ABSTRACT

Fractional calculus is the subject of evaluating derivatives and integrals of non-integer orders of a given function, while fractional differential equations (considered in this work) is the subject of studying the solution of differential equations of fractional order, which contain initial conditions. The general form of a fractional differentia equation is given by:

$$y^{(q)} = f(x, y),\ y^{(q-k)}(x_0) = y_0$$

where $k = 1, 2, \ldots, n + 1$, $n < q < n + 1$, and n is an integer number. The solution of fractional differential equations has so many difficulties in their analytic solution, therefore numerical methods may be in most cases be the suitable method of solution.

Therefore, the objective of this work is to introduce and study several approximate methods for solving fractional differential equations numerically with the cooperation of linear multistep methods for solving numerically differential equations and Riemann-Liouville formula of fractional integration.

ACKNOWLEDGEMENTS

Praise to Allah, for providing me the willingness and strength to accomplish this work.

It is pleasure to record my indebtedness to my supervisor Dr. Fadhel Subhi Fadhel for his help, appreciable advice, important comments, support and encouragement during the work.

I would like to thank the college of Science. Also, my special thanks to the staff of the Department of Mathematics, College of Science, Al-Nahrain University for all help and advice they kindly gave to me.

Finally, thanks are extended to my family and friends who gave me the all help, support, encouragement and patience.

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April, 2006

CHAPTER 1

FRACTIONAL CALCULUS

The subject of fractional calculus has a long history whose infancy dates back to the beginning of classical calculus and it is an area having interesting applications in real life problems. This type of calculus has its origins in the generalizations of the differential and integral calculus [Lubich, 1986].

This chapter consists of five sections, in section (1.1), detailed historical background about the subject of fractional differentiation and fractional integration is given with leading references about the subject are given for the interested readers.

In section (1.2), basic concepts related to the subject of fractional calculus are given which are necessary for the rest of this thesis. Among such fundamental concepts, the gamma function and the Riemann – Liouville formula.

In section (1.3), and as a tool for differentiation and integration of fractional order several types of fractional derivatives, are given and discussed in details with some basic related properties.

In order to give well understanding about the subject of fractional differentiation, some well known examples are given in section (1.4), such as the fractional differentiation of the unit function, zero function exponential function, etc.

1.1 HISTORICAL BACKGROUND

In the earlier work, the main application of fractional calculus as a technique for solving integral equations. Recently fractional derivatives have been used to model physical processes leading to the formulation of fractional differential equations. The fractional calculus may be considered as an old and yet a novel topic. It is an old topic since it's starting in 1695. L'Hospital was the first researcher who asked in a letter to Leibnitz on the possibility to performing calculations by means of fractional derivatives of order r = ½. Leibnitz answered this question looked as a Paradox to him (see [Madueno, 2002]).

In (1697), Leibnitz referring to the infinite product of Walls for π/ 2 used the notation $d^{1/2}y$ and summarized that the fractional calculus could be used t o get the same results.

The earliest more or less systematic studies seem to have been made in the beginning and middle of the 19th century by Liouville (1832), Riemann (1953), and Holmgren (1864), although Euler (1730), Lagrange (1772), and others made contributions even earlier. It was Liouville (1832) who expanded functions in series of exponentials and defined the q-th derivative of such a series by operating term-by-term as though q, where a positive integer.

Riemann in (1953), proposed a different definition that involved a definite integral and was applicable to power series with no integer exponents. Also, Grunwald in (1867), disturbed by the restriction of Liouville's approach.

Then these theoretical beginnings were a development of the applications of the fractional calculus to various problems. The first of these was discovered by Able in (1823), that the solution of the integral equation for the tautochrone could be accomplished via an integral transform. A Powerful stimulus to the use of fractional calculus to solve real life problems was provided by the development by Boole in (1844), of symbolic methods for solving linear differential equations with constant coefficients.

In the twentieth century, some notable contributions have been made to both the theory and application of fractional calculus, Weyl (1917), Hardy (1917), Hardy and Littewood (1932), Kober (1940), and Kuttner (1953), examined some rather special, but natural, properties of differintegrals of functions belonging to Lebesgue and Lipschitz classes, Erdely (1954), and Oster (1970), have given definitions of differintegrals with respect to arbitrary functions, and Post (1930) used difference quotient to define generalized differentiations for fractional operators, Riesz (1949), has developed a theory of fractional integration for functions of more than one variable, Erdely (1965), has applied the fractional calculus to integral equations and Higgins (1967), has used fractional integral operators to solve differential equations.

However, fractional calculus may be considered as a novel topic, as well as, since only from a little more than to the later fifty years, it has been an object of specialized conferences and treatises. For the first conference the merit is a scribed to B. Ross who organized the first conference on fractional calculus, and its application at the University of New Haven in June 1974.

For the first monograph the merit is ascribed to K.B Oldham and J. Spanier (1974), who after a joint collaboration started in 1968, published a book devoted to fractional calculus in 1974. The first texts and proceedings devoted solely or partly to fractional calculus and its applications are, [Davis H.T., 1927], [Evdely A., 1939], [Igor Poldlubny, 1999].

1.2 BASIC CONCEPTS

In the present section, some fundamental concepts related to the subject of fractional calculus are given in order to avoid vague notions in this subject.

1.2.1 Gamma and Beta Functions[Oldham 1974]:

Undoubtedly, one of the basic functions encountered in fractional calculus is the Euler's gamma function $\Gamma(x)$, which generalizes the ordinary definition of factorial of a positive integer number n and allows n to take also any non- integer positive or negative and even complex values.

As it is known, the gamma function $\Gamma(x)$ is defined using the following improper integral:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \; x > 0 \text{ ....} (1.1)$$

First of all, it is easy to show that the gamma function for a natural number can be proved also to satisfy:

Γ (x) = (x – 1)! And Γ (x) = (x-1)Γ (x – 1)

which enable us to calculate for any positive real x the gamma function in terms of the fractional part of x.

The expression $\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}$ may be regarded as the binomial coefficient, as follows:

$$\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} = \frac{(j-q-1)(j-q-2)...(-q+1)(-q)}{j!} = (-1)^j \binom{q}{j} \text{ ....} (1.2)$$

where; $\binom{q}{j} = \frac{q!}{j!(q-j)!}$.

Also an important functionin fractional differential equations is the beta function defined by:

$$\beta(p,q) = \int_0^1 y^{p-1}[1-y]^{q-1}\,dy, \qquad p > 0 < q,$$

1.2.2 Riemann – Liouville Formula of Fractional Derivatives:

Riemann and Liouville in (1832) introduced a differential operates of fractional order q>0 to take the from:

$$D^q{}_t y(t) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_{t_\circ}^{t} \frac{y(u)}{(t-u)^{q-m+1}} du \quad \text{.......................................(1.3)}$$

where m is an integer defined by $m - 1 < q < m$, (see [Oldham and Spanier, 1974]).

Equation (1.3) is a Volterra integral equation with singular kernel. Differential equations involving these fractional derivatives have proved to be valuable tolls in the modeling of many physical phenomena.

1.3 FRACTIONAL CALCULUS

Fractional differentiation and integration could be defined using several approaches depending on the used definition of differentiations. Therefore, this section present some of these types of differentiation.

1.3.1 Fractional Derivative:

The usual formulation of the fractional derivative, given in standard references such as [Samko, 1993], [Oldham and Spanier, 1974] is the Riemann-Liouville differential equations which require initial values expressed as fractional derivatives.

This is very inconvenient, since it is usually not clear what the physical meaning of these fractional order initial value would be and they are therefore

hard to drive from a physical system. In applications, it is often more convenient to use the formulation of the fractional derivative suggested by

Caputo1971, which is known as Grunwald derivatives which requires the same starting conditions as in ordinary differential equations of the next higher order.

The Grunwald definition of fractional derivatives is given by:

$$\frac{d^q f(t)}{dt^q} = \lim_{N\to\infty} \frac{\left(\frac{t}{N}\right)^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(t - j\left(\frac{t}{N}\right)\right) \qquad (1.4)$$

where $q < 0$ indicates fractional integration and $q>0$ indicates fractional differentiation.

The Reiman-Liouvilli definition of fractional derivative given by:

$$D^q{}_t y(t) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_{t_\circ}^{t} \frac{y(u)}{(t-u)^{q-m+1}} du \qquad (1.5)$$

Thomas J. Osler definition of fractional derivative is given by:

$$D^q{}_{t-a} y(t) = \frac{\Gamma(q+1)}{2\pi i} \int_0^{t_+} (u-t)^{-q-1} y(u) du, \qquad (1.6)$$

where he made a branch cut from t to a and the integral curve is an open which starts from a and encloses t in positive sense and return to a.

The Bertram Ross definition of fractional derivative is given by:

$$\frac{d^q}{dt^q} y(t) = \frac{\Gamma(q+1)}{2\pi i} \int_c \frac{y(u)}{(u-t)^{q+1}} du \qquad (1.7)$$

where he made a branch cut from t to infinity through the origin and integral curve C is an open contour which encloses t in the positive sense and $z \notin C$ (i.e., C is an integral curve a long that cut).

The equivalent between these formulas could be proved, but it have more computations therefore it is omitted.

1.3.2 Fractional Integration:

The common formulation for the fractional integral can derive directly from a traditional expression of the repeated integration of a function. This approach is commonly referred to as Riemann – Liouville approach.

The Riemann-Liouville definition of fractional integral is given by:

$$f_q^{+}(a,x) = \frac{1}{\Gamma(q)} \int_a^x (x-t)^{q-1} f(t)dt \text{(right hand integration)}$$

$$f_q^{-}(x,b) = \frac{1}{\Gamma(q)} \int_x^b (t-x)^{q-1} f(t)dt \text{(left hand integration)}$$

The Weyle definition of fractional integral is given by:

$$f_q^{+}(-\infty,x) = \frac{1}{\Gamma(q)} \int_{-\infty}^{x} (x-t)^{q-1} f(t)dt \quad \text{...(1.8)}$$

$$f_q^{-}(x,\infty) = \frac{1}{\Gamma(q)} \int_x^{\infty} (t-x)^{q-1} f(t)dt \quad \text{...(1.9)}$$

where f(t) is a periodic function and its mean value for one period is zero. But the formula (1.8), and (1.9), are used as the definition of the integral without any condition at the present time.

1.4 FRACTIONAL DIFFERENTIATION OF SOME WELL KNOWN FUNCTIONS

In this section, some fractional derivatives using Gruuwald definition will be evaluated as an illustrative examples to fractional differentiations. Other function could be derived, such as sinh($\sqrt{x}$), sin($\sqrt{x}$), etc.,(see [Oldham and Spanier, 1974]).

1.4.1 The Unit Functions f=1 [Oldham and Spanier, 1974]:

Consider first the differintegral to order q of the function f = 1, for which it is found convenient to reserve the special notation. This function will be referred as the unit function. Straight forward application of equation

$$\frac{d^q f}{\left[d(x-a)\right]^q} = \lim_{N\to\infty}\left\{\frac{\left[\frac{x-a}{N}\right]}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(x - j\left(\frac{x-a}{N}\right)\right)\right\}$$

to the function f = 1, gives:

$$\frac{d^q[1]}{\left[d(x-a)\right]^q} = \lim_{N\to\infty}\left\{\left[\frac{N}{x-a}\right]^q \sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}\right\}$$

Application of the following equations:

$$\sum_{j=0}^{n-1}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} = \frac{\Gamma(N-q)}{\Gamma(1-q)\Gamma(N)}$$

and

$$\lim_{j\to\infty}\left[j^{c+q+1}\frac{\Gamma(j-q)}{\Gamma(j+1)}\right] = \lim_{x\to\infty}\left[j^{c+q}\frac{\Gamma(j-q)}{\Gamma(j)}\right] = \begin{cases} +\infty, \rangle 0 \\ 1, c = 0 \\ 0, c\langle 0 \end{cases}$$

gives:

$$\frac{d^q[1]}{\left[d\left(x-a\right)\right]^q} = \lim_{N\to\infty}\left\{\left[\frac{N}{x-a}\right]^q \frac{\Gamma(N-q)}{\Gamma(1-q)\Gamma(N)}\right\} = \frac{\left[x-a\right]^{-q}}{\Gamma(1-q)}$$

Therefore:

$$\frac{d^q[1]}{\left[d\left(x-a\right)\right]^q} = \frac{\left[x-a\right]^{-q}}{\Gamma(1-q)}.$$

1.4.2 The Zero Function [Oldham, 1974]:

For a function f = c, where c is any constant including zero, this may be indicated using the differential operator representation:

$$D_0^q = \frac{d^q f(t)}{dt^q} = \lim_{N\to\infty}\left[\left(\frac{t}{N}\right)^{-q}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} f(t - j\frac{t}{N})\right]$$

One can see that:

$$\frac{d^q[c]}{[d(x-a)]^q} = c\frac{d^q[1]}{[d(x-a)]^q} = c\frac{[x-a]^{-q}}{\Gamma(1-q)}$$

Since $\frac{d^q[1]}{[d(x-a)]^q}$ does not approach infinity for x > a, it is concluded by setting c = 0, that:

$$\frac{d^q[0]}{[d(x-a)]^q} = 0 \text{, for all q} \quad \text{.........(1.10)}$$

Equation (1.10) may be appearing to be trivial or obvious. As an example of its importance, however, observe that it provides a powerful counter example to that, if:

$$\frac{d^q f}{[d(x-a)]^q} = g, \quad \text{then, } \frac{d^{-q} g}{[d(x-a)]^{-q}} = f$$

For, if gives zero on differentiation to order q, then f cannot be restored by q-order integration.

1.4.3 The Function $(x - a)^p$ [Oldham, 1974]:

The function of fractional degree we consider in this subsection is an important function given by $f = (x - a)^p$, where p is initially arbitrary, we shall see, however, that p must exceed −1 for differintegration to have the properties we demand of the operator. For integer n of either sign, one can show that:

$$\frac{d^n x^p}{dx^n} = p(p-1)\ldots(p-n+1)x^{p-n}, \; n = 0, 1, \ldots$$

from classical calculus. Our first encounter with non-integer q, will be restricted to negative q so that we may exploit the Riemann-Liouville definition. Thus:

$$\frac{d^q[x-a]^p}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)}\int_a^x \frac{[y-a]^p}{[x-y]^{q+1}}dy$$

$$= \frac{1}{\Gamma(-q)}\int_0^{x-a} \frac{v^p}{[x-a-v]^{q+1}}dv, \; q < 0$$

where v has replaced by y − a. By further replacement of v by [x − a]u. The integral may be cast into the structure of beta function form:

$$\frac{d^q[x-a]^p}{[d(x-a)]^q} = \frac{[x-a]^{p-q}}{\Gamma(-q)}\int_0^1 u^p[1-u]^{-q-1}du, q < 0 \quad \text{.............................(1.11)}$$

The integral in (1.11), will be recognized as the Beta function, $\beta(p + 1, -q)$ provided both arguments are positive, therefore:

$$\frac{d^q[x-a]^p}{[d(x-a)]^q} = \frac{[x-a]^{p-q}}{\Gamma(-q)}\beta(p+1,-q)$$

$$= \frac{\Gamma(p+1)[x-a]^{p-q}}{\Gamma(p-q+1)}, q<1, p>-1 \text{ ....(1.12)}$$

which is the fractional derivative of $[x - a]^p$.

