AN INVESTIGATION OF

AXISYMMYTRIC

VISCOELASTIC BODIES

UNDER SELF WEIGHT

A thesis

Submitted to the College of Engineering of

Al-Nahrain University in Partial Fulfillment

of the Requirements for th

of Science in Mechanical Engineering

By

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Certification

I certify that the preparation of this thesis entitled ***"An Investigation of Axisymmytric Viscoelastic Bodies under Self Weight" ,***was prepared under my direct supervision by Eng.***Uday Salah Salman*** at Al-Nahrain University / College of engineering in partial fulfillment of The requirements for the degree of Master of Science in mechanical engineering.

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دراسة ألمواد أللزجة ألمرنة تحت تأثير ألوزن ألذاتي

رسالة مقدمة

إلى كلية ألهندسة في جامعة ألنهرين وهي جزء من

متطلبات نيل درجة ماجستيرعلوم في

ألهندسة ألميكانيكية

من قبل

عدي صلاح سلمان ألكعبي

بكالوريوس 2001

محرم كانون ألثاني	1424 2004

ألخلاصة

يهتم هذا ألبحث بدراسة ألمواد أللزجة ألمرنة بأستعمال ألطرق ألنظرية و ألطرق ألعددية, أن ألطرق ألنظرية تستخدم لحل ألحالات ألبسيطة مثل ألاسطوانات ألمجوفة ألمعرظة لضغط أو تغير في درجة ألحرارة بأختلاف نوع ألحمل ألمسلط سواء كان ثابت (steady) أو متغير (Transient)

ألطرق ألعددية ألمستخدمة و هي طريقة ألعناصر المحددة (finite Element) فهي تستخدم لأعطاء قيم دقيقة للاجهادات و ألأنفعالات بالرغم من ألأخطاء ألتي قد تحدث لقيم ألانفعالات وألاجهادات عند دراسة ألمواد ألتي يقترب معامل بوزون (Poisson)من 0.5 وقد تم ألتغلب على هذه ألمشكلة بأستخدام (Isoparametric Element) وبعد ذلك تم إستخدام طريقة (Smoothing technique)لإيجاد ألنتائج على نقاط ألعقد.

أن هذا ألبحث يهتم بدراسة ألمواد أللزجة ألمرنةألمعرضة لحمل ألجاذبية و معرفة ألوقت أللازم لعودة ألأجسام ألمشوهة والمخزنة لفترات زمنية مختلفة إلى وضعها ألأصلي وذلك للحصول على أقل تشويه. [illegible] ألاشكال ذات ألسطح ألداخلي ألنجمي.وقد تم ألتركيز على ألأشكال ذات ألسطح ألداخلي ألنجمي وذلك بأخذ معاملات صلادة مختلفة [illegible] على ألنتائج ألعملية التي تم ألحصول عليها لثلاثة أنواع من [illegible] ألسطح ألداخلي للجسم المشوه أو تدوير [illegible] ب180^0وفي هذا ألبحث تم اعتماد عملية ألتدوير.في هذه ألعملية يتم عكس تأثير وزن ألجسم المشوه للحصول على أقل تشويه.

وجد أن ألزمن أللازم لتقليل ألتشويه وعودة ألجسم إلى شكله الأصلي بعد عملية ألتدوير تختلف من معدن إلى معدن حيث يعتمد على معامل ألصلادة الخاص بكل معدن فضلا عن اعتماده على ألوقت ألأصلي للتخزين.

Chapter One

Introduction and Literature Review

1.1 Definition

Viscoelastic is concerned with material, which exhibit strain rate effects in response to applied stress. [1] .The difference between viscoelastic media and more common elastic ones lies essentially in the relation between stress and strain. Whereas normal elastic analyses are based upon a (spring) constant proportionality between the two, with Young modulus as proportionality constant, the added general viscoelastic relation must allow for time [illegible] rate effects. Con[illegible]ally, it t[illegible]t the easiest way to incorporate rate effects into the mathematically model is to provide d[illegible]shpot elements in addition to the elastic spring elements. [2]

There is some conventional terminol[illegible] forms shown in Fig. 1.1; the short time ra[illegible] where the material is stiff, brittle, and elastic. At the end of the classy state a very strong dispersion region developer called the softening region is followed by the rubbery region, the creep compliance or the relaxation modulus turns horizontal against time. [3]

1.2 Solution Methods.

The solution methods can be divided into two group's analytical and numerical solution. The analytical solution used for some simple two – dimensional shapes problem, but when it is concerned with complex two-dimensional or three dimensional analysis problems, numerical method may be used. Such as finite difference method (FDM), finite element method

(FEM) and other Finite difference is of limited using because it depends on simple geometry and cannot deal with complicated geometry, most widely used method is the finite element, which can be used to construct a simple program.

1.3 Object of the Present Studies

The main objectives of the present work are:

1- Present the formulation of the theory of linear viscoelasticity, and analytical solution of typical important cases.
2- Derive the finite element equations and build a software of 2-D"plane strain and axisymmytric" to present the theory of viscoelastic problems.
3- Making an experiment for a viscoelastic torsion specimen, to get on the relaxation modulus (G (t)) for three types of viscoelastic material.
4- Using these results in study cases to show how the change in G(t) effect on this cases result.

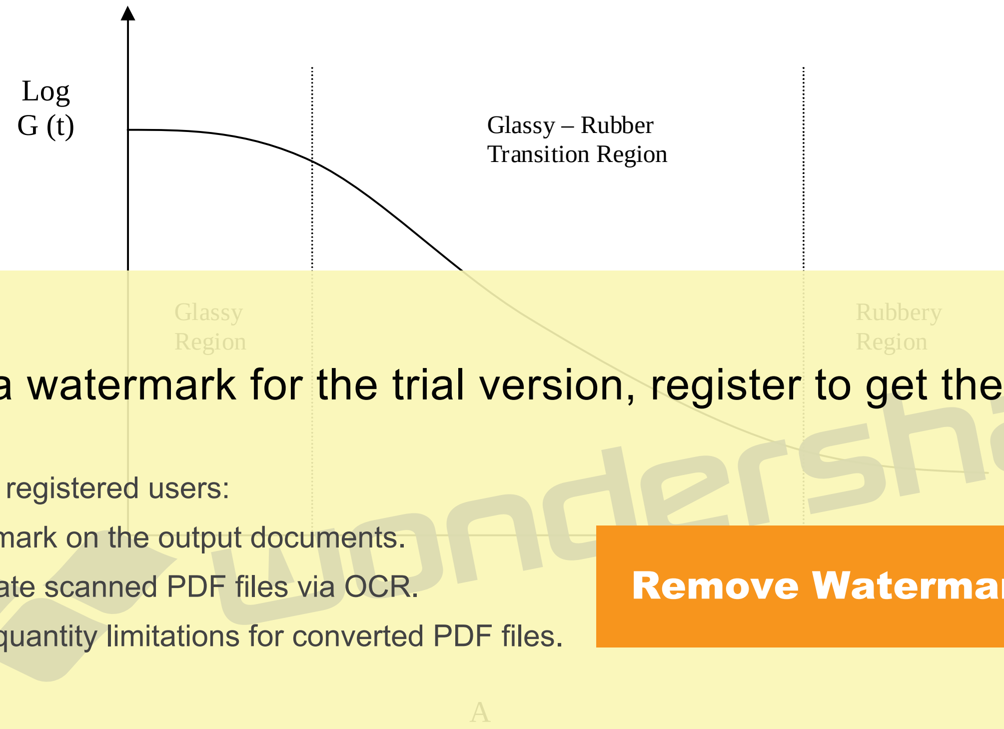


Figure 1.1A: *Shear relaxation in three regions vs. time*

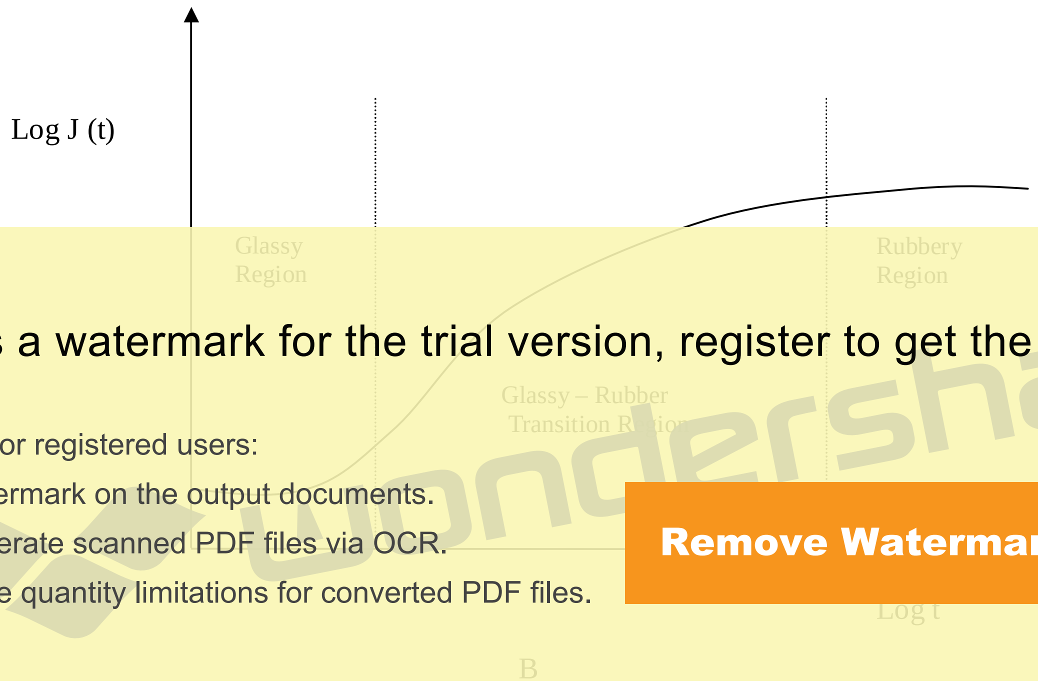


Figure 1.1B: *Creep compliance in three regions vs. time*

Literature review

1.4 Introduction

Integral transform technique such as Laplace transformation provides simple and direct methods for solving linear viscoelastic problems. Application of transform operator reduces the governing linear integrodifferential equation to a set of algebraic relations between the transforms of unknown function. And the initial and boundary conditions. Inversion, either directly or through the use of appropriate convolution theorem, provides the time domain response.

This review includes two parts, depending on the methods of solution technique, the first one is the analytical solution, and the second is the finite element solution.

1.5 Analytical Solution

Lee [4] extended the work of *Alf* [illegible] Laplace transform, formulized the concept of elastic viscoelastic correspond principle based on the solution of a viscoelastic boundary value problem can be obtained from the solution to the associated elastic problem, where the stress strain relation may be separated in two parts devioteric and volumetric part as shown below.

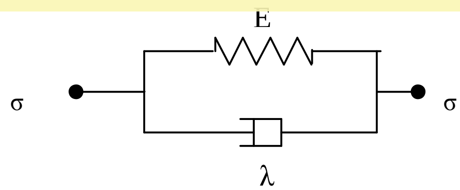
$P(S)s_{ij} = Q(S)e_{ij}$ Deviatoric relation

$P^{/}(S)s_{ij} = Q^{/}(S)e_{ij}$ Volumetric relation

And the material constant can defined as:

$$2G = \frac{Q(S)}{P(S)} \qquad ,3K = \frac{Q^{/}(S)}{P^{/}(S)}$$

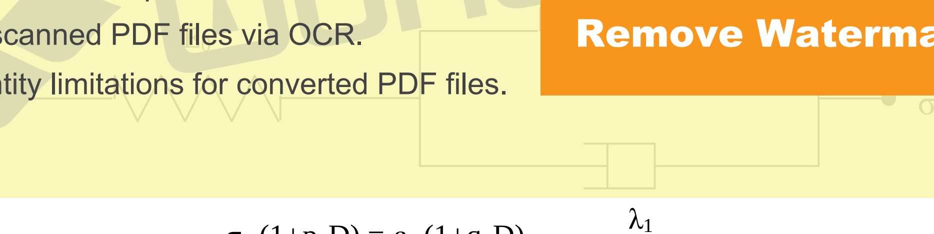
By applying the above relation on the elastic solution, the viscoelastic solution can be obtained by partial fraction or other method of inverse Laplace methods. Since inversion is much more easily carryout for low order operators P and Q, much of literature presents solution of Maxwell and Kelvin model where the viscoelastic model may represent as below.

Model	Element Arrangement	Covering Equation
Maxwell model	[illegible]	[illegible]
Voigt or Kelvin model		$\sigma_{ij}(p_1) = \varepsilon_{ij}(1+q_1D)$

Most polymer exhibit more general than Maxwell or Kelvin material model which may represent the material by a number of Kelvin models connecting in series or by a number of Maxwell models connecting in parallel which may be more difficult task of using Laplace transformation and more difficult task of inverting, therefore numerical methods are needed.

Williams [7] and ***co-workers*** [8-12] use the elastic viscoelastic corresponding principle to solve the problems of solid propellant rocket fuels.

Hussain et al [13] use the elastic viscoelastic corresponding principle by represent the material properties by the zener's model of first kind (standard linear solid), which is shown below. The material properties relaxation modulus G (t) were impose as s time dependent, then the Laplace transformation is required to modified the existing elastic solution into viscoelastic one to determine the unite responses of the mechanical or thermal loading.



$\sigma_{ij}\,(1+p_1D) = \varepsilon_{ij}\,(1+q_1D)$

Muki [14] employed an integral transform in term of convolution integral to generalize the viscoelastic solution of the previous work. Where the stress strain relation may be represent by more general hereditary integral forms of linear viscoelastic operator relation as indicated below.

$$s_{ij}(t) = 2\int_{0'}^{t} G(t-t')\,\frac{de(t')}{dt'}$$

And by using the finite difference numerical integration procedure one can get the viscoelastic response.

Rogers [15] is applied this method in moving boundary condition by consider in a circular viscoelastic hollow cylinder encased in and bonded to an elastic cylinder shell to represent a cylindrical propellant grain in a solid fuel rocket, where the inner surface may ablate at an arbitrary rate.

Park [16] and Roger [17] based their work on the Boltzman superposition integral with the unit response function used as the kernel. The existing elastic solutions are used to determined the unite responses of the corresponding viscoelastic problem based on the correspondence established between the elastic and viscoelastic unite response. The solution procedure does not required an often complex inversion step associated with an integral transform based elastic viscoelastic correspondence principle (Laplace transformation) as an illustration of the method is presented through an analysis of [illegible] displacem[illegible] viscoelastic,composite half space subjected [illegible] different temperature condition.

1.6 Finite Element Solution

Naylor [18] proposes the reduce – selective integration technique to over come the singularity due to incompressibility behavior of the polymer. He finds that in spite of gross error in the values at the centre and the edges of each element as the compressibility is reduced; all the stress components retain good accuracy at the reduced integration points using 2x2 Gauss quadrate.

Zienkiewiez [19] proposed a numerical algorithm of viscoelastic stress analysis based on elastic solution. By representing the viscoelastic behavior of a material by a number of Kelvin models connected in series and by keeping a running total of creep strain for each such model, where the constitutive equation (stress – strain relation) are represented in terms of a differential form. The total strain were separated the total strain into its elastic and creep components.

Carpenter [20] employed FEM where the constitutive equations between stress and strain are expressed in terms of high order differential equation, the advantage of using Rung – Kutta integration formulation are indicated.

Srinatha [21] discussed the solution of viscoelastic problem by FEM where the constitutive equations are represented in terms of integral form (convolution integral) instead of using the differential form.This approach have advantage that the material properties may be represent as a function of a prony series while the above two approaches are needed to represent the material properties as a differential form.

Yadagiri [22] use the preceding approach by representing the constitutive equations in terms of integra[illegible] developed an efficient special purpose co[illegible] incompressible material, by using isoparametric elements with selective integration procedure which is third order Gauss rule (3x3 Gauss or integration points) for volumetric response where the stresses and strains are computed at the (2x2) Gauss points.

Jones [23] applies the ADINA finite element code. The ADINA thermoviscoelastic material model is intended to analyze solid propellant grain during the (cool down) period. The program was used to analyze two motor cases test problems provided by the JANNAF (Joint Army Navy – NASA– Air force) interagency propulsion committee design as standard test problems. In his work he showed that ADINA results were much closer to the experimental data the best of JANNAF members result.

Hussain [24] used the approach of Finite Element formulation presented by reference [22] to develop an efficient special purpose cod “FEVES” which can be effectively employed for all the permissible values of Poisson’s ratio by using a selective integration procedure. But instead of getting the stresses and strains results at the (2x2) Gauss points the smoothing technique were used to extrapolate the stresses and strains results at the geometrical nodes. Where the results is compared with the analytical, published ones and ADINA finite element code from reference [23].

Chen [3] predicts the failure mode and failure location of the solid propellant rocket fuel of HTPB type by using MSC/NASTRAN package to get the result of thermal response and compare it with experimental result of nondestructive test using the X-ray technique.

Henrikson [25] developed a general FE formulation for the analysis of nonlinear viscoelastic materials. A single integral constitutive law proposed by [illegible] is used to describe viscoela[illegible] strain components in terms of current an[illegible] operator that includes instantaneous compliance and hereditary strain that is updated by recursive computation conveniently defines strains. Equilibrium at each time step is insured with a modified Newton-Raphson technique incorporating convergence acceleration. Multi-axial stress formulation is intuitive analogue to that of linear elasticity. Work has focused on adhesives’ response under transient load application and temperature.

1.7 Concluding Remarks

Having looked at the available literature, the following remarks can be concluded.

1-In the analytical solution the use of the integration transform in terms of convolution integral is more general than the method of using direct Laplace transform "elastic viscoelastic corresponding principle" since the difficulties associated with applying Laplace transform and the using the inverse Laplace transform are removed.

2-In the finite element methods, the methods of applying the constitutive equation in terms of convolution integral are more general of using the differential operators, since the material properties of real applications may be represented in term of Prony series in the time domain.

Chapter Two
Theory of linear Viscoelasticity

2.1 Introduction

The stress strain relations in the linear theory of viscoelasticity yield mathematical tractable representation for stress-strain –time relations which permits reasonable simple solution for many stress analyses problem. Therefore, there has been considerable activity in this area in recent years to develop new mathematical representations of linear viscoelastic behavior and new methods for linear viscoelastic analysis. [1]

Any successful constitutive relationship must be capable of describing a number of distinct features of the time dependency such as illustrated below and shown in Fig. 2.1.

I. Creep Behavior: if a stress σ is appl[illegible] the strain ε increase with time at a [illegible] the strain increases with stress, as shown in Fig. 2.1a. The behavior of an elastic material is shown as broken line.

II. Relaxation behavior: when strain is applied at zero time and held constant then the stress decreases from its value at t = 0 as time passes, shown in Fig. 2.1b.

III. Recovery behavior: if the stress is removed, either partially or entirely .the strain decreases or “recovers” as a function of time, in other words. There is delayed recovery, shown in Fig. 2.1c.

IV. Constant rate stressing behavior: constant rate of stress application results in a non – linear increase of strain with time, a linearly elastic material would give linear strain increases. If stress – strain curves are drawn for different stress states then the curve raises more steeply as strain rate increases (it is the same for all rates for an elastic material), as indicated in Fig. 2.1d.

V. Constant rate straining behavior: the same behavior obtains in that with increasing strain rate, the stress – strain curves rises more steeply, as clear in Fig.2.1e. [26].

2.1.1 Maxwell Model

This model is formed by a spring and dashpot in series, as shown in Fig. 2.2. For simple tension as $\sigma_{\circ}$ is applied at t = 0, an immediate elastic strain ε^e of the spring occurs. Then a viscous strain ε^v of dashpot is added. The total strain is equal to the same of the strain in each component. While the stress acts on them is the same. The total strain can be written as:

$$e = e^e + e^v \qquad 2.1$$

Then the strain rate is:

$$\frac{de}{dt} = \frac{de^e}{dt} + \frac{de^v}{dt}$$

But elastic strain is:

$$e^e = \frac{s}{E} \qquad 2.3$$

And

$$\frac{de^e}{dt} = \frac{1}{E}\frac{ds}{dt} \qquad 2.4$$

Whereas the viscous strain rate is given by:

$$\frac{de^v}{dt} = \frac{s}{l} \qquad 2.5$$

Thus, the governing equation of Maxwell model is:

$$\frac{de}{dt} = \frac{1}{E}\frac{ds}{dt} + \frac{s}{l} \qquad 2.6$$

It is of interest to examine the response of such a material to various stress and strain hisrories, when applied constant stresses.eq (2.6) will reduce to:

$$\frac{de}{dt} = \frac{s}{l} \qquad 2.7$$

Then by integration, it can be found that:

$$e = \frac{st}{l} + \frac{s_\circ}{E} \qquad 2.8$$

Eq. (2.8) explains that only viscous flow observed with time. After the time t_1, the stress σ is removed, an immediate recovery of elastic component of strain occurs leaving irreversible strain of viscous element, as shown in Fig. 2.3.For the case of constant strain.

$$\frac{ds}{s} = -\frac{E}{l}dt \qquad 2.9$$

By integration, as shown in Fig. 2.3, eq (2.9) will be:

$$s = s_\circ Exp(-\frac{t}{t'}) \qquad 2.10$$

Where $t' = \frac{l}{E}$ is the relaxation time.fiq (2.3) showing creep recovery and [illegible] for Maxwell model . [27]

2.1.2 Voigt or Kelvin Model

This model consists of spring and dashpot in parallel as shown in Fig. 2.4.As σ_0 is applied a dashpot prevents an instantaneous extension of the elastic spring. With time, the viscous behavior causes an increasing of strain. The total strain, elastic and viscous strain are equale.and each component support apportions of σ_0, Therefore:

$$s_\circ = s = s^e + s^v \qquad 2.11$$

This is equaled to

$$s = Ee^e + l\frac{de^v}{dt} \qquad 2.12$$

But

$$e = e^e = e^v \qquad 2.13$$

Then

$$s = Ee + l \frac{de}{dt} \qquad 2.14$$

For creep case, where the model supports to constant stress, the solution of governing eq (2.14) is:

$$e = \frac{s_{\circ}}{E}[1 - Exp(-\frac{t}{t^{//}})] \qquad 2.15$$

Where $t^{//} = \frac{l}{E}$ is the retardation time.

Comparison between Eq. (2.15) and eq (2.8) indicate that the predicated creep behavior of Kelvin model is more realistic, since the strain approach to σ_0/E as time approach to infinity. The response of Kelvin model to constant load is more readily understood by considering the recovery response, where $\sigma = 0$ then:

$$s = Ee + l \frac{de}{dt} = 0 \qquad 2.16$$

By integration

$$e = e_0 + Exp(-\frac{t}{t^{//}}) \qquad 2.17$$

Figure 2.5 shows the creep and recovery behavior of Kelvin model. Consider now Kelvin model subjected to constant strain, as shown in Fig. 2.5, then eq (2.14) will reduced to:

$$s = Ee \qquad 2.18$$

Equation (2.18) implying that the material behaves as an elastic solid. This is an inadequate for general viscoelastic behavior. [27].

Now for comparison between Maxwell and Kelvin model. It has been shown that Maxwell model gives a reasonable predication of relaxation but it has unlimited deformation.wheares Kelvin model provide a better predication for creep and recovery but it provide for a maximum displacement limited by the elastic deformation of the spring. [28].

2.1.3 The Standard Linear Solid

The simplest combination is shown in Fig. 2.6 and consists of a voigt model with spring in series. If the modulus of the additional spring is E_1 and the modulus of the spring in the voigt model is E_2then. Figure 2.7 illustrates the behavior of this model.

The differential equation is:

$$\frac{E_0+E_1}{l}\frac{1}{E_0}s+\frac{1}{E_0}\frac{ds}{dt}=\frac{de}{dt}+\frac{E_1 e}{l} \qquad 2.19$$

From which it can be seen that both creep and relaxation result in satisfactory relationships. The creep compliance function is:

$$J(t)=\frac{1}{E_1}+\frac{1}{E_2}[1-Exp(-\frac{t}{t_1})] \qquad 2.20$$

Where $t_1=\frac{l}{E_2}$ and the relaxation modulus becomes [26]

$$G(t)=E_1-\frac{E_1^{\,2}}{E_1+E_2}[1-Exp(\frac{t}{t_2})] \qquad 2.21$$

Where $t_2=\frac{l}{E_1+E_2}$

2.1.4 Generalized Maxwell and Kelvin Model

Each Maxwell and Kelvin models are not capable of describing the properties of real system. The real system is a structure of many chains, each of which may itself posses both elastic and viscous nature. The models needed to describe the behavior of real materials may be composed of many Kelvin and Maxwell elements. The behavior of these models under an entirely different set of condition provides reasonable predication of real materials. [29].

Generalized Maxwell model (GMM) proposed to describe the stress relaxation of linear viscoelastic material. This model is shown in Fig. 2.8; consist of Maxwell elements arranged in parallel.

In this arrangement the strain ε of all elements is the same, and the total stress on the system σ (t) is:

$$s\,(t) = G_0 e + e \sum_{i=1}^{n} G_i Exp[-\frac{G_i}{l_i} t] \qquad 2.22$$

And

$$G(t) = G_0 + \sum_{i=1}^{n} G_i Exp[-\frac{G_i}{l_i} t] \qquad 2.23$$

For generalized Kelvin model (GKM) which consist of a series of arrangement of Kelvin elements. Each element has a different spring modulus G_i and viscous constant λ_i as shown in Fig. 2.9. The voigt elements are connected in series with a single spring, which represent the instantaneous elastic response. Under constant stress, the elongation of the model can be expressed as:

$$e\,(t) = J_0 s + s \sum_{i=1}^{n} J_i [1 - Exp(-\frac{l_i}{J_i} t)] \qquad 2.24$$

Therefore

$$J(t) = J_0 + \sum_{i=1}^{n} J_i [1 - Exp(-\frac{l_i}{J_i} t) \qquad 2.25$$

2.2 Differential Representation Derive

Differential equations are the best and simplest relation, which describe the linear behavior of mechanical models. These models are constructed of elastic spring and viscous dashpot. [29] The constitutive equation of a rate sensitive linear material for a simple stress state, such as uniaxial stress or pure shear, may be expressed as a linear function.

$$f\,(s\,, s^{\circ}, s^{\circ\circ}, ...; e\,, e^{\circ}, e^{\circ\circ}, ...) = 0 \qquad 2.26$$

Where σ =σ (t) describes the variation of stress with the time, ε =ε (t) describes the variation of strain with the time. The dots represent the derivatives with respect to time. Equation (2.26) is commonly written in more compact forms as mentioned previous in equation (1.1). Where

$$P = \sum_{r=0}^{a} p_r \frac{\partial^r}{\partial t^r} \qquad 2.27$$

$$Q = \sum_{r=0}^{b} q_r \frac{\partial^r}{\partial t^r} \qquad 2.28$$

A differential form of the constitutive equations obtained by combining equations (2.26), (2.27) and (2.28) as follows.

$$Ps = p_0 s + p_1 \overset{\circ}{s} + p_2 \overset{\circ\circ}{s} + \ldots + p_a \frac{\partial^a s}{\partial t^a}$$
$$= q_0 e + q_1 \overset{\circ}{e} + q_2 \overset{\circ\circ}{e} + \ldots\ldots + q_b \frac{\partial^b e}{\partial t^b} \qquad 2.29$$

[illegible] derivatives with res[illegible] $p_0, q_1, q_2, \ldots\ldots$ are material constants. In eq (2.29), there is no loss of generality by assuming $p_0 = 1$, taking the Laplace transforms of eq (2.29) for zero initial conditions yields.

$$P(s)s\,(s) = (p_0 + p_1 s + p_2 s^2 + \ldots. + p_a s^a)s\,(s)$$
$$= Q(s)e\,(s) = (q_0 + q_1 s + q_2 s^2 + \ldots. + q_b s^b)e\,(s) \qquad 2.30$$

Where s is the transform variable. From eq (2.30).

$$\frac{Q(s)}{P(s)} = \frac{s\,(s)}{e\,(s)} \qquad 2.31$$

For the linear case, the p_r and q_r in eq (2.29) are independent of stress and strain, but may depend on time.

Equation (2.29) can be reduced to a special case of Maxwell model equation (2.6) by assuming that p_0=1, p_1=λ/E and q_0=0,q_1= λ, also eq(2.29) can be reduced to the Kelvin model equation(2.14) by assuming that p_0=1,q_0=E,q_1= λ

2.3 Integral Representation Derive:

Instead of the differential equations, integral may be employed as constitutive equation to describe the viscoelastic behavior of the material. The most important integral representation of viscoelastisity is given by boltzman and his theory is called Boltzman superposition theory. Boltzman proposed that:

1-The creep in specimen is a function of entire loading history, and

2-Each loading step makes an independent contribution to the final deformation, which is obtained by simple addition of each contribution. [16]

If a constant stress σ_1 is applied at t=ξ then:

$$[illegible](t) [illegible]_1 H(t-x) \qquad 2.32$$

And th[illegible] corresponding [illegible] strain will be:

$$[illegible](t) = s [illegible]) H(t-x)$$

[illegible] H(t − x) unit function, and as explained in appendix (B).

If stress σ_0 is applied at time t=0 to a linearly viscoelastic material and then at t=ξ_1 ,σ_1 is applied as shown in Fig.2.10.The strain output at any time subsequent to $\xi_1$is given by the sum of the strains at that time due to the two stresses component as thought each were acting separately, this is the Boltzman superposition principle. If the stress input σ (t) is arbitrary (i.e. variable with time) instead of constant, this arbitrary stress input can be expressed by the sum of the series of a constant stress inputs as shown in Fig. 2.11 and described by

$$s(t) = \sum_{i=1}^{r} \Delta s_i H(t-x_i) \qquad 2.34$$

The Boltzman superposition principle states that the sum of the strain outputs resulting from each component of stress input is the same as the strain output resulting from the combined stress input.

Therefore the strain output under variable stress σ (t) equals

$$e(t) = \sum_{i=1}^{r} e_i (t - x_i) \qquad 2.35$$

Or

$$e(t) = \sum_{i=1}^{r} \Delta s_i J(t - x) H(t - x) \qquad 2.36$$

If the number of the steps tends to infinity the total strain can be expressed by an integral representation as:

$$e(t) = \int_0^t J(t - x) H(t - x) d[s(x)] \qquad 2.37$$

This equation is a stieltjes integrel.If the stress history is differentiable and since the dummy variable ξ is always less than or equal to time (t), the function H (t-ξ) is therefore always unity [illegible] Hence eq (2.37) reduced to the following fo[illegible]

$$e(t) = \int_0^t J(t - x) \frac{\partial s(x)}{\partial x} dx \qquad 2.38$$

Where $d[s(x)]$has been replaced by $\frac{\partial s(x)}{\partial x} dx$ in order that time may be the independent variable.

This equation is an integral representation of creep and it can be used to describe (and to predict) the creep strains under any given stress history provided the creep compliance J (t) is known.

