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A Hybrid Approach for Solving Fuzzy Differential Equation of Second Order

A Thesis

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By

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بسم الله الرحمن الرحيم

يَـٰٓأَيُّهَا ٱلَّذِينَ ءَامَنُوٓاْ إِذَا قِيلَ لَكُمۡ تَفَسَّحُواْ فِي ٱلۡمَجَٰلِسِ
فَٱفۡسَحُواْ يَفۡسَحِ ٱللَّهُ لَكُمۡۖ وَإِذَا قِيلَ ٱنشُزُواْ فَٱنشُزُواْ يَرۡفَعِ
ٱللَّهُ ٱلَّذِينَ ءَامَنُواْ مِنكُمۡ وَٱلَّذِينَ أُوتُواْ ٱلۡعِلۡمَ دَرَجَٰتٖۚ وَٱللَّهُ
بِمَا تَعۡمَلُونَ خَبِيرٞ ١١

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Abstract

In this thesis, a hybrid approach is presented by combining fuzzy Laplace transformation method and fuzzy variational iteration methods which is employed to obtain approximate solutions of linear and nonlinear fuzzy differential equations with fuzzy initial and boundary values. Then our approach is implemented in which two approaches have been constructed according to the formula of the Lagrange multiplier obtaining the lower, upper and center solutions. The experimental results which are obtained shows every high accuracy in comparison with the exact results with less number of iterations other numerical or approximate method.

LIST OF SYMBOLS

A_r	r-Cut (r-level) set of fuzzy set $\tilde{A}, r \in [0,1]$.
$\tilde{A}, \tilde{B}, \cdots$	Fuzzy sets.
$S(\tilde{A})$, Supp($\tilde{A}$)	The support of a fuzzy set $\tilde{A}$.
$hgt(\tilde{A})$	The height of fuzzy set $\tilde{A}$.
$\tilde{x}, \tilde{y}, \cdots$	Fuzzy numbers.
$\mu_{\tilde{A}}(x)$	The membership function of the fuzzy set $\tilde{A}$.
R	Set of real numbers.
R_F	Set of all fuzzy numbers.
FDE	Fuzzy differential equation.
FLT	Fuzzy Laplace Transform.
FIVP	Fuzzy initial value problem.
VIM	Variational iteration method.
FBVP	Fuzzy boundary value problem.
λ	Lagrange multiplier.

CONTENTS

Introduction

Differential equations have been widely explored in many fields, from applications in physics, engineering, economics, and biology to theoretical mathematical developments A much newer theory, fuzzy sets theory, created to model subjective concepts whose boundaries are non-sharp, has also been explored in various fields due to its great applicability and functionality. As soon as the idea of a function with fuzzy values was born, it raised the idea of a FDE. Since then, researches given rise to different theories of FDEs. Zadeh introduced the notion of inclusion, union, intersection, complement, relation, and convexity of the Fuzzy set. The original definition of fuzzy sets is to consider a class of objects with a continuum grade of membership function which assigns to each object a grade of membership value between zero and one(Zadeh, 1965).

Chen (Chen et al., 2008) showed that under a new structure and certain conditions the two-point boundary value problem was equivalent to a fuzzy integral equation. He proved the existence of solutions to the two-point boundary value problem. He showed that this was an amendment to results of (Lakshmikantham, Murty, and Turne,r 2001) and (O'Regan, Lakshmikantham, and J. Nieto, 2003). The two-point boundary value problem for a second order FDE by using a generalized differentiability concept was introduced by (Khastan and Nieto, 2010). They presented a new concept of solutions and, utilizing the generalized differentiability.(Ahmad et al., 2014) proposed a natural way to model dynamic systems under uncertainty. A FBVPs was used and related uncertain systems. The FLT was used to find the solution of two-point boundary value under generalized Hukuhara differentiability. The FLT for the n^{th} derivative of a fuzzy valued function named as n^{th} derivative theorem were generalized by (Ahmad, Farooq, and Abdullah, 2014). The analytical solution method for the solution of an n^{th} order fuzzy initial value problem was introduced. They showed that this method was a simple approach toward the solution of the n^{th} order FIVB by the n^{th} generalized form. They solved any order of FIVP by their method.

In recent years approximate-analytical methods such as Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM), and Adomian Decomposition Method (ADM) have been used to solve FIVB which inclose ODE. The use of ADM for solving first order linear and nonlinear FIVP was introduced by (Babolian, Sadeghi, and Javadi, 2004), (Adomian, 1988) and (Allahviranloo, Khezerloo, and Mohammadzaki, 2008) While the variational iteration method (VIM) was developed by (He, 1999) ,(He, 2006) for solving nonlinear problems, it has been successful applied on initial and boundary value problems. The fuzzy variational iteration method (FVIM) has been applied by (Jafari, Saeidy, and Baleanu, 2012)to find solutions of n^{th} order linear FDE. The solution obtained by the VIM was an infinite power series, which, with appropriate initial condition, can be expressed in a closed form, i.e. the exact solution. The results presented in this contribution showed that the variational iteration method was a powerful mathematical tool to solving n^{th} order fuzzy differential equations. For linear FDE, its exact solution can be obtained by the VIM due to the fact that the Lagrange multiplier can be exactly identified.

The objective of this work is to solve the n^{th} order FDE via a combination of Laplace transforms and variational iteration method. In which the solution of the n^{th} order FDE were obtained by using fuzzy initial conditions and fuzzy boundary conditions.

The present work consists of four chapters, as well as, an introduction to the work. In chapter one, some fundamental concepts of fuzzy set theory are presented. While chapter two, the existence and uniqueness theorem of the fuzzy solution employed. Two solutions methods for fuzzy Laplace transform and the fuzzy variational iteration method and variational problem are considered are studied. Chapter three, the hybrid method combined from fuzzy Laplace transform and the fuzzy variational iteration method are used to solve the fuzzy differential equations by using the fuzzy initial conditions. Finally, the hybrid method for solving two-point fuzzy differential equations with fuzzy boundary conditions is shown in chapter four.

Chapter One

Fuzzy Basic Concepts

In this chapter, the basic concepts, definitions and theorems related to fuzzy set theory will be introduced. These concepts includes for fuzzy set theory, the r-level sets, the extension principle, fuzzy relation and fuzzy functions.

1.1 Fuzzy Set Theory:

Fuzzy set theory is a generalization of abstract set theory; it has a wider scope of applicability than abstract set theory for solving problem that involve to some degree subjective evaluation (Kandel, 1986).

Definition 1.1 (Dubois, 1980):

The universal set U is a classical set of objects and whose generic elements are denoted by x. The membership in a classical subset A of U is often viewed as a characteristic function $\mu_A(x)$ from U into $\{0,1\}$, such that:

$$\mu_A(x) = \begin{cases} 1 \ if \ x \in A \\ 0 \ if \ x \notin A \end{cases}$$

$\{0,1\}$ is called a valuation set. If the valuation set is allowed to be the real interval $[0,1]$, then it is called fuzzy set (which is denoted by $\tilde{A}$), and $\mu_A(x)$ is the grade of membership of x in $\tilde{A}$.

Summary of the fundamental and necessary concepts in fuzzy set theory are given next (Dubois, 1980).

1. The crossover point of a fuzzy set $\tilde{A}$ is that point in U whose grade of membership in $\tilde{A}$ is 0.5

2. $\tilde{A} = \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$, $\forall x \in U$.
3. $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$, $\forall x \in U$.
4. $\tilde{A}^c$ is the complement of $\tilde{A}$ with membership function

   $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)$, $\forall x \in U$
5. The empty set $\tilde{\emptyset}$ and the universal set U, have the membership functions $\mu_{\tilde{A}}(x) = 0$ and $\mu_{\tilde{A}}(x) = 1$, respectively, $\forall x \in U$.
6. $\tilde{C} = \tilde{A} \cap \tilde{B}$ is a fuzzy set with membership function:

   $\mu_{\tilde{C}}(x) = \text{Min}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$, $\forall x \in U$
7. $\tilde{D} = \tilde{A} \cup \tilde{B}$ is a fuzzy set with membership function:

   $\mu_{\tilde{D}}(x) = \text{Max}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$, $\forall x \in U$

Remark 1.1 (Pal and Dutta-Majumder, 1986):

It is important to notice that only the law of contradiction is $\tilde{A} \cup \tilde{A}^c = U$ and the law of excluded middle $\tilde{A} \cap \tilde{A}^c = \emptyset$ are broken for fuzzy sets. since $\tilde{A} \cup \tilde{A}^c \neq U$ and $\tilde{A} \cap \tilde{A}^c \neq \emptyset$. Indeed; for all $x \in U$, such that $\mu_{\tilde{A}}(x) = r$, $0 < r < 1$, then

$$\mu_{\tilde{A} \cup \tilde{A}^c}(x) = \max\{r, 1 - r\} \neq 1$$
$$\mu_{\tilde{A} \cap \tilde{A}^c}(x) = \min\{r, 1 - r\} \neq 0 \ .$$

1.1.1 The Extension Principle:

An important concept in fuzzy set theory that may be used to generalize crisp mathematical concepts to fuzzy sets is the extension principle. In the elementary form, Zadeh already implied this principle in his first contribution in 1965 (Dubois, 1980), (Zadeh, 1965).

Definition 1.2 (Dubois, 1980):

Let X be the Cartesian product of universes $X_1, X_2, \cdots X_n$ and $\tilde{A}_1, \tilde{A}_2, \cdots \tilde{A}_n$ be n-fuzzy subsets of $X_1, X_2, \cdots X_n$, respectively, f is a mapping from X into a universe Y, such that$y = f(x_1, x_2, \cdots x_n)$. Then the extension principle allows us to define a fuzzy set $\tilde{B}$ in Y by

$$\tilde{B} = \{(y, \mu_{\tilde{B}}) \mid y = f(x_1, x_2, \cdots x_n), x_1, x_2, \cdots x_n \in X\}$$

Where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup \min\{\mu_{\tilde{A}_1}(x_1), \mu_{\tilde{A}_2}(x_2), \cdots \mu_{\tilde{A}_n}(x_n)\}, & if\ f^{-1}(y) \neq \emptyset, (x_1, x_2, \cdots x_n) \in f^{-1}(y) \\ 0, & otherwise \end{cases}$$

Where f^{-1} is the inverse image of f.

1.1.2 r-Cut Sets:

This section introduces some basic and most important properties of an ordinary set that can be derived from certain fuzzy set. These sets are called the r-level sets (or r –cuts), which corresponds to any fuzzy set. The r-level sets are those sets which collect between fuzzy sets and ordinary sets, that can be used to prove most of the results that are satisfied in ordinary sets are also satisfied here to fuzzy sets and vise versa, i.e., there are also another approach in which the classical sets and fuzzy sets are connected to each other (Seikkala, 1987).

Definition 1.3 (Seikkala, 1987):

The r-level (or r –cut) set of fuzzy set $\tilde{A}$, labeled by A_r, is the crisp set of all x in U such that $\mu_{\tilde{A}}(x) \geq r$,i.e.

$$A_r = \{\, x \in U \mid \mu_{\tilde{A}}(x) \geq r\,\}, r \in [0,1]$$

One can notice that an r-level set discards those points of U whose membership value are less than r. Also, if the equality is dropped in the definition of A_r then it is called a strong r-level set and is denoted by A_{r^+} or A_{r^-}.

The following properties are satisfied for all $r \in [0,1]$,

1. $\tilde{A}$ is a fuzzy set , $r_1, r_2 \in [0,1]$, and $r_1 \le r_2$, then $A_{r_1} \supseteq A_{r_2}$.
2. $(\tilde{A} \cup \tilde{B})_r = A_r \cup B_r$.
3. $(\tilde{A} \cap \tilde{B})_r = A_r \cap B_r$.
4. $\tilde{A} \subseteq \tilde{B}$ gives $A_r \subseteq B_r$.
5. $\tilde{A} = \tilde{B}$ if and only if $A_r = B_r, \forall\, r \in [0,1]$.

Remark (1.2) (Dubois, 1980):

1. The support of $\tilde{A}$ is the crisp set (or non-fuzzy set) of all $x \in U$, such that

   $\mu_{\tilde{A}}(x) > 0$ and is denoted by $S(\tilde{A})$ or Supp($\tilde{A}$).

2. The core of $\tilde{A}$ is the crisp set (or non-fuzzy set) of all $x \in U$, such that $\mu_{\tilde{A}}(x) > 1$ and is denoted by $core(\tilde{A})$.
3. The height of fuzzy set $\tilde{A}$ (denoted by $hgt(\tilde{A})$) is the supreme of $\mu_{\tilde{A}}(x)$ over all $x \in U$. If $hgt(\tilde{A}) = 1$, then $\tilde{A}$ is normal. Otherwise, it is subnormal, and a fuzzy set may always normalized by defining the scaled membership function:

   $$\mu^*_{\tilde{A}}(x) = \frac{\mu_{\tilde{A}}(x)}{\sup \mu_{\tilde{A}}(x)}\ , \forall x \in U$$

4. A fuzzy singleton (or fuzzy point) x_r is fuzzy set whose support is a single point $x \in U$, with membership function:

   $$x_0(y) = \begin{cases} r, & if\ x = y \\ 0, & if\ x \neq y \end{cases}$$

1.1.3 Convex Fuzzy Sets:

An important property of fuzzy sets defined on R^n (for some $n \in N$) is their convexity; this property is viewed as a generalization of the classical concept of convexity of crisp sets. The definition of convexity for the fuzzy set does not necessarily mean that the membership function of a convex fuzzy set is also convex function.

Definition 1.4 (Dubois, 1980):

A fuzzy subset $\tilde{A}$ of R is convex if and only if:

$\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}, \forall \lambda \in [0,1], \forall x, y \in R.$

Remark 1.3 (Bector, Chandra, 2005):

Assume that $\tilde{A}$ is a fuzzy set, any $r \in [0,1]$, A_r is convex.

Now, for any $x, y \in A_r$ and for $\lambda \in [0,1]$:

$\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\} \geq \min\{r, r\} = r$

i.e, $(\lambda x + (1-\lambda)y \in A_r$, therefore A_r is convex for any $r \in [0,1]$, $\tilde{A}$ is convex.

1.1.4 Fuzzy Number:

A fuzzy number is a fuzzy subset of the real line that has some additional properties. The fuzzy number concept is basic for fuzzy analysis and FDE's, and a very useful tool in several applications of fuzzy sets and fuzzy logic

Definition 1.5 (Dubois, 1980) ,(Diamond and Kloeden, 2000):

A Fuzzy number $\widetilde{M}$ is a subset of the real line $\widetilde{M}: R \to [0,1]$, and satisfies the following conditions:

(i) $\widetilde{M}$ is normal, i.e. $\exists x_0 \in R$ with $\mu_{\widetilde{M}}(x_0) = 1$

(ii) $\widetilde{M}$ is a Fuzzy convex

(iii) $\widetilde{M}$ is upper semicontinuous on R, i.e, $\forall\, \varepsilon > 0\, \exists \delta > 0$ such that $\left(\mu_{\widetilde{M}}\,(x) - \mu_{\widetilde{M}}\,(x_0)\right) < \varepsilon \mid |x - x_0| < \delta$

(iv) $\widetilde{M}$ is compactly supported, i.e, $cl\{x \in R; \mu_{\widetilde{M}}\,(x) > 0\}$ is compact.

Definition 1.6 (Bector, Chandra, 2005):

A triangular Fuzzy number $\widetilde{M} = (a_1, a_2, a_3)$ is a Fuzzy number with a piecewise linear membership function$\mu_{\widetilde{M}}$ defined as follows

$$\mu_{\widetilde{M}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \le x \le a_3 \\ 0, & otherwise \end{cases} \qquad (1.1)$$

which can be represented by two end points $a_1$and a_3 and a peak point a_2.

Definition (1.7) (Bector, Chandra, 2005):

A trapezoidal Fuzzy number $\widetilde{M} = (a_1, a_2, a_3, a_4)$ is a Fuzzy number with a membership function $\mu_{\widetilde{M}}$ defined by:

$$\mu_{\widetilde{M}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \frac{a_4-x}{a_4-a_3}, & a_3 \le x \le a_4 \\ 0, & otherwise \end{cases} \qquad (1.2)$$

which can be represented by two end points $a_1$and a_4 and tolerance interval $[a_2, a_3]$.

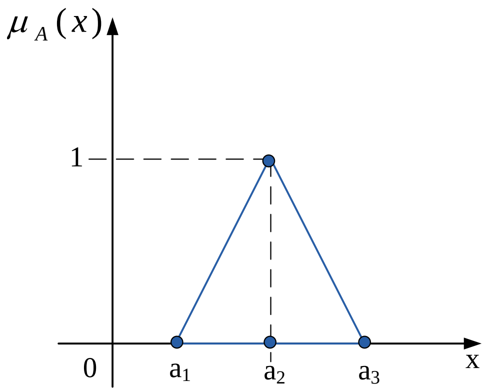

Figure (1.1): Triangular Fuzzy

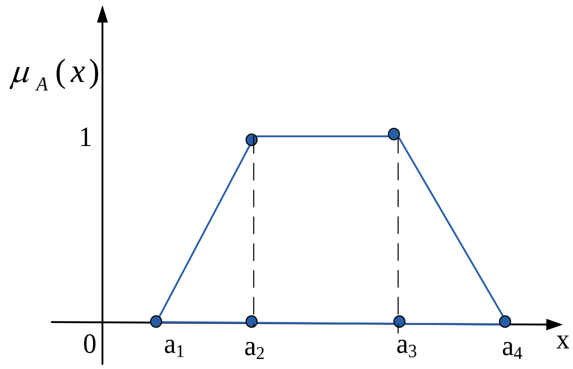

Figure (1.2): Trapezoidal Fuzzy

Remark 1.4 (Zadeh, 1965):

1. A fuzzy number $\widetilde{M}$ may be represent in terms of its r-level set, as the following closed intervals of the real line:

   $M_a = [x - \sqrt{1-r}, x + \sqrt{1-r}]$

   or

   $M_a = [rx, \frac{1}{r}x]$

   Where$x \in R$, which is called the mean value of $\widetilde{M}$ and $r \in [0,1]$. This fuzzy number may be written as $M_a = [\underline{M}, \overline{M}]$, where $\underline{M}$ refers to the greatest lower bounded of M_a, and $\overline{M}$ to the least upper bounded of M_a.
2. Fuzzy center of an arbitrary Fuzzy number $\widetilde{M} = [\underline{M}, \overline{M}]$ is defined as $M_c = \left(\frac{\underline{M}+\overline{M}}{2}\right)$.