1.4.4 The Expositional Function exp(r − cx) [Oldham,1974]:

With r and c are an arbitrary constants, then the power-series expansion is given by:

$$\exp(r-cx) = \exp(r-ca)\sum_{j=0}^{\infty}\frac{[-c(x-a)]^j}{\Gamma(j+1)}$$

which is valid for all x − a.

Differintegration term-by-term with respect to c[x − a], yields:

$$\frac{d^q \exp(r-cx)}{[d(cx-ca)]^q} = \{c(x-a)\}^{-q}\exp(r-ca)\sum_{j=0}^{\infty}\frac{\{-c(x-a)\}^j}{\Gamma(j-q+1)}$$

The sum may be expressed as an incomplete gamma function of argument −c[x − a] and parameter −q, then the final result appears as:

$$\frac{d^q \exp(r-cx)}{[d(x-a)]^q} = \frac{\exp(r-cx)}{[x-a]^q}\gamma^*(-q,-c(x-a))$$

where

$$\gamma^*(c,x) = \frac{c^{-x}}{\Gamma(x)}\int_0^c y^{x-1}\exp(-y)dy = \exp(-x)\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+c+1)}$$

where $\gamma^*(-n,y) = y^n$ for non negative integer n. The above result is seems to reduce to the well-known formula for multiple differentiation of an exponential function, reduction to the simple formula:

$$\frac{d^q \exp(\mp x)}{dx^q} = \frac{\exp(\mp x)}{x^q}\gamma^*(-q,\mp x)$$

occurs on substituting k = a = 0 and c = ±1 into the general result.

1.4.5 The Functions $\frac{x^q}{1-x}$ and $\frac{x^p}{1-x}$ [Oldham,1974]:

By using of the binomial expansion of $(1 - x)^{-1}$ and the technique of term-by-term

differentegration which is from the (linearity of differentiation), we arrive at:

$$\frac{d^q}{dx^q}\left[\frac{x^q}{1-x}\right]=\sum_{j=0}^{\infty}\frac{d^q}{dx^q}x^{j+q}$$

As a formula expressing the effect of $\frac{d^q}{dx^q}$ operator with the lower limit zero on the $\frac{x^q}{1-x}$ function.

Subject to the proviso that x not exceed unity in magnitude. Provided also that q exceed -1, the rules of subsection (1.4.3) permit differintegration of the powers of x and lead to:

$$\frac{d^q}{dx^q}\left[\frac{x^q}{1-x}\right]=\sum_{j=0}^{\infty}\frac{\Gamma(j+q+1)}{\Gamma(j+1)}x^j=\Gamma(q+1)\sum_{j=0}^{\infty}\binom{-q-1}{j}[-x]^j$$

Identification of the sum as a binomial expansion produces

$$\frac{d^q}{dx^q}\left[\frac{x^q}{1-x}\right]=\frac{\Gamma(q+1)}{[1-x]^{q+1}}$$

as the simple final result.

The technique for differintegrating $\frac{x^p}{[1-x]}$ follows such a similar result, that is, it will suffice to cite one intermediate and the final result:

$$\frac{d^q}{dx^q}\left[\frac{x^p}{1-x}\right]=x^{p-q}\sum_{j=0}^{\infty}\frac{\Gamma(j+p+1)x^j}{\Gamma(j+p-q+1)}=\frac{\Gamma(p+1)\beta_x(p-q,q+1)}{\Gamma(p-q)[1-x]^{q+1}}$$

together with the restriction, namely, $0 < x < 1$ and $p > -1$, which where assumed during the derivation.

CHAPTER

3

NUMERICAL AND APPROXIMATE METHODS FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS

Sometimes, numerical methods for solving differential equation are more reliable than analytic methods, especially in fractional differential equations, since such type of equations has some difficulties in their methods of solution, which could not be handled easily.

This chapter consists of five sections. In section 3.1, we study linear multistep methods and it's ability for solving fractional differential equations numerically. In sections 3.2, and 3.3, we modify the approach followed in linear multistep methods for solving fractional differential equations by altering the basis of the method to be cubic spline basis or using cubic spline interpolation with three node points, while in section 3.4, cubic spline basis depended on five knot points is used. Finally, in section 3.5 an illustrative example is given in order to compare between these methods.

3.1 LINEAR MULTISTEP METHODS [ABD AL-QAHAR, 2004]

This section presents an introduction to the theory of linear multistep methods (LMM's in short). Consider the initial value problem for a single first-order differential equation:

$$y'(x) = f(x, y(x)),\ y(x_0) = y_0 \qquad (3.1)$$

where f is a given continuous function and x_0, y_0 are fixed. We seek for the solution in the range $a \le x \le b$, where a and b are given and finite.

Consider the sequence of points $\{x_n\}$, defined by: $x_n = a + nh$, n = 0, 1, …, N, where h=a-b/2. The parameter h, which will always be regarded as a constant. As essential property of the majority of computational methods for the solution of equation (3.1), is that of discritization, that is, we seek for an approximate solution, not on the continuous interval $a \le x \le b$, but on the discrete set of point $\{y_n \mid n = 0, 1, \dots, N\}$. Let y_n be an approximation to the theoretical solution at x_n, that is, to $y(x_n)$, and let $f_n = f(x_n, y_n)$, [Lambert, 1973].

If a computational method for determining the sequence $\{y_n\}$ takes the form of Linear Multistep Method, of step number k, or a linear k-step method. Then the general form of LMM may thus be written as:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (3.2)$$

where α_j and β_j, are constants to be determined. We assume that $\alpha_k \neq 0$ and that not both of α_0 and β_0, equals zero. Since equation (3.2) can be multiplied on both sides by the same constant without altering the relationship, the coefficients α_j and β_j, are arbitrary to the extent of a constant multiplier. We remove this arbitrariness by assuming throughout that $\alpha_k = 1$. Thus the problem of determining the solution y(x), of the general non-linear initial value problem we replace equation (3.1) by that of finding the sequence $\{y_n\}$, which satisfies the difference equation (3.2). Note that, since f_n is in general non-linear function of y_n, then equation (3.2) is a non-linear difference equation. Such equations are no easier to handle theoretically as in linear differential equations, but they have the practical advantage of permitting us to compute the sequence $\{y_n\}$ numerically.

In order to do this, we must first supply the assistant of starting values $y_0, y_1, \ldots, y_{k-1.}$ (In the case of a one-step method, only one of such value which is y_0 is needed and we normally choose y_0 to be constant).

As a classification to the LMM we say that the LMM is explicit if $\beta_k = 0$ and implicit if $\beta_k \neq 0$. For an explicit method equation (3.2) yields the current value y_{n+k} directly in terms of previous y_{n+j}, f_{n+j}, $j = 0, 1, \ldots, k - 1$, which at this stage of the computation, have already been calculated while in implicit methods, however, will call for the solution at each stage of the computation, of the equation:

$$y_{n+k} = h\beta_k f(x_{n+k}, y_{n+k}) + g \quad \ldots\ldots (3.3)$$

where g is a known function of the previously calculated values y_{n+j}, f_{n+j}, $j = 0, 1, \ldots, k - 1$.

When the original differential equation (3.1) is linear, then equation (3.3) is also linear in y_{n+k}, and there is a unique solution for y_{n+k}, while when f is non-linear, then there is a unique solution for y_{n+k}, which can be approached arbitrarily closely by the iteration:

$$y_{n+k}^{[s+1]} = h\beta_k f(x_{n+k}, y_{n+k}^{[s]}) + g, y_{n+k}^{[0]}$$

Thus implicit methods in general entail a substantially greater computational effort than do explicit methods; on the other hand, for a given step number, implicit methods can be made more accurate than explicit ones and, moreover, enjoy more favorable stability properties. Then, the necessary and sufficient conditions for LMM to have an order p can be studied by using two associated polynomials, which are:

The First characteristic polynomial of LMM (3.2), is given by:

$$\rho(r) = \sum_{j=0}^{k} \alpha_j r^j = \alpha_k r^k + \alpha_{k-1} r^{k-1} + \ldots + \alpha_0$$

while the second characteristic polynomial is given by:

$$\sigma(r) = \sum_{j=0}^{k} \beta_j r^j = \beta_k r^k + \beta_{k-1} r^{k-1} + \ldots + \beta_0$$

Also, it is important to notice that if $\sigma(r)$ is given, then we can find a unique polynomial $\rho(r)$ of degree k such that the method has an order $p \geq k$, such that, we can consider the LMM according to the roots of the first characteristic polynomial $\rho(r)$ and whether it is explicit or implicit.

(1) If the roots of $\rho(r)$ equal to 1 and 0, then the method is called of Adam's type and if the LMM is explicit, then it is called of Adam Bashforth type, while if it is implicit then it is called of Adam-Moulton type, i.e., in Adam’s methods, we have the following:

$$\begin{aligned} \rho(r) &= r^k - r^{k-1} \\ &= r^{k-1}(r-1) = 0 \end{aligned}$$

(2) If the roots of $\rho(r)$ equals to -1, 0 and 1, then the method is called of Nystrom type if it is explicit and if the method is implicit, then it is called of Milne-Simpson type, i.e., we have:

$$\begin{aligned} \rho(r) &= r^k - r^{k-2} \\ &= r^{k-2}(r^2 - 1) \\ &= r^{k-2}(r-1)(r+1) \end{aligned}$$

Now, we explain the consistency, convergence and zero stability of LMM's, such that, a basic property which we shall demand of an acceptable LMM is that the solution $\{y_n\}$ generated by the method converges, in some sense to the theoretical solution y(x) as the step length h tends to zero. The LMM is said to be consistent with the initial value problem

$$y' = f(x, y),\ y(x_0) = y_0$$

if it has an order at least p = 1, i.e., consistent method implies that at least $C_0 = C_1 = 0$. But $C_2 \neq 0$, or:

$$\sum_{j=0}^{k} \alpha_j = 0 \quad \text{and} \quad \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$$

Finally the LMM is said to zero-stable (0-stable) if all the roots r_j's, j = 1, 2, ..., k; of $\rho(r) = 0$ satisfy the condition $|r_j| \leq 1$ and if r_j is a multiple zero of $\rho(r)$ then $|r_j| < 1$.

3.2 THE PREDICTOR-CORRECTOR METHOD [Dielthelm and Alan, 1997]

The definition of the fractional derivatives and some well known results of fractional calculus tell us that we interpret fractional differential equations such as:

$$D^q y = f(t, y(t)),\ y(t_0) = y_0,\ n < q < n + 1,\ n \in N \quad (3.4)$$

and hence upon taking D^{-q} to the both sides of (3.4), yields:

$$D^{-q}D^q y = D^{-q}f(t, y(t)),\ y(t_0) = y_0,\ n < q < n + 1,\ n \in N \quad (3.5)$$

Alternatively, we can apply fractional integral operator to the differential equation and incoorperate the initial conditions, thus converting equation (3.4), into the following equivalent equation:

$$y(t) = y(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{1}{(t-u)^{1-q}} f(u, y(u))du \quad (3.6)$$

which is a Volterra singular integral equation of the second kind.

In the following, we shall present the scheme for the numerical solution of the general fractional differential equation (3.4), [Gorenflo R., 1997].

In the development, we have in mind that these fractional differential equations are coupled with the first-order differential equation, which give us the general advice to these two algorithms in such a way that both methods are based on very similar construction principles, we thus choose an Adams-Bashforth-Moulton approach for both integrators.

The key to the derivation of the method is to replace the original fractional differential equation (3.4), by an equivalent singular Volterra integral equation (3.6), and to implement a product integration method for the latter. What we do is simply to use the trapezoidal quadrature formula with nodes t_j ($j = 0, 1, \ldots, n + 1$), taken with respect to the weighted function $(t_{n+1} - u)^{q-1}$, to replace the integral. In other words, we apply the approximation:

$$\int_{t_0}^{t_{n+1}} (t_{n+1} - u)^{q-1} g(u)du \approx \int_{t_0}^{t_{n+1}} (t_{n+1} - u)^{q-1} g_{n+1}(u)du \;\ldots\ldots\ldots\ldots\ldots\ldots\; (3.7)$$

where g_{n+1} is the piecewise linear interpolent for g whose nodes are chosen at the t_j, $j = 0, 1, \ldots, n + 1$. Then by using Legendre quadrature integration method yields that we can rewrite the integral on the right-hand side of equation (3.7) as:

$$\int_{t_o}^{t_{n+1}} (t_{n+1} - u)^{q-1} g_{n+1}(u)du = \sum_{j=0}^{n+1} a_{j,n+1}\, g(t_j) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (3.8)$$

where:

$$a_{j,n+1} = \int_{t_0}^{t_{n+1}} (t_{n+1} - u)^{q-1} \phi_{j,n+1}(u)du \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (3.9)$$

and:

$$\phi_{j,n+1}(u) = \begin{cases} (u - t_{j-1})/(t_j - t_{j-1}), \text{if}, t_{j-1} < u < t_j \\ (t_{j+1} - u)/(t_{j+1} - t_j), \text{if}, t_j < u < t_{j+1} \\ 0, \qquad \text{otherwise} \end{cases} \qquad (3.10)$$

where $a_{j,n+1}$ are termed as the coefficients of the method and $\phi_{j,n+1}(u)$ as the basis functions.

Next, we will present the derivation of equation (3.9) and (3.10).

3.2.1 Derivation of the Basis Functions $\phi_{j,n+1}$:

In order to derive the linear basis $\phi_{j,n+1}$, j = 0, 1, …, n+1, where n is the number of node points. Since for the general form of a straight line joining two points applied to $(t_{j-1}, 0)$ and $(t_j, 1)$, i.e., for $t_{j-1} < u < t_j$, we have:

$$\frac{\phi - 0}{u - t_{j-1}} = \frac{1 - 0}{t_j - t_{j-1}}$$

and hence:

$$\phi_{j,n+1} = \frac{u - t_{j-1}}{t_j - t_{j-1}}, t_{j-1} < u < t_j$$

for $t_j < u < t_{j+1}$,we have:

$$\phi_{j,n+1} = \frac{t_{j+1} - u}{t_{j+1} - t_j}, t_j < u < t_{j+1}$$

Hence, equation (3.10), is now derived. Figure (3.1) illustrate the basis function $\phi_{j,n+1}$:

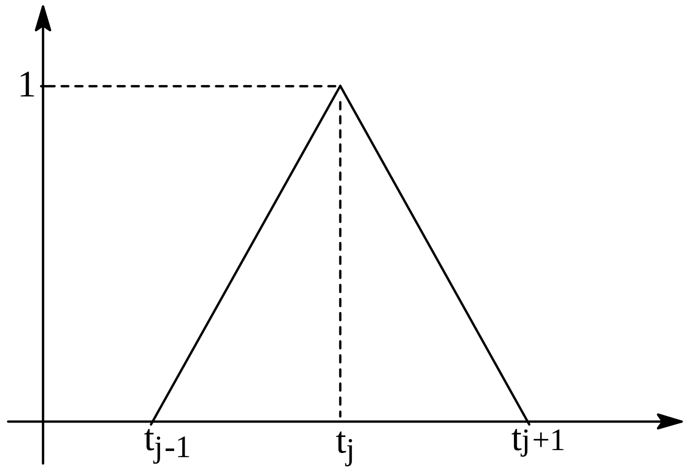

Figure (3.1) The hat function $\phi_{j,n+1}$.