An alternative form for eq (2.38) may be obtained by employing integration by parts, taking:

$$u = J(t-x); dv = \frac{\partial s(x)}{\partial x}dx \qquad 2.39$$

it will become

$$e(t) = s(t)J(0) - \int_0^t \bar{J}(t-x)s(x)dx \qquad 2.40$$

Where

$$\bar{J}(t-x) = \frac{\partial J(t-x)}{\partial x} \qquad 2.41$$

If the creep compliance J (t) is separated into a time – independent (elastic) compliance J_0 and a time – dependent creep function φ (t), eq (2.38) becomes:

$$e(t) = J_0 s(t) + \int_0^t j(t-x)\frac{\partial s(x)}{\partial x}dx \qquad 2.42$$

Exactly the same arguments apply when step changes or arbitrary changes in strain are applied and the resulting change in stress as a function of time is determined, then Boltzman superposition principle can be restated by substituting stress for strain and strain for stress. Getting on

$$s(t) = \int_0^t E(t-x)\frac{\partial e(x)}{\partial x}dx$$

Separating the relaxation modulus E (t) into a time – independent (elastic) modulus E_0 and a time-dependent stress relaxation function ψ (t) eq (2.43) becomes: [1]

$$s(t) = E_0 e(t) - \int_0^t y(t-x)\frac{\partial e(x)}{\partial x}dx \qquad 2.44$$

2.4 Material Properties

It is customary to assume that most viscoelastic materials are essentially incompressible, which is for small strains are equivalent to assuming an infinite bulk modulus or poison's ratio of one half. There are three basic approaches of approximations to the material behavior.

- The incompressibility in bulk but permitting viscoelastic shear behavior.

$$K(s) = a \qquad u = 0.5 \qquad E(s) = 3G(s) \qquad 2.45$$

- The second permits a finite value of the bulk modulus but neglect any time dependence thus replacing its actual time dependent behavior by an average constant. The shear behavior is assumed viscoelastic as before.

$$K(s) = K \qquad u(s) = \frac{3K - 2G(s)}{6K + 2sG(s)}$$

$$E(s) = \frac{9KG(s)}{3K + sG(s)} \qquad 2.46$$

- The last stage is assumed that both bulk and shear are viscoelastic.[1]

In the work reported in the thesis, it will be based on the second approach. The reduced time hypothesis is valid, linear uncoupled thermoviscoelastic formul[illegible], the material is isotropic an[illegible] between the stress and strain based on small [illegible]

The numerical package such as ADINA [23] and ANSYS [30] use the material properties in terms of shear relaxation function therefore they use the generalized Maxwell model (GMM) to represent the mechanical properties, where the ANSYS package use 10 elements as a maximum number of GMM elements.

The relaxation modulus function can be obtained from a stress relaxation test, where a specimen is instantaneously deformed to, and held at, a given strain while the stress is measured over decades of time. The stress initially reflects the relatively stiff state of the material, and then gradually decreases through the so-called transition region, settling at considerably lower value, as the material becomes more complaints .the stress relaxation test generates a curve that can be approximated numerically by a prony – dirichlet series as below.

$$G(t) = G_{\circ} + \sum_{i=1}^{m} G_i Exp[-\frac{t}{t_i}] \qquad 2.47$$

Where relaxation coefficients and t_i are their corresponding characteristic times (relaxation time). Eq (2.46) is the same of eq (2.23) where $t_i = \lambda_i / G_i$.

2.4.1 The Linking Between The material Properties In Terms of (S) Domain

The linear stress strain relations for a homogenous and isotropic viscoelastic solid, it will introduce the deviatoric components of a stress and strain through:

$$S_{ij} = s_{ij} - \frac{1}{3} d_{ij} s \qquad 2.48$$

[illegible] 2.49

$$e_{ij} = e_{ij} - \frac{1}{3} d_{ij} e \qquad 2.50$$

Where [illegible]

in which d_{ij} is the kronecker delta. Turning first to constitutive law of integral form, the relaxation integral law.

$$S_{ij}(x,t) = \int_{-a}^{t} G_1(t-\bar{t}) \frac{\partial}{\partial \bar{t}} e_{ij}(x,t) d\bar{t}$$

$$s_{ij}(x,t) = \int_{-a}^{t} G_2(t-\bar{t}) \frac{\partial}{\partial \bar{t}} e_{ij}(x,t) d\bar{t} \qquad 2.51$$

$$e_{ij}(x,t) = \int_{-a}^{t} J_1(t-\bar{t}) \frac{\partial}{\partial \bar{t}} S_{ij}(x,t) d\bar{t}$$

$$e(x,t) = \int_{-a}^{t} J_2(t-\bar{t}) \frac{\partial}{\partial \bar{t}} s_{ij}(x,t) d\bar{t} \qquad 2.52$$

Here $G_1(t), G_2(t)$ are the relaxation moduli (in shear and isotropic compression) at the uniform temperature under consideration in this connection. It is clear that stipulate that $G_i(t)=0\ , i=1,2$ for –α <t<0. Alternatively, for a medium with a finite and discrete spectrum of relaxation and retardation times, the linear constitutive law admits the differential operator representation, equations (2.27), (2.28) and (2.30).

The familiar connection between the foregoing three variants of stress strain law is obtained with the aid of the Laplace transform.

It concludes under suitable regularity assumption that eq (2.51) and (2.52) are equivalent if:

$$G_i(t)=\frac{1}{S^2 J_i(S)} \qquad 2.53$$

Similarity the equivalent of eq (2.49) and (2.30) is assured by

$$G_i(t)=\frac{Q_i(S)}{S\ P_i(S)} \qquad 2.54$$

[illegible] (2.51) [illegible] (2.30) are equivalent [illegible]

$$J_i(t)=\frac{P_i(S)}{S\ Q_i(S)} \qquad 2.55$$

Equation (2.54) and (2.55) permits the passage from a given differential operator law to an equivalent stress strain law in integral form.

In addition to the relaxation moduli in shear and isotropic compression (dilatation), it shall have occasion to make use of the “relaxation modulus extension” and of “Poisson’s ratio” for a viscoelastic material. The definition of these concepts may be used on a uniaxial tensile relaxation test at constant strain, which is characterized by.

$$\begin{aligned} e_{11} &= e_0 H(t) \quad , s_{11} = s_{11}(t) \\ s_{22} &= s_{33} = 0 \quad , s_{ij} = 0 \end{aligned} \qquad 2.56$$

The defining equations for tensile modulus E (t) and for Poisson's ratio ν (t) both of which are time dependent properties now appear as:

$$E(t) = \frac{s_{11}(t)}{e_0} \qquad , n(t) = \frac{-e_{22}}{e_0} \qquad 2.57$$

Making a connection between eq (2.56), (2.48) and by eliminating from eq (2.51) all components of stress and strain except σ_{11}(t) and ε_{22}(t) on applying the Laplace transform to the resulting pair of equations ,it is arrive at:

$$\begin{aligned} s_{11}(S) &= G_1(S)[e_0 - Se_{22}(S)] \\ s_{11}(S) &= G_2(S)[e_0 - 2Se_{22}(S)] \end{aligned} \qquad 2.58$$

In view of eq (2.55)

$$\begin{aligned} E(S) &= \frac{3G_1(S)G_2(S)}{G_1(S) + 2G_2(S)} \\ n(S) &= \frac{G_2(S) - G_1(S)}{S[G_1(S) + 2G_2(S)]} \end{aligned} \qquad 2.59$$

If the employing the usual notation for inverse transform .it will be helpful for our future purposes to particularize the [illegible] important special viscoelastic media. In [illegible] constant entering eq(2.30)have the values.$p_0=1, q_1=2\mu$ in shear and $p_0=1$,$q_1=3K$ in extension then the equation(2.30) becomes

$$\begin{aligned} S_{ij}(x,t) &= 2me_{ij}(x,t) \\ s(x,t) &= 3Ke(x,t) \end{aligned} \qquad 2.60$$

In which μ is the shear modulus and K is the bulk modulus. It will also be convenient to recall the relations.

$$\begin{aligned} K &= \frac{E}{3(1-2n)} \qquad , m = \frac{E}{2(1+n)} \\ E &= \frac{9mK}{3K+m} \qquad , n = \frac{3K-2m}{2(3K+m)} \end{aligned} \qquad 2.61$$

Where E and ν are the (constant) Young's modulus and Poisson's ratio of the elastic material. By using the differential operators of elastic model and eq (2.54), (2.55) and (2.30) therefore:

$$G_1(S)=\frac{2m}{S} \qquad ,G_2(S)=\frac{3K}{S} \qquad 2.62$$

And substituting into eq (2.59) ,then:

$$E(S)=\frac{9mK}{(m+3K)}$$

$$n(S)=\frac{3K-2m}{2S(m+3K)} \qquad 2.63$$

For Maxwell solid the differential operator values are $p_0=1/\lambda$, $p_1=1$, $q_0=0$,$q_1=2\mu$ applying the same procedure to get on

$$E(S)=\frac{9mK}{(m+3K)[S+\frac{3K}{l\,(m+3K)}]}$$

$$n(S)=\frac{(3K-2m)S+\frac{3K}{l}}{S(2m+6K)+\frac{6K}{l}S} \qquad 2.64$$

For Kelvin solid the differential operator values are $p_0=1,q_0=2\mu,q_1=2\mu\lambda$ applying the same procedure to get on

$$E(S)=\frac{9mK(\frac{1}{S}+l\,)}{mS(\frac{1}{S}+l\,)+3K}$$

$$n(S)=\frac{3K-2mS(\frac{1}{S}+l\,)}{S[2mS(\frac{1}{S}+l\,)+6K]} \qquad 2.65$$

2.4.2 Evaluating the Stress Relaxation Modulus

When the experimental results are available then it easy to find the G (t) by using a prony series by the following. [31]

I. Fitting eq (2.47) to a single experimental curve, it is sufficient to choose m relaxation times per decades [31] in the times interval of the experimental curve.At time t_k, the calculated relaxation is

$$G_c(t_k) = G_0 + \sum_{i=1}^{m} G_i Exp[-\frac{t_k}{t_i}] \qquad 2.66$$

Where the transient relaxation is expressed in the form of prony series and t_i is the relaxation time. If there are n data points in the experimental curve then eq (2.66) becomes:

$$G_c(t_k) = \sum_{k=1}^{n} G_0 + \sum_{k=1}^{n} \sum_{i=1}^{m} G_i Exp[-\frac{t_k}{t_i}]$$

II. Defining a residual ***R*** such that

$$R = \sum_{k=1}^{n} [G_c(t_k) - G_e(t_k)]^2$$

III. Finding the linear parameter by

The least square technique is to find the known linear parameters G_0 and G_i (i=1……….m) which minimized R, i.e.

$\frac{\partial R}{\partial G_0} = 0$ And $\frac{\partial R}{\partial G_i} = 0$ for all i , As G_e has no dependence on the unknowns, then

$$\frac{\partial R}{\partial G_0} = \sum_{k=1}^{n} [G_c(t_k) - G_e(t_k)] \frac{\partial G_c(t_k)}{\partial G_0} = 0 \qquad 2.69$$

Where

$$\frac{\partial G_c(t_k)}{\partial G_0} = 1 \qquad 2.70$$

Similarly,

$$\frac{\partial R}{\partial G_i} = \sum_{k=1}^{n}[G_c(t_k) - G_e(t_k)] \; \frac{\partial G_c(t_k)}{\partial G_i} = 0 \qquad 2.71$$

Where

$$\frac{\partial G_c(t_k)}{\partial G_i} = Exp(-\frac{t_k}{t_i}) \qquad 2.72$$

From eq (2.70) and (2.71), it can be shown that

$$\sum_{k=1}^{n} G_c(t_k) = \sum_{k=1}^{n} G_e(t_k) \qquad 2.73$$

By substituting eq (2.67) into eq (2.73), the following expression is obtained:

$$nG_0 + \sum_{k=1}^{n}\sum_{i=1}^{m} G_i Exp(-\frac{t_k}{t_i}) = \sum_{k=1}^{n} G_e(t_k) \qquad 2.74$$

Similarly eq (2.71) and (2.72), the following can be deduced

$$\sum_{k=1}^{n} G_c(t_k) Exp(-\frac{t_k}{t_i}) = \sum_{k=1}^{n} G_e(t_k) Exp(-\frac{t_k}{t_i}) \qquad 2.75$$

By substituting eq (2.67) into eq (2.75), the following expression is obtained:

$$G_0 \sum_{k=1}^{n} Exp(-\frac{t_k}{t_i}) + \sum_{k=1}^{n}[\sum_{i=1}^{m} G_i Exp(\text{[illegible]}) \text{[illegible]} = \sum_{k=1}^{n} G_e(t_k) Exp(-\frac{t_k}{t_i})$$

IV. Making expression in a matrix form for equations (2.74) and (2.76) as follow:

$$[A]X = Y$$

Where X is the set of unknowns G_0 and G_i and Y is a vector calculated from the experimental data points. Therefore:

$$X^T = \{G_0 G_1 G_2 \ldots\ldots\ldots\ldots\ldots G_m\}$$
$$Y^T = \{\sum_{i=1}^{n} G_e(t_i) \sum_{i=1}^{n} G_e(t_i) e^{t_i/t_1} \ldots\ldots\ldots \sum_{i=1}^{n} G_e(t_i) e^{t_i/} \qquad 2.77$$

$$A = \begin{vmatrix} n & \sum_{i=1}^{n} e^{-t_i/t_1} & \sum_{i=1}^{n} e^{-t_i/t_m} \ldots\ldots & \sum_{i=1}^{n} e^{-t_i/t_m} \\ \sum_{i=1}^{n} e^{-t_i/t_1} & \sum_{i=1}^{n} e^{-2t_i/t_1} & \sum_{i=1}^{n} e^{-t_i/t_1} e^{-t_i/t_2} \ldots & \sum_{i=1}^{n} e^{-t_i/t_1} e^{-t_i/t_m} \\ \sum_{i=1}^{n} e^{-t_i/t_2} & \sum_{i=1}^{n} e^{-t_i/t_1} e^{-t_i/t_2} & \sum_{i=1}^{n} e^{-2t_i/t_2} \ldots\ldots & \sum_{i=1}^{n} e^{-t_i/t_2} e^{-t_i/t_m} \\ \sum_{i=1}^{n} e^{-t_i/t_m} & \sum_{i=1}^{n} e^{-t_i/t_1} e^{-t_i/t_m} & \sum_{i=1}^{n} e^{-t_i/t_2} e^{-t_i/t_m} & \sum_{i=1}^{n} e^{-2t_i/t_m} \end{vmatrix}$$

2.5 Time -Temperature Effect on Mechanical Behavior

In general mechanical properties of viscoelastic material depend not only on time (or load duration) but also on temperature; especially some of mechanical properties of amorphous polymers (see appendix C) have a strong dependence on temperature. The linear thermoviscoelasticity theory does not [illegible] the mechanical properties to vary with temperature and the properties corresponding to a particular [illegible] throughout the solution procedure; otherw[illegible] linear in that the material property functions depend on material temperature that again only can be determined when material properties function are known. However, there exists a special class of material whose temperature dependence of mechanical properties is trials is referred to as being thermorheologically and the corresponding description of temperature-dependent properties was first proposed by leader man. [23]

The simplifying feature of the thermorheologically simple materials is that when material property (e.g. relaxation modulus or creep compliance) curves measured at different constant temperatures are all plotted against time on logarithmic scales, the curves can be superposed so as to form a single curve (called a master curve) corresponding to an arbitrary fixed temperature (called reference temperature) by means of horizontal shift only as shown in Fig. 2.12. The horizontal shifts between master curve and the isothermal curve are independent of time but depend only on temperature, this feature has a significant consequence in that the dependent of the material property on both time and temperature can be represented by dependence on single variable called reduced time, and the feature is often referred to as the time – temperature superposition.

The φ (T) is a temperature – dependent material shifting function and reflects the influence of temperature on the internal viscosity of the material as defined below. [30]

For the temperature above the class transition temperature of material, the shift factor φ (T) for thermorheogically simple material usually is expressed in the following form:

$$\log j(T) = \frac{-c_1(T-T_R)}{[illegible](T-T_R)} \qquad 2.78$$

Where [illegible] constants [illegible] q (2.48) is [illegible] shown in Fig. 2.13. [30]

2.5.1 Modification of the Constitutive Law

The constitutive laws discussed in the preceding section rest on the assumption that the entire body is permanently maintained at a uniform temperature. Accordingly the response function G (t) and J (t) entering the previous equations as well as material parameters p and q are to be regarded as having been determined at the relevant fixed "Base temperature".[14]

It now turns to the modification arising in the stress strain law if the temperature field is variable with position and time. With a view toward the analytical formulation of the temperature time equivalent hypothesis it confine our attention at first to the effect of a uniform temperature change, i.e., a change in the base temperature, and consider merely the temperature dependence of the relaxation moduli.to this end let G (t, T) at the constant temperature T so that, in accordance to:

$$G_i(t,T) = f[\log t + y\,(T)] \qquad 2.79$$

Where the “shift function” $y\,(T)$ obeys

$$y\,(T_0) = 0 \qquad \frac{dy}{dT} > 0 \qquad 2.80$$

Setting

$$y\,(T) = \log j\,(T) \qquad 2.81$$

Where $j\,(T)$ is the shift factor and according to eq (2.80) is conform to

$$j\,(T_0) = 1 \quad j\,(T) > 0 \quad \frac{dj}{dT} > 0 \qquad 2.82$$

And is thus a positive, monotone increasing function of T throughout the range of validity of eq (2.80). Also, this [illegible] (2.81) yield

$$G_i(t,T) = G_i(x) \qquad 2.83$$

Provided the “reduced time” ξ is defined by

$$x = t\,j\,(T) \qquad 2.84$$

Consequently, the entire one-parameter family of response-function pairs G_i (t, T) is completely determined by its single member $G_i(t_0)$=G_i(t, T) once ψ (T) is known for the temperature range in problem. The shift function ψ (T), and hence the shift factor φ (T), in turn, represent an intrinsic property of the material under consideration.

Equations (2.83), (2.84) enable us to pass from eq (2.51), which holds at the base temperature T_0, to the corresponding relaxation integral law applicable at any constant temperature T. this transition is evidently effected by replacing $G_i(t-t')$in eq(2.51)with $G_i(\xi-\xi')$,where ξ is given by eq(2.84) and $\xi'=t'$ φ(T),provided the body (in the absence of the load) is considered to be in the unstrained stat at the uniform temperature T.

Next suppose the medium is under the influence of a ***variable temperature field*** T(t,T) .In this event the foregoing modification of the constitutive law eq(2.51) requires a twofold amendment .First , to allow for the temperature dependence of the response functions in the presence of a time-dependent temperature distribution, the definition in eq(2.84) of the reduced time ξ must be generalized consistent with the postulated temperature time equivalence.Scond,thermal expansion must be taken into account in the equation governing the dilatational response of the material. In this manner Morland and Lee [14] arrived at the generalized relaxation integral law.

$$S_{ij}(x,t)=\int_{-a}^{t} G_1(x-x')\frac{\partial}{\partial t'}e_{ij}(x,t')dt' \quad \text{[illegible]}$$

$$s(x,t)=\int_{-a}^{t} G_2(x-x')\frac{\partial}{\partial t'}[e_{kk}(x,t)-3\text{[illegible]}$$

Where

$$x = f(x,t)=\int_0^t j\,[T(x,t)]dt \quad , \quad x' = f(x,t') \qquad 2.86$$

While the "θ" is defined by

$$q(x,t)=\frac{1}{a_0}\int_{T_0}^{T(x,t)} a(T')dT' \quad , a_0 = a(T_0) \qquad 2.87$$

Here α (T) is the temperature dependent coefficient thermal expansion and α_0 its value at the base temperature T_0,G_1(t) and G_2(t),as before ,are the relaxation moduli measured at the base temperature .

Equations (2.86), in contrast to eq (2.84), imply a dependence of the reduced time ξ upon both position and the physical time.

If, in particular,α(T)is constant,eq(2.85)become α(T)= α_0 , θ(x,t)=T(t,T)-T_0.further,in case T(x,t) is constant, the educed time ξ given in eq(2.86) coincides with that defined in eq(2.84).finally, eq(2.85) degenerate into eq(2.51) when T(x,t)=T_0 .examining the structure of eq(2.85),and bearing in mind eq(2.86),(2.87) ,one notes that the temperature enters eq(2.85) both through the reduced time ξ and the temperature function θ.

Analogous consideration applies to the generalization of the creep law eq (2.52), which-under no isothermal conditions, assumes the modified form.

$$e_{ij}(x,t)=\int^{t} J_1(x-x^{/})\frac{\partial}{\partial t^{/}}S_{ij}(x,t)dt^{/} \quad (x,t)$$

$$e(x,t)=\int^{t} J_2(x-x^{/})\frac{\partial}{\partial t^{/}}s_{ij}(x,t)dt^{/}+3a_0 q(x,t)$$

Here [illegible] are the creep compliances at the [illegible]

to eliminate any explicit reference to t fro[illegible]

reduced time ξ .for this purpose. it note from eq(2.86),with the aid of eq(2.82) that the reduced time is taken for fixed(x_1,x_2,x_3),is a monotone increasing function of t .Hence f(x,t) in eq(2.86) may be inverted with respect to t ,so that

$$t=g(x,x) \quad , \quad t^{/}=g(x,x^{/}) \qquad 2.89$$

Also, by eqs (2.86) and (2.89).

$$\frac{\partial x}{\partial t}=j[T(x,t)] \quad , \frac{\partial t}{\partial x}=\left(\frac{\partial x}{\partial t}\right) \qquad 2.90$$

Suppose F (x, t) is any function of position and time. Then to avoid ambiguity, we shall consistently adopt the notation

$$F(x,t)=F^{/}(x,x)=F[x,g(x,x)] \qquad 2.91$$

It should be emphasized that F and $F^{/}$ distinct functions unless ξ=t, i.e., unless φ(T)=1 ,moreover, $F^{/}$ needs to be distinguished from F(x,t),the values of which are obtained by replacing the argument t in F(x,t) with ξ ,rather than by subjecting t to the first of the transformations eq(2.89).Using this equation .Eqs(2.85),(2.88) may be changed to the following forms.

$$S^{/}_{ij}(x,x) = \int_{-a}^{t} G_1(x - x^{/}) \frac{\partial}{\partial x^{/}} e^{/}_{ij}(x,x)dx^{/}$$
$$s^{/}(x,x) = \int_{-a}^{t} G_2(x - x^{/}) \frac{\partial}{\partial x^{/}} [e^{/}_{ij}(x,x) - 3aq^{/}(x,x^{/})]dx^{/} \qquad 2.92$$

And

$$e^{/}_{ij}(x,x) = \int_{-a}^{t} J_1(x - x^{/}) \frac{\partial}{\partial x^{/}} S^{/}_{ij}(x,x^{/})dx^{/} \quad (x,x^{/})$$
$$e^{/}(x,x) = \int_{-a}^{t} J_2(x - x^{/}) \frac{\partial}{\partial x^{/}} s^{/}_{ij}(x,x^{/})dx^{/} + 3a_0 q^{/}(x,x) \qquad 2.93$$

The extent on of the constitutive law eq (2.30) to non-isothermal conditions (for a thermorheologically simple mat[illegible] generalization may be reached economicall[illegible] of eq (2.92) with respect to ξ and by subsequent use of eq (2.54).On applying the corresponding inverse transforms, one thus the modified differential operator law may be written as:

$$P_1(D^{/})S^{/}_{ij}(x,x) = Q_1(D^{/})e^{/}_{ij}(x,x)$$
$$P_2(D^{/})s^{/}_{ij}(x,x) = Q_2(D^{/})[e^{/}_{ij}(x,x) - 3a_0 q^{/}(x,x)] \qquad 2.94$$

In which

$$D^{/} = \frac{\partial}{\partial x} \qquad 2.95$$

And the polynomial operators P_i,Q_i ,as before, are given by eq(2.30),when(x,t)=$T_{0.}$

Eq (2.94) may be reduced to eq (2.30). If the medium is elastic, the differential operators have the values p_0=1,q_1=2μ in shear and p_0=1 ,q_1=3K in extension, fail to evolve the reduced time, whence the response is independent of the temperature in this special instance. Elastic materials with temperature dependent characteristics thus don't belong to the class of thermorheologically simple viscoelastic solid, i.e. the assumption that the "elastic constants" vary with temperature is inconsistent with the temperature - time equivalence hypothesis.

The temperature - time equivalence hypothesis implies that the temperature dependence of the response of the material is governed by a single function of the temperature, namely by the shift – factor φ (T).

For Maxwell and Kelvin models and by using the differential operators of these models and accordance with eqs (2.6) and (2.14), and using eqs (2.89), (2.90) and (2.91), one obtains for a thermorheologically simple Maxwell and Kelvin solid.

$$\frac{\partial}{\partial t} S_{ij}(x,t) + \frac{1}{\lambda(T)} S_{ij}(x,t) = 2\mu \frac{\partial}{\partial t} e_{ij}(x,t$$

$$S_{ij}(x,t) = 2\mu\lambda(T) \frac{\partial}{\partial t} e_{ij}(x,t) + e_{ij}(x,$$

Where

$$\lambda(T) = \frac{\lambda_0}{\varphi(T)} \qquad 2.97$$

Comparing eq (2.96) with eqs (2.6) and (2.14), we identify the function λ (T) in eqs (2.96) as a temperature – dependent relaxation time and retardation time respectively may be identified.

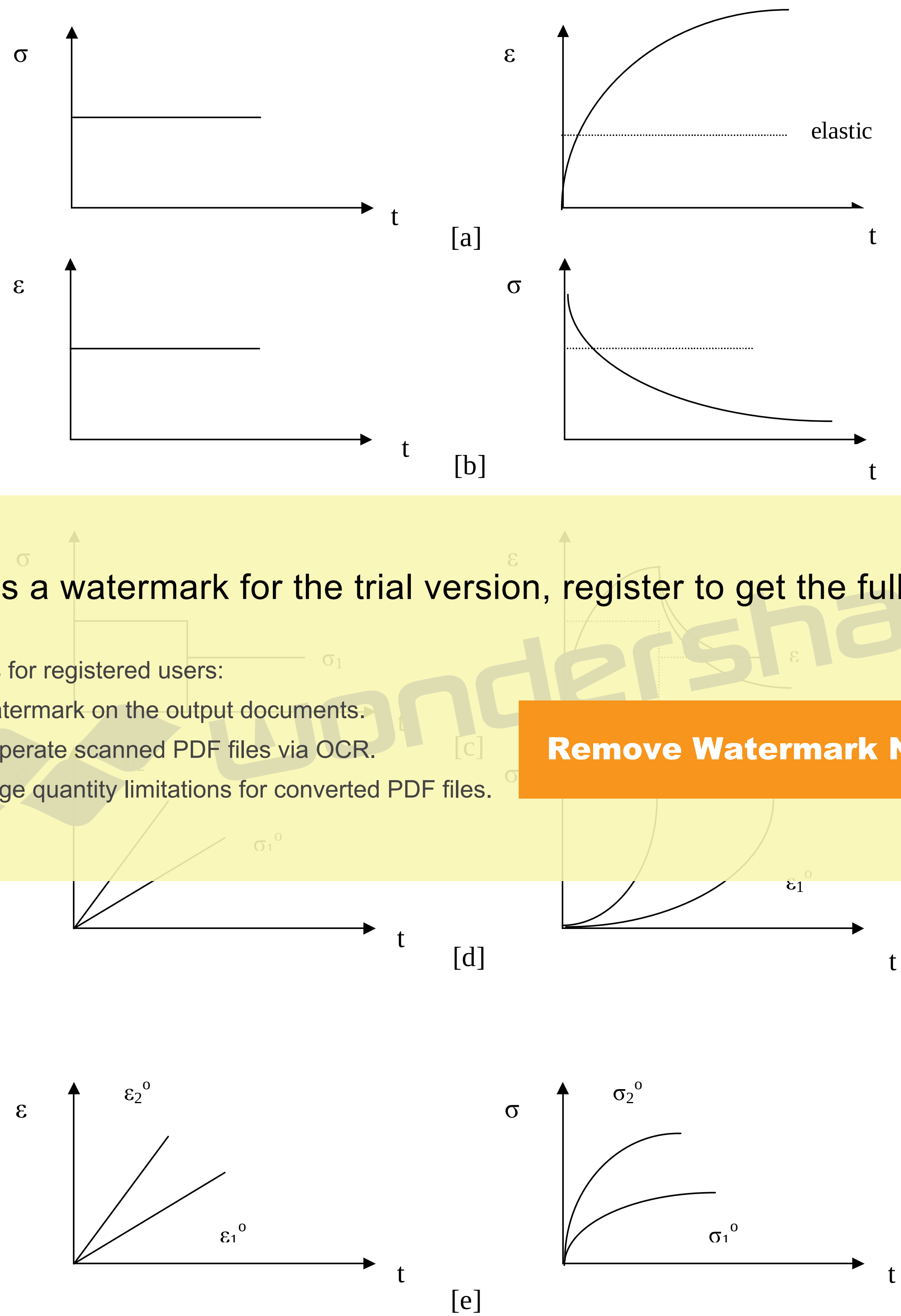


***Figure 2.1**: Stress-strain behavior a-Creep b-Relaxation c-Recovery d-Constant rate stressing e-Constant rate straining*