Definition 1.8 (Fard, Borzabadi, and Heidari, 2014):

The gH-difference of two fuzzy numbers $u, v \in R_F$, is the fuzzy number w, if it exists such that

$$u \ominus_{\text{gH}} v = w \Leftrightarrow \begin{cases} (i) u = v + w \\ or\ (ii) v = u + (-1)w \end{cases} \quad (1.3)$$

If $w = u \ominus_{\mathrm{gH}} v$ exist as a fuzzy number, its level sets $[\underline{w}(r), \overline{w}(r)]$ are obtained by

$$\underline{w}(r) = \min\{\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)\}$$

and

$$\overline{w}(r) = \max\{\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)\}, \forall r \in [0,1].$$

1.1.5 Fuzzy Function on Fuzzy Sets (Dubois, 1980):

A fuzzy function is a generalization of the classical function in which a classical function f is a mapping from the domain D of definition of the function into a space S; $f(D) \subseteq S$ is called the range of f. Different features of the classical concept of a function can be rated to be fuzzy rather than crisp. Therefore, different 'degrees' of fuzzification of the classical notion of a function are imaginable:

1. There can be a crisp mapping from a fuzzy set, which carries along the fuzziness of the domain and therefore generates a fuzzy set. The image of a crisp argument would again be crisp.
2. Ordinary function can have fuzzy properties or be constrained by fuzzy constraints.

Definition 1.9 (Dubois, 1980):

A classical function $f: X \to Y$, f have fuzzy domain $\tilde{A}$ and fuzzy range $\tilde{B}$ if:

$$\mu_{\tilde{B}}\big(f(x)\big) \geq \mu_{\tilde{A}}(x), \forall x \in X$$

Definition 1.10 (Dubois, 1980):

Let X and Y be two universal and $\tilde{P}(Y)$ be the set of all fuzzy subsets of Y (power set), $\tilde{f}: X \rightarrow \tilde{P}(Y)$ is a mapping, Then $\tilde{f}$ is a fuzzy function if and only if:

$$\mu_{\check{f}(\mathrm{x})}(y) = \mu_{\check{R}}(x, y), \forall (x, y) \in X \times Y$$

where $\mu_{\check{R}}(x, y)$ is the membership function of a fuzzy relation.

Definition 1.11 (Wu, 2000):

Let R_F be a set of all fuzzy numbers, $cl(R_F)$ be a set of all closed fuzzy number and $b(R_F)$ be a set of all boundary Fuzzy numbers. Then:

(i) $f(x)$ is a fuzzy-valued function if $f: U \rightarrow R_F$

(ii) $f(x)$ is a closed-fuzzy-valued function if $f: U \rightarrow cl(R_F)$

(iii) $f(x)$ is a bounded-fuzzy-valued function if $f: U \rightarrow b(R_F)$

Definition 1.12 (Fard, Borzabadi, and Heidari, 2014):

Let $x_0 \in (a, b)$ and h be such that $x_0 + h \in (a, b)$, then the gH-derivative of a function $f: (a, b) \rightarrow R_F$ at x_0 is defined as:

$$f'_{gH}(x_0) = \lim_{h \rightarrow 0} \frac{1}{h} [f(x_0 + h) \ominus_{\mathrm{gH}} f(x_0)] \qquad (1.4)$$

f is gH-differentiable at x_0 if $f'_{gH}(x_0) \in R_F$ satisfying equation (1.4).

Definition 1.13 (Fard, Borzabadi, and Heidari, 2014):

Let $f: [a, b] \rightarrow R_F$ and $x_0 \in (a, b)$ wih $\underline{f}'(x, r)$ and $\overline{f}'(x, r)$ both differentiable at x_0 if:

- f is (i)-gH-differentiable at x_0 if:

$$\text{(i)}f'_{gH}(x) = \left[\underline{f'}(x_0, r), \overline{f}'(x_0, r)\right], \forall\, r \in [0,1]$$

- f is (ii)-gH-differentiable at x_0 if:

$$\text{(ii)}f'_{gH}(x) = \left[\overline{f}'(x_0, r), \underline{f}'(x_0, r)\right], \forall\, r \in [0,1]$$

Theorem 1.1(Fard, Borzabadi, and Heidari, 2014):

Let $f: (a, b) \to R_F$ be such that:

$f(x)_r = [\underline{f}(x, r), \overline{f}(x, r)]$

Suppose that functions $\underline{f}(x, r)$ and $\overline{f}(x, r)$ are real-value functions, differentiability with respect to x, uniformly in $r \in [0,1]$. Then the function $f(x)$ is gH-differentiable at a fixed $x \in [a, b]$ if and only if one of the following two conditions holds:

a. $\underline{f}'(x, r)$ is increasing, $\overline{f}'(x, r)$ is decreasing as function of r, and

   $\underline{f'}(x, 1) \leq \overline{f'}(x, 1)$

or

b. $\underline{f}'(x, r)$ is decreasing, $\overline{f}'(x, r)$ is increasing as function of r, and

   $\overline{f}'(x, 1) \leq \underline{f}'(x, 1)$.

Also $\forall\, r \in [0,1]$ have:

$$f'_{gH}(x) = \left[\min\{\underline{f}'(x, r), \overline{f}'(x, r)\}, \max\{\underline{f}'(x, r), \overline{f}'(x, r)\}\right]$$

Chapter Two

Fuzzy Differential Equations

In this chapter, solutions methods for fuzzy Laplace transform and the fuzzy variation iteration method which are used to build our proposed method are presented.

2.1 Existence and uniqueness Theorem of Fuzzy Differential Equation:

Assume that $f: I \times E^n \to E^n$ is level wise continuous, Where the interval $I = \{t: |t - t_0| \leq \delta \leq a\}$. Consider the fuzzy differential equation

$$\tilde{x}'(t) = f\big(t, \tilde{x}(t)\big), \qquad \tilde{x}(t_0) = \tilde{x}_0 \tag{2.1}$$

where $\tilde{x}_0 \in E^n$. We denote $J_0 = I \times B(\tilde{x}_0, b)$, where $a > 0 >, b > 0, \tilde{x}_0 \in E^n$,

$$B(\tilde{x}_0, b) = \{\tilde{x} \in E^n \mid D(\tilde{x}, \tilde{x}_0) \leq b\} \tag{2.2}$$

Definition (2.1) (Park and Han, 1999):

A mapping $\tilde{x}: I \to E^n$ is a solution to the problem (2.1) if it is level wise continuous and satisfies the integral equation:

$$\tilde{x}(t) = \tilde{x}_0 + \int_{t_0}^{t} f\big(s, \tilde{x}(s)\big)ds, \text{ foe all } t \in I \tag{2.3}$$

Theorem (2.1) (Park and Han, 1999):

Assume that:

I. A mapping $f: J_0 \to E^n$ is level wise continuous.
II. For any pair $(t, \tilde{x}),(t, \tilde{y}) \in J_0$, satisfied contraction condition, such that:

$$d([f(t, \tilde{x})]^{\propto}, [f(t, \tilde{y})]^{\propto}) \leq Ld([\tilde{x}]^{\propto}, [\tilde{y}]^{\propto}) \tag{2.4}$$

where $0 \leq L < 1$ is a given constant and for any $\propto \in [0,1]$.

Then there exists a unique solution $\tilde{x} = \tilde{x}(t)$ of problem (2.1) defined on the interval:

$|t - t_0| \leq \delta = \min\{a, b/M\}$ (2.5)

where $M = D(f(t,\tilde{x}), \tilde{0})$, $\tilde{0} \in E^n$, such that $\tilde{0}(t) = 1$, for $t = 0$ and 0 otherwise foe any $(t, \tilde{x}) \in J_0$.

Moreover, there exists a fuzzy set-valued mapping $\tilde{x}: I \to E^n$, such that $D(\tilde{x}_n(t), \tilde{x}(t)) \to 0$ on $|t - t_0| \leq \delta$, as $n \to \infty$, with $[\tilde{x}]^{\propto} = 0$, $\propto \in [0,1]$.

2.2 Basic Methods:

In this section, the basic methods which are needed to combine them to get a fuzzy Hybrid method are presented.

2.2.1 Fuzzy Laplace Transform:

Suppose that f is a fuzzy-valued function and s is a real parameter, then according to (Salahshour and Allahviranloo, 2013),(Allahviranloo and Ahmadi, 2010) Fuzzy Laplace Transform (FLT) of the function f is defined as follows

Definition (2.2) (Ahmad, Farooq, and Abdullah, 2014):

Classical Fuzzy Laplace Transform can be defined by considering the

fuzzy-valued function in which the lower and upper functions of FLT are as bellows:

$$\hat{F}(s;r) = \mathcal{L}[f(t;r) = \left[\mathcal{L}\left(\underline{f}(t;r)\right), \mathcal{L}\left(\overline{f}(t;r)\right)\right] \tag{2.6}$$

Where

$$\mathcal{L}\left[\underline{f}(t;r)\right] = \int_0^{\infty} e^{-st}\underline{f}(t;r)\,dt = \lim_{\tau\to\infty} \int_0^{\tau} e^{-st}\,\underline{f}\,(t)dt \tag{2.7}$$

and

$$\mathcal{L}[\overline{f}(t;r)] = \int_0^\infty e^{-st}\overline{f}(t;r)dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st}\,\overline{f}\,(t)dt. \qquad (2.8)$$

2.2.1.1 Fuzzy Initial Value of n^th Order (Allahviranloo and Ahmadi 2010):

In the present section, an n^th order of FIVP's under gH-differentiability is:

$$y^{(n)}(t) = f\left(t, y(t), y'(t), y''(t), \dots, y^{(n-1)}(t)\right) \qquad (2.9)$$

The initials values of equation (2.9) are:

$$\left.\begin{array}{c} y(x_0) = y_0 \\ y'(x_0) = y_0' \\ y''(x_0) = y_0'' \\ \vdots \\ y^{(n-1)}(x_0) = y_0^{n-1} \end{array}\right\} \qquad (2.10)$$

The upper and lower values for equation (2.9) is:

$$y(t) = \left(\underline{y}(t,r), \overline{y}(t,r)\right), y'(t) = \left(\underline{y}'(t,r), \overline{y}'(t,r)\right),\ y''(t) = \left(\underline{y}''(t,r), \overline{y}''(t,r)\right),$$

Continuing the process, the (n-1)^th order for upper and lower values are:
$$y^{(n-1)}(t) = \left(\underline{y}^{(n-1)}(t,r), \overline{y}^{(n-1)}(t,r)\right)$$

are fuzzy-valued function for t, where $f(t, y(t), y'(t), y''(t), \dots, y^{(n-1)}(t))$ is a continuous fuzzy-valued function

Theorem (2.2) (Ahmad, Farooq, and Abdullah 2014):

Let $f, f', \dots, f^{(n-1)}$ be continuous fuzzy-valued functions on $[0,\infty)$ and of exponential order and that f^n is piecewise continuous fuzzy-valued function on $[0,\infty)$, then:

if $f, f' \dots f^{(n-1)}$ are (i)-gH-differentiable

$$\mathcal{L}\big(f^{n}(t)\big) = s^{n}\mathcal{L}\big(f(t)\big) \ominus s^{n-1}f(0) \ominus s^{n-2}f^{'}(0) \ominus s^{n-3}f^{''}(0) \dots \ominus f^{(n-1)}(0) \tag{2.11}$$

if $f, f' \dots f^{(n-2)}$ are (i)-gH-differentiable and $f^{(n-1)}$ is (ii)-gH-differentiable

$$\mathcal{L}\left(f^{(n)}(t)\right) = \ominus f^{(n-1)}(0) \ominus (-s^{n})\mathcal{L}\big(f(t)\big) \ominus s^{n-1}f(0) \ominus s^{n-2}f'(0) \ominus \dots \ominus s^{n-(n-1)}f^{(n-2)}(0) \tag{2.12}$$

if $f, f' \dots f^{n-3}$ are (i)-gH-differentiable and $f^{(n-1)}, f^{(n-2)}$ are (ii)-gH-differentiable

$$\mathcal{L}\left(f^{(n)}(t)\right) = \ominus s^{n-(n-1)}f^{(n-2)} \ominus f^{(n-1)}(0) \ominus (-s^{n})\mathcal{L}\big(f(t)\big) \ominus s^{n-1}f(0) \ominus s^{n-2}f'(0) \dots \ominus s^{n-(n-2)}f^{(n-3)}(0) \tag{2.13}$$

Similarly: if $f, f' \dots f^{(n-1)}$ are (i)-gH-differentiable and f is (ii)-gH-differentiable, then:

$$\mathcal{L}\left(f^{(n)}(t)\right) = \ominus s^{n-1}f(0) \ominus (-s^{n})\mathcal{L}\big(f(t)\big) \ominus s^{n-2}f^{'}(0) \ominus \dots \ominus f^{(n-1)}(0) \tag{2.14}$$

The process continues until the 2^{n} system of differential equations is obtained, hence the last equation is:

$$\mathcal{L}\left(f^{(n)}(t)\right) = s^{n}\mathcal{L}\big(f(t)\big) \ominus s^{n-1}f(0) \ominus s^{n-2}f'(0) \ominus s^{n-3}f''(0) \dots \ominus f^{(n-1)}(0) \tag{2.15}$$

Under the condition $f, f', \dots, f^{n-1}$ are (ii)-gH-differentiable (Allahviranloo and Ahmadi, 2010).

2.2.1.2 Constructing Solutions of FIVP (Ahmad, Farooq, and Abdullah, 2014):

Consider the following n[th] order FIVP in general form:

$$y^{(n)}(t) = f\left(t, y(t), y'(t), \dots, y^{(n-1)}(t)\right) \tag{2.16}$$

Subject to the initial conditions $y(0) = \left(\underline{y}(0;r), \overline{y}(0;r)\right), y'(0) = \left(\underline{y}'(0;r), \overline{y}'(0;r)\right), y''(0) = \left(\underline{y}''(0;r), \overline{y}''(0;r)\right), \cdots, y^{(n-1)}(0) = \left(\underline{y}^{(n-1)}(0;r), \overline{y}^{(n-1)}(0;r)\right)$.

When using FLT to equation (2.5) and theorem (2.2) yield:

$$\mathcal{L}\left(y^{(n)}(t)\right) = \mathcal{L}(f(t, y(t), y'(t), \dots, y^{(n-1)}(t)) \tag{2.17}$$

$$s^2\mathcal{L}\big(y(t)\big) \ominus s^{n-1}y(0) \ominus s^{n-2}y'(0) \dots \ominus y^{(n-1)}(0) = \mathcal{L}(f(t, y(t), y'(t), \dots, y^{(n-1)}(t)) \tag{2.18}$$

The lower value of equation (2.18) is:

$$\mathcal{L}\left(\underline{y}(t;r)\right) = s^n\mathcal{L}(\underline{y}(t;r) - s^{n-1}\underline{y}(0;r) - s^{n-2}\underline{y}'(0;r) - \cdots - \underline{y}^{(n-1)}(0;r) = \mathcal{L}\underline{f}\left\{t, y(0;r), y'(0,r), \dots, y^{(n-1)}(0;r)\right\} \tag{2.19}$$

While, the upper value of the equation is:

$$\mathcal{L}\big(\overline{y}(t;r)\big) = s^n\mathcal{L}(\overline{y}(t;r) - s^{n-1}\overline{y}(0;r) - s^{n-2}\overline{y}'(0;r) - \cdots - \overline{y}^{(n-1)}(0;r) = \mathcal{L}\overline{f}\left\{t, y(0;r), y'(0,r), \dots, y^{(n-1)}(0;r)\right\} \tag{2.20}$$

It is assumed that A(s;r) and B(s;r) are the solution of equations (2.19) and (2.20):

Let the left hand side of equation (2.19) equals to:

$$\mathcal{L}\left(\underline{y}(t;r)\right) = A(s;r) \quad (2.21)$$

Also, it can be write the left hand side of equation (2.20) as follows:

$$\mathcal{L}\left(\overline{y}(t;r)\right) = B(s;r). \quad (2.22)$$

Using inverse Laplace transform for equations (2.21) and (2.22) yields:

$$\left(\underline{y}(t;r)\right) = \mathcal{L}^{-1}\left(A(s;r)\right) \quad (2.23)$$

and

$$\left(\overline{y}(t;r)\right) = \mathcal{L}^{-1}\left(B(s;r)\right) \quad (2.24)$$

2.2.2 Basic Concepts of Calculus of Variation:

The Subject of calculus of variation is concerned with solving extremal problem for a functional. This meant that the maximum and minimum problem for functions whose domain contains functions $Y(x)$(or $Y(x_0, \cdots x_1)$, or n-tuples of functions). The range of functional will be the real numbers R,(Douglas, 1941)

Definition (2.3) (Nucci and Arthurs, 2010):

Let Ω a set of function and R be the set of real numbers. Then the function $J\colon \Omega \rightarrow$ R is called a functional.

Setting

$v(y) = \int_{x_0}^{x_1} F\big(x, y(x), y'(x)\big)dx$

and for given $y_0, y_1 \in R$

$Y = \{y \in C^1[x_0, x_1]: y(x_0) = y_0, y(x_1) = y_1\}$

where $-\infty < x_0 < x_1 < \infty$ and F is sufficiently regular. One of the basic problems in the calculus of variation is

$$\min_{y \in Y} v(y) \qquad (2.25)$$

that is

$y \in Y: v(y_1) \leq v(y_2),\ \forall y \in Y$

All solutions of the variational problem (2.20) above satisfy the following so-called Euler-Lagrange equations:

$F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') = 0,\ \forall x \in (x_0, x_1)$

with the boundary conditions $y(x_0) = y_0, y(x_1) = y_1$.

Theorem (2.3) (Nucci and Arthurs, 2010):

A necessary condition for

$J(y) = \int_{x_0}^{x_1} F(x, y, y')dx$

with $y(x_0) = y_0$ and $y(x_1) = y_1$to have an extremum at y is that y is a solution of

$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$

with $x_0 < x < x_1$ and $F = F(x, y, y')$. This is known as the Euler-Lagrange equation

2.2.2.1 Variational Iteration Method:

Now, to illustrate the basic concept of VIM consider the following general equation:

$$\frac{d^m y}{dt^m} + Ry(t) + Ny(t) = g(t) \tag{2.26}$$

Where R, N are linear and nonlinear operators reespectively, and $g(t)$ is inhomogeneous source term. The general Lagrange multiplier method was modified to an iteration method, which is known as correction functional. The basic character of the method is to construct a correction functional for equation (2.26), which reads as

$$\tilde{y}_{n+1}(t) = \tilde{y}_n(t) + \int_0^t \lambda(s;t)\left(\frac{d^m \tilde{y}}{dt^m} + R\tilde{y}_n(s) + N\tilde{y}_n(s) - g(s)\right)ds \tag{2.27}$$

Where, λ is general Lagrange multiplier, which can be identified optimally via the variational theory, it is to be noted that λ may be constant or a function.