3.2.2 Derivation of $a_{j,\,n+1}$:

Depending on the final form of the basis functions $\phi_{j,n+1}$, we can derive $a_{j,n+1}$ as follows:

From (3.9), we have:

$$a_{j,n+1} = \int_{t_0}^{t_{n+1}} (t_{n+1} - u)^{q-1} \phi_{j,n+1}(u)du$$

$$= \int_{t_{j-1}}^{t_j} (t_{n+1} - u)^{q-1} \frac{u - t_{j-1}}{t_j - t_{j-1}} du + \int_{t_j}^{t_{j+1}} (t_{n+1} - u)^{q-1} \frac{t_{j+1} - u}{t_{j+1} - t_j} du$$

$$= \frac{1}{h}\left[\int_{t_{j-1}}^{t_j} (t_{n+1} - u)^{q-1}(u - t_{j-1})du + \int_{t_j}^{t_{j+1}} (t_{n+1} - u)^{q-1}(t_{j+1} - u)du \right]$$

.... (3.11)

Now, for the first integral in equation (3.11), and upon using the method of integration by parts, we have:

$$\int_{t_{j-1}}^{t_j} (t_{n+1}-u)^{q-1}(u-t_{j-1})du = \int_{t_{j-1}}^{t_j} (t_{n+1}-u)^{q-1}udu -$$

$$t_{j-1}\int_{t_{j-1}}^{t_j} (t_{n+1}-u)^{q-1}du$$

$$=\frac{h^{q+1}}{q(q+1)}\left[(n-j+2)^{q+1}-(n+1-j)^q\left((n+q)+(2-j)\right)\right] \text{......} (3.12)$$

Now, to the second integral in equation (3.11), we proceed similarly as in the first integration to get the following result:

$$\int_{t_j}^{t_{j+1}} (t_{n+1}-u)^{q-1}(t_{j+1}-u)du = \frac{h^{q+1}}{q(q+1)}\left[(n-j)^{q+1}+\right.$$

$$\left.(n+1-j)^q\left((q-n)+j\right)\right] \text{............} (3.13)$$

Hence, substituting (3.12) and (3.13) in (3.11), gives for $1 \le j \le n$:

$$a_{j,n+1} = \frac{1}{h}[\int_{t_{j-1}}^{t_j} (t_{n+1}-u)^{q-1}(u-t_{j-1})du + \int_{t_j}^{t_{j+1}} (t_{n+1}-u)^{q-1}(t_{n+1}-u)du]$$

$$=\frac{1}{h}\left[\frac{h^{q+1}}{q(q+1)}\left((n-j+2)^{q+1}\right)-(n+1-j)^q\left((q-n)+j\right)\right]$$

$$=\frac{h^q}{q(q+1)}\left[(n-j+2)^{q+1}+(n-j)^{q+1}\left(n+1-j)^q(-2n-2+2j\right)\right]$$

$$=\frac{h^q}{q(q+1)}\left[(n-j+2)^{q+1}(n+1-j)^q\left(n+1-j\right)+(n-j)^{q+1}\right]$$

$$=\frac{h^q}{q(q+1)}\left[(n-j+2)^{q+1}+(n-j)^{q+1}\left(n+1-j)^q(-2n-2+2j\right)\right]$$

$$\text{..............................} (3.14)$$

Equation (3.14), could be applied to the interior node points, t_1, t_2…, t_n, while for the boundary node points t_1 and t_{n+1}, we have:

If j = 0, then:

$$a_{0,n+1} = \int_{t_0}^{t_1} (t_{n+1} - u)^{q-1} \phi_{0,n+1}(u) du$$

$$= \int_{t_0}^{t_1} (t_{n+1} - u)^{q-1} \frac{t_1 - u}{t_1 - t_0} du$$

$$= \frac{h^{q+1}}{q(q+1)} \left[n^{q+1} - (n - j)\left(n + 1\right)^q \right] \text{ .... (3.15)}$$

Similarly, for j = n+1

$$a_{n+1,n+1} = \int_{t_0}^{t_{n+1}} (t_{n+1} - u)^{q-1} \phi_{n+1,n+1}(u) du$$

$$= \frac{h^q}{q(q+1)} \text{ .... (3.16)}$$

Therefore, the final form of the coefficients is given by:

$$a_{j,n+1} = \begin{cases} \dfrac{h^q}{q(q+1)}\left((n^{q+1} - (n-q)(n+1)^q\right), & j = 0 \\ \dfrac{h^q}{q(q+1)}((n-j+2)^{q+1} - 2(n-j+1)^{q+1} + (n-j)), & j = 1, 2, ..., n \\ \dfrac{h^q}{q(q+1)}, & j = n+1 \end{cases}$$

.... (3.17)

3.2.3 Adam's Method for Solving Fractional Differential Equations:

Using Adam-Moulton implicit method to solve the singular integral equation (3.6), with the cooperation of quadratic integration methods, the following formula is obtained:

$$y_{n+1} = y_0 + \frac{1}{\Gamma(q)}\left[\sum_{j=0}^{n} a_{j,n+1} f(t_j, y_j) + a_{n+1,n+1} f(t_{n+1}, y_{n+1}^{p})\right] \text{.......} (3.18)$$

Now, the problem is the determination of the predictor formula that we require to calculate the value y_{n+1}^{p}. The idea we use is to generalize the one-step Adams-Bashforth method which is the same as that one described above for the Adams-Mouton technique. We replace the integral on the right-hand side of equation (3.6), by any quadrature rule, i.e.,

$$\int_{t_0}^{t_{n+1}} (t_{n+1} - u)^{q-1} g(u)du \approx \int_{j=0}^{n} b_{j,n+1} g(t_j) \text{ ....} (3.19)$$

Similarly as in subsection (3.3.2), we have:

$$b_{j,n+1} = \int_{t_j}^{t_{j+1}} (t_{n+1} - u)^{q-1} du = \frac{1}{q}\left((t_{n+1} - t_j)^q - (t_{n+1} - t_{j+1})^q\right) \text{ .....} (3.20)$$

Again, for equispaced case, we have the simpler expression:

$$b_{j,n+1} = \frac{h^q}{q}\left((n+1-j)^q - (n-j)^q\right) \text{ ....} (3.21)$$

Thus, the predictor value of y_{n+1}^{p}, is given by:

$$y_{n+1}^{p} = y_0 + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j) \text{ ....} (3.22)$$

This completes the description of our basic algorithm, which is the fractional version of the one-step Adams-Bshforth-Moulton method.

Recapitulating, one can see that, we first have to calculate the predictor y_{n+1}^{p} according to equation (3.22), then evaluate $f(t_{n+1}, y_{n+1}^{p})$, and using this to determine the corrector value of y_{n+1} by means of equation (3.18), and finally evaluate $f(t_{n+1}, y_{n+1})$, which is then used in the next integration step. Therefore, methods of this type are frequently called predictor- corrector or, more precisely, PECE (Predict, Evaluate, Correct, and Evaluate) method.

3.3 APPROXIMATE SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS USING CUBIC SPLINE INTERPOLATION WITH 3-NODE POINTS

Suppose that we have m + 1 data points p_0, p_1, …, p_m through which we have to draw a curve such as that shown in figure (3.2) (in which m = 6), [Bartels, 1987].

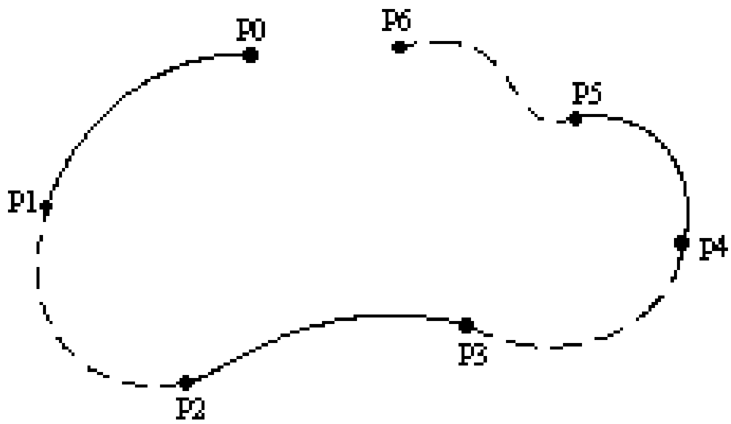

Figure (3.2) An interpolating cubic splin.

Each successive pair of data points is connected by distinct curve segment. The i^{th} segment runs from p_i to p_{i+1}, and we will assume that the parameter $\bar{u}$ runs correspondingly from the knot $\bar{u}_i$ to the knot $\bar{u}_{i+1}$ to generate this segment. Since each such segment $\phi(\bar{u})$ is represented parametrically as

$(X_i(\bar{u}), Y_i(\bar{u}))$, we are indeed concerned with how the $X_i(\bar{u})$ and $Y_i(\bar{u})$ are determined by the points $p_i = (x_i, y_i)$.

In general, the x-coordinates $X(\bar{u})$ of points on a curve are determined solely by the x-coordinates $x_0, x_1, \ldots, x_m$, of the data points, and similarly $Y(\bar{u})$ is determined solely by the y-coordinates of the data points, since both $X(\bar{u})$ and $Y(\bar{u})$ are treated in the same way, we will discuss only $Y(\bar{u})$.

For ease of computation, we will limit ourselves to the use of polynomials in defining $X_i(u)$ and $Y_i(u)$. Indeed cubic polynomials usually provide sufficient flexibility for many applications at reasonable cost.

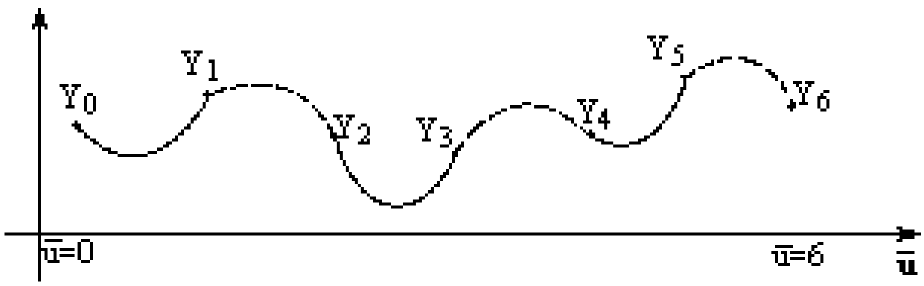

Figure (3.3) $Y(\bar{u})$ for the curve shown in figure (3.2) above.

It will be easiest to continue the discussion by reparametrizing each segment Y_i separatedly by substituting u for $\bar{u}$ as was described earlier, This means that $u = \bar{u}_i - i$ for the knot sequence given in figure (3.3). Each $Y_i(u)$ is a cubic polynomial in the parameter u. It is known that:

$$Y_i(u) = a_i + b_i u + c_i u^2 + d_i u^3$$

where a_i, b_i, c_i and d_i are constants have to be evaluated for each i = 1, 2, …, m; and hence:

$$Y_i(0) = y_i = a_i$$

$$Y_i(1) = y_{i+1} = a_i + b_i + c_i + d_i$$

Where $Y_i(0)$ stands for the left hand limit and $Y_i(1)$ for the right hand limit.

Because we have four coefficients to be determined, we need two other constraints are needed to completely determine a particular $Y_i(u)$. One easy way to do this is to simply pick, arbitrarily, first derivatives D_i of $Y(u)$ at each knot $\bar{u}_i$, so that:

$$Y_i^{(1)}(0) = D_i = b_i$$

$$Y_i^{(1)}(1) = D_{i+1} = b_i + 2c_i + 3d_i$$

These four equations how can be solved analytically, once and for all, to yield:

$$a_i = Y_i,\ b_i = D_i,\ c_i = 3(y_{i+1}-y_i) - 2D_i - D_{i+1},\ d_i = 2(y_i - y_{i-1}) + D_i + D_{i+1}$$

Since we use D_i as the derivative at the left end of the i^{th} segment (i.e., as $Y_i^{(1)}(0)$) and at the right of the $(i - 1)^{th}$ segment (as $Y_{i-1}^{(1)}(1)$), $Y(u)$ has continuous first derivative. This technique is called Hermite interpolation. It can be generalized to higher-order polynomials. [Bartels, 1987].

A question may arise which is how are the D_i specified ?. One possibility is to compute them automatically, perhaps by fitting a parabola through y_{i-1}, y_i and y_{i+1}, and using its derivative at y_i as D_i, arbitrary (such as 0) can be used at the end points, or one can use for D_i the y component of a weighted average of the vector from p_{i-1} to p_i and the vector from p_{i+1} to p_i, or the user may specify derivative vectors directly.

It is possible to arrange that successive segments match second as well as first derivatives at joints, using only cubic polynomials. Suppose, as above, that we want to interpolate the $(m + 1)$ points $p_0, p_1, \ldots, p_m$ by such a curve. Each of the m segments $Y_0(u), Y_1(u), \ldots, Y_{m-1}(u)$ is a cubic polynomial determined by four coefficients. Hence, we have 4m unknown values to determined. At each of the $(m - 1)$ interior knots $\bar{u}_1, \bar{u}_2\ldots, \bar{u}_{m-1}$ (where two segments meet), we have four conditions:

$$\left.\begin{array}{l} Y_{i-1}(1) = y_i, Y_{i-1}^{(1)}(1) = Y_i^{(1)}(0) \\ Y_i(0) = y_i, Y_{i-1}^{(2)}(1) = Y_i^{(2)}(0) \end{array}\right\} \qquad (3.23)$$

Since we also require that:

$Y_0(0) = y_0$

$Y_{m-1}(1) = y_m$

Then we have a total of $4(m-1) + 2 = 4m - 2$ conditions from which to determine our 4m knows. Thus, we need two more conditions. These may be chosen in a variety of ways. A common choice is simply to require that the second derivatives at the endpoints $\overline{u}_0$ and $\overline{u}_m$ both are equals to zero; these conditions yield what is called a natural cubic spline.

Now, we have to derive a natural cubic spline equation using 3-node points as a knot points for each bases. Let us derive it in general.