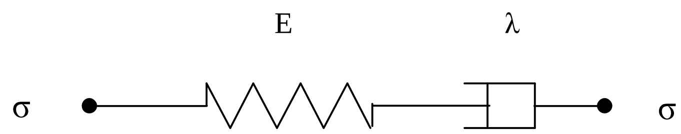


Figure 2.2: *Maxwell Model*

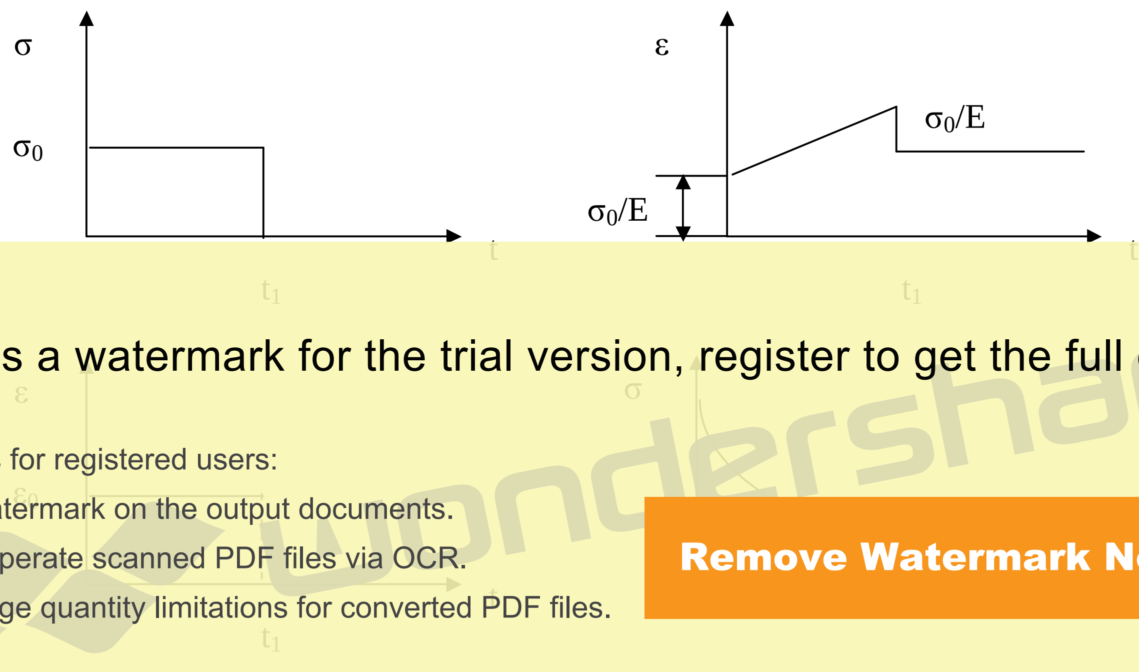


Figure 2.3: *Creeps, Recovery and Relaxation Behavior of Maxwell Model*

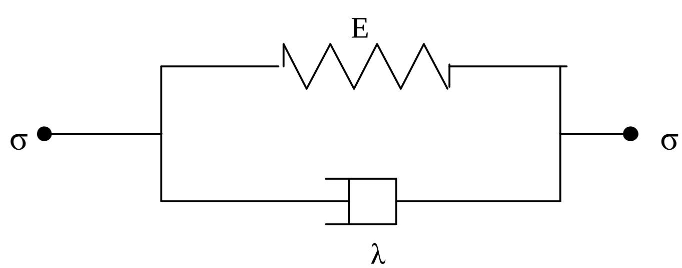


Figure 2.4: *Kelvin Model*

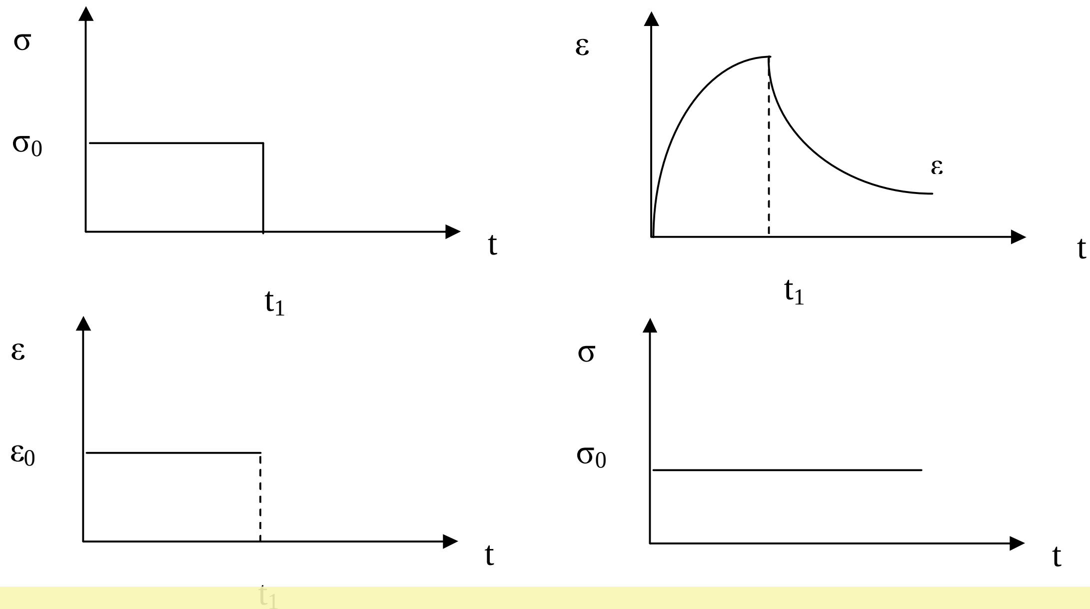


Figure 2.5: Creeps, Recovery and Relaxation Behavior of Kelvin Model

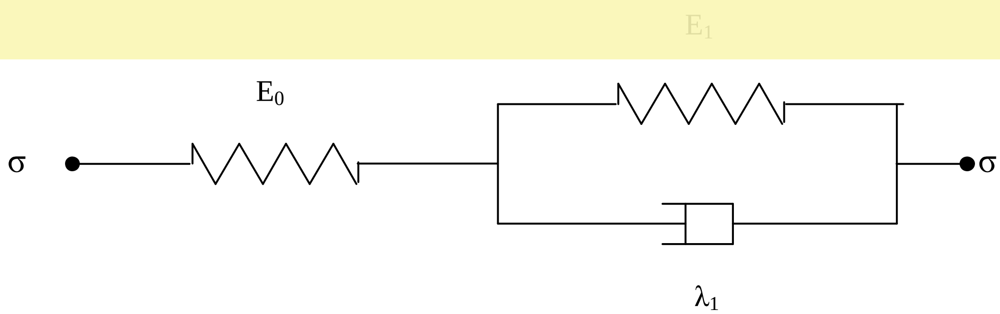


Figure 2.6:*Standard Linear Solid Model*

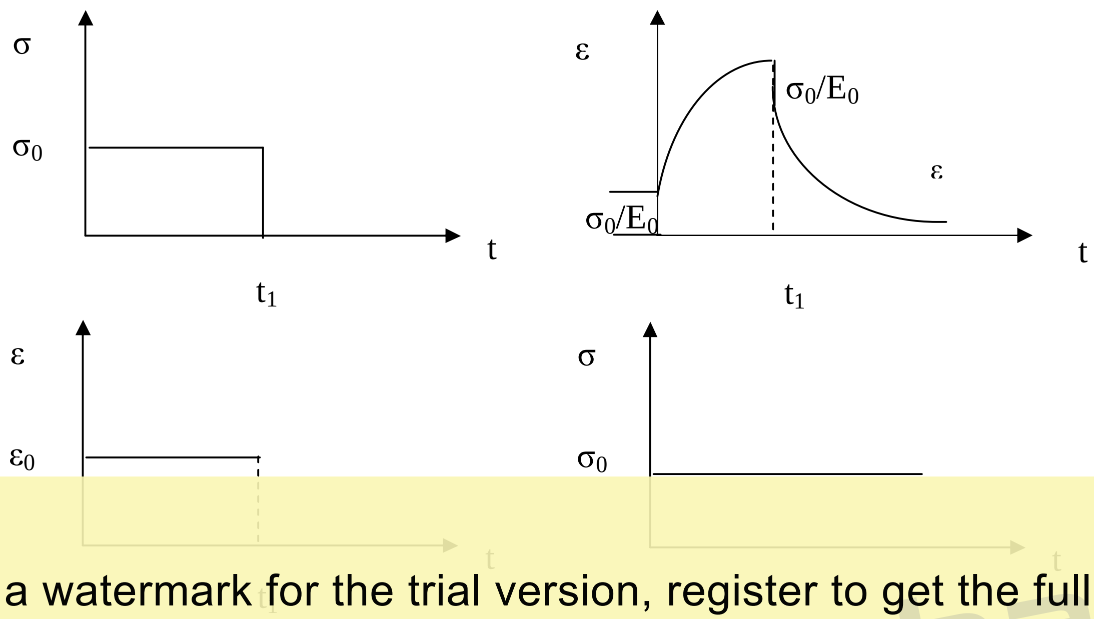


Figure 2.7: Creeps, Recovery and Relaxation Behavior of Standard Linear Solid Model

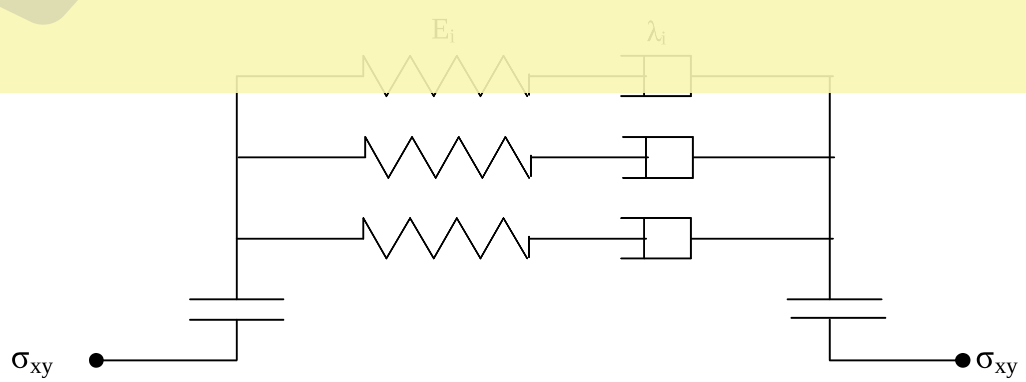


Figure 2.8: *Generalized Maxwell model*

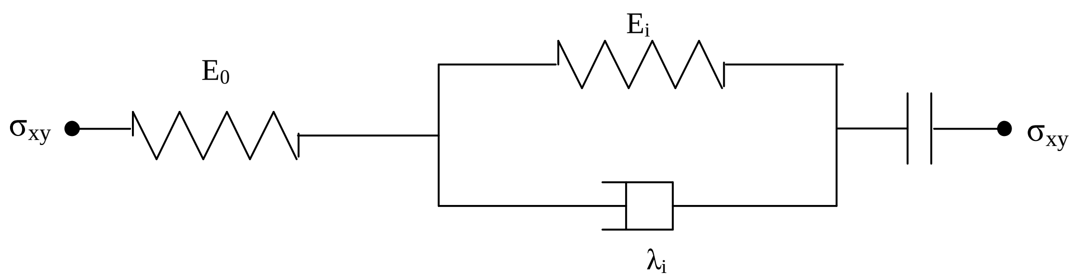


Figure 2.9: *Generalized Kelvin model*

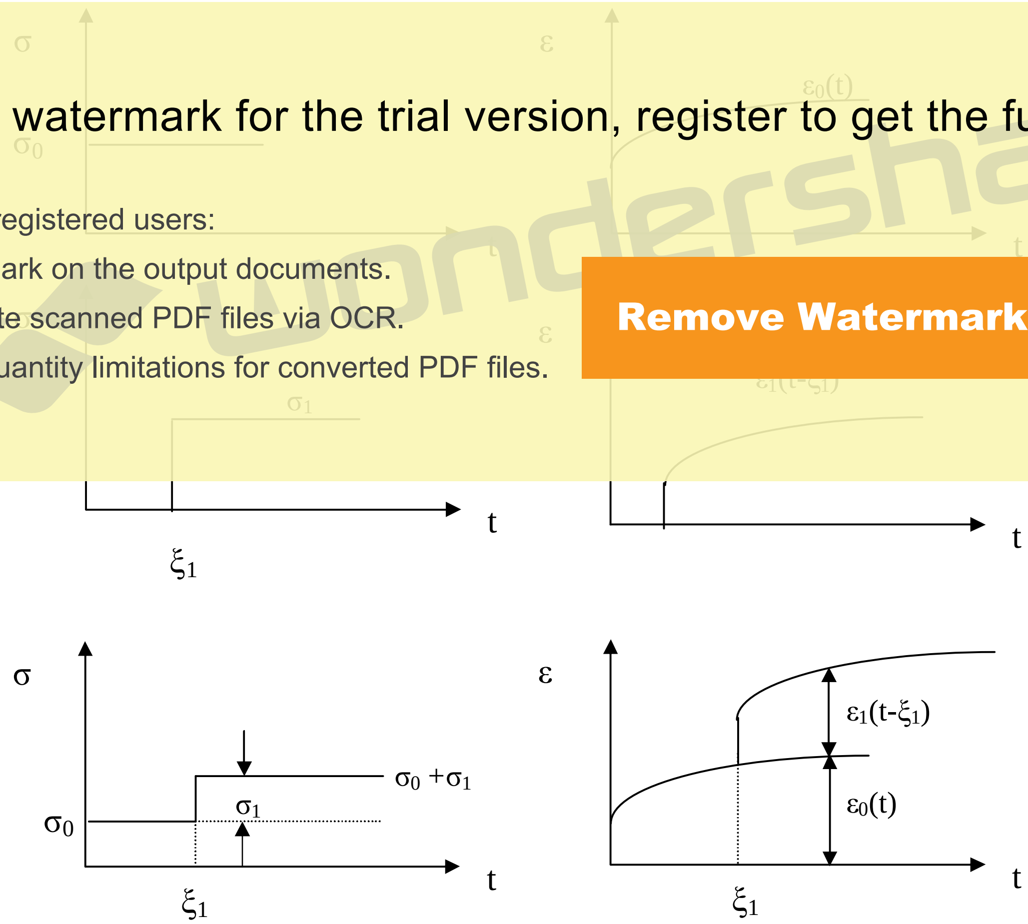


Figure 2.10: *Boltzman Superposition Principle*

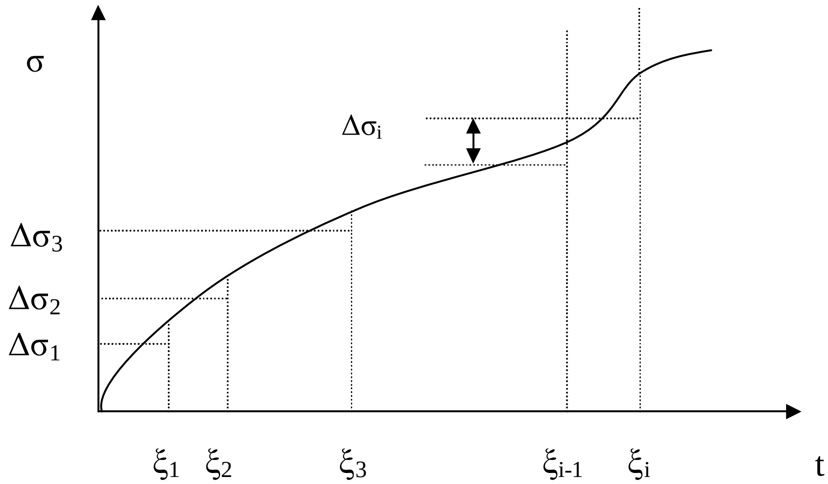


Figure 2.11: *Variable Stress Input Approximated By the Sum of a Series of Constant Stress Input*

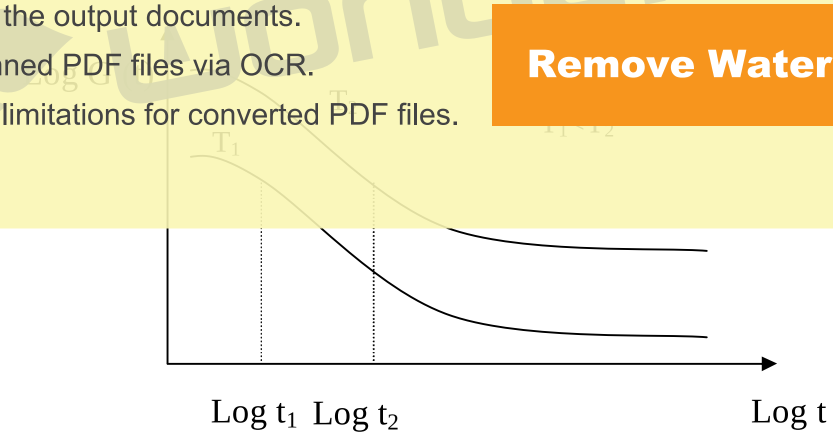


Figure 2.12: *Shifting of Shear Relaxation by Cooling or Heating*

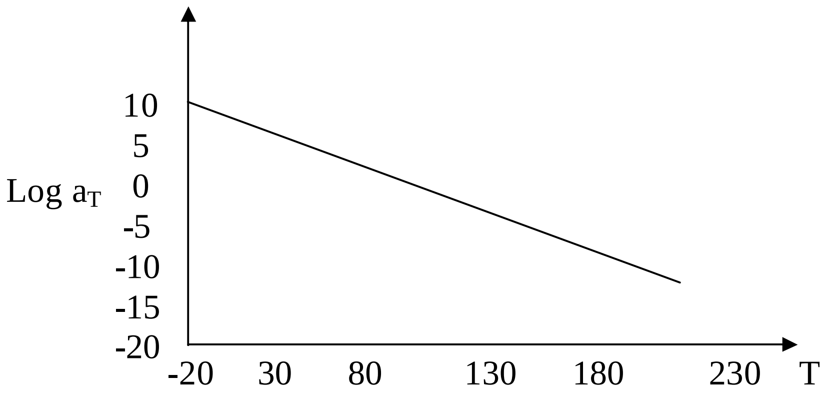


Figure 2.13: *Time Temperature Superposition Function Vs temperature*

Chapter Three
Finite Element Formulation for Thermoviscoelastic Analysis

3.1 Introduction

The Finite Element Method "FEM" has become a powerful tool for the numerical solution of a wide range of engineering problems. In this method of analysis a complex region-defining a continuum is disecritized into simple geometric shape called elements, which are connected, at a finite number of points known as nodal points. The material properties and the governing relationships are considered over these elements and expressed in terms of unknown values at element corners. An assembly processed, duly considering the loading and constraints, results in a set of equations. Solution of these equations gives the approximate behavior of the continuum.

Analytical solution [illegible] preferred to [illegible] governed by differential or integral equation[illegible] the analytical solution is difficult to obtain, especially when the region under consideration is irregular and mathematically impossible to describe the boundary of the problem domain or the material is represented by a series of Maxwell (GMM) to represent as a Prony series, which is more difficult to solve analytically.

In this chapter the Finite Element technique, which has been demonstrated to provide an excellent analysis tool of problems with a complex geometrical configuration subjected to gravitational loading, has been extended to provide analysis for a linear viscoelastic solids.

3.2 Thermoviscoelastic Stress-Strain Relations

The Thermoviscoelastic stress-strain relation may be obtained by using the elastic stress strain relation. The general thermoelastic stress-strain relation can be written as [32]: -

$$s_{ij}(x)=2Ge_{ij}(x)+d_{ij}\left(k-\frac{2}{3}G\right)e_{kk}(x)-3d_{ij}a_oK\Delta T(x) \qquad (3.1)$$

The elastic viscoelastic corresponding principle can be applied to deduce the following stress component in Laplace domain [33].as illustrated in appendix (B).

$$s_{ij}(x,s)=2sG(s)e_{ij}(x,s)+d_{ij}\left(K(s)-\frac{2}{3}G(s)\right)se_{kk}(x,s)- \qquad (3.2)$$
$$3d_{ij}a_osK(s)\Delta T(x,s)$$

Inverse Laplace transformation (convolution integral) can be employed to obtain the stress variation with the time domain:

$$[illegible](x,t)=2\int_{0}^{t}G(z-z')\frac{\partial e_{ij}(x,t')}{\partial t'}d[illegible]$$
$$[illegible]\frac{2}{3}d_{ij}\int_{0}^{t}G(z-z')\frac{\partial e_{kk}(x,t')}{\partial t'}dt'-3d_{ij}Ka_o\Delta T(x,t)$$

The discontinuity at t=0, may be eliminated from the above expression to obtain the following expression:

$$s_{ij}(x,t)=2G(z)e_{ij}(x,0)+\int_{0^+}^{t}G(z-z')\frac{\partial e_{ij}(x,t')}{\partial t'}+$$
$$d_{ij}\left[K-\frac{2}{3}G(z-z')\right]e_{kk}(x,0)-\frac{2}{3}d_{ij}\int_{0^+}^{t}G(z-z')\frac{\partial e_{kk}(x,t')}{\partial t'}dt' \qquad (3.4)$$
$$-3d_{ij}Ka_o\Delta T(x,t)$$

Integration by parts can be applied to the second and fourth terms to simplify the above equation into:

$$s_{ij}(t) = 2G(0)e_{ij}(x,t) - \int_{0^+}^{t} \frac{\partial G(z - z')}{\partial t'} e_{ij}(x,t')dt' +$$

$$d_{ij}\left[K - \frac{2}{3}G(0)\right]e_{kk}(x,t) - \frac{2}{3}d_{ij}\int_{0^+}^{t}\frac{\partial G(z - z')}{\partial t'}e_{kk}(x,t')dt' \tag{3.5}$$

$$-3d_{ij}Ka_o\Delta T(x,t')$$

Equation (3.5) can be written in matrix form as shown below where the spatial variables x in the arguments are suppressed for simplicity:

$$|s(t)| = [D_0]e(t) + [D_1]£\{e(t)\} - 3aK(\Delta T(t))\{I\} \tag{3.6}$$

Where

$$£\{e(t)\} = \int_{0^+}^{t}\frac{\partial G(z - z')}{\partial t'}\{e(t')\}dt' \tag{3.7}$$

Where for

1. plane strain problems

$$[D_0] = \begin{bmatrix} K + \frac{4}{3}G(0) & K - \frac{2}{3}G(0) & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots \end{bmatrix}$$

$$|D_1| = \begin{bmatrix} -\frac{4}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & -\frac{4}{3} & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{3.9}$$

$$\{I\}^T = \{1 \quad 1 \quad 0\} \tag{3.10}$$

2. Axisymmetric problems

$$\boldsymbol{D}_0=\begin{bmatrix} \boldsymbol{K}+\frac{4}{3}\boldsymbol{G}(0) & \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & 0 \\ \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & \boldsymbol{K}+\frac{4}{3}\boldsymbol{G}(0) & \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & 0 \\ \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & \boldsymbol{K}+\frac{4}{3}\boldsymbol{G}(0) & 0 \\ 0 & 0 & 0 & \boldsymbol{G}(0) \end{bmatrix} \tag{3.11}$$

$$[\boldsymbol{D}_1]=\begin{bmatrix} -\frac{4}{3} & \frac{2}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & -\frac{4}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{2}{3} & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.12}$$

$\{I\}=\{1$ [illegible] (3.13)

3. Three-dimensional problems

$$[D_0]=\begin{bmatrix} K+\frac{4}{3}G(0) & K-\frac{2}{3}G(0) & K-\frac{2}{3}G(0) & 0 & 0 & 0 \\ K-\frac{2}{3}G(0) & K+\frac{4}{3}G(0) & K-\frac{2}{3}G(0) & 0 & 0 & 0 \\ K-\frac{2}{3}G(0) & K-\frac{2}{3}G(0) & K+\frac{4}{3}G(0) & 0 & 0 & 0 \\ 0 & 0 & 0 & G(0) & 0 & 0 \\ 0 & 0 & 0 & 0 & G(0) & 0 \\ 0 & 0 & 0 & 0 & 0 & G(0) \end{bmatrix} \tag{3.14}$$

$$[\boldsymbol{D}_1]=\begin{bmatrix} -\frac{4}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{2}{3} & -\frac{4}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & -\frac{4}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.15}$$

$$\{1\}^T = \{1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0\} \tag{3.16}$$

3.3 Reduce time

The shifted time z in eq. (3.5) is related to the real time t, throw the relation [22].

$$z = z(t) = \int_{0^+}^{t} \frac{dt'}{A_T(T(t'))} \tag{3.17}$$

where A_T is the shift function evaluated by

as

$$\log a_T(T) = \frac{C_1(T - T_R)}{C_2 + (T - T_R)} = -h(T) \tag{3.18}$$

$$\frac{1}{A_T[T(t)]} = 10^{h(T)} \tag{3.19}$$

For the cases of transient temperature loading the shifted time at t_k, z_k can be calculated numerically suing trapezoidal rule from eq. (3.17) to give the following [23]

$$z_k = z_{k-1} + \int_{t_{k-1}}^{t_k} \frac{dt'}{A_T(T(t_{k-1}, t_k))} \tag{3.20}$$

While for the case of steady state temperature loading, it can be shown that [23]:

$$z_k = t * 10^{h(T)} \tag{3.21}$$

3.4 Finite Element Formulation

A detail of the Finite Element Method "displacement method" is presented by many workers [34-37]. The displacement and coordinate geometry at any point inside an isoparamertic element can be related to the nodal displacement and coordinates using the shape function as follow [38]:

$$\{X\}=\{N\}^T\{X_i\} \tag{3.22}$$

$$\{d\}=\{N\}^T\{d_i\} \tag{3.23}$$

The strain-displacement relation may be written as:

$$\{e\}=[B]\{d_i\} \tag{3.24}$$

The principle of minim total potential energy is used to drive the Finite element equations. The minimum total potential energy principle states that [34]: "of all possible displacements that satisfy the given boundary conditions, those that satisfy the equilibrium equations, make the potential energy assume a stationary value. For stable equilibrium, the potential energy is minimum." The total potential energy X is defined as:

$$X = U - W \tag{3.25}$$

Where

$$W=\{d\}^T\{F\} \tag{3.26.a}$$

$$U=\frac{1}{2}\int_v \{d\}^T[B]^T\{s\}dv \tag{3.26.b}$$

But

$$\{e\}^T=\{d\}^T[B]^T \tag{3.27}$$

Therefore

$$X=\frac{1}{2}\int_v \{d\}^T[B]^T\{s\}dv-\{d\}^T\{F\} \tag{3.28}$$

Applying the principle of minimum total potential energy, it can be shown that [34]:

$$\frac{dX}{dd}=\int_v [B]^T[s]dv-\{F\}=0 \tag{3.29}$$

Or

$$\int_v [B]^T [s] dv = \{F\} \qquad (3.30)$$

By substituting eq. (3.6) into eq. (3.30), it can be proved that:

$$\int_v [B]^T [D_0][B]\{d\} dv + \int_v [B]^T [D_1]\{e(t)\} dv - \{F_m(t)\} = \{F_T(t)\} \qquad (3.31)$$

Where:

$$\{F_T(t)\} = 3a_0 K \Delta T(t) \int_v [B]^T [I] dv \qquad (3.32)$$

$\{\boldsymbol{F}_T(\boldsymbol{t})\}$ is the thermal load vector. $\{\boldsymbol{F}_m(\boldsymbol{t})\}$ is the mechanical load vector due to nodal force and / or surface traction and/ or centrifugal force and / or gravity load.[39]

More details of numerical schemes of mesh discretization, numerical integration, assembly and solution of system of equations can be found in references [34-37].

3.4.1 Time Marching Scheme:-

In order to obtain the response at time t_k, the time axis is discretized into k_{th} equal time intervals, i.e.

$$t_k = \sum_{n=0}^{k} n.\Delta t \qquad (3.33)$$

And in order to solve eq.(3.31), one has to approximate the time variation of the field quantities in addition to the usual approximation of the spatial variation. For this purpose a linear interpolation function is used which is described with the resulting time stepping algorithms.

The field variables (i.e. displacements) are assumed to vary linearly during a time step. Employing the trapezoidal rule for time domain, eq. (3.7) can be written for k_{th} time step as:

$$\pounds\{e(t)\} = \frac{1}{2}(G(0) - G(z_k - z_{k-1}))[B]\{d(t_k)\} +$$
$$\frac{1}{2}(G(0) - G(z_k - z_{k-1}))[B]\{d(t_{k-1})\} + \qquad (3.34.a)$$
$$\sum_{j=1}^{k-2}\left(G\left(z_k - z_{j+1}\right) - G\left(z_k - z_j\right)\right)[B]\left\{d*\left(t_{j+1}\right)\right\}$$

And by employing eq.(3.34) and the second term of eq. (3.31) one can get the following relation:

$$\int_v [B]^T [D_1].\{e(t)\}dv = \frac{1}{2}(G(0) - G(z_k - z_{k-1}))[k_1]\{d(t_k)\} +$$
$$\frac{1}{2}(G(0) - G(z_k - z_{k-1}))[k_1]\{d(t_{k-1})\} + \qquad (3.34.b)$$
$$\sum_{j=1}^{k-2}\left(G\left(z_k - z_{j+1}\right) - G\left(z_k - z_j\right)\right)[K_1]\left\{d*\left(t_{j+1}\right)\right\}$$

Where

$$\left\{d*\left(t_{j+1}\right)\right\} = \frac{1}{2}\left(\left\{d\left(t_j\right)\right\} + \left\{d\left(t_{j+1}\right)\right\}\right) \qquad (3.35)$$

$$[K_1] = \int_v \left[B^T\right][D_1][B]dv \quad [illegible]$$

The above eq. (3.34.a) has been performi[illegible] integration as shown in the appendix (B). Substituting eq. (3.34.a) the elemental equilibrium equation can be deduced as:

$$\left[[K_0] + \frac{1}{2}(G(0) - G(z_k - z_{k-1}))[k_1]\right]\{q(t_k)\} = \qquad (3.37)$$
$$\{F_m(t_k)\} + \{F_T(t_k)\} + \{M(t_k)\}$$

Where

$$[K_0] = \int_v \left[B^T\right][D_0][B]dv \qquad (3.38)$$
$$\{M(t_k)\} = [K_1]\{\Phi(t_k)\} \qquad (3.39.a)$$

And

$$\{\Phi(t_k)\} = \frac{1}{2}\left(G(0) - G(z_k - z_{k-1})\right)\{d(t_{k-1})\} + \sum_{j=1}^{k-2}\left(G(z_k - z_{j+1}) - G(z_k - z_j)\right)\{d^*(t_{j+1})\} \quad (3.39.b)$$

3.4.2 Solution Procedure:-

For the first time step, which will give the elastic solution at time t=0, the equilibrium equation i.e. eq.(3.37) can be reduced into:

$$[K_0]\{d(0)\} = \{F_m(0)\} + \{F_T(0)\} \quad (5.40)$$

For the second, third etc. time steps equilibrium equation will written as eq. (3.37), where $[k_0]$ and $[k_1]$ are constant matrices all that necessary at every time step is just to update $\{m(t_k)\}$ "memory load vector", and update the values of the load vector $\{F_m(t)\}, \{F_T(t)\}$ for transient loading and then solve the algebraic eq. (3.37).

The output results from the solution procedure will be the nodal displacement vector using eq.(3.40) for the elastic solution and eq.(3.37) for k_{th} time steps. The strains are obtained by using eq. (3.24) [illegible] from eq. (3.5) as

$$\{s(t_k)\} = [A_1]\{e(t_k)\} + \{A_2\} - 3a_0 K\Delta T(t_k)\{I\} \quad (3.41)$$

Where

$$[A_1] = [D_0] + \frac{1}{2}\left(G(0) - G(z_k - z_{k-1})\right)[D_1] \quad (3.42)$$

$$\{A_2\} = [D_1][B]\{\Phi(t_k)\} \quad (3.43)$$

3.4.3 Incompressibility Consideration

Most viscoelastic materials are assumed to be incompressible or nearly incompressible solids (i.e. Poisson's ratio approaching to one-half). Application of the usual finite element method (displacement method) for the analysis of such solids yields severally oscillating in the stress and strain across the elements. This aspect has been studied for elastic materials and is well document in literature [18]. This oscillation may be overcome by using the following steps:

1. Using selective integration procedure, which is exact (3×3) Gauss integration points for the shear component and approximate (2×2) Gauss integration for the bulk components of the elastic stiffness matrix[37].
2. Using 8-nodes serendipity isoparametric elements for plane strain and axisymmetric solid and 20-node for three-dimensional solid quadratic isoparametric element [23] as shown in Fig.3.1.
3. The location of stress and strain output [18], i.e. the sampling position, within the element can be selected at the (2×2) gauss points, which are favored and give a best results of stresses and strains. Whiles the results at the geometrical nodes or (3×3) gauss points are given poor and unreasonable result.