The subscript n denotes the n^{th} approximation, and $\tilde{y}_n$ is the restricted variation $i.e.\ \delta\tilde{y}_n = 0$.

To determine the Lagrange multiplier λ that can be identified optimally via integration by parts and using a restricted variation (Jafari and Alipoor, 2011), (Wazwaz, 2010), that a general formula for λ for the n^{th} order differential equation is:

$$\tilde{y}^{(m)} + f\left(\tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \ldots, \tilde{y}^{(m)}(t)\right) = 0 \tag{2.28}$$

Lagrange multiplier can be proved as in the form bellows:

$$\lambda(s;t) = (-1)^m \frac{(s-t)^{m-1}}{(m-1)!} \tag{2.29}$$

Take the determined of the Lagrange multiplier, the successive approximation $\tilde{y}_{n+1}$ is calculated by using any initial function $\tilde{y}_0$. Consequently, the solution obtained by taking the limit, as follows:

$$\tilde{y} = \lim_{n\to\infty} \tilde{y}_n \qquad (2.30)$$

The correction functional (2.29) give sequence of approximation, and the exact solution is obtained at the limit of the successive approximations. The solution of equation (2.27) is considered as a fixed point of the following functional under a suitable choice of the initial term of $\tilde{y}_0(t)$.

Theorem (2.4) Banach's Fixed Point (Grabiec, 1988), (Jafari, Saeidy, and Baleanu, 2011):

Assume that X is Banach space and $A: X \to X$ is a nonlinear mapping, and suppose that:

$\|A(y) - A(v)\| \leq k\|y - v\|$ and $y, v \in X$ for some constants $k < 1$.

Then A has a unique fixed point. Furthermore, the sequence $y_{n+1} = A[y_n]$, with an arbitrary choice of $y_0 \in X$, converges to the fixed point of A.

According to Theorem (2.3), for the nonlinear mapping

$$A(\tilde{y}(t)) = \tilde{y}(t) + \int_0^t \lambda(s;t)(\frac{d^m\tilde{y}}{dt^m} + R\tilde{y}(s) + N\tilde{y}(s) - g(s))\,dt \qquad (2.31)$$

A sufficient condition for convergence of the (VIM) is strict convergence of A.Furthermore the sequence (2.31) converges to the fixed point of A which is also the solution of problem (2.26).

2.2.2.2 Variation Iteration Method for n^{th} Order fuzzy differential equations:

Consider the following n^{th} order FDE:

$$\tilde{y}^{(m)} + f\left(t, \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \dots, \tilde{y}^{(m)}(t)\right) = 0 \qquad (2.32)$$

where $t \in [0,1]$, with the initial conditions :

$\tilde{y}^{(i)}(0) = \left(g_i(r), k_i(r)\right), \quad i = 0,1,2, \dots, n-1.$

Then the upper and lower functions for equation (2.32) are:

$$\left.\begin{aligned} \underline{\tilde{y}}_{n+1}(t,r) &= \underline{\tilde{y}}_n(t,r) + \int_0^t \underline{\lambda}(s;t)\left\{\frac{d^m}{ds^m}\underline{\tilde{y}}_n + f\left(t, \underline{\tilde{y}}_n(s), \underline{\tilde{y}}'_n(s), \dots, \underline{\tilde{y}}_n^{(m)}(s)\right)\right\} ds \\ \overline{\tilde{y}}_{n+1}(t,r) &= \overline{\tilde{y}}_n(t,r) + \int_0^t \overline{\lambda}(s;t)\left\{\frac{d^m}{ds^m}\overline{\tilde{y}}_n + f\left(t, \overline{\tilde{y}}_n, \overline{\tilde{y}}'_n(s), \dots, \overline{\tilde{y}}_n^{(m)}(s)\right)\right\} ds \end{aligned}\right\} \qquad (2.33)$$

where $n \geq 0$.

From equation (2.33) it can be obtain the Lagrange multipliers in the form of: $\underline{\lambda}(s,t) = \overline{\lambda}(s,t) = (-1)^m \frac{(s-t)^{m-1}}{(m-1)!}$ (2.34)

It is mentioned that, if f is a linear operator according to the Euler-Lagrange differential equations then the right-hand side of equation (2.34) is exact value of the Lagrange multipliers (Jafari and Alipoor, 2011).

Substituting equation (2.34) in equations (2.33) yields:

$$\left.\begin{aligned} \underline{\tilde{y}}_{n+1}(t,r) &= \underline{\tilde{y}}_n(t,r) + \int_0^t (-1)^m \frac{(s-t)^{m-1}}{(m-1)!}\left\{\frac{d^m}{ds^m}\underline{\tilde{y}}_n + f(t, \underline{\tilde{y}}_n(s), \underline{\tilde{y}}'_n(s), \dots, \underline{\tilde{y}}_n^{(m)}(s))\right\} ds \\ \overline{\tilde{y}}_{n+1}(t,r) &= \overline{\tilde{y}}_n(t,r) + \int_0^t (-1)^m \frac{(s-t)^{m-1}}{(m-1)!}\left\{\frac{d^m}{ds^n}\overline{\tilde{y}}_n + f(t, \overline{\tilde{y}}_n(s), \overline{\tilde{y}}'_n(s), \dots, \overline{\tilde{y}}_n^{(m)}(s))\right\} ds \end{aligned}\right\}$$

(2.35)

Using Banach's fixed point theorem to equation (2.35), the converge condition for the sequences solutions can be obtained.

Theorem (2.5) (Jafari and Alipoor, 2011):

The nonlinear mapping is defined as:

$T[\underline{\tilde{y}}(t,r)] =$

$$\underline{\tilde{y}}(t,r) + \int_0^t (-1)^m \frac{(s-t)^{m-1}}{(m-1)!} \left\{ \frac{d^m}{ds^m} \underline{\tilde{y}}_n + f\left(t, \underline{\tilde{y}}(s), \underline{\tilde{y}}'_n(s), \ldots, \underline{\tilde{y}}^{(m)}(s)\right)\right\} ds \quad (2.36)$$

$S[\overline{\tilde{y}}(t,r)] =$

$$\overline{\tilde{y}}(t,r) + \int_0^t (-1)^m \frac{(s-t)^{m-1}}{(m-1)!} \left\{ \frac{d^m}{ds^m} \overline{\tilde{y}} + f\left(t, \overline{\tilde{y}}(s), \overline{y'}(s), \ldots, \overline{\tilde{y}}^{(m)}(s)\right)\right\} ds \quad (2.37)$$

The nonlinear mappings T and S are contraction by getting sufficient condition for convergence of the iteration sequences $\underline{\tilde{y}}_n(t)$ and $\overline{\tilde{y}}_n(t)$ in equation (2.35).

Furthermore, the sequence of equation (2.35) converges to the fixed point of T and S, which can be the solution of equation (2.31). Therefore, by choosing an initial approximation $\tilde{y}(0) = (g_0(r), k_0(r))$ for equation (2.35), the successive approximations of the solution to (2.32) can be obtained. Which leads to an exact solution can be exactly identified of the linear fuzzy differential equation (Jafari and Alipoor, 2011), (Goetschel and Voxman, 1986).

Chapter Three

Hybrid Method

In this chapter, present an approximate solution of linear and nonlinear Fuzzy differential equations by combined of the Laplace transform and the variational iteration method.

3.1 Hybrid Method:

In this work, building a hybrid method based on the philosophy of combing Laplace transform method and variational iteration method presented in (Wu and Baleanu, 2013) and (Wu, 2013). Since the variational iteration method depend on the value of the Lagrange multiplier, two different approaches for calculating the Lagrange multiplier are used to get more efficient method.

First Approach:

1. After taking the Laplace transform of equation (2.26) yields:

$$\mathcal{L}\left(\frac{d^m y}{dt^m}\right) + \mathcal{L}(Ry(t) + Ny(t)) = \mathcal{L}\big(g(t)\big) \tag{3.1}$$

By using the differential properties of Laplace transform and initial conditions that shown in section (2.2.1.2) yields:

$$\underline{\tilde{Y}}_{n+1}(s;r) = \underline{\tilde{Y}}_n(s) + \underline{\lambda}(s)\left(s^m \underline{\tilde{Y}}_n(s;r) - s^{m-1}\underline{\tilde{y}}(0) - \cdots - \underline{\tilde{y}}^{(m-1)}(0) + \mathcal{L}\left[R\left(\underline{\tilde{y}}_n\right) + N\left(\underline{\tilde{y}}_n\right) - g(t)\right]\right) \tag{3.2}$$

$$\overline{\tilde{Y}}_{n+1}(s;r) = \overline{\tilde{Y}}_n(s;r) + \overline{\lambda}(s;r)\left(s^m \overline{Y}_n(s;r) - s^{m-1}\overline{y}(0) - \cdots - \overline{y}^{(m-1)}(0) + \mathcal{L}\left[R\left(\overline{\tilde{y}}_n\right) + N\left(\overline{\tilde{y}}_n\right) - g(t)\right]\right) \tag{3.3}$$

2. The iteration formula of equations (3.2) and (3.3) can be used to suggest the main scheme involving the Lagrange multiplier. Consider the terms $\mathcal{L}\left[R\left(\tilde{\overline{y}}_n\right)+N\left(\tilde{\overline{y}}_n\right)-g(t)\right]$ in equation (3.2) and $\mathcal{L}\left[R\left(\tilde{\overline{y}}_n\right)+N\left(\tilde{\overline{y}}_n\right)-g(t)\right]$ in equation (3.3) as restricted variations, this makes equation (3.2) and (3.3) stationary with respect to $\tilde{\underline{Y}}_n$ and $\tilde{\overline{Y}}_n$:

$$\delta\tilde{\underline{Y}}_{n+1}(s)=\delta\tilde{\underline{Y}}_n(s)+\underline{\lambda}(s)\left(s^m\delta\tilde{\underline{Y}}_n(s)\right) \quad (3.4)$$

$$\delta\tilde{\overline{Y}}_{n+1}(s)=\delta\tilde{\overline{Y}}_n(s)+\overline{\lambda}(s)\left(s^m\delta\tilde{\overline{Y}}_n(s)\right) \quad (3.5)$$

where δ is the classical variation, The optimality condition for the extreme $\frac{\delta\tilde{Y}_{n+1}}{\delta\tilde{Y}_n}=0$, Equations (3.4) and (3.5) lead to :

$$\lambda(s)=\underline{\lambda}(s)=\overline{\lambda}(s)=-\frac{1}{s^m} \quad (3.6)$$

3. The sequential approximations are obtained by taking the inverse Laplace transform to equations (3.2) and (3.3) after substituting $\lambda(s)$ yields:

$$\tilde{\underline{y}}_{n+1}(t;r)=\tilde{\underline{y}}_n(t;r)-\mathcal{L}^{-1}\left[\frac{1}{s^m}\left(s^m\tilde{\underline{y}}_n(s;r)-s^{m-1}\underline{y}(0)-\cdots-\underline{y}^{(m-1)}(0)+\mathcal{L}\left[R\left(\tilde{\underline{y}}_n\right)+N\left(\tilde{\underline{y}}_n\right)-g(t)\right]\right)\right] \quad (3.7)$$

$$\tilde{\overline{y}}_{n+1}(t;r)=\tilde{\overline{y}}_n(t;r)-\mathcal{L}^{-1}\left[\frac{1}{s^m}\left(s^m\tilde{\overline{y}}_n(s;r)-s^{m-1}\overline{y}(0)-\cdots-\overline{y}^{(m-1)}(0)+\mathcal{L}\left[R\left(\tilde{\overline{y}}_n\right)+N\left(\tilde{\overline{y}}_n\right)-g(t)\right]\right)\right] \quad (3.8)$$

Rearrange equations (3.7) and (3.8) yields:

$$\tilde{\underline{y}}_{n+1}(t;r)=\mathcal{L}^{-1}\left(\frac{\underline{y}(0)}{s}+\cdots+\frac{\underline{y}^{(m-1)}(0)}{s^m}\right)+\mathcal{L}^{-1}\left[\frac{1}{s^m}\left(\mathcal{L}\left[R\left(\underline{y}_n\right)+N\left(\underline{y}_n\right)-g(t)\right]\right)\right] \quad (3.9)$$

$$\overline{y}_{n+1}(t;r) = \mathcal{L}^{-1}\left(\frac{\overline{y}(0)}{s} + \cdots + \frac{\overline{y}^{(m-1)}(0)}{s^m}\right) + \mathcal{L}^{-1}\left[\frac{1}{s^m}\left(\mathcal{L}\left[R\left(\overline{y}_n\right) + N\left(\overline{y}_n\right) - g(t)\right]\right)\right] \quad (3.10)$$

With initial approximation to equations (3.9) and (3.10) the following equation can be considered:

$$\underline{y}_0(t;r) = \mathcal{L}^{-1}\left(\frac{\underline{y}(0)}{s} + \cdots + \frac{\underline{y}^{(m-1)}(0)}{s^m}\right) \quad (3.11)$$

$$\overline{y}_0(t;r) = \mathcal{L}^{-1}\left(\frac{\overline{y}(0)}{s} + \cdots + \frac{\overline{y}^{(m-1)}(0)}{s^m}\right) \quad (3.12)$$

after applying Laplace inverse to equations (3.11) and (3.12) yields:

$$\underline{y}_0(t;r) = \underline{y}(0) + \underline{y}'(0)t + \cdots + \frac{\underline{y}^{(m-1)}(0)t^{m-1}}{(m-1)!} \quad (3.13)$$

$$\overline{y}_0(t;r) = \overline{y}(0) + \overline{y}'(0)t + \cdots + \frac{\overline{y}^{(m-1)}(0)t^{m-1}}{(m-1)!} \quad (3.14)$$

4. Finally, the values of $y_1, y_2, y_3, \ldots, y_n$ are obtained, and the solution of equation (2.26) is:

$$\underline{y}(\mathrm{t};\mathrm{r}) = \lim_{n\to\infty} \underline{y}_n(t) \quad (3.15)$$

$$\overline{y}(\mathrm{t};\mathrm{r}) = \lim_{n\to\infty} \overline{y}_n(t) \quad (3.16)$$

Second Approach

According to equation (2.21)assuming that

$$R[\tilde{y}(t;r)] = \sum_{i=0}^{m-1}\left(a_i\tilde{y}^{(i)}(t) + b_i(t)\tilde{y}^{(i)}(t)\right) \quad (3.17)$$

where a_i constant and b_i variable coefficient

1. Take the Laplace transform on (2.21), then the iteration formula becomes

$\underline{\tilde{Y}}_{n+1}(s;r) =$

$\underline{\tilde{Y}}_n(s;r) + \lambda\left[s^m \underline{\tilde{Y}}_n(s;r) - \sum_{k=0}^{m-1} \underline{\tilde{y}}^{(i)}(0)s^{m-k-1} + \mathcal{L}\left(\sum_{i=0}^{m-1} a_i \underline{\tilde{Y}}_n^{(i)}\right) +\right.$

$$\left.\mathcal{L}\left(\sum_{i=0}^{m-1} b_i(t)\underline{\tilde{Y}}_n^{(i)}\right) + \mathcal{L}\left(N\left[\underline{\tilde{Y}}_n\right] - g(t)\right)\right] \tag{3.18}$$

$\overline{\tilde{Y}}_{n+1}(s;r) =$

$\overline{y}_n(s;r) + \lambda\left[s^m \overline{\tilde{Y}}_n(s;r) - \sum_{k=0}^{m-1} \overline{\tilde{y}}^{(i)}(0)s^{m-k-1} + \mathcal{L}\left(\sum_{i=0}^{m-1} a_i \overline{\tilde{Y}}_n^{(i)}\right) +\right.$

$$\left.\mathcal{L}\left(\sum_{i=0}^{m-1} b_i(t)\overline{\tilde{Y}}_n^{(i)}\right) + \mathcal{L}\left(N\left[\overline{\tilde{Y}}_n\right] - g(t)\right)\right] \tag{3.19}$$

2. Consider the terms $\mathcal{L}\left(\sum_{i=0}^{m-1} b_i(t)\underline{\tilde{Y}}_n^{(i)}\right)$ and $\mathcal{L}(N[\underline{\tilde{Y}}_n])$ in equation (3.18) and $\mathcal{L}\left(\sum_{i=0}^{m-1} b_i(t)\overline{\tilde{Y}}_n^{(i)}\right)$ and $\mathcal{L}\left(N\left[\overline{\tilde{Y}}_n\right]\right)$ in equation (3.19) as restricted variation. make equations (3.18) and (3.19) stationary with respect to $\underline{\tilde{Y}}_n$ and $\overline{\tilde{Y}}_n$

$$\delta\underline{\tilde{Y}}_{n+1}(s) = \delta\underline{\tilde{Y}}_n(s) + \underline{\lambda}\left[s^m \delta\underline{\tilde{Y}}_n(s) + \sum_{i=0}^{m-1} a_i s^i \delta\underline{\tilde{Y}}_n(s)\right] \tag{3.20}$$

$$\delta\overline{\tilde{Y}}_{n+1}(s) = \delta\overline{\tilde{Y}}_n(s) + \overline{\lambda}\left[s^m \delta\overline{\tilde{Y}}_n(s) + \sum_{i=0}^{m-1} a_i s^i \delta\overline{\tilde{Y}}_n(s)\right] \tag{3.21}$$

from equations (3.20) and (3.21) determined the Lagrange multiplier as follows

$$\lambda(s) = \underline{\lambda}(s) = \overline{\lambda}(s) = -\frac{1}{\sum_{i=0}^{m-1} a_i s^i}\ , a_m = 1 \tag{3.22}$$

3. Take the Laplace inverse for equations (3.18) and (3.19) then the iteration formula obtained as

$$\underline{\tilde{y}}_{n+1}(t) = \underline{\tilde{y}}_n(s) + \mathcal{L}^{-1}\left[\lambda(s)\left[s^m \underline{\tilde{y}}_n(s) - \sum_{k=0}^{m-1} \underline{\tilde{y}}^{(i)}(0)s^{m-k-1} + \mathcal{L}\left(\sum_{i=0}^{m-1} a_i \underline{\tilde{y}}_n^{(i)}\right) + \mathcal{L}\left(\sum_{i=0}^{m-1} b_i(t)\underline{\tilde{y}}_n^{(i)}\right) + \mathcal{L}\left(N\left[\underline{\tilde{y}}_n\right] - g(t)\right)\right]\right] \quad (3.23)$$

$$\overline{\tilde{y}}_{n+1}(t) = \overline{\tilde{y}}_n(s) + \mathcal{L}^{-1}\left[\lambda(S)\left[s^m \overline{\tilde{y}}_n(s) - \sum_{k=0}^{m-1} \overline{\tilde{y}}^{(i)}(0)s^{m-k-1} + \mathcal{L}\left(\sum_{i=0}^{m-1} a_i \overline{\tilde{y}}_n^{(i)}\right) + \mathcal{L}\left(\sum_{i=0}^{m-1} b_i(t)\overline{\tilde{y}}_n^{(i)}\right) + \mathcal{L}\left(N[\overline{\tilde{y}}_n] - g(t)\right)\right]\right] \quad (3.24)$$

$$\underline{\tilde{y}}_{n+1}(t) = \underline{\tilde{y}}_0 + \mathcal{L}^{-1}\left(\sum_{i=0}^{m-1} b_i(t)\underline{\tilde{y}}_n^{(i)}\right) + N\left[\underline{\tilde{y}}_n\right] \quad (3.25)$$

$$\overline{\tilde{y}}_{n+1}(t) = \overline{\tilde{y}}_0 + \mathcal{L}^{-1}\left(\sum_{i=0}^{m-1} b_i(t)\overline{\tilde{y}}_n^{(i)}\right) + N[\overline{\tilde{y}}_n] \quad (3.26)$$

where the initial iteration value can be determined as

$$\underline{\tilde{y}}_0(t) = \mathcal{L}^{-1}\left[\lambda(s)\left[-\sum_{k=0}^{m-1} \underline{\tilde{y}}^{(i)}(0)s^{m-k-1} + \mathcal{L}\left(\sum_{i=0}^{m-1} a_i \underline{\tilde{y}}_n^{(i)}\right) + +\mathcal{L}[g(t)]\right]\right] \quad (3.27)$$

$$\overline{\tilde{y}}_0(t) = \mathcal{L}^{-1}\left[\lambda(s)\left[-\sum_{k=0}^{m-1} \overline{\tilde{y}}^{(i)}(0)s^{m-k-1} + \mathcal{L}\left(\sum_{i=0}^{m-1} a_i \overline{\tilde{y}}_n^{(i)}\right) + +\mathcal{L}[g(t)]\right]\right] \quad (3.28)$$

4. Let $y_n = \sum_{j=0}^{n} v_j$ and apply the Adomian decomposition method (ADM) (Adomian, 1988) to expand the term $N[\tilde{y}_n]$ as $\sum_{j=0}^{n} A_j$ then the iteration formula

$$\begin{cases} v_{j+1} = \mathcal{L}^{-1}\left(\lambda(s)\mathcal{L}\left(\sum_{i=0}^{m-1} b_i(t)v_j^{(i)} + A_j\right)\right) \\ \qquad v_0 = \tilde{y}_0 \end{cases} \quad (3.29)$$

where A_j is the famous Adomian decomposition series.