If j = i, then:

$$\phi_j(u) = a_j + b_j(u - t_j) + c_j(u - t_j)^2 + d_j(u - t_j)^3$$

and letting ϕ_{j-1} and to ϕ_j be the left and right parts censuring ϕ_j, respectively, as it is shown in figure (3.4):

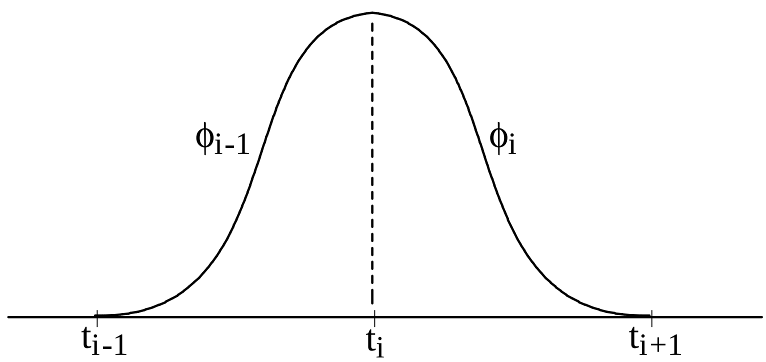

Figure (3.4) Cubic spline basis $\phi_{j,n+1}$ of 3-nodes points.

Also, recall from the definition of cubic spline interpolation of a function f defined on [a, b] on a set of numbers, called nodes, $a = t_0 < t_1 < \ldots < t_n = b$, then a cubic spline interpolation denoted by ϕ for f is a function thus satisfies the following conditions:

(a) ϕ is a cubic polynomial denoted by ϕ_j on the subinterval $[t_j, t_{j+1}]$, for each $j = 0, 1, \ldots, n-1$.

(b) $\phi(t_j) = f(t_j)$, for each $j = 0, 1, \ldots, n-1$.

(c) $\phi_{j+1}(t_{j+1}) = \phi_j(t_{j+1})$, for each $j = 0, 1, \ldots, n-1$.

(d) $\phi'_{j+1}(t_{j+1}) = \phi'_j(t_{j+1})$, for each $j = 0, 1, \ldots, n-1$.

(e) $\phi''_{j+1}(t_{j+1}) = \phi''_j(t_{j+1})$, for each $j = 0, 1, \ldots, n-1$.

(f) One of the following sets of boundary conditions is satisfied:

(i) $\phi''(t_0) = \phi''(t_n) = 0$ (free boundary) .... (3.24)

(ii) $\phi'(t_0) = f'(t_0)$ and $\phi'(t_n) = f'(t_n)$ (Hermite boundary) .... (2.25)

To construct the cubic spline interpolate for a given function f with $\phi''(a) = \phi''(b) = 0$, the above conditions can be applied to the cubic polynomials:

$$\phi_j(u) = a_j + b_j(u - t_j) + c_j(u - t_j)^2 + d_j(u - t_j)^3$$

for each $j = 0, 1, \ldots, n-1$.

Clearly, $\phi_j(t_j) = a_j = f(t_j)$ and if condition (c) is applied, then for each $j = 0, 1, \ldots, n-2$:

$$a_{j+1} = \phi_{j+1}(t_{j+1}) = \phi_j(t_{j+1}) = a_j + b_j(t_{j+1} - t_j) + c_j(t_{j+1} - t_j)^2 + d_j(t_{j+1} - t_j)^3$$

$$= a_j + b_j h + c_j h^2 + d_j h^3 \quad \text{.......................... (3.26)}$$

where $h = t_{j+1} - t_j$.

In a similar manner, define $b_n = \phi'(t_n)$ and observe that:

$$\phi'_j(u) = b_j + 2c_j(u - t_j) + 2d_j(u - t_j)^2$$

implies $\phi'_j(t_j) = b_j$, for each j = 0, 1, …, n – 1. Applying condition (d):

$$b_{j+1} = b_j + 2c_jh + 3d_jh^2 \qquad (3.27)$$

for each j = 0, 1, …, n – 1.

Another relation between the coefficients of ϕ_j can be obtained by defining $c_n = \phi''(t_n)/2$ and applying condition (e). In this case:

$$c_{j+1} = c_j + 3d_jh \qquad (3.28)$$

for each j = 0, 1, …, n – 1.

Solving for d_j in equation (3.28) and substituting this value into equations (3.26) and (3.27) gives the new equations:

$$a_{j+1} = a_j + b_jh + \frac{h^2}{3}(2c_j + c_{j+1}) \qquad (3.29)$$

and

$$b_{j+1} = b_j + h(c_j + c_{j+1}) \qquad (3.30)$$

for each j = 0, 1, …, n – 1.

The final relationship involving the coefficients in obtained by solving the appropriate equation in the form of equation (3.29), first for b_j.

$$b_j = \frac{1}{h}(a_{j+1} - a_j) - \frac{h}{3}(2c_j + c_{j+1}) \qquad (3.31)$$

Substituting these values into the equation derived from (3.30), when the index is reduced by one, gives the linear system of equations:

$$hc_{j-1} + 4hc_j + hc_{j+1} = \frac{3}{h}(a_{j+1} - a_j) - \frac{3}{h}(2c_j + c_{j+1}) \qquad (3.32)$$

for each j = 1, 2, …, n − 1. This system involves as unknowns only $\{c_j\}_{j=0}^{n}$. Since the values of $\{a_j\}_{j=0}^{n}$ are given by spacing of the nodes $\{t_j\}_{j=0}^{n}$ and the values of f at the nodes. When evaluating $\{c_j\}_{j=0}^{n}$, then $\{b_j\}_{j=0}^{n}$ and $\{d_j\}_{j=0}^{n}$ could be evaluated from (3.31) and (3.28), respectively.

Then on order to construct the cubic spline interpolation with three node points, namely, t_{i-1}, t_i and t_{i+1} with $\phi_{i-1}(t_{i-1}) = 0$, $\phi_{i-1}(t_i) = \phi_i(t_i) = 1$ and $\phi(t_{i+1}) = 0$, then the above conditions will applied to these three points as follows:

$$\phi_{j-1}(u) = a_0 + b_0(u - t_{j-1}) + c_0(u - t_{j-1})^2 + d_0(u - t_{j-1})^3 \quad (3.33)$$

$$\phi_j(u) = a_1 + b_1(u - t_j) + c_1(u - t_j)^2 + d_1(u - t_j)^3 \quad (3.34)$$

since $a_j = f(t_j)$, then $a_0 = a_2 = 0$ and $a_1 = 1$. Then the following system is obtained $Ax = b$, where:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ h & 4h & h \\ 0 & 0 & 1 \end{bmatrix},\ x = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ \frac{3}{h}(a_2 - a_1) - \frac{3}{h}(a_1 - a_0) \\ 0 \end{bmatrix}$$

After solving this system, we obtained $c_0 = c_2 = 0$ and $c_1 = -1.5/h^2$. Then from equation (3.31), we have $b_0 = 1.5/h$ and $b_1 = 0$. Also, from (3.28), the values of d_0 and d_1 are evaluated to be $d_0 = -0.5/h^3$ and $d_1 = 0.5/h^3$.

Therefore, the final form of $\varphi_{i-1}(u)$ and $\phi_i(u)$ are given by:

$$\phi_{i-1}(u) = \frac{1.5}{h}(u - t_{i-1}) - \frac{0.5}{h^3}(u - t_{i-1})^3, \text{ if } t_{i-1} \le u \le t_i$$

$$\phi_i(u) = 1 - \frac{1.5}{h^2}(u - t_i)^2 + \frac{0.5}{h^3}(u - t_i)^3, \text{ if } t_i \le u \le t_{i+1}$$

Now, in order to evaluate the constants $a_{j,n+1}$, for all j = 0, 1, …, n+1, we proceed as follows:

Depending on the final from of the basis function $\phi_{j,n+1}$, we can derive $a_{j,n+1}$ as follows:

For j = i

$$a_{i,n+1} = \int_{t_{i-1}}^{t_i} (t_{n+1} - u)^{q-1} \phi_{j,n+1}(u)du$$

$$= \int_{t_{i-1}}^{t_i} (t_{n+1} - u)^{q-1} \left[\frac{1.5}{h}(u - t_{i-1}) - \frac{0.5}{h^3}(u - t_{i-1})^3 \right] du +$$

$$\int_{t_i}^{t_{i+1}} (t_{n+1} - u)^{q-1} \left[1 - \frac{1.5}{h^2}(u - t_i)^2 + \frac{0.5}{h^3}(u - t_i)^3 \right] du$$

$$= \frac{h^q}{q(q+1)} \left[1.5\{(n+2-i)^{q+1} + (n-i)^{q+1}\} \right] +$$

$$\left[\frac{3\left[2(n+1-i)^{q+3} - (n+2-i)^{q+3} - (n-i)^{q+3}\right]}{(q+2)(q+3)} \right]$$

Similarly, for j = 0 and j = n + 1, we have:

$$a_{0,n+1} = \frac{h^q}{q(q+1)} \left[(n+1)^q (q+1) + 1.5n^{q+1} - \frac{3(n+1)^{q+2}}{q+2} + \frac{3\left[(n+1)^{q+3} - n^{q+3}\right]}{(q+2)(q+3)} \right]$$

and

$$a_{n+1,n+1} = \frac{h^q}{q(q+1)} \left[1.5 - \frac{0.5}{(q+2)(q+3)} \right]$$

3.4 APPROXIMATE SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS USING CUBIC SPLINE INTERPOLATION WITH 5-NODE POINTS

Using a little foresight we can modify the basis functions to be of five node points. For this purpose, suppose each basis function to be non zero over four successive intervals (which for convenience all are assumed to have length one), as shown in figure (3.5), and ask that with in each interval abases function be defined by a cubic polynomial:

$p_3(u) = a_j + b_j u + c_j u^2 + d_j u^3,\ i - 3 \le j \le i$

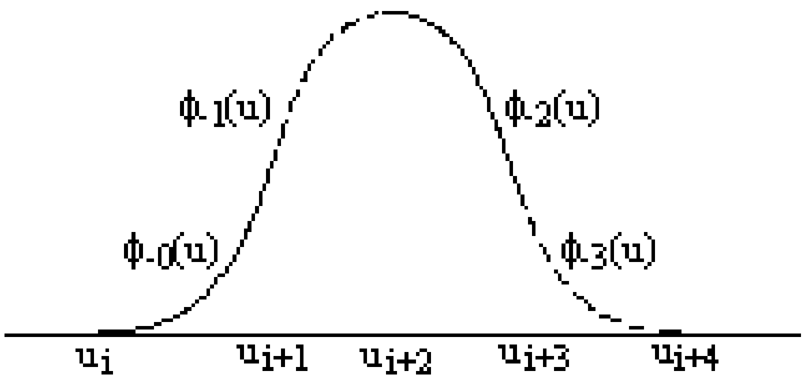

Figure (3.5) The uniform cubic B-spline $\phi_i(u)$ is cubic C^2 basis function centered at u_{i+2}.

Since the nonzero portion of our cubic basis function $\phi(u)$ consists (from left to right) of for basis segments $\phi_0(u)$, $\phi_{-1}(u)$, $\phi_{-2}(u)$ and $\phi_{-3}(u)$, and since each segment has four coefficients to be determined, there are sixteen coefficients to be determined. The basis- function $\phi_i(u)$ is identically zero for $u \le u_i$ and for $u \ge u_{i+4}$, so the first and second derivatives $\phi_i^{(1)}(u)$ and $\phi_i^{(2)}(u)$ are also identically zero outside the interval (u_i, u_{i+4}).

Similarly, as in section (3.3), the basis functions $\phi_j(u)$ with 5-node points (see figure (3.6)), as follows:

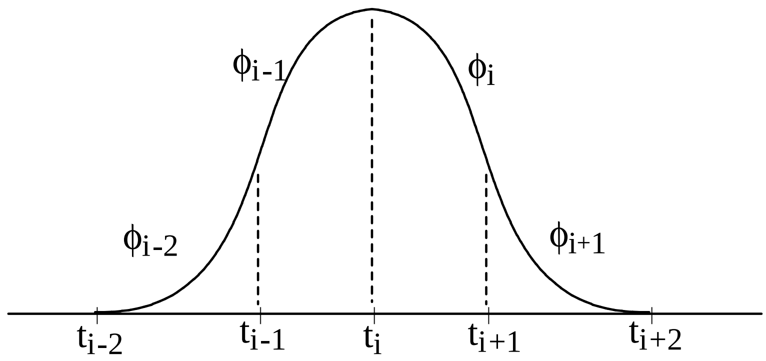

Figure (3.6) Cubic basis $\phi_{i,n+1}$ of 5-nods points.

$\phi_{i-2}(u) = a_0 + b_0(u - t_{i-2}) + c_0(u - t_{i-2})^2 + d_0(u - t_{i-2})^3$.................. (3.35)

$\phi_{i-1}(u) = a_1 + b_1(u - t_{i-1}) + c_1(u - t_{i-1})^2 + d_1(u - t_{i-1})^3$.................. (3.36)

$\phi_i(u) = a_2 + b_2(u - t_i) + c_2(u - t_i)^2 + d_2(u - t_i)^3$............................. (3.37)

$\phi_{i+1}(u) = a_3 + b_3(u - t_{i+1}) + c_3(u - t_{i+1})^2 + d_3(u - t_{i+1})^3$................ (3.38)

Since $a_j = f(t_j)$, then $a_0 = a_4 = 0$, $a_1 = a_3 = ½$ and $a_2 = 1$, and the following system for evaluating c_j's is obtained:

$Ax = b$

Where:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ h & 4h & h & 0 & 0 \\ 0 & h & 4h & h & 0 \\ 0 & 0 & h & 4h & h \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},\ x = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ \frac{3}{h}(a_2 - a_1) - \frac{3}{h}(a_1 - a_0) \\ \frac{3}{h}(a_3 - a_2) - \frac{3}{h}(a_2 - a_1) \\ \frac{3}{h}(a_4 - a_3) - \frac{3}{h}(a_3 - a_2) \\ 0 \end{bmatrix}$$

After solving this system, we obtained $c_0 = c_4 = 0$, $c_1 = c_3 = 0.214/h^2$ and $c_2 = -0.857/h^2$. Then from equation (3.31), we have $b_0 = 1.5/h$ and $b_1 = 0$. Also, from (3.28), the values of b_i's and d_i's are evaluated to be:

$b_0 = 0.429/h$, $b_1 = 0.643/h$, $b_2 = 0$ and $b_3 = -0.643/h$

$d_0 = 0.071/h^3$, $d_1 = -0.357/h^3$, $d_2 = 0.357/h^3$ and $d_3 = -0.0.71/h^3$

Then:

$$\phi_{i-2,n+1}(u) = \frac{0.429}{h}(u - t_{i-2}) + \frac{0.071}{h^3}(u - t_{i-2})^3, \text{ if } t_{i-2} \le u \le t_{i-1}$$

$$\phi_{i-1,n+1}(u) = \frac{1}{2} + \frac{0.643}{h}(u - t_{i-1}) + \frac{0.214}{h^2}(u - t_{i-1})^2 - \frac{0.357}{h^3}(u - t_{i-1})^3, \text{ if } t_{i-1} \le u \le t_i$$

$$\phi_{i,n+1}(u) = 1 - \frac{0.857}{h^2}(u - t_i)^2 + \frac{0.357}{h^3}(u - t_i)^3, \text{ if } t_i \le u \le t_{i+1}$$

$$\phi_{i+1,n+1}(u) = \frac{1}{2} - \frac{0.643}{h}(u - t_{i+1}) + \frac{0.214}{h^2}(u - t_{i+1})^2 - \frac{0.071}{h^3}(u - t_{i+1})^3, \text{ if } t_{i+1} \le u \le t_{i+2}$$