According to first assumption the elastic sti[illegible] be split into shear and bulk components [37] as shown below

$$[K_0] = [k_0^v] + [K_0^s] \tag{3.44}$$

Where

$$[K_0^v] = \int_v [B]^T [D_0^v][B] dv \tag{3.45}$$

$$[K_0^s] = \int_v [B]^T [D_0^s][B] dv \tag{3.46}$$

Where

1. plane strain problems

$$[D_0]=\begin{bmatrix} 2G(0) & 0 & 0 \\ 0 & 2G(0) & 0 \\ 0 & 0 & 2G(0) \end{bmatrix}+\begin{bmatrix} K-\frac{2}{3}G(0) & K-\frac{2}{3}G(0) & 0 \\ K-\frac{2}{3}G(0) & K-\frac{2}{3}G(0) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (3.47)$$

Where

$$[\boldsymbol{D}_0^s]=\begin{bmatrix} 2\boldsymbol{G}(0) & 0 & 0 \\ 0 & 2\boldsymbol{G}(0) & 0 \\ 0 & 0 & 2\boldsymbol{G}(0) \end{bmatrix} \qquad (3.48)$$

$$[\boldsymbol{D}_0^v]=\begin{bmatrix} \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & 0 \\ \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (3.49)$$

2. Axisymmetric problems

$$[\boldsymbol{D}_0^s]=\begin{bmatrix} 2\boldsymbol{G}(0) & 0 & 0 & 0 \\ 0 & 2\boldsymbol{G}(0) & 0 & 0 \\ 0 & 0 & 2\boldsymbol{G}(0) & 0 \\ 0 & 0 & 0 & 2\boldsymbol{G}(0) \end{bmatrix} \qquad (3.50)$$

$$[\boldsymbol{D}_0^v]=\begin{bmatrix} \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & 0 \\ \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & 0 \\ \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & \boldsymbol{K}-\frac{2}{3}\boldsymbol{G}(0) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (3.51)$$

3. three dimensional problems

$$[D_0^s] = \begin{bmatrix} 2G(0) & 0 & 0 & 0 & 0 & 0 \\ 0 & 2G(0) & 0 & 0 & 0 & 0 \\ 0 & 0 & 2G(0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G(0) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0\,2G(0) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G(0) \end{bmatrix} \qquad (3.52)$$

$$[D_0^v] = \begin{bmatrix} K - \frac{2}{3}G(0) & K - \frac{2}{3}G(0) & K - \frac{2}{3}G(0) & 0 & 0 & 0 \\ K - \frac{2}{3}G(0) & K - \frac{2}{3}G(0) & K - \frac{2}{3}G(0) & 0 & 0 & 0 \\ K - \frac{2}{3}G(0) & K - \frac{2}{3}G(0) & K - \frac{2}{3}G(0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (3.53)$$

The integration of $[k_0^s]$ can be performed [illegible] (2×2) G[illegible] Legender formula, while the $[k_0^v]$ can be e[illegible] Gauss - Legender formula to overcome the singularity due to the value of Poisson's ratio, when it reaches to 0.5. The above (2×2) gauss point is called the reduced or selective integration technique and is recommended only when the Poisson's ratio "v" reaches to 0.5, for other values it found that (3×3) gives more accurate results.

3.4.4 Local Smoothing of Stresses and Strains:

The geometrical nodes of the finite element mesh, which are the most useful out put locations for stresses and strains, appear to be the worst sample points for incompressible (or nearly incompressible) materials. It has been shown that the integration points "(2×2) gauss points" are the best stresses and strains sample points but the stresses and strain will be as discontinuous between the elements, to solve this problem one can use local smoothing technique [38] as shown in Fig. 3.2. First the smoothing may be performed separately over each individual element and this will be called local smoothing, and then taken the average of stresses and strains of the nodal of all elements meeting at common node. The smoothing function of 2-D problems "plane strain and axisymmetric problems" is shown below:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{bmatrix} = \begin{bmatrix} 1+\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1-\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1+\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1-\frac{\sqrt{3}}{2} \\ 1-\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1+\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1-\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1+\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \sigma_I \\ \sigma_{II} \\ \sigma_{III} \\ \sigma_{IV} \end{bmatrix} \quad (3.54)$$

3.5 Computer Programming:

The software "FEVES" is developed for linear viscoelastic analysis can effectively employed for all permissible value of Poisson's ratio. The software can be effectively used for the analysis of compressible structures throw the use of a control parameter, which changes the computational flow from the selective integration "(2×2) gauss point integration" to a third order "(3×3) gauss rule for the computation of both the deviatoric and volumetric components of the stiffness matrix. The function of each subroutine Fig. 3.3 is described as follows:

1. DATA: - the main job of this subroutine is to enter geometrical and control data: No. of elements & nodes, boundary condition, and the material properties represented in term of Prony series.

2. LOAD: - the job of this subroutine is to determine the type of loading "point load, surface truncations, thermal load, gravity load, and centrifugal force".
3. ASSEO: - this subroutine is responsible for evaluating the global elastic stiffness matrix $[\boldsymbol{K}_0]$which is written into tow (shear and bulk) components [34] as shown in eq. (3.45) and (3.46) respectively.
4. ASSEI: - this subroutine is responsible for evaluating the global viscoelastic stiffness matrix$[\boldsymbol{K}_1]$.
5. ASSET :- this subroutine is responsible for evaluating the total elastic and viscoelastic stiffness matrix as below

$$\left[[K_0]+\frac{1}{2}(G(0)-G(z_k-z_{k-1}))[k_1]\right]$$

6. MMAT: - the memory load vector eq. (3.39.a) is evaluating in this subroutine.
7. REDUCER: - the job of this subro[illegible] boundary conditions, to reduce the [illegible] stiffness matrix.
8. SOLVER: - in this subroutine the gauss-elimination method is employed to solve the reduced system of equation.
9. DISPL: - the job of this subroutine is formatted the displacement and printed it.
10. STRESS: - the purpose of this subroutine is to evaluate the nodal stresses eq. (3.41) of compressible material and at the (2×2) gauss points for the incompressible and nearly incompressible material and then using smoothing technique to extrapolate the stresses from (2×2) gauss point to the corner nodal points.

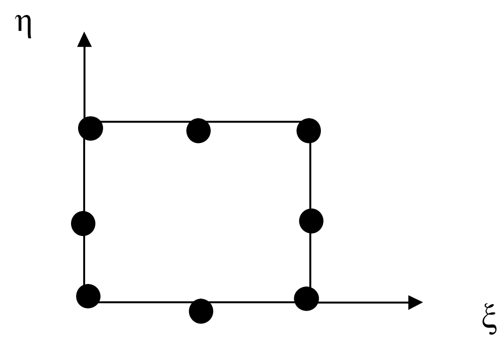
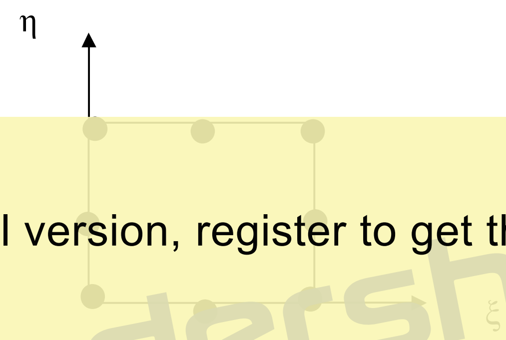
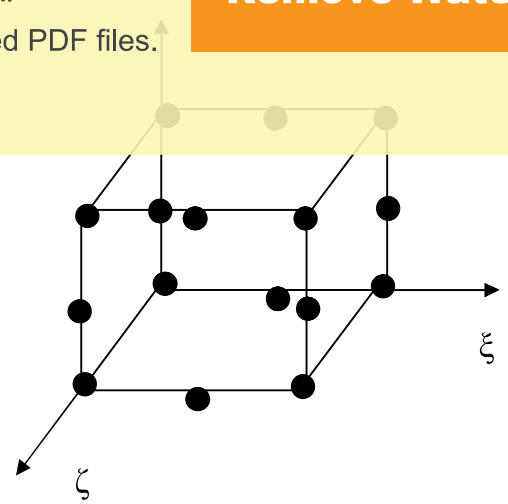
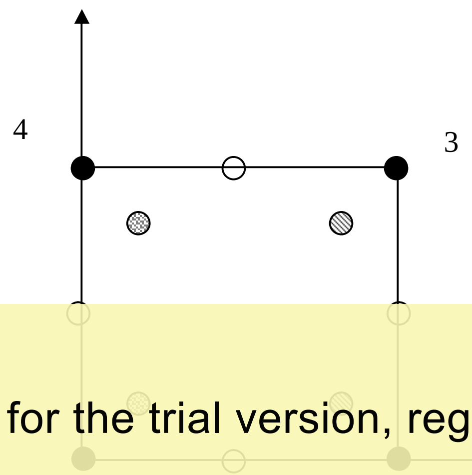
Element type	Intrinsic coordinate
Plane strain element	
Axisymmytric Element	
3-d Solid Element	

Figure3.1: *8-nodes serendipity isoparametric elements*



○ Inside node

◉ 2 X 2 Gauss integration

Figure 3.2: *Locations of 2X2 gauss points*

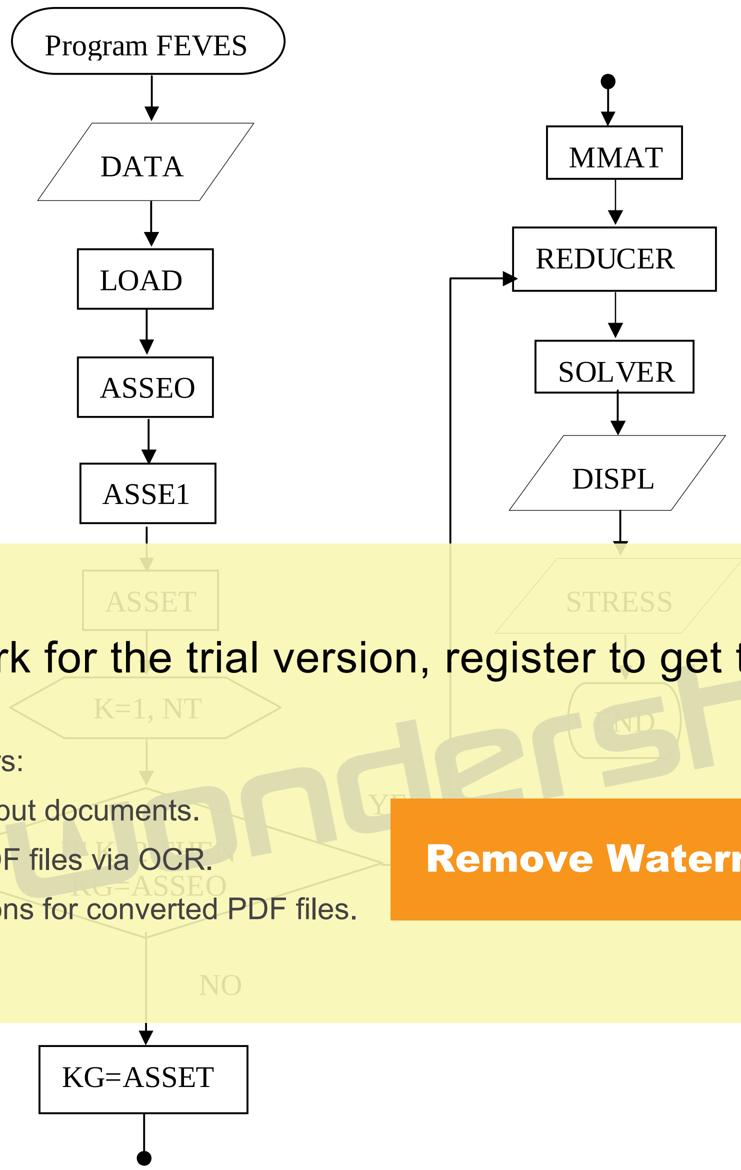


Figure 3.3: *Block diagram of FEVES code*

Chapter Four

Experimental Work

4.1 Introduction

In this chapter, experimental tests have been carried out with two-fold aim in mind. One of these tests has focused on evaluating the instantaneous response of a viscoelastic specimen to an applied angle of twist. On the other hand, the second test was designed to assess the time dependent behavior of such a speicemen.the measure that was used for such an assessment is the shear relaxation modulus that was computed as an outcome for the experiment. Both of the tests were conducted on three different types of polymeric material. The testing was made under 37^0C temperature.

4.2 The apparatus

As shown in Figs. 4.1a and 4.1 b. The apparatus consists mainly of:

- Loading device (1) with scale and re[illegible] measurement.
- Torque measurement unit (2) with digital torque meter (6).
- Calibration device (3).

The specimen (4) is mounted between the loading device (1) and the Torque measurement unit (2) into hexagon sockets. All components are mounted on a track base (5).

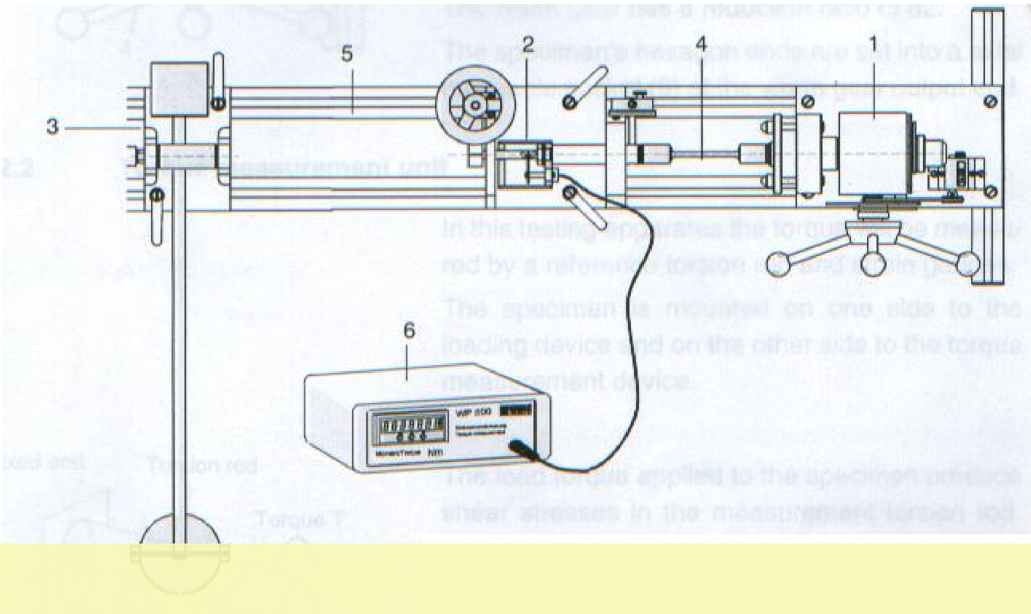


Figure 4.1b: *The torsion test apparatus*

4.2.1 Loading device

The torsional loading is transmitted to the specimen by a worm gear (1) and a hand wheel (4). The twisting angle at the output and the input read off by two 360^0 scales (2,3). At the input side of the gear there is in addition a five digit revolution counter (5) which shows the input revolutions 1:1.the worm gear has a reduction ratio of 62.The specimen's hexagon ends are set into a axial moveable socket (6) at the worm gear output end. As shown in Fig.

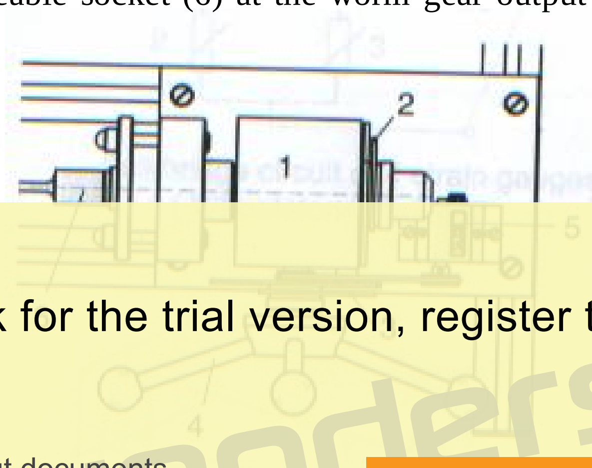


Figure 4.2: loading de

4.2.2 Torque measurement unit

In this testing apparatuses the torque will be measured by a reference torsion rod and strain gauges. The specimen is mounted on one side to the loading device and one on the other side to the torque measurement device. The load torque applied to the specimen produces shear stresses in the measurement torsion rod. These shear stresses are proportional to the load torque. Strain gauges are used for detecting the shear stresses. Because the strain gauges can only measure strain but not twisting they must be applied in the direction of the maximum principal stresses. This case of pure torsion the maximum of principal stress will occur at an angle of 45^0 to the axis of the torsion rod.

Due to the arrangement of four strain gauges in form of a full bridge circuit the distortion influences of additional bending and direct stresses is minimized. The signal of the gauges is conditioned by a measuring amplifier with a digitally read out. The amplifier also delivers the supply voltage for the bridge circuit.

A lever and a threaded spindle at the fixed side of the torsion rod can compensate the deformation. A dial gauge at the side of the specimen holder can control the compensating.

4.3 The specimen

As is illustrated in Fig. 4.3a and 4.3b, the set of dimensions defining any *specimen* under test are five numbers. These are the gauge length of the specimen L_G and the tentative length of the specimen L, radius of the specimen R, the radius of fillet r and the diameter of the holder D. for each studied material, two specimens were tested with two different radii.

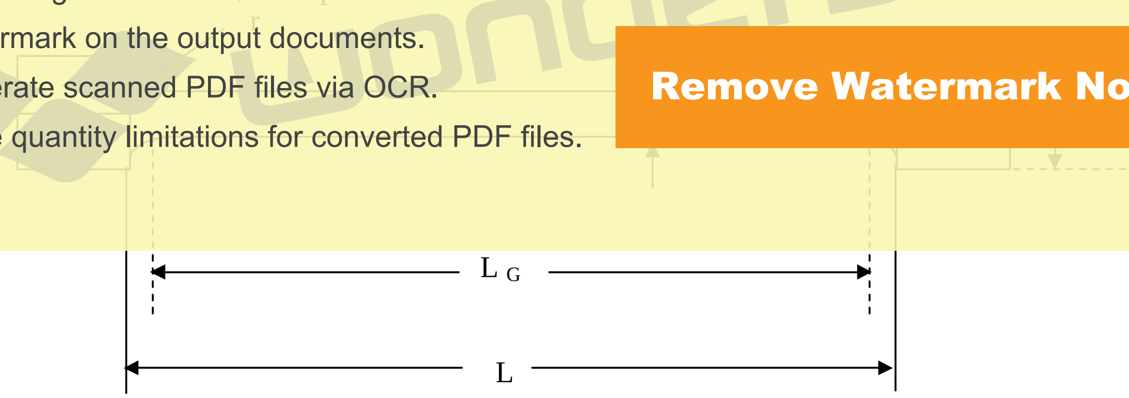


Figure 1 *the specimen geometry*

L_G=70 mm
L=80 mm
r =5 mm
R=6 mm
D=15 mm

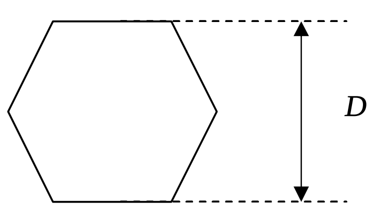


Figure 2: *Side view of the holder*
Figure 4.3a: *Main dimension of the specimen*

Figure 4.3b: Actual specimens picture

4.4 Testing procedure

These tests consist mainly of two experimental tests as follows:

Test one: the first test is conducted by fitting [illegible] device and reading torque against angle of t[illegible]

Test two: the experimental procedure follows the following pattern. After making the necessary calibration, the specimen is subjected to the test by adjusting the twist angle to 30^0.then torque value is read of every five minutes for the specimen having a 12-mm radius. Linear time –dependent theory is used to find the shear modulus at each time station. Thus the following equation is utilized.

$$\frac{T}{J} = \frac{Gq}{L} = \frac{t}{R} \qquad 4.1$$

Where J is the second polar moment of area. J= (π/2) R^4.

Time dependent parameters "shear relaxation coefficient" is obtained by fitting the resulting data to a Prony series representation of G(t) as follow:

$$G(t) = G_0 + \sum_{i=1}^{m=2} G_i Exp[-\frac{t}{t_i}]$$

$$G(t) = G_0 + G_1 \exp(-\frac{t}{t_1}) + G_2 \exp(-\frac{t}{t_2}) \qquad 4.2$$

It was elected to evaluate the above representation up to 5 relaxation parameters G_i (i=0, 1, 2) and λ_j (j=1, 2). This is done by substituting 5 experimental readings in equation (4.2) and solving the resulting algebraic equation set of equations simultaneously.

4.5 Experimental results

Materials to be tested are outlined below.

1-Material type .One

The above procedure is conducted for material type one [illegible] order to get the shear relaxation modulus. The experimental results are listed in Tables (4.1), (4.2) while Figs. 4.4 and 4.8 show [illegible] behavior of the material [illegible] By considering a five readings from Table (4.1) and applying eq (4.1) for each one to get the shear relaxation modulus at five different time steps as follow.

$$at \quad t = 0\sec \quad G(0) = \frac{3.45 * 0.07}{p / 2(0.012)^4 * 0.523} = 1442997$$

$$at \quad t = 300\sec \quad G(300) = \frac{2.15 * 0.07}{p / 2(0.012)^4 * 0.523} = 8992590$$

$$at \quad t = 1500\sec \quad G(1500) = \frac{1.5 * 0.07}{p / 2(0.012)^4 * 0.523} = 6273900$$

$$at \quad t = 3300\sec \quad G(3300) = \frac{1.25 * 0.07}{p / 2(0.012)^4 * 0.523} = 5228250$$

$$at \quad t = 3600\sec \quad G(3600) = \frac{1.20 * 0.07}{p / 2(0.012)^4 * 0.523} = 5019120$$

Applying the above results in equation 4.2, as follow:

$$G(0) = G_0 + G_1 \exp(-\frac{0}{t_1}) + G_2 \exp(-\frac{0}{t_2}) = 1442997$$

$$G(300) = G_0 + G_1 \exp(-\frac{300}{t_1}) + G_2 \exp(-\frac{300}{t_2}) = 8992590$$

$$G(1500) = G_0 + G_1 \exp(-\frac{1500}{t_1}) + G_2 \exp(-\frac{1500}{t_2}) = 6273900$$

$$G(3300) = G_0 + G_1 \exp(-\frac{3300}{t_1}) + G_2 \exp(-\frac{3300}{t_2}) = \text{[illegible]}$$

$$G(3600) = G_0 + G_1 \exp(-\frac{3600}{t_1}) + G_2 \exp(-\frac{3600}{t_2}) = 5019120$$

***Table 4.1:** the results of the sample of torsion test*
$\theta=30^{\circ}$, T=37°C and R=12mm

No	***Time (sec)***	***Torque (N.m)***	***Shear Relaxation (N/m^2)***
1	0	3.45	14429971
2	300	2.15	8992590
3	600	1.85	7737810.707
4	900	1.70	7110420.650
5	1200	1.60	6692160.612
6	1500	1.50	6273900.574
7	1800	1.40	5855640.535
8	2100	1.35	5646510.516
9	2400	1.35	5646510.516
10	2700	1.30	5437380.497
11	3000	1.25	[illegible]
12	3300	1.25	5228250.478
13	3600	1.20	5019120.459

[illegible] results of the sa[illegible]
R=6mm and T=[illegible]

No	***angle of twist (Deg)***	***Torque (N.m)***
1	0	0
2	5	0.3
3	10	0.5
4	15	0.65
5	20	0.90
6	25	1.05
7	30	1.10
8	35	1.15
9	40	1.18
10	45	1.20
12	50	1.25

The shear relaxation parameters obtained are as listed in Table (4.7). Equations (4.1) and (4.2) and table (4.7) are used to express the behavior of theoretical torque with (time, angle of twist) and shear relaxation G (t) in terms of prony series and make a comparison between experimental and Prony series results as shown in the Figs. 4.7 and 4.8.

2-Material type .two:

The same procedure applied to materials types two by taking five experimental results and using equations (4.1) and (4.2) to get on the results as follows:

$$at \quad t = 0\sec \quad G(0) = \frac{5.85 * 0.07}{p/2(0.012)^4 * 0.523} = 25403225$$

$$at \quad t = 300\sec \quad G(300) = \frac{2.54 * 0.07}{p/2(0.012)^4 * 0.523} = 10638957$$

$$at \quad t = 1500\sec \quad G(1500) = \frac{1.70 * 0.07}{p/2(0.012)^4 * 0.523} = 7382134$$

$$at \quad t = 3300\sec \quad G(3300) = \frac{1.[illegible]}{p/2(0[illegible]} [illegible]$$

$$at \quad t = 3600\sec \quad G(3600) = \frac{1.40 * 0.07}{p/2(0.012)^4 * 0.523} = 6079404$$

Applying the above results in equation (4.2), as follows:

$$G(0) = G_0 + G_1 \exp(-\frac{0}{t_1}) + G_2 \exp(-\frac{0}{t_2}) = 25403225$$

$$G(300) = G_0 + G_1 \exp(-\frac{300}{t_1}) + G_2 \exp(-\frac{300}{t_2}) = 10638985$$

$$G(1500) = G_0 + G_1 \exp(-\frac{1500}{t_1}) + G_2 \exp(-\frac{1500}{t_2}) = 7382134$$

$$G(3300) = G_0 + G_1 \exp(-\frac{3300}{t_1}) + G_2 \exp(-\frac{3300}{t_2}) = 6296526$$

$$G(3600) = G_0 + G_1 \exp(-\frac{3600}{t_1}) + G_2 \exp(-\frac{3600}{t_2}) = 6079404$$

The experimental results are listed in Tables (4.3) and (4.4) while Figs. 4.9 and 4.13 show the behavior of the material properties.

Table 4.3: *the results of the sample of torsion test*
$\theta=30^{\circ}$, $T=37^{\circ}C$ and R=12mm

No	***Time (sec)***	***Torque (N.m)***	***Shear Relaxation (N/m^2)***
1	0	5.85	25403225
2	300	2.45	10638957
3	600	2.05	8901985
4	900	1.90	8250620
5	1200	1.75	7599255
6	1500	1.70	7382134
7	1800	1.65	7165012
8	2100	1.60	6947890
9	2400	1.55	6730769
10	2700	1.50	6513647
11	3000	1.50	6513647
12	3300	1.45	62965[illegible]6
13	[illegible]	1.40	[illegible]

Table 4.4: the results of the sa[illegible]
R=6mm and $T=37^{o}c$

No	***angle of twist (Deg)***	***Torque (N.m)***
1	0	0
2	5	0.45
3	10	0.65
4	15	0.95
5	20	1.20
6	25	1.30
7	30	1.40
8	35	1.60
9	40	1.65
10	45	1.70
12	50	1.75

2-Material type .Three:

The same procedure applied to materials types three by taking five experimental results and using equations (4.1) and (4.2) to get on the results as follow:

$$at \quad t=0\sec \quad G(0)=\frac{4.54*0.07}{p/2(0.012)^4*0.523}=19714640$$

$$at \quad t=300\sec \quad G(300)=\frac{1.90*0.07}{p/2(0.012)^4*0.523}=8250620$$

$$at \quad t=1500\sec \quad G(1500)=\frac{1.45*0.07}{p/2(0.012)^4*0.523}=6296526$$

$$at \quad t=3300\sec \quad G(3300)=\frac{1.25*0.07}{p/2(0.012)^4*0.523}=5428039$$

$$at \quad t=3600\sec \quad G(3600)=\frac{1.20*0.07}{p/2(0.012)^4*0.523}=5210918$$

Applying the above results in equation (4.2), as follow:

$$G(0)=G_0+G_1\exp(-\frac{0}{t_1})+G_2\exp(-\frac{0}{t_2})=19714640$$

$$G(300)=G_0+G_1\exp(-\frac{300}{t_1})+G_2\exp(-\frac{300}{t_2})=8250620$$

$$G(1500)=G_0+G_1\exp(-\frac{1500}{t_1})+G_2\exp(-\frac{1500}{t_2})=6296526$$

$$G(3300)=G_0+G_1\exp(-\frac{3300}{t_1})+G_2\exp(-\frac{3300}{t_2})=5428039$$

$$G(3600)=G_0+G_1\exp(-\frac{3600}{t_1})+G_2\exp(-\frac{3600}{t_2})=5210918$$

The experimental results are listed in Tables (4.5) and (4.6) while Figs. 4.14 and 4.18 show the behavior of the material properties.

Table 4.5: *the results of the sample of torsion test*
θ=30°, T=37°C and R=12mm

No	*Time (sec)*	*Torque (N.m)*	*Shear Relaxation (N/m^2)*
1	0	4.54	19714640
2	300	1.90	8250620
3	600	1.70	7382133
4	900	1.55	6730769
5	1200	1.50	6513647
6	1500	1.45	6296526
7	1800	1.40	6079404
8	2100	1.35	5862282
9	2400	1.35	5862282
10	2700	1.30	5645161
11	3000	1.30	5645161
12	3300	1.25	5428039
13	3600	1.20	5210918

Table 4.6: *the results of the sa*[illegible]
R=6mm and T=[illegible]

No	*angle of twist (Deg)*	*Torque (N.m)*
1	0	0
2	5	0.25
3	10	0.50
4	15	0.65
5	20	0.80
6	25	0.95
7	30	1.10
8	35	1.15
9	40	1.25
10	45	1.30
12	50	1.35

__Table 4.7:__ Shear relaxation parameters

Material type	**G_0**	**G_1**	**G_2**	**t_1**	**t_2**
one	4786751	4563498	5080506	22.113	2.273
two	6273722	5950405	13179072	14.057	0.003
three	5961238	13704571	747375	3.104	22.431

The theoretical curves plotted in Figs. 4.7, 4.11 and 4.14 have been derived in appendix A in equations (A.21).

