(t) + M\, p(t) = N[Kp](t) + \sigma(t)$.

Also $\alpha_n(0) = \alpha_n(T)$, that is $p(0) = p(T)$. Thus p is a solution of eq.(2.19b). In the same manner, one can easily prove that r is a solution of eq.(2.19). The proofs that p and r are the extremal solutions of eq.(2.19) is similar to that in theorem(2.7). ■

2.3 Existence of the Extremal Solutions for the Periodic Boundary Value Problem for the Nonlinear Integro-Differential Equations:

In this section we give some theorems to guarantee the existence and the uniqueness of the solutions for special types of the periodic boundary value problems of the first order nonlinear integro-differential equation:

$$u'(t) = f(t, u(t), [Ku](t)),\ t \in J = [0, T] \qquad \text{(2.21a)}$$

together with the periodic boundary conditions

$$u(0) = u(T) \qquad \text{(2.21b)}$$

where $T > 0$, $f : J\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function and $K : C(J) \longrightarrow C(J)$ is an integral operator defined by

$$[Ku](t) = \int_0^T k(t, s)u(s)\, ds$$

Moreover, the existence of the extremal solutions for the periodic boundary value problem given by eq.(2.21) is discussed.

We start this section by the following theorem. This theorem appeared in [Nieto J., et al., 2000] without proof; here we give its proof.

Theorem (2.9):

Consider the boundary value problem for the first order non-linear integro-differential which consists of the integro-differential equation

$$u'(t) + Mu(t) = g(t, u(t), [Ku](t)) \quad \text{...(2.22a)}$$

together with the boundary condition

$$u(0) = u(T) + \lambda \quad \text{.. (2.22b)}$$

where $T > 0$, $\lambda \in \mathbb{R}$, $g \in C(J\times\mathbb{R}, \mathbb{R})$, $M \in \mathbb{R}\backslash\{0\}$, $J = [0, T]$ and $K : C(J) \longrightarrow C(J)$ is an integral operator defined by

$$[Ku](t) = \int_0^T k(t, s)u(s)\, ds$$

If $u \in C(J)$ is a solution of eq. (2.22) then

$$u(t) = \int_0^T G(t, s)g(s, u(s), [Ku](s))ds + h_\lambda(t)$$

where G is defined previously.

Moreover if $u\in C^1(J)$ satisfies eq.(2.22),

then it is a solution of eq.(2.22).

Proof:

Multiply eq.(2.22a) by G(t, s) and integrating the resulting integro-differential equation from 0 to T, to get:

$$\int_0^T [u'(s) + Mu(s)]G(t, s)\, ds = \int_0^T G(t, s)\, g(s, u(s), [Ku](s))\, ds$$

Then from the definition of G(t,s), the above equation reduces to:

$$\frac{1}{1-e^{-MT}}\left[\int_0^t u'(s)e^{-M(t-s)}ds + \int_0^t Mu(s)e^{-M(t-s)}ds + \int_t^T u'(s)e^{-M(T+t-s)}ds + \int_t^T Mu(s)e^{-M(T+t-s)}ds\right] = \int_0^T G(t, s)g(s, u(s), [Ku](s))\, ds$$

Thus

$$u(t) + \frac{e^{-Mt}}{1-e^{-MT}}[-u(0) + u(T)] = \int_0^T G(t, s)\, g(s, u(s), [Ku](s))\, ds$$

But u(0) = u(T) + λ, then

$$u(t) = \int_0^T G(t, s)\, g(s, u(s), [Ku](s))\, ds + \frac{\lambda e^{-Mt}}{1-e^{-MT}}$$

$$= \int_0^T G(t, s)\, g(s, u(s), [Ku](s))ds + h_\lambda(t)$$

is a solution of eq.(2.22).

Conversely, if $u \in C(J)$ satisfies eq.(2.22), then from the definition of G(t, s) and $h_\lambda(t)$,the solution

$$u(t) = \int_0^T G(t, s)g(s, u(s), [Ku](s))ds + h_\lambda(t$$

can be written as:

$$u(t)=\frac{1}{1-e^{-MT}}\left[\int_0^t e^{-M(t-s)}g(s,u(s),[Ku](s))\,ds+\right.$$

$$\left.\int_t^T e^{-M(T+t-s)}\left[g(s,u(s),[Ku](s))\right]ds\right]+\frac{\lambda e^{-Mt}}{1-e^{-MT}}$$