The values of $a_{j,n+1}$ could be evaluated depending on the final form of basis functions $\phi_j(u)$, similarly in section (3.3) to get the following final form:

If $j = i$, then:

$$a_{i,n+1} = \frac{h^q}{q}\left[\frac{-1.285(n+2-i)^{q+1} + 0.429(n+3-i)^{q+1} + 0.428(n-1-i)^{q+1}}{(q+1)} + \frac{-0.426(n+2-i)^{q+2} - 0.002(n-1-i)^{q+2}}{(q+1)(q+2)} + \frac{0.426\{(n+3-i)^{q+3} + (n-1-i)^{q+3}\}}{(q+1)(q+2)(q+3)} - \frac{2.568(n+2-i)^{q+3} - 4.284(n+1-i)^{q+3}}{(q+1)(q+2)(q+3)}\right]$$

similarly if j = 0 and j = n+1,

$$a_{0,n+1} = \frac{h^q}{q}\left[(n+1)^q + \frac{0.428(n-1)^{q+1}}{(q+1)} + \frac{0.002(n-1)^{q+2} - 1.714(n+1)^{q+2}}{(q+1)(q+2)}\right.$$

$$\left. + \frac{2.142(n+1)^{q+3} - 2.568n^{q+3} + 0.246(n-1)^{q+3}}{(q+1)(q+2)(q+3)}\right]$$

$$a_{n+1,n+1} = \frac{h^q}{q}\left[\frac{0.001 + 0.429(2)^{q+1}}{(q+1)} + \frac{0.002}{(q+1)(q+2)} + \frac{1.714 + 0.426(2)^{q+3}}{(q+1)(q+2)(q+3)}\right]$$

3.5 ILLUSTRATIVE EXAMPLE

In this section, we give an example as a test problem to check the accuracy of the results using the above three approaches discussed previously in sections (3.2), (3.3) and (3.4), respectively.

Example:

Consider the fractional differential equation:

$$y^{(1/2)} = -y + t^2 + 2\frac{2t^{3/2}}{\Gamma(5/2)},\ y(0) = 0$$

where the exact solution is given by $y(t) = t^2$.

Then the results obtained upon using the three approaches are given in table (3.1) with step size $h = 0.1$ with its comparison with the exact solution.

Table (3.1).

x	***Linear***	***error***	***Cubic (3-points)***	***error***	***Cubic (5-points)***	***error***
0	0.014	0.004	0.016	0.006	0.018	0.008
0.1	0.037	0.003	0.014	0.004	0.016	0.006
0.2	0.073	0.017	0.042	0.002	0.037	0.003
0.3	0.127	0.033	0.147	0.013	0.161	0.001
0.4	0.201	0.049	0.235	0.015	0.261	0.011
0.5	0.295	0.065	0.346	0.014	0.491	0.0131
0.6	0.414	0.076	0.511	0.021	0.472	0.018
0.7	0.581	0.059	0.667	0.027	0.662	0.022
0.8	0.725	0.085	0.875	0.065	0.842	0.032
0.9	1.128	0.128	1.116	0.116	1.11	0.11

From the above obtained results, one can see the accuracy of the results in which the approximate solution of the solution of fractional differential equation using cubic spline interpolation with 3-nod pointes is more accurate than the solution obtained by using linear approximate. Also, the result obtained by using cubic spline interpolation with 5-nod pointes is more accurate than the results obtained by using cubic spline interpolation of 3-nod points.

CHAPTER 2

THEORY OF FRACTIONAL DIFFERENTIAL EQUATIONS

Analytic solution of fractional differential equations is so difficult and very limited, therefore, analytic methods for solving fractional differential equations dose not work in all cases, but they may be powerful in some cases and may not work in other.

This chapter presents some of the most fundamental and popular methods for solving fractional differential equations analytically, such as the inverse operator method and Laplace transformation method.

In addition this chapter presents some of the most basic concepts in fractional differential equations as well as the statement and proof of the existence and uniqueness theorem which has its basis on Schauder fixed point theorem.

2.1 PROPERTIES OF FRACTIONAL DIFFERENTIATION AND INTEGRATION

In this section, some properties related to fractional differentiation and integration are explained, those properties which will provide our primary means of understanding and utilizing fractional differential equations.

We start with those properties of most importance:

2.1.1 Linearity:

By linearity of the differintegral operator, by which we mean:

$$D^q[f_1 + f_2] = D^q f_1 + D^q f_2 \quad \text{...(2.1)}$$

where f_1 and f_2 are any two functions and q is a fractional number, and:

$$D^q[cf] = cD^q f \qquad (2.2)$$

where f is an arbitrary function while c is an arbitrary constant.

2.1.2 Scale Change:

By a scale change of the function f with respect to a lower limit a, we mean its replacement by f (βx − βa + a), where β is a constant termed the scaling factor, and hence the fractional derivative of order q with Y = βy − βa + a, and X = x + (a − βa)/β, is given by:

$$\frac{d^q f(\beta x)}{[d(x-a)]^q} = \frac{d^q f(\beta x - \beta a + a)}{[d(x-a)]^q}$$

$$= \frac{1}{\Gamma(-q)} \int_a^x \frac{f(\beta y - \beta a + a)}{[x-y]^{q+1}} dy$$

$$= \frac{1}{\Gamma(-q)} \int_a^{\beta X} \frac{f(Y)[dY/\beta]}{\{[\beta X - Y]/\beta\}^{q+1}}$$

$$= \frac{\beta^q}{\Gamma(-q)} \int_a^{\beta X} \frac{f(Y)}{[\beta X - Y]^{q+1}} dY$$

$$= \beta^q \frac{d^q f(\beta X)}{[d(\beta X - a)]^q} \qquad (2.3)$$

2.1.3Leibniz's Rule:

The rule for differentiation of a product of two functions f and g is a familiar result in elementary calculus. It states that for a positive integer n:

$$\frac{d^n[fg]}{dx^n} = \sum_{j=0}^{n} \binom{n}{j} \frac{d^{n-j} f}{dx^{n-j}} . \frac{d^j g}{dxj} \qquad (2.4)$$

The following product rule for multiple integrals is also satisfied

$$\frac{d^{-n}[fg]}{[d(x-a)]^{-n}}=\sum_{j=0}^{\infty}\binom{-n}{j}\frac{d^{-n-j}f}{[d(x-a)]^{-n-j}}.\frac{d^{j}g}{[d(x-a)]^{j}} \quad \text{.......................(2.5)}$$

Now, when we observe that the finite sum in (2.4) could be equally well extend to infinity (since $\binom{n}{j}=0$ for all j > n), we might expect the product rule to be generalized to an arbitrary order q as:

$$\frac{d^{q}[fg]}{[d(x-a)]^{q}}=\sum_{j=0}^{\infty}\binom{q}{j}\frac{d^{q-j}f}{[d(x-a)]^{q-j}}.\frac{d^{j}g}{[d(x-a)]^{j}}$$

Thus such a generalization is indeed valid for real order q and is called the Leibniz rule.

Further generalization of Leibniz's rule due to Osler (1972) is the integral form (see [Oldham and Spanier, 1974]):

$$\frac{d^{q}[fg]}{dx^{q}}=\int_{-\infty}^{\infty}\frac{\Gamma(q+1)}{\Gamma(q-\gamma-\lambda+1)\Gamma(\gamma+\lambda+1)}.\frac{d^{q-\gamma-\lambda}f}{dx^{q-\gamma-\lambda}}.\frac{d^{\gamma+\lambda}g}{dx^{\gamma+\lambda}}d\lambda$$

In which a discrete sum is replaced by an integral.

2.1.4 The Chain Rule [Oldham and Spanier, 1974]:

The Chain rule for the first order differentiation is given by:

$$\frac{d}{dx}g\left(f(x)\right)=\frac{d}{df(x)}g\left(f(x)\right)\frac{d}{dx}f(x)$$

which tacks a simple counterpart in the integral calculus.

Indeed if there were such a counterpart, the process of integration would pose no greater difficulty than does differentiation. Since any general formula for $d^{q}g\left(f(x)\right)/[d(x-a)]^{q}$ must encompass integration as a special case of little hope that can be held out for a useful chain rule for arbitrary q. Nevertheless, a formal chain rule in fractional orders may be derived quit simply, which takes the form:

$$\frac{d^q\Phi}{[d(x-a)]^q}=\frac{[x-a]^{-q}}{\Gamma(1-q)}\Phi+\sum_{j=1}^{\infty}\binom{q}{j}\frac{[x-a]^{j-q}}{\Gamma(j-q+1)}.\frac{d^j\Phi}{dx^j}$$

Now, we consider $\Phi = \Phi(f(x))$ and evaluate $d^j\Phi\left(f(x)\right)/dx^j$, in the second term of the last equation as follows:

$$\frac{d^j}{dx^j}\Phi\left(f(x)\right)=j!\sum_{m=1}^{j}\Phi^{(m)}\sum\left(\prod_{k=1}^{j}\frac{1}{p_k!}\left[\frac{f^{(k)}}{k!}\right]^{p_k}\right)$$

where Σ extends over all combinations of nonnegative integer values of p_1, p_2, ..., p_j, such that:

$$\sum_{k=1}^{j}kp_k=j \quad \text{and} \quad \sum_{k=1}^{j}p_k=m$$

Thus:

$$\frac{d^q}{[d(x-a)]^q}\Phi\left(f(x)\right)=\frac{[x-a]^{-q}}{\Gamma(1-q)}\Phi\left(f(x)\right)+$$
$$\sum_{j=1}^{\infty}\binom{q}{j}\frac{[x-a]^{j-q}}{\Gamma(j-q+1)}j!\sum_{m=1}^{j}\Phi^{(m)}\sum\left(\prod_{k=1}^{j}\frac{1}{p_k!}[\tfrac{f^{(k)}}{k!}]^{p_k}\right)$$

The complexity of this result will inhibit its general utility. We see on inserting q = –1 that even for the case of a single integration:

$$\int_a^x\Phi\left(f(x)\right)dy=[x-a]\Phi\left(f(x)\right)+$$
$$\sum_{j=1}^{\infty}[-1]^j\frac{[x-a]^{j+1}}{j+1}\sum_{m=1}^{j}\Phi^{(m)}\sum\left(\prod_{k=1}^{j}\frac{1}{p_k!}\left[\frac{f^{(k)}}{k!}\right]^{p_k}\right)$$

The chain rule gives an infinite series that offers little hope of begin expressible in closed form, except for trivially simple instances of the functions f and Φ.

2.1.5Composition Rule [Oldham, 1974]:

In seeking a general composition rule for the operator $d^q/[d(x-a)]^q$, we search for the relationship between: $\frac{d^q}{[d(x-a)]^q}.\frac{d^Q f}{[d(x-a)]^Q}$ and $\frac{d^{q+Q}f}{[d(x-a)^{q+Q}}$, which we temporarily abbreviated for simplicity as $d^q d^Q f$ and $d^{q+Q} f$ of course, if these symbols are to be general meaningful then we need to assume not only that f is a differintegerable but that $d^Q f$ is differintegerable as well.

The most general nonzero differintegerable series is a finite sum of differintegerable units, each having the form:

$$f_u = [x-a]^p \sum_{j=0}^{\infty} a_j [x-a]^j, p > -1, a_0 \neq 0 \text{ ....}(2.6)$$

We shall see that the composition rule may be valid for some units of f but possibly not for others. It follows from the linearity of differintegral operators that:

$$d^q d^Q f = d^{q+Q} f \text{ ....}(2.7)$$

If:

$$d^q d^Q f_u = d^{q+Q} f_u \text{ ....}(2.8)$$

For every unit f_u of f. accordingly, we shall first assess the validity of the composition rule (2.8) for differintegrable series unit function f_u.

Obviously, if $f_u = 0$, then $d^Q f_u = 0$ for every Q.

$$d^q d^Q [0] = d^{q+Q}[0] = 0$$

While the composition rule is trivially satisfied for the differintegrable function $f_u = 0$, we shall see that the possibility $f_u \neq 0$ but $d^Q f_u = 0$, is exactly the condition that prevents the composition rule (2.8), and there for (2.7), from being satisfied generally. Having dealt with the case $f_u = 0$, we now assume $f_u \neq 0$, and use the following equation

$$\frac{d^q f_u}{[d(x-a)]^q} = \sum_{j=0}^{\infty} \frac{a_j \Gamma(p+j+1)}{\Gamma(p-q+j+1)} [x-a]^{p+j-q}, q \leq 0$$

To evaluate $d^Q f_u$, we have:

$$d^Q f_u = \sum_{j=0}^{\infty} a_j d^q [x-a]^{p+j} = \sum_{j=0}^{\infty} \frac{a_j \Gamma(p+j+1)[x-a]^{p+j-Q}}{\Gamma(p+j-Q+1)} \quad \text{.............(2.9)}$$

Furthermore, we note that since $p > -1$, it follows that $p + j > -1$, so that $\Gamma(p + j + 1)$ is always finite but nonzero. Individual terms in $d^Q f_u$ will vanish. Therefore, only when the coefficient a_j is zero or when the denominator gamma function $\Gamma(p + j + 1 - Q)$ is infinite. We, see, then, that a necessary and sufficient condition for $d^Q f_u \neq 0$ is:

$\Gamma(p + j + 1 - Q)$ is finite for each j for which $a_j \neq 0$ ....(2.10)

The last condition may be shown to be equivalent to:

$f_u - d^{-Q} d^Q f_u = 0$ ....(2.11)

That is, to the condition that the differintegrable unit f_u be regenerated upon to application, first of d^Q, then d^{-Q}. Assuming (2.11) temporarily, we find that d^q may then be applied to equation (2.9) to give:

$$d^q d^Q f_u = \sum_{j=0}^{\infty} \frac{a_j \Gamma(p+j+1)\Gamma(p+j-Q+1)[x-a]^{p+j-Q-q}}{\Gamma(p+j-Q+1)\Gamma(p+j-Q-q+1)} \quad \text{...........(2.12)}$$

With the condition (2.11) or equivalently (2.10) in effect, we may safely cancel the $\Gamma(p + j - Q + 1)$ factors in (2.12), arriving at:

$$d^q d^Q f_u = \sum_{j=0}^{\infty} \frac{a_j \Gamma(p+j+1)[x-a]^{p+j-Q-q}}{\Gamma(p+j-Q-q+1)} \text{..................................(2.13)}$$

On the other hand, the same technique shows that:

$$d^{q+Q} f_u = \sum_{j=0}^{\infty} a_j d^{q+Q} f_u$$

$$= \sum_{j=0}^{\infty} \frac{a_j \Gamma(p+j+1)[x-a]^{p+j-Q-q}}{\Gamma(p+j-Q-q+1)} = d^q d^Q f_u$$

Thus the composition rule (2.8), is obeyed for the unit f_u as long as condition (2.11), is satisfied. However, when (2.11), is violated, $d^Q f_u = 0$
so that $d^q d^Q f_u = 0$, and on the other hand, it is not necessarily the case that $d^{q+Q} f_u = 0$.