The solution of equation (2.26) can be in three cases $\underline{y}, \overline{y}$ and y^c as it is shown later in examples (3.1) and (3.2).

Now, we present the converges of our approach, by rewritten equation (3.1) as follow (Matinfar, Saeidy, and Ghasemi, 2013):

$$\mathcal{L}\left(\frac{d^m y}{dt^m}\right) + \Re\tilde{y}(s) + \mathcal{N}\tilde{y}(s) = h(s) \qquad (3.30)$$

After applying Laplace transform to the equation (2.26), the following variables can be defined as follows: $\Re$ is a new linear operator, $\mathcal{N}$ is new nonlinear operator and $h(s)$ is inhomogeneous source term of the mentioned equation.

Based on what illustrated in variation Iteration Method (VIM), the optimal value of Lagrange multiplier equation (2.24) can be found to equation (3.30), as shown bellows:

$$\tilde{y}_{n+1}(t) = \tilde{y}_n(t) + \int_0^t (-1)^n \frac{(s-x)^{n-1}}{(n-1)!}\left[\mathcal{L}\left(\frac{d^m\tilde{y}}{dt^m}\right) + \Re\tilde{y}(s) + \mathcal{N}\tilde{y}(s) - h(s)\right] ds \qquad (3.31)$$

Now, define the operation $A[\tilde{y}_n]$, and $\tilde{y}(t) = w(t)$ as bellows:

$$A[w] = \int_0^t (-1)^n \frac{(s-x)^{n-1}}{(n-1)!}\left[\mathcal{L}\left(\frac{d^m w}{dt^m}\right) + \Re w(s) + \mathcal{N}w(s) - h(s)\right] ds \qquad (3.32)$$

And define the components v_k , $k = 0,1,2, \dots$, as bellows:

$$\begin{cases} v_0 = w_0 \\ v_1 = A[v_0] \\ v_2 = A[v_0 + v_1] \\ \quad\vdots \\ v_{k+1} = A[v_0 + v_1 + \cdots + v_k] \end{cases} \qquad (3.33)$$

Consequently, the limit of $W(t)$ equals the summation of v_k from zero to infinity, as shown in the equation bellows:

$$W(t) = \lim_{k\to\infty} w_k(t) = \sum_{k=0}^{\infty} v_k \qquad (3.34)$$

Applying the inverse Laplace to equation (3.34), the solution of (3.30) can be obtained as follows:

$$\tilde{y}(t) = \mathcal{L}^{-1}(W(t)) \qquad (3.35)$$

To complete the proof we need the following theorem

Theorem (3.1) (Matinfar, Saeidy, and Ghasemi 2013):

Let $A[w]$ defined in (3.32), be an operator from a Hilbert space H to H. The series solution $w(t) = \sum_{k=0}^{\infty} v_k$ that defined in equation (3.34) converges if $\exists\ 0 < \gamma < 1$ such that $\|v_{k+1}\| \le \gamma \|v_k\|$ for some $k \in \mathbb{N} \cup \{0\}$.

3.2Numerical examples:

Example (3.1):

The example solved in (Jafari, Saeidy, and Baleanu, 2011) can be resolved by the hybrid VIM that proposed by this work.

Consider the following second-order linear fuzzy differential equation.

$$\begin{cases} y''(t) + y(t) = -t \qquad t \in [0,1] \\ y(0) = (0.1r - 0.1, 0.1 - 0.1r) \\ y'(0) = (0.088 + 0.1r, 0.288 - 0.1r) \end{cases} \qquad (3.36)$$

with exact solutions :

$$\underline{y}(t,r) = (0.1r - 0.1)cost + (1.088 + 0.1r)sint - t$$

$$\overline{y}(t,r) = (0.1 - 0.1r)cost + (1.288 + 0.1r)sint - t$$

$\tilde{y}_c(t,r) = 1.188 \sin t - t$

The solution of equation (3.69) by using hybrid method is as follows:

First approach:

Take the Laplace transform to the equation yields:

$$s^2Y(s) - sy(0) - y'(0) = \mathcal{L}[-Y - t] \tag{3.37}$$

Obtain the iteration formula of equation (3.70) yields:

$$\underline{\tilde{Y}}_{n+1}(s;r) = \underline{\tilde{Y}}_n(s;r) + \underline{\lambda}(s)\left[s^2\underline{\tilde{Y}}_n(s) - s\underline{y}(0) - \underline{y}'(0) + \underline{\tilde{Y}}_n(s) + \frac{1}{s^2}\right] \tag{3.38}$$

$$\overline{\tilde{Y}}_{n+1}(s;r) = \overline{\tilde{Y}}_n(s;r) + \overline{\lambda}(s)\left[s^2\overline{\tilde{Y}}_n(s) - s\overline{y}(0) - \overline{y}'(0) + \overline{\tilde{Y}}_n(s) + \frac{1}{s^2}\right] \tag{3.39}$$

$$\tilde{Y}_{c_{n+1}}(s;r) = \tilde{Y}_{c_n}(s;r) + \lambda_c(s)\left[s^2\tilde{Y}_{c_n}(s) - s\tilde{y}_c(0) - \tilde{y}_c{}'(0) + \tilde{Y}_{c_n}(s) + \frac{1}{s^2}\right] \tag{3.40}$$

Consider the term $\left(\underline{\tilde{Y}}_n(s) + \frac{1}{s^2}\right)$, $\left(\overline{\tilde{y}}_n(s) + \frac{1}{s^2}\right)$ and $\left(\tilde{Y}_{c_n}(s) + \frac{1}{s^2}\right)$ as restricted variation, then:

$$\delta\underline{\tilde{Y}}_{n+1}(s;r) = \delta\underline{\tilde{Y}}_n(s;r) + \underline{\lambda}(s)\left(s^2\delta\underline{\tilde{Y}}_n(s)\right)$$

$$\delta\overline{\tilde{Y}}_{n+1}(s;r) = \delta\overline{\tilde{Y}}_n(s;r) + \overline{\lambda}(s)\left(s^2\delta\overline{\tilde{Y}}_n(s)\right)$$

$$\delta\tilde{Y}_{c_{n+1}}(s;r) = \delta\tilde{Y}_{c_n}(s;r) + \lambda_c(s)\left(s^2\delta\tilde{Y}_{c_n}(s)\right)$$

With the Lagrange multiplier:

$$\underline{\lambda}(s) = \overline{\lambda}(s) = \lambda_c(s) = -\frac{1}{s^2}$$

Taking the Laplace inverse transform to equations (3.38), (3.39) and (3.40):

$$\underline{\tilde{y}}_{n+1}(t;r) = \underline{\tilde{y}}_n(t;r) - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\left[s^2\underline{\tilde{y}}_n(s;r) - s\underline{\tilde{y}}(0) - \underline{\tilde{y}}'(0) + \underline{\tilde{y}}_n(s;r) + \frac{1}{s^2}\right]\right\}$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \overline{\tilde{y}}_n(t;r) - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\left[s^2\overline{\tilde{y}}_n(s;r) - s\overline{\tilde{y}}(0) - \overline{\tilde{y}}'(0) + \overline{\tilde{y}}_n(s;r) + \frac{1}{s^2}\right]\right\}$$

$$\tilde{Y}_{c_{n+1}}(t;r) = \tilde{Y}_{c_n}(t;r) - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\left[s^2\overline{\tilde{Y}_{c_n}}(s;r) - s\tilde{Y}_c(0) - \tilde{Y}_c{}'(0) + \tilde{Y}_{c_n}(s;r) + \frac{1}{s^2}\right]\right\}$$

$$\underline{\tilde{y}}_{n+1}(t;r) = \mathcal{L}^{-1}\left[\frac{\underline{\tilde{y}}(0)}{s} + \frac{\underline{\tilde{y}}'(0)}{s^2}\right] + L^{-1}\left\{\frac{1}{s^2}\left[\underline{\tilde{y}}_n(s;r) + \frac{1}{s^2}\right]\right\} \tag{3.41}$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \mathcal{L}^{-1}\left[\frac{\overline{\tilde{y}}(0)}{s} + \frac{\overline{\tilde{y}}'(0)}{s^2}\right] + L^{-1}\left\{\frac{1}{s^2}\left[\overline{\tilde{y}}_n(s;r) + \frac{1}{s^2}\right]\right\} \tag{3.42}$$

$$\tilde{Y}_{c_{n+1}}(t;r) = \mathcal{L}^{-1}\left[\frac{\tilde{Y}_c(0)}{s} + \frac{\tilde{Y}_c{}'(0)}{s^2}\right] + L^{-1}\left\{\frac{1}{s^2}\left[\tilde{Y}_{c_n}(s;r) + \frac{1}{s^2}\right]\right\} \tag{3.43}$$

Therefore,

$$\underline{\tilde{y}}_{n+1}(t;r) = \underline{\tilde{y}}(0) + \underline{\tilde{y}}'(0)t + \mathcal{L}^{-1}\left\{\frac{1}{S^2}\left[\underline{\tilde{y}}_n(s) + \frac{1}{s^2}\right]\right\} \tag{3.44}$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \overline{\tilde{y}}(0) + \overline{\tilde{y}}'(0)t + \mathcal{L}^{-1}\left\{\frac{1}{S^2}\left[\overline{\tilde{y}}_n(s) + \frac{1}{s^2}\right]\right\} \tag{3.45}$$

$$\tilde{Y}_{c_{n+1}}(t;r) = \tilde{Y}_c(0) + \tilde{Y}_c{}'(0)t + \mathcal{L}^{-1}\left\{\frac{1}{S^2}\left[\tilde{Y}_{c_n}(s) + \frac{1}{s^2}\right]\right\} \tag{3.46}$$

With initial iteration

$$\underline{\tilde{y}}_0(t;r) = \underline{\tilde{y}}(0) + \underline{\tilde{y}}'(0)t \tag{3.47}$$

$$\overline{\tilde{y}}_0(t;r) = \overline{\tilde{y}}(0) + \overline{\tilde{y}}'(0)t \tag{3.48}$$

$$\tilde{Y}_{c_0}(t;r) = \tilde{Y}_c(0) + \tilde{Y}_c{}'(0)t \tag{3.49}$$

then

$$\underline{\tilde{y}}_0(t;r) = (0.1r - 0.1) + (0.088 + 0.1r)t$$

$$\overline{\tilde{y}}_0(t;r) = (0.1 - 0.1r) + (0.288 - 0.1r)t$$

$$\tilde{Y}_{c_0}(t;r) = 0.188t$$

$$\underline{\tilde{y}}_1(t;r) = (0.1r - 0.1) + (0.088 + 0.1r\,)t + \frac{(0.1-0.1r)}{2!}t^2 - \frac{(0.088+0.1r)}{3!}t^3 - \frac{t^3}{3!}$$

$$\overline{\tilde{y}}_1(t;r) = (0.1 - 0.1r) + (0.288 - 0.1r)t - \frac{(0.1-0.1r)}{2!}t^2 - \frac{(0.288-0.1r)}{3!}t^3 - \frac{t^3}{3!}$$

$$\tilde{Y}_{c_1}(t;r) = 0.188t - \frac{0.188}{3!}t^3 - \frac{t^3}{3!}$$

$$\underline{\tilde{y}}_2(t;r) = (0.1r - 0.1) + (0.088 + 0.1r\,)t - \frac{(0.1r-0.1)}{2!}t^2 - \frac{(0.088+0.1r\,)}{3!}t^3 + \frac{(0.1r-0.1)}{4!}t^4 + \frac{(0.088+0.1r\,)}{5!}t^5 - \frac{t^3}{3!} + \frac{t^5}{5!}$$

$$\overline{\tilde{y}}_2(t;r) = (0.1 - 0.1r) + (0.288 - 0.1r)t - \frac{(0.1-0.1r)}{2!}t^2 - \frac{(0.288-0.1r)}{3!}t^3 + \frac{(0.1-0.1r)}{4!}t^4 + \frac{(0.288-0.1r)}{5!}t^5 - \frac{t^3}{3!} + \frac{t^5}{5!}$$

$$\tilde{Y}_{c_2}(t;r) = 0.188t - \frac{0.188}{3!}t^3 + \frac{0.188}{5!}t^5 - \frac{t^3}{3!} + \frac{t^5}{5!}$$

$$\underline{\tilde{y}}_3(t;r) = (0.1r - 0.1) + (0.088 + 0.1r\,)t - \frac{(0.1r-0.1)}{2!}t^2 - \frac{(0.088+0.1r\,)}{3!}t^3 + \frac{(0.1r-0.1)}{4!}t^4 + \frac{(0.088+0.1r\,)}{5!}t^5 - \frac{(0.1r-0.1)}{6!}t^6 - \frac{(0.088+0.1r\,)}{7!}t^7 - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}$$

$$\overline{\tilde{y}}_3(t;r) = (0.1 - 0.1r) + (0.288 - 0.1r)t - \frac{(0.1-0.1r)}{2!}t^2 - \frac{(0.288-0.1r)}{3!}t^3 + \frac{(0.1-0.1r)}{4!}t^4 + \frac{(0.288-0.1r)}{5!}t^5 - \frac{(0.1-0.1r)}{6!}t^6 - \frac{(0.288-0.1r)}{7!}t^7 - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}$$

$$\tilde{Y}_{c_3}(t;r) = 0.188t - \frac{0.188}{3!}t^3 + \frac{0.188}{5!}t^5 - \frac{0.188}{7!}t^7 - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}$$

Continuing the solution to n, yields

$$\underline{\tilde{y}}_n(t;r) = (0.1r-0.1) - \frac{(0.1r-0.1)}{2!}t^2 + \frac{(0.1r-0.1)}{4!}t^4 - \frac{(0.1r-0.1)}{6!}t^6 + \cdots (-1)^n \frac{(0.1r-0.1)}{2n!}t^{2n} + (0.088+0.1r\,)t - \frac{(0.088+0.1r\,)}{3!}t^3 + \frac{(0.088+0.1r\,)}{5!}t^5 - \frac{(0.088+0.1r\,)}{7!}t^7 + \cdots (-1)^n \frac{(0.088+0.1r\,)}{2n+1!}t^{2n+1} - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots (-1)^n \frac{t^{2n+1}}{2n+1!}$$

$$\overline{\tilde{y}}_n(t;r) = (0.1-0.1r) - \frac{(0.1-0.1r)}{2!}t^2 + \frac{(0.1-0.1r)}{4!}t^4 - \frac{(0.1-0.1r)}{6!}t^6 + \cdots (-1)^n \frac{(0.1-0.1r)}{2n!}t^{2n} + (0.288-0.1r)t - \frac{(0.288-0.1r)}{3!}t^3 + \frac{(0.288-0.1r)}{5!}t^5 - \frac{(0.288-0.1r)}{7!}t^7 + \cdots (-1)^n \frac{(0.288-0.1r)}{2n+1!}t^{2n+1} - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots (-1)^n \frac{t^{2n+1}}{2n+1!}$$

$$\tilde{Y}_{c_n}(t;r) = 0.188t - \frac{0.188}{3!}t^3 + \frac{0.188}{5!}t^5 - \frac{0.188}{7!}t^7 + \cdots (-1)^n \frac{0.188}{2n+1!}t^{2n+1} - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots (-1)^n \frac{t^{2n+1}}{2n+1!}$$

As $n \to \infty$

$\underline{\tilde{y}}(t;r)$ converge to $(0.1r-0.1)\cos t + (1.088+0.1r\,)\sin t - t$

$\overline{\tilde{y}}(t;r)$ converge to$(0.1-0.1r)\cos t + (1.288-0.1r)\sin t - t$

$\tilde{y}_{c_n}(t;r)$ converge to $1.188\sin t - t$

Second approach:

Take the Laplace transform to the equation (3.36) yields:

$$s^2Y(s) - sY(0) - y'(0) + \mathcal{L}[Y+t] = 0 \qquad (3.50)$$

Obtain the iteration formula of equation (3.50) yields:

$$\underline{\tilde{Y}}_{n+1}(s;r) = \underline{\tilde{Y}}_n(s;r) + \underline{\lambda}(s)\left[s^2\underline{\tilde{Y}}_n(s;r) - s\underline{\tilde{y}}(0) - \underline{\tilde{y}}'(0) + \underline{\tilde{Y}}_n(s;r) + \frac{1}{s^2}\right] \quad (3.51)$$

$$\overline{\tilde{Y}}_{n+1}(s;r) = \overline{\tilde{Y}}_n(s;r) + \overline{\lambda}(s)\left[s^2\overline{\tilde{Y}}_n(s;r) - s\overline{\tilde{Y}}(0) - \overline{\tilde{Y}}'(0) + \overline{\tilde{Y}}_n(s;r) + \frac{1}{s^2}\right] \quad (3.52)$$

$$\tilde{Y}_{c_{n+1}}(s;r) = \tilde{Y}_{c_n}(s) + \lambda_c(s)\left[s^2\tilde{Y}_{c_n}(s;r) - s\tilde{Y}_c(0) - \tilde{Y}_c{}'(0) + \tilde{Y}_{c_n}(s;r) + \frac{1}{s^2}\right] \quad (3.53)$$

Consider the term $\frac{1}{s^2}$ as restricted variation

$$\delta\underline{\tilde{Y}}_{n+1}(s;r) = \delta\underline{\tilde{Y}}_n(s) + \underline{\lambda}(s)(s^2-1)\delta\underline{\tilde{Y}}_n(s)$$

$$\delta\overline{\tilde{Y}}_{n+1}(s;r) = \delta\overline{\tilde{Y}}_n(s) + \overline{\lambda}(s)(s^2-1)\delta\overline{\tilde{Y}}_n(s)$$

$$\delta\tilde{Y}_{c_{n+1}}(s;r) = \delta\tilde{Y}_{c_n}(s) + \lambda_c(s)(s^2-1)\delta\tilde{Y}_{c_n}(s)$$

with the Lagrange multiplier

$$\underline{\lambda}(s) = \overline{\lambda}(s) = \lambda_c(s) = -\frac{1}{s^2+1}$$

substitute $\lambda(s)$ in equations (3.51), (3.52) and (3.53) and Take the Laplace inverse, yields:

$$\underline{\tilde{y}}_{n+1}(t;r) = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\left[s\underline{\tilde{y}}(0) + \underline{\tilde{y}}'(0)\right]\right] + \mathcal{L}^{-1}\left[\frac{-1}{(s^2+1)}\left(\frac{1}{s^2}\right)\right] \quad (3.54)$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\left[s\overline{\tilde{y}}(0) + \overline{\tilde{y}}'(0)\right]\right] + \mathcal{L}^{-1}\left[\frac{-1}{(s^2+1)}\left(\frac{1}{s^2}\right)\right] \quad (3.55)$$

$$\tilde{y}_{c_{n+1}}(t;r) = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\left[s\tilde{y}_c(0) + \tilde{y}_c{}'(0)\right]\right] + \mathcal{L}^{-1}\left[\frac{-1}{(s^2+1)}\left(\frac{1}{s^2}\right)\right] \quad (3.56)$$

then

$$\underline{\tilde{y}}_{n+1}(t;r) = \underline{\tilde{y}}_0 + \mathcal{L}^{-1}\left[\frac{-1}{s^2} + \frac{1}{(s^2+1)}\right]$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \overline{\tilde{y}}_0 + \mathcal{L}^{-1}\left[\frac{-1}{s^2} + \frac{1}{(s^2+1)}\right]$$

$$\tilde{y}_{c_{n+1}}(t;r) = \tilde{y}_{c_0} + \mathcal{L}^{-1}\left[\frac{-1}{s^2} + \frac{1}{(s^2+1)}\right]$$

then

$$\underline{\tilde{y}}_0(t;r) = (0.1r - 0.1)\cos(t) + (0.088 + 0.1r\,)\sin(t)$$

$$\overline{\tilde{y}}_0(t;r) = (0.1 - 0.1r)\cos(t) + (0.288 - 0.1r)\sin(t)$$

$$\tilde{y}_{c_0}(t;r) = 0.188\sin(t)$$

$$\underline{\tilde{y}}_1(t;r) = (0.1r - 0.1)\cos(t) + (1.088 + 0.1r\,)\sin(t) - t$$

$$\overline{\tilde{y}}_1(t;r) = (0.1 - 0.1r)\cos(t) + (1.288 - 0.1r)\sin(t) - t$$

$$\tilde{y}_{c_1}(t;r) = 1.88\sin(t) - t$$

$$\underline{\tilde{y}}_1(t;r) = \underline{\tilde{y}}_2(t;r) = \underline{\tilde{y}}_2(t;r) = \cdots = \underline{\tilde{y}}_n(t;r)$$

$$\overline{\tilde{y}}_1(t;r) = \overline{\tilde{y}}_2(t;r) = \overline{\tilde{y}}_3(t;r) = \cdots = \overline{\tilde{y}}_n(t;r)$$

$$\tilde{y}_{c_1}(t;r) = \tilde{y}_{c_2}(t;r) = \tilde{y}_{c_3}(t;r) = \cdots = \tilde{y}_{c_n}(t;r)$$

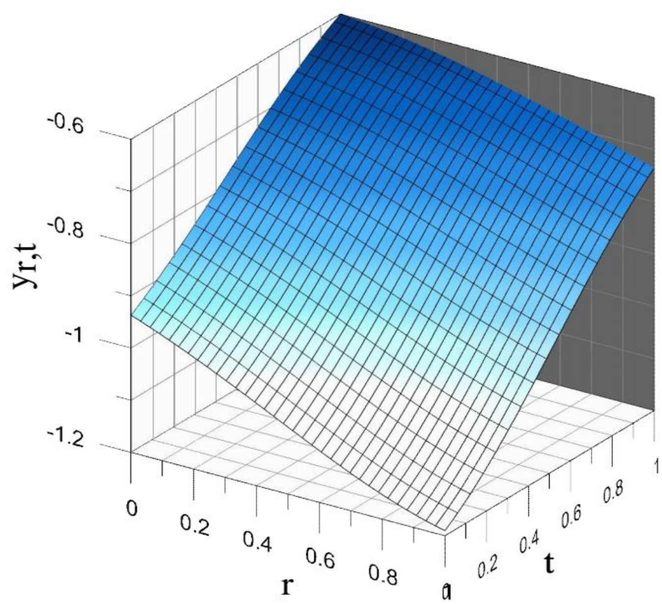

Figure (3.1): The exact solution of $\overline{\tilde{y}}(t;r)$

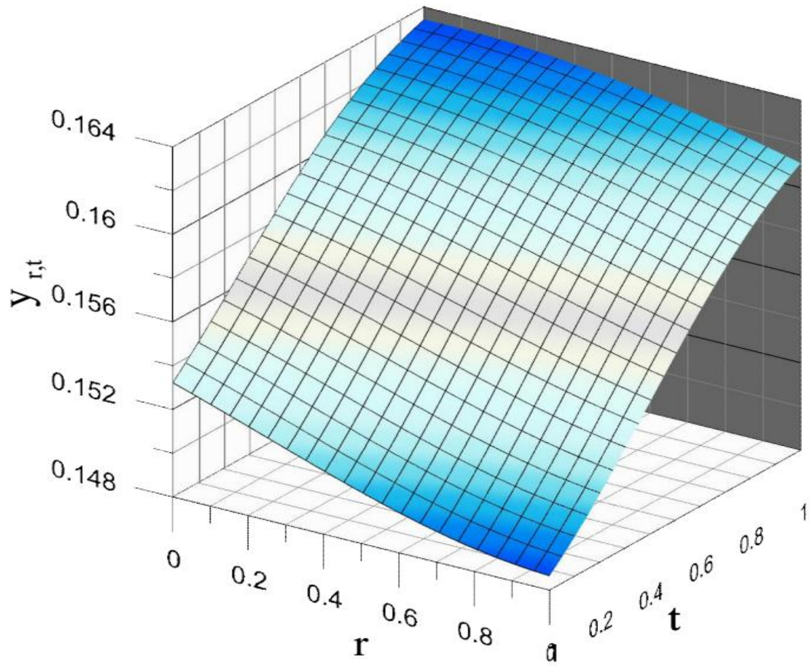

Figure (3.2): The exact solution of $\overline{\overline{y}}(t;r)$

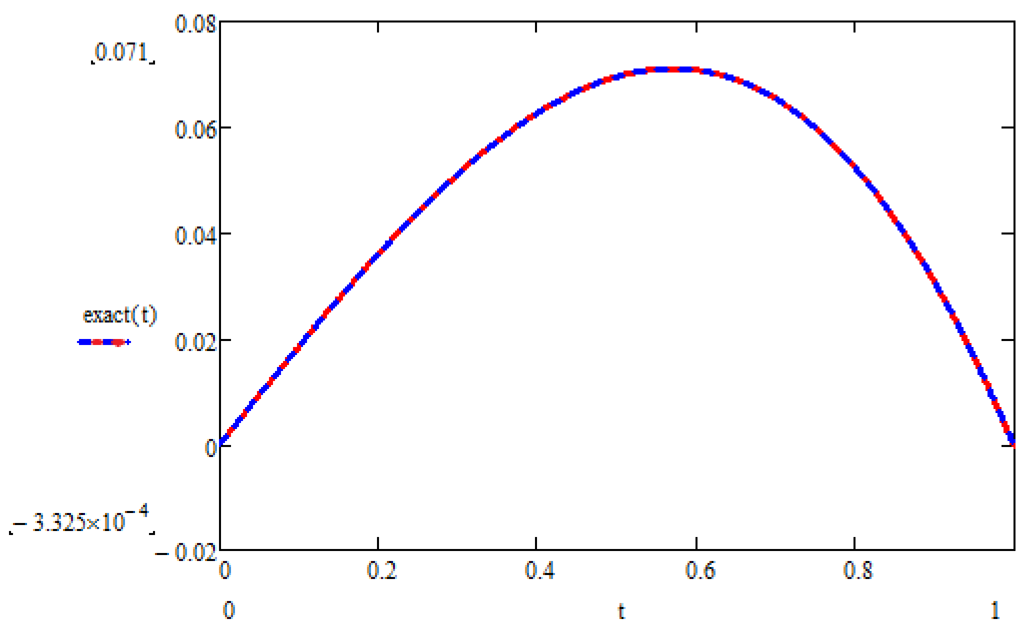

Figure (3.3): The exact solution of $\tilde{y}_c(t;r)$

Example (3.2):

The example solved in (Jameel, 2014) can be resolved by the hybrid VIM that proposed by this work.

Consider the following second-order non-linear fuzzy differential equation.

$$\begin{cases} y''(t) + y'(t) = -y(t)^3 \quad t \geq 0 \\ y(0) = [0.9 + 0.1r, 1.1 - 0.1r] \\ y'(0) = [0.9 + 0.1r, 1.1 - 0.1r] \end{cases} \tag{3.57}$$

The solution of equation (3.57) is using hybrid method is as follows:

Take the Laplace transform to the equation yields:

$$s^2Y(s) - sy(0) - y'(0) + \mathcal{L}[Y' + Y^3] = 0 \tag{3.58}$$

Obtain the iteration formula of equation (3.92) yields:

$$\underline{\tilde{Y}}_{n+1}(s;r) = \underline{\tilde{Y}}_n(s;r) + \underline{\tilde{\lambda}}(s)\left[s^2\underline{\tilde{Y}}_n(s;r) - s\underline{\tilde{y}}(0) - \underline{\tilde{y}}'(0) + \mathcal{L}\left[\underline{\tilde{Y}}_n{}' + \underline{\tilde{Y}}_n{}^3\right]\right] \tag{3.59}$$

$$\overline{\tilde{Y}}_{n+1}(s;r) = \overline{\tilde{Y}}_n(s;r) + \overline{\tilde{\lambda}}(s)\left[s^2\overline{\tilde{y}}_n(s;r) - s\overline{\tilde{y}}(0) - \overline{\tilde{y}}'(0) + \mathcal{L}\left[\overline{\tilde{Y}}_n{}' + \overline{\tilde{Y}}_n{}^3\right]\right] \tag{3.60}$$

$$\tilde{Y}_{c_{n+1}}(s;r) = \tilde{Y}_{c_n}(s;r) + \tilde{\lambda}_c(s)\left[s^2\tilde{y}_{c_n}(s;r) - s\tilde{y}_c(0) - \tilde{y}_c{}'(0) + \mathcal{L}\left[\tilde{Y}_{c_n}{}' + \tilde{Y}_{c_n}{}^3\right]\right] \tag{3.61}$$

Consider the term $\mathcal{L}\left[\underline{\tilde{Y}}_n{}' + \underline{\tilde{Y}}_n{}^3\right]$, $\mathcal{L}\left[\overline{\tilde{Y}}_n{}' + \overline{\tilde{Y}}_n{}^3\right]$ and $\mathcal{L}\left[\tilde{Y}_{c_n}{}' + \tilde{Y}_{c_n}{}^3\right]$ in the above equations as restricted variation, then:

$$\delta\underline{\tilde{Y}}_{n+1} = \delta\underline{\tilde{Y}}_n(s) + \underline{\tilde{\lambda}}(s)\left(s^2\delta\underline{\tilde{Y}}_n(s)\right)$$

$$\delta\overline{\tilde{Y}}_{n+1} = \delta\overline{\tilde{Y}}_n(s) + \overline{\tilde{\lambda}}(s)\left(s^2\delta\overline{\tilde{Y}}_n(s)\right)$$

$$\delta\tilde{Y}_{c_{n+1}}(s;r) = \delta\tilde{Y}_{c_n}(s) + \tilde{\lambda}_c(s)\left(s^2\delta\tilde{Y}_{c_n}(s;r)\right)$$

With the Lagrange multiplier:

$$\underline{\tilde{\lambda}}(s) = \overline{\tilde{\lambda}}(s) = \tilde{\lambda}_c(s) = -\frac{1}{s^2}$$

Taking the Laplace inverse transform to equations (3.59), (3.60) and (3.61):

$$\underline{\tilde{y}}_{n+1}(t;r) = \underline{\tilde{y}}_n(t;r) - \mathcal{L}^{-1}\left(\frac{1}{s^2}\left[s^2\underline{\tilde{y}}_n(s;r) - s\underline{\tilde{y}}(0) - \underline{\tilde{y}}'(0) + \mathcal{L}\left[\underline{\tilde{Y}}_n{}' + \underline{\tilde{Y}}_n{}^3\right]\right]\right)$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \overline{\tilde{y}}_n(t;r) - \mathcal{L}^{-1}\left(\frac{1}{s^2}\left[s^2\overline{\tilde{y}}_n(s;r) - s\overline{\tilde{y}}(0) - \overline{\tilde{y}}'(0) + \mathcal{L}\left[\overline{\tilde{Y}}_n{}' + \overline{\tilde{Y}}_n{}^3\right]\right]\right)$$

$$\tilde{y}_{c_{n+1}}(t;r) = \tilde{y}_{c_n}(t;r) - \mathcal{L}^{-1}\left(\frac{1}{s^2}\left[s^2\tilde{y}_{c_n}(s;r) - s\tilde{y}_c(0) - \tilde{y}_c{}'(0) + \mathcal{L}\left[\tilde{Y}_{c_n}{}' + \tilde{Y}_{c_n}{}^3\right]\right]\right)$$

$$\underline{\tilde{y}}_{n+1}(t;r) = \mathcal{L}^{-1}\left[\frac{\underline{\tilde{y}}(0)}{s} + \frac{\underline{\tilde{y}}'(0)}{s^2}\right] + \mathcal{L}^{-1}\left\{\frac{-1}{s^2}\mathcal{L}\left[\underline{\tilde{Y}}_n{}' + \underline{\tilde{Y}}_n{}^3\right]\right\}$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \mathcal{L}^{-1}\left[\frac{\overline{\tilde{y}}(0)}{s} + \frac{\overline{\tilde{y}}'(0)}{s^2}\right] + \mathcal{L}^{-1}\left\{\frac{-1}{s^2}\mathcal{L}\left[\overline{\tilde{Y}}_n{}' + \overline{\tilde{Y}}_n{}^3\right]\right\}$$

$$\tilde{y}_{c_{n+1}}(t;r) = \mathcal{L}^{-1}\left[\frac{\tilde{y}_c(0)}{s} + \frac{\tilde{y}_c{}'(0)}{s^2}\right] + \mathcal{L}^{-1}\left\{\frac{-1}{s^2}\mathcal{L}\left[\tilde{Y}_{c_n}{}' + \tilde{Y}_{c_n}{}^3\right]\right\}$$

Therefore,

$$\underline{\tilde{y}}_{n+1}(t;r) = \underline{\tilde{y}}_0(t) + \mathcal{L}^{-1}\left\{\frac{-1}{s^2}\mathcal{L}\left[\underline{\tilde{Y}}_n{}' + \underline{\tilde{Y}}_n{}^3\right]\right\} \quad (3.62)$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \overline{\tilde{y}}_0(t) + \mathcal{L}^{-1}\left\{\frac{-1}{s^2}\mathcal{L}\left[\overline{\tilde{Y}}_n{}' + \overline{\tilde{Y}}_n{}^3\right]\right\} \quad (3.63)$$

$$\tilde{y}_{c_{n+1}}(t;r) = \tilde{y}_{c_0}(t) + \mathcal{L}^{-1}\left\{\frac{-1}{s^2}\mathcal{L}\left[\tilde{Y}_{c_n}{}' + \tilde{Y}_{c_n}{}^3\right]\right\} \quad (3.64)$$

With initial iteration

$$\underset{\sim}{\tilde{y}}_0(t;r) = \underline{\tilde{y}}(0) + \underline{\tilde{y}}'(0)t \quad (3.65)$$

$$\overline{\tilde{y}}_0(t;r) = \overline{\tilde{y}}(0) + \overline{\tilde{y}}'(0)t \quad (3.66)$$

$$\tilde{y}_{c_0}(t;r) = \tilde{y}_c(0) + \tilde{y}_c{}'(0)t \quad (3.67)$$

then

$$\underline{\tilde{y}}_0(t;r) = (0.9 + 0.1\text{r}) + (0.9 + 0.1\text{r})t$$

$$\overline{\tilde{y}}_0(t;r) = (1.1 - 0.1\text{r}) + (1.1 - 0.1\text{r})\text{t}$$

$$\tilde{y}_{c_0}(t;r) = 1 + t$$

$$\underline{\tilde{y}}_1(t;r) = (0.9 + 0.1\text{r}) + (0.9 + 0.1\text{r})t + \frac{(0.9 + 0.1\text{r})^3 + (0.9 + 0.1\text{r})}{2!}t^2 + \frac{3(0.9 + 0.1\text{r})^3}{3!}t^3 + \frac{3(0.9 + 0.1\text{r})^3}{4!}t^4 + \frac{(0.9 + 0.1\text{r})^3}{5!}t^5$$

$$\overline{\tilde{y}}_1(t;r) = (1.1 - 0.1\text{r}) + (1.1 - 0.1\text{r})\text{t} + \frac{(1.1 - 0.1\text{r})^3 + (1.1 - 0.1\text{r})}{2!}t^2 + \frac{3(1.1 - 0.1\text{r})^3}{3!}t^3 + \frac{3(1.1 - 0.1\text{r})^3}{4!}t^4 + \frac{(1.1 - 0.1\text{r})^3}{5!}t^5$$

$$\tilde{y}_{c_{0_1}}(t;r) = 1 + t + \frac{2}{2!}t^2 + \frac{3}{3!}t^3 + \frac{3}{4!}t^4 + \frac{1}{5!}t^5$$