Then

$$u'(t)=\frac{1}{1-e^{-MT}}\left[-M\int_0^t e^{-M(t-s)}g(s,u(s),[Ku](s))ds+\right.$$

$$g(t,u(t),[Ku](t))]-M\int_t^T e^{-M(T+t-s)}\left[g(s,u(s),[Ku](s))\right]ds-$$

$$\left.e^{-MT}\left[g(t,u(t),[Ku](t))\right]\right]-\frac{M\lambda e^{-Mt}}{1-e^{-MT}}$$

$$=g(t,u(t),[Ku](t))-\frac{M}{1-e^{-MT}}\int_0^t e^{-M(t-s)}\{g(s,u(s),[Ku](s))\}ds\ -$$

$$\frac{M}{1-e^{-MT}}\int_t^T e^{-M(T+t-s)}\left\{g(s,u(s),[Ku](s))\right\}ds-\frac{M\lambda e^{-Mt}}{1-e^{-MT}}$$

$$=\ g(t,u(t),[Ku](t))\ -M\int_0^t G(t,s)\left\{g(s,u(s),[Ku](s))\right\}ds\ -\ \frac{M\lambda e^{-Mt}}{1-e^{-MT}}$$

Thus

$$u'(t)+Mu(t)=g(t,u(t),[Ku](t))\ -M\int_0^t G(t,s)\left\{g(s,u(s),[Ku](s))\right\}ds\ +$$

$$M\int_0^t G(t,s)\left\{g(s,u(s),[Ku](s))\right\}ds$$

$$=\ g(t,u(t),[Ku](t))$$

which means that

$$u(t) = \int_0^T G(t, s)g(s, u(s), [Ku](s))ds + h_\lambda(t)$$

is a solution of eq.(2.22a). Moreover

$$u(0) = \frac{1}{1-e^{-MT}} \int_0^T e^{-M(T-s)}\{g(s, u(s), [Ku](s))\}ds + \frac{\lambda}{1-e^{-MT}}$$

and

$$u(T) = \frac{1}{1-e^{-MT}} \int_0^T e^{-M(T-s)}\{g(s, u(s), [Ku](s))\}ds + \frac{\lambda e^{-MT}}{1-e^{-MT}}$$

Thus

$$u(T) + \lambda = \frac{1}{1-e^{-MT}} \int_0^T e^{-M(T-s)}\{g(s, u(s), [Ku](s))\}ds + \frac{\lambda e^{-MT}}{1-e^{-MT}} + \lambda$$

$$= \frac{1}{1-e^{-MT}} \int_0^T e^{-M(T-s)}\{g(s, u(s), [Ku](s))\}ds + \frac{\lambda}{1-e^{-MT}} = u(0)$$

which means that

$$u(t) = \int_0^T G(t, s)g(s, u(s), [Ku](s))ds + h_\lambda(t)$$

is a solution of eq.(2.22b). Thus, the function u defined above is a solution of eq.(2.22).

Corollary (2.4):

Consider the periodic boundary value problem for the first order non-linear integro-differential which consists of the integro-differential equation

$u'(t) + Mu(t) = g(t, u(t), [Ku](t))$ ....(2.23a)

together with the periodic boundary condition

$u(0) = u(T)$ .... (2.23b)

where $T > 0, g \in C(J \times \mathbb{R}, \mathbb{R})$, $M \in \mathbb{R} \backslash \{0\} = [0, T]$ and $K : C(J) \longrightarrow C(J)$ is an integral operator defined by

$$[Ku](t) = \int_0^T k(t, s)u(s)\, ds$$

Then if $u \in C^1(J)$ is a solution of eq. (2.23) then

$$u(t) = \int_0^T G(t, s)g(s, u(s), [Ku](s))ds \quad \text{.. (2.24)}$$

where G is defined previously.

Moreover if $u \in C(J)$ satisfies eq.(2.24), then it is a solution of eq.(2.23).

<u>Remarks (2.2):</u>

1. Its easy to check that the fixed points of the operator $A : C(J) \longrightarrow C(J)$ which is defined by

$$[Au](t) = \int_0^T G(t, s)g(s, u(s), [Ku](s))ds + h_\lambda(t),\ t \in J = [0, T]$$

are precisely the solutions of the boundary value problem given by eq.(2.22) where G and h_λ are defined previously.

2. Its easy to check that the fixed points of the operator $A : C(J) \longrightarrow C(J)$ which is defined by

$$[Au](t) = \int_0^T G(t, s)g(s, u(s), [Ku](s))ds, \ t \in J = [0, T]$$

are precisely the solutions of the periodic boundary value problem given by eq.(2.23) where G is defined previously.

Next, in the following theorem we give sufficient conditions to ensure the existence and the uniqueness of solution for eq.(2.22). This theorem appeared in [Nieto J., et al., 2000]. Here we give the details of its proof.

Theorem (2.10), [Nieto J., et al., 2000]:

Consider eq.(2.22). Suppose that g satisfies the Lipschitz condition with respect to, i.e., there exist L_1, $L_2 \geq 0$, such that

$$|g(t, x, u) - g(t, y, v)| \leq L_1 |x - y| + L_2 |u - v|, \ t \in J, \ x, y, u, v \in \mathbb{R} \qquad \text{.........................(2.25a)}$$

and

$$\|k\|_\infty < \frac{|M| - L_1}{TL_2} \qquad \text{.. (2.26)}$$

Then eq.(2.22) has a unique solution.

Proof:

We show that the operator A defined by:

$$[Au](t) = \int_0^T G(t, s)g(s, u(s), [Ku](s))ds + h_\lambda(t)$$

is a contraction operator.