For example, we may choose $f_u = x^{-1/2}$, $a = 0$, $Q = 1/2$, and $q = -1/2$, then:

$$f_u - d^{-Q} d^Q f_u = x^{-1/2} - d^{-1/2} d^{1/2} x^{-1/2} = x^{-1/2} - d^{-1/2} \frac{\Gamma(1/2)}{\Gamma(0)} x^{-1} = x^{-1/2} \neq 0$$

So that condition (2.11), is certainly violated. Therefore $d^Q f_u = 0$ and $d^q d^Q f_u = 0$ while $d^{q+Q} f_u = d^0 x^{-1/2} = x^{-1/2} \neq 0$.

In generalizing, we easily see the relationship between $d^q d^Q f_u$ and $d^{q+Q} f_u$ in the case $f_u - d^{-Q} d^Q f_u \neq 0$, to be:

$$0 = d^q d^Q f_u = d^{q+Q} f_u - d^{q+Q} \{f_u - d^{-Q} d^Q f_u\} \text{..................................(2.14)}$$

While equation (2.14) is a trivial identity for differintegrable units, we shall see that it is less trivial and, therefore, more useful for general differentegrable series f if and only if equation (2.8) is valid for every differintegrable unit f_u of f, it is straightforward to apply the theory just developed for units f_u to obtain the composition rule for general f. The only difference is that while the conditions:

$$f_u \neq 0 \text{ and } f_u - d^{-Q} d^Q f_u = 0 \text{..(2.15)}$$

For units f_u guaranteed that $d^Q f_u \neq 0$, this is no longer the case for arbitrary f. The reason, of course, is that some units of f may satisfy (2.15) while others do not. This will make it possible to violate the composition rule (2.7), even though $f \neq 0$ and $d^Q f \neq 0$. The condition:

$$f - d^{-Q} d^Q f = 0 \qquad (2.16)$$

For general differintegrable series f is however, still necessary and sufficient to guarantee (2.7), we mention in passing that for general differintegrable f, as was the case for differintegrable units f_u. $d^q d^Q f = d^{q+Q} f$, At least when $Q < 0$, and even when $Q < 1$, for functions f bounded at $x = a$. We have noticed previously that, in case where the composition rule is violated the equation:

$$d^q d^Q f = d^{q+Q} f - d^{q+Q}\{f - d^{-Q} d^Q f\}.$$

2.2 FRACTIONAL DIFFERENTIAL EQUATIONS

A relationship involving one or more derivatives of an unknown function y with respect to its independent variable x is known as an ordinary differential equation. Similar relationships involving at least one differentegral of non integer order may be termed as extraordinary differential equations.

As with ordinary differential equations, the situation of extraordinary differential equations often involves integrals and contains arbitrary constants.

The differential equations may involve Riemann - Liouville differential operators of fractional order q>0, which takes the form:

$$D^q_{x_0} y(x) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_{x_o}^{x} \frac{y(u)}{(x-u)^{q-m+1}} du, x \neq u \qquad (2.17)$$

where m is an integer number defined by $m-1 < q \leq m$. Differential equations involving these fractional derivatives have proved to be valuable tools in the modeling of many physical problems. Also, D^q has an m-dimensional kernel, and therefore we need to specify m initial conditions in order to obtain a unique solution to the fractional differential equation:

$$D^q y(x) = f(x, y(x)) \quad \text{.......(2.18)}$$

with some given function f. In the standard mathematical theory, the initial conditions corresponding to (2.18) must be of the form:

$$\frac{d^{q-k}}{dt^{q-k}} y(x)\Big|_{x=a} = b_k,\ k = 1, 2, \ldots, m \quad \text{.......(2.19)}$$

with given values b_k. In other words, we must, specify some fractional derivatives of the function y.

In practical applications, these values are frequently not available and so Caputo (1967) suggested that one should incorporate derivatives of integer-order of the function y as they are commonly used in initial value problems with integer-order equations, into the fractional-order equation, given:

$$D^q[y - T_{m-1}(y)]\,(x) = f\,(x, y(x)) \quad \text{.......(2.20)}$$

where $T_{m-1}(y)$ is the Taylor polynomial of order (m–1) for y, centered at 0. Then, one can specify the initial conditions in the classical form:

$$y^{(k)}(0) = y_0^{(k)} \quad , k = 0, 1, \ldots, m - 1 \quad \text{.......(2.21)}$$

As a classification, fractional differential equations may be classified to be either linear or non-linear, homogenous or non-humongous, etc. In which equation (2.18) is linear if it does not contain terms of independent variable alone, otherwise it is non-homogenous. Also fractional differential equations are said to be linear if the dependent variable y(x) appears linearly in the fractional differential equation, otherwise it is non-linear.

2.3 THE EXISTENCE AND UNIQUENESS THEOREM [Diethelm, 1999]:

Looking at the questions of existence and uniqueness of the solution, we can present the following results that are very similar to the corresponding classical theorems known in the case of first - order ordinary differential equations. Only the scalar setting will be discussed explicitly; the generalization to vector - valued functions is straight forward.

Theorem (2.3.1) (Existence):

Assume that $D := [0, \chi^*] \times [y_0^{(0)} - \alpha, y_0^{(0)} + \alpha]$ with some real number $\chi^* > 0$ and some $\alpha > 0$, and let the function $f : D \longrightarrow R$, be a continuous function. Furthermore, define:

$$\chi := \min\{\chi^*, (\alpha\Gamma(q+1) / \|f\|_\infty)^{1/q}\}$$

Then, there exists a function $y : [0, \chi] \longrightarrow R$, solving the initial value problem (2.20)-(2.21).

Theorem (2.3.2) (Uniqueness):

Assume that $D := [0, \chi^*] \times [y_0^{(0)} - \alpha, y_0^{(0)} + \alpha]$, with some real number $\chi^* > 0$ and some $\alpha > 0$. Furthermore, let the function $f : D \longrightarrow R$, be bounded function on D and fulfill a Lipschitz condition with respect to the second variable, i.e.,

$$|f(x, y) - f(x, z)| \leq L|y - z|$$

with some constant $L > 0$ independent of x, y and z.

Then there exists at most one function $y : [0, \chi] \longrightarrow R$, solving the initial value problem (2.20)-(2.21).

In order to prove these two theorems, we shall use the following very simple result.

It can be proved easily by applying the integral operator of order q, given by:

$$I^{q}(\Phi)(x)=\frac{1}{q}\int_{0}^{x}(x-z)^{q-1}\Phi(z)dz$$

Lemma (2.3.3):

If the function f is continuous, then the initial value problem (2.20)-(2.21) is equivalent to the nonlinear singular Volterra integral equation of the second kind:

$$y(x)=\sum_{k=0}^{m-1}\frac{x^{k}}{k!}y^{(k)}(0)+\frac{1}{\Gamma(q)}\int_{0}^{x}(x-z)^{q-1}f(z,y(z))dz \text{ ....(2.22)}$$

with $m-1<q<m$. In other words, every solution of the Volterra equation (2.22), is also a solution of our original initial value problem (2.20)-(2.21), and vice versa.

We may therefore focus our attention on equation (2.22). This equation is weakly singular if $0 < q < 1$, and regular for $q \geq 1$. Thus in the latter case, the claims of the two theorems follow immediately from the classical results in the theory of Volterra integral equations.

However, in the former case (which is the case required in most practical applications), we must give explicit proofs.

The Proof of Theorem (2.3.1):

In particular, we use the same operator as:

$$\left(Ay\right)(x)=y_{0}^{(0)}+\frac{1}{\Gamma(q)}\int_{0}^{x}(x-z)^{q-1}f(z,y(z))dz \text{ ....(2.23)}$$

and recall that it maps the nonempty, convex and closed set:

$$U:=\{y\in C[0,x]:\left\|y-y_{0}^{(0)}\right\|_{\infty}\leq\alpha\} \text{ to itself.}$$

We shall now prove that A is a continuous operator.

A stronger result is to prove that A satisfies:

$$\left\|A^n y - A^n \tilde{y}\right\|_{L\infty[0,x]} \leq \frac{\left(Lx^q\right)^n}{\Gamma(1+qn)}\left\|y-\tilde{y}\right\|_{L\infty[0,x]}, \forall y,\tilde{y} \in U \text{ ....(2.24)}$$

Then since f is continuous on the compact set D, it is uniformly continuous there. Thus, given an arbitrary $\varepsilon > 0$, we can find $\delta > 0$, such that:

$$\left|f(x,y)-f(x,z)\right| \leq \frac{\varepsilon}{x^q}\Gamma(q+1), \text{whenever}, \left|y-z\right| < \delta \text{ ....(2.25)}$$

Now, let $y,\tilde{y} \in U$ such that $\left\|y-\tilde{y}\right\| < \delta$. Then, in view of (2.25), one gets:

$$\left|f(x,y(x))-f(x,\tilde{y}(x))\right| < \frac{\varepsilon}{x^q}\Gamma(q+1) \text{ ....(2.26)}$$

for all $x \in [0, \chi]$, hence:

$$\left|(Ay)(x)-(A\tilde{y})(x)\right| = \frac{1}{\Gamma(q)}\left|\int_0^x (x-z)^{q-1}\left(f(z,y(z))-f(z,\tilde{y}(z))\right)\right|dz$$

$$\leq \frac{\Gamma(q+1)\varepsilon}{\chi^q\Gamma(q)}\int_0^x (x-z)^{q-1}dz$$

$$= \frac{\varepsilon x^q}{\chi^q} \leq \varepsilon$$

Hence A is continuous operator. Then we look at the set of functions:

$A(U) := \{Ay : y \in U\}$

and for $z \in A(U)$, we find that, for all $x \in [0, \chi]$

$$\left|z(x)\right| = \left|(Ay)(x)\right| \leq \left|y_0^{(0)}\right| + \frac{1}{\Gamma(q)}\int_0^x (x-z)^{q-1}\left|f\left(z,y(z)\right)\right|dz$$

$$\leq \left|y_0^{(0)}\right| + \frac{1}{\Gamma(q+1)}\left\|f\right\|_\infty x^q$$

This means that A(U) is bounded in pointwise sense.

Moreover, for $0 \le x_1 \le x_2 \le \chi$, we have:

$$\left|(Ay)(x_1) - (Ay)(x_2)\right| = \frac{1}{\Gamma(q)}\left|\int_0^{x_1}(x_1 - z)^{q-1} f(z, y(z))dz - \int_0^{x_2}(x_2 - z)^{q-1} f(z, y(z))dz\right|$$

$$\le \frac{\|f\|_\infty}{\Gamma(q+1)}[2(x_2 - x_1)^q + x_1^q - x_2^q] \text{ ....(2.27)}$$

Thus, if $|x_2 - x_1| < \delta$, then:

$$\left|(Ay)(x_1) - (Ay)(x_2)\right| \le 2\frac{\|f\|_\infty}{\Gamma(q+1)}\delta^*$$

Noting that the expression on the right-hand side is independent of y, we see that the set A(U) is equicontinuous. Then every sequence of functions from A(U) has got a uniformly convergent subsequence, and therefore A(U) is relatively compact. Then, Schauder's fixed point theorem asserts that A has got a fixed point. By construction, a fixed point of A is a solution of the initial value problem (2.21). ■

Remark (2.3.1):

The proof of the uniqueness theorem is based on the following generalization of Banach fixed point theorem [Weissinger J., 1952].

Theorem (2.3.4):

Let U be a nonempty closed subset of a Banach space E, and let $\alpha_n \ge 0$ for every n and such that $\sum_{n=0}^{\infty} \alpha_n$ converges. Moreover, let the mapping $A : U \longrightarrow U$, satisfy the inequality:

$$\|A^n u - A^n v\| \le \alpha_n \|u - v\|$$

for every $n \in N$ and every $u, v \in U$. Then, A has a uniquely defined fixed point u*. Furthermore, for any $u_0 \in U$, the sequence $\{A^n u_0\}_{n=1}^{\infty}$ converges to this point u*.

The Proof of Theorem (2.3.2):

As we identified previously, we need only to discuss the case when $0 < q < 1$. In this situation, the Volterra integral equation (2.22) reduces to:

$$y(x) = y_0^{(0)} + \frac{1}{\Gamma(q)} \int_0^x (x-z)^{q-1} f(z, y(z)dz \quad \text{.................................(2.28)}$$

we thus introduce the set U defined previously.

Now, since the constant function $y \equiv y_0^{(0)}$ is in U, we also see that U is non empty.

On U we define the operator A by:

$$(Ay)(x) := y_0^{(0)} + \frac{1}{\Gamma(q)} \int_0^x (x-z)^{q-1} f(z, y(z)dz \quad \text{............................(2.29)}$$

Using this operator, equation (2.29), could be rewritten as y = Ay**,** and in order to prove our desired uniqueness result, we have to show that A has a unique fixed point. Let us therefore investigate the properties of the operator A. From (2.28), recall that:

$$\left|(Ay)(x_1) - (Ay)(x_2)\right| \le \frac{\|f\|_\infty}{\Gamma(q+1)}[2(x_2 - x_1)^q + x_1^q - x_2^q]$$

Proving that Ay is a continuous function. Moreover, for $y \in U$ and $x \in [0, X]$, we have:

$$\begin{aligned}
\left|Ay(x) - y_0^{(0)}\right| &= \frac{1}{\Gamma(q)}\left|\int_0^x (x-z)^{q-1} f(z, y(z))dz\right| \\
&\le \frac{1}{\Gamma(q+1)}\|f\|_\infty x^q \\
&\le \frac{1}{\Gamma(q+1)}\|f\|_\infty \frac{\alpha\Gamma(q+1)}{\|f\|_\infty} = \alpha
\end{aligned}$$

Thus, we have shown that $Ay \in U$, i.e. A maps the set U to itself. The next step is to prove that, for every $n \in N$, where N is the set of natural numbers and every $x \in [0, \chi]$, we have:

$$\left\|A^n y - A^n \tilde{y}\right\|_{L_\infty[0,x]} \leq \frac{(Lx^q)^n}{\Gamma(1+qn)}\left\|y-\tilde{y}\right\|_{L_\infty[0,x]} \quad \text{.............................(2.30)}$$

This can be seen by induction. In case of n = 0, the statement is trivially true. For the induction steps from n – 1 to n, we write:

$$\left\|A^n y - A^n \tilde{y}\right\|_{L_\infty[0,x]} = \left\|A(A^{n-1}y) - A(A^{n-1}\tilde{y})\right\|_{L_\infty[0,x]}$$

$$= \frac{1}{\Gamma(q)} \sup_{0\leq w\leq x}\left|\int_0^w (w-z)^{q-1}[f(z, A^{n-1}y(z)) - f(z, A^{n-1}\tilde{y}(z))]dz\right|$$

$$\leq \frac{L}{\Gamma(q)} \sup_{0\leq w\leq x}\int_0^w (w-z)^{q-1}\left|A^{n-1}y(w) - A^{n-1}\tilde{y}(w)\right|dz$$

$$\leq \frac{L}{\Gamma(q)}\int_0^x (x-z)^{q-1} \sup_{0\leq w\leq z}\left|A^{n-1}y(w) - A^{n-1}\tilde{y}(w)\right|dz$$

$$\leq \frac{L^n}{\Gamma(q)\Gamma(1+q(n-1))}\int_0^x (x-z)^{q-1}z^{q(n-1)} \sup_{0\leq w\leq x}\left|y(w)-\tilde{y}(w)\right|$$

$$\leq \frac{L^n}{\Gamma(q)\Gamma(1+q(n-1))} \sup_{0\leq w\leq x}\left|y(w)-\tilde{y}(w)\right|\int_0^x (x-z)^{q-1}z^{q(n-1)}dz$$

$$= \frac{L^n}{\Gamma(q)\Gamma(1+q(n-1))}\left\|y-\tilde{y}\right\|_{L\infty[0,x]}\frac{\Gamma(q)\Gamma(1+q(n-1))}{\Gamma(1+qn)}x^{qn}$$

which is the desired result (2.30).