Second approach:

Take the Laplace transform to the equation (3.57) yields:

$$s^2Y(S) - sy(0) - y'(0) + sy(0) - y'(0) + L[Y^3] = 0 \quad (3.68)$$

Obtain the iteration formula of equation (3.68) yields:

$$\underline{\tilde{Y}}_{n+1}(s;r) = \underline{\tilde{Y}}_n(s;r) + \underline{\tilde{\lambda}}(s)\Big[s^2\underline{\tilde{Y}}_n(S;r) - s\underline{\tilde{y}}(0) - \underline{\tilde{y}}'(0) + s\underline{\tilde{Y}}_n(r;t) - y'(0) + \mathcal{L}\big[\underline{\tilde{Y}}_n{}^3\big]\Big] \quad (3.69)$$

$$\overline{\tilde{Y}}_{n+1}(s;r) = \overline{\tilde{Y}}_n(s;r) + \overline{\tilde{\lambda}}(s)\Big[s^2\overline{\tilde{Y}}_n(S;r) - s\overline{\tilde{y}}(0) - \overline{\tilde{y}}'(0) + s\overline{\tilde{Y}}(s;r) - \overline{\tilde{y}}'(0) + \mathcal{L}\big[\underline{\tilde{Y}}_n{}^3\big]\Big] \quad (3.70)$$

$$\tilde{Y}_{c_{n+1}}(s;r) = \tilde{Y}_{c_n}(s;r) + \tilde{\lambda}_c(s)\Big[s^2\tilde{Y}_{c_n}(S;r) - s\tilde{y}_c(0) - \tilde{y}_c{}'(0) + s\tilde{Y}_{c_n}(s;r) - \tilde{Y}_{c_n}{}'(0) + \mathcal{L}\big[\tilde{Y}_{c_n}{}^3\big]\Big] \quad (3.71)$$

Consider the term $\mathcal{L}\left[\tilde{Y}_n{}^3\right]$ as restricted variation

$$\delta\underline{\tilde{Y}}_{n+1}(s;r) = \delta\underline{\tilde{Y}}_n(s;r) + \underline{\tilde{\lambda}}(s)(s^2 - s)\delta\underline{\tilde{Y}}_n(s;r)$$

$$\delta\overline{\tilde{Y}}_{n+1}(s;r) = \delta\overline{\tilde{Y}}_n(s;r) + \overline{\tilde{\lambda}}(s)(s^2 - s)\delta\overline{\tilde{Y}}_n(s;r)$$

$$\delta\tilde{Y}_{c_{n+1}}(s;r) = \delta\tilde{Y}_{c_n}(s;r) + \tilde{\lambda}_c(s)(s^2 - s)\delta\tilde{Y}_{c_n}(s;r)$$

with the Lagrange multiplier

$$\underline{\tilde{\lambda}}(s) = \overline{\tilde{\lambda}}(s) = \tilde{\lambda}_c(s) = -\frac{1}{s^2+s}$$

substitute $\lambda(s)$ in equations (3.69), (3.70) and (3.71) then Take the Laplace inverse, yields:

$$\underline{\tilde{y}}_{n+1}(t;r) = \mathcal{L}^{-1}\left[\frac{1}{s^2+s}\left[s\underline{\tilde{y}}(0) + \underline{\tilde{y}}'(0) + \underline{\tilde{y}}(0)\right]\right] + \mathcal{L}^{-1}\left[\frac{-1}{(s^2+s)}\mathcal{L}\left[\underline{\tilde{Y}}_n{}^3\right]\right] \quad (3.\ 72)$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \mathcal{L}^{-1}\left[\frac{1}{s^2+s}\left[s\overline{\tilde{y}}(0) + \overline{\tilde{y}}'(0) + \overline{\tilde{y}}(0)\right]\right] + \mathcal{L}^{-1}\left[\frac{-1}{(s^2+s)}\mathcal{L}\left[\overline{\tilde{Y}}_n{}^3\right]\right] \quad (3.\ 72)$$

$$\tilde{y}_{c_{n+1}}(t;r) = \mathcal{L}^{-1}\left[\frac{1}{s^2+s}\left[s\tilde{y}_c(0) + \tilde{y}_c{}'(0) + \tilde{y}_c(0)\right]\right] + \mathcal{L}^{-1}\left[\frac{-1}{(s^2+s)}\mathcal{L}\left[\tilde{Y}_{c_n}{}^{3}\right]\right] \quad (3.72)$$

then

$$\underline{\tilde{y}}_{n+1}(t;r) = \underline{\tilde{y}}_0(t;r) + \mathcal{L}^{-1}\left[\frac{-1}{(s^2+s)}\mathcal{L}\left[\underline{\tilde{Y}}_n{}^{3}\right]\right]$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \overline{\tilde{y}}_0(t;r) + \mathcal{L}^{-1}\left[\frac{-1}{(s^2+s)}\mathcal{L}\left[\overline{\tilde{Y}}_n{}^{3}\right]\right]$$

$$\tilde{y}_{c_{n+1}}(t;r) = \tilde{y}_{c_0}(t;r) + \mathcal{L}^{-1}\left[\frac{-1}{(s^2+s)}\mathcal{L}\left[\tilde{Y}_{c_n}{}^{3}\right]\right]$$

where

$$\underline{\tilde{y}}_0 = \underline{\tilde{y}}(0) + (1-e^{-t})\underline{\tilde{y}}'(0)$$

$$\overline{\tilde{y}}_0 = \overline{\tilde{y}}(0) + (1-e^{-t})\overline{\tilde{y}}'(0)$$

$$\tilde{y}_{c_0} = \tilde{y}_c(0) + (1-e^{-t})\tilde{y}_c{}'(0)$$

Let $\underline{y}_n = \sum_{j=0}^{n} v_j$ and apply the Adomian decomposition method (ADM) to expand the term $\underline{y}_n{}^{3}$ as $\sum_{j=0}^{n} A_j$ then the iteration formula

$$\begin{cases} \tilde{v}_{j+1} = \mathcal{L}^{-1}\left[\frac{-1}{(s^2+s)}\mathcal{L}[A_j]\right] \\ \tilde{v}_0 = \tilde{y}_0 \end{cases}$$

where A_j is the famous Adomian decomposition series

$$\begin{cases} A_0 = \tilde{v}_0^3 \\ A_1 = 3\tilde{v}_0^2\tilde{v}_1 \\ A_2 = 3\tilde{v}_0\tilde{v}_1^2 + 3\tilde{v}_0^2\tilde{v}_2 \\ \qquad\vdots \end{cases}$$

then

$\underline{\tilde{v}}_0 = 2(0.9 + 0.1\mathrm{r}) - (0.9 + 0.1\mathrm{r})e^{-t}$

$\overline{\tilde{v}}_0 = 2(1.1 - 0.1\mathrm{r}) - (1.1 - 0.1\mathrm{r})e^{-t}$

$\tilde{v}_{c_0} = 2 - e^{-t}$

$$\underline{\tilde{v}}_1 = -2.3333\,(0.9 + 0.1\mathrm{r})^3 + 8\,(0.9 + 0.1\mathrm{r})^3 t - 19.166\,(0.9 + 0.1\mathrm{r})^3 e^{-t} + 4\,(0.9 + 0.1\mathrm{r})^3\,e^{-2t} - 0.833(0.9 + 0.1\mathrm{r})^3 e^{-3t} + 8(0.9 + 0.1\mathrm{r})^3\,te^{-t}$$

$$\overline{\tilde{v}}_1 = -2.3333\,(1.1 - 0.1\mathrm{r})^3 + 8\,(1.1 - 0.1\mathrm{r})^3 t - 19.166\,(1.1 - 0.1\mathrm{r})^3 e^{-t} + 4\,(1.1 - 0.1\mathrm{r})^3\,e^{-2t} - 0.833(1.1 - 0.1\mathrm{r})^3 e^{-3t} + 8(1.1 - 0.1\mathrm{r})^3\,te^{-t}$$

$\tilde{v}_{c_1} = -2.3333 + 8\,t - 19.166\,e^{-t} + 4\,e^{-2t} - 0.833e^{-3t} + 8\,te^{-t}$

Then

$$\underline{\tilde{y}}_1(t;r) = 2.3333\,(0.9 + 0.1\mathrm{r})^3 + 2(0.9 + 0.1\mathrm{r}) + 8\,(0.9 + 0.1\mathrm{r})^3 t - \big(19.166\,(0.9 + 0.1\mathrm{r})^3 + (0.9 + 0.1\mathrm{r})\big)e^{-t} + 4\,(0.9 + 0.1\mathrm{r})^3\,e^{-2t} - 0.833(0.9 + 0.1\mathrm{r})^3 e^{-3t} + 8(0.9 + 0.1\mathrm{r})^3\,te^{-t}$$

$$\overline{\tilde{y}}_1(t;r) = -2.3333\,(1.1 - 0.1\mathrm{r})^3 + 2(1.1 - 0.1\mathrm{r}) + 8\,(1.1 - 0.1\mathrm{r})^3 t - \big(19.166\,(1.1 - 0.1\mathrm{r})^3 + 2(1.1 - 0.1\mathrm{r})\big)e^{-t} + 4\,(1.1 - 0.1\mathrm{r})^3\,e^{-2t} - 0.833(1.1 - 0.1\mathrm{r})^3 e^{-3t} + 8(1.1 - 0.1\mathrm{r})^3\,te^{-t}$$

$\tilde{y}_{c_1}(t;r) = -0.3333 + 8\,t - 20.166\,e^{-t} + 4\,e^{-2t} - 0.833e^{-3t} + 8\,te^{-t}$

Chapter Four

Fuzzy Boundary Value Problem

In this chapter, the hybrid method of variation iteration method implemented previously to give approximate solutions for Fuzzy boundary value problems.

4.1 Fuzzy boundary value problem:

Consider the two-point fuzzy boundary value problem:

$$\tilde{y}''(t) = f\big(t, \tilde{y}(t), \tilde{y}'(t)\big), \qquad t \in [0, T] \tag{4.1}$$

subject to the boundary conditions

$$\tilde{y}(0) = \emptyset_1 , \quad \tilde{y}(T) = \emptyset_2 \tag{4.2}$$

where $\tilde{y}(t)$ is a fuzzy function of crisp variable t on,$f : [0, T] \times R_F^2 \to R_F$ is continuous and $\emptyset_1, \emptyset_2 \in R_F$.

Let $\tilde{y}(t)$ is fuzzy valued function on $[0, T]$ represented by $\left[\underline{\tilde{y}}(t, r), \overline{\tilde{y}}(t, r)\right]$. Let $\tilde{y}(t), \tilde{y}'(t)$ be gH-differentiable on $[0, T]$ then equations (4.1) and (4.2) my be transformed to the following boundary value problems (Armand and Gouyandeh 2013), (Khastan and Nieto 2010)

1. if $y(t)$ and $y'(t)$ are (i)-gH-differentiable on $[0, T]$, then

$$\begin{cases} \underline{\tilde{y}}''(t; r) = \underline{f}(t, \tilde{y}(t; r), \tilde{y}'(t, r)) \\ \overline{\tilde{y}}''(t; r) = \overline{f}(t, \tilde{y}(t; r), \tilde{y}'(t, r)) \\ \underline{\tilde{y}}(0; r) = \underline{\emptyset}_1 , \overline{\tilde{y}}(0; r) = \overline{\emptyset}_1 \\ \underline{\tilde{y}}(T; r) = \underline{\emptyset}_2 , \overline{\tilde{y}}(T; r) = \overline{\emptyset}_2 \end{cases} \tag{4.3}$$

2. if $\tilde{y}(t)$ is (i)-gH-differentiable and $\tilde{y}'(t)$ are (ii)-gH-differentiable on $[0, T]$, then

$$\begin{cases} \underline{\tilde{y}}''(t;r) = \overline{f}(t,\tilde{y}(t;r),\tilde{y}'(t,r) \\ \overline{\tilde{y}}''(t;r) = \underline{f}(t,\tilde{y}(t;r),\tilde{y}'(t,r) \\ \underline{\tilde{y}}(0;r) = \underline{\emptyset}_1 , \overline{\tilde{y}}(0;r) = \overline{\emptyset}_1 \\ \underline{\tilde{y}}(T;r) = \underline{\emptyset}_2 , \overline{\tilde{y}}(T;r) = \overline{\emptyset}_2 \end{cases} \tag{4.4}$$

3. if $\tilde{y}(t)$ is (ii)-gH-differentiable and $\tilde{y}'(t)$ are (i)-gH-differentiable on $[0,T]$, then

$$\begin{cases} \underline{\tilde{y}}''(t;r) = \overline{f}(t,\tilde{y}(t;r),y'(t,r) \\ \overline{\tilde{y}}''(t;r) = \underline{f}(t,\tilde{y}(t;r),y'(t,r) \\ \underline{\tilde{y}}(0;r) = \underline{\emptyset}_1 , \overline{\tilde{y}}(0;r) = \overline{\emptyset}_1 \\ \underline{\tilde{y}}(T;r) = \underline{\emptyset}_2 , \overline{\tilde{y}}(T;r) = \overline{\emptyset}_2 \end{cases} \tag{4.5}$$

4. if $y(t)$ and $\tilde{y}'(t)$ are (ii)-gH-differentiable on $[0,T]$, then

$$\begin{cases} \underline{\tilde{y}}''(t;r) = \underline{f}(t,y(t;r),y'(t,r) \\ \overline{\tilde{y}}''(t;r) = \overline{f}(t,y(t;r),y'(t,r) \\ \underline{\tilde{y}}(0;r) = \underline{\emptyset}_1 , \overline{\tilde{y}}(0;r) = \overline{\emptyset}_1 \\ \underline{\tilde{y}}(T;r) = \underline{\emptyset}_2 , \overline{\tilde{y}}(T;r) = \overline{\emptyset}_2 \end{cases} \tag{4.6}$$

Now, Constructing Solution by using FLT (Ahmad et al., 2014)

Taking FLT for equation (4.1)

$$\mathcal{L}\left[\tilde{y}''(t) = \mathcal{L}\tilde{y}[f(t,(t),\tilde{y}'(t))]\right] \tag{4.7}$$

By using the differential properties of Laplace transform and initial conditions

$$s^2\mathcal{L}[\tilde{y}(t)] \ominus s\tilde{y}(0) \ominus \tilde{y}'(0) = \mathcal{L}[f(t,y(t),y'(t))] \tag{4.8}$$

The classical form of FLT is given as follows

$$s^2\mathcal{L}\left[\underline{\tilde{y}}(t;r)\right] \ominus s\underline{\tilde{y}}(0;r) \ominus \underline{\tilde{y}}'(0;r) = \mathcal{L}\left[\underline{f}(t,\tilde{y}(t;r),\tilde{y}'(t;r))\right] \tag{4.9}$$

$$s^2\mathcal{L}\left[\overline{\tilde{y}}(t;r)\right] \ominus s\overline{\tilde{y}}(0;r) \ominus \overline{\tilde{y}}'(0;r) = \mathcal{L}\left[\overline{f}(t,\tilde{y}(t;r),\tilde{y}'(t;r))\right] \tag{4.10}$$

Replace the unknown value $y'(0;r)$ by constant $\gamma = (\gamma_1, \gamma_2)$ where $\gamma_1$in lower case and $\gamma_2$in upper case. Then find these values by applying the boundary conditions in equation (4.2).