To do this, consider

$$\|Au - Av\| = \left\| \int_0^T G(t,s)g(s,u(s),[Ku](s))ds + h_\lambda(t) - \int_0^T G(t,s)(g(s,v(s),[Kv](s))ds + h_\lambda(t) \right\|$$

$$= \max_{t\in[0,T]} \left| \int_0^T G(t,s)\left[g(s,u(s),[Ku](s)) - g(s,v(s),[Kv](s))\right] ds \right|$$

$$\leq \max_{t\in[0,T]} \int_0^T |G(t,s)| \left[L_1 \|u-v\| + L_2 \|Ku - Kv\| \right] ds$$

$$\leq \max_{t\in[0,T]} \int_0^T |G(t,s)| \left[L_1 \|u-v\| + L_2 \|k\|_\infty \|u-v\| \right] ds$$

$$= \frac{1}{|M|} \left[L_1 + L_2 \|k\|_\infty T \right] \|u-v\| \qquad (2.27)$$

But $\|k\|_\infty \leq \frac{|M| - L_1}{TL_2}$, thus:

$$\frac{1}{|M|}\{L_1 + L_2\|k\|_\infty T\} < \frac{1}{|M|}\{L_1 + L_2\frac{|M| - L_1}{TL_2}T\}$$

$$= \frac{1}{|M|}|M| = 1$$

Therefore A is a contraction operator and hence A has a unique fixed point which is the unique solution of eq.(2.22). ■

The proof of the following corollary is clear and thus we omitted it.

Corollary (2.5):

Consider eq.(2.21). Assume the same hypothesis as in theorem (2.10) then the periodic boundary value problem given by eq.(2.21) has a unique solution.

Now, the following theorem shows that under certain conditions the existence of the extremal solutions for the periodic boundary value problem given by eq.(2.21) This theorem appeared in [Nieto J., et al., 2000]. Here we give the details of its proof.

Theorem (2.11):

Consider the periodic boundary value problem given by eq.(2.21). Suppose that $\alpha, \beta \in C^1(J)$, such that $\alpha < \beta$ on J. Assume in addition that:

(1) $\alpha'(t) \leq f(t, \alpha(t), [K\alpha](t))$, $t \in J$ and $\alpha(0) \leq \alpha(T)$.

(2) $\beta'(t) \geq f(t, \beta(t), [K\beta](t))$, $t \in J$ and $\beta(0) \geq \beta(T)$.

(3) f is continuous and there exist constants $M > 0$ and $N > 0$, such that:

$$f(t, x, y) - f(t, u, v) \geq -M(x-u) - N(y-v)$$

for each $t \in J$, $\alpha(t) \leq u \leq \beta(t)$, and $[K\alpha](t) \leq v \leq y \leq [K\beta](t)$.

(4) The kernel $k \in C(J\times J)$ is such that $k \geq 0$ on $J\times J$ and satisfies the inequality:

$$\|k\|_\infty \leq \frac{M}{2NT}\left(\sqrt{\left(\frac{e^{MT}-1}{MT}\right)^2 + 4} - \frac{e^{MT}-1}{MT}\right)$$

Then there exist monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ with $\alpha_0 = \alpha, \beta_0 = \beta$ such that $\lim\limits_{n\to\infty} \alpha_n(t) = p(t)$, $\lim\limits_{n\to\infty} \beta_n(t) = r(t)$ uniformly and monotonically on $[0, 2\pi]$ and that p, r are the minimal and the maximal solutions of the periodic boundary value problem given by eq.(2.21).

Proof:

For any $\eta \in [\alpha, \beta] = \{\eta \in C[[0, T], \square\,] : \alpha(t \leq \eta(t) \leq \beta(t), t \in [0,T]\}$, we define the modified linear periodic boundary value problem which consists of the first order linear ordinary differential equation:

$u'(t) + Mu(t) = -N[Ku](t) + \sigma_\eta(t),\ t \in J$ ....(2.28a)

together with the periodic boundary conditions:

$u(0) = u(T)$ .... (2.28b)

where $\sigma_\eta(t) = M\eta(t) + N[K\eta](t) + f(t, \eta(t), [K\eta](t))$.

Thus by using theorem (2.2), a unique solution exists for eq.(2.28).

Define the solution operator $A : [\alpha, \beta] \longrightarrow C(J)$ such that $A\eta = u$, where u is the unique solution for eq.(2.28).

We shall show that A possessing the following properties:

(a) If $\alpha \le \eta \le \beta$, then $\alpha \le A\eta \le \beta$.

(b) If $\alpha \le \eta_1 \le \eta_2 \le \beta$, then $\alpha \le A\eta_1 \le A\eta_2 \le \beta$.

To show (a), we must prove $A\eta \ge \alpha$ and $A\eta \le \beta$

First, we prove $A\eta \ge \alpha$. To do this, consider $v_1 = A\eta - \alpha$.