We have now shown that the operator A fulfils the assumptions of theorem (2.3.4), with $\alpha_n = (Lx^q)^n / \Gamma(1+qn)$

In order to apply that theorem, we only need to verify that the series $\sum_{n=0}^{\infty} \alpha_n$ converges. This however is a well known result, since:

$$\sum_{n=0}^{\infty}\frac{(L\chi^{q})^{n}}{\Gamma(1+qn)} =: E_{q}(L\chi^{q})$$

which is the Mittag-Leffler function of order q, evaluated at $L\chi^{q}$.

Therefore, we may apply the fixed point theorem and deduce the uniqueness of the solution of our differential equation. ■

Remark (2.3.2):

Without the assumption of Lipschitze condition on f, then the solution need not to be unique. To see this, consider the following simple one-dimensional example:

$$D^{q}y = y^{k}$$

with initial condition y(0) = 0. Consider 0 < k < 1, so that the function on the right-hand side of the differential equation is continuous, but the Lipschitz condition is violated, obviously the zero function is a solution of this initial value problem, however, setting $p_j(x) := x^j$, we recall that:

$$D^{q}p_{j}(x) = \frac{\Gamma(j+1)}{\Gamma(j+1-q)}p_{j-q}(x)$$

Thus, the function:

$$y(x) = \sqrt[k]{\Gamma(j+1)/\Gamma(j+1-q)x^{j}}\text{, with } j = q/(1-k)$$

Also solves the problem, proving that the solution is not unique.

2.4 ANALYTICAL METHODS FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS

In the present section, some analytical methods and presented which has the utility of solving fractional differential equations theoretically.

2.4.1 Inverse Operator Method:

This method is based on considering perhaps the simplest of all fractional differential equations:

$$\frac{d^Q f}{dx^Q} = F \text{ ....}(2.31)$$

where Q is arbitrary, F is a known function. It is tempting to apply the operator d^{-Q}/dx^{-Q} to both sides of equation (2.31), and perform "inversion"

$$f = \frac{d^{-Q} F}{dx^{-Q}}$$

But this is not the most general solution. In fact, referring to our discussion of the composition law:

$$D_x^{\alpha} D_x^{\beta} f(x) = D_x^{\beta} D_x^{\alpha} f(x) = D_x^{\alpha+\beta} f(x)$$

We recall that it is precisely the condition:

$$f - \frac{d^{-Q}}{dx^{-Q}} \frac{d^Q f}{dx^Q} = 0$$

which guarantees obedience to the composition rule for general differintegrable series f. The difference $f - \frac{d^{-Q}}{dx^{-Q}} . \frac{d^Q f}{dx^Q}$, will not in general, vanish, but will consist of those portions of the differentegrable series units f_u in f that are sent to zero under the action of $\frac{d^Q}{dx^Q}$. We decompose f into differintegrable units $f_{u,i}$, where:

$$f_{u,i} \equiv x^{p_i} \sum_{j=0}^{\infty} a_{ij} x^j, p_i > -1, a_{io} \neq 0, i = 1, 2, ..., n \text{ ....}(2.32)$$

and investigate the conditions of $f_{u,i}$ required to give:

$$f - \frac{d^{-Q}}{dx^{-Q}} \frac{d^{Q} f}{dx^{Q}} \neq 0 \text{ ....(2.33)}$$

The condition (Γ(p + j + 1 – Q)) or ($f_u - d^{-Q}d^{Q}f_u = 0$), to which are equivalent, tell us that condition (2.33) obtains if and only if, for some in the range $1 \leq i \leq n$,

$$\Gamma(p_i - Q + 1) \text{ ....(2.34)}$$

is infinite. This condition can occur, however, only when $p_i - Q + 1 = 0, -1, \ldots,$ that is, when $p_i = Q - 1, Q - 2, \ldots,$ and putting these facts together shows that, in the most general case:

$$f - \frac{d^{-Q}}{dx^{-Q}} \cdot \frac{d^{Q} f}{dx^{Q}} = c_1 x^{Q-1} + c_2 x^{Q-2} + \ldots + c_m x^{Q-m}$$

where $c_1, c_2, \ldots, c_m$, are arbitrary constants and $0 < Q \leq m < Q + 1$. For $Q \leq 0$ the right-hand member of the equation is zero. Thus:

$$f - c_1 x^{Q-1} - c_2 x^{Q-2} \ldots\ldots - c_m x^{Q-m} = \frac{d^{-Q}}{dx^{-Q}} \cdot \frac{d^{Q} f}{dx^{Q}} = \frac{d^{-Q} f}{dx^{-Q}}$$

and the most general solution of equation (2.31) is:

$$f = \frac{d^{-Q} F}{dx^{-Q}} + c_1 x^{Q-1} + c_2 x^{Q-2} + \ldots\ldots + c_m x^{Q-m} \text{ ....(2.35)}$$

Next, consider the fractional equation:

$$\frac{d^{Q} f}{dx^{Q}} + A \frac{d^{Q-1} f}{dx^{Q-1}} = F(x) \text{ ....(2.36)}$$

where Q is again arbitrary, A is a known constant, and F is known function of x. Application of the operator $\frac{d^{1-Q}}{dx^{1-Q}}$ to the both sides of equation (2.36), yields, by techniques like those discussed in connection with the inversion of equation (2.31):

$$\frac{df}{dx} + Af = \frac{d^{1-Q} F}{dx^{1-Q}} + c_1 x^{Q-2} + c_2 x^{Q-3} + \ldots\ldots + c_m x^{Q-m-1}$$

A first-order ordinary differential equation for f whose solution may be accomplished by standard methods [Murphy, 1960].

For the two extraordinary differential equations just treated were quite special, the solution of the even slightly more general equation:

$$\frac{d^q f}{dx^q} + \frac{d^Q f}{dx^Q} = F$$

encounters very great difficulties except when the difference $q - Q$ is integer or half-integer.

2.4.2 Semi Fractional Differential Equations:

By a semi differential equation we shall understand a relationship involving differintegrals of an unknown function. Each differintegral order occurring as some multiple of 1/2, at least one.

For example, the equation:

$$\frac{d^6 f}{dx^6} + \sin(x)\frac{d^{\frac{3}{2}} f}{dx^{\frac{3}{2}}} = \exp(x)$$

is a semi fractional differential equation, and also:

$$\frac{d^{\frac{3}{2}} f}{dx^{\frac{3}{2}}} - \frac{d^{\frac{-1}{2}} f}{dx^{\frac{-1}{2}}} + 2f = 0$$

is a semi differential equation, while:

$$\frac{d^2 f}{dx^2} + \frac{df}{dx} = \frac{d^{\frac{1}{2}} F}{dx^{\frac{1}{2}}}$$

is not if F is regarded as a known function. We shall discover by examples that two principal techniques are available for solving semi fractional differential equations, namely:

1) Transformation to an ordinary differential equation.
2) Laplace Transformation method.

As it is often the case when dealing with the fractional calculus, we are not able to discuss solutions of every general semi fractional differential equations but are forced to content ourselves with examples intended to reveal solution techniques.

2.4.3 Laplace Transform Method[Oldham, 1974]:

In this subsection, we seek for a Laplace transform $d^q f/dx^q$ for all q and differintegrable f, i.e., we wish to relate:

$$\mathcal{L}\left\{\frac{d^q f}{dx^q}\right\}=\int_0^\infty \exp(-sx)\frac{d^q f}{dx^q}dx$$

to the Laplace transform $\mathcal{L}\{f\}$ of the differintegrable function, Let us first recall the well-known transforms on integer-order derivatives:

$$\mathcal{L}\left\{\frac{d^q f}{dx^q}\right\}=s^q\mathcal{L}\{f\}-\sum_{k=0}^{q-1}s^{q-1-k}\frac{d^k f(0)}{dx^k},q=1,2,...$$

and multiple integrals:

$$\mathcal{L}\left\{\frac{d^q f}{dx^q}\right\}=s^q\mathcal{L}\{f\},q=0,-1,-2,... \qquad ....(2.37)$$

and note that both formulas are embraced by:

$$\mathcal{L}\left\{\frac{d^q f}{dx^q}\right\}=s^q\mathcal{L}\{f\}-\sum_{k=0}^{q-1}s^k\frac{d^{q-1-k}f(0)}{dx^{q-1-k}},q=0,\mp1,... \quad ....(2.38)$$

Also, formula (2.38), can be generalized to include non integer q by the simple extension:

$$\mathcal{L}\left\{\frac{d^q f}{dx^q}\right\} = s^q \mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k} f(0)}{dx^{q-1-k}}, \text{ for all } q \text{ ....(2.39)}$$

where n is integer such than $n - 1 < q \le n$. The sum vanishes when $q \le 0$. In proving (2.39), we first consider $q < 0$, so that the Riemann-Liouville definition:

$$\frac{d^q f}{dx^q} = \frac{1}{\Gamma(-q)} \int_0^x \frac{f(y)}{[x-y]^{q+1}} dy, q < 0, x \ne y$$

may be adopted and upon direct application of the convolition theorem [Churchill, 1948]:

$$\mathcal{L}\left\{\int_0^x f_1(x-y) f_2(y) dy\right\} = \mathcal{L}\{f_1\}\mathcal{L}\{f_2\}$$

Then gives:

$$\mathcal{L}\left\{\frac{d^q f}{dx^q}\right\} = \frac{1}{\Gamma(-q)} \mathcal{L}\left\{x^{-1-q}\right\}\mathcal{L}\{f\} = s^q \mathcal{L}\{f\}, q < 0 \text{ ....(2.40)}$$

so that equation (2.37) unchanged generalizes for negative q.

For non integer q, we use the result:

$$\left[\frac{d^q f}{dx^q}\right] = \frac{d^n}{dx^n}\left[\frac{d^{q-n} f}{dx^{q-n}}\right]_{L-R}$$

$$\frac{d^q f}{dx^q} = \frac{d^n}{dx^n} \frac{d^{q-n} f}{dx^{q-n}}$$

where n is an integer number such that $n - 1 < q < n$. Now, on application of the formula (2.38), we find that:

$$\mathcal{L}\left\{\frac{d^q f}{dx^q}\right\} = \mathcal{L}\left\{\frac{d^n}{dx^n}\left[\frac{d^{q-n} f}{dx^{q-n}}\right]\right\}$$

$$= s^n \mathcal{L}\left\{\frac{d^{q-n} f}{dx^{q-n}}\right\} - \sum_{k=0}^{n-1} s^k \frac{d^{n-1-k}}{dx^{n-1-k}}\left[\frac{d^{q-n} f}{dx^{q-n}}\right](0)$$

The difference $q - n$ being negative, the first right-hard term may be evaluated by use of equation (2.40), since $q - n < 0$, the composition rule may be applied to the terms within the summation. The result:

$$\mathcal{L}\left\{\frac{d^q f}{dx^q}\right\} = s^q \mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k} f(0)}{dx^{q-1-k}}, 0 < q \neq 1, 2, \ldots$$

follows from these two operations and is seen to be incorporated in (2.39). The transformation (2.39) is a very simple generalization of the classical formula for the Laplace transform of the derivative or integral of f. No similar generalization exists, however, for the classical formulas [Oldham, 1974]:

$$\mathcal{L}\left\{\frac{-f}{x}\right\} = \frac{d^{-1}\mathcal{L}\{f\}}{ds^{-1}}(s) - \frac{d^{-1}\mathcal{L}\{f\}}{ds^{-1}}(\infty)$$

$$\mathcal{L}\{-xf\} = \frac{d\mathcal{L}\{f\}}{ds}$$

$$\mathcal{L}\left\{[-x]^n f\right\} = \frac{d^n \mathcal{L}\{f\}}{ds^n}, \; n = 1, 2, \ldots \ldots\ldots\ldots\ldots\ldots\ldots(2.41)$$

As a final result of this section we shall establish the useful formula:

$$\mathcal{L}\left\{\exp(-kx)\frac{d^q}{dx^q}[fe^{kx}]\right\} = [s+k]^q \mathcal{L}\{f\}, q \leq 0 \ldots\ldots\ldots\ldots(2.42)$$

of which equation (2.39), may be regarded as the $k = 0$ instance.

The linear fractional ordinary differential equations with constant coefficients, so let us consider the equation:

$$\sum_{i=0}^{n} c_i \frac{d^{q_i} f(x)}{dx^{q_i}} = g(x), \text{where } -1 < q_i \le n_i \quad \text{.....................................(2.43)}$$

by taking the Laplace transformation to the both sides of the above equation, we get:

$$\mathrm{L}\left\{\sum_{i=0}^{n} c_i \frac{d^{q_i} f(x)}{dx^{q_i}}\right\} = \mathrm{L}\{g(x)\}$$

Now, using the homogeneous and linear properties of the Laplace transformation, to get:

$$\sum_{i=0}^{n} c_i \mathrm{L}\left\{\frac{d^{q_i} f(x)}{dx^{q_i}}\right\} = \mathrm{L}\{g(x)\}$$

Using equation (2.39) with f(x) is defined for all $x \in (0, \infty)$, we can find:

$$\mathrm{L}\{f(x)\} = G(x) \quad \text{...(2.44)}$$

So taking the Laplace transform to the both sides of equation (2.43), will give the solution of equation (2.43).

In this method, the following initial conditions are needed:

$$\frac{d^{q-1-k} f(0)}{dx^{q-1-k}} = 0,\ k = 0, 1, \ldots, n-1$$

where $m - 1 < q \le m$, if the initial, conditions are non-homogeneous, then the shift property could be used to transform to the origin.