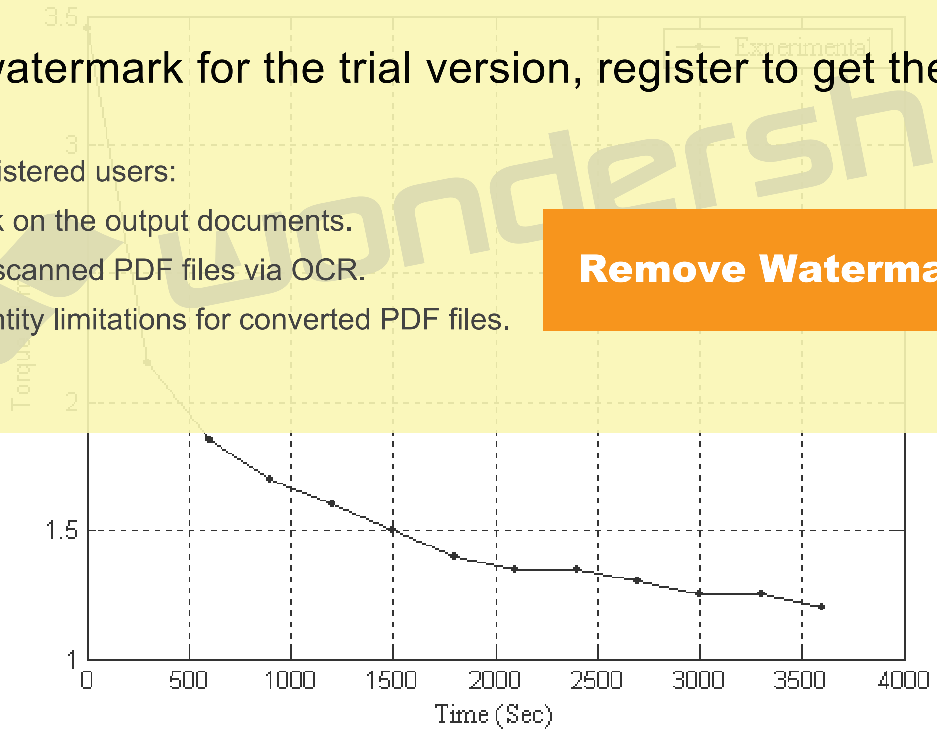


***Figure 4.4:** Experimental results of torque vs. time*

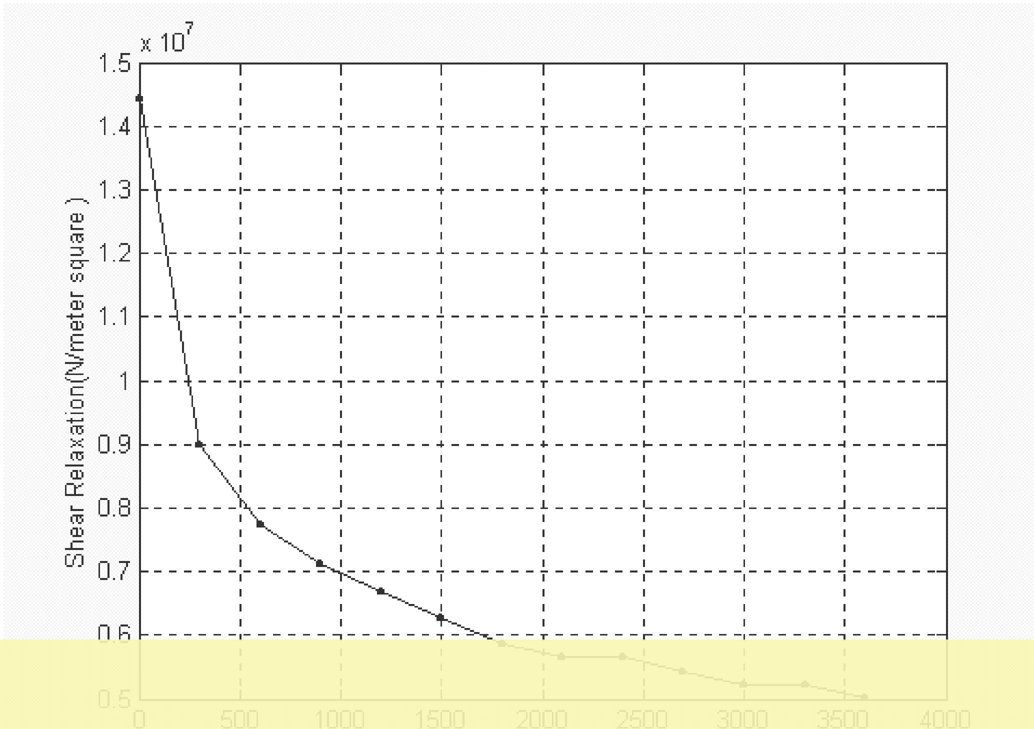


Figure 4.5: *Experimental Shear Relaxation modulus vs. time*

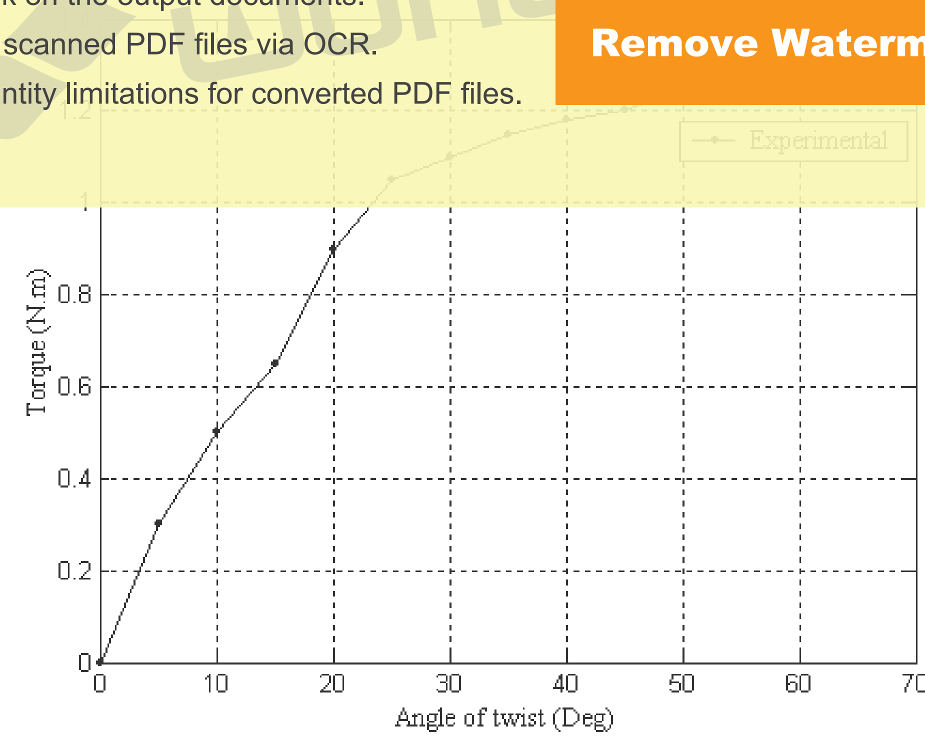


Figure 4.6: *Experimental results of torque vs. angle of twist*

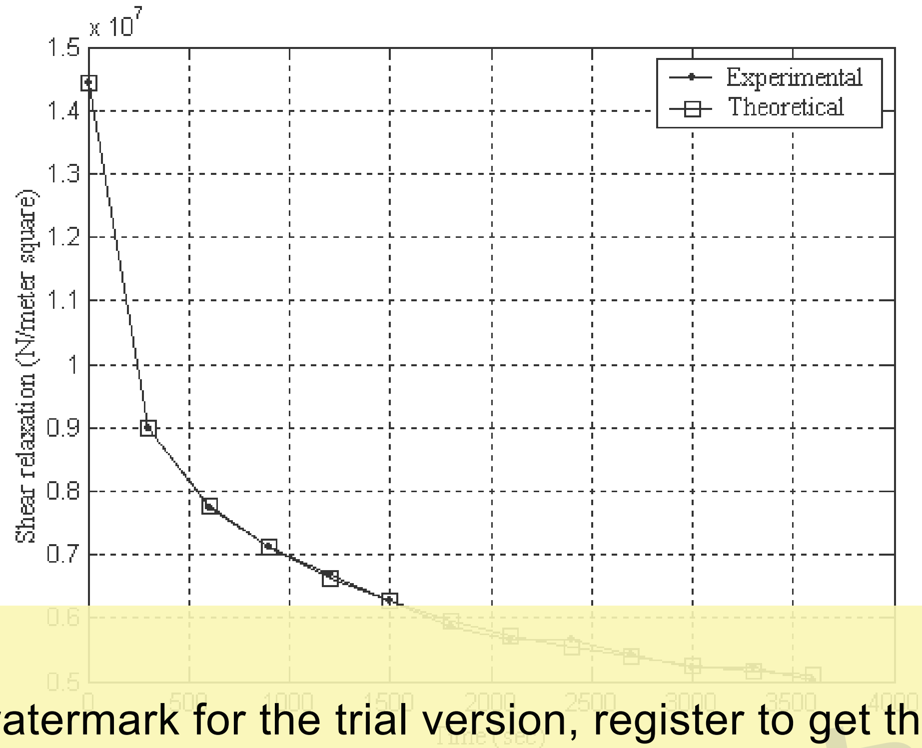


Figure 4.7: Experimental and theoretical shear relaxation vs. time

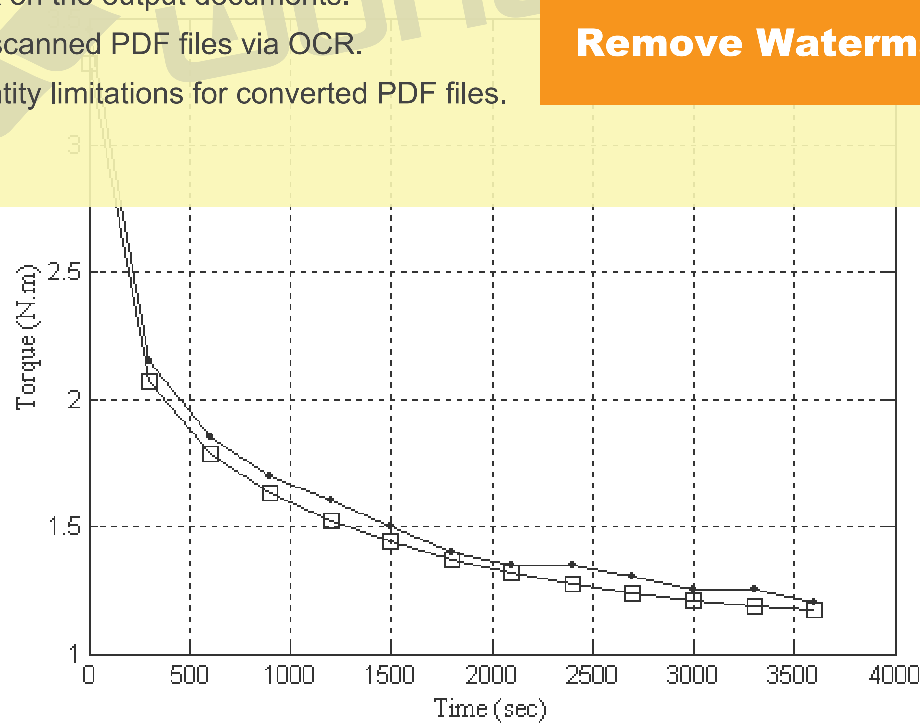


***Figure 4.8:** Experimental and theoretical torque vs. time*

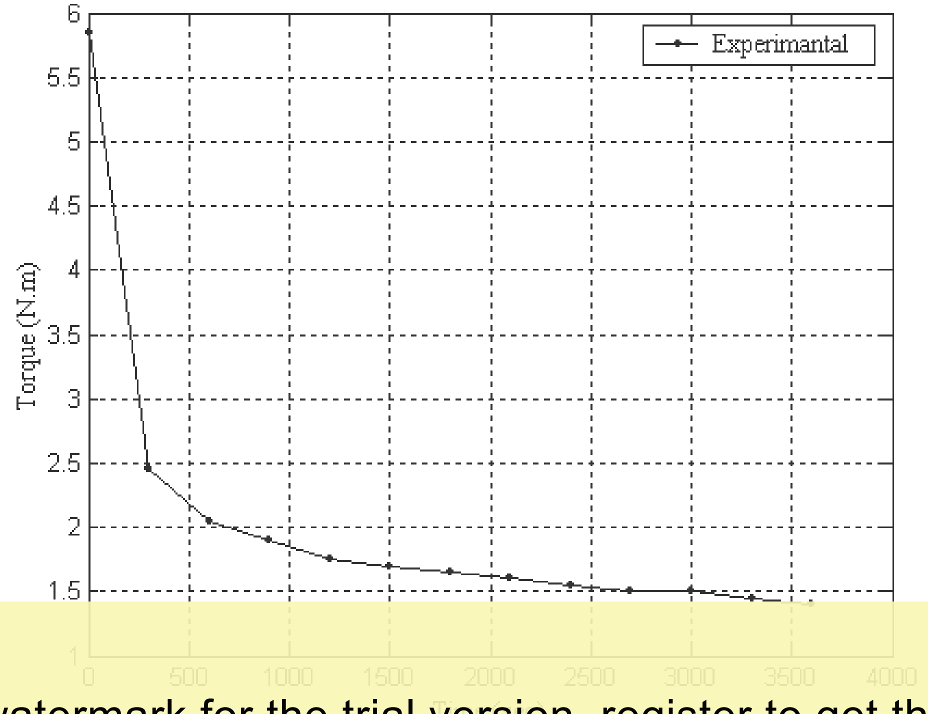


Figure 4.9: Experimental results of torque vs. time

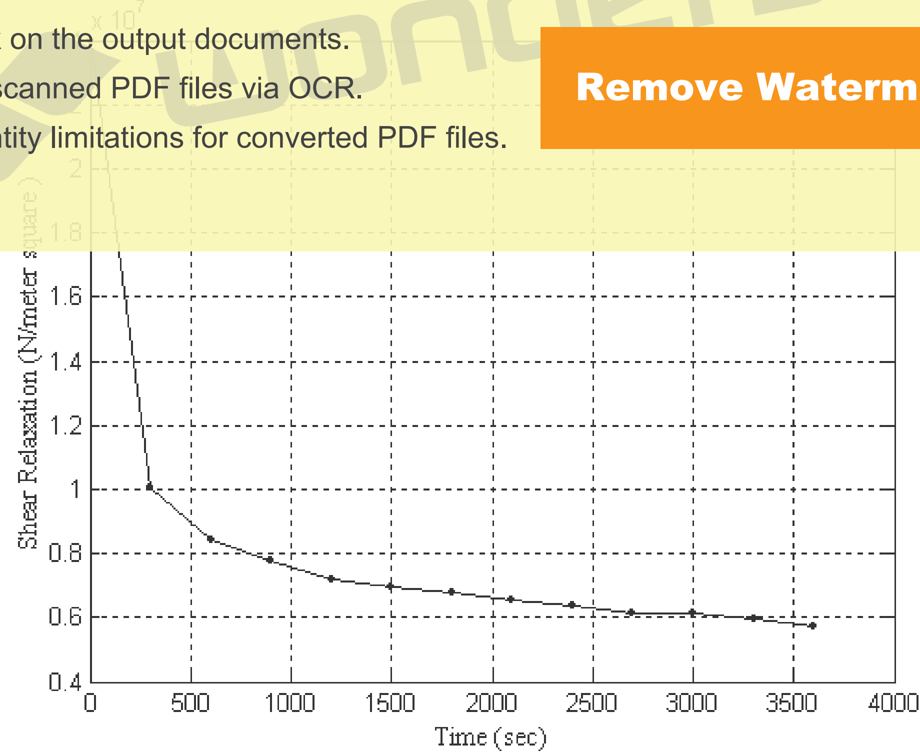


Figure 4.10: *Experimental Shear Relaxation modulus vs. time*

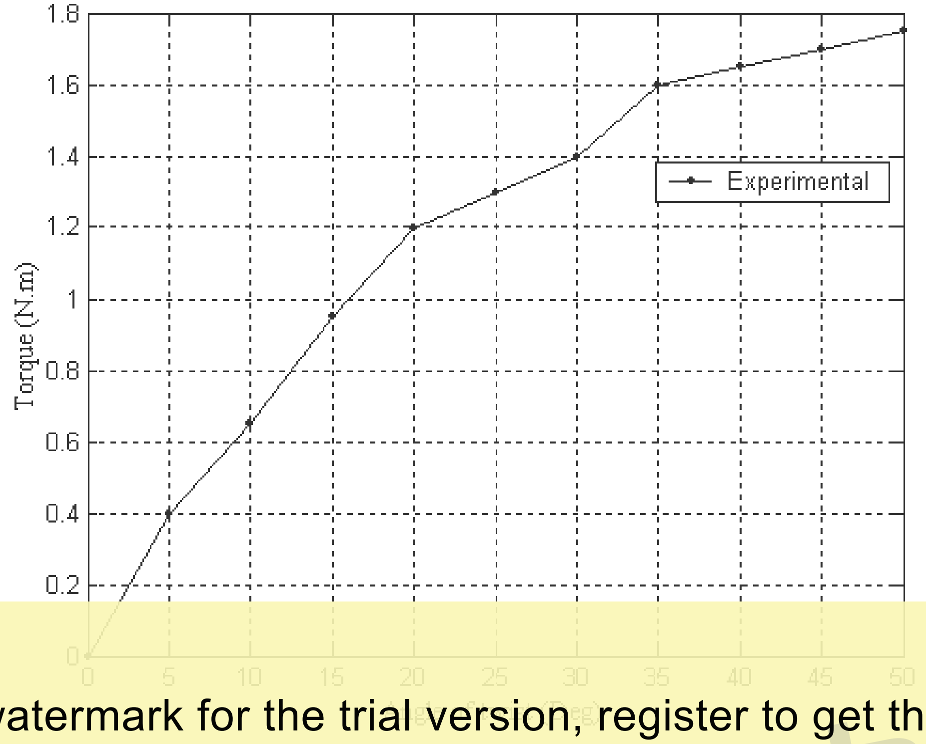


Figure 4.11: *Experimental results of torque vs. angle of twist*

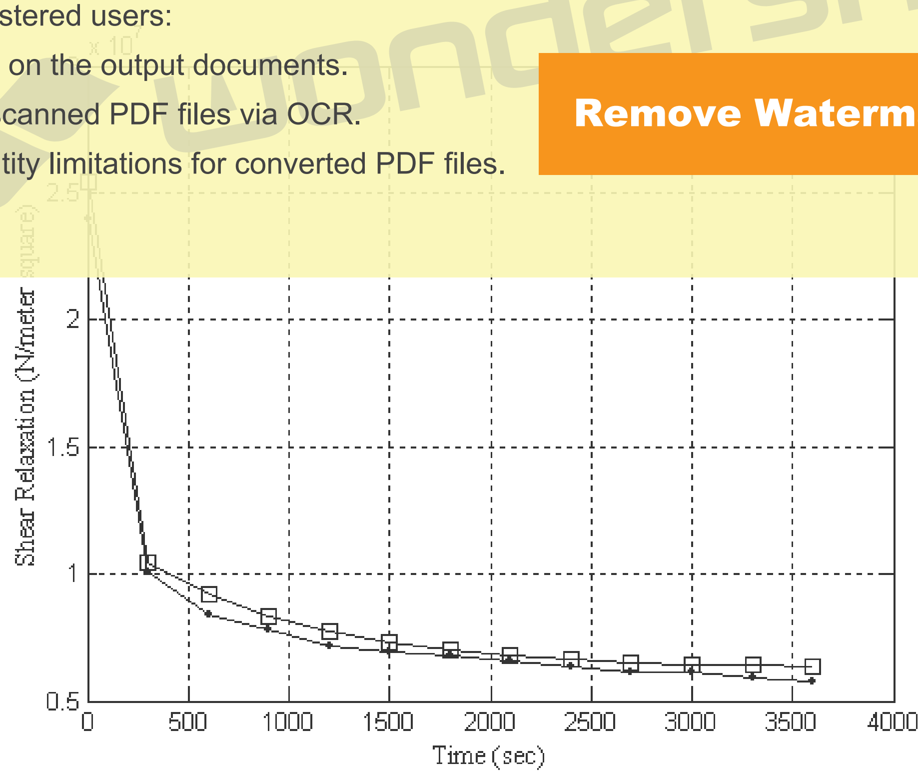


Figure 4.12: *Shear relaxations vs. time*

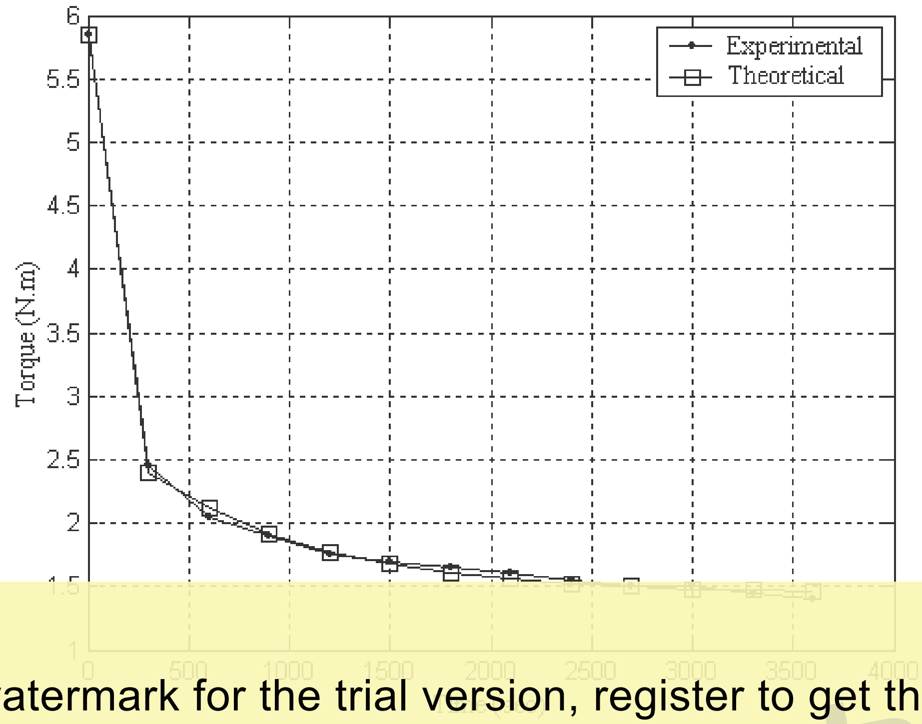


Figure 4.13: Experimental and theoretical torque vs. time

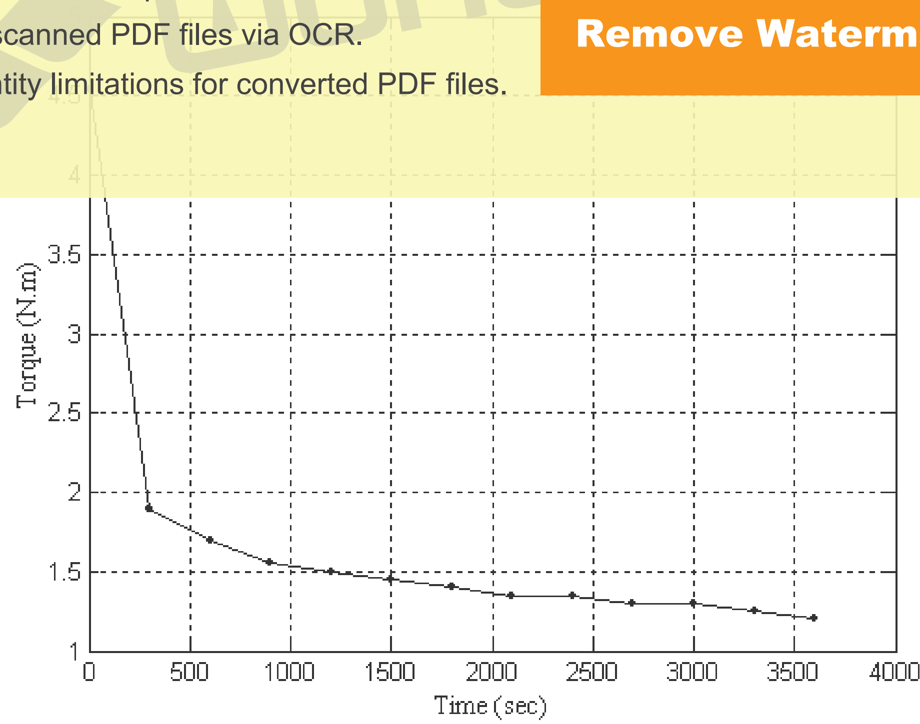


Figure 4.14: *Experimental results of torque vs. time*

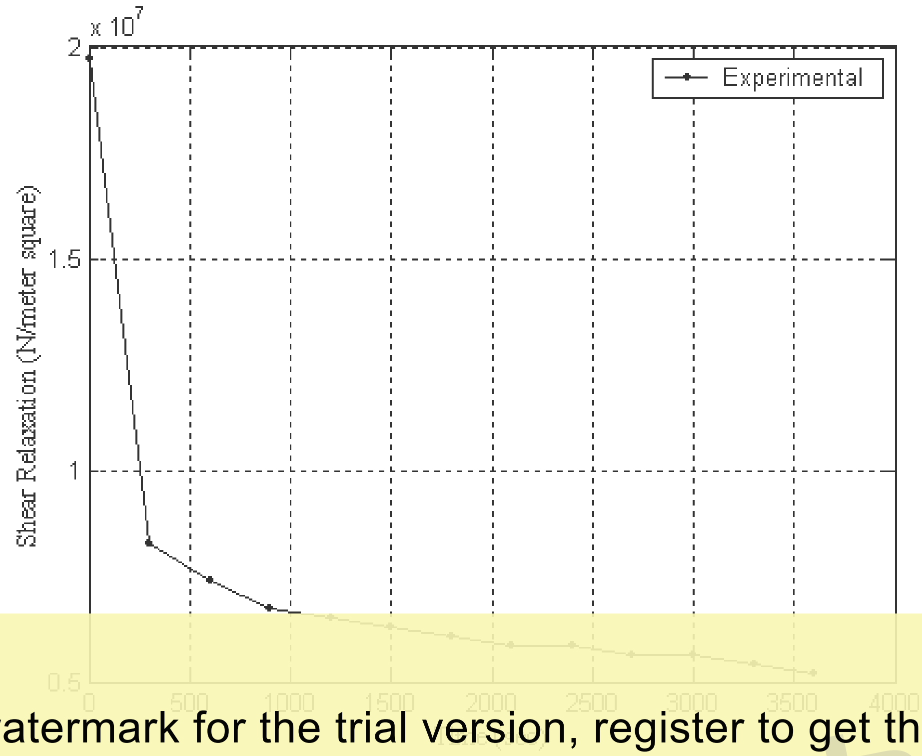


Figure 4.15: Experimental Shear Relaxation modulus vs. time

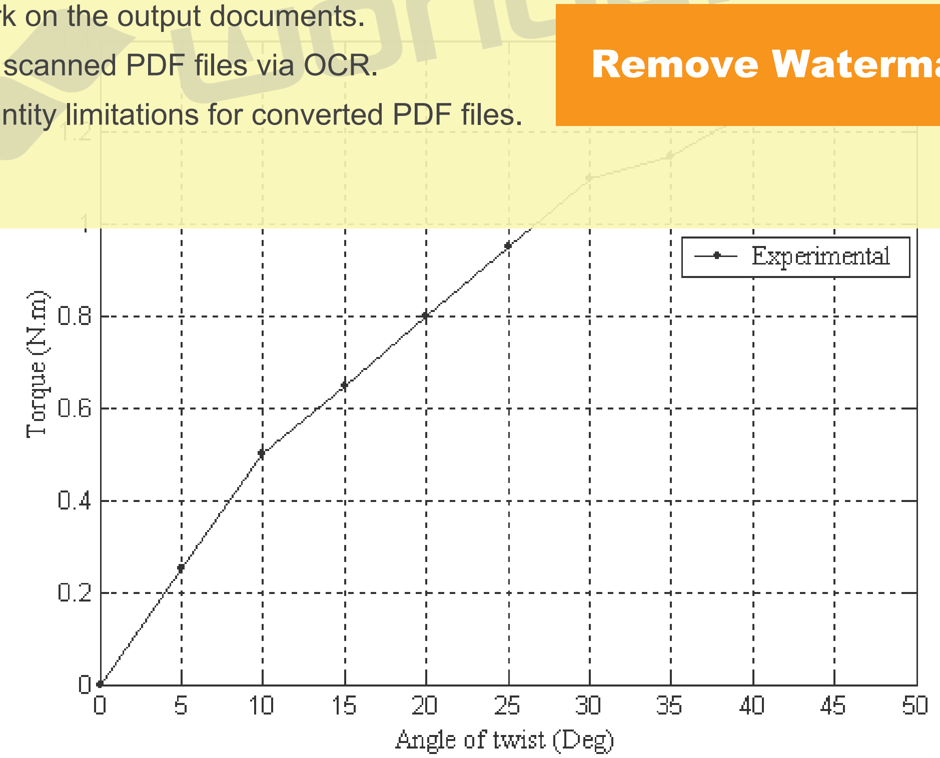


Figure 4.16: *Experimental results of torque vs. angle of twist*

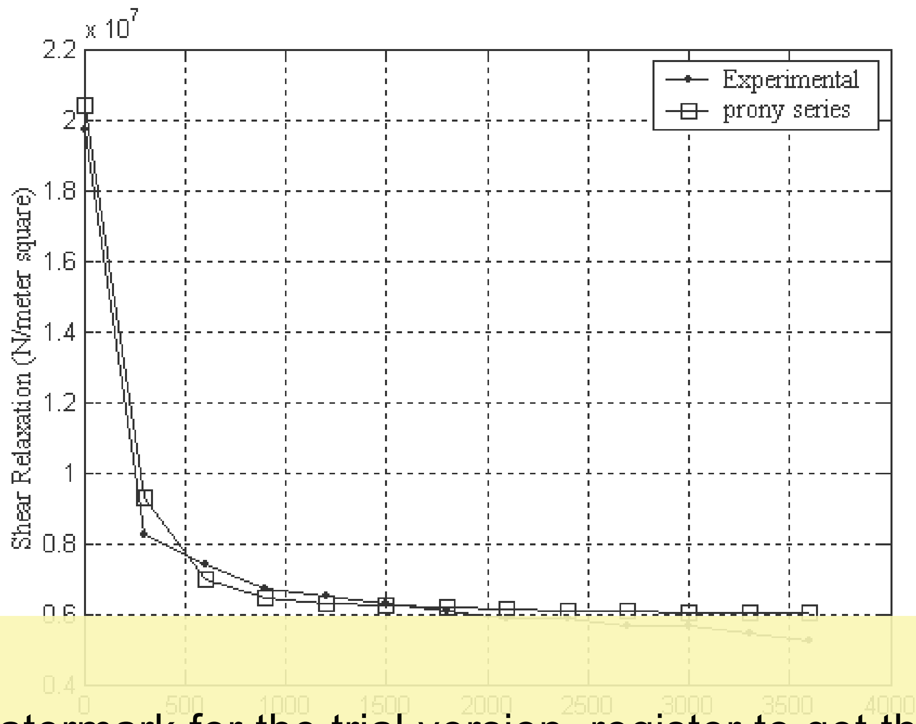


Figure 4.17: *Experimental and theoretical shear relaxation vs. time*

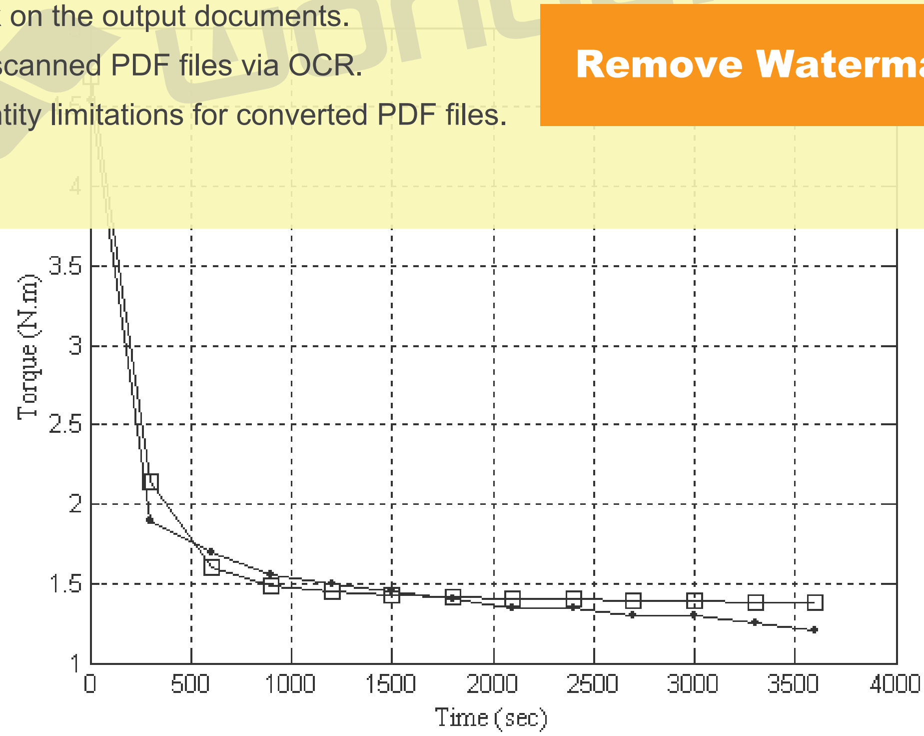


Figure 4.18: *Experimental and theoretical torque vs. time*

Chapter Five

Results

5.1 Introduction

This chapter deal with the verification of study cases that has been carried out by comparing the results obtained from the present packages with those obtained from analytical solution or with those obtained from other finite element packages.