Thus $v_1 = u_1 - \alpha$, where u_1 is the unique solution of eq.(2.28). Hence:

$$v_1'(t)+Mv_1(t)+N[Kv_1](t) = u'_1(t)-\alpha'(t) + M(u_1(t) - \alpha(t)) + N[K(u_1 - \alpha)](t)$$

But K is a linear operator, then:

$$\begin{aligned} v_1'(t) + Mv_1(t) + N[Kv_1](t) &= u_1'(t) - \alpha'(t) + Mu_1(t) - M\alpha(t)) + N[Ku_1](t) \\ &\quad - N[K\alpha](t) \\ &= u_1'(t) + Mu_1(t) + N[Ku_1](t) - \alpha'(t) - M\alpha(t) - N[K\alpha](t) \\ &= M\eta(t)+N[K\eta](t)+f(t,\eta(t),[K\eta](t))-\alpha'(t)-M\alpha(t)-N[K\alpha](t) \end{aligned}$$

but from the condition (1) one can get

$$\begin{aligned} v_1'(t) + Mv_1(t) + N[Kv_1](t) \ge\ & M\eta(t)+N[K\eta](t)+f(t,\eta(t), [K\eta](t)) - \\ & f(t,\alpha(t),[K\alpha](t)) - M\alpha(t) - N[K\alpha](t) \end{aligned}$$

$$= M(\eta(t) - \alpha(t)) + f(t,\eta(t), [K\eta](t)) - f(t,\alpha(t),[K\alpha](t)) + N([K\eta](t) - N[K\alpha](t)$$

Now since $k \geq 0$ and $\alpha(t) \leq \eta(t)$, then $k(t, s)\alpha(s) \leq k(t, s)\eta(s)$, $(t, s) \in J\times J$

and hence $[K\alpha](t) = \int_0^T k(t, s)\alpha(s)\, ds \leq \int_0^T k(t, s)\eta(s)\, ds = [K\eta](t)$

On the other hand, since $k\geq 0$ and $\eta(t)\leq\beta(t)$, then

$$[K\eta](t) = \int_0^T k(t, s)\eta(s)ds \leq \int_0^T k(t, s)\beta(s)ds = [K\beta](t)$$

Therefore, for $\alpha(t) \leq \alpha(t) \leq \eta(t) \leq \beta(t)$ and $[K\alpha](t) \leq [K\alpha](t) \leq [K\eta](t) \leq [K\beta](t)$,. one can get:

$$f(t, \eta(t),[K\eta](t)) - f(t, \alpha(t),[K\alpha](t)) \geq -M(\eta(t) - \alpha(t)) - N([K\eta](t) - [K\alpha](t))$$

Therefore,

$$v_1'(t) + Mv_1(t) + N[Kv_1](t) \geq M(\eta(t) - \alpha(t)) + N([K\eta](t) - [K\alpha](t)) - M(\eta(t) - \alpha(t)) - N([K\eta](t) - [K\alpha](t))$$

$$= 0$$

Also

$$v_1(0) = u_1(0) - \alpha(0) = u_1(T) - \alpha(0) \geq u_1(T) - \alpha(T) = v_1(T)$$

Therefore $v_1(0) - v_1(T) \geq 0$

Then by using theorem (2.5)we have $v_1(t) \geq 0$, for each $t \in J$ and $A\eta(t) \geq \alpha(t)$ on J.

Second we prove $A\eta\leq\beta$. To do this, we consider $v_2 = \beta - A\eta$, thus

$$v_2 = \beta - u_2$$

where u_2 is the unique solution of eq.(2.28). Hence

$$v_2'(t) + Mv_2(t) + N[Kv_2](t) = \beta'(t) - u_2'(t) + M\beta(t) - Mu_2(t) + N[K\beta](t) + N[Ku_2](t)$$

But from the condition (2), one can obtain

$$v_2'(t) + Mv_2(t) + N[Kv_2](t) \geq f(t, \beta(t), [K\beta](t)) + M\beta(t) + N[K\beta](t) + N[Ku_2](t) - M\eta(t) - N[K\eta](t) - f(t,\eta(t), [K\eta](t))$$

Now, since $\alpha(t) \leq \eta(t) \leq \beta(t) \leq \beta(t)$ and $[K\alpha](t) \leq [K\eta](t) \leq [K\beta](t) \leq [K\beta](t)$, then from condition (3), one can have

$$f(t, \beta(t), [K\beta](t)) - f(t, \eta(t), [K\eta](t)) \geq -M(\beta(t) - \eta(t)) - N([K\beta](t) - [K\eta](t))$$

Thus

$$v_2'(t) + Mv_2(t) + N[Kv_2](t) \geq -M(\beta(t) - \eta(t)) - N([K\beta](t) - [K\eta](t)) + M(\beta(t) - \eta(t)) + N([K\beta](t) - [K\eta](t)) = 0$$

Also, $v_2(0) = \beta(0) - u_2(0) \geq \beta(T) - u_2(T) = v_2(T)$

Therefore, $v_2(0) - v_2(T) \geq 0$

Then by using theorem (2.5), we have $v_2(t) \geq 0$, for each $t \in J$ and hence $A\eta \leq \beta$.