2.5 ILLUSTRATIVE EXAMPLES

Some illustrative examples for solving fractional differential equation using the methods which had been discussed earlier are given next.

Example (2.5.1):

Consider the fractional differential equation:

$$\frac{d^{\frac{3}{2}}f}{dx^{\frac{3}{2}}} = x^5 \quad \text{...}(2.45)$$

with initial conditions $\frac{d^{\frac{1}{2}}f(0)}{dx^{\frac{1}{2}}} = k_0$, and $\frac{d^{\frac{-1}{2}}f(0)}{dx^{\frac{-1}{2}}} = k_1$.

Now, we applying $d^{-3/2}f / dx^{-3/2}$, for both sides of equation (2.45), yields:

$$\frac{d^{\frac{-3}{2}}}{dx^{\frac{-3}{2}}} \cdot \frac{d^{\frac{3}{2}}f}{dx^{\frac{3}{2}}} = \frac{d^{\frac{-3}{2}}f}{dx^{\frac{-3}{2}}}$$

Hence:

$$f = \frac{d^{\frac{-3}{2}}x^5}{dx^{\frac{-3}{2}}} + c_1 x^{\frac{1}{2}} + c_2 x^{\frac{-1}{2}}$$

$$= \frac{\Gamma(5+1)}{\Gamma(5+\frac{3}{2}+1)} x^{5+\frac{3}{2}} + c_1 x^{\frac{1}{2}} + c_2 x^{\frac{-1}{2}}$$

$$= \frac{\Gamma(6)}{\Gamma(\frac{15}{2})} x^{\frac{13}{2}} + c_1 x^{\frac{1}{2}} + c_2 x^{\frac{-1}{2}} \quad \text{...}(2.46)$$

Now, taking the first initial condition and applying equation (2.46), gives:

$$\frac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}} = \frac{\Gamma(6)}{\Gamma(\frac{15}{2})} \frac{d^{\frac{1}{2}}x^{\frac{13}{2}}}{dx^{\frac{1}{2}}} + c_1 \frac{d^{\frac{1}{2}}x^{\frac{1}{2}}}{dx^{\frac{1}{2}}} + c_2 \frac{d^{\frac{1}{2}}x^{\frac{-1}{2}}}{dx^{\frac{1}{2}}}$$

$$=\frac{\Gamma(6)}{\Gamma(\frac{15}{2})}\frac{\Gamma(\frac{13}{2}+1)x^{\frac{13}{2}-\frac{1}{2}}}{\Gamma(\frac{13}{2}-\frac{1}{2}+1)}+c_1\frac{\Gamma(\frac{3}{2})}{\Gamma(1)}+c_2\frac{\Gamma(\frac{-1}{2}+1)x^{-1}}{\Gamma(0)}$$

Hence:

$$\frac{d^{\frac{1}{2}}f(0)}{dx^{\frac{1}{2}}}=c_1\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)}\Rightarrow k_0=c_1\Gamma\left(\frac{3}{2}\right)\Rightarrow c_1=\frac{k_0}{\Gamma\left(\frac{3}{2}\right)}$$

Now, if we take the second initial condition, we obtain:

$$\frac{d^{-\frac{1}{2}}f}{dx^{\frac{-1}{2}}}=\frac{\Gamma(6)}{\Gamma(\frac{15}{2})}\frac{d^{\frac{-1}{2}}x^{\frac{13}{2}}}{dx^{\frac{-1}{2}}}+c_1\frac{d^{\frac{-1}{2}}x^{\frac{1}{2}}}{dx^{\frac{-1}{2}}}+c_1\frac{d^{\frac{-1}{2}}x^{\frac{-1}{2}}}{dx^{\frac{-1}{2}}}$$

$$=\frac{\Gamma(6)}{\Gamma\left(\frac{15}{2}\right)}\frac{\Gamma\left(\frac{13}{2}+1\right)}{\Gamma\left(\frac{13}{2}+\frac{1}{2}+1\right)}x^{\frac{13}{2}+\frac{1}{2}}+c_1\frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+1\right)}x+c_2\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}$$

and therefore:

$$\frac{d^{\frac{-1}{2}}f(0)}{dx^{\frac{-1}{2}}}=c_2\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}$$

which implies that:

$$c_2=\frac{k_1}{\Gamma\left(\frac{1}{2}\right)}$$

Then: $$f=\frac{\Gamma(6)x^{\frac{13}{2}}}{\Gamma(\frac{15}{2})}+c_1x^{\frac{1}{2}}+c_2x^{\frac{-1}{2}}$$

$$= \frac{\Gamma(6)x^{\frac{13}{2}}}{\Gamma(\frac{15}{2})} + \frac{k_o}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}} + \frac{k_1}{\Gamma(\frac{1}{2})}x^{\frac{-1}{2}}$$

Example (2.5.2):

Consider the semi differential equation:

$$\frac{d^{\frac{1}{2}}f(x)}{dx^{\frac{1}{2}}} + \frac{d^{\frac{-1}{2}}f(x)}{dx^{\frac{-1}{2}}} + 2f(x) = \frac{2}{\sqrt{\pi x}} + 6\sqrt{\frac{x}{\pi}} + \frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}} + 2x + 4 \text{ ....(2.47)}$$

and in order to solve this equation using Laplace transformation method, first we take the Laplace transformation to the both sides of equation (2.47):

$$L\left\{\frac{d^{\frac{1}{2}}f(x)}{dx^{\frac{1}{2}}}\right\} + L\left\{\frac{d^{\frac{-1}{2}}f(x)}{dx^{\frac{-1}{2}}}\right\} + 2L\{f(x)\} = \frac{2}{\sqrt{\pi}}L\left\{\frac{1}{\sqrt{x}}\right\} + \frac{6}{\sqrt{\pi}}L\{\sqrt{x}\}$$

$$+\frac{4}{3\sqrt{\pi}}L\left\{x^{\frac{3}{2}}\right\} + 2L\{x\} + L\{4\}$$

or equivalently in its final form:

$$L(f) = \frac{2s^2 + 3s + 1 + 2\sqrt{s} + 4s\sqrt{s}}{s^2(s+1+\sqrt{s})}$$

$$= \frac{(2s+1)+(s+1+2\sqrt{s})}{s^2(s+1+2\sqrt{s})}$$

$$= \frac{2}{s} + \frac{1}{s^2}$$

Then upon using the inverse Laplace transform, we have:

f(x) = 2 + x

as the solution of the fractional differential equation.

CONTENTS

Examining Committee's Certification

We certify that we read this thesis entitied "*Approximate Method for Solving Some Fractional Differential Equations*" and as examining committee examined the student, *Esraa Habib Khalil* in its contents and in what it connected with, and that is in our opinion it meet the standard of thesis for the degree of Master of Science, in Mathematics.

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Republic of Iraq
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Al-Nahrain University,
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Department of Mathematics and Computer Applications.

Numerical Approximations To the Gamma Cumulative Distribution Function With Random Varietes Generation By Using Monte-Carlo Simulation

A Thesis submitted to
The Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University, as a partial Fulfillment of the Requirements for the Degree of Master of Science in Applied Mathematics

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March 2006 *Saphar 1427*

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بِسِم الله الرحمن الرحيم
نرفع درجات من نشاء وفوق كل ذي علم عليم

سورة يوسف
جزء من آيه(٧٦

الإهداء

إلى...

من أحيا القلوب بعد مماتها وأنارها بعد ظلمتها وألف بينها بعد شتاتها الحبيب المصطفى محمد(صلى الله عليه وآله الغر الميامين).

من أفتخر بالإنتساب اليه...مثلي الأعلى والدي العزيز

حضن الحنان والحب الساكن في صدري والدتي العزيزه

من أحاطتني بالمحبه والدعاء المرحومه جدتي

من يبعث وجودهم الأمن والطمأنينه في قلبي أخوتي وأخواتي الأعزاء

من نوروا لي الطريق... أساتذتي الأعزاء

رفاق الدرب... زملائي وزميلاتي

من قدم لي نصيحه أو مساعده... ولم أذكره بالأسم

هدى

Acknowledgment

Above all else, I want to express my great thanks to Allah for the uncountable gifts and for helping me to present this thesis.

The honor is mine to express my sincere thanks and gratitude to my supervisors Dr. Akram M. Al-Abood for his guidance and sustained efforts throughout this work.

I'm greatly indebted for the assistance given to me by Head and staff of Mathematical and Computer Applications Department, College of Science, Al-Nahrain University.

Special thanks is devoted to Ms. Mushtaqe

Finally to all my friends………………. I present my thanks.

Huda

March ,2006

Abstract

In this thesis the gamma distn. Is considered for the reason of it appearance in many statistical fields of applications. Some mathematical and statistical properties of the distn. Are collected and unified. Moments and higher moments are illustrated and two methods of estimation for the distn. Parameters are discussed theoretically and assessed practically.

A new method of approximation to the cumulative distn. Function is drived and compared with four well-known method of appri\oximation and it shown a high performance.

Finally five procedure for generating random variates from gamma distn. Are discussed and their efficiencies are compared theoretically and pratically by Monte-Carlo simulation.

Notations And Abbreviations

r.v= random variable

r.s= random sample

Distn. =Distribution

p.d.f= Probability density function

c.d.f= Cumulative density function

m.l.e= maximum likelihood estimate

M.L.E= Maximum Likelihood Estimator

MC= Monte-Carlo

IT= Inverse Transform

INTRODUCTION

Fractional calculus is that field of mathematics of study which grows out of the traditional definitions of the calculus integral and derivative operators in which the same by fractional exponents is an outgrowth of exponents with integral value. According to our primary school ideas an exponents provides a short notation for what is essentially a repeated multiplication of numerical value. These concept in itself is easy to grasp and straight forward. However, this physical definitions can clearly become confused when considering exponents of non-integer value, [Loverro, A., 2004].

Oldham and Spanier [Oldham, 1974], who wrote in this field or subject had began their work in 1968 with the realization that the use of half-order derivatives and integrals leads to a formulation of certain electro chemical problems which is more economical and useful than the classical approaches. This discovering stimulated our interest, not only in the applications of notions of the derivative and integral to arbitrary order, but also in the basic mathematical properties of these fascinating operators. Their collaboration since 1968, has taken us far beyond the original motivation and has produced a wealth of material some of which are believe to be original. As benefits a cooperative effort between a mathematician Spanier and the chemist Oldham, their work attempts to expose not only the theory underlying the properties of the generalized operator, but also to illustrate the wide variety of fields to which these ideas may be applied with profit. They do not presume to present an exhaustive survey of the subject, but our aim has been to introduce as many readers as possible to the benty and utility of this material. Accordingly, they have made a deliberate attempt to keep the mathematical discussions as simple, as possible. For example, we have not used techniques of modern functional analysis to deal with d^q/dx^q, $q \in R$, from an operator-theoretic point of view. This latter approach, which has been taken to some extent by Feller (1952), and Hille (1939, 1948) should prove to be very fruitful but is properly the subject of a much more advanced work. Now we have sought to incorporate the fractional calculus into the larger field of symbolic, operational mathematics (Boole, 1844; Heaviside, 1893, 1920; Mikusinki, 1959; Fridman, 1969; Bourlet, 1897).

This thesis consists of three chapters.

In chapter one, we study the fundamental concepts and definitions related to fractional calculus including historical background, fundamental concepts, while the main objective of this chapter is to give an overview about fractional differentiation and integration including differentiation of unit function, zero function, the function $(x - a)^p$, etc.

In chapter two, we present the basic theory of fractional differential equations including two aspects. The first aspect is the formulation of fractional differential equations and its relationship with initial conditions, as well as, analytical methods for solving fractional differential equations including the inverse operator method, solution of semi- fractional differential equations, Laplace transformation method, as well as, with some illustrative examples. The second aspect of this chapter is to give the statement and proof of the existence and uniqueness theorem of fractional differential equation using Schauder fixed point theorem.

Chapter three presents the numerical and approximate methods for solving fractional differential equations, since numerical methods may be sometimes the most reliable and applicable method for solving differential equations, in general, and fractional differential equations, in particular. Therefore, in this chapter, several numerical and approximate methods are derived for solving fractional differential equations using linear, cubic with three node points, cubic with five nod points spline basis functions for solving fractional differential equations, which are examined using an illustrative example.

It is important to notice that, the computer programs are written using the mathematical software MATHCAD, 2001 (i), and the results are given in tabulated form.

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Supervisors Certification

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الإهداء

إلى ...

رَسُولِ اللهِ خَيْرِ خَلقِ اللهِ البشيرِ النَّذيرِ ... النبي (ص)

اللذين هما فرحة الوجود وسبب أبتسامتي وصبري ... والدي ووالدتي.

القلب الكبير ... أخي قتيبة.

سندي في الحياة ... أخوتي وأخواتي.

رفاق الدرب الطويل ... زملائي وزميلاتي.

كل من يهمه أمري.

أهدي جهدي المتواضع هذا

إسراء

المستخلص

يعتبر التفاضل الكسري من المواضيع المهمة والحديثة في حساب المشتقات والتكاملات ذات الرتب الكسرية لدوال معينة. بينما يعتبر موضوع المعادلات التفاضلية الكسرية (موضوع هذا البحث) هو ذلك الحقل الذي يهتم بدراسة طرق ايجاد حلول المعادلات التفاضليةالكسرية تحليلياً وعددياً. حيث أن الصيغة العامة للمعادلة التفاضلية الكسرية:

$$y^{(q)} = f(x, y),\ y^{(q-k)}(x_0) = y_0$$

حيث أن $k = 1, 2, \ldots, n + 1$ و $n < q < n + 1$، وان n عدد صحيح.

كما وان حلول المعادلات التفاضلية الكسرية يتضمن الكثير من الصعوبات والمشاكل عند أيجاد الحل تحليلياً ولذلك يمكن أعتبار الطرق العددية من اكثر الطرق شيوعاً عند أيجاد الحلول.

هدف هذه الرسالة هو لاستحداث ودراسة أساليب تقريبية لحل المعادلات التفاضلية الكسرية وبمساهدة الطرائق متعددة الخطوات وصيغة ريمان ليوفيلي للتكاملات الكسرية لحل المعادلات التفاضلية الكسرية.

بِسمِ اللّهِ الرَّحمنِ الرَّحيم

نَرْفَعُ دَرَجاتٍ مَنْ نَشاءُ وَفَوْقَ كُلِّ ذي
عِلْمٍ عَليمُ

صَدَقَ اللّهُ العظيم

سورة يوسف

جزء من اية (٧٦)

fجمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعـة النهرين
كلية العلوم
قسم الرياضيات وتطبيقات الحاسوب

طريقة تقريبية لحل المعادلات التفاضلية الكسورية

رسالة مقدمة الى
كلية العلوم- جامعةالنـهرين- كجزء من متطلبات نيل درجة الماجستير في علوم الرياضيات

من قِبَلْ
إسراء حبيب خليل
(بكالوريوس علوم، جامعة النـهرين، ٢٠٠٣)

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نيسان ٢٠٠٦