5.2 Cases of study

Several numerical examples are investigated to verify the analytical solution and software.

5.2.1 [illegible]

The geometry of this case is shown in Fig.5.1. Where b=2a. The material properties of this material are as follow:

1-The rheological model are the Zener m[illegible] [illegible] elastic i[illegible] in Fig.5.2.

2-The constant parameters of this model are shown in the following table (5.1).

Table: 5.1

Parameters	G_0 *(Mpa)*	G_1 *(Mpa)*	λ	***k*** *(Mpa)*
Values	480	160	1600	1280

3-The problem solved as a plane strain problem and according to symmetry around the X and Y-axis,due to symmetry only one quarter of the cylinder will considered in the finite element analysis as shown in Fig.5.3.

According to the calculated parameters of this model, so$\boldsymbol{G_0}$=G_0G_1/G_0+G_1 $\boldsymbol{G_1}$=G_0^2/G_0+G_1 and $\boldsymbol{\lambda_1}$= G_0+G_1/λ then: **G(t)** =120+360exp (-0.4 t), the shear relaxation function verses time is shown in Fig.5.4.

Two runs are investigated for this case .first one, when the hollow cylinder subjected to mechanical load only whilst in the second run the hollow cylinder subjected to thermal gradient load only.

Run 1: A pressurized viscoelastic hollow cylinder

The thick viscoelastic hollow cylinder subjected to a steady state and transient internal pressure loadings as shown in Fig.5.5.The exact solution of steady and transient loading for the displacement and the radial stress distribution for inner and outer surfaces at different time steps are given in equations (A.5), (A.6), (A.11) and (A.12) and represented in Figs. (5.6), (5.7), (5.8) and (5.9) respectively.

Run 2: A thermoviscoelastic hollow

A thick walled viscoelastic hollow cylinder subjected to a steady and transient thermal loading, as shown in Fig.5.10 is studied. Assuming that body is permanently maintained at a uniform temperature (temperature difference loading), therefore two types of solution according to the loading input is derived and illustrated in appendix A in equations (A.16) and (A.17). The results of this case are shown in Figs 5.11 and 5.12.

The above example shows how the analytical solution of the viscoelastic materials can be derived.

5.2.2 Density effect in viscoelastic materials

The gravity effect on the viscoelastic media is a serious problem in solid propellant engineering.therefor the following typical cases will be investigated.

Case one: Solid mass slump problem

The problem is studied for a simple example of rectangular prism structure in [37] .This problem is solved as a plane strain problem, and the main dimension, loading, boundery conditions and finite element mesh are shown in Fig.5.13a the materials properties and the shear relaxation parameters are given in table (5.2).

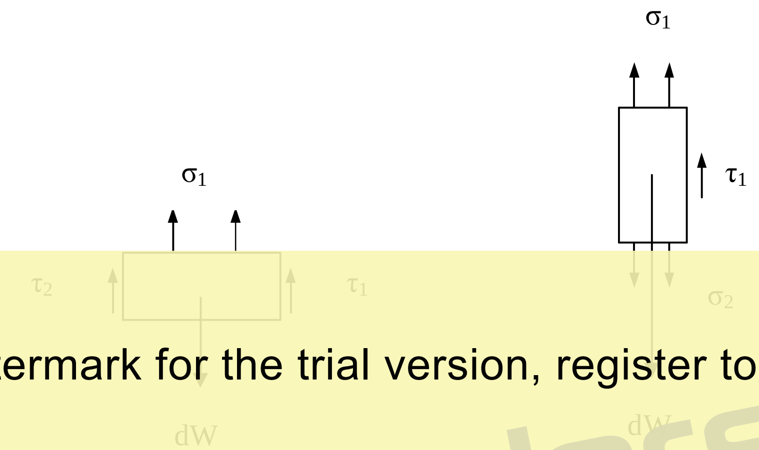
Table: 5.2

G_0 (Mpa)	G_1 (Mpa)	G_2 (Mpa)	λ_1	λ_2	ρ (Density)	ν	k (Mpa)
0.022	0.03	[illegible]	0.0025	0.007	[illegible]	[illegible]	[illegible]

The example is nearly incompressible and the plot of the shear relaxation with respect to time is shown in Fig.5.14. The results of this case for displacement in horizontal and vertical direction for upper edge are shown in Figs 5.15 and 5.16. Figure 5.13b show that, the maximum deflection occurs at upper and lower edges, therefore the displacement of these edges will study for a different ratio of (a/b) to recognize the effect of this supporting on the important deformation as shown in Figs 5.17a to 5.17g. These results are approximately similar to the finite element results given in Ref [37].

Case two: Gravity effect on viscoelastic hollow cylinder

The geometry, the finite element mesh and load details are given in Fig. 5.18 and the material properties are taken as in previous example [37]. This problem is solved as a plane strain problem; the cylinder will undergo a dimensional deviation at inner and outer surface due to its own weight. Fig 5.19 illustrates that, the maximum deviation from the unreformed form occurs at point C, and the vertical displacement of a three selected points with respect to time is shown in Fig.5.20 .it can be concluded that element force analysis at point C has two direct stress in negative Y direction whilst at point A and B has one direct stress in negative Y direction, as indicated below.



Case three: Gravity effect in solid propellant grains

The material properties are shown in table (5.3) solid propellant grains subjected to gravity force is a serious problem in solid propellant engineering. The material being in nature viscoelastic. The propellant grains stored for long time undergoes dimensional deviation due to their own weight.Normelly the grains supported by a casing. It is expected that the slumping can be minimized by supporting the grain at the inner surface .In general it shown that the displacement due to the material own weight is very small, but it's accumulation for long time may give a significant amount, which, in turn may cause a danger deformation in design. [37].

Table: 5.3

G_0	G_1	G_2	λ_1	λ_2	ρ (Density) (kg/mm^2)	v	k (Mpa)
0.022	0.03	0.048	0.0025	0.016	2.3e-6	0.499	1.078e-3

Usually this grain stored horizontally and the outer surface is assumed rigidity fixed (plane strain condition) as shown in Fig.5.21. Due to symmetry of the geometry and gravity load about the Y axis, only one half of the grain need to solve by the finite element analysis as shown in Fig.5.22. The vector of the deformation shape for 10 Minute storage times is shown in Fig.5.23 and the displacement deviation is shown in Fig.5.24. From Fig.5.23 and 5.24, it is clear that the [illegible] load while the lo[illegible] subjected to com[illegible] points A B C D E for ten minutes are shown in Figs.25a ,5.25b and 5.25c respectively .these Figures show that point C is a critical point and have the maximum deflection. Points B and D have the same value of strain in Y axis, but reversed in X axis, and point A and E have the same value of strain in X and Y axes.

5.3 Minimizing the density load effect

To minimize the density load effect (slumping effect), rotate the grained propellant by 180^0 to reflect the gravitational load and reflect the grain geometry to it's original shape as shown in Fig.5.26.from Fig 5.26,it is clear that some important region ,that the viscoelastic material will behave in this procedure. As follow.

Elastic region (e): this region indicates the elastic behavior of the viscoelastic material.

Viscous region (V): this region indicates the viscous behavior of the viscoelastic material.

Starting point of reflecting (P): point represent the point at which reflecting of the geometry begins.

First recovery time: the time required to return the geometry to it's original position after the first reflection.

Second recovery time: the time required to return the geometry to it's original position after the second reflection.

5.4 Appling the experimental results to the solid propellant grain

After making a preview for some results of analytical solution for many cases and making a studying for slumping effect in grain geometry [illegible]ated in previous examples.

1-Material type .One: When applying [illegible] material to grain propellant case. The [illegible] displacement of point A B C D E is display in Fig.5.27. It is clear that the effective maximum vertical displacement is at point C, therefore it is a significant point .Reflecting the load will apply to this case as shown in Fig.5.28. The vertical displacement value of point C at t=100 Min is equal to 0.123e-6 mm and the time required to restore this vertical displacement to it's original value is equal to t=11.5 min as shown in Fig.5.29.

2-Material type .Two:

When applying the experimental results for this material to point A B C D E, the following results for vertical displacement is display below in Figs.5.30, 5.31 and 5.32. Point C required 18 min to restore to it's original position for 100 min through storage. At t=100 min U_y=0.962e-7mm. the deformation shape of this case is shown in Fig 5.34.

3-Material type .Three:

When applying the experimental results for this material to grain case. Point A B C D E, the following results for vertical displacement is display below in Figs. 5.33, 5.34 and 5.35. Point C required 15 min to restore to it's original position for 10 min through storage. At t=100 min U_y=1.21e-7.

Table (5.4) will give the details about the three investigated materials. it is clear that the time required to return the geometry to the original shape (minimum distortion) is the same for different time storage.

Table: 5.4

Material type	Storage time (min)	First recovery time (min)	Second recovery time (min)
One	[illegible]	[illegible]	[illegible]
	100	[illegible]	[illegible]
	150	11.5	10
Two	50	18.3	15.5
	100	18.3	15.5
	150	18.3	15.5
Three	50	15.5	13.2
	100	15.5	13.2
	150	15.5	13.2

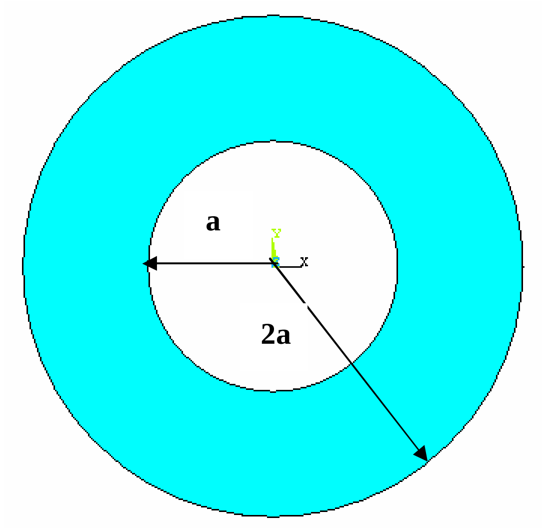


Figure 5.1: *Hollow cylinder geometry*

Figure 5.2: *Rheological model*

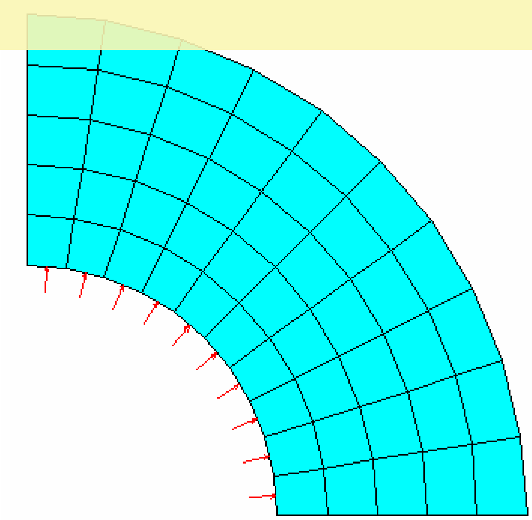


Figure 5.3: *Finite element and load details*

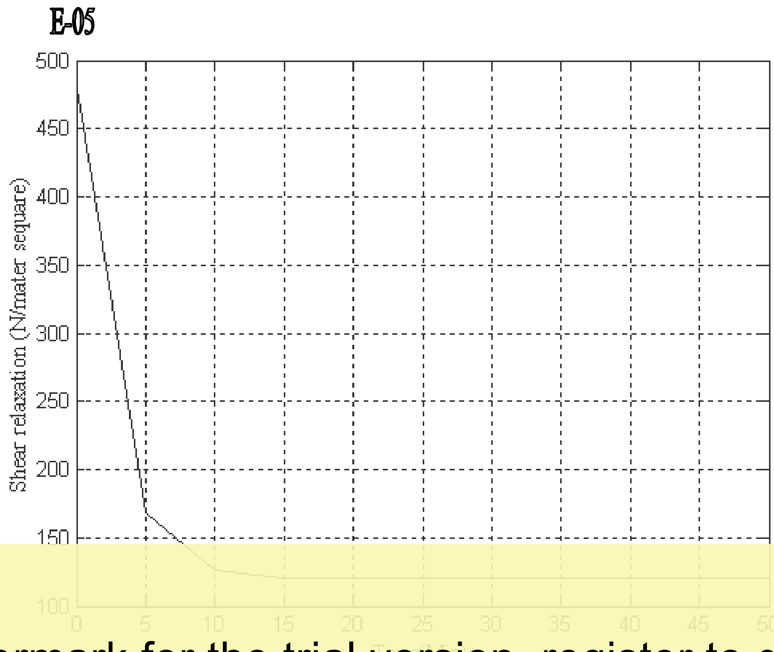


Figure 5.4: Shear relaxation vs. time

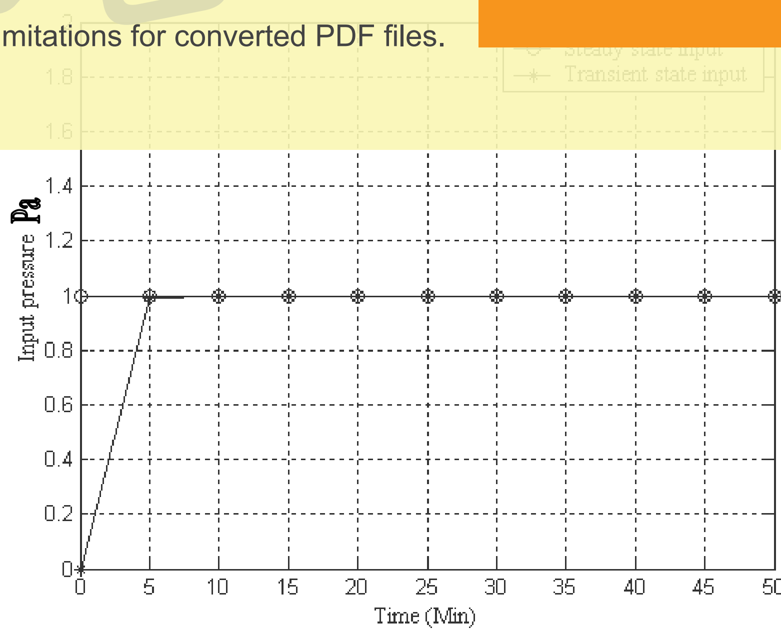


Figure 5.5: *input pressure vs. time*

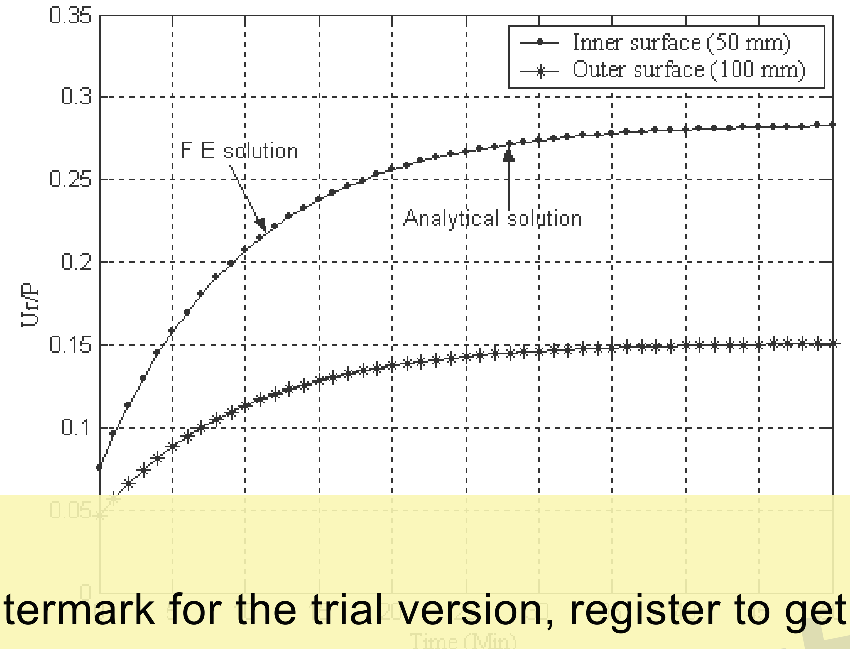


Figure 5.6: Displacement vs. time

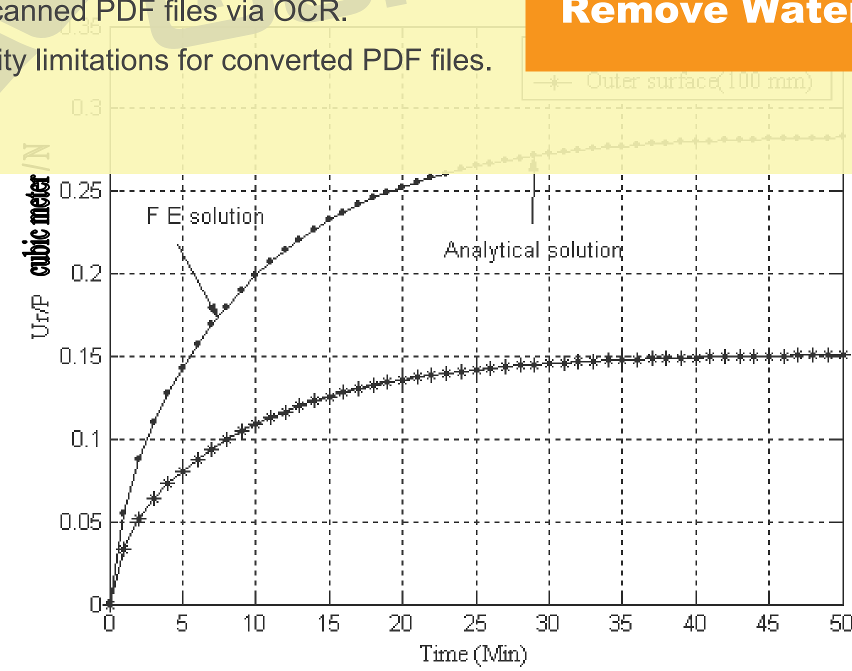


Figure 5.7: *Displacement vs. time*

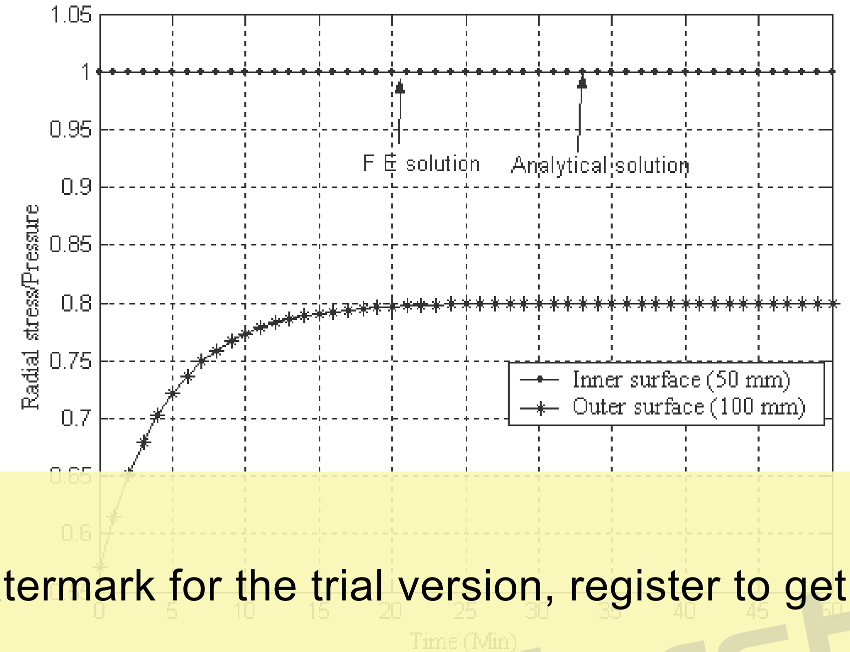


Figure 5.8: Radial stress vs. time

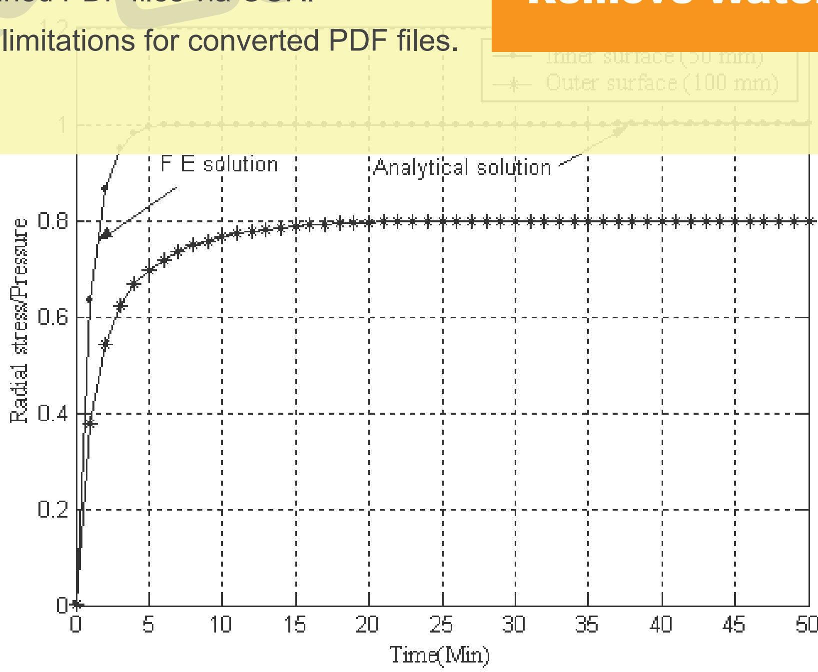


***Figure 5.9:** Radial stress vs. time*

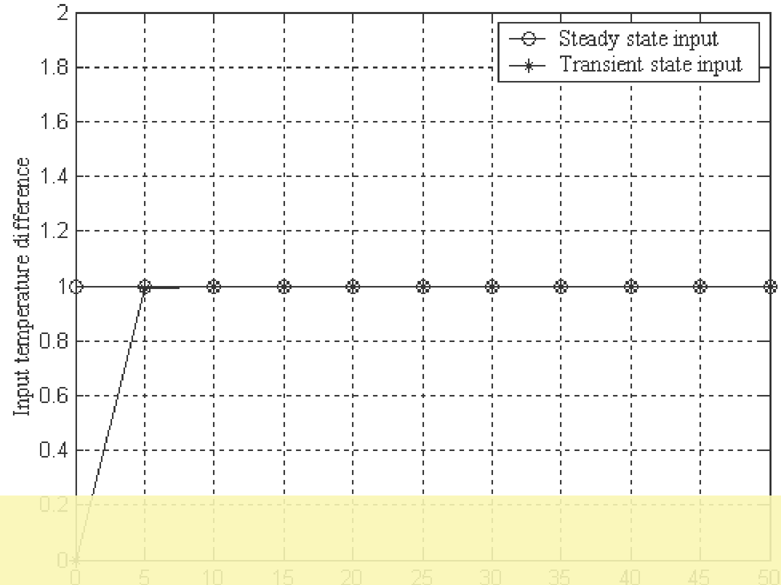


Figure 5.10: *input temperature vs. time*

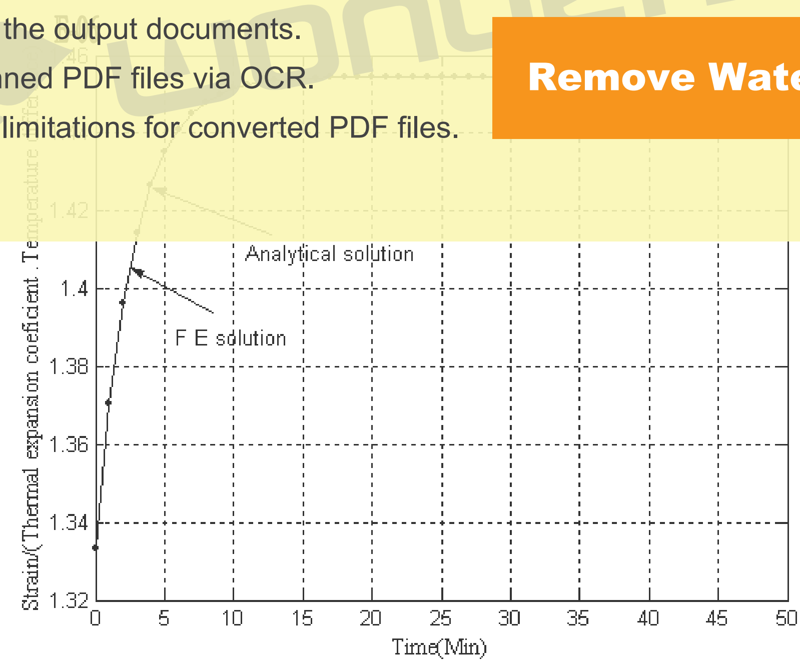


Figure 5.11: *strain vs. time*

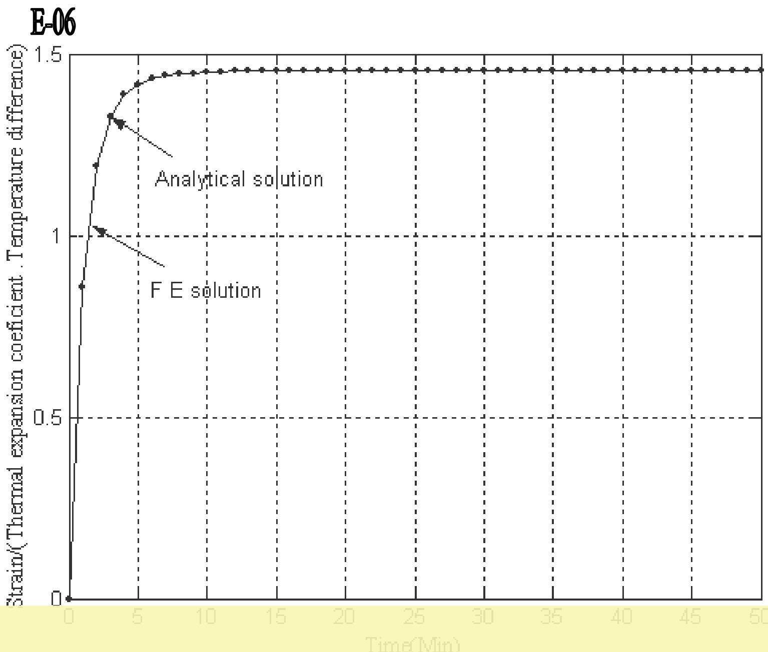


Figure 5.12: strain vs. time

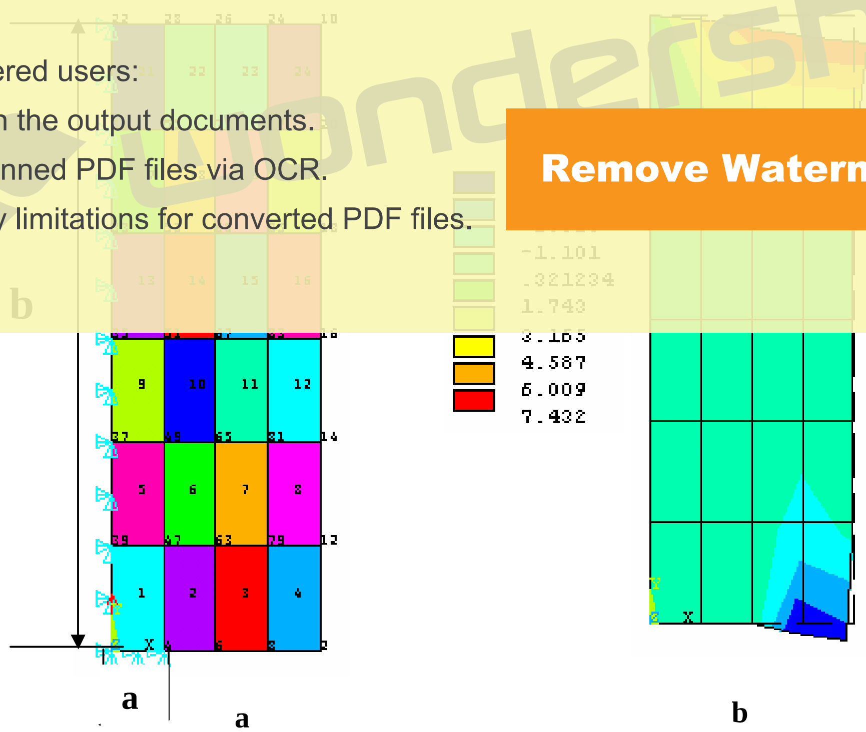


Figure 5.13a: *The geometry and finite element meshing*
Figure 5.13 b: *Deformation shape for a/b=0.50*