In order to prove (b), let $\eta_1, \eta_2 \in [\alpha, \beta]$, such that $\eta_1 \leq \eta_2$. Consider $v_3 = A\eta_2 - A\eta_1$. Thus $v_3(t) = u_2(t) - u_1(t)$, where u_1 and u_2 are the unique solution of eq.(2.28) with respect to η_1 and η_2 respectively. Hence:

$$v_3'(t) + Mv_3(t) + N[Kv_3](t) = u'_2(t) - u'_1(t) + Mu_2(t) - Mu_1(t) + N[Ku_2](t) - N[Ku_1](t)$$

Thus

$v_3'(t) + Mv_3 (t) + N[Kv_3](t) = M\eta_2(t) + N[K\eta_2](t) + f(t, \eta_2(t), [K\eta_2](t)) -$

$$M\eta_1(t) - N[K\eta_1](t) - f(t, \eta_1(t), [K\eta_1](t))$$

Next, since $\alpha \le \eta_2(t) \le \beta$ and $\alpha \le \eta_1(t) \le \beta$, then from the part (a), one can have $\alpha \le A\eta_2(t) \le \beta$ and $\alpha \le A\eta_1(t) \le \beta$. Also, since $\eta_1 \le \eta_2$, then $[K\eta_1](t) \le [K\eta_2](t)$.

Hence, for $\alpha(t) \le \eta_1(t) \le \eta_2(t) \le \beta(t)$ and $[K\alpha](t) \le [K\eta_1](t) \le [K\eta_2](t) \le [K\beta](t)$, one can have:

$$f(t, \eta_2(t), [K\eta_2](t)) - f(t, \eta_1(t), [K\eta_1](t)) \ge -M\eta_2(t) + M\eta_1(t) - N[K\eta_2](t) + N[K\eta_1](t)$$

Therefore:

$$\begin{aligned} v_3'(t) + Mv_3(t) + N[Kv_3](t) &\ge M(\eta_2(t) - \eta_1(t)) + N([K\eta_2](t) - [K\eta_1](t)) - \\ &\quad M(\eta_2(t) - \eta_1(t)) - N([K\eta_2](t) - [K\eta_1](t)) \\ &= 0 \end{aligned}$$

Also

$$v_3(0) = u_2(0) - u_1(0) = u_2(T) - u_1(T) = v_3(T),$$

thus, $v_3(0) - v_3(T) = 0$

Hence by theorem (2.5) one can get $v_3(t) \ge 0$, for each $t \in J$ and hence $A\eta_2 \ge A\eta_1$. Therefore, it follows that we can define the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ defined by $\alpha_0 = \alpha$, $\beta_0 = \beta$, such that $\alpha_n = A\alpha_{n-1}$,

$\beta_n = A\beta_{n-1}$, n=1,2,

From theorem (1.1), one can get:

$$\alpha = \alpha_0 \le \alpha_1 \le \ldots \le \alpha_n \le \beta_n \le \ldots \le \beta_1 \le \beta_0 = \beta$$

and $\lim\limits_{n\to\infty} \alpha_n(t) = p(t)$, $\lim\limits_{n\to\infty} \beta_n(t) = \eta(t)$ uniformly and monotonically on J. We will show that p and r are solutions of eq.(2.21).

Since $\alpha_n = A\alpha_{n-1}$, then

$$\alpha'_n(t) + M\alpha_n(t) = -N[K\alpha_n](t) + M\alpha_{n-1}(t) + N[K\alpha_{n-1}](t) + f(t,\alpha_{n-1}(t),[K\alpha_{n-1}](t)),\ t \in J$$

Taking limit as $n \longrightarrow \infty$, one can have

$$p'(t) + Mp(t) = -N[Kp](t) + Mp(t) + N[Kp](t) + f(t, p(t), [Kp](t)),\ t \in J$$

that is

$$p'(t) = f(t, p(t), [Kp](t)),\ t \in J$$

also, since $\alpha_n(0) = \alpha_n(T)$, then $p(0) = p(T)$

Therefore, p is a solution of eq.(2.21). Similarly, we can easily check that r is another solution of eq.(2.21). The proof that p and r are the extremal solutions of eq.(2.21) is similar to that in theorem (1.1). ■

CONCLUSIONS AND RECOMMENDATIONS

From the present study, we can conclude the following:

1. One of the motivations for the study of the periodic boundary value problems for the integro-differential equations is the application to higher order mixed boundary value problems using a suitable change of variable to reduce the order.
2. Finding the extremal solutions of the periodic boundary value problems of the nonlinear ordinary differential equation is based on transforming it into one which is linear ordinary differential equations with the same order.
3. Finding the extremal solutions of the periodic boundary value problems of the linear and nonlinear integro-differential equation depends on reducing it into one which is linear integro-differential equation with the same order.
4. The expansion methods are powerful methods that can be used to solve the linear and nonlinear periodic boundary value problems for n-th order ordinary integro-differential equations.