To complete the solution of equations (4.1) and (4.2), assume that A(s;r) and B(s;r) are the solution of equations (4.9) and (4.10) respectively; then the above equation becomes

$$\mathcal{L}\left(\underline{\tilde{y}}(t;r)\right) = A(s;r) \quad (4.11)$$

$$\mathcal{L}\left(\overline{\tilde{y}}(t;r)\right) = B(s;r). \quad (4.12)$$

Using Inverse Laplace Transform yields upper and lower solutions for equations (4.1) and (4.2):

$$\left(\underline{\tilde{y}}(t;r)\right) = \mathcal{L}^{-1}\big(A(s;r)\big) \quad (4.13)$$

$$\left(\overline{\tilde{y}}(t;r)\right) = \mathcal{L}^{-1}\big(B(s;r)\big) \quad (4.14)$$

4.2 Hybrid Method with Boundary Conditions:

Constructing solution by using the hybrid method that shown in section (3.2)

First approach:

Consider the following general nonlinear equation

$$\frac{d^2\tilde{y}}{dt^2} + R[\tilde{y}(t)] + N[\tilde{y}(t)] = g(t) \quad (4.15)$$

With boundary conditions as in equation (4.2)

Where R is a linear operator, N is nonlinear operator, and g(t) is a nonhomogeneous

1. After taking the Laplace transform for equation (4.15) yields

$$s^2\mathcal{L}[\tilde{y}(t)] - s\tilde{y}(0) - \tilde{y}'(0) = \mathcal{L}[f(t, \tilde{y}(t), \tilde{y}'(t))] \qquad (4.16)$$

2. using the iteration formula in equation (3.31) and setting $m = 2$ and $\lambda = -\frac{1}{s^2}$, yields

$$\underline{\tilde{y}}_{n+1}(t;r) = \underline{\tilde{y}}_n(t;r) - \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(s^2\underline{\tilde{y}}_n(s) - s\underline{\tilde{y}}(0) - \underline{\tilde{y}}'(0) + \mathcal{L}\left[R\left(\underline{\tilde{y}}_n\right) + N\left(\underline{\tilde{y}}_n\right) - g(t)\right]\right)\right] \qquad (4.17)$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \overline{\tilde{y}}_n(t;r) - \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(s^2\overline{\tilde{y}}_n(s) - s\overline{\tilde{y}}(0) - \overline{\tilde{y}}'(0) + \mathcal{L}\left[R\left(\tilde{\overline{y}}_n\right) + N\left(\tilde{\overline{y}}_n\right) - g(t)\right]\right)\right] \qquad (4.18)$$

Rearrange equations (4.17) ad (4.18), have

$$\underline{\tilde{y}}_{n+1}(t;r) = \mathcal{L}^{-1}\left(\frac{\underline{\tilde{y}}(0)}{s} + \frac{\underline{\tilde{y}}'(0)}{s^2}\right) + \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(\mathcal{L}\left[R\left(\underline{\tilde{y}}_n\right) + N\left(\underline{\tilde{y}}_n\right) - g(t)\right]\right)\right] \qquad (4.19)$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \mathcal{L}^{-1}\left(\frac{\overline{\tilde{y}}(0)}{s} + \frac{\overline{\tilde{y}}'(0)}{s^2}\right) + \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(\mathcal{L}\left[R\left(\tilde{\overline{y}}_n\right) + N\left(\tilde{\overline{y}}_n\right) - g(t)\right]\right)\right] \qquad (4.20)$$

3. using the initial approximation y_0 in equations (3.42) and (3.43) by setting $m = 2$, yields

$$\underline{\tilde{y}}_0(t;r) = \underline{\tilde{y}}(0) + \underline{\tilde{y}}'(0)t \qquad (4.21)$$

$$\overline{\tilde{y}}_0(t;r) = \overline{\tilde{y}}(0) + \overline{\tilde{y}}'(0)t \qquad (4.22)$$

Second approach:

in equation (4.15)Assuming that

$$R[\tilde{y}(t)] = \sum_{i=0}^{2}\left(a_i\tilde{y}^{(i)}(t) + b_i\tilde{y}^{(i)}(t)\right) \qquad (4.23)$$

Where a_i constant and b_i variable coefficient

1. Take the Laplace transform on (4.22), then the iteration formula becomes

$$\tilde{Y}_{n+1}(s;r) = \tilde{Y}_n(s;r) + \lambda\left[s^2\tilde{Y}_n(s;r) - s\tilde{Y}(0) - \tilde{Y}'(0) + \mathcal{L}\left(\sum_{i=0}^{2} a_i\tilde{Y}_n^{(i)}\right) + \mathcal{L}\left(\sum_{i=0}^{2} b_i(t)\tilde{Y}_n^{(i)}\right) + \mathcal{L}\left(N[\tilde{Y}_n] - g(t)\right)\right] \quad (4.24)$$

2. using the iteration formula in equations (3.53) and (3.54) setting $m = 2$ and $\lambda(s) = -\frac{1}{\sum_{i=0}^{2} a_i s^i}$, $a_2 = 1$, yields

$$\underline{\tilde{y}}_{n+1}(t;r) = \underline{\tilde{y}}_n(t;r) + \mathcal{L}^{-1}\lambda\left[s^2\underline{\tilde{y}}_n(S) - s\underline{\tilde{y}}(0) - \underline{\tilde{y}}'(0) + \mathcal{L}\left(\sum_{i=0}^{2} a_i\underline{\tilde{y}}_n^{(i)}\right) + \mathcal{L}\left(\sum_{i=0}^{2} b_i(t)\underline{\tilde{y}}_n^{(i)}\right) + \mathcal{L}\left(N\left[\underline{\tilde{y}}_n\right] - g(t)\right)\right] \quad (4.25)$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \overline{\tilde{y}}_n(t;r) + \mathcal{L}^{-1}\lambda\left[s^2\overline{\tilde{y}}_n(S) - s\overline{\tilde{y}}(0) - \overline{\tilde{y}}'(0) + \mathcal{L}\left(\sum_{i=0}^{2} a_i\overline{\tilde{y}}_n^{(i)}\right) + \mathcal{L}\left(\sum_{i=0}^{2} b_i(t)\overline{\tilde{y}}_n^{(i)}\right) + \mathcal{L}\left(N[\overline{\tilde{y}}_n] - g(t)\right)\right] \quad (4.26)$$

Rearrange equations (4.25) ad (4,26) , yields

$$\underline{\tilde{y}}_{n+1}(t;r) = \underline{\tilde{y}}_0 + \mathcal{L}^{-1}\left(\sum_{i=0}^{2} b_i(t)\underline{\tilde{y}}_n^{(i)}\right) + N\left[\underline{\tilde{y}}_n\right] \quad (4.27)$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \overline{\tilde{y}}_0 + \mathcal{L}^{-1}\left(\sum_{i=0}^{2} b_i(t)\overline{\tilde{y}}_n^{(i)}\right) + N[\overline{\tilde{y}}_n] \quad (4.28)$$

Where the initial iteration value can be determined as

$$\tilde{y}_0(t;r) = \mathcal{L}^{-1}\left[\lambda(s)\left[-s\underline{\tilde{y}}(0) - \underline{\tilde{y}}'(0) + \mathcal{L}\left(\sum_{i=0}^{m-1} a_i\underline{\tilde{y}}_n^{(i)}\right) + +\mathcal{L}[g(t)]\right]\right] \quad (4.29)$$

$$\tilde{y}_0(t;r) = \mathcal{L}^{-1}\left[\lambda(s)\left[-s\overline{\tilde{y}}(0) - \overline{\tilde{y}}'(0) + \mathcal{L}\left(\sum_{i=0}^{m-1} a_i\overline{\tilde{y}}_n^{(i)}\right) + +\mathcal{L}[g(t)]\right]\right] \quad (4.30)$$

3. Let $y_n = \sum_{j=0}^{n} v_j$ and apply the Adomian decomposition method (ADM) to expand the term $N[y_n]$ as $\sum_{j=0}^{n} A_j$ then the iteration formula

$$\begin{cases} v_{j+1} = \mathcal{L}^{-1}\left(\mathcal{L}\left(\sum_{i=0}^{2} b_i(t) v_j^{(i)}\right) + A_j\right) \\ \quad v_0 = \tilde{y}_0 \end{cases} \tag{4.31}$$

Where A_j is the famous Adomian decomposition series.

Now, replace the unknown value $y'(0)$ by constant γ_1 in lower case and γ_2 in upper case. Then find these values by applying the boundary conditions in equations (4.2).

4.3 Numerical example:

Example (4.1):

The example solved in (Guo, Shang, and Lu 2013) can be resolved by the hybrid VIM that proposed by this work.

Consider the following second-order linear fuzzy differential equation.

$$\begin{cases} \tilde{y}''(t) - 4\tilde{y}' + 4\tilde{y} = 4t - 4 \qquad t \geq 0 \\ \tilde{y}(0) = (3 + 2r, 6 - 5r) \\ \tilde{y}(1) = ((-1 + 5r)e^2 + 1, (7 - 5r)e^2 + 1) \end{cases} \tag{4.32}$$

with exact solutions

$$\underline{\tilde{y}}(t, r) = (-1 + r)te^{2t} + (1 + r)e^{2t} + t$$

$$\overline{\tilde{y}}(t, r) = (1 - r)te^{2t} + (2 - r)e^{2t} + t$$

$$\tilde{y}_c(t, r) = (-5 + 3r)te^{2t} + (4.5 - 1.5r)e^{2t} + t$$

First approach:

The solution of equation (4.32) is using the hybrid method is as follows:

Take the Laplace transform to the equation yields:

$$s^2 y(s) - sy(0) - y'(0) = \mathcal{L}[4y' - 4y + 4t - 4] \tag{4.33}$$

Obtain the iteration formula of equation (4.33) yields:

$$\underline{\tilde{Y}}_{n+1}(s;r) = \underline{\tilde{Y}}_n(s;r) + \underline{\lambda}(s)\left[s^2\underline{\tilde{Y}}_n(s) - s\underline{\tilde{Y}}(0) - \underline{\tilde{Y}}'(0) + \mathcal{L}[-4\underline{\tilde{Y}}_n' + 4\underline{\tilde{Y}}_n - 4t + 4]\right] \quad (4.34)$$

$$\overline{\tilde{Y}}_{n+1}(s;r) = \overline{\tilde{Y}}_n(s;r) + \overline{\lambda}(s)\left[s^2\overline{\tilde{Y}}_n(s) - s\overline{\tilde{Y}}(0) - \overline{\tilde{Y}}'(0) + \mathcal{L}[-4\overline{\tilde{Y}}_n' + 4\overline{\tilde{Y}}_n - 4t + 4]\right] \quad (4.35)$$

$$\tilde{Y}_{c_{n+1}}(s;r) = \tilde{Y}_{c_n}(s;r) + \lambda_c(s)\left[s^2\tilde{Y}_{c_n}(s) - s\tilde{Y}_c(0) - \tilde{Y}_c{}'(0) + \mathcal{L}[-4\tilde{Y}_{c_n}{}' + 4\tilde{Y}_{c_n} - 4t + 4]\right] \quad (4.36)$$

Consider the term $(\mathcal{L}[-4y_n' + 4y_n - 4t + 4])$ in the above equations as restricted variation, then

$$\delta\underline{\tilde{Y}}_{n+1}(s;r) = \delta\underline{\tilde{Y}}_n(s;r) + \underline{\lambda}(s)(s^2\delta\underline{\tilde{Y}}_n(s;r))$$

$$\delta\overline{\tilde{Y}}_{n+1}(s;r) = \delta\overline{\tilde{Y}}_n(s;r) + \overline{\lambda}(s)(s^2\delta\overline{\tilde{Y}}_n(s;r))$$

$$\delta\tilde{Y}_{c_{n+1}}(s;r) = \delta\tilde{Y}_{c_n}(s;r) + \lambda_c(s)(s^2\delta\tilde{Y}_{c_n}(s;r))$$

then

$$\underline{\lambda}(s) = \overline{\lambda}(s) = \lambda_c(s) = -\frac{1}{s^2}$$

Taking the inverse Laplace transform to equations (4.34), (4.35) and (4.36):

$$\underline{\tilde{y}}_{n+1}(t;r) = \underline{\tilde{y}}_n(t;r) - \mathcal{L}^{-1}\left[\frac{1}{s^2}\left[s^2\underline{\tilde{y}}_n(s) - s\underline{\tilde{y}}(0) - \underline{\tilde{y}}'(0) + \mathcal{L}\left[-4\underline{\tilde{Y}}_n' + 4\underline{\tilde{Y}}_n - 4t + 4\right]\right]\right]$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \overline{\tilde{y}}_n(t;r) - \mathcal{L}^{-1}\left[\frac{1}{s^2}\left[s^2\overline{\tilde{y}}_n(s) - s\overline{\tilde{y}}(0) - \overline{\tilde{y}}'(0) + \mathcal{L}\left[-4\overline{\tilde{Y}}_n' + 4\overline{\tilde{Y}}_n - 4t + 4\right]\right]\right]$$

$$\tilde{y}_{c_{n+1}}(t;r) = \tilde{y}_{c_n}(t;r) - \mathcal{L}^{-1}\left[\frac{1}{s^2}\left[s^2\tilde{y}_{c_n}(s) - s\tilde{y}_c(0) - \tilde{y}_c{}'(0) + \mathcal{L}\left[-4\tilde{Y}_{c_n}' + 4\tilde{Y}_{c_n} - 4t + 4\right]\right]\right]$$

then

$$\underline{\tilde{y}}_{n+1}(t;r) = \mathcal{L}^{-1}\left(\frac{\underline{\tilde{y}}(0)}{s} + \frac{\underline{\tilde{y}}'(0)}{s^2}\right) + \mathcal{L}^{-1}\left[\frac{1}{s^2}\left[\mathcal{L}\left[-4\underline{\tilde{Y}}_n' + 4\underline{\tilde{Y}}_n - 4t + 4\right]\right]\right]$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \mathcal{L}^{-1}\left(\frac{\overline{\tilde{y}}(0)}{s} + \frac{\overline{\tilde{y}}'(0)}{s^2}\right) + \mathcal{L}^{-1}\left[\frac{1}{s^2}\left[\mathcal{L}\left[-4\overline{\tilde{Y}}_n' + 4\overline{\tilde{Y}}_n - 4t + 4\right]\right]\right]$$

$$\tilde{y}_{c_{n+1}}(t;r) = \mathcal{L}^{-1}\left(\frac{\tilde{y}_c(0)}{s} + \frac{\tilde{y}_c{}'(0)}{s^2}\right) + \mathcal{L}^{-1}\left[\frac{1}{s^2}\left[\mathcal{L}\left[-4\tilde{Y}_{c_n}' + 4\tilde{Y}_{c_n} - 4t + 4\right]\right]\right]$$

Therefore,

$$\underline{\tilde{y}}_{n+1}(t;r) = \underline{\tilde{y}}(0) + \underline{\tilde{y}}'(0)t + \mathcal{L}^{-1}\left[\frac{1}{s^2}\left[\mathcal{L}\left[-4\underline{\tilde{Y}}_n' + 4\underline{\tilde{Y}}_n - 4t + 4\right]\right]\right] \quad (4.37)$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \overline{\tilde{y}}(0) + \overline{\tilde{y}}'(0)t + \mathcal{L}^{-1}\left[\frac{1}{s^2}\left[\mathcal{L}\left[-4\overline{\tilde{Y}}_n' + 4\overline{\tilde{Y}}_n - 4t + 4\right]\right]\right] \quad (4.38)$$

$$\tilde{y}_{c_{n+1}}(t;r) = \tilde{y}_c(0) + \tilde{y}_c{}'(0)t + \mathcal{L}^{-1}\left[\frac{1}{s^2}\left[\mathcal{L}\left[-4\tilde{Y}_{c_n}' + 4\tilde{Y}_{c_n} - 4t + 4\right]\right]\right] \quad (4.39)$$

With initial iteration

$$\underline{\tilde{y}}_0(t;r) = \underline{\tilde{y}}(0) + \underline{\tilde{y}}'(0)t$$

$$\overline{\tilde{y}}_0(t;r) = \overline{\tilde{y}}(0) + \overline{\tilde{y}}'(0)t$$

$$\tilde{y}_{c_0}(t;r) = \tilde{y}_c(0) + \tilde{y}_c{}'(0)t$$

then

$$\underline{\tilde{y}}_0(t;r) = (3 + 2r) + [(-2 - e^2) + (5e^2 - 2)r]t$$

$$\overline{\tilde{y}}_0(t;r) = (6 - 5r) + [(7e^2 - 5) + (5 - 5e^2)r]t$$

$\tilde{y}_{c_0}(t;r) = (4.5 - 1.5r) + (3e^2 - 3.5 + 1.5r)t$

$\underline{\tilde{y}}_1(t;r) =$

$(3 + 2r) + \left[(-2 - e^2) + \left(5e^2 - 2\right)r\right]t + \frac{4\left[(-2-e^2)+\left(5e^2-2\right)r-(3+2r)-1\right]}{2!}t^2$

$-\frac{4\left[(-2-e^2)+(5e^2-2)r+1\right]}{3!}t^3$

$\overline{\tilde{y}}_1(t;r) =$

$(6 - 5r) + [(7e^2 - 5) + (5 - 5e^2)r]t + \frac{4[(7e^2-5)+(5-5e^2)r-(6-5r)-1]}{2!}t^2 -$
$\frac{4[(7e^2-5)+(5-5e^2)r+1]}{3!}t^3$

$\tilde{y}_{c_1}(t;r) =$

$(4.5 - 1.5r) + (3e^2 - 3.5 + 1.5r)t + \frac{4[(4.5-1.5r)+(3e^2-3.5+1.5r)-1]}{2!}t^2 -$
$\frac{4(3e^2-2.5+1.5r)}{3!}t^3$

$\underline{\tilde{y}}_2(t;r) =$

$(3 + 2r) + [(-2 - e^2) + (5e^2 - 2)r]t + \frac{4[(-2-e^2)+(5e^2-2)r-(3+2r)-1]}{2!}t^2 +$
$\frac{12[(-2-e^2)+(5e^2-2)r+1]-16(3+2r)}{3!}t^3 - \frac{16[(-2-e^2)+(5e^2-2)r]+8(3+2r)}{4!}t^4 +$
$\frac{16[(-2-e^2)+(5e^2-2)r+1]}{3!5!}t^5$

$\overline{\tilde{y}}_2(t;r) =$

$(6 - 5r) + [(7e^2 - 5) + (5 - 5e^2)r]t + \frac{4[(7e^2-5)+(5-5e^2)r-(6-5r)-1]}{2!}t^2 +$
$\frac{12[(7e^2-5)+(5-5e^2)r-1]-16(6-5r)}{3!}t^3 - \frac{16[(7e^2-5)+(5-5e^2)r]-8(6-5r)}{4!}t^4 -$
$\frac{4[(7e^2-5)+(5-5e^2)r-1]}{3!5!}t^5$

$\tilde{y}_{c_1}(t;r) =$

$(4.5 - 1.5r) + (3e^2 - 3.5 + 1.5r)t + \frac{4[(4.5-1.5r)+(3e^2-3.5+1.5r)-1]}{2!}t^2 -$
$\frac{4(3e^2-2.5+1.5r)-16(4.5-1.5r)}{3!}t^3 - \frac{16(3e^2-3.5+1.5r)-8(4.5-1.5r)}{4!}t^4 + \frac{16(3e^2-2.5+1.5r)}{5!}t^5$

⋮

Second approach:

Take the Laplace transform the equation (4.32) yields:

$$s^2y(s) - sy(0) - y'(0) - 4sy(s) + 4y(0) + sy(s) = \mathcal{L}[4t - 4] \qquad (4.40)$$

Obtain the iteration formula of equation (4.40) yields:

$$\underline{\widetilde{Y}}_{n+1}(s;r) = \underline{\widetilde{Y}}_{n}(s;r) + \underline{\lambda}(s)\left[s^2\underline{\widetilde{Y}}_{n}(s;r) - s\underline{\widetilde{Y}}(0) - \underline{\widetilde{Y}}'(0) + -4s\underline{\widetilde{Y}}_{n}(s;r) + 4\underline{\widetilde{Y}}(0) + 4\underline{\widetilde{Y}}_{n}(s;r) - \mathcal{L}[4t-4]\right] \qquad (4.41)$$

$$\overline{\widetilde{Y}}_{n+1}(s;r) = \overline{\widetilde{Y}}_{n}(s;r) + \overline{\lambda}(s)\left[s^2\overline{\widetilde{Y}}_{n}(s;r) - s\overline{\widetilde{Y}}(0) - \overline{\widetilde{Y}}'(0) + -4s\overline{\widetilde{Y}}_{n}(s;r) + 4\overline{\widetilde{Y}}(0) + 4\overline{\widetilde{Y}}_{n}(s;r) - \mathcal{L}[4t-4]\right] \qquad (4.42)$$

$$\widetilde{Y}_{c_{n+1}}(s;r) = \widetilde{Y}_{c_n}(s;r) + \lambda_c(s)\left[s^2\widetilde{Y}_{c_n}(s;r) - s\widetilde{Y}_c(0) - \widetilde{Y}_c{}'(0) + -4s\widetilde{Y}_{c_n}(s;r) + 4\widetilde{Y}_c(0) + 4\widetilde{Y}_{c_n}(s;r) - \mathcal{L}[4t-4]\right] \qquad (4.43)$$