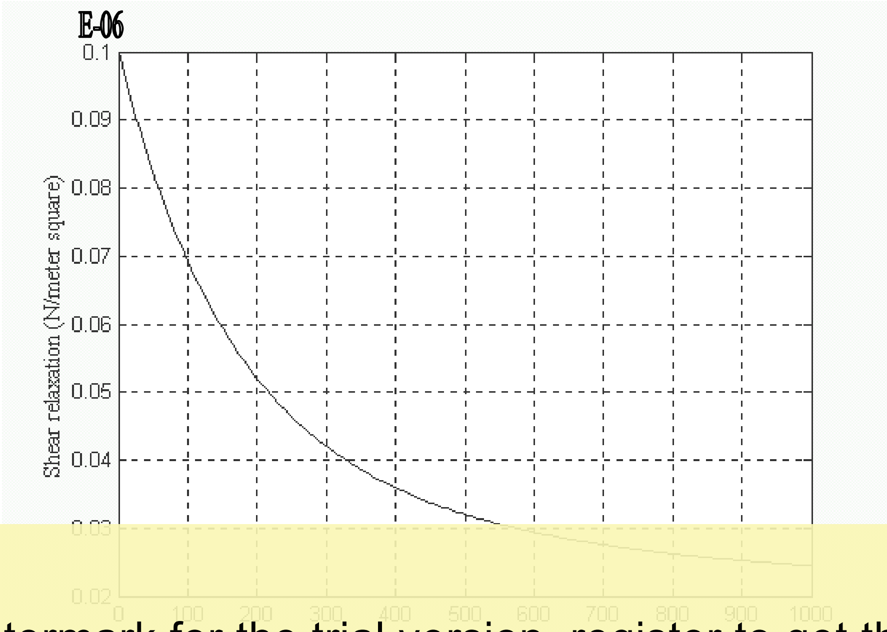


Figure 5.14: *Shear relaxation vs. time*

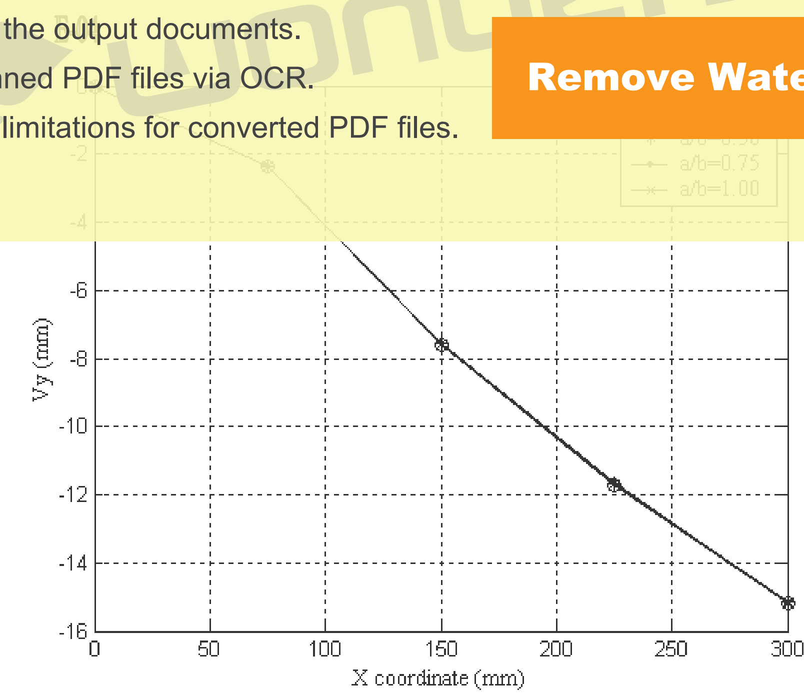


Figure 5.15: *Vertical displacement for upper edge*

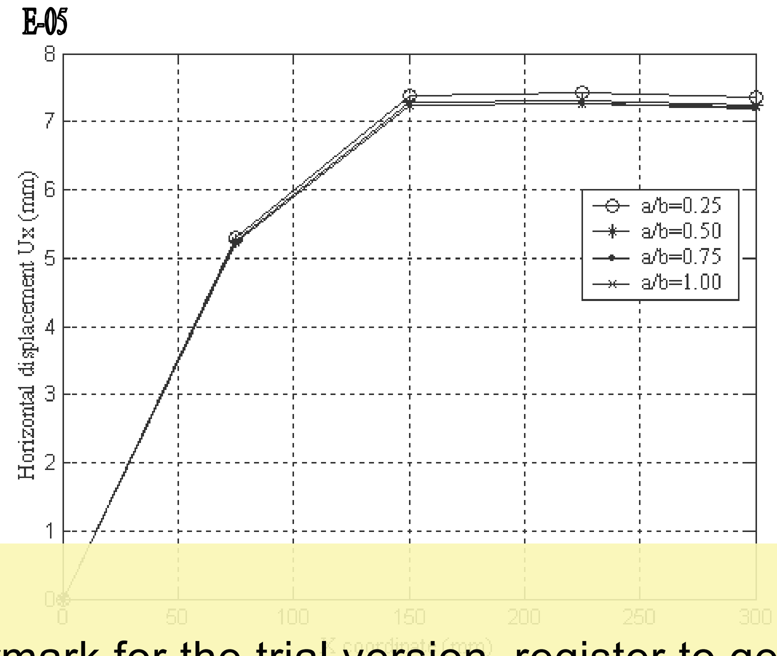


Figure 5.16: *Horizontal displacement for upper edge*

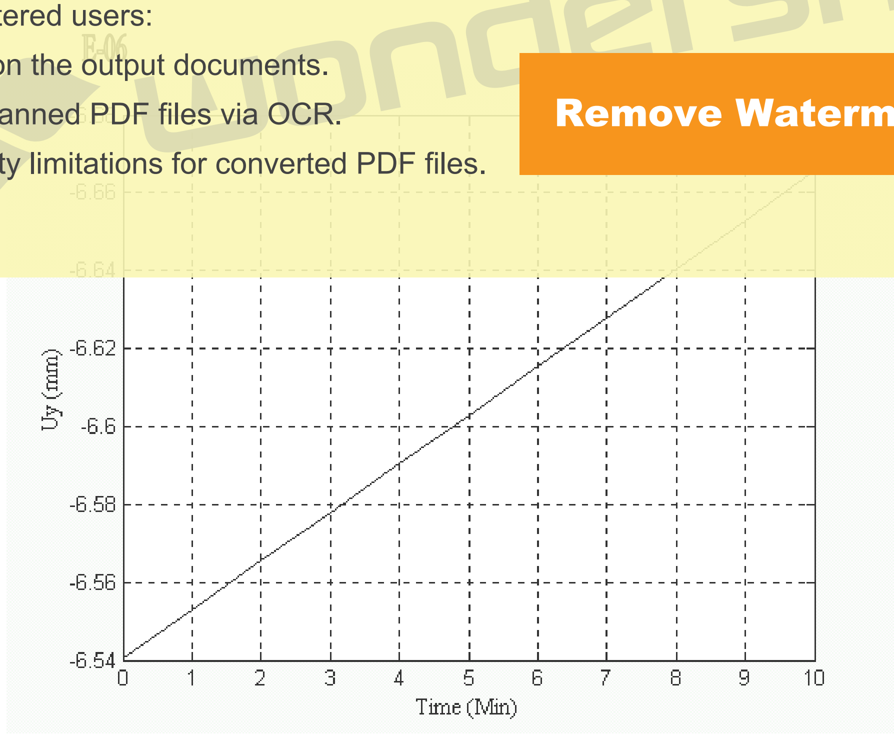


Figure 5.17a: *Vertical displacement for lower edge at a/b=0.25*

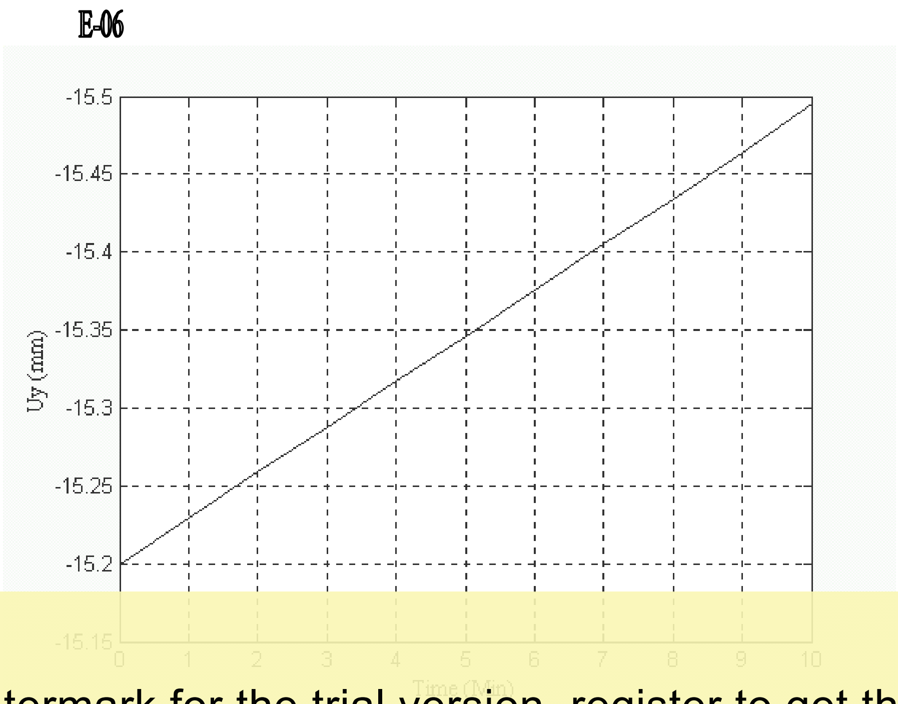


Figure 5.17b: Vertical displacement for upper edge at a/b=0.25

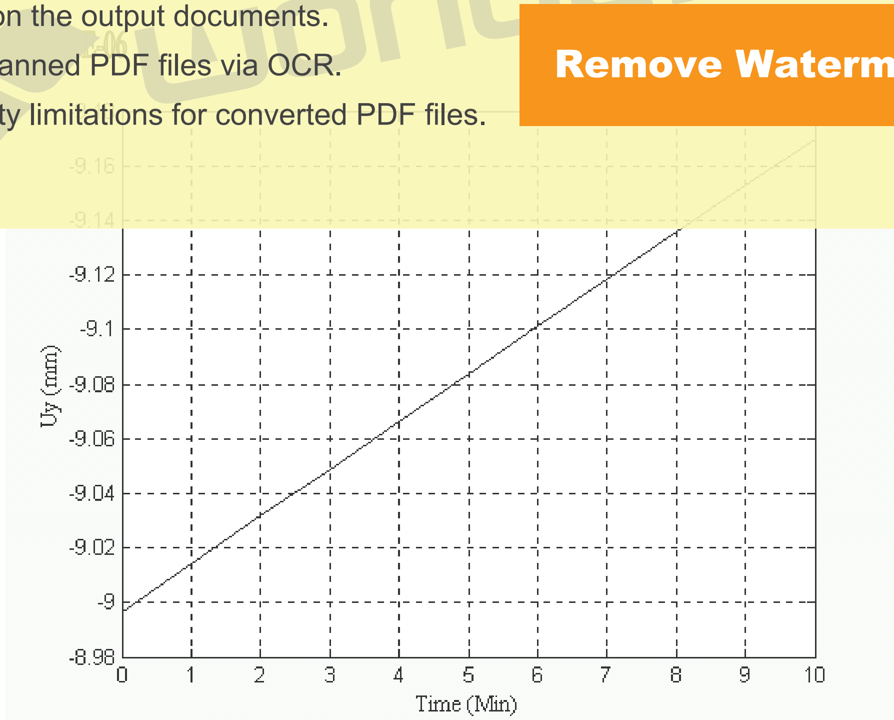


Figure 5.17 c: *Vertical displacement for lower edge at a/b=0.50*

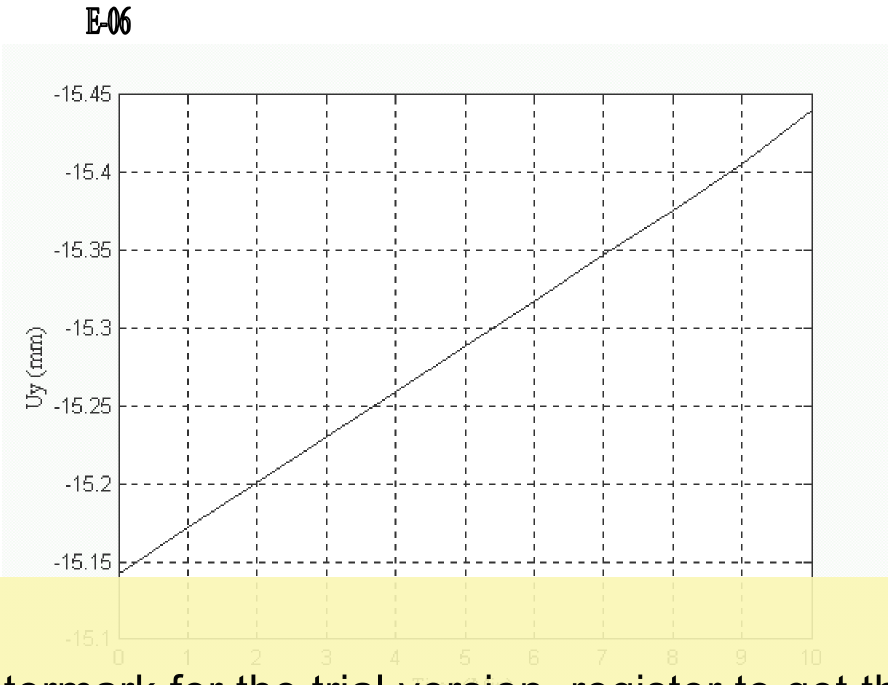


Figure 5.17 d :*Vertical displacement for upper edge at a/b=0.5*

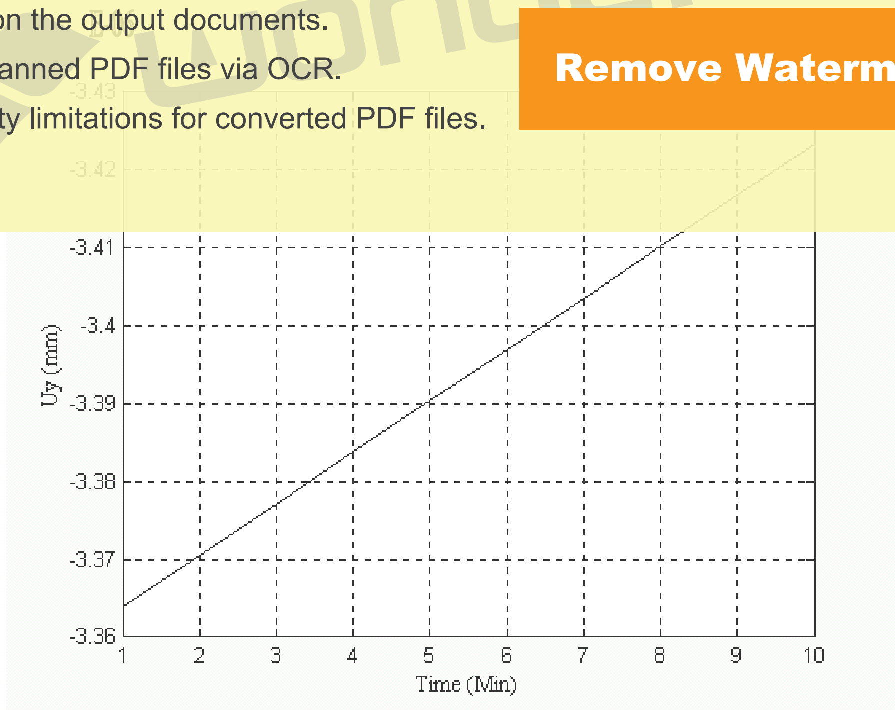


Figure 5.17 e: *Vertical displacement for lower edge at a/b*=0.75

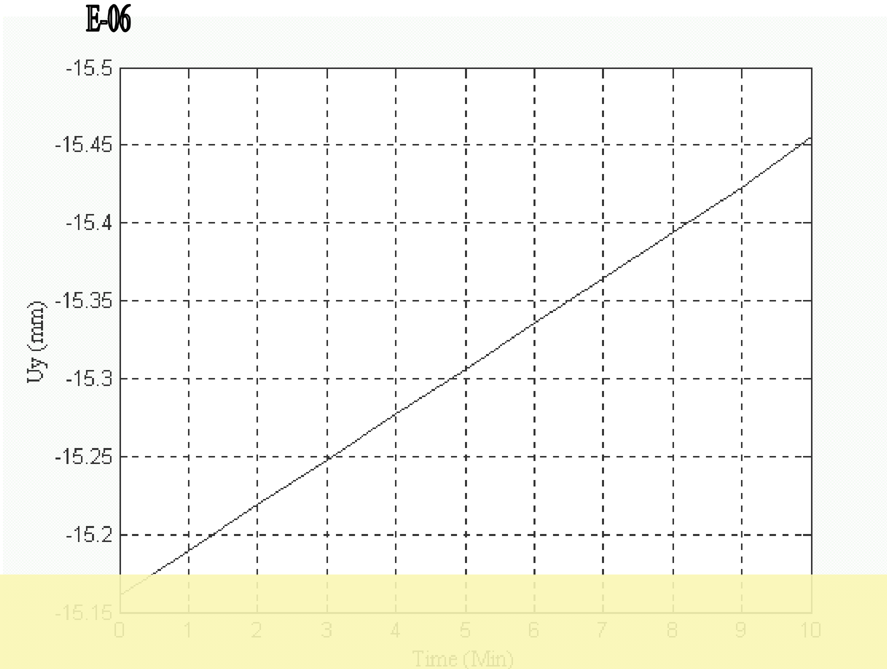


Figure 5.17 f: *Vertical displacement for upper edge at a/b=0.7*

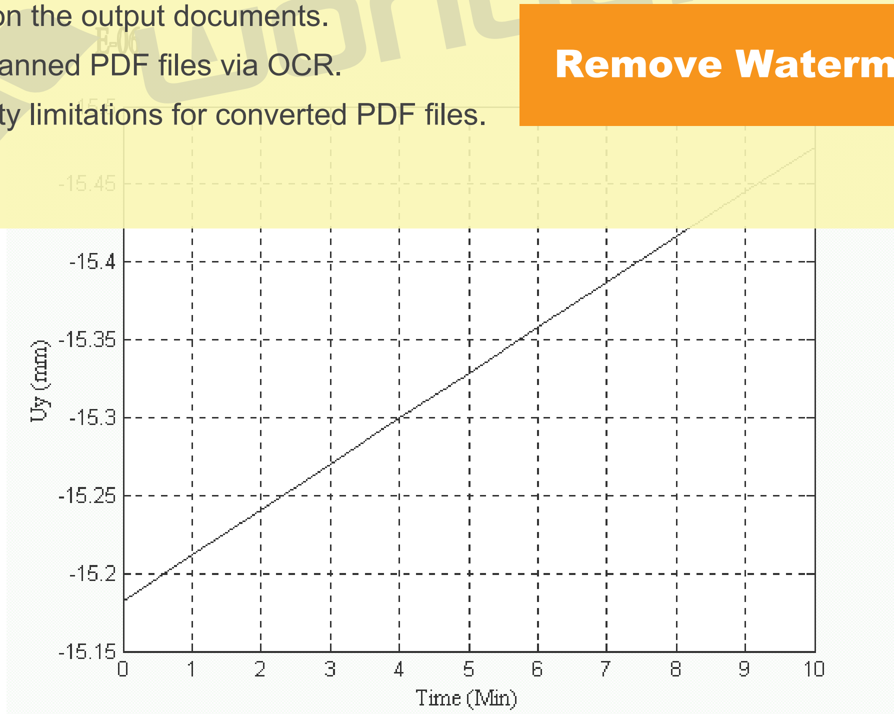


Figure 5.17 g: *Vertical displacement for upper edge at a/b*=1.00

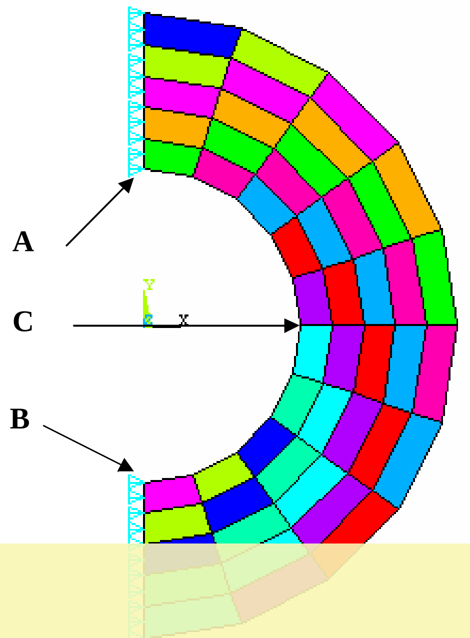


Figure 5.18: Geometry and finite element meshing

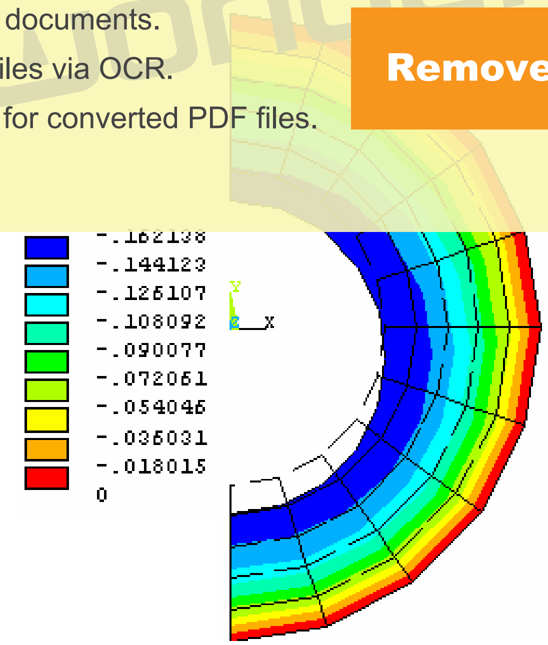


Figure 5.19: *Deformation shape and vertical displacement values*

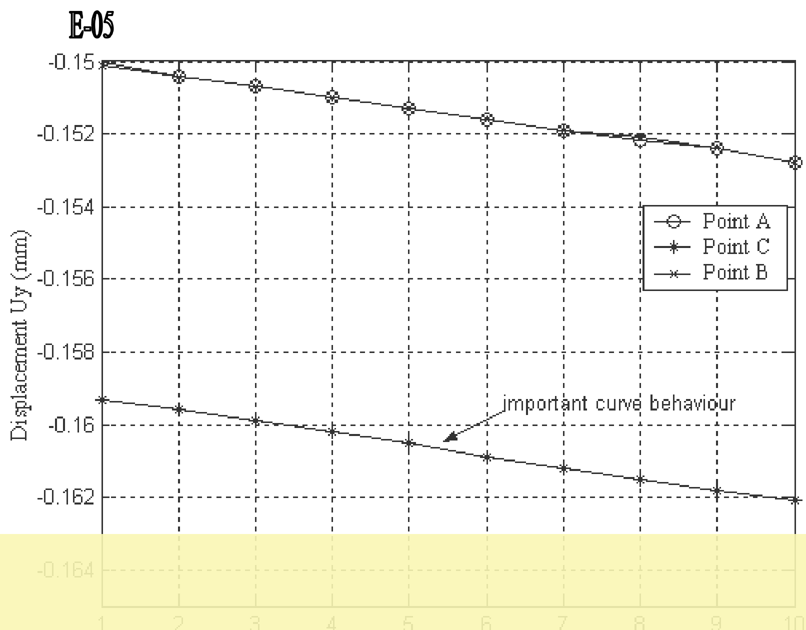


Figure 5.20: Vertical displacement vs. tim

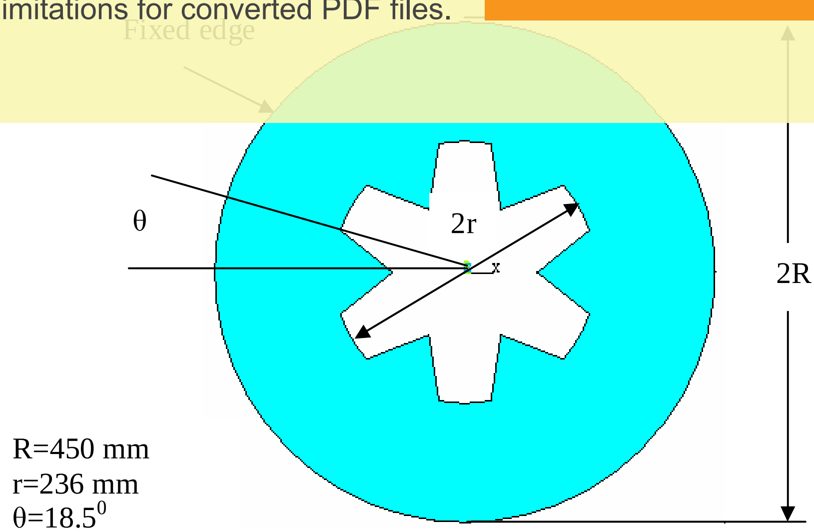


Figure 5.21:*Main dimension and geometry of the solid propellant grain*

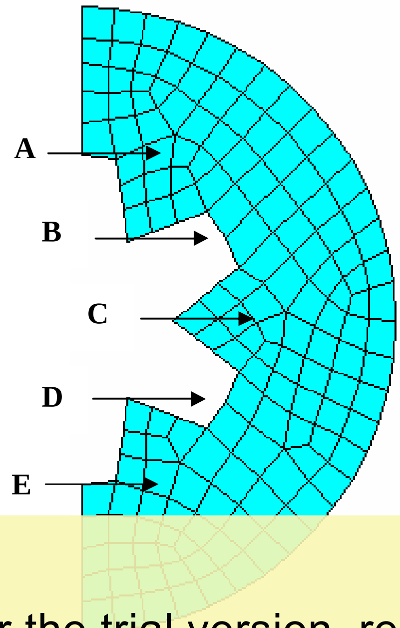


Figure 5.22: *Finite element for one half section of grained geometry*

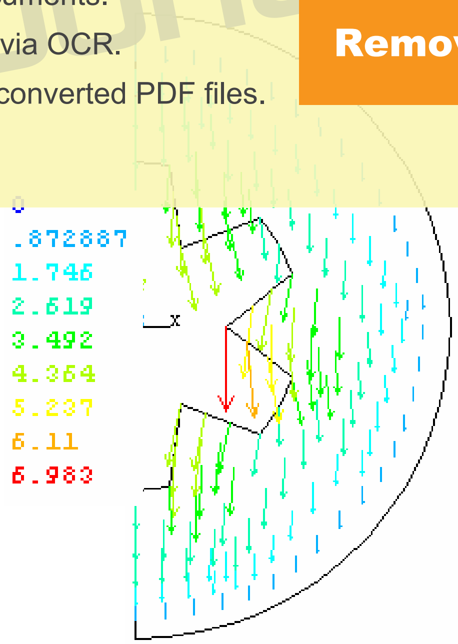


Figure 5.23: *Deformation shape vector in Y direction*

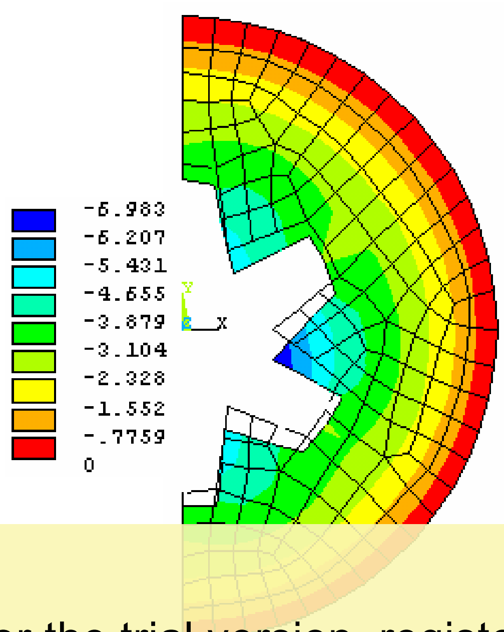


Figure 5.24: Deformation shape for the vertical displacement

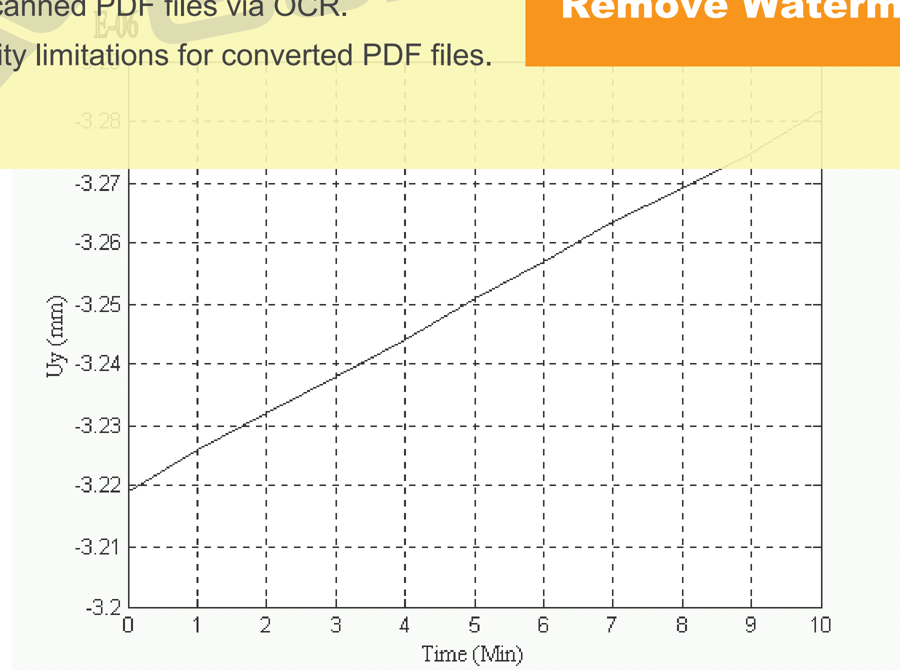


***Figure 5.25a:** Vertical displacement of Point A*

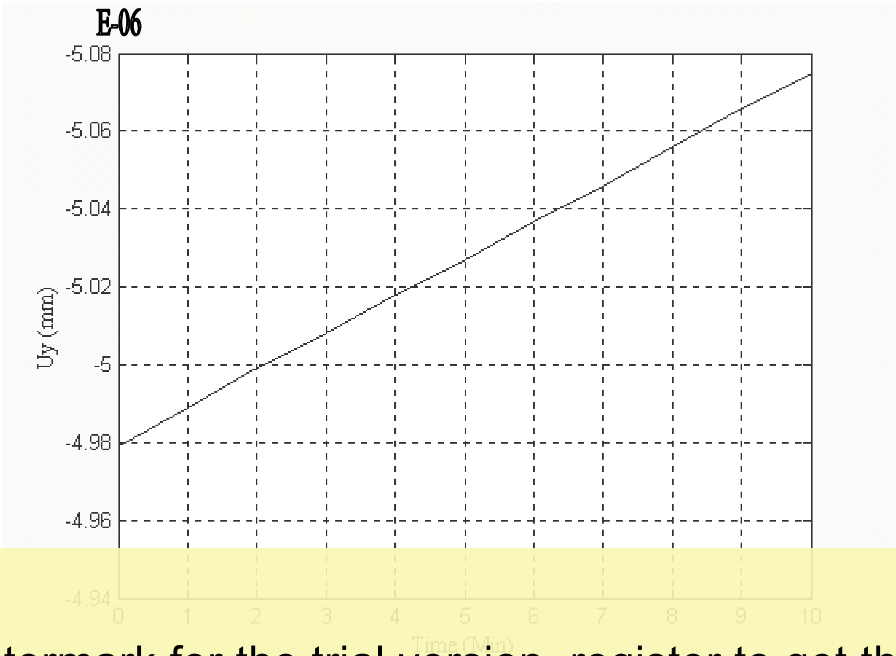


Figure 5.25 b: *Vertical displacement of Point B*

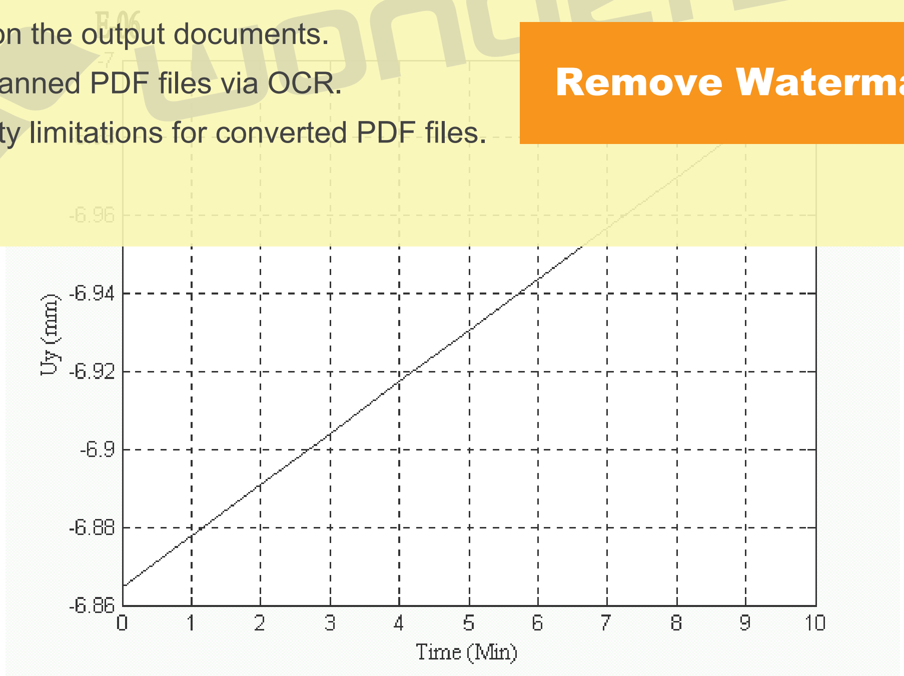
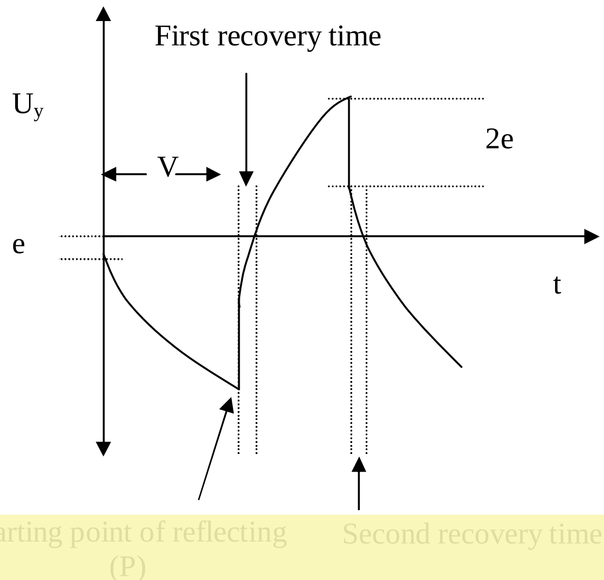