Also for future work, we can recommend the introduction of the following open problems:

1. Finding the extremal solutions for the periodic boundary value problems of partial differential equations and partial integro-differential equations.
2. Devote some numerical and approximation methods to find the extremal solutions of the periodic boundary value problems.

3. Generalize the previous study to include the linear ordinary differential equation with non-constant coefficients and the integro-differential equations with order greater than one.
4. Solve some real life applications in which its mathematical modeling can be reduced to the periodic boundary value problems.

CONTENTS

INTRODUCTION

One of the most important, sources in mathematics is the subject of the boundary value problems, such type of problems arise in many fields of daily life, such as biology (the rate of growth of microorganism), [Aiba S., 1965], chemical engineering (an exothermic chemical reaction, heat conduction associated with radiation effects, deformation of shells), [Kamenetskii D., 1966], hydrodynamics (flow of non-Newtonian fluids on a vertical plate, effect of fluid motion on a free surface shape), [Bender C. & Orszage S., 1999], classical mechanics (calculation of N-body trajectories, nonlinear oscillation, stress analysis of solid propellant grains), [Courant R. & Hilbert D., 1953] and so on. Therefore, the boundary value problems (with or without parameters) have been a subject of study for more than two centuries ago.

Generally speaking, a boundary value problem (with or without parameters) consists of an equation (linear or nonlinear) this equation may be (differential equation, integro-differential equation, delay differential equations) with boundary conditions (linear or nonlinear). Because of this variety, a considerable amount of theoretical and numerical study has been devoted to this type of problems.

For the boundary value problems (without parameters), there are many authors who study this type of problems such as [Ince E., 1944] and [Coddington E. & Levinson N., 1955] gave some theorems for the existence and uniqueness of solution for special types of linear boundary value problems in ordinary differential equations, [Bernfeld S. & Lakshmikanntham V., 1974] used some techniques such as differential inequalities and shooting type methods to ensure the existence and uniqueness of solution for special type of nonlinear boundary value problems. [Maha A., 1983] treated with the existence and uniqueness theory of the solution for the boundary value problems of fractional order. [Kubicek M., 1983] gave some methods for solving some types of the linear and nonlinear boundary value problems in ordinary differential equations like the method of adjoints and the Green's function method, [Burden R., 1985] used the shooting method and the finite-difference method to solve the linear and nonlinear second order differential boundary value problems.

Recall that, in a boundary value problems if the coefficients of the differential equation and / or of the boundary conditions depend upon a parameters, it is frequently of interest to determine the value or values of the parametes for which such nontrivial solutions exist. These parameters are called eigenvalues and the corresponding solutions are called eigenfunctions and in this case these problems are said to be eigenvalue problems or boundary value problems with parameters.

The boundary value problems, especially the periodic value problems have many real life applications in stability theory, dynamical systems, physics, engineering and mathematical biology, [Nieto J., 1991].

Many researchers studied the periodic boundary value problems, say [Krylov., 1929] gave some approximated methods for solving the periodic boundary value problems for the ordinary differential equations.[Berernes Jand Schmitt K., 1973] studied the periodic value problems value problems for systems of second order ordinary differential equations. [Nieto J., 1988] studied non-linear second order periodic boundary value problems.[Rachůnkobá I., Tvrdý M., 2002] gave the periodic boundary value problems for nonlinear second order differential equations with impulses. [Seda V. & Nieto J. and Lois M., 1992] devoted periodic boundary value problems for non-linear higher-order ordinary differential equations. [Liz., 1996] devoted the periodic boundary value problems for a class of functional differential equations. [Xu H. & Nieto J., 1997] discussed the extremal solutions of a class of non-linear integro-differential equations in Banach spaces. [Nieto J., 2002] studied the periodic boundary value problems for the first-order impulsive ordinary differential equations. [Yongxiang L., 2004] studied the existence of positive solutions of higher-order periodic boundary value problems.

The main purpose of this work is to study the periodic boundary value problem for the integro-differential equations.

This study includes the existence and the uniqueness of the solution for the periodic boundary value problem of the first order linear and nonlinear integro-differential equations. Also, the existence of the extremal solutions for such periodic boundary value problem is discussed.

Moreover, some approximated methods are devoted to solve such types of the periodic boundary value problems.

This thesis consists of three chapters. In chapter one, we give some theorems that guarantee the existence of the extremal solution of the periodic boundary value problem for the ordinary differential equations. This chapter consists of three sections.

In section one, the existence of the extremal solutions for the periodic boundary value problem of the first order ordinary differential equation is discussed. In section two, we give some necessary conditions that ensure the existence of the extremal solutions of the periodic boundary value problem of the second order ordinary differential equations. In section three, the same previous study is devoted for the third order ordinary differential equations.