Consider the term ($\mathcal{L}[-4t + 4]$) in the above equations as restricted variation, then

$$\delta\underline{\widetilde{Y}}_{n+1}(s;r) = \delta\underline{\widetilde{Y}}_{n}(s;r) + \underline{\lambda}(s)((s^2 - 4s + 4)\delta\underline{\widetilde{Y}}_{n}(s;r))$$

$$\delta\overline{\widetilde{Y}}_{n+1}(s;r) = \delta\overline{\widetilde{Y}}_{n}(s;r) + \overline{\lambda}(s)((s^2 - 4s + 4)\delta\overline{\widetilde{Y}}_{n}(s;r))$$

$$\delta\widetilde{Y}_{c_{n+1}}(s;r) = \delta\widetilde{Y}_{c_n}(s;r) + \lambda_c(s)((s^2 - 4s + 4)\delta\widetilde{Y}_{c_n}(s;r))$$

then:

$$\underline{\lambda}(s) = \overline{\lambda}(s) = \lambda_c(s) = -\frac{1}{s^2 - 4s + 4}$$

Substitute $\lambda(s)$ in equations (4.41), (4.42) and (4.43) and taking the Laplace inverse yields:

$$\underline{\widetilde{y}}_{n+1}(t;r) = \left[\frac{\underline{\widetilde{y}}'(0)}{(s-2)^2} + \underline{\widetilde{y}}(0)\left[\frac{1}{(s-2)} - \frac{2}{(s-2)^2}\right] + \mathcal{L}^{-1}[\frac{1}{s^2} - \frac{1}{(s-2)^2}]\right] \qquad (4.44)$$

$$\overline{\tilde{y}}_{n+1}(t;r) = \left[\frac{\overline{\tilde{y}}'(0)}{(s-2)^2} + \overline{\tilde{y}}(0)\left[\frac{1}{(s-2)} - \frac{2}{(s-2)^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2} - \frac{1}{(s-2)^2}\right]\right] \qquad (4.45)$$

$$\tilde{y}_{c_{n+1}}(t;r) = \left[\frac{\tilde{y}_c{}'(0)}{(s-2)^2} + \tilde{y}_c(0)\left[\frac{1}{(s-2)} - \frac{2}{(s-2)^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2} - \frac{1}{(s-2)^2}\right]\right] \qquad (4.46)$$

With initial iteration

$$\underline{\tilde{y}}_0(t;r) = \left(\underline{\tilde{y}}'(0) - 2\underline{\tilde{y}}(0)\right)te^{2t} + e^{2t}\underline{\tilde{y}}(0)$$

$$\overline{\tilde{y}}_0(t;r) = \left(\overline{\tilde{y}}'(0) - 2\overline{\tilde{y}}(0)\right)te^{2t} + e^{2t}\overline{\tilde{y}}(0)$$

$$\tilde{y}_{c_0}(t;r) = \left(\tilde{y}_c{}'(0) - 2\tilde{y}_c(0)\right)te^{2t} + e^{2t}\tilde{y}_c(0)$$

then

$$\underline{\tilde{y}}_0(t;r) = (-4 - 3 + e^{-2})te^{2t} + (3 + 2r)e^{2t}$$

$$\underline{\tilde{y}}_1(t;r) = (-6 - 3 + e^{-2})te^{2t} + (3 + 2r)e^{2t} + t$$

$$\overline{\tilde{y}}_0(t;r) = (1 + e^{-2})te^{2t} + (6 - 5r)e^{2t}$$

$$\overline{\tilde{y}}_1(t;r) = (-1 + e^{-2})te^{2t} + (6 - 5r)e^{2t} + t$$

$$\tilde{y}_{c_0}(t;r) = (1.5 + 1.5 + e^{-2})te^{2t} + (4.5 - 1.5r)e^{2t}$$

$$\tilde{y}_{c_1}(t;r) = (-0.5 + 1.5 + e^{-2})te^{2t} + (4.5 - 1.5r)e^{2t} + t$$

$$\underline{\tilde{y}}_1(t;r) = \underline{\tilde{y}}_2(t;r) = \underline{\tilde{y}}_2(t;r) = \cdots = \underline{\tilde{y}}_n(t;r)$$

$$\overline{\tilde{y}}_1(t;r) = \overline{\tilde{y}}_2(t;r) = \overline{\tilde{y}}_3(t;r) = \cdots = \overline{\tilde{y}}_n(t;r)$$

$$\tilde{y}_{c_1}(t;r) = \tilde{y}_{c_2}(t;r) = \tilde{y}_{c_3}(t;r) = \cdots = \tilde{y}_{c_n}(t;r)$$

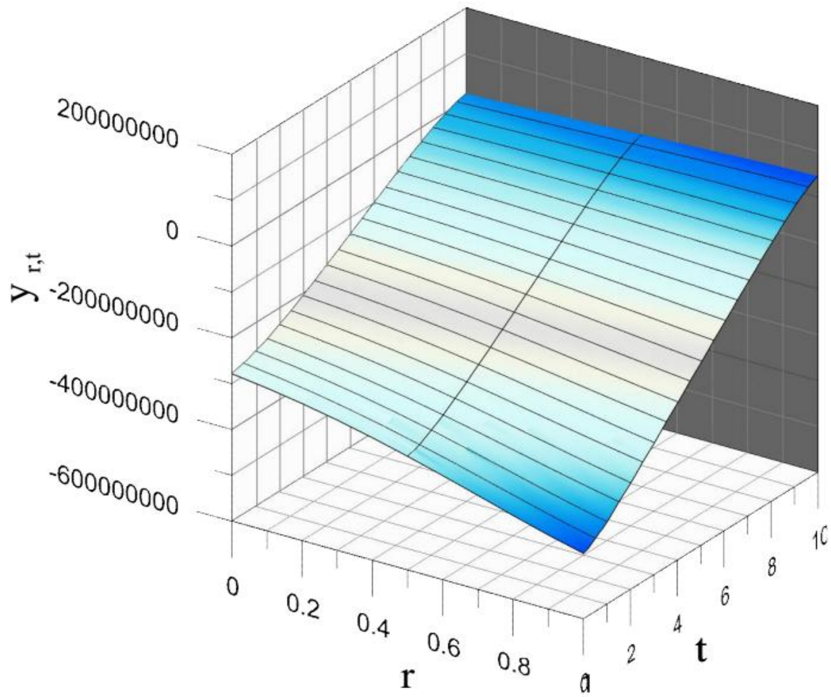

Figure (4.1): The exact solution of $\underline{\tilde{y}}(t;r)$

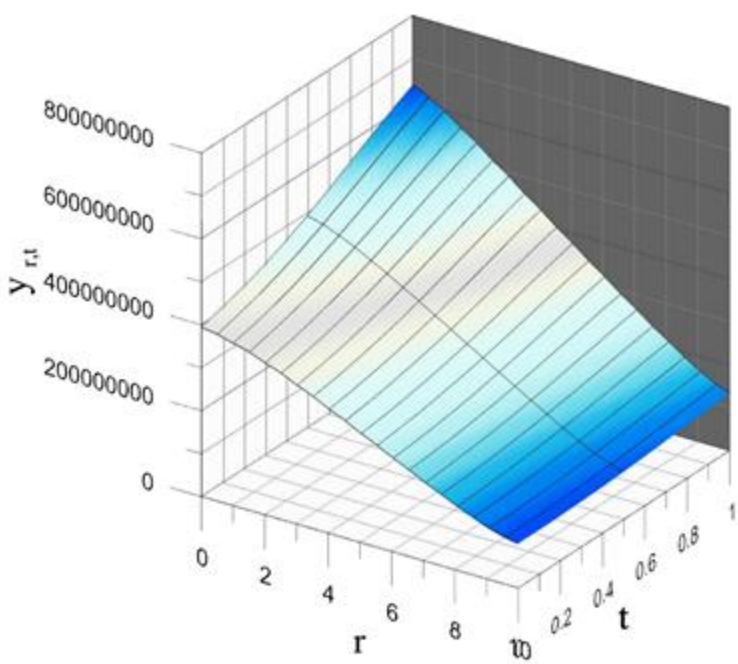

Figure (4.2): the exact solution of $\overline{\tilde{y}}(t;r)$

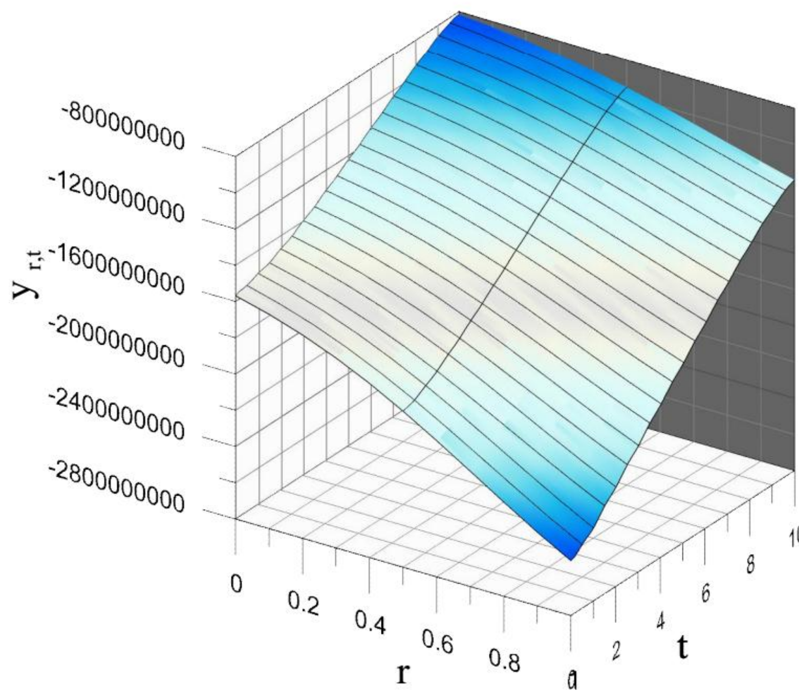

Figure (4.3): the exact solution of $\widetilde{y}_c(t;r)$

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Conclusions and Future Work

The fuzzy Laplace transformation and fuzzy VIM are used to obtain the approximate solution of linear and nonlinear fuzzy differential equations. In this work, two approaches are used according to the type of the identification of the Lagrange multipliers, where the second approach consumes more simple mathematical operations. Comparing with the exact solutions our result had been shown more accurate with less number of iterations compared with the results obtained in (Jafari, Saeidy, and Baleanu 2011), (Jameel 2014), (Guo, Shang, and Lu 2013).

Also we recommend the following problems for future work:

1. Using the proposed approximate methods for solving fuzzy differential equations with fuzzy coefficients.
2. Using our Hybrid method for solving a system of fuzzy differential equations.
3. Using our Hybrid method for solving partial fuzzy differential equations.

References

Adomian G.,"A Review of the Decomposition Method in Applied Mathematics", Journal of Mathematical Analysis and Applications 135 (2): 501–44, 1988.

Ahmad L., Muhammad F. and Saleem A., "Solving Nth Order Fuzzy Differential Equation by Fuzzy Laplace Transform", arXiv:1403.0242v1 [math.GM], 2 Mar 2014.

Ahmad L., Muhammad F., Saif U. and Saleem A.,"Solving Fuzzy Two-Point Boundary Value Problem Using Fuzzy Laplace Transform", arXiv:1403.0571v1 [math.GM], 2 Mar 2014.

Allahviranloo T. and Barkhordari A., "Fuzzy Laplace Transforms" Soft Computing 14 (3): 235, 2010.

Armand A. and Gouyandeh Z., "Solving Two-Point Fuzzy Boundary Value Problem Using the Variational Iteration Method", Communications on Advanced Computational Science with Applications 2013: 1–10.

Babolian E., Sadeghi H. and Javadi Sh., "Numerically Solution of Fuzzy Differential Equations by Adomian Method", Applied Mathematics and Computation 149 (2): 547–57, 2004.

Bector C. and Suresh C., "Fuzzy Mathematical Programming and Fuzzy Matrix Games", Vol. 169. Springer, 2005.

Chen M., Wu C., Xue X. and Liu G., "On Fuzzy Boundary Value Problems", Information Sciences, 178 (7): 1877–92, 2008.

Diamond P. and Peter K., "Metric Topology of Fuzzy Numbers and Fuzzy Analysis", In: Dubois D., Prade H. (eds) Fundamentals of Fuzzy Sets. The Handbooks of Fuzzy Sets Series, vol 7. Springer, Boston, MA, 2000.

Douglas J., “Solution of the Inverse Problem of the Calculus of Variations”, Transactions of the American Mathematical Society 50 (1): 71–128, 1941.

Dubois D., “Fuzzy Sets and Systems: Theory and Applications”, Academic Press, 1980.

Fard O., Borzabadi A. and Heidari M.,“On Fuzzy Euler-Lagrange Equations”, Ann. Fuzzy Math. Inform 7 (3): 447–461, 2014.

Gao S., Zhang Z. and Cao C., “Multiplication Operation on Fuzzy Numbers”, Journal of Software 4 (4): 331–338, 2009.

Goetschel R. and Voxman W., “Elementary Fuzzy Calculus”, Fuzzy Sets and Systems 18 (1): 31–43, 1986.

Grabiec M., “Fixed Points in Fuzzy Metric Spaces”, Fuzzy Sets and Systems 27 (3): 385–89, 1988.

Guo X., Shang D. and Lu X., “Fuzzy Approximate Solutions of Second-Order Fuzzy Linear Boundary Value Problems”, Boundary Value Problems (1): 212, 2013.

JH He, “Variational Iteration Method – a Kind of Non-Linear Analytical Technique: Some Examples.” International Journal of Non-Linear Mechanics 34 (4): 699–708, 1999.

JH He, “Some Asymptotic Methods for Strongly Nonlinear Equations”, International Journal of Modern Physics B 20 (10): 1141–99, 2006.

Jafari H. and Alipoor A., “A New Method for Calculating General Lagrange Multiplier in the Variational Iteration Method”, Numerical Methods for Partial Differential Equations 27 (4): 996–1001, 2011.

Jafari H., Saeidy M. and Baleanu D., “The Variational Iteration Method for Solving N-Th Order Fuzzy Differential Equations”, Open Physics 10 (1): 76–85, 2011.

Jameel A., “The Variational Iteration Method for Solving Fuzzy Duffing’s Equation”, Journal of Interpolation and Approximation in Scientific Computing, Volume 2014 (2014), Article ID jiasc-00053, 14 Pages.

Kandel A., “Fuzzy Mathematical Techniques with Applications”, Addison-Wesley, Reading, MA, xiv+ 274 pp. ISBN 0-201-11752-5, 1986.

Khastan A. and Juan J., “A Boundary Value Problem for Second Order Fuzzy Differential Equations”, Nonlinear Analysis: Theory, Methods & Applications 72 (9–10): 3583–93, 2010.

Lakshmikantham V., Murty K. and Turner J., “Two-Point Boundary Value Problems Associated with Non-Linear Fuzzy Differential Equations”, Mathematical Inequalities & Applications, no. 4: 527–33, 2001.

Matinfar M., Saeidy M. and Ghasemi M., “The Combined Laplace-Variational Iteration Method for Partial Differential Equations”, International Journal of Nonlinear Sciences and Numerical Simulation 14 (2): 93–101, 2013.

Nucci M. and Arthurs A., “On the Inverse Problem of Calculus of Variations for Fourth-Order Equations”, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, March, rspa20090618, 2010.

O’Regan D., Lakshmikantham V. and Nieto J., “Initial and Boundary Value Problems for Fuzzy Differential Equations”, Nonlinear Analysis: Theory, Methods & Applications 54 (3): 405–15, 2003.

Pal S. and Majumder D., “Fuzzy Mathematical Approach to Pattern Recognition”, New York, NY, USA: Halsted Press, 1986.

Prk, J. Y. and Han, H. K.: "Existence and Uniqueness Theorem for a Solution of Fuzzy Differential Equations", Internet J. Math and Math. Sci, Vol.22, No.2, (1999), pp.271-279.

Salahshour S. and Allahviranloo T.,“Applications of Fuzzy Laplace Transforms”, Soft Computing 17 (1): 145–58, 2013.

Seikkala S., “On the Fuzzy Initial Value Problem”, Fuzzy Sets and Systems, Fuzzy Numbers, 24 (3): 319–30, 1987.

Wazwaz A., “The Variational Iteration Method for Solving Linear and Nonlinear Volterra Integral and Integro-Differential Equations”, International Journal of Computer Mathematics 87 (5): 1131–41, 2010.

Wu G., “Challenge in the Variational Iteration Method – A New Approach to Identification of the Lagrange Multipliers”, Journal of King Saud University - Science 25 (2): 175–78, 2013.

Wu G. and Baleanu D., “Variational Iteration Method for Fractional Calculus - a Universal Approach by Laplace Transform”, Advances in Difference Equations 2013 (1): 18, 2013.

Zadeh L. “Fuzzy Sets”, Information and Control 8 (3): 338–53, 1965.

الملخص

في هذا البحث تم أستنباط أسلوب هجيني من خلال ربط طريقتي تحويلات لابلاس الضبابية (FLT) وطريقة التغاير التكرارية الضبابية (FVIT) لحل المعادلات التفاضلية الخطيه وغير الخطيه ذات القيم الابتدائية الضبابية و الحدودية الضبابية.

و قد تم تنفيذ أسلوبنا الهجيني من خلال أستنباط أسلوبين حسب العلاقة الرياضية في أستخراج مضروبات لاكرانج للحصول على الحلول. النتائج التجربيه التي تم الحصول عليها تظهر دقه عاليه مع عدد اقل من التكرارات.

جمهوريه العراقيه

وزارة التعليم العالي والبحث العلمي

جامة النهرين

كلية العلوم

قسم الرياضيات وتطبيقات الحاسوب

طريقة مهجنه لحل المعادلات التفاضلية الضبابية من الرتبة الثانية

رسالة

مقدمة الى مجلس كلية العلوم / جامعة النهرين

كجزء من متطلبات نيل درجة الماجستير في علوم الرياضيات

من قبل

بسمه عبد الهادي نعمه

(بكالوريوس علوم /جامعة النهرين، 2013)

اشراف

أ. د. علاء الدين نوري احمد

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