Figure 5.25 c: *Vertical displacement of Point C*



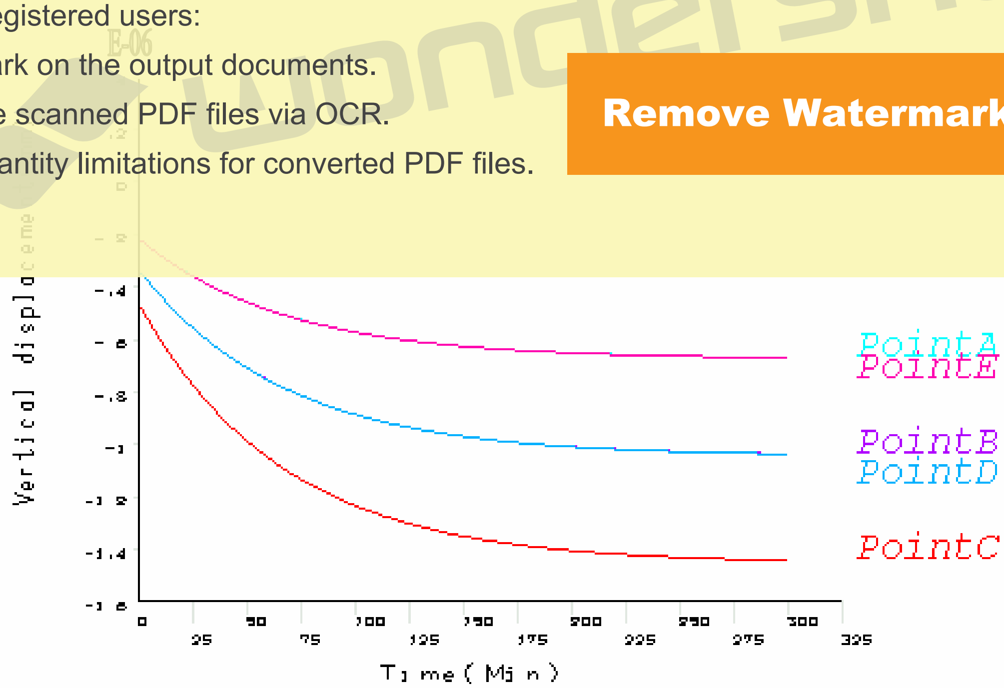


Figure 5.27: *Vertical displacement of point A B C D E*

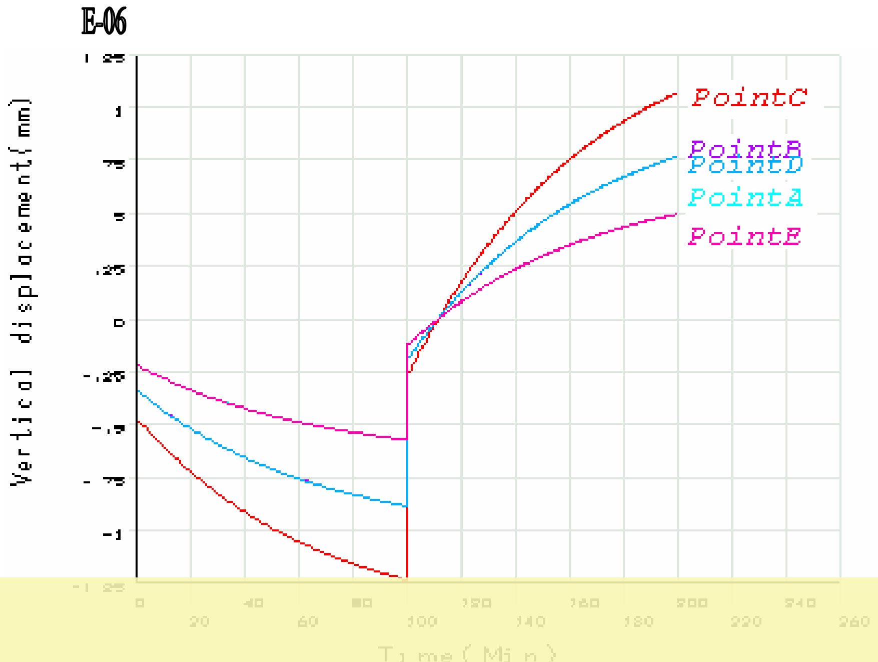


Figure 5.28: Reflecting load by 180°

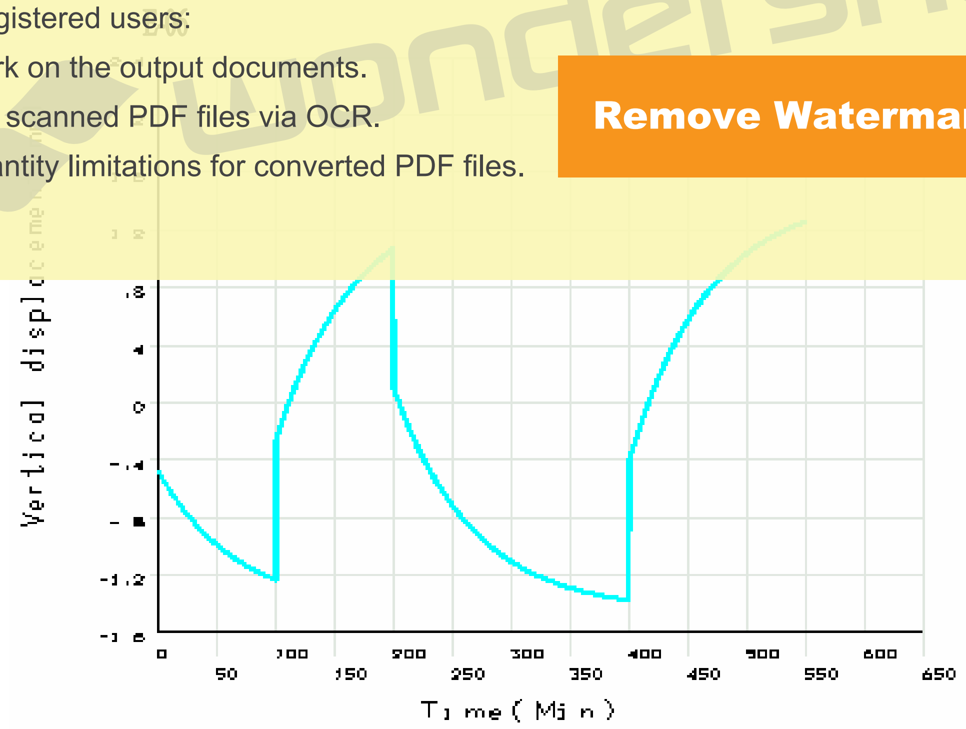


Figure 5.29: *Reflecting load for three reflecting steps of point C*

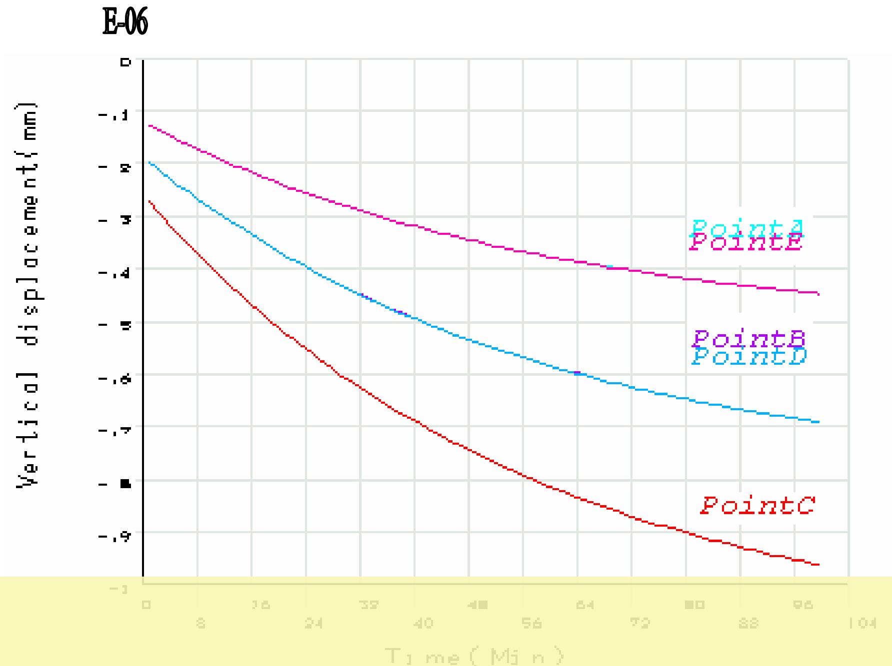


Figure 5.30: Vertical displacement of point A B C D E (mm)

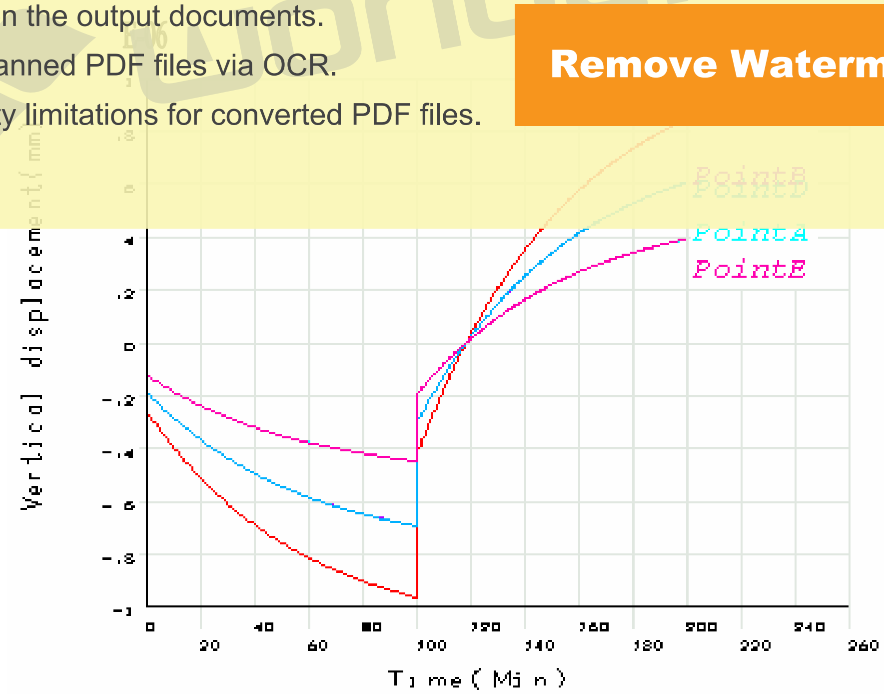


***Figure 5.31:** Reflecting load by 180^0*

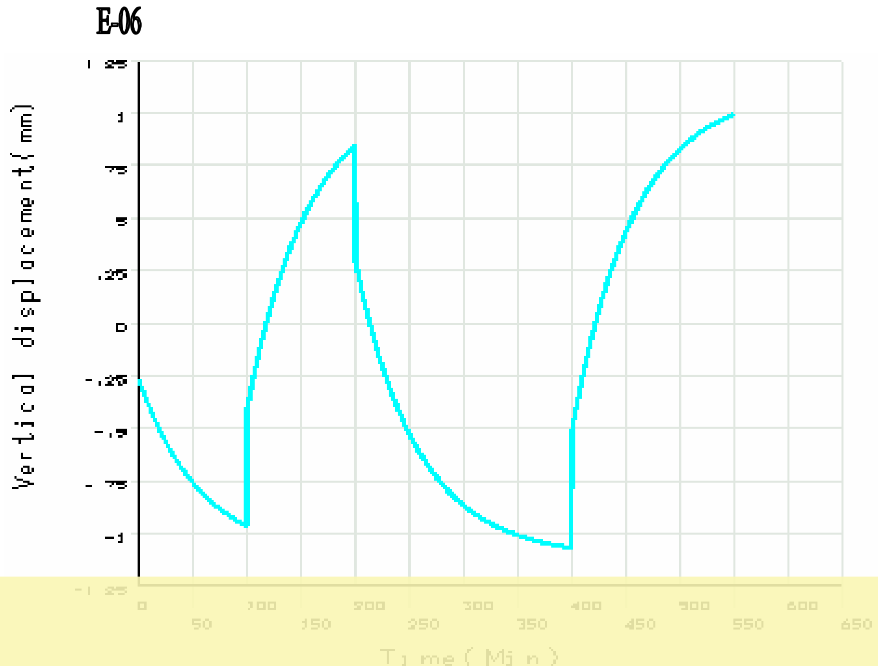


Figure 5.32: Reflecting load for three reflecting steps of point C

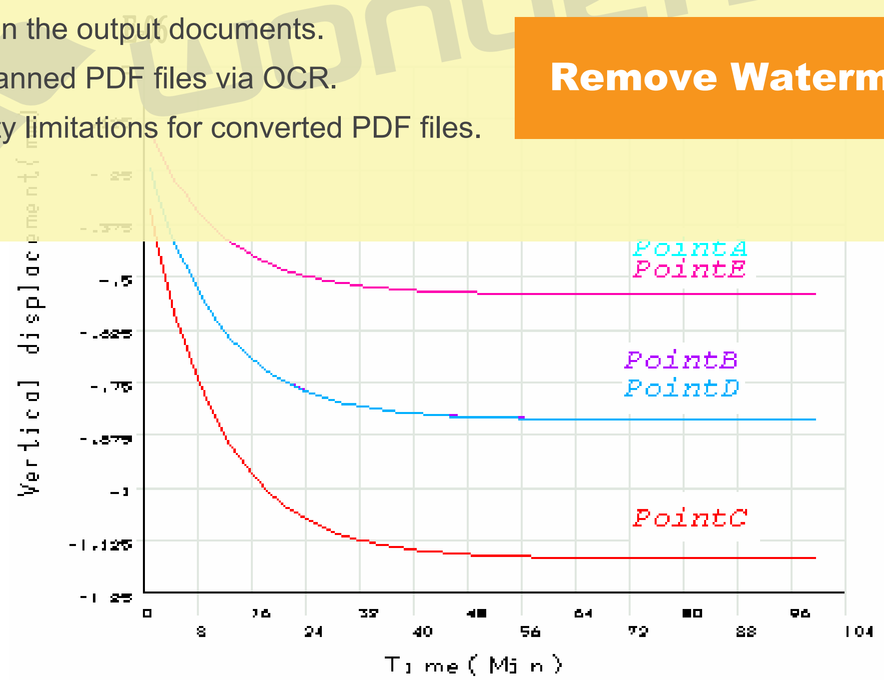


***Figure 5.33:** Vertical displacement of point A B C D E (mm)*

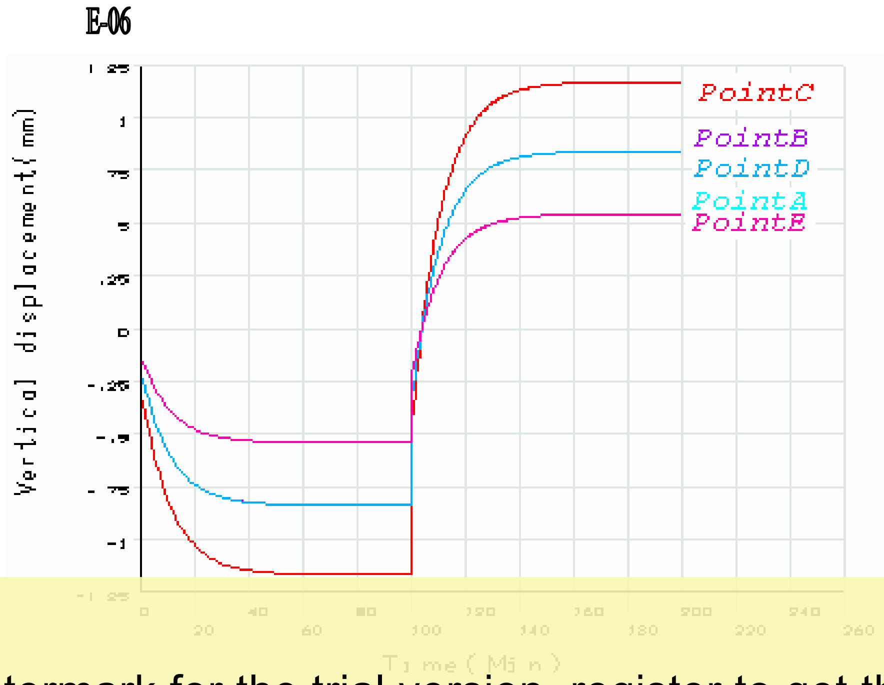


Figure 5.34: Reflecting load by 180^0

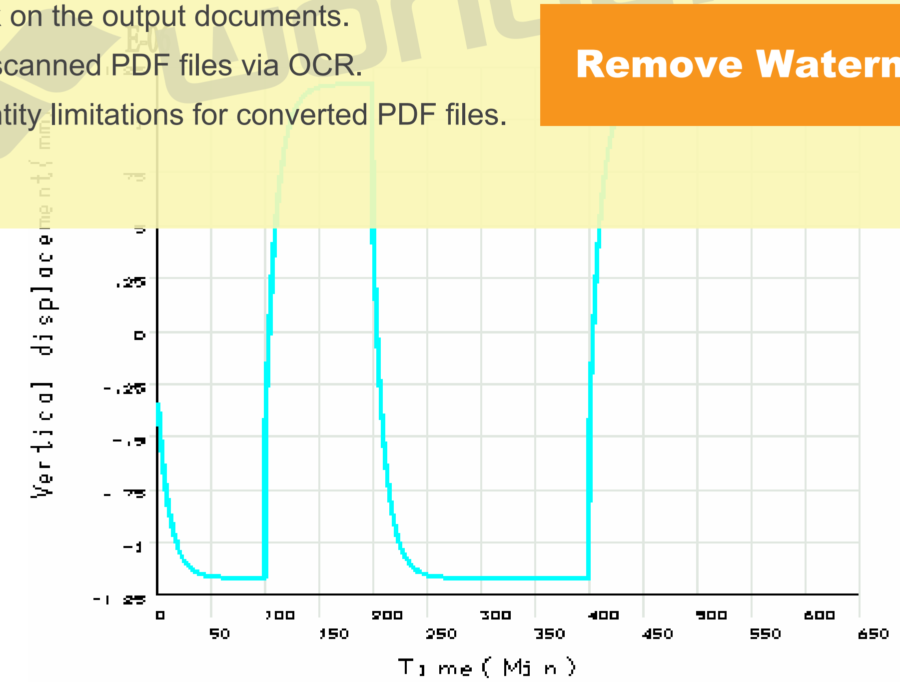


Figure 5.35: *Reflecting load for three reflecting steps of point C*

Chapter Six

Conclusions and Suggestion for Future work

6.1 Conclusion

1-Analytical solution for linear viscoelastic axisymmytric bodies has been derived using the elastic viscoelastic corresponding principle, which gives a good solution for this material.

2-The vertical displacement component U_y in viscoelastic bodies under self weight depends on the behavior of shear relaxation modulus G(t).

3-Results of Finite Element method illustrate that viscoelastic material have an effective term which gives a big difference in the material behavior which is the memory load term .

4-Analytical and numerical solutions for simple study cases gave approximately the same results.

5- the experimental work that was carried o

(t`) the behavior of G(t) dropped suddenly

the device to the small deviation of the reading .

6-the required time to get a minimum deformation is equal for different storage times of the same material because of the linearity behavior of the stress _strain considered in this study.

6.2 Suggestion for Future work

1-The non linear solution for the tested material may be considered to give accurate investigation.

2-Extent the present software to 3-D modeling to get an accurate modeling, geometrical and boundary conditions.

3-It is very useful to employ the boundary element method BEM which is suitable for infinite domain.

4-Body forces loading such as gravity load, centrifugal force and thermal loading should be solved in other numerical methods such as boundary Element.

References

1. William N.F ,James S.L "Creep and relaxation of non linear viscoelastic material"(1976).
2. William M.L,"structure analysis of viscoelastic materials"(1976).
3. Sanal K.V.R. & Sunil G." Viscoelastic Characterization of Solid Propellant Using Tubular Test ", Second Int.High Energy Materials Conference and Exihibit, December, (1998).
4. Chen J.T.&Leu S.Y.,"Finite Element Analysis, Design and Experiment on Solid Propellant Motors With A Sees Reliever ",J. Finite element In Analysis an Design,Vol.29,(1998).
5. Alfery T., Non-homogeneous Stresses In Viscoelastic Media ,[illegible] Appl.Math.2,113-119 (1944).
6. Tsien H.S.,"A Generalized Of Alfery's Theorem For Viscoelastic [illegible] Quart.Appl.Math.2,113-119 [illegible]
7. [illegible] Williams M.L., Structure Analysis Of Viscoelastic Materis .AIAA J.,Vol.2,No.5,May, (1964).
8. Boyed A.B.,Burkes, &Medford J.E.,"Grain Design and Development Problems For Very Large Rocket Motors",Solid Propellant Rocket Reseach.An American Rocket Society Seres,Vol.11,(1960).
9. Freudenal A. M &Henry L.A.,"On "Poisson's Ratio" In Linear Viscoelastic propellant", Solid Propellant Rocket Research.An American Rocket Society Series,Vol.11,(1960).
10. Williams M.L.,"Mechanical Properties and the design Of solid Popellant Roket Motors,Solid propellant Rocket Research. An American Rocket Society Series,Vol.11,(1960).

11.Vandenkerckhov J.& Lampens G.,"Stress and Strain Aalysis Of Cylinderical Case-Bonded Grains", Solid propellant Rocket Research. An American Rocket Society Series,Vol.11,(1960).

12.Au N.N"A Method Of Strength Analysis Of Solid Propellant Rocket Grain", Solid propellant Rocket Research. An American Rocket Society Series,Vol.11,(1960).

13.Imad Ahmed Hussain,Mohammed Qasim Abdullah,Laith Abed Sabrie,"Analytical Solution Of Transient and Steady Thermal stress In Linear Viscoelastic Solids",1st National Conference Of Mechanical Engineering,Kuffa University(2000).

14.Muki R.,Sternberg E.,"On Transient Thermal Stresses In Viscoelastic Materials With Temperature-Dependent Properties",J.Applied Mechanics,June,(1961).

15.Roger T.G.,Lee T.H.,"The Cylinder Problem In Viscoelastic Stress Analysis",Quart.Appl.Math 117-1[illegible]

16.Sun W.P.& Richard K.Y.,"Analysis [illegible] With Transient Temperatures",J. Mechanical Engineering,February,223-231,(1998).

17.Roger," Layered Viscoelastic Media With Transient Temperatures Analysis",September,(2000).

18.Naylor D.J.,"Stress In Nearly Incompressible Materials By Finite Element With ApplicationToThe Calculation Of Excess pore Pressure".Int.J.Num.Meth.Engng 8,443-460(1974).

19.Zienkiewiez O.C.,Watson M.,& King I.P.,"A Numerical Method Of Viscoelastic Stress Analysis",Int.J.Mech.Sci.Vol.10,(1968).

20.CarpenterW.C,"ViscoelascStressAnalysis".Int.J.Num.Meth.Engng.4,357-366(1972).

21. Srinatha H.R & Lewise R.W.,"A Finite Method For ThermoviscoelasticAnalysisOfPlaneProblems",J.CMAE,No.25,Holland ,(1981).

22. Yadagiri S.,Redy C.P."Viscoelastic Analysis In nearly Incompressible Solids"Comp.Stuct.5,817-825,(1985).

23. Jones I.W.,Louis E.P.,"A Linear Thermoviscoelastic Material Model For Solid Rocket Motor Structure Analysis".Comp.Stuct.21,No.1/2,253-243(1985).

24. Imad Ahmed Hussain ,M.Rijab,LaithAbed Sabrie."Finite Element Of Linear Viscoelastic Solids".First National Conference Of Mechanical Engineering,Kuffa University.(2000).

25. Henrikson,M.,"Nonlinear Viscoelastic Stress Analysis (A Finite Element Approach",Comp.Stuct.18,133-139(1984).

26. Bland D. R. "The theory of linear viscoelasticity". Pergamon Press (1960).

27. Hearn S.J.,"Mechanics of Materials", [illegible]

28. Christensen R. M.,"Theory of Viscoelasticity"2nd Edition.

29. Flugge W.,"Viscolasticity".Second Edition,Springer-verlag (1975).

30. ANSYS Manual helps Version 5.4.

31. Za'ibel A.H.,"Creep,Recovery and Stress Relaxation of Plastics Under combined load systems",Ph.D Thesis,(1999).

32. Timoshenko S.P. &Gooder J.N.,"theory of Elasticity",M_cGrraw-Hill(1970).

33. Lee E.H. "Stress analysis of viscoelastic bodies" Quart .apple. math. 13, 183-190(1955).

34. Zienkiewiez O.C.,Taylor R.L.,"The Finite Element method"(1980).

35. Chandrupalta T.R.,Bele gundu A.D.,"Introduction to the finite element in engineering"prentice,Hall of India(1996).

36.Hinton E.Owen D.R.J.,"Finite element programming",Academic press Inc.London (1977).

37.Cooks R.D.,"concepts and applications of finite element analysis", M_cGrraw-Hill (1974).

38. (1983).

39.Hinton E.,Campbell J.S.,"local and global smoothing of discontinuous F.Efunctionusingaleastsquaremethod",Int.J.Num.Meth.Engineering,8,461-480 (1974).

40.John A.Schey "Introduction to manufacturing processes" Second edition.(1987).

Abstract

This work is interest with the viscoelastic bodies that has been deflected by the self weighting and stored for variant storage time to finding the time required to restore these bodies to it's original shape. Some shape has been study such as rectangular, cylindrical and grain shape, therefore our study focused on the grained shape geometry by taking a different shear relaxation modulus that has been got from the experimental tests for different types of polymer.

There are many techniques to minimize the deformation to minimum values such as fixing the inner surface or rotating the body by 180^{o} .and the rotating technique is considered as efficient technique.

Finally it is found that the time required to restore the deflected body to it's original position is different from one types of polymer to another according to the shear relaxation modulus which is used in it, but is the same for the same type of polymer which are used in th [illegible] time of storage.

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Chapter Three

Finite element formulation for viscoelastic analysis

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Nomenclature

1. Matrices and Vectors

Symbol	Definition
[A]	Stress coefficient matrix
[B]	Strain displacement matrix
$[D_0]$	Elastic matrix
$[D_1]$	Viscoelastic matrix
$[K_0]$	Elastic stiffness matrix
$[K_1]$	Viscoelastic stiffness matrix
$\{A_2\}$	Stress coefficient matrix
{I}	Identity vector
{M(t)}	Memory load vector
{X}	Coordinate vector
$\{\delta\}$	Nodal displacement vector
{F}	Nodal force vector
$\{\sigma\}$	Stress [illegible]
$\{\varepsilon\}$	Strain [illegible]

2. Simple variables

Symbol	Definition	Units
a_T , A_T	WLF shift factor	-------
C_1	WLF equation constant	-------
C_2	WLF equation constant	-------
E	Elastic modulus	N/mm^2
G(t)	Shear relaxation modulus	N/mm^2
G(0)	Initial shear modulus	N/mm^2
G_i	Shear relaxation coefficients	N/mm^2
G_c	Calculated shear relaxation coefficients	N/mm^2
G_e	Experimental shear relaxation coefficients	N/mm^2

$J(t)$	Creep compliance	mm^2/N
$J(0)$	Initial creep compliance	mm^2/N
J_i	Creep compliance coefficients	mm^2/N
k	Bulk modulus	N/mm^2
P	Pressure	N/mm^2
Q , P	Material constants	--------
S	Laplace operator	--------
T	Current temperature	^0C
T_g	Glassy temperature	^0C
t , t_K	Current time	min
$t^/$	Relaxation time	min
$t^{//}$	Retardation time	min
U_x U_y U_z	Displacement in x ,y and z axis	mm
U	Strain energy per unit volume	N/mm^3
R	Residual error	[illegible]
W	Work done on the body by external forces	N.mm
ξ , ζ	Natur[illegible] [illegible]y linear c[illegible]	[illegible]
[illegible]	Reduced tim[illegible]	[illegible]
[illegible]	Damping coeffic[illegible]	[illegible]
χ	Total potential energy	N.mm
£	Convolution integral symbol	------
N	Poisson ratio	------
σ	Normal stress	N/mm^2
σ_0	Initial stress	N/mm^2
σ^e	Elastic stress	N/mm^2
σ^v	Viscose stress	N/mm^2
$\sigma_I , \sigma_{\Pi} , \sigma_{III} , \sigma_{\Pi\Pi}$	Stress at gauss points	N/mm^2
Ρ	Mass density	Kg/mm^2
α	Linear thermal expansion	$mm/mm.^0C$
ε	Normal strain	------
ε^e	Elastic strain	------
ε^v	Viscose strain	------
δ	Displacement	mm

τ	Past time	Min
ΔT	Temperature difference	^0C
Δt	Time difference	min
Ψ	Time – dependent relaxation function	-------
Φ	Time – dependent creep function	