In chapter two, we devote the existence of the extremal solutions for the periodic boundary value problem of the first order linear and nonlinear integro-differential equation. This chapter consists of three sections. In section one; we give some basic concepts of the integro-differential equation. In section two, we give some existence and uniqueness theorems for the periodic boundary value problem of the first order linear integro-differential equation. Also, the existence theorems of the extremal solution for the periodic boundary value problem of the first order linear ordinary integro-differential equation are presented. In section three, we generalize the above study to be valid for the periodic boundary value problem of the first nonlinear ordinary integro-differential equation. This section constitutes the main part of our work and to the best of our knowledge is seems to be new.

In chapter three, we give some approximation methods, namely the expansion methods to be used to solve the periodic boundary value problem for the first order and nonlinear integro-differential equation. This chapter consists of two sections. In section one, we give the expansion methods, say the collocation, Galerkian and least square methods to solve the periodic boundary value problems of the first-order linear integro-differential equations with some illustrate examples. In section two, we use the same above methods to solve the periodic boundary value problems of the second-order linear integro-differential equations with some illustrate examples.

الإهـــداء

إلى الذي بذل جهد السنين سخياً....

وصاغ من الأيام سلالم العلم ...لارتقي بها إلى ذرى الحياة ....

أبي العزيز

إلى القلب الكبير الذي حمل أسراري ...

إلى الحنان الذي منحني الدفيء والاستقرار ...

والى من أخص الله الجنة تحت قدميها...

أمي الحنون

إلى العيون البريئة التي تنظر ألي بحب....

أخواتي الثلاثة

إلى الذين واكبوا معي طريق السنين بإخلاص...

أصدقائي وزملائي الأعزاء

إلى الشموع التي أضاءت طريق العلم...

أساتذتي

نور قاسم

المستخلص

الهدف الرئيسي من هذا العمل مصنف إلى أربع محاور والتي يمكن تلخيصها كالآتي:

الهدف الأول: دراسة مبرهنات وجود ووحدانية الحلول لمسائل القيم الحدودية الدورية للمعادلات التفاضلية الاعتيادية .

الهدف الثاني: تبني مبرهنات الوجود للحلول الحرجة لمسائل القيم الحدودية الدورية السابقة .

الهدف الثالث: اعطاء مبرهنات وجود ووحدانية الحلول لمسائل القيم الحدودية الدورية للمعادلات التكاملية التفاضلية الخطية واللاخطية الاعتيادية . كذلك اعطاء مبرهنات الوجود للحلول الحرجة لمسائل القيم الحدودية الدورية للمعادلات التكاملية التفاضلية .

الهدف الرابع: حل مسائل القيم الحدودية الدورية للمعادلات التكاملية التفاضلية باستخدام طرق المفكوك .

REFERENCES

[1] Aftabizadeh A., Gupta C. & Xuj, "Periodic Boundary Value Problems for Third Order Ordinary Differential Equations", Nonlinear Anal., Vol.14, pp.1-10, (1990).

[2] Agarwal R., "Boundary Value Problems for Higher Order Differential Equations", World Scientific, Singapore, (1986).

[3] Aiba S., Humphrey A. & Mills N., "Biochemical Engineering", Academic Press, New York, (1965).

[4] Bender C. & Orszage S., "Advanced Mathematical Methods for Scientists and Engineers I", Springer-Verlag, New York, Inc., (1999).

[5] Bernfeld S. & Lakshmikanntham V., "An Introduction to Nonlinear Boundary Value Problems", Prentice-Hall, Inc., (19 74).

[6] Boyce D., "Elementary differential equations And Boundary Value Problems", John Wiley& Sons, Inc., (2001).

[7] Burden R. & Faires J., "Numerical Analysis", Prindle, Weber & Schmidit Publishers, (1985).

[8] Chambers L., "Integral Equations; A Short Course", International Text Book Company Limited, (1976).

[9] Coddington E. & Levinson N., "Theory of Ordinary Differential equations", McGraw-Hill Book Company, (1955).

[10] Courant R. & Hilbert D., "Methods of Mathematical Physics", Vol.I, New York, (1953).

[11] Delves L. & Mohamed J., "Computational Methods for Integral Equations ", Cambridge University Press (1985).

[12] Guenther R., Lee J., "Boundary Value Problems for a Class of Integro-differential equations and Applications', Office of Naval Research, (1999).

[13] Kubick M. & Hlavacek V. "Numarical solution of Nonlinear Boundary Value Problems with applications ", Prentice-Hall, Inc., (1983).

[14] Guenther R., Lee J., "Boundary Value Problems for a Class of Integro-differential equations and Applications', Office of Naval Research, (1999).

[15] Ince E., "Ordinary Differential Equations", Dover, New York, (1944).

[16] Kamenetskii D., "Diffusion and Heat Transfer in Chemical Kinetics', Plenum Press, New York, (1966).

[17] Lakshmikamtham V. & Hu S., "Periodic Boundary Value Problems for Integro-Differential Equations of Volterra Type", Nonlinear Analysis: TMA 10, pp.1203-1208, (1986).

[18] Lakshmikamtham V. & Leela S., "Existence and Monotone Method for Periodic Solutions of First Order Differential Equations", Journal of Mathematical Analysis and Applications, Vol.91, pp.237-243, (1983).

[19] Lakshmikamtham V. & Leela S., "Remarks of First and second Order Periodic Boundary Value Problems", Nonlinear Analysis, Theory, Methods & Applications: Vol.8, No.3, pp.281-287, (1984).

[20] Levitem P. and Zhikov V., "Almost Periodic Functions and Differential Equations", (1978).

[21] Maha A., "Existence and Uniqueness Theorems for Differential Equations of Fractional Order", M.Sc. Thesis, Baghdad University, (1983).

[22] Narie, N.," Variational Formulation of Delay Differential Equations", M.Sc. Thesis, Department of Mathematics, College of Education Ibn Al-Haithem, Baghdad University, Baghdad, Iraq, (2001).

[23] Nieto J., "Periodic Solutions for Third Order Ordinary Differential Equations", Comment. Math. Univ. Carolina., Vol. 32, No.3, pp.495-499, (1991).

[24] Nieto J., "Periodic Boundary Value Problems for Second Order Integro-Ordinary Differential Equations with General Kernal and Carathéodory Nonlinearities", Internet. J. Math. Math. Sci., Vol.18, pp.757-764, (1995)

[25] Nieto J., "Periodic Boundary Value Problems for First Order Impulsive Ordinary Differential Equations", Nonlinear Analysis, Vol.51, (2002)..

[26] Nieto, J. & Leela S., "Nonlinear Second Order Periodic Boundary Value Problems", J. Math. Anal. Appl., Vol.130, pp.22-29, (1988).

[27] Nieto J. & Liz E. "Periodic Boundary Value Problem for Integro-Differential Equations with General Kernal", Dynam. Systems Appl., Vol.3, No.297-304, (1994).

[28] Nieto J. & Liz E." Periodic Boundary Value Problem for a Class of Functional Differential Equations", Journal of Mathematical Analysis and Applications, Vol.200, pp.680-686, (1996).

[29] .Nieto J. and Liz E. and Franco D." Periodic Boundary Value Problem for Nonlinear First Order Ordinary Integro Differential Equations", Integro and Integral Equations, (2000).

[30] Poorkarimi H. & Wiener J., "Bounded Solutions of Nonlinear Hyperbolic Equations with Delay", Nonlinear Anal. and Appl., Marcel Dekker Inc., (1987).

[31] Poorkarimi H. & Wiener J., "Bounded Solutions of Nonlinear Hyperbolic Equations with Delay", Nonlinear Anal. and Appl., Marcel Dekker Inc., (1987).

[32] Poorkarimi H. & Wiener J., "Bounded Solutions of Nonlinear Parabolic Equations with Time Delay", Elect. J. Diff. Eq., Conference 02, pp. 87-91, (1999).

[33] Rachůnková I. & Tvrdý M., "Periodic Boundary Value Problems for Nonlinear Second Order Differential Equations with Impulses-Part II", (to Appear).

[34] Rama M., "Ordinary Differential Equations ", (1980).

[35] Ravi P. & Donal O., "Integral and Integral Equations Theory, Methods and Applications", Gordon and Breach Science Publishing, Canada, (2000).

[36] Saaty T., "Modern Nonlinear Mathematics", Dover, New York, (1967).

[37] Salih S. "Approximated Solution of Linear Delay Differential Equations", M.Sc. Thesis, School of Applied Sciences, Technology University, Baghdad, Iraq, (2004).

[38] Smith G., "Numerical Solution of Partial Differential Equations", Oxford University Press, (1965).

[39] Vandewall S. & Gander M., "Optimized Over Lapping Schwarz Methods for Parabolic PDE's with Time-Delay", Conference on Domain Decomposition Methods, Berlin, Germany, (2003).

[40] Wu J., "Theory and applications of Partial Differential Equations", Springer-Verlag, New York, (1996).

[41] Xu H &Nieto J.., "Extremal Solutions of a Class of Nonlinear Integro Differential Equations in Banach Spaces", Proceedings of the American Mathematical Society, Vol.125, No.9, (1997), pp.2605-2614

[42] Yongxiang L., "Positive Solutions of Higher-Order Periodic Boundary Value Problems", Computers and Mathematics with Applications, Vol. 1, No. 48, (2002).

[43] Zeena H., "About the Solutions of the Generalized Linear and Nonlinear Eigenvalue Problems", M.Sc. Thesis, Department of Mathematics, College of Science, Al-Nahrain University, (2002).

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المفاتيح الاستدلالية :-, integro differential equation

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A Thesis

Submitted to the College of Science of Al-Nahrain University, as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

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SUPERVISOR CERTIFICATION

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In view of the available recommendations, I forward this thesis for debate by the examining committee.

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Date: / / 2006

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التكاملية التفاضلية الاعتيادية

رسالة مقدمه إلى
كلية العلوم في جامعة النهرين كجزء من متطلبات نيل درجة ماجستير
علوم في الرياضيات

من قبل

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